Feedback Linearization

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1 Introduction

Consider a class of single-input-single-output (SISO) nonlinear systems of the form

$$\dot{x} = f(x) + g(x)u \tag{1}$$
$$y = h(x) \tag{2}$$

where $x \in \mathcal{D} \subset \mathbb{R}^n$, $u \in \mathbb{R}^1$, $f : \mathcal{D} \to \mathbb{R}^n$, $g : \mathcal{D} \to \mathbb{R}^n$, and the domain $\mathcal{D}$ contains the origin. [1]

In this lecture, we shall answer the following two (tightly related) questions:

1. Does there exist a nonlinear change of variables $z = \begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x)$, and a control input $u = \alpha(x) + \beta(x)v$ that would transform the system (1)-(2) into the following partially linear form?
$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi)\\ \dot{\xi} &= A\xi + Bv\\ y &= C\xi\end{aligned}$$

2. Does there exist a nonlinear change of variables $z = T(x)$ and a control input $u = \alpha(x) + \beta(x)v$ that would transform the system (1) into the following fully linear form?
$$\dot{z} = \tilde{A}z + \tilde{B}v$$

[1] This condition is not a limitation to the theory of feedback linearization and the results can be extended to the multiple-input-multiple-output (MIMO) cases

If the answer to question 2. is positive, then we say that the system (1) is feedback linearizable. If the answer to question 1. is positive, then we say that the system (1)-(2) is input-output linearizable. Both scenarios are very attractive from the control design point of view, since in either case we can rely on linear design techniques to render the closed-loop system stable.

Since we cannot expect every nonlinear system to possess such properties of feedback linearization, it is interesting to understand the structural properties that the nonlinear system should possess that render it feedback linearizable.

Before we dive into the theoretical developments of feedback linearization, let us first look into two simple examples.

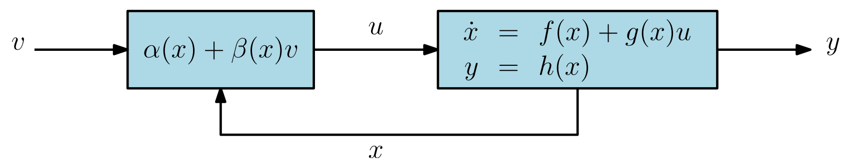

Figure 1: General idea of feedback linearization

Example 1 *Consider the example of the pendulum that we have seen in previous lectures. The dynamics are given by*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\left[\sin\left(x_1 + \delta\right) - \sin\delta\right] - bx_2 + cu \end{aligned}$$

If we choose the control

$$u = \frac{a}{c}\left[\sin\left(x_1 + \delta\right) - \sin\delta\right] + \frac{v}{c}$$

we can cancel the nonlinear term $a\left[\sin\left(x_1 + \delta\right) - \sin\delta\right]$. *The resulting linear system is given by*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -bx_2 + v \end{aligned}$$

As such, the stabilization problem for the nonlinear system has been reduced to a stabilization problem for a controllable linear system. We can proceed to design a stabilizing linear state feedback control

$$v = -k_1 x_1 - k_2 x_2$$

that renders the closed-loop system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -k_1 x_1 - (k_2 + b)\, x_2\end{aligned}$$

asymptotically stable. The overall state feedback control law comprises linear and nonlinear parts

$$u = \left(\frac{a}{c}\right)[\sin(x_1+\delta) - \sin\delta] - \frac{1}{c}(k_1 x_1 + k_2 x_2)$$

Example 2 *Consider the system*

$$\begin{aligned}\dot{x}_1 &= a \sin x_2\\ \dot{x}_2 &= -x_1^2 + u\end{aligned}$$

We cannot simply choose u to cancel the nonlinear term $a \sin x_2$. However, let us first apply the following change of variables

$$\begin{aligned}z_1 &= x_1\\ z_2 &= a \sin x_2 = \dot{x}_1\end{aligned}$$

Then, the new variables z_1 and z_2 satisfy

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= a \cos x_2 . \dot{x}_2 = a \cos\left(\sin^{-1}\frac{z_2}{a}\right)\left(u - z_1^2\right)\end{aligned}$$

and the nonlinearities can be canceled by the control

$$u = x_1^2 + \frac{1}{a \cos x_2} v$$

which is well defined for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. The state equation in the new coordinates can be found by inverting the transformation to express (x_1, x_2) in the terms of (z_1, z_2), that is,

$$\begin{aligned}x_1 &= z_1\\ x_2 &= \sin^{-1}\left(\frac{z_2}{a}\right)\end{aligned}$$

which is well defined for $-a < z_2 < a$. The transformed state equation is given by

