Mathematical Programming Representation of Discrete-Event Simulation

ABSTRACT

1. Introduction

Discrete Event Simulation (DES) is one of the most used tool for performance evaluation of discrete event dynamic systems, and hence, simulation optimization algorithms are widely developed for optimizing the parameters of such systems. Most of the state-of-the-art simulation optimization algorithms consider DES as a black-box function, and the structure of DES has been seldom studied. Under the black-box setting, simulation optimization algorithms always require huge amount of iterations to explore the search region of the optimization problem, and can be computationally inefficient. On the contrary, a minority of the simulation optimization literature explores the structure of the DES, and that research is referred to as white-box simulation optimization. The benefit of white-box simulation optimization is saving of simulation budget since optimization procedure is guided by the information contained in the structure of DES. However, the barrier of white-box simulation optimization is how to model DES as white box, so that it eventually favors the optimization. In this work, we present how to establish an equivalent Mathematical Programming Representation (MPR) for certain types of DES. We deploy this work with the modeling approach, conditions under which the approach can be applied and some examples.

Chan and Schruben (2008) is the first work proposing a modeling framework to translate DES into MPR in a general sense. Their modeling framework is based on the Event Relationship Graph (ERG) of the system dynamics. To derive the MP model, an ERG of the discrete event system has to be constructed and expanded to an elementary ERG (EERG) model, and a routined approach can be applied to translate the EERG model into an MP model. However, the work of Chan and Schruben (2008) has some limitations. First, deriving an ERG is not an easy task, and the user has to pay quite much attention to detect all the event relationships and complete the triggering conditions between each pair of related events. That limits the wide spread of such approach. Second, Chan and Schruben (2008) proposes the modeling approach case-by-case, which means the user has to first identify which situations he faces by analyzing the EERG and then choose the appropriate model. This is quite a burden, since EERG is an expansion of ERG and could be huge. This work proposes a user-friendly modeling approach without plotting the ERG and the modeling procedure can be automatically generated in a general-purpose programming language. Despite

taking different paths, the MPRs proposed in this work and Chan and Schruben (2008) lead to the equivalent results, which are both equivalent to a simulation realization.

When there is already a DES model, the benefit of developing an MPR is not obvious due to the high complexity of solving MPR. However, the MPR of a simulation model favors the optimal design and control of the DES thanks to the vast theory and methodological works in MP. Many works in literature show the potentiality of this research direction. For instance, the gradient can be conveniently estimated from the simulation model, if the MPR is approximated into Linear Programming (LP) and solving the dual of the LP (Chan & Schruben, 2008; Zhang, Matta, & Alfieri, 2020). Moreover, if some of the parameters in the MPR are changed to decision variables, the MPR becomes an integrated simulation optimization model. Solving the integrated model provides the optimal solution of the optimization problem (Matta, 2008). MP-based algorithms, such as linear programming approximation (Alfieri & Matta, 2012), Benders decomposition (Weiss & Stolletz, 2015), column generation (Alfieri, Matta, & Pastore, 2020), has been applied to improve the efficiency of solving the integrated MP models.

The application of MPR-based simulation optimization approaches is usually found in operations management of manufacturing and service systems. The integrated simulation optimization model has proved itself to be well suited in solving the buffer allocation problem. The most efficient approach to finding the sample-path global optimal of the buffer allocation problem of serial production line is developed based on it (Zhang, Pastore, Matta, & Alfieri, n.d.). Thanks to the flexibility of DES in evaluating complex systems, the buffer allocation problem of production systems with complex blocking mechanism, such as kanban control, base stock control, extended kanban control, can be managed (Pedrielli, Alfieri, & Matta, 2015). Thanks to the flexibility of MPR in modeling optimization problems, problems involving real-valued decision variables such as optimal production rate (Tan, 2015), bottleneck detection (Zhang & Matta, 2018) and throughput improvement problem (Zhang & Matta, 2020) have all been well addressed and the sample-path global optimal solution can be obtained. It is worthy to notice that, before the above mentioned works were proposed, there were many state-of-the-art heuristic approaches addressing those problems, but without guarantee of global or local optimality. Thus, the development of MPR-base simulation optimization has made much contribution in the research area of manufacturing and service system optimization.

2. Resource allocation problem of queueing systems

2.1. Mathematical programming representation of simulation model

We study only the system that the occurrence of an event will lead to the increment or decrement of one unit of the state variables. A simulation model is called a *natural* simulator if the following assumptions all hold:

(1) An event e^ξ can be triggered if the state variables $\mathbf{s}$ satisfy specific conditions *at that time*, regardless of the history of the state or event occurrence, and the condition is not changed along time, i.e., condition is static. It could be possible to define more state variables in case of history dependence and variant triggering conditions.
(2) Natural triggering relationship: if and only if e^ξ is an s-increment event, e^ξ triggers an s-decrement event, vice versa.

(3) Natural triggering condition: the condition for triggering an e^{ξ} is that each state variable s must within its predefined domain, i.e., $\mathbf{l} \leq \mathbf{s} \leq \mathbf{u}$, regardless of event type ξ.

(4) For all e^{ξ}, the number of execution N^{ξ} is known before simulation, and the simulation terminate when all types of events have been triggered for that number.

Assumptions for a variable x to be resource-type:

(1) For all e^{ξ}, $\mathbf{u}$ is monotonically increasing on x, and $\mathbf{l}$ is monotonically decreasing on x.

Formulate the MP model of simulation:

$e_i^{\xi} \geq 0$	time of the i-th occurrence of event e^{ξ}.
$\tau_l^{s+} \geq 0$	time of the l-th occurrence of events that increments state variable s.
$\tau_l^{s-} \geq 0$	time of the l-th occurrence of events that decrements state variable s.
$x_{i,l}^{\xi,s+} \in \{0,1\}$	equal to 1 if e_i^{ξ} is the l-th increment of s.
$x_{i,l}^{\xi,s-} \in \{0,1\}$	equal to 1 if e_i^{ξ} is the l-th decrement of s.

