Mathematical Programming Representation of Discrete-Event Simulation

ARTICLE HISTORY
Compiled August 28, 2020

ABSTRACT

1. Introduction

Discrete Event Simulation (DES) is one of the most used tool for performance evaluation of complex systems and, hence, simulation–optimization algorithms are widely used when performance evaluation has to be coupled with optimization, i.e., when the best system configuration, according to some criteria, has to be found meanwhile guaranteeing a given value of some performance measure. Most of the state–of–the–art simulation–optimization algorithms consider DES as a *black–box* function, and the structure of DES models has been seldom studied. Under the black–box setting, simulation–optimization algorithms work in an iterative way, alternating simulation and optimization procedures, and always require many iterations to explore the feasible region of the optimization problem, thus possibly leading to computational inefficiency. On the contrary, a minority of the simulation–optimization literature explores the structure of the DES models, and such research is referred to as *white–box* simulation–optimization. The benefit of white–box simulation–optimization is the saving of simulation budget due to the fact that the optimization procedure is guided by the information contained in the structure of the DES model. However, the barrier to the use of white–box simulation–optimization is how to model DES as white–box, so that it eventually favors optimization. This work proposes a procedure to establish a white–box simulation model, which is equivalent Mathematical Programming Representation (MPR), based on the well-known event–scheduling logic for certain types of DES models. Specifically, the modeling procedure, the conditions under which it can be applied and some examples are discussed.

Chan and Schruben (2008) proposed a modeling framework to translate a DES model into an MPR model in a general sense. Their modeling framework is based on the Event Relationship Graph (ERG) of the system dynamics. To derive the MPR, an ERG of the discrete event system has to be constructed and expanded to an elementary ERG (EERG) model, and a routine procedure can be applied to translate the EERG model into an MPR. However, this procedure has some limitations. First, deriving an ERG is not an easy task, and the user has to pay quite much attention to detect all the event relationships and complete the triggering conditions between each pair of related events. The difficulty of developing ERG limits the wide spread of this procedure. Second, the modeling procedure is case–by–case, which means the user has to first identify which situations he/she faces by analyzing the EERG, and then choose the appropriate model, including the variables and constraints. This is quite difficult, since EERG is an expansion of ERG and the resulting graph could be huge and writing down the complete MPR model could be even impossible. This work proposes a procedure that does not need the ERG and can be used to automatically generate in a general–purpose programming language. Despite being different, the MPRs proposed in this work and Chan and Schruben (2008) lead to equivalent results, which, in turn, are both equivalent to a simulation realization.

The benefit of developing an MPR might be not obvious (especially when there is already a DES model) due to the high complexity of solving MPR. However, the MPR of a simulation model favors the optimal design and control of the discrete event systems and can use the vast theory and methodological developing in the mathematical programming (MP) field. Many works in the literature show the potentiality of this research direction. For instance, the gradient can be conveniently estimated from the simulation model, if the MPR is approximated into Linear Programming (LP) and the dual can be conveniently obtained (Chan & Schruben, 2008; Zhang, Matta, & Alfieri, 2020). Moreover, if some of the parameters in the MPR are changed to decision variables, the MPR becomes an integrated simulation–optimization model. Solving the integrated model provides the optimal solution of the optimization problem (Matta, 2008). MP–based algorithms, such as linear programming approximation (Alfieri & Matta, 2012), Benders decomposition (Weiss & Stolletz, 2015), column generation (Alfieri, Matta, & Pastore, 2020), have been applied to improve the efficiency of integrated MP model solution.

The application of MPR–based simulation–optimization approaches is usually found in operations management of manufacturing and service systems. The integrated simulation–optimization model has proved itself to be well suited in solving the buffer allocation problem (Zhang, Pastore, Matta, & Alfieri, n.d.). Thanks to the flexibility of DES in evaluating complex systems, the buffer allocation problem of production systems with complex blocking mechanism, such as kanban control, base stock control, extended kanban control, can be managed (Pedrielli, Alfieri, & Matta, 2015). Thanks to the flexibility of MPR in modeling optimization problems, problems involving real–valued decision variables such as optimal production rate (Tan, 2015), bottleneck detection and throughput improvement problem (Zhang & Matta, 2020) have all been well addressed and the sample–path global optimal solution can be obtained. Before the above mentioned works were proposed, there were many state–of–the–art heuristic approaches addressing those problems, but without any guarantee of global or local optimality. Thus, the development of MPR–based simulation–optimization has made its contribution in the research area of manufacturing and service system optimization.

The rest of the paper is organized as follows.

2. MPR generation procedure

2.1. *Event-scheduling execution logic of DES*

The event-scheduling approach is the logic behind all major simulation software and used by practitioners when developing simulation codes with general purpose languages (Law, 2014). The logic is briefly shown in Figure 1. The fundamental elements are the system states and the events. The system state is a collection of state variables to describe the system at a particular time. An event execution can lead to the change of system state. The event list contains the scheduled but not executed events together with occurring times. When simulation is launched, the system state and the event list is initialized with user-defined values, and the simulation clock is set to zero. The event with the earliest occurring time in the event list will be executed, and the system evolves into a new state together with the simulation clock. The new state may enable to schedule new events, i.e., adding new event together with execution time to the event list, or cancel events, i.e., removing some event executions from the event list. There is usually a delay between time when an event is added to the event list and the time when the event is executed, but the delay might be equal to zero or positive. In the remaining of this work, we refer time when an event is added to the event list as the *scheduling time*. The algorithm will terminate with certain conditions.

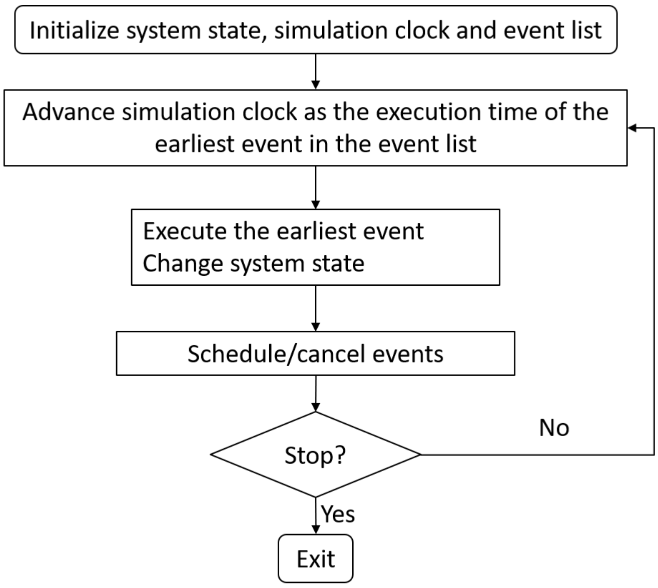

Figure 1. Event-scheduling simulation algorithm.

In this section, a procedure to translating DES models into MRP is introduced. Before presenting the procedure, the conditions that the DES model has to satisfy in order to have the procedure applicable, are described.

2.2. Assumptions

To apply the procedure proposed in this work, the following assumptions must be satisfied.

(1) State variables are integer.
(2) For all the events e^ξ of type ξ, the *scheduling conditions* are in the form $a_s^\xi \leq s \leq c_s^\xi$ for certain state variable s, when multiple state variables are involved, they are combined using logical operator "AND".
(3) The delay between the scheduling time and the execution time of event e^ξ are independent and identically distributed random variables.
(4) When more than one event in the event list have the same execution time, the execution sequence is immaterial, i.e., each execution sequence leads to the same new system state and new event list when the simulation clock is advanced.

(5) Simulation terminates when the number of execution of each event e^{ξ} reaches N^{ξ}, which is known, i.e., the number of entities in the system is known.

The first assumption says that the state variables should be integer. Integer variables widely exist in DES models, such as number of jobs in buffers, idle servers, and binary variables to model system behavior and control. A discrete state variable can be translated into an integer variable or a set of binary variables. For real-valued state variables, it can be approximately discretized. Thus, the first assumption is fairly general.

The second condition is, instead, more strict. However, if one model does not satisfy this condition, one can consider to introduce extra binary variables to satisfy assumption (2). For instance, if the condition to schedule event e^{ξ} is $s \leq a^{\xi}\ OR\ s \geq b^{\xi}$, a binary variable $\tilde{s}$ is introduced to the model, and $\tilde{s}$ is equal to one if and only if $s \leq a^{\xi}\ OR\ s \geq b^{\xi}$.

As for the third condition, if the delay is not an iid random variate, it should be splitted into several events so that each event has iid delay. For instance, if the distribution of service time depends on the job type, the event of *finishing* a job should be splitted such that the *finishing* of each job type is represented by one event.

The forth condition, in practice, says that the execution time is the only attribute of priority for the event executions in the event list. If one would like to specify the priority to some events having the same execution time, he/she can specify the priority by adding the event with lower priority to the event list after the execution of the event with higher priority, which can be done by introducing extra binary variables.

The fifth condition specifies the termination condition. Specifically, for queueing systems, the termination condition refers to the number of jobs passing through each station.

2.3. Mathematical programming model

The MPR represents the dynamics of the simulated system. Specifically, event scheduling time, event execution time and state variable changes during simulation can all be seen in the MPR as decision variables. The i-th scheduling time and the i-th execution time of event e^{ξ} are denoted by $e_i^{\xi,0}$ and $e_i^{\xi,1}$, respectively. The system state changes with a series of event executions, and $\mathcal{E}_k$ denotes the time of the k-th execution. The index i represents the sequence of scheduling or execution of a specific event type, and the index k represents the sequence of execution of general event types. The simulation clock is initialized to zero with $\mathcal{E}_0 = 0$. $e_i^{\xi,0}$, $e_i^{\xi,1}$ and $\mathcal{E}_k$ are all real-valued. The notation u_k^s is used to denote the value of state variable s just after the k-th execution, i.e., just after $\mathcal{E}_k$. The domain of variable u_k^s should be integer. The initial system state is given as u_0^s. Some binary variables are also used in the MPR, and they are introduced as the model is explained in details.

To implement the event-schedule algorithm and generate MPR, for each event type, the conditions to schedule, the conditions to cancel, the distribution of delay between scheduling and execution and the state variable changes upon execution should be provided. In the event list, multiple executions of the same event type are allowed. For instance, in a G/G/m queue system, when all the servers are occupied, there are m executions of departure event in the event list. The number of executions of event e^{ξ}, denoted by β^{ξ}, is also mandatory to develop the MPR.

The notations of some sets are as follows. The set of all event types is denoted by Ξ. I^{ξ} denotes the set of $\{1, ..., N^{\xi}\}$, which is the number of executions of event e^{ξ}, and

K denotes the set of $\{1, ..., \sum_{\xi\in\Xi} N^{\xi}\}$, which is the number of executions of all events types. S denotes the set of state variables. S^{ξ} and $S^{\bar{\xi}}$ denote the set of state variables which is relevant to scheduling and cancellation condition of event e^{ξ}, respectively.

2.3.1. Event execution time

The first group of constraints, denoted by group A, are the constraints binding executions $e_i^{\xi,1}$ and $\mathcal{E}_k$. Binary variables $w_{i,k}^{\xi}$ are used, and $e_i^{\xi,1}$ and $\mathcal{E}_k$ are binded if and only if $w_{i,k}^{\xi}$ is equal to one, as shown in constraints (A1) and (A2). Constraints (A3) and (A4) state that each $e_i^{\xi,1}$ can be binded to one and only one $\mathcal{E}_k$. Constraints (A5) imply that the $\mathcal{E}_k$ are temporally sequenced with index k. Constraints (A6) show that the i-th execution of event e^{ξ} cannot be binded to an execution earlier than $(i-1)$-th execution of event e^{ξ}.

$$
\begin{aligned}
e_i^{\xi,1} - \mathcal{E}_k \geq M(w_{i,k}^{\xi} - 1) &\quad \forall\, \xi \in \Xi, i \in I^{\xi}, k \in K \quad (A1)\\
\mathcal{E}_k - e_i^{\xi,1} \geq M(w_{i,k}^{\xi} - 1) &\quad \forall\, \xi \in \Xi, i \in I^{\xi}, k \in K \quad (A2)\\
\sum_{k\in K} w_{i,k}^{\xi} = 1 &\quad \forall\, \xi \in \Xi, i \in I^{\xi} \quad (A3)\\
\sum_{\xi\in\Xi}\sum_{i\in I^{\xi}} w_{i,k}^{\xi} = 1 &\quad \forall\, k \in K \quad (A4)\\
\mathcal{E}_k - \mathcal{E}_{k-1} \geq 0 &\quad \forall\, k \in K \quad (A5)\\
\sum_{k\in K} k w_{i,k}^{\xi} - \sum_{k\in K} k w_{i-1,k}^{\xi} \geq 1 &\quad \forall\, \xi \in \Xi, i \in I^{\xi} \quad (A6)
\end{aligned}
$$

2.3.2. Multiple-execution events

The second group of constraints, denoted by group B, are the constraints binding event scheduling $e_i^{\xi,0}$ and event execution $e_{i'}^{\xi,1}$. If the event is single–execution, i.e., the maximal number of executions in the event list is equal to one, the i-th execution $e_i^{\xi,1}$ is binded to the i-th scheduling with a delay t_i^{ξ}, as in constraints (B1), where t_i^{ξ} is a sample of delay between scheduling and execution. Thus, the variable $e_i^{\xi,1}$ can be replaced by $e_i^{\xi,0} + t_i^{\xi}$ and the MPR model is reduced.