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = a\cos\left(\sin^{-1}\left(\frac{z_2}{a}\right)\right)\left(-z_1^2 + u\right)$$

which is in the required form to use state feedback. Finally, the control input that we would use is of the following form

$$u = x_1^2 + \frac{1}{a\cos x_2}(-k_1 z_1 - k_2 z_2) = x_1^2 + \frac{1}{a\cos x_2}(-k_1 x_1 - k_2 a\sin(x_2))$$

2 Input-Output Linearization

Consider the single-input-single-output system (1)-(2), where the vector fields $f : \mathcal{D} \to \mathbb{R}^n$ and $g : \mathcal{D} \to \mathbb{R}^n$ and the function $h : \mathcal{D} \to \mathbb{R}$ are sufficiently smooth.

Definition 1 *f is called smooth if $f \in C^\infty$, that is, f is continuous and all its derivatives of all orders are continuous.*

The goal is to derive conditions under which the input-output map can be rendered linear. *The idea is to take a sufficient number of derivatives of the output, until the input appears.*

We proceed by taking the derivative of y, which is given by

$$\dot{y} = \frac{\partial h}{\partial x}\left[f(x) + g(x)u\right] = L_f h(x) + L_g h(x) u$$

where $L_f h(x) \triangleq \frac{\partial h}{\partial x} f(x)$ is called the Lie Derivative of h with respect to (along) f. If $L_g h(x) = 0$, then $\dot{y} = L_f h(x)$, independent of u and we repeat the differentiation process again. Calculating the second derivative of y, denoted by $y^{(2)}$, we obtain

$$y^{(2)} = \frac{\partial (L_f h)}{\partial x}\left[f(x) + g(x)u\right] = L_f^2 h(x) + L_g L_f h(x) u$$

Once again, if $L_g L_f h(x) = 0$, then $y^{(2)} = L_f^2 h(x)$ is independent of u and we repeat the process. Actually, we repeat the processes of taking derivatives of the output until we see that $h(x)$ satisfies

$$L_g L_f^{i-1} h(x) = 0, i = 1, 2, ..., \rho - 1; \quad L_g L_f^{\rho-1} h(x) \neq 0$$

Therefore, u does not appear in the expressions of y, $\dot{y}$,...,$y^{(\rho-1)}$ and appears in the expression of $y^{(\rho)}$ with a nonzero coefficient, i.e.,

$$y^{(\rho)} = L_f^{\rho} h\left(x\right) + L_g L_f^{\rho-1} h\left(x\right) u \tag{3}$$

We can clearly see from (3), that the system is input-output linearizable, since the state feedback control

$$u = \frac{1}{L_g L_f^{\rho-1} h\left(x\right)}\left(-L_f^{\rho} h\left(x\right) + v\right) \tag{4}$$

reduces the input-output map (in some domain $\mathcal{D}_0 \subset \mathcal{D}$) to

$$y^{(\rho)} = v$$

which is a chain of ρ integrators. In this case, the integer ρ is called the ***relative degree*** of the system.

Definition 2 *The nonlinear system* (1)-(2) *is said to have a relative degree ρ, $1 \leq \rho \leq n$, in the region $\mathcal{D}_0 \subset \mathcal{D}$ if*

$$\begin{cases} L_g L_f^i h(x) = 0, \ i = 1, \cdots, \rho - 2 \\ L_g L_f^{\rho-1} h(x) \neq 0 \end{cases} \tag{5}$$

for all $x \in \mathcal{D}_0$.

Example 3 *Consider the controlled van der Pol equation*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \varepsilon\left(1 - x_1^2\right) x_2 + u \end{aligned}$$

with output $y = x_1$. Calculating the derivatives of the output, we obtain

$$\begin{aligned} \dot{y} &= \dot{x}_1 = x_2 \\ \ddot{y} &= \dot{x}_2 = -x_1 + \varepsilon\,(1 - x_1^2)\, x_2 + u \end{aligned}$$

Hence, the system has relative degree two in $\mathbb{R}^2$. For the output $y = x_1 + x_2^2$, we have that

$$\dot{y} = x_2 + 2x_2\left[-x_1 + \varepsilon\left(1 - x_1^2\right) x_2 + u\right]$$

and the system has relative degree one in $\mathcal{D}_0 = \{x \in \mathbb{R}^2 | x_2 \neq 0\}$. As such, we can see that the procedure of input-output linearization is dependent on the choice of the output map $h(x)$.