The MP model of simulation is

$$\min\{\textstyle\sum_{\xi,i} e_i^{\xi}\} \tag{1}$$

$$s.t. \tag{2}$$

$$\tau_l^{s+} - \tau_{l+s_0-u_s}^{s-} \geq 0 \qquad \forall\, s \tag{3}$$

$$\tau_l^{s-} - \tau_{l-s_0+l_s}^{s+} \geq 0 \qquad \forall\, s \tag{4}$$

$$\tau_l^{s+} - e_i^{\xi} \geq M(x_{i,l}^{\xi,s+} - 1) \quad \forall\, s \text{ and } e^{\xi} \text{ with increment of } s \tag{5}$$

$$\tau_l^{s-} - e_i^{\xi} \geq M(x_{i,l}^{\xi,s-} - 1) \quad \forall\, s \text{ and } e^{\xi} \text{ with decrement of } s \tag{6}$$

$$e_i^{\xi} - \tau_l^{s+} \geq M(x_{i,l}^{\xi,s+} - 1) \quad \forall\, s \text{ and } e^{\xi} \text{ with increment of } s \tag{7}$$

$$e_i^{\xi} - \tau_l^{s-} \geq M(x_{i,l}^{\xi,s-} - 1) \quad \forall\, s \text{ and } e^{\xi} \text{ with decrement of } s \tag{8}$$

$$\textstyle\sum_{\xi,i} x_{i,l}^{\xi,s+} = 1 \qquad \forall\, s,l \tag{9}$$

$$\textstyle\sum_{\xi,i} x_{i,l}^{\xi,s-} = 1 \qquad \forall\, s,l \tag{10}$$

$$\textstyle\sum_{s,l} x_{i,l}^{\xi,s+} = 1 \qquad \forall\, \xi,i \tag{11}$$

$$\textstyle\sum_{s,l} x_{i,l}^{\xi,s-} = 1 \qquad \forall\, \xi,i \tag{12}$$

$$\tag{13}$$

2.2. *Mathematical programming representation of simulation model - V2*

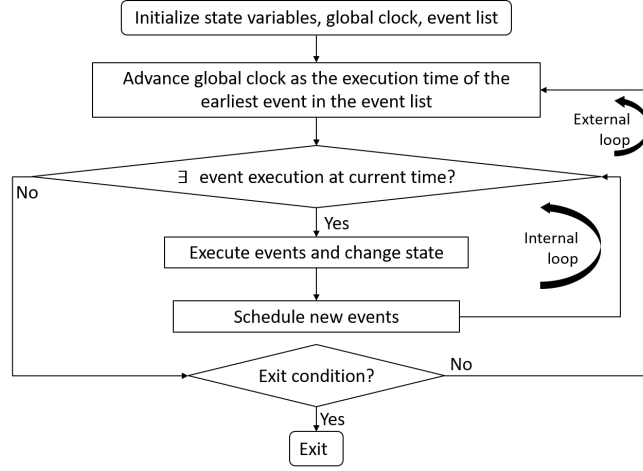Revise event-based simulation algorithm.

Figure 1. Event-based simulation algorithm.

An equivalent mathematical programming model exists if the following assumptions are satisfied:

(1) State variables are integer.
(2) For all event e^ξ, the *scheduling conditions* are in the form of $a_s^\xi \leq s \leq b_s^\xi$ combined with logic operator "AND", where s is a state variable, and a_s^ξ and b_s^ξ are lower and upper bounds.
(3) The scheduling conditions is independent of the history and not changed along time. (It could be possible to define more state variables in case of history dependence and time-variant scheduling conditions.)
(4) An event execution of e^ξ leads to integer increment or decrement equal to Δ_s^ξ of certain state variables s, and Δ_s^ξ is not changed along time.
(5) The delay between scheduling and execution time of an event e^ξ, denoted by t^ξ, is random variate. They can be generated independently from the simulation run. (*This point is different from ERG. In ERG, the delay is dependent on the edge, i.e, a couple of events, but I consider delay dependent on a single event.*)
(6) For all events e^ξ, the number of executions I^ξ is known before simulation.

Preparation Event e^ξ is expanded into a series of events $e^{\xi,0}$, $e^{\xi,1}$, ..., e^{ξ,Δ^ξ}, where Δ^ξ is equal to the maximum among Δ_s^ξ for all $s \in \Theta^\xi$. The expansion is conducted as follows. First, event $e^{\xi,0}$ is executed as soon as all the scheduling conditions are satisfied, and the state variables $s \in \Theta^\xi$ are not changed. Then, event $e^{\xi,1}$ is executed after t^ξ time unit after an execution of $e^{\xi,0}$. For all $s \in \Theta^\xi$, if $\Delta_s^\xi \geq \delta$, $e^{\xi,\delta}$ will increase or decrease s by one, for all $\delta = 1, ..., \Delta^\xi$. The i-th execution of event $e^{\xi,\delta}$ for $\delta = 1, ..., \Delta^\xi$ are simultaneous.

Constraints (A) The constraints below imply that event $e^{\xi,1}$ is scheduled to execute with a delay t^ξ, after an execution of $e^{\xi,0}$.

$$e_{i'}^{\xi,1} - e_i^{\xi,0} \geq t_i^\xi + M(x_{i,i'}^\xi - 1) \qquad \forall \xi, i, i' = 1, ..., I^\xi \qquad (14)$$

e^ξ	event of type ξ
s	state variable
S	set of all state variables
S^ξ	set of state variables whose value is conditioned for scheduling event e^ξ.
$\Theta^{\xi+}$	the set of state variables that event e^ξ will increase its value.
$\Theta^{\xi-}$	the set of state variables that event e^ξ will decrease its value.
Θ^ξ	$\Theta^{\xi+} \cap \Theta^{\xi-}$
E^{s+}	set of events whose execution increases the value of state variable s.
E^{s-}	set of events whose execution decreases the value of state variable s.
Δ_s^ξ	increment or decrement of state variable s when event e^ξ is executed.
I^ξ	total number of executions of event e^ξ
L^{s+}	total number of times that state variable s is increased.
L^{s-}	total number of times that state variable s is decreased.
t^ξ	delay between scheduling and execution of event e^ξ.
t_i^ξ	delay between i-th scheduling and its execution of event e^ξ.