$$e_i^{\xi,1} - e_i^{\xi,0} = t_i^{\xi} \quad \forall \xi \in \Xi, i \in I^{\xi} \quad (B1)$$

If event e^{ξ} is a multiple-execution event, i.e., the maximal number of executions in the event list β^{ξ} is at least two, a late scheduled execution could overtake an early scheduled one, since the delay time between scheduling and execution is a random variate. For instance, in a G/G/m queue, the first starting job may be not the first leaving job, if its service time is pretty long. Thus, the binary variable $y_{i,i'}^{\xi}$ is introduced, and the i-th scheduling of event e^{ξ} is the i'-th execution when $y_{i,i'}^{\xi}$ is equal to one, as in constraints (B2) and (B3). Constraints (B4) and (B5) show that each $e_{i'}^{\xi,1}$ can be binded to one and only one $e_i^{\xi,0}$. Constraints (B6) imply that the i-th scheduling cannot be binded to an execution earlier than the $(i+\beta^{\xi})$-th, since at most β^{ξ} executions are allowed to be in the list at the same time. For the same reason, constraints (B7) state that $i^{'}$-th execution cannot be binded to the scheduling later than the $(i^{'} - \beta^{\xi})$-th. For instance, in a G/G/2 queue, the third arrival job cannot be the first departure job, since its service starts after the first departure.

$$e_{i'}^{\xi,1} - e_{i}^{\xi,0} \geq t_i^{\xi} + M(y_{i,i'}^{\xi} - 1) \qquad \forall \xi \in \Xi, i, i^{'} \in I^{\xi} \qquad (B2)$$

$$e_{i}^{\xi,0} - e_{i'}^{\xi,1} \geq -t_i^{\xi} + M(y_{i,i'}^{\xi} - 1) \qquad \forall \xi \in \Xi, i, i^{'} \in I^{\xi} \qquad (B3)$$

$$\sum_{i \in I^{\xi}} y_{i,i'}^{\xi} = 1 \qquad \forall \xi \in \Xi, i^{'} \in I^{\xi} \qquad (B4)$$

$$\sum_{i' \in I^{\xi}} y_{i,i'}^{\xi} = 1 \qquad \forall \xi \in \Xi, i \in I^{\xi} \qquad (B5)$$

$$\sum_{i'=i+\beta^{\xi}}^{N^{\xi}} y_{i,i'}^{\xi} = 0 \quad \forall \xi \in \Xi, 1 \leq i \leq N^{\xi} - \beta^{\xi} \quad (B6)$$

$$\sum_{i=1}^{i'-\beta^{\xi}} y_{i,i'}^{\xi} = 0 \quad \forall \xi \in \Xi,\ \beta^{\xi} + 1 \leq i^{'} \leq N^{\xi} \quad (B7)$$

2.3.3. Constraints for scheduling new events

The third group of constraints, denoted by group C, state that an execution of event e^{ξ} can be scheduled right after $\mathcal{E}_k$ if the condition for scheduling an event e^{ξ} is true. Binary variables $x_{i,k}^{\xi}$ is used, and $x_{i,k}^{\xi}$ equal to one represents that the i-th scheduling of event e^{ξ} is enabled by $\mathcal{E}_k$, as in constraints (C1) and (C2). As stated in Section 3.1, the condition for scheduling event e^{ξ} is a set of inequalities in the form $a_s^{\xi} \leq s \leq c_s^{\xi}$ combined with operator "AND". Binary variables z_k^{ξ} are used to verify that condition, and if z_k^{ξ} is equal to one, the condition is true, as in constraints (C3) and (C4). Moreover, a set of binary variables $v_k^{\xi,s,0}$ and $v_k^{\xi,s,1}$ are used to verify if the condition is false. Specifically, constraints (C5) state that if $v_k^{\xi,s,a}$ is equal to one, u_k^s will be smaller than $a^{\xi,s}$, and hence, the inequality $a_s^{\xi} \leq s$ is violated. Similar for constraints (C6), if $v_k^{\xi,s,b}$ is equal to one, $s \leq c_s^{\xi}$ is violated. Constraints (C7) imply that z_k^{ξ} is equal to one if and only if $a_s^{\xi} \leq s \leq c_s^{\xi}$, and that z_k^{ξ} is equal to zero if and only if $a_s^{\xi} \leq s \leq c_s^{\xi}$ is violated. Constraints (C8) state that if the condition to schedule e^{ξ} is true, an execution of e^{ξ} has to be added into the event list. Constraints (C9) show that each scheduling is enabled by one and only one $\mathcal{E}_k$. Constraints (C10) show that if the i-th scheduling of event e^{ξ} is scheduled by execution $\mathcal{E}_k$, the $(i-1)$-th execution cannot be scheduled by an execution later than $\mathcal{E}_k$.

$$
\begin{aligned}
e_i^{\xi,0} - \mathcal{E}_k &\geq M(x_{i,k}^{\xi} - 1) \quad \forall\, \xi \in \Xi, k \in K, i \in I^{\xi} && (C1)\\
\mathcal{E}_k - e_i^{\xi,0} &\geq M(x_{i,k}^{\xi} - 1) \quad \forall\, \xi \in \Xi, k \in K, i \in I^{\xi} && (C2)\\
u_k^s - a^{\xi,s} &\geq M(z_k^{\xi} - 1) \quad \forall\, \xi \in \Xi, k \in K, s \in S && (C3)\\
b^{\xi,s} - u_k^s &\geq M(z_k^{\xi} - 1) \quad \forall\, \xi \in \Xi, k \in K, s \in S && (C4)\\
(a^{\xi,s} - 1) - u_k^s &\geq M(v_k^{\xi,s,0} - 1) \quad \forall\, \xi \in \Xi, k \in K, s \in S && (C5)\\
u_k^s - (b^{\xi,s} + 1) &\geq M(v_k^{\xi,s,1} - 1) \quad \forall\, \xi \in \Xi, k \in K, s \in S && (C6)\\
1 - z_k^{\xi} &\leq \sum_{s \in S^{\xi}} v_k^{\xi,s,0} + \sum_{s \in S^{\xi}} v_k^{\xi,s,1} \qquad \forall\, \xi \in \Xi, k \in K && (C7)\\
\sum_{i \in I^{\xi}} x_{i,k}^{\xi} &= z_k^{\xi} \qquad \forall\, \xi \in \Xi, k \in K && (C8)\\
\sum_{k \in K} x_{i,k}^{\xi} &= 1 \qquad \forall\, \xi \in \Xi, i \in I^{\xi} && (C9)\\
\sum_{k \in K} k x_{i,k}^{\xi} - \sum_{k \in K} k x_{i-1,k}^{\xi} &\geq 1 \qquad \forall\, \xi \in \Xi, i \in I^{\xi} && (C10)\\
\sum_{k \in K} k x_{i+1,k}^{\xi} - \sum_{k \in K} k w_{i,k}^{\xi} &\geq 0 \quad \forall\, \xi \in \Xi, i \in I^{\xi}, t^{\xi} = 0 && (C11)\\
x_{i,k}^{\xi} &= x_{i,k}^{\tilde{\xi}} \quad \forall\, \xi \in \Xi, i \in I^{\xi}, t^{\xi} > 0 && (C12)
\end{aligned}
$$

Different rules are applied to zero-delay events and positive-delay events. First, all the zero-delay events are single-execution, because when the delay between scheduling and execution of an event is zero, multiple executions is equivalent to sequential scheduling of a single-execution event. Thus, for zero-delay events, constraints (C11) are relevant, which state that a new event can be scheduled only after the previous execution is conducted. For the event e^{ξ} with strictly positive delay, the following routine should be followed to formulate a correct MPR. A *counting* events $e^{\tilde{\xi}}$, which has the same scheduling condition as e^{ξ} and is zero-delay should be included artificially. The execution of $e^{\tilde{\xi}}$ will increment the number of executions of e^{ξ}, which is a state variable, by one. Constraints of group A and constraints (B1) (C1), (C2), (C10) and (C11) should also be applied to all counting events. Since each time the condition to schedule e^{ξ} is true, event $e^{\tilde{\xi}}$ can also be scheduled, the i-th scheduling of event e^{ξ} and $e^{\tilde{\xi}}$ should be enabled by the same execution, as stated in constraints (C12), and there is no need to repeat constraints (C3) to (C9) for event $e^{\tilde{\xi}}$. In many cases, an event can play the role of counting event for another event, so there is no need to create a duplicating event. For instance, a DES model of G/G/m queue is composed of three events, which are arrival, start and finish. The condition to schedule both start and finish events is an idle server and a job in the queue, and start event is zero-delay and increases the number of busy server by one, which is equivalent to the number executions of finish events. Thus, the start event can be used as the counting event for finish event

2.3.4. Constraints for event cancellation

The forth group of constraints, denoted by group D, state that executions of event e^{ξ} in the event list can be canceled right after $\mathcal{E}^k$ if the cancellation condition is true. Similar to constraints (C3) to (C7), constraint (D1) to (D5) show that binary variables $z_k^{\bar{\xi}}$ are equal to one if the cancellation condition of e^{ξ} is true right after $\mathcal{E}^k$, where binary variable $z_k^{\bar{\xi}}$, $v_k^{\bar{\xi},s,0}$ and $v_k^{\bar{\xi},s,1}$ are the counter part of z_k^{ξ}, $v_k^{\xi,s,0}$ and $v_k^{\xi,s,1}$, but for event cancellation other than event scheduling. Binary variables $\theta_{i,k}^{\xi}$ are used to represent cancellation an execution. Specifically, $\theta_{i,k}^{\xi}$ is equal to one if and only if the i-th execution of event e^{ξ} is canceled after execution $\mathcal{E}_k$. For the i-th execution of event e^{ξ}, integer variables $k_i^{\xi,0}$ represent the index of execution that scheduled it and $k_i^{\xi,1}$ represent its execution sequence. The definition of $k_i^{\xi,0}$ and $k_i^{\xi,1}$ is shown in constraints (D6) to (D8). The i-th execution of event e^{ξ} is canceled after execution $\mathcal{E}_k$ if the cancellation condition is true at a certain time between its scheduling $k_i^{\xi,0}$ and execution $k_i^{\xi,1}$, i.e., there exist k between $k_i^{\xi,0}+1$ and $k_i^{\xi,1}-1$ such that $z_k^{\bar{\xi}}$ is equal to one. Binary variables $\theta_{i,k}^{\xi}$ is equal to one if and only if the i-th execution of event e^{ξ} is canceled since $z_k^{\bar{\xi}}$ is equal to one as in constraints (D9) to (D13), where binary variables $\phi_{i,k}^{\xi,0}$ and $\phi_{i,k}^{\xi,1}$ represent if k is smaller than $k_i^{\xi,0}-1$ or greater than $k_i^{\xi,1}+1$, respectively. Binary variables $\gamma_{i,k}^{\xi}$ equal to one show that the i-th execution of event e^{ξ} is actually executed as the k-th execution $\mathcal{E}_k$, without cancellation, which is implied with constraints (D14) and (D15).

$$u_k^s - a^{\bar{\xi},s} \geq M(z_k^{\bar{\xi}} - 1) \quad \forall\, \xi \in \Xi, k \in K, s \in S^{\bar{\xi}} \quad (D1)$$
$$b^{\bar{\xi},s} - u_k^s \geq M(z_k^{\bar{\xi}} - 1) \quad \forall\, \xi \in \Xi, k \in K, s \in S^{\bar{\xi}} \quad (D2)$$
$$(a^{\bar{\xi},s} - 1) - u_k^s \geq M(v_k^{\bar{\xi},s,0} - 1) \quad \forall\, \xi \in \Xi, k \in K, s \in S^{\bar{\xi}} \quad (D3)$$
$$u_k^s - (b^{\bar{\xi},s} + 1) \geq M(v_k^{\bar{\xi},s,1} - 1) \quad \forall\, \xi \in \Xi, k \in K, s \in S^{\bar{\xi}} \quad (D4)$$
$$1 - z_k^{\bar{\xi}} \leq \sum_{s \in S^{\bar{\xi}}} v_k^{\bar{\xi},s,0} + \sum_{s \in S^{\bar{\xi}}} v_k^{\bar{\xi},s,1} \quad \forall\, \xi \in \Xi, k \in K \quad (D5)$$
$$k_i^{\xi,1} = \sum_{k \in K} k w_{k,i}^{\xi} \quad \forall\, \xi, i \quad (D6)$$
$$k_i^{\xi,0} \geq k + M(y_{i',i}^{\xi} - 1) + M(x_{i',k}^{\xi} - 1) \quad \forall\, \xi \in \Xi, i \in I^{\xi} \quad (D7)$$
$$k_i^{\xi,0} \leq k + M(1 - y_{i',i}^{\xi}) + M(1 - x_{i',k}^{\xi}) \quad \forall\, \xi \in \Xi, i \in I^{\xi} \quad (D8)$$
$$k z_k^{\bar{\xi}} - k_i^{\xi,0} - 1 \geq M(\theta_{i,k}^{\xi} - 1) \quad \forall\, \xi \in \Xi, i \in I^{\xi} \quad (D9)$$
$$k_i^{\xi,1} - 1 - k z_k^{\bar{\xi}} \geq M(\theta_{i,k}^{\xi} - 1) \quad \forall\, \xi \in \Xi, i \in I^{\xi}, k \in K \quad (D10)$$
$$k_i^{\xi,0} - 1 - k z_k^{\bar{\xi}} \geq M(\phi_{i,k}^{\xi,0} - 1) \quad \forall\, \xi \in \Xi, i \in I^{\xi}, k \in K \quad (D11)$$
$$k z_k^{\bar{\xi}} - k_i^{\xi,1} - 1 \geq M(\phi_{i,k}^{\xi,1} - 1) \quad \forall\, \xi \in \Xi, i \in I^{\xi}, k \in K \quad (D12)$$
$$1 - \theta_{i,k}^{\xi} \leq \phi_{i,k}^{\xi,0} + \phi_{i,k}^{\xi,1} \quad \forall\, \xi \in \Xi, i \in I^{\xi}, k \in K \quad (D13)$$
$$\gamma_{i,k}^{\xi} \geq w_{i,k}^{\xi} - \sum_{k' \in K} \theta_{i,k'}^{\xi} \quad \forall\, \xi \in \Xi, i \in I^{\xi}, k \in K \quad (D14)$$
$$w_{i,k}^{\xi} - \sum_{k' \in K} \theta_{i,k'}^{\xi} \geq M(\gamma_{i,k}^{\xi} - 1) \quad \forall\, \xi \in \Xi, i \in I^{\xi}, k \in K \quad (D15)$$