Continuing with our derivations, we now let

$$z = T(x) = \begin{bmatrix} T_1(x) \\ --- \\ T_2(x) \end{bmatrix} \triangleq \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix} \triangleq \begin{bmatrix} \phi(x) \\ --- \\ \psi(x) \end{bmatrix} \triangleq \begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix} \tag{6}$$

where $\phi_1(x)$ to $\phi_{n-\rho}(x)$ are chosen such that

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad 1 \leq i \leq n-\rho \tag{7}$$

which ensures that the η-dynamics

$$\dot{\eta} = \frac{\partial \phi}{\partial x}[f(x) + g(x)u] = \frac{\partial \phi}{\partial x} f(x)\bigg|_{x=T^{-1}(z)}$$

are independent of u. Note also that our definition of $T_2(x)$ in (6) results in $\xi = \begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(\rho-1)} \end{bmatrix}$.

Of course, it is crucial to know at this point if such functions ϕ exist in order to define the transformation T; the next result shows that.

Definition 3 *A continuously differentiable transformation T with a continuously differential inverse is a called a diffeomorphism.*

Theorem 1 *Consider the system* (1)-(2)*, and suppose that it has a relative degree $\rho \leq n$ in $\mathcal{D}$. If $\rho = n$, then for every $x_0 \in \mathcal{D}$, a neighborhood N of x_0 exists such that the map*

$$T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1}h(x) \end{bmatrix}$$

restricted to N is a diffeomorphism on N. If $\rho < n$, then for every $x_0 \in \mathcal{D}$, a neighborhood N of x_0 and smooth maps $\phi_1(x)$, $\cdots$, $\phi_{n-\rho}(x)$ exist such that the map such that the map

$$T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix}$$

restricted to N, is a diffeomorphism on N.

We can now apply the change of variables (see the Appendix) $z = T(x)$ to transform the system (1)-(2). into

$$\dot{\eta} = f_0(\eta, \xi) \tag{8}$$
$$\dot{\xi} = A_c\xi + B_c\gamma(x)[u - \alpha(x)] \tag{9}$$
$$y = C_c\xi \tag{10}$$

where $\xi \in \mathbb{R}^\rho$, $\eta \in \Re^{n-\rho}$, (A_C, B_C, C_C) is a canonical form representation of a chain of ρ integrators, i.e.,

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & & 0 & 1 \\ 0 & \cdots & & 0 & 0 \end{bmatrix}, \; B_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, C_c = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$f_0(\eta, \xi) = \left.\frac{\partial \phi}{\partial x} f(x)\right|_{x=T^{-1}(z)} \tag{11}$$
$$\gamma(x) = L_g L_f^{\rho-1}h(x) \tag{12}$$
$$\alpha(x) = -\frac{L_f^\rho h(x)}{L_g L_f^{\rho-1}h(x)} \tag{13}$$

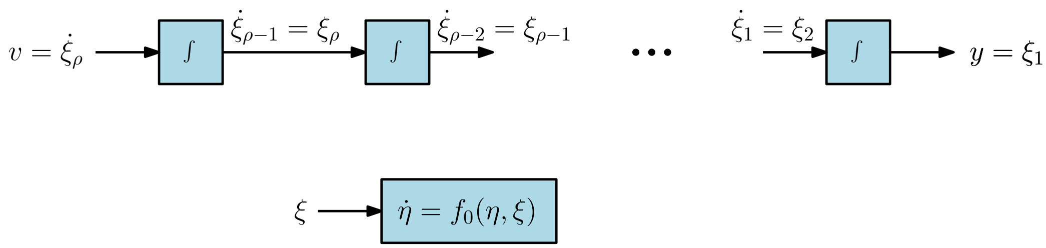

Figure 2: Transformed system (8)-(10) and the input defined as in (4)

We have kept α and γ expressed in the original coordinates. These two functions are uniquely determined in terms of f, g, and h and are independent of the choice of ϕ. They can be expressed in the new coordinates by setting

$$\alpha_0(\eta, \xi) = \alpha\left(T^{-1}(z)\right), \quad \gamma_0(\eta, \xi) = \gamma\left(T^{-1}(z)\right) \tag{14}$$

but now they are dependent on the choice of the functions ϕ. Regarding the definition of the equilibrium point for the transformed system, assume that $\bar{x}$ is the open-loop equilibrium of the system (1), then

$$\bar{\eta} = \phi(\bar{x}), \quad \bar{\xi} = \begin{bmatrix} h(\bar{x}) & 0 & \cdots & 0 \end{bmatrix}^T \tag{15}$$

