Matrix notation*

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Consider the definition of a linear projection model with the linear projection coefficient β, the unique minimizer such that $\beta = \arg\min_\beta \mathbb{E}(y - x'\beta)$ under Assumption 2.1[1].

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}, y = x'\beta + e$$

$$\mathbb{E}(xe) = 0$$

$$\beta = \mathbb{E}xx'^{-1}\mathbb{E}xy$$

Assume we have n sample for (x, y). Denote ith observation of x as $x_i, 1 \leq i \leq n$. Also define the corresponding matrix X stacking x_i.

$$x_i = \begin{pmatrix} x_{1,i} \\ x_{2,i} \\ \vdots \\ x_{k,i} \end{pmatrix}, X = \begin{pmatrix} x_{1,1} & x_{2,1} & \cdots & x_{k,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{k,2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1,i} & x_{2,i} & \cdots & x_{k,i} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{k,n} \end{pmatrix} = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_i \\ \vdots \\ x'_n \end{pmatrix}$$

Defining the projection error e for each x_i denoted ϵ, and stacking n observations of y_i column-wise, we have $Y = [y_1, y_2, \cdots y_n]'$. Very often, it is convenient to write the model and the estimator using matrix notation involving X and Y. Matrices can make notations for theory concise as well as making computation of quantities simple. For a sample point i, we have $y_i = x'_i\beta + e_i$. The Least Squares Estimator is $\hat{\beta}_{OLS} = (\sum_{i=1}^n x_i x'_i)^{-1}(\sum_{i=1}^n x_i y_i)$. Stacking observations column-wise

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[1] $\mathbb{E}y^2 < \infty, \mathbb{E}\|x\|^2 < \infty, \mathbb{E}xx'$ positive definite; Assumption 2.1 of Hansen (2020).

and using the definition of X and Y, we have the following.

$$Y = \begin{pmatrix} y_1 \\ y_i \\ \vdots \\ y_n \end{pmatrix} = X\beta + \epsilon = \begin{pmatrix} x_{1,1} & x_{2,1} & \cdots & x_{k,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{k,2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{k,n} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

Lemma 1. $\sum_{i=1}^{n} x_i x_i' = X'X$

Proof. Using simple algebra, it is clear that (1) and (2) are identical.

$$\sum_{i=1}^{n} x_i x_i' = \sum_{i=1}^{n} \begin{pmatrix} x_{1,i} \\ x_{2,i} \\ \vdots \\ x_{k,i} \end{pmatrix} \begin{pmatrix} x_{1,i} & x_{2,i} & \cdots & x_{k,i} \end{pmatrix} = \sum_{i=1}^{n} \begin{pmatrix} x_{1,i}^2 & x_{1,i}x_{2,i} & \cdots & x_{1,i}x_{k,i} \\ x_{2,i}x_{1,i} & x_{2,i}^2 & \cdots & x_{2,i}x_{k,i} \\ \vdots & \vdots & \vdots & \vdots \\ x_{k,i}x_{1,i} & x_{k,i}x_{2,i} & \cdots & x_{k,i}^2 \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{n} x_{1,i}^2 & \sum_{i=1}^{n} x_{1,i}x_{2,i} & \cdots & \sum_{i=1}^{n} x_{1,i}x_{k,i} \\ \sum_{i=1}^{n} x_{2,i}x_{1,i} & \sum_{i=1}^{n} x_{2,i}^2 & \cdots & \sum_{i=1}^{n} x_{2,i}x_{k,i} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^{n} x_{k,i}x_{1,i} & \sum_{i=1}^{n} x_{k,i}x_{2,i} & \cdots & \sum_{i=1}^{n} x_{k,i}^2 \end{pmatrix} \tag{1}$$

$$X'X = \begin{pmatrix} x_{1,1} & x_{2,1} & \cdots & x_{k,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{k,2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1,i} & x_{2,i} & \cdots & x_{k,i} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{k,n} \end{pmatrix}' \begin{pmatrix} x_{1,1} & x_{2,1} & \cdots & x_{k,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{k,2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1,i} & x_{2,i} & \cdots & x_{k,i} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{k,n} \end{pmatrix}$$

$$= \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,i} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,i} & \cdots & x_{2,n} \\ \vdots & \vdots & \vdots & \vdots & & \\ x_{k,1} & x_{k,2} & \cdots & x_{k,i} & \cdots & x_{k,n} \end{pmatrix} \begin{pmatrix} x_{1,1} & x_{2,1} & \cdots & x_{k,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{k,2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1,i} & x_{2,i} & \cdots & x_{k,i} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{k,n} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{n} x_{1,i}^2 & \sum_{i=1}^{n} x_{1,i}x_{2,i} & \cdots & \sum_{i=1}^{n} x_{1,i}x_{k,i} \\ \sum_{i=1}^{n} x_{2,i}x_{1,i} & \sum_{i=1}^{n} x_{2,i}^2 & \cdots & \sum_{i=1}^{n} x_{2,i}x_{k,i} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^{n} x_{k,i}x_{1,i} & \sum_{i=1}^{n} x_{k,i}x_{2,i} & \cdots & \sum_{i=1}^{n} x_{k,i}^2 \end{pmatrix} \tag{2}$$

Q.E.D.

References

Hansen, B. (2020): "ECONOMETRICS," *Department of Economics, University of Wisconsin.*