Table 1. Notations

$e_i^{\xi,\delta} \geq 0$	time of i-th execution of event $e^{\xi,\delta}$
$\tau_l^{s+} \geq 0$	time when state variable s is increased for the l-th time.
$\tau_l^{s-} \geq 0$	time when state variable s is decreased for the l-th time.
$x_{i,i'}^\xi \in \{0,1\}$	equal to 1 if the i' execution of event e^ξ is the i-th scheduled one.
$y_{i,l}^{\xi,\delta,s+} \in \{0,1\}$	equal to 1 if the i-th execution of event e^ξ is the l-th time that state variable s in incre
$y_{i,l}^{\xi,\delta,s-} \in \{0,1\}$	equal to 1 if the i-th execution of event e^ξ is the l-th time that state variable s in decr
$z_{i,l}^{\xi,s+} \in \{0,1\}$	equal to 1 if the i-th scheduling of event e^ξ is later than τ_l^{s+}.

Table 2. Decision variables

It should be noticed that, if multiple executions of the same event e^ξ are allow to exist in the future event list simultaneously, the execution of $e^{\xi,1}$ scheduled by the i-th execution of $e^{\xi,0}$ may be not the i-th execution of $e^{\xi,1}$. Thus, binary variables $x_{i,i'}^\xi$ are introduced, and it is equal to one if the i' execution of event $e^{\xi,1}$ is scheduled by the i-th execution of event $e^{\xi,0}$. Since each execution of $e^{\xi,0}$ can schedule one and only one execution of $e^{\xi,1}$, the following constraints should also be satisfied:

$$\sum_{i=1}^{N^\xi} x_{i,i'}^\xi = 1 \qquad \forall\, \xi, i' = 1, ..., I^\xi \tag{15}$$

$$\sum_{i'=1}^{N^\xi} x_{i,i'}^\xi = 1 \qquad \forall\, \xi, i = 1, ..., I^\xi \tag{16}$$

If up to α^ξ multiple executions of event e^ξ are allowed, the following constraints can be added:

$$e_i^{\xi,0} - e_{i-\alpha^\xi}^{\xi,1} \geq 0 \qquad \forall\, \xi,\; i = \alpha^\xi + 1, ..., I^\xi \tag{17}$$

$$\sum_{i'=i+\alpha^\xi}^{I^\xi} x_{i,i'}^\xi = 0 \qquad \forall\, \xi,\; i = 1, ..., I^\xi - \alpha^\xi \tag{18}$$

$$\sum_{i=1}^{i'-\alpha^\xi} x_{i,i'}^\xi = 0 \qquad \forall\, \xi,\; i' = \alpha^\xi + 1, ..., I^\xi \tag{19}$$

If α^ξ is equal to one, the constraints (A) are reduced to:

$$e_i^{\xi,1} - e_i^{\xi,0} \geq t_i^\xi \qquad \forall \xi, i = 1, ..., I^\xi \tag{20}$$

$$e_i^{\xi,0} - e_{i-1}^{\xi,1} \geq 0 \qquad \forall\, \xi,\; i = 2, ..., I^\xi \tag{21}$$

Constraints (B) Binding $e_i^{\xi,\delta}$ and τ_l^{s+}, τ_l^{s-}:

$$\tau_l^{s+} - e_i^{\xi,\delta} \geq M(y_{i,l}^{\xi,\delta,s+} - 1) \qquad \forall\, s \in S, e^\xi \in E^{s+}, \delta = 1, .., \Delta_s^\xi, i = 1, .., I^\xi, l = 1, .., L^{s+} \tag{22}$$

$$e_i^{\xi,\delta} - \tau_l^{s+} \geq M(y_{i,l}^{\xi,\delta,s+} - 1) \qquad \forall\, s \in S, e^\xi \in E^{s+}, \delta = 1, .., \Delta_s^\xi, i = 1, .., I^\xi, l = 1, .., L^{s+} \tag{23}$$

$$\tau_l^{s-} - e_i^{\xi,\delta} \geq M(y_{i,l}^{\xi,\delta,s-} - 1) \qquad \forall\, s \in S, e^\xi \in E^{s-}, \delta = 1, .., \Delta_s^\xi, i = 1, .., I^\xi, l = 1, .., L^{s-} \tag{24}$$

$$e_i^{\xi,\delta} - \tau_l^{s-} \geq M(y_{i,l}^{\xi,\delta,s-} - 1) \qquad \forall\, s \in S, e^\xi \in E^{s-}, \delta = 1, .., \Delta_s^\xi, i = 1, .., I^\xi, l = 1, .., L^{s-} \tag{25}$$

$$\sum_{\substack{\xi: e^\xi \in E^{s+} \\ i=1,...,I^\xi \\ \Delta=1,...,\Delta_s^\xi}} y_{i,l}^{\xi,\delta,s+} = 1 \qquad \forall\, s \in S, l = 1, ..., L^{s+} \tag{26}$$

$$\sum_{l=1,...,L^{s+}} y_{i,l}^{\xi,\delta,s+} = 1 \qquad \forall\, \xi, s \in \Theta^{\xi+}, i = 1, ..., I^\xi, \delta = 1, ..., \Delta_s^\xi \tag{27}$$

$$\sum_{\substack{\xi: e^\xi \in E^{s-} \\ i=1,...,I^\xi \\ \Delta=1,...,\Delta_s^\xi}} y_{i,l}^{\xi,\delta,s-} = 1 \qquad \forall\, s \in S, l = 1, ..., L^{s-} \tag{28}$$

$$\sum_{l=1,...,L^{s-}} y_{i,l}^{\xi,\delta,s-} = 1 \qquad \forall\, \xi, s \in \Theta^{\xi-}, i = 1, ..., I^\xi, \delta = 1, ..., \Delta_s^\xi \tag{29}$$

$$\tag{30}$$

Binary variables $y_{i,l}^{\xi,\delta,s+} \in \{0,1\}$ are equal to one if the i-th execution of event $e^{\xi,\delta}$ is the l-th time that state variable s in increased. Since events $e^{\xi,1}, ..., e^{\xi,\Delta_s^\xi}$ are expanded from one event e^ξ, and they are executed simultaneously, the following constraints are also added:

$$e_i^{\xi,\delta} = e_i^{\xi,1} \qquad \forall\, \xi,\; i = 1, ..., I^\xi, \delta = 1, ..., \Delta^\xi \tag{31}$$

$$y_{i,l+\delta-1}^{\xi,\delta,s+} = y_{i,l}^{\xi,1,s+} \qquad \forall\, \xi,\; s \in \Theta^{\xi+},\; \delta = 1, ..., \Delta_s^\xi,\; i = 1, ..., I^\xi,\; l = 1, ..., L^{s+} - \Delta_s^\xi \tag{32}$$

$$y_{i,l+\delta-1}^{\xi,\delta,s-} = y_{i,l}^{\xi,1,s-} \qquad \forall\, \xi,\; s \in \Theta^{\xi-},\; \delta = 1, ..., \Delta_s^\xi,\; i = 1, ..., I^\xi,\; l = 1, ..., L^{s-} - \Delta_s^\xi \tag{33}$$

Constraints (C) To trigger event $e^{\xi,0}$, the conditions $a_s^\xi \le s \le b_s^\xi$ for all state variable $s \in S^\xi$ should be satisfied. $s \in S^\xi$ can be categorized into one of the following three situations:

- event e^ξ does not change the value of s, i.e., $s \notin \Theta^\xi$.
- event e^ξ increases the value of s, i.e., $s \in \Theta^{\xi+}$.
- event e^ξ decreases the value of s, i.e., $s \in \Theta^{\xi-}$.