2.3.5. Constraints for state evolution

Constraints (E1) represent the evolution of state variables. If the $\mathcal{E}_k$ is of event type ξ, the state variable s is changed by function $f^{\xi}(u_{k-1}^s)$. Constraints (E2) to (E4) show the evolution of the state variable s^{ξ}, which counts the number of executions of e^{ξ} in the event list. If $z_k^{\bar{\xi}}$ is equal to one, s^{ξ} is set to zero, as in (E2). Otherwise, it is increased by one if a new execution is added and decreased by one if one execution is conducted.

$$u_k^s = \sum_{\xi \in \Xi} \sum_{i \in I^{\xi}} \gamma_{i,k}^{\xi} f^{\xi}(u_{k-1}^s) \quad \forall\, s \in S, k \in K \quad (E1)$$
$$u_k^{s^{\xi}} \leq \beta^{\xi}(1 - z_k^{\bar{\xi}}) \quad \forall\, \xi \in \Xi, k \in K \quad (E2)$$
$$u_k^{s^{\xi}} \leq u_{k-1}^{s^{\xi}} + z_{k-1}^{\xi} - \sum_{i \in I^{\xi}} \gamma_{i,k}^{\xi} + \beta^{\xi} z_k^{\bar{\xi}} \quad \forall\, \xi \in \Xi, k \in K \quad (E3)$$
$$u_k^{s^{\xi}} \geq u_{k-1}^{s^{\xi}} + z_{k-1}^{\xi} - \sum_{i \in I^{\xi}} \gamma_{i,k}^{\xi} - \beta^{\xi} z_k^{\bar{\xi}} \quad \forall\, \xi \in \Xi, k \in K \quad (E4)$$

If all the events changes the state variables with a fixed increment or decrement equal to $\Delta^{\xi,s}$, constraints (E1) will be changed to (E5), which are linear constraints, and the MPR is a MILP.

$$u_k^s = u_{k-1}^s + \sum_{\xi \in \Xi} \sum_{i \in I^\xi} \gamma_{i,k}^\xi \Delta^{\xi,s} \quad \forall\ s \in S, k \in K \quad (E5)$$

2.3.6. Objective function

With the constraints above, there is a unique feasible solution in terms of event occurring times, but the binary variables could be multiple, since there could be more than one simultaneous events. Thus, the definition of objective function is quite flexible.

3. Examples

In this section, several examples are presented. For each example, the necessary information to be provided, and then the derivation of MPR are shown.

3.1. G/G/1 queue

The first example is a G/G/1 queue. Table 1 shows the events composing the DES model of G/G/1, i.e., the arrival e^a, the start of service e^r and the finish of service e^f. The state variables includes the state of server m, the number of jobs in the queue q. Server state m equal to zero represents that the server is idle, and m equal to one implies that the server is occupied. Start event can be used as the counting events of finish event, and state variable m equal to zero or one is equivalent to the number of executions of e^f in the event list. Arrival event is positive-delay, a counting event $e^{\tilde{a}}$ and the counter of arrival events in the event list u^a are included. If the simulation run includes N jobs, the number of executions of events e^a, $e^{\tilde{a}}$, e^r, e^f are all equal to N, i.e, $I^a = I^{\tilde{a}} = I^r = I^f = \{1, ..., N\}$ and $K = \{1, ..., 4N\}$.

Variable	Event	Condition to schedule	Delay	β^ξ	State change
e^a	Arrival	$u^a \leq 0$	t^a	1	$q++, u^a--$
$e^{\tilde{a}}$	Counting arrival	$u^a \leq 0$	0	1	u^a++
e^r	Start	$1 \leq q, m \leq 0$	0	1	$m++, q--$
e^f	Finish	$1 \leq q, m \leq 0$	t^f	1	$m--$

Table 1. Events to simulate G/G/1 system.

The MPR proposed in this work is as follows:

$$
\begin{aligned}
&(A1)-(A6),(B1),(C1),(C2),(C8)-(C10) && \\
z_k^a &= 1 - u_k^a && \forall k && (1)\\
u_k^q &\geq z_k^r && \forall k && (2)\\
-u_k^q &\geq M(v_k^{r,q,0} - 1) && \forall k && (3)\\
-u_k^m &\geq M(z_k^r - 1) && \forall k && (4)\\
u_k^m - 1 &\geq M(v_k^{r,m,0} - 1) && \forall k && (5)\\
1 - z_k^r &\leq v_k^{r,q,0} + v_k^{r,m,0} && \forall k && (6)\\
z_k^f &= z_k^r && \forall k && (7)\\
z_k^a &= z_k^{\tilde{a}} && \forall k && (8)\\
\sum_{k\in K} kx_{i+1,k}^r - \sum_{k\in K} kw_{i,k}^r &\geq 0 && \forall i && (9)\\
u_k^a &= u_{k-1}^a - \sum_i w_{i,k}^a + \sum_i w_{i,k}^{\tilde{a}} && \forall k && (10)\\
u_k^m &= u_{k-1}^m - \sum_i w_{i,k}^f + \sum_i w_{i,k}^r && \forall k && (11)\\
u_k^q &= u_{k-1}^q + \sum_i w_{i,k}^a - \sum_i w_{i,k}^r && \forall k && (12)\\
\mathcal{E}_0 &= 0, u_0^q = u_0^m = 0 && && \\
u_k^a &\in \{0,1\},\ m \in \{0,1\},\ q \in \mathbb{N} && && (13)
\end{aligned}
$$

Group-A constraints are the same as presented in section XXX. All the events are single-execution, so constraints (B1) is applied, and $e_{i,1}^{\xi}$ can be replaced by $e_{i,0}^{\xi} + t_i^{\xi}$ for all event types. For group (C) constraints, (C1), (C2) and (C8) to (C10) are the same as presented in Section XXX. Constraints (C3) to (C7) are expanded. For the arrival events, the scheduling condition is that the execution in the event list is less than one, as in constraints (1). For the start event, condition $1 \leq q$ is verified with constraints (2) and (3), and condition $m \leq 0$ is verified with constraints (4) and (5). Constraints (6) guarantee that a start event must be scheduled once the scheduling condition is true. Start event can be used as the counting events of finish event, and state variable m is equivalent to the number of executions of e^f in the event list, so constraints (7) are included. Arrival event is positive-delay, a counting event $e^{\tilde{a}}$, the counter of arrival events in the event list u^a and constraints (8) are included. Start events is zero-delay, so constraints (9) are applied. Constraints (10) to (12) show the state variable evolution. The objective function of the model is missing, because it can be arbitrary as stated above.

The MPR model proposed in Chan and Schruben (2008) is as follows:

$$\min\{\sum_{i=1}^{N}(e^a_{i,1}+e^r_{i,1}+e^f_{i,1})\}$$

$$e^a_{i,1}-e^a_{i-1,1}=t^a_i \qquad \forall\ i \tag{14}$$

$$e^f_{i,1}-e^r_{i,1}=t^f_i \qquad \forall\ i \tag{15}$$

$$e^r_{i,1}-e^a_{i,1}\geq 0 \qquad \forall\ i \tag{16}$$

$$e^r_{i,1}-e^f_{i-1,1}\geq 0 \qquad \forall\ i \tag{17}$$

To explain the MPR model, the ERG of G/G/1 queue is first shown in Figure 2. Each triggering relationship in the ERG, i.e., two connected nodes is represented by one constraint. Constraints (14) to (17) state the relationships of $e^a \to e^a$, $e^s \to e^f$, $e^a \to e^s$ and $e^f \to e^s$, respectively. The objective function is the sum of all the event execution time. As can be seen, the model is a linear programming.

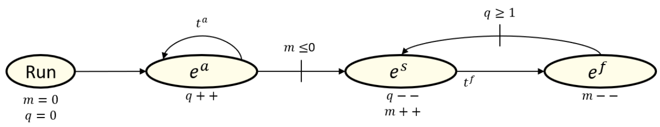

Figure 2. ERG of G/G/1 queue.

3.2. Single server merge

A queueing system composed of three servers within a merge architecture is presented in this section. Jobs enter the system at server 1 or server 2, and the buffer in front of the two servers has infinite capacity. It is assumed that all the jobs arrive at time zero. After processing a job, server 1 or 2 can release the job to the finite buffer in front of server 3, if the buffer is not full. The blocking policy is block-after-service. If there is only one space available in the buffer, and both servers 1 and 2 are holding a finished job, server 1 has higher priority to release. After processing a job, server 3 releases the job from the system immediately.

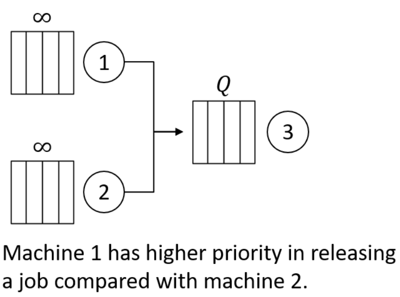

Figure 3. Example: single server merge.

State variables have to be defined in the first place. Variable m^i and q represents the state of server i and the available space in buffer 3, respectively. Both server 1 and 2 have three states, namely *idle*, *working* and *holding a finished job*, and each state refers to m^i equal to 0, 1 and 2 respectively. Server 3 has only two states, *idle* and *working*, and refers to m^3 equal to 0 and 1, respectively. The servers are initially idle, and buffer 3 is empty, i.e., $m^1 = m^2 = m^3 = 0$ and $q = Q$, where Q denotes the capacity of buffer 3.

The events composed the DES model are then defined as in Table 2. Event $e^{r,1}$ represents the starting of service of server 1, the scheduling of such event is enabled if server 1 is idle, i.e., $m^1 \leq 0$. Executing $e^{r,1}$ will increase state variable m^1 from 0 to 1. Event $e^{f,1}$ represents the finishing of server 1, scheduling of it requires that server 1 is idle. The delay between scheduling and execution is random variate t^1, which is the service time. Executing $e^{f,1}$ makes the server hold a finished job and hence increase state variable m^1 from 1 to 2. Events $e^{r,2}$ and $e^{f,2}$ are similar to events $e^{r,1}$ and $e^{f,1}$. The departure events of server 1 and server 2 differ from each other. As long as there is a space available in buffer 3 and server 1 holds a finished job, a departure event of server 1 $e^{d,1}$ can be scheduled. For the departure event of server 2 $e^{d,2}$, one more condition to verify is that server 1 does not hold a finished job due to the priority rule. The execution of departure event of server 1 or 2 will occupy an available space in buffer 3, and make the server idle. The start event of server 3 $e^{r,3}$ is enabled if it is idle and there is at least one job waiting in buffer 3. After execution of $e^{r,3}$, the server becomes busy and the available space in buffer 3 is increased by one. The departure event $e^{d,3}$ can be scheduled if server 3 is working with delay t^3, which is a random variate representing the service time of server 3. Execution of $e^{d,3}$ makes server 3 idle. Since all the stations are composed of a single server, all events are single-execution.

Variable	Event	Condition to schedule	Delay	β^{ξ}	State change
$e^{r,1}$	Start m1	$m^1 \leq 0$	0	1	$m^1 ++$
$e^{f,1}$	Finish m1	$m^1 \leq 0$	t^1	1	$m^1 ++$
$e^{d,1}$	Depart m1	$m^1 \geq 2$ & $q \geq 1$	0	1	$m^1 = m^1 - 2,\ q --$
$e^{r,2}$	Start m2	$m^2 \leq 0$	0	1	$m^2 ++$
$e^{f,2}$	Finish m2	$m^2 \leq 0$	t^2	1	$m^2 ++$
$e^{d,2}$	Depart m2	$m^2 \geq 2$ & $q \geq 1$& $m^1 \leq 1$	0	1	$m^2 = m^2 - 2,\ q --$
$e^{r,3}$	Start m3	$m^3 \leq 0$ & $q \leq Q - 1$	0	1	$m^3 ++,\ q ++$
$e^{d,3}$	Depart m3	$m^3 \leq 0$ & $q \leq Q - 1$	t^3	1	$m^3 --$

Table 2. Events to simulate a merge queueing system.