The transformed system (8)-(10) is said to be in the normal form. This form decomposes the system into an external part ξ and an internal part η. The external part is linearized by the state feedback control

$$u = \alpha(x) + \beta(x)v \tag{16}$$

where $\beta(x) = \gamma^{-1}(x)$, while the internal part is made unobservable by the same control (see Figure 2). Setting $\xi = 0$ in the internal dynamics (8), results in

$$\dot{\eta} = f_0(\eta, 0) \tag{17}$$

which is called the ***zero dynamics*** of the system. If the zero dynamics of the system are (globally) asymptotically stable, the system is called ***minimum phase***.

Finally, the linearized system may then be stabilized by the choice of an appropriate state feedback:

$$v = -K\xi.$$

2.1 Relationship to Linear Systems

The notions of relative degree and minimum phase can also be found in linear systems. Consider a linear system represented by the transfer function

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_0} \tag{18}$$

where $m < n$ and $b_m \neq 0$. We can realize the transfer function in (18) in state-space form as

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{19}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \\ 0 & & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & & & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \cdots & b_m & 0 & \cdots & 0 \end{bmatrix}$$

This linear space model is a special case of

$$\dot{x} = f(x) + g(x)\,u, \quad y = h(x)$$

where $f(x) = Ax$, $g = B$, and $h(x) = Cx$. We know that the relative degree of the transfer function is given by $n - m$. However, let us verify that this is the case. We take the derivative of the output, i.e.,

$$\dot{y} = C\dot{x} = CAx + CBu \tag{20}$$

Now, if $m = n - 1$, then $CB = b_{n-1} \neq 0$ and the system has relative degree one. In general we have that

$$y^{(n-m)} = CA^{(n-m)}x + CA^{(n-m-1)}Bu \tag{21}$$

and we have the conditions that

$$CA^{(i-1)}B = 0, \forall i = 1, 2, \cdots, n-m-1, \text{ and } CA^{(n-m-1)}B = b_m \neq 0 \tag{22}$$

and the relative degree of the system is $n-m$, i.e., the difference between the order of the denominator and the numerator of the transfer function $G(s)$.

Moreover, we know that a minimum phase system means that the zeros of the transfer function $G(s)$ lie in the open left-half complex plane. We can show as well that there is a correspondence of the zeros of $H(s)$ and the normal form zero dynamics.

3 Full State Feedback Linearization

We know from the previous section, that the system (1) is feedback linearizable if we can find a sufficiently smooth function $h(x)$ in a domain $\mathcal{D}$ such that the system (1)-(2) has relative degree n within some region $\mathcal{D}_0 \subset \mathcal{D}$. This is because the normal form would have no zero dynamics. In fact, we can show that this result holds as an if and only if statement. As such, it remains to show that such a function $h(x)$ exists.

We begin with some definitions. For any two vector fields f and g on $\mathcal{D} \subset \mathbb{R}^n$, the **Lie bracket** $[f, g]$ is a third vector field that is defined as

$$[f, g](x) \triangleq \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) = L_f g(x) - L_g f(x) \tag{23}$$

Taking Lie brackets can be repeated, as such we define the following notation:

$$\begin{aligned} ad_f^0 g(x) &= g(x) \\ ad_f g(x) &= [f, g](x) \\ ad_f^k g(x) &= [f, ad_f^{k-1} g](x), \quad k \geq 1 \end{aligned}$$

Example 4 *Consider the two vector fields $f(x) = Ax$ and g is constant. Then, $ad_f g(x) = [f, g](x) = -Ag$, $ad_f^2 g(x) = [f, ad_f g](x) = -A(-Ag) = A^2 g$, and $ad_f^k g = (-1)^k A^k g$.*

Definition 4 (Distribution) *Consider the vector fields $f_1, \cdots, f_k$ on a domain $\mathcal{D} \subset \mathbb{R}^n$. The distribution, denoted by*

$$\Delta = \{f_1, f_2, \cdots, f_k\} \tag{24}$$

is the collection of all vectors spaces $\Delta(x)$ for $x \in \mathcal{D}$, where

$$\Delta(x) = span\{f_1(x), f_2(x), \cdots, f_k(x)\}$$

is the subspace of $\mathbb{R}^n$ spanned by the vectors $f_1(x), \cdots\ f_k(x)$.