If event e^ξ does not change the value of s, or if it is executed after being scheduled with positive delay, i.e., $s \notin \Theta^\xi$ or $t^\xi > 0$, the following constraints are applied:

$$e_i^{\xi,0} - \tau_l^{s+} \le M z_{i,l}^{\xi,s+} \quad \forall\, \xi, i = 1, ..., I^\xi,\ s \in S^\xi, l = 1, ..., L^{s+} \tag{34}$$

$$e_i^{\xi,0} - \tau_l^{s-} \le M z_{i,l}^{\xi,s-} \quad \forall\, \xi,\ i = 1, ..., I^\xi,\ s \in S^\xi, l = 1, ..., L^{s-} \tag{35}$$

$$e_i^{\xi,0} - \tau_l^{s-} \ge -M \hat{z}_{i,l}^{\xi,s-} \quad \forall\, \xi,\ i = 1, ..., I^\xi,\ s \in S^\xi, l = 1, ..., L^{s-} \tag{36}$$

$$e_i^{\xi,0} - \tau_l^{s+} \ge -M \hat{z}_{i,l}^{\xi,s+} \quad \forall\, \xi,\ i = 1, ..., I^\xi,\ s \in S^\xi, l = 1, ..., L^{s+} \tag{37}$$

$$z_{i,l}^{\xi,s+} + \hat{z}_{i,s_0+l-b_s^\xi}^{\xi,s-} \le 1 \qquad\qquad ?? \tag{38}$$

$$z_{i,l}^{\xi,s-} + \hat{z}_{i,-s_0+l+a_s^\xi}^{\xi,s+} \le 1 \qquad\qquad ?? \tag{39}$$

$$\tag{40}$$

$$e_i^{\xi,0} - \tau_l^{s+} \ge M(z_{i,l}^{\xi,s+} - 1) \quad \forall\, \xi, i = 1, ..., I^\xi,\ s \in S^\xi, l = 1, ..., L^{s+} \tag{41}$$

$$\tau_l^{s+} - e_i^{\xi,0} > -M z_{i,l}^{\xi,s+} \quad \forall\, \xi, i = 1, ..., I^\xi,\ s \in S^\xi, l = 1, ..., L^{s+} \tag{42}$$

$$e_i^{\xi,0} - \tau_l^{s-} \ge M(z_{i,l}^{\xi,s-} - 1) \quad \forall\, \xi, i = 1, ..., I^\xi,\ s \in S^\xi, l = 1, ..., L^{s-} \tag{43}$$

$$\tau_l^{s-} - e_i^{\xi,0} > -M z_{i,l}^{\xi,s-} \quad \forall\, \xi, i = 1, ..., I^\xi,\ s \in S^\xi, l = 1, ..., L^{s-} \tag{44}$$

$$\tag{45}$$

If $e_i^{\xi,0}$ is executed after $\tau_l^{s+}(\tau_l^{s-})$, $z_{i,l}^{\xi,s+}(z_{i,l}^{\xi,s-})$ is equal to one. If $e_i^{\xi,0}$ is executed before $\tau_l^{s+}(\tau_l^{s-})$, $\hat{z}_{i,l}^{\xi,s+}(\hat{z}_{i,l}^{\xi,s-})$ is equal to one.

If event e^ξ increases the value of s, and it is executed immediately when scheduled, i.e., $s \in \Theta^{\xi+}$ and $t^\xi = 0$, the following constraint should be applied:

$$e^{\xi,0} - \tau_{s_0+l-1-b}^{s-} \ge M(y_{i,l}^{\xi,1,s+} - 1) \tag{46}$$

$$e^{\xi,0} - \tau_{s_0+l-1-a}^{s-} \le M(1 - y_{i,l}^{\xi,1,s+}) \tag{47}$$

If event e^ξ decreases the value of s, and it is executed immediately when scheduled, i.e., $s \in \Theta^{\xi-}$ and $t^\xi = 0$, the following constraint should be applied:

$$e^{\xi,0} - \tau_{-s_0+l-1+a}^{s+} \ge M(y_{i,l}^{\xi,1,s-} - 1) \tag{48}$$

$$e^{\xi,0} - \tau_{-s_0+l-1+b}^{s+} \le M(1 - y_{i,l}^{\xi,1,s-}) \tag{49}$$

2.3. Mathematical programming representation of simulation model - V3

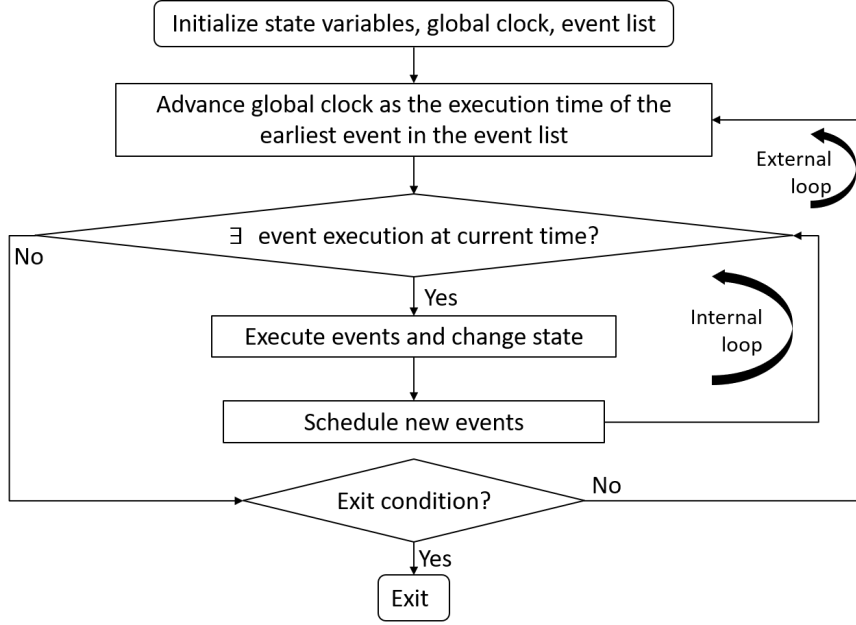Event-scheduling algorithm for DES.