The MPR proposed in this work is as follows:

$$
\begin{aligned}
&(A1)-(A6),(B1),(C1),(C2),(C8)-(C10) && && \\
&-u_k^{m^1} \geq 2(z_k^{r,1}-1) && \forall\ k && (18)\\
&u_k^{m^1}-1 \geq v_k^{(r,1),m^1,1}-1 && \forall\ k && (19)\\
&1-z_k^{r,1} \leq v_k^{(r,1),m^1,1} && \forall\ k && (20)\\
&z_k^{f,1} = z_k^{r,1} && \forall\ k && (21)\\
&-u_k^{m^2} \geq 2(z_k^{r,2}-1) && \forall\ k && (22)\\
&u_k^{m^2}-1 \geq v_k^{(r,2),m^2,1}-1 && \forall\ k && (23)\\
&1-z_k^{r,2} \leq v_k^{(r,2),m^2,1} && \forall\ k && (24)\\
&z_k^{f,2} = z_k^{r,2} && && (25)
\end{aligned}
$$

$$u_k^{m^1} - 2 \geq 2(z_k^{d,1} - 1) \qquad \forall\ k \tag{26}$$

$$1 - u_k^{m^1} \geq v_k^{(d,1),m^1,0} - 1 \qquad \forall\ k \tag{27}$$

$$u_k^q - 1 \geq z_k^{d,1} - 1 \qquad \forall\ k \tag{28}$$

$$-u_k^q \geq Q(v_k^{(d,1),q,0} - 1) \qquad \forall\ k \tag{29}$$

$$1 - z_k^{d,1} \leq v_k^{(d,1),m^1,0} + v_k^{(d,1),q,0} \qquad \forall\ k \tag{30}$$

$$u_k^{m^2} - 2 \geq 2(z_k^{d,2} - 1) \qquad \forall\ k \tag{31}$$

$$1 - u_k^{m^2} \geq v_k^{(d,2),m^2,0} - 1 \qquad \forall\ k \tag{32}$$

$$u_k^q - 1 \geq z_k^{d,2} - 1 \qquad \forall\ k \tag{33}$$

$$-u_k^q \geq Q(v_k^{(d,2),q,0} - 1) \qquad \forall\ k \tag{34}$$

$$1 - u_k^{m^1} \geq z_k^{d,2} - 1 \qquad \forall\ k \tag{35}$$

$$u_k^{m^1} - 2 \geq 2(v_k^{(d,2),m^1,1} - 1) \qquad \forall\ k \tag{36}$$

$$1 - z_k^{d,2} \leq v_k^{(d,2),m^2,0} + v_k^{(d,2),q,0} + v_k^{(d,2),m^1,1} \qquad \forall\ k \tag{37}$$

$$-u_k^{m^3} \geq z_k^{r,3} - 1 \qquad \forall\ k \tag{38}$$

$$u_k^{m^3} - 1 \geq v_k^{(r,3),m^3,1} - 1 \qquad \forall\ k \tag{39}$$

$$Q - 1 - u_k^q \geq z_k^{r,3} - 1 \qquad \forall\ k \tag{40}$$

$$u_k^q - Q \geq Q(v_k^{(r,3),q,1} - 1) \qquad \forall\ k \tag{41}$$

$$1 - z_k^{r,3} \leq v_k^{(r,3),m^3,1} + v_k^{(r,3),q,1} \qquad \forall\ k \tag{42}$$

$$z_k^{d,3} = z_k^{r,3} \qquad \forall\ k \tag{43}$$

$$\sum_k k x_{i+1,k}^{\xi} - \sum_k k w_{i,k}^{\xi} \geq 0$$

$$\forall \xi \in \{(s,1),(s,2),(s,3),(d,1),(d,2)\}, \forall\ i \tag{44}$$

$$u_k^{m^1} = u_{k-1}^{m^1} + \sum_i w_{i,k}^{r,1} + \sum_i w_{i,k}^{f,1} - 2\sum_i w_{i,k}^{d,1} \qquad \forall\ k \tag{45}$$

$$u_k^{m^2} = u_{k-1}^{m^2} + \sum_i w_{i,k}^{r,2} + \sum_i w_{i,k}^{f,2} - 2\sum_i w_{i,k}^{d,2} \qquad \forall\ k \tag{46}$$

$$u_k^{m^3} = u_{k-1}^{m^3} + \sum_i w_{i,k}^{r,3} - \sum_i w_{i,k}^{d,3} \qquad \forall\ k \tag{47}$$

$$u_k^q = u_{k-1}^q - \sum_i w_{i,k}^{d,1} - \sum_i w_{i,k}^{d,2} + \sum_i w_{i,k}^{r,3} \qquad \forall\ k \tag{48}$$

$$\mathcal{E}_0 = 0, u_0^{m^1} = u_0^{m^2} = u_0^{m^3} = 0, u_0^q = Q$$

$$u_k^{m^1},\ u_k^{m^2} \in \{0,1,2\},\ u_k^{m^3} \in \{0,1\},\ q \in \{0,...,Q\} \qquad \forall\ k \tag{49}$$

If the simulation run includes N_1 jobs from server 1 and N_2 jobs from server 2, the number of executions of events $e^{r,1}$, $e^{f,1}$, $e^{d,1}$ are equal to N_1, the number of executions of events $e^{r,2}$, $e^{f,2}$, $e^{d,2}$ are equal to N_2, the number of executions of events $e^{r,3}$, $e^{d,3}$ are equal to $N_1 + N_2$, and total number of executions is equal to $5N_1 + 5N_2$. Group-A constraints are the same as presented in section XXX. All the events are single-execution, so constraints (B1) is applied, and $e_{i,1}^{\xi}$ can be replaced by $e_{i,0}^{\xi} + t_i^{\xi}$ for all event types. For group (C) constraints, (C1), (C2) and (C8) to (C10) are the same as presented in Section XXX. Constraints (C3) to (C7) are expanded as follows. For event $e^{r,1}$, the condition on m^1 is verified with constraints (18) to (20). Event $e^{r,1}$ is also a counting event of $e^{f,1}$, so constraints (21) are included for scheduling $e^{f,1}$. The same constraints are applied to events $e^{r,2}$ and $e^{f,2}$, with constraints (22) to (25). The scheduling conditions of event $e^{d,1}$ and $e^{d,2}$ are verified with constraints (26) to (30) and constraints (31) to (37), respectively. For event $e^{r,3}$, the scheduling condition is verified with constraints (38) ro (42). Event $e^{r,3}$ is also a counting event of $e^{d,3}$, so constraints (43) are included for scheduling $e^{d,3}$. For all the zero-delay events, constraints (44) are relevant. Constraints (45) to (48) state the state variable evolution.

The MPR model proposed in Chan and Schruben (2008) is as follows:

$$\min\{\sum_{i=1}^{N_1}(e_i^{s,1}+e_i^{f,1}+e_i^{d,1})+\sum_{i=1}^{N_2}(e_i^{s,2}+e_i^{f,2}+e_i^{d,2})+\sum_{i}e_i^{d,2,1}+ \sum_{i=1}^{N_1+N_2}(e_i^{s,3}+e_i^{d,3}-e_i^{w:(d,1),(d,2)})+\sum_{i=1}^{N_1+N_2}\sum_{k=1}^{N_1}e_{i,k,i-k}^{p:(d,1),(d,2)}\}$$

$$e_i^{s,1}-e_{i-1}^{d,1}=0 \tag{50}$$

$$e_i^{f,1}-e_i^{s,1}=t_i^1 \tag{51}$$

$$e_i^{s,2}-e_{i-1}^{d,2}=0 \tag{52}$$

$$e_i^{f,2}-e_i^{s,2}=t_i^2 \tag{53}$$

$$e_i^{d,3}-e_i^{s,3}=t_i^3 \tag{54}$$

$$e_i^{d,1}-e_i^{f,1}\geq 0 \tag{55}$$

$$e_i^{s,3}-e_{i-1}^{d,3}\geq 0 \tag{56}$$

$$e_i^{d,2} - e_j^{d,2,1} \geq M(\sigma_{(d,2,1):j,(d,2):i} - 1) \tag{57}$$

$$e_i^{d,2} - e_j^{d,2,1} \geq m(1 - \sigma_{(d,2,1):j,(d,2):i}) \tag{58}$$

$$e_j^{d,2,1} \geq e_k^{d,1} - M(1 - \zeta_{(d,1):k,(d,2,1):j}) \tag{59}$$

$$e_k^{d,1} \geq e_j^{d,2,1} + m\zeta_{(d,1):k,(d,2,1):j} + (1 - \zeta_{(d,1):k,(d,2,1):j})\varepsilon \tag{60}$$

$$e_{k+1}^{f,1} \geq e_j^{d,2,1} - M(1 - \eta_{(d,2,1):j,(f,1):k+1}) \tag{61}$$

$$e_j^{d,2,1} \geq e_{k+1}^{f,1} + m\eta_{(d,2,1):j,(f,1):k+1} \tag{62}$$

$$\gamma_{(d,1):k,(d,2,1):j} - (\zeta_{(d,1):k,(d,2,1):j} + \eta_{(d,2,1):j,(f,1):k+1}) + 1 \geq 0 \tag{63}$$

$$2\gamma_{(d,1):k,(d,2,1):j} - (\zeta_{(d,1):k,(d,2,1):j} + \eta_{(d,2,1):j,(f,1):k+1}) \leq 0 \tag{64}$$

$$\sum_k \gamma_{(d,1):k,(d,2,1):j} - \sum_i \sigma_{(d,2,1):j,(d,2):i} \geq 0 \tag{65}$$

$$n\sum_i \sigma_{(d,2,1):j,(d,2):i} - \sum_k \gamma_{(d,1):k,(d,2,1):j} \geq 0 \tag{66}$$

$$\sum_i \sigma_{(d,2,1):j,(d,2):i} \leq 1 \tag{67}$$

$$\sum_j \sigma_{(d,2,1):j,(d,2):i} \leq 1 \tag{68}$$

$$\sum_{j=i}^{n} \sigma_{(d,2,1):j,(d,2):i} \geq \sum_{j=i+1}^{n} \sigma_{(d,2,1):j,(d,2):i+1} \tag{69}$$

$$\sum_{p=j+1}^{n} \sum_{q=1}^{i-1} \sigma_{(d,2,1):p,(d,2):q} \geq min\{i-1, n-j\}(1 - \sigma_{(d,2,1):j,(d,2):i}) \tag{70}$$

$$e_i^{d,2,1} - e_j^{s,3} \geq M(\sigma_{(s,3):j,(d,2,1):i} - 1) \tag{71}$$

$$e_i^{d,2,1} - e_j^{s,3} \geq m(1 - \sigma_{(s,3):j,(d,2,1):i}) \tag{72}$$

$$e_j^{s,3} \geq e_k^{f,2} - M(1 - \zeta_{(f,2):k,(s,3):j}) \tag{73}$$

$$e_k^{f,2} \geq e_j^{s,3} + m\zeta_{(f,2):k,(s,3):j} + (1 - \zeta_{(f,2):k,(s,3):j})\varepsilon \tag{74}$$

$$e_{k+1}^{d,2} \geq e_j^{s,3} - M(1 - \eta_{(s,3):j,(d,2):k+1}) \tag{75}$$

$$e_j^{s,3} \geq e_{k+1}^{d,2} + m\eta_{(s,3):j,(d,2):k+1} \tag{76}$$

$$\gamma_{(f,2):k,(s,3):j} - (\zeta_{(f,2):k,(s,3):j} + \eta_{(s,3):j,(d,2):k+1}) + 1 \geq 0 \tag{77}$$

$$2\gamma_{(f,2):k,(s,3):j} - (\zeta_{(f,2):k,(s,3):j} + \eta_{(s,3):j,(d,2):k+1}) \leq 0 \tag{78}$$

$$\sum_k \gamma_{(f,2):k,(s,3):j} - \sum_i \sigma_{(s,3):j,(d,2,1):i} \geq 0 \tag{79}$$

$$n\sum_i \sigma_{(s,3):j,(d,2,1):i} - \sum_k \gamma_{(f,2):k,(s,3):j} \geq 0 \tag{80}$$

$$\sum_i \sigma_{(s,3):j,(d,2,1):i} \leq 1 \tag{81}$$

$$\sum_j \sigma_{(s,3):j,(d,2,1):i} \leq 1 \tag{82}$$

$$\sum_{j=i}^{n} \sigma_{(s,3):j,(d,2,1):i} \geq \sum_{j=i+1}^{n} \sigma_{(s,3):j,(d,2,1):i+1} \tag{83}$$

$$\sum_{p=j+1}^{n} \sum_{q=1}^{i-1} \sigma_{(s,3):p,(d,2,1):q} \geq min\{i-1, n-j\}(1 - \sigma_{(s,3):j,(d,2,1):i}) \tag{84}$$

$$e_i^{w:(d,1),(d,2)} \leq e_{i,k,i-k}^{p:(d,1),(d,2)} \tag{85}$$

$$e_k^{d,1} \geq e_{i-k}^{d,2} - M(1 - \alpha_{i,k,i-k}^{(d,1),(d,2)}) \tag{86}$$

$$e_{i-k}^{d,2} \geq e_{i-k}^{d,1} + m\alpha_{i,k,i-k}^{(d,1),(d,2)} + (1 - \alpha_{i,k,i-k}^{(d,1),(d,2)})\varepsilon \tag{87}$$

$$e_{i,k,i-k}^{p:(d,1),(d,2)} \geq e_k^{d,1} - M(1 - \alpha_{i,k,i-k}^{(d,1),(d,2)}) \tag{88}$$

