Verifying Parallel Programs with MPI-Spin
Part 4: Numerical Computation

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Overview

1. Goal: prove that program computes correct result
2. Symbolic execution
    - Performing symbolic arithmetic in MPI-SPIN
3. Functional equivalence
4. Three types of numerical equivalence
5. Diffusion revisited
6. Dealing with branches: the path condition

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  - a trusted sequential version of program
- our method
  - use MPI-Spin to prove the sequential and parallel program are functionally equivalent
    - i.e., produce same output for any given input
  - reduces the problem of verifying the correctness of a parallel numerical program to the problem of verifying the correctness of a sequential numerical program
  - uses symbolic execution to model floating-point computation

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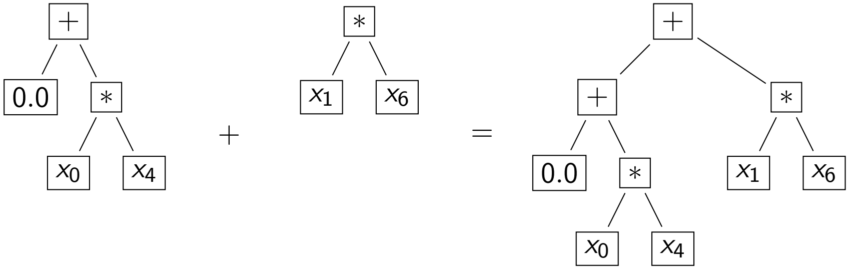
Input: symbolic constants $x_0, x_1, \ldots$
Output: symbolic expressions in the x_i



$$0.0 + (x_0 x_4) \quad + \quad x_1 x_6 \quad = \quad (0.0 + (x_0 x_4)) + x_1 x_6$$

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  3. `SYM_FALSE`: the boolean value `false`
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- basic functions
  1. `MPI_Symbolic SYM_intConstant(int n)`
     - returns symbolic expression with single node containing n
     - e.g., $\boxed{4}$
  2. `MPI_Symbolic SYM_symbolicConstant(int i)`
     - returns symbolic expression with single node containing x_i
     - e.g., $\boxed{x_4}$
     - usually used to model input
     - also used to represent floating-point constants (π, e, ...)

Forming new symbolic expressions from old

- the following return an `MPI_Symbolic` of numeric type
  1. `SYM_add(MPI_Symbolic x, MPI_Symbolic y)`
  2. `SYM_subtract(MPI_Symbolic x, MPI_Symbolic y)`
  3. `SYM_multiply(MPI_Symbolic x, MPI_Symbolic y)`
  4. `SYM_divide(MPI_Symbolic x, MPI_Symbolic y)`
  5. `SYM_sqrt(MPI_Symbolic x)`
  6. `SYM_abs(MPI_Symbolic x)`
  7. `SYM_if(MPI_Symbolic b, MPI_Symbolic x, MPI_Symbolic y)`
     - `(b ? x : y)`

Forming new symbolic expressions from old

- the following return an `MPI_Symbolic` of boolean type
    1. `SYM_equals(MPI_Symbolic x, MPI_Symbolic y)`
        - `/* x == y */`
    2. `SYM_nequals(MPI_Symbolic x, MPI_Symbolic y)`
        - `/* x != y */`
    3. `SYM_lessThan(MPI_Symbolic x, MPI_Symbolic y)`
    4. `SYM_greaterThan(MPI_Symbolic x, MPI_Symbolic y)`
    5. `SYM_lessThanOrEquals(MPI_Symbolic x, MPI_Symbolic y)`
    6. `SYM_greaterThanOrEquals(MPI_Symbolic x, MPI_Symbolic y)`
    7. `SYM_conjunct(MPI_Symbolic p, MPI_Symbolic q)`
        - `/* p && q */`
    8. `SYM_negate(MPI_Symbolic p)`
        - `/* !p */`

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Solution: Value numbering

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  - every expression has a unique ID number
- in the model...
  - replace all floating-point values with ID numbers
  - replace all floating-point operations with symbolic operations
    - to evaluate $x + y$:
    - is $x + y$ already in the table?
    - if *yes*, return its ID number
    - if *no*, create new table entry and return new ID number

Symbolic expression table: example

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  - prove sequential and parallel programs are functionally equivalent
- **Method**
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     - input: $\mathbf{x} = (x_1, \ldots, x_n)$, output: $\mathbf{y}$
  2. construct symbolic model M_{par} of parallel program
     - input: $\mathbf{x} = (x_1, \ldots, x_n)$, output: $\mathbf{y}'$
     - using same symbolic table

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Numerical Issues

Problem: distinct symbolic expressions should be considered "equivalent" in some cases

Example: real equivalence

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- $((x_3 + x_1) + x_2) + x_0$ and $((x_0 + x_1) + x_2) + x_3$
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- $((x_3 + x_1) + x_2) + x_0$ and $((x_0 + x_1) + x_2) + x_3$
- if computer arithmetic were real arithmetic then evaluating these expressions would yield identical results for any values of $x_0, x_1, \ldots$
- computer arithmetic is not real arithmetic
  - e.g., floating-point addition can never be associative
  - **What every computer scientist should know about floating-point arithmetic**
    - **David Goldberg**
    - **ACM Computing Surveys 23(1), 1991**

Numerical Issues, cont.