Figure 2. Event-based simulation algorithm.

An equivalent mathematical programming model exists if the following assumptions are satisfied:

(1) For all event e^ξ, the *scheduling conditions* are in the form of $a_s^\xi \leq s \leq b_s^\xi$ combined with logic operator "AND", where s is a state variable, and a_s^ξ and b_s^ξ are lower and upper bounds.

(2) The scheduling conditions is independent of the history and not changed along time. (It could be possible to define more state variables in case of history dependence and time-variant scheduling conditions.)

(3) An event execution of e^ξ leads to (integer) increment or decrement equal to Δ_s^ξ of certain state variables s, and Δ_s^ξ is not changed along time. (A direct evaluation can be modeled in this way.)

(4) The delay between scheduling and execution time of an event e^ξ, denoted by t^ξ, is random variate. They can be generated independently from the simulation run. (*This point is different from ERG. In ERG, the delay is dependent on the edge, i.e, a couple of events, but I consider delay dependent on a single event.*)

(5) For all events e^ξ, the number of executions N^ξ is known before simulation.

$$e_i^{\xi,0} \geq 0 \qquad i=1,...,I^\xi \qquad \text{the } i\text{-th scheduling time of event } e^\xi.$$
$$e_i^{\xi,1} \geq 0 \qquad i=1,...,I^\xi \qquad \text{the } i\text{-th execution time of event } e^\xi.$$
$$\mathcal{E}_k \geq 0 \qquad k=0,...,K \qquad \text{time of the } k\text{-th execution of any events.}$$
$$u_k^s \in \mathbb{Z} \qquad k=0,...,K \qquad \text{value of state variable } s \text{ just after the } k\text{-th event.}$$
$$w_{i,k}^\xi \in \{0,1\} \qquad k=1,...,K \qquad \text{binding } e_i^{\xi,1} \text{ and } \mathcal{E}_k.$$
$$x_{i,k}^\xi \in \{0,1\} \qquad k=0,...,K \qquad \text{equal to one if } \mathcal{E}_k \text{ schedules } e^{\xi,0}.$$
$$y_{i,i'}^\xi \in \{0,1\} \qquad \qquad \text{binding } e_i^{\xi,0} \text{ and } e_{i'}^{\xi,1} \text{ in case of overtaking.}$$
$$z_k^\xi \in \{0,1\} \qquad k=0,...,K \qquad \text{equal to one if the condition for scheduling } e^\xi \text{ is true right after } \mathcal{E}_k.$$
$$v_k^{\xi,s,0} \in \{0,1\} \qquad k=0,...,K \qquad \text{equal to one if } s_k \leq a^{\xi,s} - 1$$
$$v_k^{\xi,s,1} \in \{0,1\} \qquad k=0,...,K \qquad \text{equal to one if } s_k \geq b^{\xi,s} - 1$$
$$r_k^\xi \in \mathbb{Z} \qquad k=0,...,K \qquad \text{number of existing parallel executions of } e_i^{\xi,1} \text{ after } \mathcal{E}_k \text{ before scheduling.}$$
$$n_k^\xi \in \mathbb{Z} \qquad k=0,...,K \qquad \text{number of scheduled executions of } e_i^{\xi,1} \text{ after } \mathcal{E}_k \text{ before scheduling.}$$

Table 3. Notation

Constraints (A): binding $e_i^{\xi,1}$ and $\mathcal{E}_k$:

$$e_i^{\xi,1} - \mathcal{E}_k \geq M(w_{i,k}^\xi - 1) \quad A1 \quad \forall \, \xi, i, k \tag{50}$$

$$\mathcal{E}_k - e_i^{\xi,1} \geq M(w_{i,k}^\xi - 1) \quad A2 \quad \forall \, \xi, i, k \tag{51}$$

$$\sum_k w_{i,k}^\xi = 1 \quad A3 \quad \forall \, \xi, i \tag{52}$$

$$\sum_{\xi,i} w_{i,k}^\xi = 1 \quad A4 \quad \forall \, k \tag{53}$$

$$\sum_k k w_{i+1,k}^\xi - \sum_k k w_{i,k}^\xi \geq 1 \quad A5 \quad \forall \, \xi, i \tag{54}$$

Constraints (B): binding $e_i^{\xi,0}$ and $e_{i'}^{\xi,1}$, where α^ξ is the maximal number of executions existing simultaneously in the event list:

$$e_{i'}^{\xi,1} - e_i^{\xi,0} \geq t_i^\xi + M(y_{i,i'}^\xi - 1) \quad B1 \quad \forall \xi, i, i' = 1,...,N^\xi \tag{55}$$

$$e_i^{\xi,0} - e_{i'}^{\xi,1} \geq -t_i^\xi + M(y_{i,i'}^\xi - 1) \quad B2 \quad \forall \xi, i, i' = 1,...,N^\xi \tag{56}$$

$$\sum_{i=1}^{N^\xi} y_{i,i'}^\xi = 1 \quad B3 \quad \forall \, \xi, i' = 1,...,N^\xi \tag{57}$$

$$\sum_{i'=1}^{N^\xi} y_{i,i'}^\xi = 1 \quad B4 \quad \forall \, \xi, i = 1,...,N^\xi \tag{58}$$

$$\sum_{i'=i+\alpha^\xi}^{N^\xi} y_{i,i'}^\xi = 0 \quad B5 \quad \forall \, \xi, \, i = 1,...,N^\xi - \alpha^\xi \tag{59}$$

$$\sum_{i=1}^{i'-\alpha^\xi} y_{i,i'}^\xi = 0 \quad B6 \quad \forall \, \xi, \, i' = \alpha^\xi + 1,...,N^\xi \tag{60}$$

If $\alpha^\xi = 1$, variables $y_{i,i'}^\xi$ are redundant and constraints (B) are reduced to:

$$e_i^{\xi,1} - e_i^{\xi,0} = t_i^\xi \quad B1 \quad \forall \xi, i = 1, ..., N^\xi \tag{61}$$

Number of executions of event e^ξ waiting in the event list can be a state variable n^ξ, and one condition for scheduling an e^ξ is $n^\xi \leq \alpha^\xi$. Thus, it can be managed as a generic scheduling condition.