$$e_k^{d,1} \geq e_{i,k,i-k}^{p:(d,1),(d,2)} + m(1 - \alpha_{i,k,i-k}^{(d,1),(d,2)}) \tag{89}$$

$$e_{i,k,i-k}^{p:(d,1),(d,2)} \geq e_k^{d,2} - M\alpha_{i,k,i-k}^{(d,1),(d,2)} \tag{90}$$

$$e_k^{d,2} \geq e_{i,k,i-k}^{p:(d,1),(d,2)} + m\alpha_{i,k,i-k}^{(d,1),(d,2)} \tag{91}$$

$$e_i^{d,2,1} - e_j^{f,2} \geq M(\sigma_{(f,2):j,(d,2,1):i} - 1) \tag{92}$$

$$e_i^{d,2,1} - e_j^{f,2} \geq m(1 - \sigma_{(f,2):j,(d,2,1):i}) \tag{93}$$

$$e_j^{f,2} \geq e_k^{s,3} - M(1 - \zeta_{(s,3):k,(f,2):j}) \tag{94}$$

$$e_k^{s,3} \geq e_j^{f,2} + m\zeta_{(s,3):k,(f,2):j} + (1 - \zeta_{(s,3):k,(f,2):j})\varepsilon \tag{95}$$

$$e_{k+Q}^{w:(d,1),(d,2)} \geq e_j^{f,2} - M(1 - \eta_{(f,2):j,(w:(d,1),(d,2)):k+Q}) \tag{96}$$

$$e_j^{f,2} \geq e_{k+Q}^{w:(d,1),(d,2)} + m\eta_{(f,2):j,(w:(d,1),(d,2)):k+Q} \tag{97}$$

$$\gamma_{(s,3):k,(f,2):j} - (\zeta_{(s,3):k,(f,2):j} + \eta_{(f,2):j,(w:(d,1),(d,2)):k+Q}) + 1 \geq 0 \tag{98}$$

$$2\gamma_{(s,3):k,(f,2):j} - (\zeta_{(s,3):k,(f,2):j} + \eta_{(f,2):j,(w:(d,1),(d,2)):k+Q}) \leq 0 \tag{99}$$

$$\sum_k \gamma_{(s,3):k,(f,2):j} - \sum_i \sigma_{(f,2):j,(d,2,1):i} \geq 0 \tag{100}$$

$$n \sum_i \sigma_{(f,2):j,(d,2,1):i} - \sum_k \gamma_{(s,3):k,(f,2):j} \geq 0 \tag{101}$$

$$\sum_i \sigma_{(f,2):j,(d,2,1):i} \leq 1 \tag{102}$$

$$\sum_j \sigma_{(f,2):j,(d,2,1):i} \leq 1 \tag{103}$$

$$\sum_{j=i}^{n} \sigma_{(f,2):j,(d,2,1):i} \geq \sum_{j=i+1}^{n} \sigma_{(f,2):j,(d,2,1):i+1} \tag{104}$$

$$\sum_{p=j+1}^{n} \sum_{q=1}^{i-1} \sigma_{(f,2):p,(d,2,1):q} \geq min\{i-1, n-j\}(1 - \sigma_{(f,2):j,(d,2,1):i}) \tag{105}$$

$$e_i^{s,3} \geq e_i^{w:(d,1),(d,2)} \tag{106}$$

$$e_i^{d,1} - e_j^{f,1} \geq M(\sigma_{(f,1):j,(d,1):i} - 1) \tag{107}$$

$$e_i^{d,1} - e_j^{f,1} \geq m(1 - \sigma_{(f,1):j,(d,1):i}) \tag{108}$$

$$e_j^{f,1} \geq e_k^{s,3} - M(1 - \zeta_{(s,3):k,(f,1):j}) \tag{109}$$

$$e_k^{s,3} \geq e_j^{f,1} + m\zeta_{(s,3):k,(f,1):j} + (1 - \zeta_{(s,3):k,(f,1):j})\varepsilon \tag{110}$$

$$e_{k+Q}^{d,2} \geq e_j^{f,1} - M(1 - \eta_{(f,1):j,(d,2):k+Q}) \tag{111}$$

$$e_j^{f,1} \geq e_{k+Q}^{d,2} + m\eta_{(f,1):j,(d,2):k+Q} \tag{112}$$

$$\gamma_{(s,3):k,(f,1):j} - (\zeta_{(s,3):k,(f,1):j} + \eta_{(f,1):j,(d,2):k+Q}) + 1 \geq 0 \tag{113}$$

$$2\gamma_{(s,3):k,(f,1):j} - (\zeta_{(s,3):k,(f,1):j} + \eta_{(f,1):j,(d,2):k+Q}) \leq 0 \tag{114}$$

$$\sum_k \gamma_{(s,3):k,(f,1):j} - \sum_i \sigma_{(f,1):j,(d,1):i} \geq 0 \tag{115}$$

$$n \sum_i \sigma_{(f,1):j,(d,1):i} - \sum_k \gamma_{(s,3):k,(f,1):j} \geq 0 \tag{116}$$

$$\sum_i \sigma_{(f,1):j,(d,1):i} \leq 1 \tag{117}$$

$$\sum_j \sigma_{(f,1):j,(d,1):i} \leq 1 \tag{118}$$

$$\sum_{j=i}^{n} \sigma_{(f,1):j,(d,1):i} \geq \sum_{j=i+1}^{n} \sigma_{(f,1):j,(d,1):i+1} \tag{119}$$

$$\sum_{p=j+1}^{n} \sum_{q=1}^{i-1} \sigma_{(f,1):p,(d,1):q} \geq min\{i-1, n-j\}(1 - \sigma_{(f,1):j,(d,1):i}) \tag{120}$$

To explain the MPR model, the ERG of the merge system is first shown in Figure 4. Each triggering relationship in the ERG is represented by one or more constraints. Since the modeling framework requires that there is at most one condition on each arc, event $e^{d,2,1}$ is introduced to split the composed conditions from $e^{f,2}$ and $e^{s,3}$ to $e^{d,3}$. The state variables m^1 and m^2 are also varied, and they are equal to one if the server is blocked. The new definition is equivalent to the one we used to derived the model proposed in this work, but simplify the model under the framework of Chan and Schruben (2008). Constraints (50) to (55) state the triggering relationship of $e^{d,1} \to e^{s,1}$, $e^{s,1} \to e^{f,1}$, $e^{d,2} \to e^{s,2}$, $e^{s,1} \to e^{f,1}$, $e^{s,3} \to e^{d,3}$ and $e^{s,3} \to e^{d,1}$, respectively. Constraints (56) show the triggering relationship between $e^{d,1} \to e^{s,3}$ and also $e^{d,2} \to e^{s,3}$. Constraints (57) to (70) imply the triggering relationship from $e^{d,2,1}$ to $e^{d,2}$. Constraints (71) to (84) imply the triggering relationship from $e^{d,3}$ to $e^{d,2,1}$. Constrains (85) to (91) imply the convolution of $e^{d,1}$ and $e^{d,2}$. The real-valued variables $e_i^{w:(d,1).(d,2)}$ represent the i-th execution of $e^{d,1}$ or $e^{d,2}$, and $e_{i,k,i-k}^{p:(d,1),(d,2)}$ are equal to the maximum between $e_k^{d,1}$ and $e_{i-k}^{d,2}$. Constrains (92) to (105) imply the triggering relationship from $e^{f,2}$ to $e^{d,2,1}$. Constraints (106) show the triggering relationship from $e^{d,3}$ to $e^{s,3}$. Constraints (107) to (120) show the triggering relationship from $e^{f,1}$ to $e^{d,1}$. The variables ζ, σ, η, γ, α are all binary, and the notations are the same as in Chan and Schruben (2008). The objective function is also defined as proposed in Chan and Schruben (2008), i.e., minimizing the sum of all event execution times, minimizing the real-valued variables bounded from below (i.e. $e^{p:(d,1),(d,2)}$), and maximizing the real-valued variables bounded from top (i.e., $e^{w:(d,1),(d,2)}$).

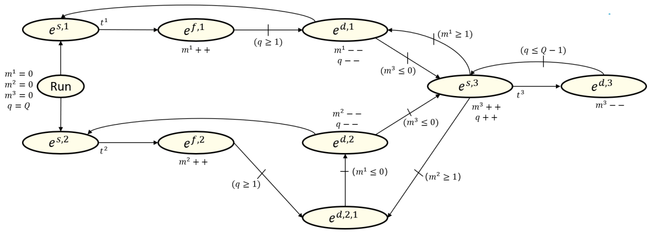

Figure 4. ERG of merge queueing system.

It can be seen that the model proposed in Chan and Schruben (2008) requires to derive different constraints from each arcs according to the condition on the arc and the state changes of several events. Furthermore, the constraints bound the event execution time from below, which indicate that the event *can* be executed, and the objective function drives the event to be executed as soon as possible. However, the DES model *must* execute the event once the conditions are true, which cannot be guaranteed with the model. To guarantee the equivalence between the MPR and the simulation implementation, the multiplier of each term of the objective function has to be carefully chosen. Using the model proposed in this work, the objective function can be arbitrary, i.e., any performance indicator can be the objective function.

3.3. Multiple-server merge

A multiple-server merge queueing system is shown in Figure 5. The multiple-server merge queue is a generalization of the system presented in Section 4.1, where the number of parallel servers in station 1, 2 and 3 is equal to s^1, s^2 and s^3, respectively. The state variables are changed accordingly. To describe the state of multiple-server station j, two state variables g^j and h^j are needed to represent the number of idle servers and the number of finished jobs of each the station. The state variable q is used to represent the number of available space in buffer 3.

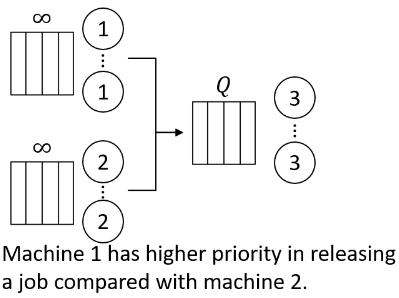

Figure 5. Example: multi-server merge.

The events composing the DES model are shown in Table 3. The start event of station 1 and 2 is scheduled when there are at least one empty server, and their execution decreases the number of empty servers by one. The scheduling condition of finish event $e^{f,j}$ is the same as $e^{s,j}$, but its execution will increase the number of finished jobs by one. Event $e^{f,j}$ is a multi-execution event with positive delays, an event to count the number of executions of it should be introduced. However, the event $e^{s,j}$ plays that role and the number of executions in the event list is equal to $(s^j - g^j)$. Similarly with single-server system, the departure of station 1 requires that there is at least one finished job in the station and there is at least one space available in buffer 3, but the departure of station 2 also requires that there is no finished job in station 1. The departure of station 1 and 2 will increase the number of empty servers by one, decrease the number of finished jobs by one and decrease the number of available space by one. As for station 3, the start and departure event can be scheduled if there is at least one empty server and one job in buffer 3. Thus, $e^{s,3}$ is used to count the number executions in the event list of $e^{d,3}$. Execution of $e^{s,3}$ will decrease the number of empty server by one and increase the available space in buffer 3 by one.

Variable	Event	Condition to schedule	Delay	β^{ξ}	State change
$e^{s,1}$	Start 1	$1 \leq g^1$	0	1	$g^1 --$
$e^{f,1}$	Finish 1	$1 \leq g^1$	t^1	s^1	$h^1 ++$
$e^{d,1}$	Depart 1	$1 \leq h^1 \& \ q \geq 1$	0	1	$g^1 ++, h^1 --,\ q --$
$e^{s,2}$	Start 2	$1 \leq g^2$	0	1	$g^2 --$
$e^{f,2}$	Finish 2	$1 \leq g^2$	t^2	s^2	$h^2 ++$
$e^{d,2}$	Depart 2	$1 \leq h^2 \ \& \ q \geq 1 \ \& \ h^1 \leq 0$	0	1	$g^2 ++, h^2 --,\ q --$
$e^{s,3}$	Start 3	$1 \leq g^3 \ \& \ q \leq Q-1$	0	1	$g^3 --,\ q ++$
$e^{d,3}$	Depart 3	$1 \leq g^3 \ \& \ q \leq Q-1$	t^3	s^3	$g^3 ++$

Table 3. Events to simulate a multi-server merge queueing system.