Definition 5 *A distribution Δ is called involutive, if*

$$g_1 \in \Delta, \ \textit{and } g_2 \in \Delta \Rightarrow [g_1, g_2] \in \Delta$$

And now we are ready to state the main result of this section.

Theorem 2 (Feedback Linearization) *The SISO system (1) is feedback linearizable if and only if*

1. *the matrix $\mathcal{M}(x) = [g(x), ad_f g(x), \cdots, ad_f^{n-1} g(x)]$ has full rank ($= n$) for all $x \in \mathcal{D}_0$, and*
2. *the distribution $\Delta = span\{g, ad_f g, \cdots, ad_f^{n-2} g\}$ is involutive on $\mathcal{D}_0$.*

Example 5 *Consider the system*

$$\dot{x} = f(x) + gu = \begin{bmatrix} a\sin(x_2) \\ -x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

We compute the Lie bracket

$$ad_f g = [f, g](x) = -\frac{\partial f}{\partial x} g = \begin{bmatrix} -a\cos(x_2) \\ 0 \end{bmatrix}$$

Accordingly, the matrix

$$\mathcal{M}(x) = [g, ad_f g] = \begin{bmatrix} 0 & -a\cos(x_2) \\ 1 & 0 \end{bmatrix}$$

has full rank for any x such that $\cos(x_2) \neq 0$. Moreover, the distribution $\Delta = span\{g\}$ is involutive. Therefore, the conditions of Theorem (2) are satisfied and the system is feedback linearizable in the domain $\mathcal{D}_0 = \{x \in \mathbb{R}^2 \mid \cos(x_2) \neq 0\}$. Let us now find the function $h(x)$ for which the system is feedback linearizable. $h(x)$ should satisfy the following conditions

$$\frac{\partial h}{\partial x} g = 0, \quad \frac{\partial (L_f h)}{\partial x} g \neq 0, \quad h(0) = 0$$

Now, $\frac{\partial h}{\partial x} g = \frac{\partial h}{\partial x_2} = 0$ which implies that $h(x)$ is independent of x_2. Therefore, $L_f h(x) = \frac{\partial h}{\partial x_1} a\sin(x_2)$. The condition $\frac{\partial (L_f h)}{\partial x} g = \frac{\partial (L_f h)}{\partial x_2} = \frac{\partial h}{\partial x_1} a\cos(x_2) \neq 0$ is satisfied in $\mathcal{D}_0$ for any choice of h that satisfies $\frac{\partial h}{\partial x_1} \neq 0$. There are many such choices for h, for example, $h(x) = x_1$ or $h(x) = x_1 + x_1^3$.

4 Stability

Consider again the input-output linearized system in normal form (8)-(10), and assume that we have designed the feedback control law

$$u = \alpha(x) + \beta(x)v$$

where $\beta(x) = \gamma^{-1}(x)$, and $v = -K\xi$, such that $(A_c - B_cK)$ is Hurwitz. The resulting system reduces to the following dynamics

$$\dot{\eta} = f_0(\eta, \xi) \tag{25}$$

$$\dot{\xi} = (A_c - B_cK)\xi \tag{26}$$

Theorem 3 *The origin of the system* (25)-(25) *is asymptotically stable, if the origin of the zero dynamics* $\dot{\eta} = f_0(\eta, 0)$ *is asymptotically stable (minimum phase).*

Proof: The idea is to construct a special Lyapunov function. Since the zero dynamics are asymptotically stable, there exists (by converse Lyapunov theorem) a continuously differentiable function $V_\eta(\eta)$ such that

$$\frac{\partial V_\eta}{\partial \eta} f(\eta, 0) < -\alpha(\|\eta\|)$$

in some neighborhood of $\eta = 0$, where α is a strictly increasing continuous function with $\alpha(0) = 0$. Let $P = P^T > 0$ be the solution of the Lyapunov equation.

$$P(A_c - B_cK) + (A_c - B_cK)^T P = -I$$

Consider the function

$$V(\eta, \xi) = V_\eta(\eta) + k\sqrt{\xi^T P \xi}, \quad k > 0 \tag{27}$$

The derivative of V is given by

$$\begin{aligned}
\dot{V} &= \frac{\partial V_1}{\partial \eta} f_0(\eta, \xi) + \frac{k}{2\sqrt{\xi^T P \xi}} \xi^T \underbrace{[P(A_C - B_CK) + (A_C - B_CK)P]}_{-I} \xi \\
&= \frac{\partial V_1}{\partial \eta} f(\eta, 0) + \frac{\partial V_1}{\partial \eta}(f_0(\eta, \xi) - f_0(\eta, 0)) \frac{k}{2\sqrt{\xi^T P \xi}} \xi^T \xi \\
&\le -\alpha_3(\|\xi\|) + k_1(\|\xi\|) - kk_2(\|\xi\|)
\end{aligned}$$

for $k_1, k_2 > 0$[2]. By choosing k large enough, we guarantee $\dot{V} < 0$, and the result follows.