Constraints (C): event e^ξ can be scheduled right after $\mathcal{E}_k$ if all state variables s satisfies condition $a_s^\xi \leq s_k \leq b_s^\xi$.

$$e_i^{\xi,0} - \mathcal{E}_k \geq M(x_{i,k}^\xi - 1) \quad C1 \quad \forall\, \xi, k, i \tag{62}$$

$$\mathcal{E}_k - e_i^{\xi,0} \geq M(x_{i,k}^\xi - 1) \quad C2 \quad \forall\, \xi, k, i \tag{63}$$

$$\sum_k x_{i,k}^\xi = 1 \quad C3 \quad \forall\, \xi, i \tag{64}$$

$$b^{\xi,s} - u_k^s \geq M(z_k^\xi - 1) \quad C4 \quad \forall\, \xi, k, s \tag{65}$$

$$u_k^s - a^{\xi,s} \geq M(z_k^\xi - 1) \quad C5 \quad \forall\, \xi, k, s \tag{66}$$

$$u_k^s - (b^{\xi,s} + 1) \geq M(v_k^{\xi,s,1} - 1) \quad C6 \quad \forall\, \xi, k, s \tag{67}$$

$$(a^{\xi,s} - 1) - u_k^s \geq M(v_k^{\xi,s,0} - 1) \quad C7 \quad \forall\, \xi, k, s \tag{68}$$

$$1 - z_k^\xi \leq \sum_{s \in S^\xi} v_k^{\xi,s,0} + \sum_{s \in S^\xi} v_k^{\xi,s,1} + v_k^{\xi,r} + v_k^{\xi,N} \quad C8 \quad \forall\, \xi, k \tag{69}$$

$$\sum_{i=1}^{N^\xi} x_{i,k}^\xi = z_k^\xi \quad C9 \quad \forall\, \xi, k \tag{70}$$

$$\sum_k k x_{i+1,k}^\xi - \sum_k k x_{i,k}^\xi \geq 1 \quad C10 \quad \forall\, \xi, i \tag{71}$$

Constraints (D): evolution of state variables

$$u_k^s = u_{k-1}^s + \sum_\xi \sum_{i=1}^{N^\xi} w_{i,k}^\xi \Delta^{\xi,s} \quad D1 \quad \forall\, s, k \tag{72}$$

$$r_k^\xi = r_{k-1}^\xi + X_{k-1}^\xi - \sum_i w_{i,k}^\xi \quad D2 \quad \forall\, \xi, k \tag{73}$$

$$R^\xi - r_k^\xi \geq z_k^\xi \quad D3 \quad \forall\, \xi, k \tag{74}$$

$$r_k^\xi \geq R^\xi v_k^{\xi,r} \quad D4 \quad \forall\, \xi, k \tag{75}$$

$$n_k^\xi = n_{k-1}^\xi + X_{k-1}^\xi \quad D5 \quad \forall\, \xi, k \tag{76}$$

$$N^\xi - n_k^\xi \geq z_k^\xi \quad D6 \quad \forall\, \xi, k \tag{77}$$

$$n_k^\xi \geq N^\xi v_k^{\xi,N} \quad D7 \quad \forall\, \xi, k \tag{78}$$

Constraints (E): others

$$\mathcal{E}_0 = 0 \quad E1 \tag{79}$$

$$\mathcal{E}_k - \mathcal{E}_{k-1} \geq 0 \quad E1 \quad \forall\, k \tag{80}$$

Objective function: with the constraints above, there is a unique solution in terms of event occurring times (solution of the binary variables could be multiple in case of multiple simultaneous events). Thus, the objective can be any function of event occurring time. I tried minimize/maximize the sum of $\mathcal{E}_k$, and they give the same solution.

Conditions for a variable x to be *resource-type* are not valid any more.

(1) $\forall \xi$ and s, upper bound b_s^ξ is monotonically increasing on x.
(2) $\forall \xi$ and s, lower bound a_s^ξ is monotonically decreasing on x.

The reason is that increasing b_s^ξ or decreasing a_s^ξ will tighten constraints C6 and C7. To be simple, we consider b only.

$$b - u_k^s \geq M(z_k^\xi - 1) \quad C4 - b \quad \forall \, \xi, k, s \tag{81}$$

$$u_k^s - (b+1) \geq M(v_k^{\xi,s,1} - 1) \quad C6 - b \quad \forall \, \xi, k, s \tag{82}$$

(when $u = b + 1$, event e^ξ cannot be scheduled.)
If b is increased to $b + 1$:

$$(b+1) - u_k^s \geq M(z_k^\xi - 1) \quad C4 - (b+1) \quad \forall \, \xi, k, s \tag{83}$$

$$u_k^s - (b+2) \geq M(v_k^{\xi,s,1} - 1) \quad C6 - (b+1) \quad \forall \, \xi, k, s \tag{84}$$

(when $u = b + 1$, event e^ξ must be scheduled.)
A group of relaxed constraints are:

$$(b+1) - u_k^s \geq M(z_k^\xi - 1) \quad C4 - (b+1) \quad \forall \, \xi, k, s \tag{85}$$

$$u_k^s - (b+1) \geq M(v_k^{\xi,s,1} - 1) \quad C6 - (b) \quad \forall \, \xi, k, s \tag{86}$$

(when $u = b + 1$, event e^ξ can be scheduled or not.)
Todo:

(1) What kind of performance indicators can be used? (Regular function of time, in scheduling area. Weighted sum, maximum. Refer to book on scheduling.)
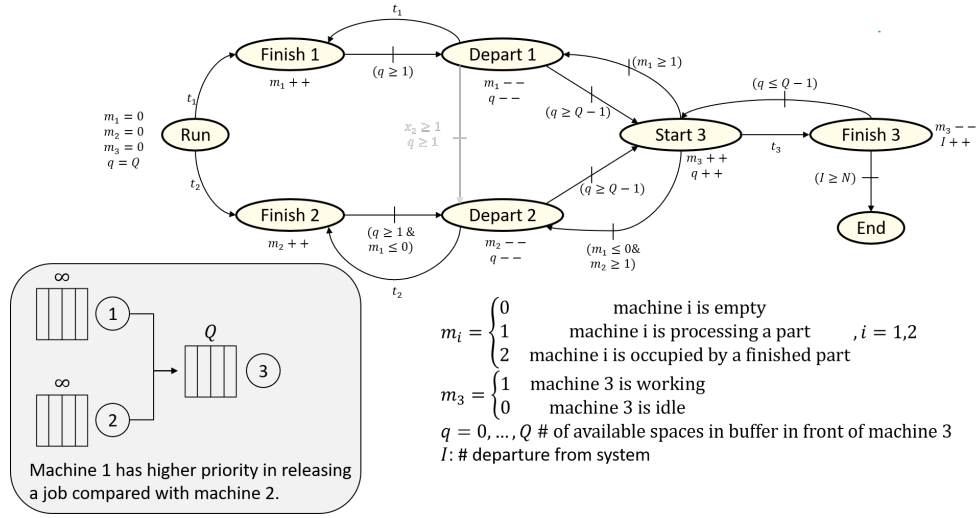
2.4. Merge

Figure 3. Example: merge.

Variable	Event	Condition to schedule	Delay	# executions	State change
$e^{s,1}$	Start m1	$m^1 \leq 0$	0	1	$m^1 + +$
$e^{f,1}$	Finish m1	$1 \leq m^1 \leq 1$	t^1	1	$m^1 + +$
$e^{d,1}$	Depart m1	$m^1 \geq 2\ AND$ $q \geq 1$	0	1	$m^1 = m^1 - 2,\ q - -$
$e^{s,2}$	Start m2	$m^2 \leq 0$	0	1	$m^2 + +$
$e^{f,2}$	Finish m2	$1 \leq m^2 \leq 1$	t^2	1	$m^2 + +$
$e^{d,2}$	Depart m2	$m^2 \geq 2\ AND$ $q \geq 1\ AND$ $m^1 \leq 1$	0	1	$m^2 = m^2 - 2,\ q - -$
$e^{s,3}$	Start m3	$m^3 \leq 0\ AND$ $q \leq Q - 1$	0	1	$m^3 + +,\ q + +$
$e^{d,3}$	Depart m3	$m^3 \geq 1$	t^3	1	$m^3 - -$

Table 4. Merge-S3M111

12

MP model:

$$\min \sum_k \mathcal{E}_k \tag{87}$$

$$e_i^{(\xi,j),1} - \mathcal{E}_k \geq M(w_{i,k}^{\xi,j} - 1) \qquad \xi \in \{s,f,d\}, j \in \{1,2,3\}, \forall i, k \tag{88}$$

$$\mathcal{E}_k - e_i^{(\xi,j),1} \geq M(w_{i,k}^{\xi,j} - 1) \qquad \xi \in \{s,f,d\}, j \in \{1,2,3\}, \forall i, k \tag{89}$$

$$\sum_k w_{i,k}^{\xi,j} = 1 \qquad \forall \, \xi \in \{s,f,d\}, j \in \{1,2,3\}, i \tag{90}$$

$$\sum_{(\xi,j),i} w_{i,k}^{\xi,j} = 1 \qquad \forall \, k \tag{91}$$

$$\sum_k k w_{i+1,k}^{\xi,j} - \sum_k k w_{i,k}^{\xi,j} \geq 1 \qquad \forall \, \xi \in \{s,f,d\}, j \in \{1,2,3\}, i \tag{92}$$

$$e_i^{s,j,1} - e_i^{s,j,0} \geq 0 \qquad j = 1,2,3, \forall \, i \tag{93}$$

$$e_i^{f,j,1} - e_i^{f,j,0} \geq t_i^j \qquad j = 1,2, \forall \, i \tag{94}$$

$$e_i^{d,j,1} - e_i^{d,j,0} \geq 0 \qquad j = 1,2, \forall \, i \tag{95}$$

$$e_i^{d,3,1} - e_i^{d,3,0} \geq t_i^3 \qquad \forall \, i \tag{96}$$

$$e_i^{\xi,j,0} - \mathcal{E}_k \geq M(x_{i,k}^{\xi,j} - 1) \qquad \forall \, \xi \in \{s,f,d\}, j = 1,2,3, k, i \tag{97}$$

$$\mathcal{E}_k - e_i^{\xi,j,0} \geq M(x_{i,k}^{\xi,j} - 1) \qquad \forall \, \xi \in \{s,f,d\}, j = 1,2,3, k, i \tag{98}$$

$$m_k^j = m_{k-1}^j + \sum_{i=1}^{N^j} (w_{i,k}^{s,j} + w_{i,k}^{f,j} - 2w_{i,k}^{d,j}) \qquad j = 1,2, \forall \, k \tag{99}$$

$$m_k^3 = m_{k-1}^3 + \sum_{i=1}^{N^3} (w_{i,k}^{s,3} - w_{i,k}^{d,3}) \qquad \forall \, k \tag{100}$$

$$q_k = q_{k-1} + \sum_{i=1}^{N^3} w_{i,k}^{s,3} - \sum_{i=1}^{N^1} w_{i,k}^{d,1} - \sum_{i=1}^{N^2} w_{i,k}^{d,2} \tag{101}$$

$$m_k^j \geq M(z_k^{s,j} - 1) \qquad j = 1,2,3, \forall \, k \tag{102}$$

$$1 - m_k^j \geq M(z_k^{f,j} - 1) \qquad j = 1,2, \forall \, k \tag{103}$$

$$m_k^j - 1 \geq M(z_k^{f,j} - 1) \qquad j = 1,2, \forall \, k \tag{104}$$

$$m_k^j - 2 \geq M(z_k^{d,j} - 1) \qquad j = 1,2, \forall \, k \tag{105}$$

$$q_k - 1 \geq M(z_k^{d,j} - 1) \qquad j = 1,2, \forall \, k \tag{106}$$

$$1 - m_k^1 \geq M(z_k^{d,2} - 1) \qquad \forall \, k \tag{107}$$

$$m_k^3 - 1 \geq M(z_k^{d,3} - 1) \qquad \forall \, k \tag{108}$$

$$(Q - 1) - q_k \geq M(z_k^{s,3} - 1) \qquad \forall \, k \tag{109}$$

$$\sum_k x_{i,k}^{\xi,j} = 1 \qquad \forall \, \xi \in \{s,f,d\}, j = 1,2,3, \forall \, i, k \tag{110}$$

$$\sum_{i=1}^{N^j} x_{i,k}^{\xi,j} \leq z_k^{\xi,j} \qquad \forall \, \xi \in \{s,f,d\}, j = 1,2,3, k \tag{111}$$

$$\sum_k k x_{i+1,k}^{\xi,j} - \sum_k k x_{i,k}^{\xi,j} \geq 0 \qquad \forall \, \xi \in \{s,f,d\}, j = 1,2,3, i \tag{112}$$