4. Draft

MP model of single server merge model:

$$\min \sum_k \mathcal{E}_k \tag{121}$$

$$e_i^{(\xi,j),1} - \mathcal{E}_k \geq M(w_{i,k}^{\xi,j} - 1) \qquad \xi \in \{s,f,d\}, j \in \{1,2,3\}, \forall i,k \tag{122}$$

$$\mathcal{E}_k - e_i^{(\xi,j),1} \geq M(w_{i,k}^{\xi,j} - 1) \qquad \xi \in \{s,f,d\}, j \in \{1,2,3\}, \forall i,k \tag{123}$$

$$\sum_k w_{i,k}^{\xi,j} = 1 \qquad \forall\ \xi \in \{s,f,d\}, j \in \{1,2,3\}, i \tag{124}$$

$$\sum_{(\xi,j),i} w_{i,k}^{\xi,j} = 1 \qquad \forall\ k \tag{125}$$

$$\sum_k k w_{i+1,k}^{\xi,j} - \sum_k k w_{i,k}^{\xi,j} \geq 1 \qquad \forall\ \xi \in \{s,f,d\}, j \in \{1,2,3\}, i \tag{126}$$

$$e_i^{s,j,1} - e_i^{s,j,0} \geq 0 \qquad j = 1,2,3, \forall\ i \tag{127}$$

$$e_i^{f,j,1} - e_i^{f,j,0} \geq t_i^j \qquad j = 1,2, \forall\ i \tag{128}$$

$$e_i^{d,j,1} - e_i^{d,j,0} \geq 0 \qquad j = 1,2, \forall\ i \tag{129}$$

$$e_i^{d,3,1} - e_i^{d,3,0} \geq t_i^3 \qquad \forall\ i \tag{130}$$

$$e_i^{\xi,j,0} - \mathcal{E}_k \geq M(x_{i,k}^{\xi,j} - 1) \qquad \forall\ \xi \in \{s,f,d\}, j = 1,2,3,k,i \tag{131}$$

$$\mathcal{E}_k - e_i^{\xi,j,0} \geq M(x_{i,k}^{\xi,j} - 1) \qquad \forall\ \xi \in \{s,f,d\}, j = 1,2,3,k,i \tag{132}$$

$$m_k^j = m_{k-1}^j + \sum_{i=1}^{N^j} (w_{i,k}^{s,j} + w_{i,k}^{f,j} - 2w_{i,k}^{d,j}) \qquad j = 1,2, \forall\ k \tag{133}$$

$$m_k^3 = m_{k-1}^3 + \sum_{i=1}^{N^3} (w_{i,k}^{s,3} - w_{i,k}^{d,3}) \qquad \forall\ k \tag{134}$$

$$q_k = q_{k-1} + \sum_{i=1}^{N^3} w_{i,k}^{s,3} - \sum_{i=1}^{N^1} w_{i,k}^{d,1} - \sum_{i=1}^{N^2} w_{i,k}^{d,2} \tag{135}$$

$$m_k^j \geq M(z_k^{s,j} - 1) \qquad j = 1,2,3, \forall\ k \tag{136}$$

$$1 - m_k^j \geq M(z_k^{f,j} - 1) \qquad j = 1,2, \forall\ k \tag{137}$$

$$m_k^j - 1 \geq M(z_k^{f,j} - 1) \qquad j = 1,2, \forall\ k \tag{138}$$

$$m_k^j - 2 \geq M(z_k^{d,j} - 1) \qquad j = 1,2, \forall\ k \tag{139}$$

$$q_k - 1 \geq M(z_k^{d,j} - 1) \qquad j = 1,2, \forall\ k \tag{140}$$

$$1 - m_k^1 \geq M(z_k^{d,2} - 1) \qquad \forall\ k \tag{141}$$

$$m_k^3 - 1 \geq M(z_k^{d,3} - 1) \qquad \forall\ k \tag{142}$$

$$(Q-1) - q_k \geq M(z_k^{s,3} - 1) \qquad \forall\ k \tag{143}$$

$$\sum_k x_{i,k}^{\xi,j} = 1 \qquad \forall\ \xi \in \{s,f,d\}, j = 1,2,3, \forall\ i,k \tag{144}$$

$$\sum_{i=1}^{N^j} x_{i,k}^{\xi,j} \leq z_k^{\xi,j} \qquad \forall\ \xi \in \{s,f,d\}, j = 1,2,3,k \tag{145}$$

$$\sum_k k x_{i+1,k}^{\xi,j} - \sum_k k x_{i,k}^{\xi,j} \geq 0 \qquad \forall\ \xi \in \{s,f,d\}, j = 1,2,3,i \tag{146}$$

5. Resource allocation problem of queueing systems

5.1. Mathematical programming representation of simulation model

We study only the system that the occurrence of an event will lead to the increment or decrement of one unit of the state variables. A simulation model is called a *natural* simulator if the following assumptions all hold:

(1) An event e^ξ can be triggered if the state variables $\mathbf{s}$ satisfy specific conditions *at that time*, regardless of the history of the state or event occurrence, and the condition is not changed along time, i.e., condition is static. It could be possible to define more state variables in case of history dependence and variant triggering conditions.
(2) Natural triggering relationship: if and only if e^ξ is an s-increment event, e^ξ triggers an s-decrement event, vice versa.
(3) Natural triggering condition: the condition for triggering an e^ξ is that each state variable s must within its predefined domain, i.e., $\mathbf{l} \leq \mathbf{s} \leq \mathbf{u}$, regardless of event type ξ.
(4) For all e^ξ, the number of execution N^ξ is known before simulation, and the simulation terminate when all types of events have been triggered for that number.

Assumptions for a variable x to be resource-type:

(1) For all e^ξ, $\mathbf{u}$ is monotonically increasing on x, and $\mathbf{l}$ is monotonically decreasing on x.

Formulate the MP model of simulation:

$e_i^\xi \geq 0$	time of the i-th occurrence of event e^ξ.
$\tau_l^{s+} \geq 0$	time of the l-th occurrence of events that increments state variable s.
$\tau_l^{s-} \geq 0$	time of the l-th occurrence of events that decrements state variable s.
$x_{i,l}^{\xi,s+} \in \{0,1\}$	equal to 1 if e_i^ξ is the l-th increment of s.
$x_{i,l}^{\xi,s-} \in \{0,1\}$	equal to 1 if e_i^ξ is the l-th decrement of s.

The MP model of simulation is

$$
\begin{aligned}
&\min\{\textstyle\sum_{\xi,i} e_i^{\xi}\} && && (147)\\
&s.t. && && (148)\\
&\tau_l^{s+} - \tau_{l+s_0-u_s}^{s-} \geq 0 && \forall\ s && (149)\\
&\tau_l^{s-} - \tau_{l-s_0+l_s}^{s+} \geq 0 && \forall\ s && (150)\\
&\tau_l^{s+} - e_i^{\xi} \geq M(x_{i,l}^{\xi,s+} - 1) && \forall\ s\ and\ e^{\xi}\ with\ increment\ of\ s && (151)\\
&\tau_l^{s-} - e_i^{\xi} \geq M(x_{i,l}^{\xi,s-} - 1) && \forall\ s\ and\ e^{\xi}\ with\ decrement\ of\ s && (152)\\
&e_i^{\xi} - \tau_l^{s+} \geq M(x_{i,l}^{\xi,s+} - 1) && \forall\ s\ and\ e^{\xi}\ with\ increment\ of\ s && (153)\\
&e_i^{\xi} - \tau_l^{s-} \geq M(x_{i,l}^{\xi,s-} - 1) && \forall\ s\ and\ e^{\xi}\ with\ decrement\ of\ s && (154)\\
&\textstyle\sum_{\xi,i} x_{i,l}^{\xi,s+} = 1 && \forall\ s,l && (155)\\
&\textstyle\sum_{\xi,i} x_{i,l}^{\xi,s-} = 1 && \forall\ s,l && (156)\\
&\textstyle\sum_{s,l} x_{i,l}^{\xi,s+} = 1 && \forall\ \xi,i && (157)\\
&\textstyle\sum_{s,l} x_{i,l}^{\xi,s-} = 1 && \forall\ \xi,i && (158)\\
& && && (159)
\end{aligned}
$$

5.2. Mathematical programming representation of simulation model - V2

Revise event-based simulation algorithm.

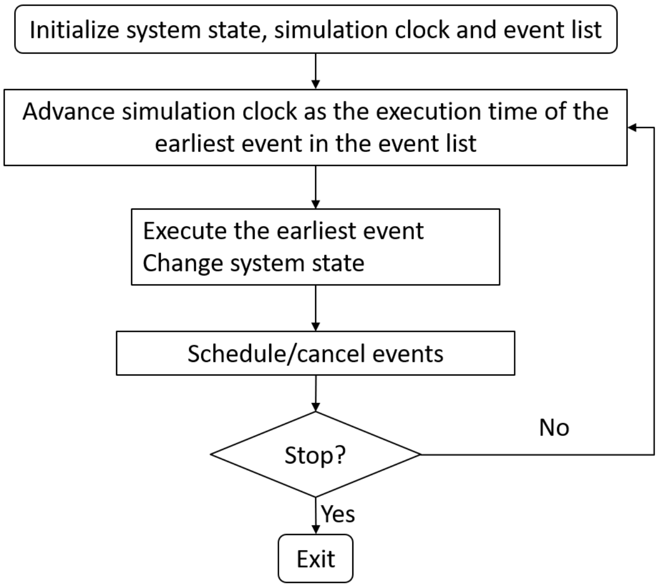

Figure 6. Event-based simulation algorithm.

An equivalent mathematical programming model exists if the following assumptions are satisfied:

(1) State variables are integer.
(2) For all event e^ξ, the *scheduling conditions* are in the form of $a_s^\xi \leq s \leq c_s^\xi$ combined with logic operator "AND", where s is a state variable, and a_s^ξ and c_s^ξ are lower and upper bounds.
(3) The scheduling conditions is independent of the history and not changed along time. (It could be possible to define more state variables in case of history dependence and time-variant scheduling conditions.)
(4) An event execution of e^ξ leads to integer increment or decrement equal to Δ_s^ξ of certain state variables s, and Δ_s^ξ is not changed along time.
(5) The delay between scheduling and execution time of an event e^ξ, denoted by t^ξ, is random variate. They can be generated independently from the simulation run. (*This point is different from ERG. In ERG, the delay is dependent on the edge, i.e, a couple of events, but I consider delay dependent on a single event.*)
(6) For all events e^ξ, the number of executions I^ξ is known before simulation.

Preparation Event e^ξ is expanded into a series of events $e^{\xi,0}$, $e^{\xi,1}$, ..., e^{ξ,Δ^ξ}, where Δ^ξ is equal to the maximum among Δ_s^ξ for all $s \in \Theta^\xi$. The expansion is conducted as follows. First, event $e^{\xi,0}$ is executed as soon as all the scheduling conditions are satisfied, and the state variables $s \in \Theta^\xi$ are not changed. Then, event $e^{\xi,1}$ is executed after t^ξ time unit after an execution of $e^{\xi,0}$. For all $s \in \Theta^\xi$, if $\Delta_s^\xi \geq \delta$, $e^{\xi,\delta}$ will increase or decrease s by one, for all $\delta = 1, ..., \Delta^\xi$. The i-th execution of event $e^{\xi,\delta}$ for $\delta = 1, ..., \Delta^\xi$ are simultaneous.

Constraints (A) The constraints below imply that event $e^{\xi,1}$ is scheduled to exe-

e^{ξ}	event of type ξ
s	state variable
S	set of all state variables
S^{ξ}	set of state variables whose value is conditioned for scheduling event e^{ξ}.
$\Theta^{\xi+}$	the set of state variables that event e^{ξ} will increase its value.
$\Theta^{\xi-}$	the set of state variables that event e^{ξ} will decrease its value.
Θ^{ξ}	$\Theta^{\xi+} \cap \Theta^{\xi-}$
E^{s+}	set of events whose execution increases the value of state variable s.
E^{s-}	set of events whose execution decreases the value of state variable s.
Δ_s^{ξ}	increment or decrement of state variable s when event e^{ξ} is executed.
I^{ξ}	total number of executions of event e^{ξ}
L^{s+}	total number of times that state variable s is increased.
L^{s-}	total number of times that state variable s is decreased.
t^{ξ}	delay between scheduling and execution of event e^{ξ}.
t_i^{ξ}	delay between i-th scheduling and its execution of event e^{ξ}.

Table 4. Notations

$e_i^{\xi,\delta} \geq 0$	time of i-th execution of event $e^{\xi,\delta}$
$\tau_l^{s+} \geq 0$	time when state variable s is increased for the l-th time.
$\tau_l^{s-} \geq 0$	time when state variable s is decreased for the l-th time.
$x_{i,i'}^{\xi} \in \{0,1\}$	equal to 1 if the i' execution of event e^{ξ} is the i-th scheduled one.
$y_{i,l}^{\xi,\delta,s+} \in \{0,1\}$	equal to 1 if the i-th execution of event e^{ξ} is the l-th time that state variable s in incre
$y_{i,l}^{\xi,\delta,s-} \in \{0,1\}$	equal to 1 if the i-th execution of event e^{ξ} is the l-th time that state variable s in decr
$z_{i,l}^{\xi,s+} \in \{0,1\}$	equal to 1 if the i-th scheduling of event e^{ξ} is later than τ_l^{s+}.