- in some cases, knowing the sequential and parallel programs are "real equivalent" is good enough
- if testing yields slightly different results...
  - ...but you know programs are real equivalent
  - then you know the only reason results differ is due to vagaries of floating-point arithmetic
  - and not to some error in your parallel program

Three equivalence relations

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  - goal is for each equivalence class to have at most one representative in table
  - this is not always achievable with 100% precision
  - estimation is always conservative
    - if MPI-Spin says two expressions are equivalent then they are equivalent
    - if MPI-Spin says they are not equivalent then they might be equivalent

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- numeric operations are treated as uninterpreted functions
- example: x and $x + 0$ are not Herbrand equivalent
- Herbrand equivalence is the strongest form of equivalence
  - if two programs are Herbrand equivalent then they will produce the same result no matter how numeric operations are implemented
    - as long as the numeric operations are deterministic functions!
- in many complex examples, sequential and parallel versions are Herbrand equivalent
  - complexity lies elsewhere (e.g., in distribution of data, coordination of processes)

IEEE equivalence

- two expressions are IEEE equivalent if one can be reduced to the other using the following identities:
  - $x + y = y + x$
  - $x + 0 = x = 0 + x$
  - $x - x = 0$
  - $xy = yx$
  - $1x = x = x1$
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- rationale
  - all of these identities are guaranteed to hold on any platform conforming to the IEEE-754 standard
- IEEE equivalence is weaker than Herbrand equivalence
- two IEEE equivalent programs are guaranteed to produce the exact same results if executed on an IEEE-754-compliant platform
  - but results may differ on non-compliant platforms

Real equivalence

- two expressions are real equivalent if one can be reduced to the other using any of the field identities
  - all of the IEEE identities
  - associativity of addition and multiplication
  - $x(1/x) = 1$

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  - suppose program uses `MPI_Reduce` or `MPI_Allreduce`
  - reduction operations is floating-point addition
  - **MPI Standard** permits MPI implementation to perform additions in any order
    - order used on one execution could be different than order used on another execution
  - prevents any possibility of IEEE equivalence

Numerical model of diffusion2d

Composite model:

- `diffusion/diffusion_sym.prom`
- `diffusion/diffusion_sym.c`

The path correspondence problem

- the programs may contain branches on expressions that involve the symbolic variables
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- the programs may contain branches on expressions that involve the symbolic variables
  - **if** $(x_0 \neq 0)\ \{\ldots\}$ **else** $\{\ldots\}$
- only want to compare the result of an execution path in the parallel program to the result of a *corresponding* path in the sequential program

Path conditions and domains

- enumerate all paths through the sequential program
  - keeping track of the path condition for each path

$$\mathbf{y} = \begin{cases} f_1(\mathbf{x}) & \text{if } p_1(\mathbf{x}) \\ f_2(\mathbf{x}) & \text{if } p_2(\mathbf{x}) \\ \vdots & \vdots \\ f_n(\mathbf{x}) & \text{if } p_n(\mathbf{x}) \end{cases}$$

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- each p_i determines a path domain $D_i = \{\mathbf{x} \mid p_i(\mathbf{x})\}$
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Solution to path correspondence problem:

1. discover path conditions/domains automatically
2. for each domain D_i: compare symbolic results of sequential and parallel programs for all inputs in D_i

Modeling conditional statements

To model the statement **if** $(x_0 \neq 0)$ $\{\ldots\}$ **else** $\{\ldots\}$

$p \leftarrow$ true; /* path condition */
$\vdots$
$b \leftarrow \mu(p, x_0 \neq 0)$;
if $(b = -1)$ {
  if $(\mathit{choose}())$ {
    $b \leftarrow 1$; $p \leftarrow p \wedge (x_0 \neq 0)$;
  } **else** {
    $b \leftarrow 0$; $p \leftarrow p \wedge (x_0 = 0)$;
  }
}
if $(b = 1)$ { ... } **else** { ... }

$$\mu(p, q) = \begin{cases} 1 & \text{if } p \Rightarrow q \\ 0 & \text{if } p \Rightarrow \neg q \\ -1 & \text{if don't know} \end{cases}$$

for boolean-valued symbolic expressions p, q

The method: incorporating path condition

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Example: Gaussian elimination

Step 1 Locate the leftmost column of A that does not consist entirely of zeros, if one exists. The top nonzero entry of this column is the pivot.

Step 2 Interchange the top row with the pivot row, if necessary, so that the entry at the top of the column found in Step 1 is nonzero.

Step 3 Divide the top row by pivot in order to introduce a leading 1.

Step 4 Add suitable multiples of the top row to all other rows so that all entries above and below the leading 1 become zero.

Repeat.

Gaussian elimination

transforms a matrix to its reduced row-echelon form:

$$\mathbf{x} = \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} \quad \rightarrow \quad \mathbf{y} = \begin{pmatrix} y_0 & y_1 \\ y_2 & y_3 \end{pmatrix}$$

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$$\mathbf{y} = \begin{cases} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \text{if } x_0 = 0 \wedge x_2 = 0 \wedge x_1 = 0 \wedge x_3 = 0 \\ \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) & \text{if } x_0 = 0 \wedge x_2 = 0 \wedge x_1 = 0 \wedge x_3 \neq 0 \\ \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) & \text{if } x_0 = 0 \wedge x_2 = 0 \wedge x_1 \neq 0 \\ \left(\begin{smallmatrix} 1 & x_3/x_2 \\ 0 & 0 \end{smallmatrix}\right) & \text{if } x_0 = 0 \wedge x_2 \neq 0 \wedge x_1 = 0 \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{if } x_0 = 0 \wedge x_2 \neq 0 \wedge x_1 \neq 0 \\ \left(\begin{smallmatrix} 1 & x_1/x_0 \\ 0 & 0 \end{smallmatrix}\right) & \text{if } x_0 \neq 0 \wedge x_3 - x_2(x_1/x_0) = 0 \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{if } x_0 \neq 0 \wedge x_3 - x_2(x_1/x_0) \neq 0 \end{cases}$$

Numerical model of Gaussian Elimination

See

- `mpi-spin/examples/gausselim/source/`
- `mpi-spin/examples/gausselim/model/`