□

Remark 1 *The result in the last theorem is local and does not extend to the global setup, even if the zero dynamics are globally asymptotically stable. In order to make the result global, we have to impose more restrictive requirements regarding the zero dynamics, namely the notion of input-to-state stability of the zero dynamics (which is beyond the current scope of the course).*

5 Robustness

Feedback linearization is based on exact cancellation of the nonlinearities in the system dynamics, which is practically very difficult to achieve. In a realistic setup, we would have only approximations $\hat{\alpha}$, $\hat{\beta}$ and $\hat{T}(x)$ of the true α, β, and $T(x)$. The feedback control law has then the form

$$u = \hat{\alpha}(x) + \hat{\beta}(x)v = \hat{\alpha}(x) - \hat{\beta}(x)K\xi = \hat{\alpha}(x) - \hat{\beta}(x)K\hat{T}_2(x)$$

Accordingly, the closed-loop system of the normal form is given by

$$\dot{\eta} = f(\eta, \xi) \tag{28}$$

$$\dot{\xi} = A\xi + B\gamma(x)\left[(\hat{\alpha}(x) - \alpha(x)) - \hat{\beta}(x)K\hat{T}_2(x)\right] \tag{29}$$

Adding and subtracting $BK\xi$ to the $\dot{\xi}$ equation we obtain

$$\dot{\eta} = f(\eta, \xi) \tag{30}$$

$$\dot{\xi} = (A - BK)\xi + B\delta(z) \tag{31}$$

where

$$\delta(z) = \gamma(x)\left[(\hat{\alpha}(x) - \alpha(x)) - \left(\hat{\beta}(x) - \beta(x)\right)KT - \hat{\beta}(x)K\left(\hat{T}_2(x) - T(x)\right)\right]\Big|_{x=T^{-1}(z)}$$

Hence, the local closed loop system differs from the nominal one by an additive perturbation. Thus, we can show that locally the closed-loop system remains stable, despite small uncertainties in the feedback linearization maps.

For more details on feedback linearization concepts and extensions, see [1] and [2, Ch.13]

[2] $k_1 > 0$ follows from local continuity and differentiability of the function f_0, and $k_2 > 0$ follows from P

References

[1] A. Isidori. *Nonlinear control systems*. Springer-Verlag, 3rd edition, 1995.

[2] H. K. Khalil. *Nonlinear systems*. Prentice hall, 3rd edition, 2002.

Appendix

The η-dynamics can be derived as follows

$$\begin{aligned}\dot{\eta} = \dot{\phi}(x) &= \frac{\partial \phi}{\partial x}\dot{x} = \frac{\partial \phi}{\partial x}(f(x) + g(x)u) \\ &= \frac{\partial \phi}{\partial x} f(x) = \frac{\partial \phi}{\partial x} f(x)\bigg|_{x=T^{-1}(x)} \triangleq f_0(\eta, \xi)\end{aligned}$$

The ξ-dynamics can be derived as follows

$$\begin{aligned}\dot{\xi} &= \begin{bmatrix}\dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_\rho\end{bmatrix} = \frac{d}{dt}\begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(\rho-1)}\end{bmatrix} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(\rho)}\end{bmatrix} = \begin{bmatrix}\xi_2 \\ \vdots \\ \xi_\rho \\ y^{(\rho)}\end{bmatrix} \\ &= A_c\xi + B_c y^{(\rho)} = A_c\xi + B_c(L_g L_f^{\rho-1}h(x))\left[(u - \left(-\frac{L_f^\rho h(x)}{L_g L_f^{\rho-1}h(x)}\right)\right]\end{aligned}$$

we can now define

$$\gamma(x) \triangleq L_g L_f^{\rho-1}h(x), \quad \alpha(x) \triangleq -\frac{L_f^\rho h(x)}{L_g L_f^{\rho-1}h(x)}$$

Finally, our choice of the transformation $T_2(x)$ yields

$$y = h(x) = \xi_1 = C_c\xi$$