Table 5. Decision variables

cute with a delay t^{ξ}, after an execution of $e^{\xi,0}$.

$$e_{i'}^{\xi,1} - e_i^{\xi,0} \geq t_i^{\xi} + M(x_{i,i'}^{\xi} - 1) \qquad \forall \xi, i, i' = 1, ..., I^{\xi} \tag{160}$$

It should be noticed that, if multiple executions of the same event e^{ξ} are allow to exist in the future event list simultaneously, the execution of $e^{\xi,1}$ scheduled by the i-th execution of $e^{\xi,0}$ may be not the i-th execution of $e^{\xi,1}$. Thus, binary variables $x_{i,i'}^{\xi}$ are introduced, and it is equal to one if the i' execution of event $e^{\xi,1}$ is scheduled by the i-th execution of event $e^{\xi,0}$. Since each execution of $e^{\xi,0}$ can schedule one and only one execution of $e^{\xi,1}$, the following constraints should also be satisfied:

$$\sum_{i=1}^{N^{\xi}} x_{i,i'}^{\xi} = 1 \qquad \forall\ \xi, i' = 1, ..., I^{\xi} \tag{161}$$

$$\sum_{i'=1}^{N^{\xi}} x_{i,i'}^{\xi} = 1 \qquad \forall\ \xi, i = 1, ..., I^{\xi} \tag{162}$$

If up to α^{ξ} multiple executions of event e^{ξ} are allowed, the following constraints can be added:

$$e_i^{\xi,0} - e_{i-\alpha^\xi}^{\xi,1} \geq 0 \qquad \forall\ \xi,\ i = \alpha^\xi + 1, ..., I^\xi \tag{163}$$

$$\sum_{i'=i+\alpha^\xi}^{I^\xi} x_{i,i'}^{\xi} = 0 \qquad \forall\ \xi,\ i = 1, ..., I^\xi - \alpha^\xi \tag{164}$$

$$\sum_{i=1}^{i'-\alpha^\xi} x_{i,i'}^{\xi} = 0 \qquad \forall\ \xi,\ i' = \alpha^\xi + 1, ..., I^\xi \tag{165}$$

If α^ξ is equal to one, the constraints (A) are reduced to:

$$e_i^{\xi,1} - e_i^{\xi,0} \geq t_i^\xi \qquad \forall \xi, i = 1, ..., I^\xi \tag{166}$$

$$e_i^{\xi,0} - e_{i-1}^{\xi,1} \geq 0 \qquad \forall\ \xi,\ i = 2, ..., I^\xi \tag{167}$$

Constraints (B) Binding $e_i^{\xi,\delta}$ and $\tau_l^{s+},\ \tau_l^{s-}$:

$$\tau_l^{s+} - e_i^{\xi,\delta} \geq M(y_{i,l}^{\xi,\delta,s+} - 1) \qquad \forall\ s \in S, e^\xi \in E^{s+}, \delta = 1, .., \Delta_s^\xi, i = 1, .., I^\xi, l = 1, .., L^{s+} \tag{168}$$

$$e_i^{\xi,\delta} - \tau_l^{s+} \geq M(y_{i,l}^{\xi,\delta,s+} - 1) \qquad \forall\ s \in S, e^\xi \in E^{s+}, \delta = 1, .., \Delta_s^\xi, i = 1, .., I^\xi, l = 1, .., L^{s+} \tag{169}$$

$$\tau_l^{s-} - e_i^{\xi,\delta} \geq M(y_{i,l}^{\xi,\delta,s-} - 1) \qquad \forall\ s \in S, e^\xi \in E^{s-}, \delta = 1, .., \Delta_s^\xi, i = 1, .., I^\xi, l = 1, .., L^{s-} \tag{170}$$

$$e_i^{\xi,\delta} - \tau_l^{s-} \geq M(y_{i,l}^{\xi,\delta,s-} - 1) \qquad \forall\ s \in S, e^\xi \in E^{s-}, \delta = 1, .., \Delta_s^\xi, i = 1, .., I^\xi, l = 1, .., L^{s-} \tag{171}$$

$$\sum_{\substack{\xi: e^\xi \in E^{s+} \\ i=1,...,I^\xi \\ \Delta=1,...,\Delta_s^\xi}} y_{i,l}^{\xi,\delta,s+} = 1 \qquad \forall\ s \in S, l = 1, ..., L^{s+} \tag{172}$$

$$\sum_{l=1,...,L^{s+}} y_{i,l}^{\xi,\delta,s+} = 1 \qquad \forall\ \xi, s \in \Theta^{\xi+}, i = 1, ..., I^\xi, \delta = 1, ..., \Delta_s^\xi \tag{173}$$

$$\sum_{\substack{\xi: e^\xi \in E^{s-} \\ i=1,...,I^\xi \\ \Delta=1,...,\Delta_s^\xi}} y_{i,l}^{\xi,\delta,s-} = 1 \qquad \forall\ s \in S, l = 1, ..., L^{s-} \tag{174}$$

$$\sum_{l=1,...,L^{s-}} y_{i,l}^{\xi,\delta,s-} = 1 \qquad \forall\ \xi, s \in \Theta^{\xi-}, i = 1, ..., I^\xi, \delta = 1, ..., \Delta_s^\xi \tag{175}$$

$$\tag{176}$$

Binary variables $y_{i,l}^{\xi,\delta,s+} \in \{0, 1\}$ are equal to one if the i-th execution of event $e^{\xi,\delta}$ is the l-th time that state variable s in increased. Since events $e^{\xi,1}, ..., e^{\xi,\Delta_s^\xi}$ are expanded from one event e^ξ, and they are executed simultaneously, the following constraints are also added:

$$e_i^{\xi,\delta} = e_i^{\xi,1} \qquad \forall\ \xi,\ i = 1, ..., I^\xi, \delta = 1, ..., \Delta^\xi \tag{177}$$

$$y_{i,l+\delta-1}^{\xi,\delta,s+} = y_{i,l}^{\xi,1,s+} \qquad \forall\ \xi,\ s \in \Theta^{\xi+},\ \delta = 1, ..., \Delta_s^\xi,\ i = 1, ..., I^\xi,\ l = 1, ..., L^{s+} - \Delta_s^\xi + 1 \tag{178}$$

$$y_{i,l+\delta-1}^{\xi,\delta,s-} = y_{i,l}^{\xi,1,s-} \qquad \forall\ \xi,\ s \in \Theta^{\xi-},\ \delta = 1, ..., \Delta_s^\xi,\ i = 1, ..., I^\xi,\ l = 1, ..., L^{s-} - \Delta_s^\xi + 1 \tag{179}$$

Constraints (C) To trigger event $e^{\xi,0}$, the conditions $b_s^\xi \leq s \leq c_s^\xi$ for all state variable $s \in S^\xi$ should be satisfied. $s \in S^\xi$ can be categorized into one of the following three situations:

- event e^ξ does not change the value of s, i.e., $s \notin \Theta^\xi$.
- event e^ξ increases the value of s, i.e., $s \in \Theta^{\xi+}$.
- event e^ξ decreases the value of s, i.e., $s \in \Theta^{\xi-}$.

If event e^ξ does not change the value of s, or if it is executed after being scheduled with positive delay, i.e., $s \notin \Theta^\xi$ or $t^\xi > 0$, the following constraints are applied:

$$e_i^{\xi,0} - \tau_l^{s+} \leq M z_{i,l}^{\xi,s+} \quad \forall\ \xi, i = 1, ..., I^\xi,\ s \in S^\xi, l = 1, ..., L^{s+} \tag{180}$$
$$e_i^{\xi,0} - \tau_l^{s-} \leq M z_{i,l}^{\xi,s-} \quad \forall\ \xi,\ i = 1, ..., I^\xi,\ s \in S^\xi, l = 1, ..., L^{s-} \tag{181}$$
$$e_i^{\xi,0} - \tau_l^{s-} \geq -M \hat{z}_{i,l}^{\xi,s-} \quad \forall\ \xi,\ i = 1, ..., I^\xi,\ s \in S^\xi, l = 1, ..., L^{s-} \tag{182}$$
$$e_i^{\xi,0} - \tau_l^{s+} \geq -M \hat{z}_{i,l}^{\xi,s+} \quad \forall\ \xi,\ i = 1, ..., I^\xi,\ s \in S^\xi, l = 1, ..., L^{s+} \tag{183}$$
$$z_{i,l}^{\xi,s+} + \hat{z}_{i,s_0+l-b_s^\xi}^{\xi,s-} \leq 1 \quad ?? \tag{184}$$
$$z_{i,l}^{\xi,s-} + \hat{z}_{i,-s_0+l+a_s^\xi}^{\xi,s+} \leq 1 \quad ?? \tag{185}$$
$$\tag{186}$$

$$e_i^{\xi,0} - \tau_l^{s+} \geq M(z_{i,l}^{\xi,s+} - 1) \quad \forall\ \xi, i = 1, ..., I^\xi,\ s \in S^\xi, l = 1, ..., L^{s+} \tag{187}$$
$$\tau_l^{s+} - e_i^{\xi,0} > -M z_{i,l}^{\xi,s+} \quad \forall\ \xi,\ i = 1, ..., I^\xi,\ s \in S^\xi, l = 1, ..., L^{s+} \tag{188}$$
$$e_i^{\xi,0} - \tau_l^{s-} \geq M(z_{i,l}^{\xi,s-} - 1) \quad \forall\ \xi,\ i = 1, ..., I^\xi,\ s \in S^\xi, l = 1, ..., L^{s-} \tag{189}$$
$$\tau_l^{s-} - e_i^{\xi,0} > -M z_{i,l}^{\xi,s-} \quad \forall\ \xi,\ i = 1, ..., I^\xi,\ s \in S^\xi, l = 1, ..., L^{s-} \tag{190}$$
$$\tag{191}$$

If $e_i^{\xi,0}$ is executed after $\tau_l^{s+}(\tau_l^{s-})$, $z_{i,l}^{\xi,s+}(z_{i,l}^{\xi,s-})$ is equal to one. If $e_i^{\xi,0}$ is executed before $\tau_l^{s+}(\tau_l^{s-})$, $\hat{z}_{i,l}^{\xi,s+}(\hat{z}_{i,l}^{\xi,s-})$ is equal to one.

If event e^ξ increases the value of s, and it is executed immediately when scheduled, i.e., $s \in \Theta^{\xi+}$ and $t^\xi = 0$, the following constraint should be applied:

$$e^{\xi,0} - \tau_{s_0+l-1-b}^{s-} \geq M(y_{i,l}^{\xi,1,s+} - 1) \tag{192}$$
$$e^{\xi,0} - \tau_{s_0+l-1-a}^{s-} \leq M(1 - y_{i,l}^{\xi,1,s+}) \tag{193}$$

If event e^ξ decreases the value of s, and it is executed immediately when scheduled, i.e., $s \in \Theta^{\xi-}$ and $t^\xi = 0$, the following constraint should be applied:

$$e^{\xi,0} - \tau_{-s_0+l-1+a}^{s+} \geq M(y_{i,l}^{\xi,1,s-} - 1) \tag{194}$$
$$e^{\xi,0} - \tau_{-s_0+l-1+b}^{s+} \leq M(1 - y_{i,l}^{\xi,1,s-}) \tag{195}$$

5.3. Mathematical programming representation of simulation model - V3

Event-scheduling algorithm for DES.

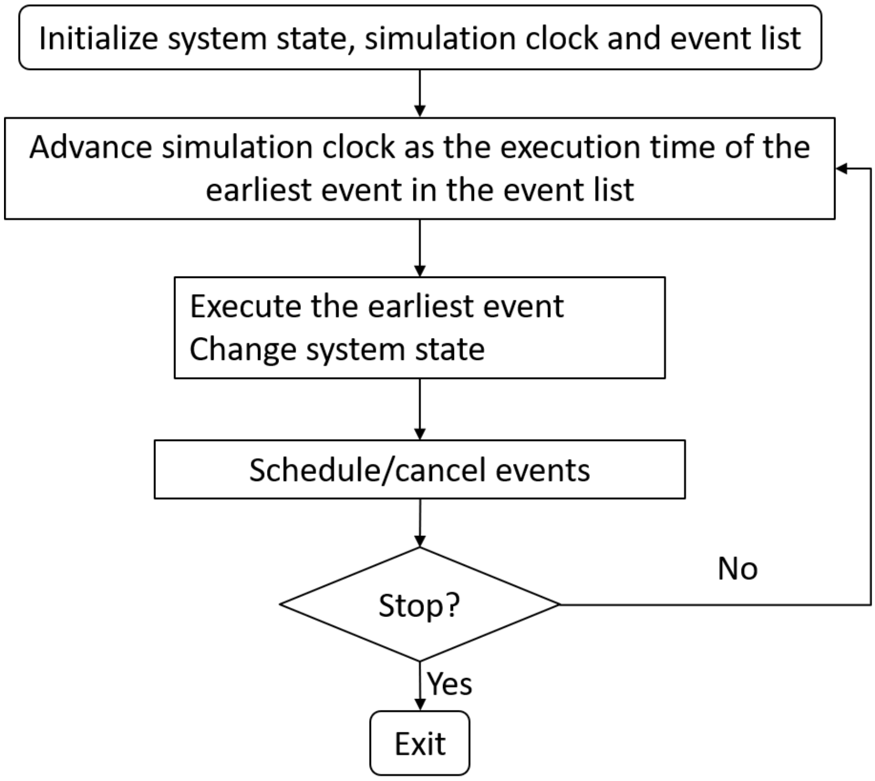

Figure 7. Event-based simulation algorithm.

An equivalent mathematical programming model exists if the following assumptions are satisfied:

(1) For all event e^{ξ}, the *scheduling conditions* are in the form of $b_s^{\xi} \leq s \leq c_s^{\xi}$ combined with logic operator "AND", where s is a state variable, and b_s^{ξ} and c_s^{ξ} are lower and upper bounds.

(2) The scheduling conditions is independent of the history and not changed along time. (It could be possible to define more state variables in case of history dependence and time-variant scheduling conditions.)

(3) An event execution of e^{ξ} leads to (integer) increment or decrement equal to Δ_s^{ξ} of certain state variables s, and Δ_s^{ξ} is not changed along time. (A direct evaluation can be modeled in this way.)