2.5. Merge - 2 machines in station 3

Variable	Value	Initialization	Description
e^1	0,1,2	2	number of empty machines in station 1
f^1	0,1,2	0	number of finished jobs in station 1
e^2	0,1	1	number of empty machines in station 2
f^2	0,1	0	number of finished jobs in station 2
e^3	0,1	1	number of empty machines in station 3
q	0,...,Q	Q	number of available spaces in queue

Table 5. State variables: Merge-S3M211

Variable	Event	Condition to schedule	Delay	# executions	State change
ai $e^{f,1}$	Finish m1	$1 \leq e^1 \leq 2$	t^1	2	f^1++
$e^{d,1}$	Depart m1	$1 \leq f^1 \leq 2\ AND$ $1 \leq q \leq Q$	0	1	$e^1++,\ f^1--,\ q--$
$e^{s,2}$	Start m2	$1 \leq e^2 \leq 1$	0	1	e^2--
$e^{f,2}$	Finish m2	$1 \leq e^2 \leq 1$	t^2	1	f^2++
$e^{d,2}$	Depart m2	$1 \leq f^2 \leq 1\ AND$ $1 \leq q \leq Q\ AND$ $0 \leq f^1 \leq 0$	0	1	$f^2--, e^2++,\ q--$
$e^{s,3}$	Start m3	$1 \leq e^3 \leq 1\ AND$ $0 \leq q \leq Q-1$	0	1	$e^3--,\ q++$
$e^{d,3}$	Depart m3	$1 \leq e^3 \leq 1\ AND$ $0 \leq q \leq Q-1$	t^3	1	e^3++

Table 6. Events: Merge-S3M211

2.6. Failure

2.7. Jobshop

2.8. Identifying Resource-type variables

3. Gradient-based approximate cut

3.1. Gradient estimation

3.2. Gradient-based feasibility cut

4. Combinatorial cut generation

4.1. Combinatorial cut

4.2. Heuristic for tightening Exact combinatorial cut

5. Feasibility-cut-based algorithm

The complete algorithm for solving RAP–PC is summarized in Algorithm 1. The resource capacities are initialized to the lower bound. The searching region of RAP–PC–MIP is initialized to $\mathbb{X}$, and the lower and upper bounds of the objective function, C^L and C^U, respectively, are set considering the upper bound and lower bound of the capacity of each resource. Lines 7 to 11 show that approximate cuts are generated and used in the model when infeasible solutions are found. Once a feasible solution is found, the upper bound C^U, which is also the incumbent solution, can be updated after comparing the value of the found feasible solution and that of the current incumbent. Then, all the currently used approximate cuts are replaced by exact cuts of the DIS. If there are only exact cuts in RAP–PC–MIP, the solution is the new lower bound C^L. The algorithm terminates when the gap between the upper bound and lower bound is within a tolerance or the time limit is exceeded.

6. Numerical analysis

7. Conclusion

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Algorithm 1 MIP–based algorithm.

Input:

Lower bound $\mathbf{a} = [a_1, ..., a_J]$ and upper bound $\mathbf{b} = [b_1, ..., b_J]$ of resource capacity $\mathbf{x}$, such that $a_j \leq x_j \leq b_j \; \forall \; j = 1, ..., J$.
Tolerance of optimality gap ε_{opt}.
Optional input: time limit of the algorithm T_{lim}.

Ensure:

Sample-path global optimal $\mathbf{x}^*$.

1: Initialize system with lower bound $\mathbf{x} \leftarrow \mathbf{a}$
2: Initialize incumbent with upper bound $\mathbf{x}^* \leftarrow \mathbf{b}$.
3: Initialize lower bound of the objective $C^L \leftarrow \mathbf{c}^T \mathbf{a}$.
4: Initialize upper bound of the objective $C^U \leftarrow \mathbf{c}^T \mathbf{b}$.
5: Add initial constraints which defines $\mathbb{X}$ to the RAP–PC–MIP.
6: **while** $C^U - C^L > \varepsilon_{opt}$ and T_{lim} is not exceeded. **do**
7:     **while** There exists at least one violated performance constraint **do**
8:         Generate one approximate cut $CA(\bar{\mathbf{x}}, l)$ for each violated constraints l and add all the generated cuts to the RAP–PC–MIP.
9:         $\bar{\mathbf{x}} \leftarrow$ solution of the RAP–PC–MIP.
10:        Simulate the system of $\bar{\mathbf{x}}$.
11:    **end while**
12:    Update upper bound and incumbent $C^U \leftarrow \mathbf{c}^T \bar{\mathbf{x}}$, $\mathbf{x}^* \leftarrow \bar{\mathbf{x}}$ if $\mathbf{c}^T \bar{\mathbf{x}} < C^U$.
13:    **if** There exist approximate cuts in RAP–PC–MIP **then**
14:        For all the currently used approximate cuts $CA(\bar{\mathbf{x}}^r, l)$, find dominating infeasible solution $\bar{\mathbf{x}}_d(\bar{\mathbf{x}}^r)$ and replace approximate cuts $CA(\bar{\mathbf{x}}^r, l)$ by exact cuts $CE(\bar{\mathbf{x}}_d(\bar{\mathbf{x}}^r), l)$ of the DIS.
15:        $\bar{\mathbf{x}} \leftarrow$ solution of the RAP–PC–MIP.
16:        Simulate the system of $\bar{\mathbf{x}}$.
17:        Update lower bound $C^L \leftarrow \max\{\mathbf{c}^T \bar{\mathbf{x}}, \; C^L\}$.
18:    **end if**
19: **end while**

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