(4) The delay between scheduling and execution time of an event e^{ξ}, denoted by t^{ξ}, is random variate. They can be generated independently from the simulation run. (*This point is different from ERG. In ERG, the delay is dependent on the edge, i.e, a couple of events, but I consider delay dependent on a single event.*)

(5) For all events e^{ξ}, the number of executions N^{ξ} is known before simulation.

$e_i^{\xi,0} \geq 0$	i=1,...,I^ξ	the i-th scheduling time of event e^ξ.
$e_i^{\xi,1} \geq 0$	i=1,...,I^ξ	the i-th execution time of event e^ξ.
$\mathcal{E}_k \geq 0$	k=0,...,K	time of the k-th execution of any events.
$u_k^s \in \mathbb{Z}$	k=0,...,K	value of state variable s just after the k-th event.
$w_{i,k}^\xi \in \{0,1\}$	k=1,...,K	binding $e_i^{\xi,1}$ and $\mathcal{E}_k$.
$x_{i,k}^\xi \in \{0,1\}$	k=0,...,K	equal to one if $\mathcal{E}_k$ schedules $e^{\xi,0}$.
$y_{i,i'}^\xi \in \{0,1\}$		binding $e_i^{\xi,0}$ and $e_{i'}^{\xi,1}$ in case of overtaking.
$z_k^\xi \in \{0,1\}$	k=0,...,K	equal to one if the condition for scheduling e^ξ is true right after $\mathcal{E}_k$.
$v_k^{\xi,s,0} \in \{0,1\}$	k=0,...,K	equal to one if $s_k \leq a^{\xi,s} - 1$
$v_k^{\xi,s,1} \in \{0,1\}$	k=0,...,K	equal to one if $s_k \geq b^{\xi,s} - 1$
$r_k^\xi \in \mathbb{Z}$	k=0,...,K	number of existing parallel executions of $e_i^{\xi,1}$ after $\mathcal{E}_k$ before scheduling.
$n_k^\xi \in \mathbb{Z}$	k=0,...,K	number of scheduled executions of $e_i^{\xi,1}$ after $\mathcal{E}_k$ before scheduling.

Table 6. Notation

Constraints (A): binding $e_i^{\xi,1}$ and $\mathcal{E}_k$:

$$e_i^{\xi,1} - \mathcal{E}_k \geq M(w_{i,k}^\xi - 1) \quad A1 \quad \forall\, \xi, i, k \tag{196}$$

$$\mathcal{E}_k - e_i^{\xi,1} \geq M(w_{i,k}^\xi - 1) \quad A2 \quad \forall\, \xi, i, k \tag{197}$$

$$\sum_k w_{i,k}^\xi = 1 \quad A3 \quad \forall\, \xi, i \tag{198}$$

$$\sum_{\xi,i} w_{i,k}^\xi = 1 \quad A4 \quad \forall\, k \tag{199}$$

$$\sum_k k w_{i+1,k}^\xi - \sum_k k w_{i,k}^\xi \geq 1 \quad A5 \quad \forall\, \xi, i \tag{200}$$

Constraints (B): binding $e_i^{\xi,0}$ and $e_{i'}^{\xi,1}$, where α^ξ is the maximal number of executions existing simultaneously in the event list:

$$e_{i'}^{\xi,1} - e_i^{\xi,0} \geq t_i^\xi + M(y_{i,i'}^\xi - 1) \quad B1 \quad \forall \xi, i, i' = 1, ..., N^\xi \tag{201}$$

$$e_i^{\xi,0} - e_{i'}^{\xi,1} \geq -t_i^\xi + M(y_{i,i'}^\xi - 1) \quad B2 \quad \forall \xi, i, i' = 1, ..., N^\xi \tag{202}$$

$$\sum_{i=1}^{N^\xi} y_{i,i'}^\xi = 1 \quad B3 \quad \forall\, \xi, i' = 1, ..., N^\xi \tag{203}$$

$$\sum_{i'=1}^{N^\xi} y_{i,i'}^\xi = 1 \quad B4 \quad \forall\, \xi, i = 1, ..., N^\xi \tag{204}$$

$$\sum_{i'=i+\alpha^\xi}^{N^\xi} y_{i,i'}^\xi = 0 \quad B5 \quad \forall\, \xi,\ i = 1, ..., N^\xi - \alpha^\xi \tag{205}$$

$$\sum_{i=1}^{i'-\alpha^\xi} y_{i,i'}^\xi = 0 \quad B6 \quad \forall\, \xi,\ i' = \alpha^\xi + 1, ..., N^\xi \tag{206}$$

If $\alpha^\xi = 1$, variables $y^\xi_{i,i'}$ are redundant and constraints (B) are reduced to:

$$e_i^{\xi,1} - e_i^{\xi,0} = t_i^\xi \quad B1 \quad \forall \xi, i = 1, ..., N^\xi \tag{207}$$

Number of executions of event e^ξ waiting in the event list can be a state variable n^ξ, and one condition for scheduling an e^ξ is $n^\xi \leq \alpha^\xi$. Thus, it can be managed as a generic scheduling condition.

Constraints (C): event e^ξ can be scheduled right after $\mathcal{E}_k$ if all state variables s satisfies condition $a^\xi_s \leq s_k \leq b^\xi_s$.

$$e_i^{\xi,0} - \mathcal{E}_k \geq M(x^\xi_{i,k} - 1) \quad C1 \quad \forall\, \xi, k, i \tag{208}$$

$$\mathcal{E}_k - e_i^{\xi,0} \geq M(x^\xi_{i,k} - 1) \quad C2 \quad \forall\, \xi, k, i \tag{209}$$

$$\sum_k x^\xi_{i,k} = 1 \quad C3 \quad \forall\, \xi, i \tag{210}$$

$$b^{\xi,s} - u^s_k \geq M(z^\xi_k - 1) \quad C4 \quad \forall\, \xi, k, s \tag{211}$$

$$u^s_k - a^{\xi,s} \geq M(z^\xi_k - 1) \quad C5 \quad \forall\, \xi, k, s \tag{212}$$

$$u^s_k - (b^{\xi,s} + 1) \geq M(v_k^{\xi,s,1} - 1) \quad C6 \quad \forall\, \xi, k, s \tag{213}$$

$$(a^{\xi,s} - 1) - u^s_k \geq M(v_k^{\xi,s,0} - 1) \quad C7 \quad \forall\, \xi, k, s \tag{214}$$

$$1 - z^\xi_k \leq \sum_{s \in S^\xi} v_k^{\xi,s,0} + \sum_{s \in S^\xi} v_k^{\xi,s,1} + v_k^{\xi,r} + v_k^{\xi,N} \quad C8 \quad \forall\, \xi, k \tag{215}$$

$$\sum_{i=1}^{N^\xi} x^\xi_{i,k} = z^\xi_k \quad C9 \quad \forall\, \xi, k \tag{216}$$

$$\sum_k k x^\xi_{i+1,k} - \sum_k k x^\xi_{i,k} \geq 1 \quad C10 \quad \forall\, \xi, i \tag{217}$$

Constraints (D): evolution of state variables

$$u^s_k = u^s_{k-1} + \sum_\xi \sum_{i=1}^{N^\xi} w^\xi_{i,k} \Delta^{\xi,s} \quad D1 \quad \forall\, s, k \tag{218}$$

$$r^\xi_k = r^\xi_{k-1} + X^\xi_{k-1} - \sum_i w^\xi_{i,k} \quad D2 \quad \forall\, \xi, k \tag{219}$$

$$R^\xi - r^\xi_k \geq z^\xi_k \quad D3 \quad \forall\, \xi, k \tag{220}$$

$$r^\xi_k \geq R^\xi v_k^{\xi,r} \quad D4 \quad \forall\, \xi, k \tag{221}$$

$$n^\xi_k = n^\xi_{k-1} + X^\xi_{k-1} \quad D5 \quad \forall\, \xi, k \tag{222}$$

$$N^\xi - n^\xi_k \geq z^\xi_k \quad D6 \quad \forall\, \xi, k \tag{223}$$

$$n^\xi_k \geq N^\xi v_k^{\xi,N} \quad D7 \quad \forall\, \xi, k \tag{224}$$

Constraints (E): others

$$\mathcal{E}_0 = 0 \quad E1 \tag{225}$$

$$\mathcal{E}_k - \mathcal{E}_{k-1} \geq 0 \quad E1 \quad \forall\, k \tag{226}$$

Objective function: with the constraints above, there is a unique solution in terms of event occurring times (solution of the binary variables could be multiple in case of multiple simultaneous events). Thus, the objective can be any function of event occurring time. I tried minimize/maximize the sum of $\mathcal{E}_k$, and they give the same solution.

Conditions for a variable x to be *resource-type* are not valid any more.

(1) $\forall\ \xi$ and s, upper bound c_s^ξ is monotonically increasing on x.
(2) $\forall\ \xi$ and s, lower bound b_s^ξ is monotonically decreasing on x.

The reason is that increasing c_s^ξ or decreasing b_s^ξ will tighten constraints C6 and C7. To be simple, we consider b only.

$$b - u_k^s \geq M(z_k^\xi - 1) \quad C4 - b \quad \forall\ \xi, k, s \tag{227}$$
$$u_k^s - (b+1) \geq M(v_k^{\xi,s,1} - 1) \quad C6 - b \quad \forall\ \xi, k, s \tag{228}$$

(when $u = b + 1$, event e^ξ cannot be scheduled.)
If b is increased to $b + 1$:

$$(b+1) - u_k^s \geq M(z_k^\xi - 1) \quad C4 - (b+1) \quad \forall\ \xi, k, s \tag{229}$$
$$u_k^s - (b+2) \geq M(v_k^{\xi,s,1} - 1) \quad C6 - (b+1) \quad \forall\ \xi, k, s \tag{230}$$

(when $u = b + 1$, event e^ξ must be scheduled.)
A group of relaxed constraints are:

$$(b+1) - u_k^s \geq M(z_k^\xi - 1) \quad C4 - (b+1) \quad \forall\ \xi, k, s \tag{231}$$
$$u_k^s - (b+1) \geq M(v_k^{\xi,s,1} - 1) \qquad C6 - (b) \qquad \forall\ \xi, k, s \tag{232}$$

(when $u = b + 1$, event e^ξ can be scheduled or not.)
Todo:

(1) What kind of performance indicators can be used? (Regular function of time, in scheduling area. Weighted sum, maximum. Refer to book on scheduling.)

5.4. Merge

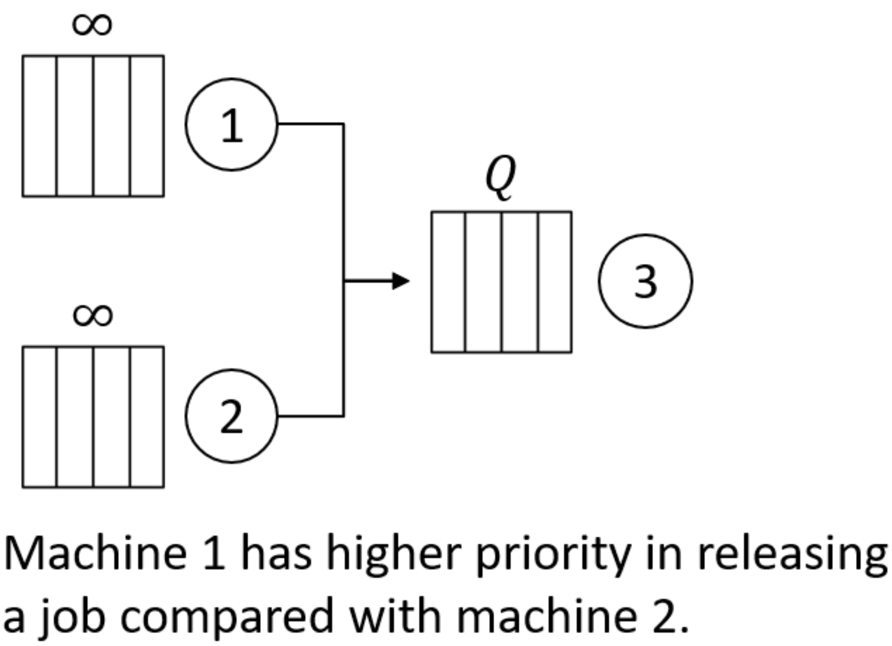

Figure 8. Example: merge.

Variable	Event	Condition to schedule	Delay	# executions	State change
$e^{s,1}$	Start m1	$m^1 \leq 0$	0	1	$m^1 ++$
$e^{f,1}$	Finish m1	$1 \leq m^1 \leq 1$	t^1	1	$m^1 ++$
$e^{d,1}$	Depart m1	$m^1 \geq 2\ AND$ $q \geq 1$	0	1	$m^1 = m^1 - 2,\ q --$
$e^{s,2}$	Start m2	$m^2 \leq 0$	0	1	$m^2 ++$
$e^{f,2}$	Finish m2	$1 \leq m^2 \leq 1$	t^2	1	$m^2 ++$
$e^{d,2}$	Depart m2	$m^2 \geq 2\ AND$ $q \geq 1\ AND$ $m^1 \leq 1$	0	1	$m^2 = m^2 - 2,\ q --$
$e^{s,3}$	Start m3	$m^3 \leq 0\ AND$ $q \leq Q - 1$	0	1	$m^3 ++,\ q ++$
$e^{d,3}$	Depart m3	$m^3 \geq 1$	t^3	1	$m^3 --$

Table 7. Merge-S3M111

MP model:

$$\min \sum_k \mathcal{E}_k \tag{233}$$

$$e_i^{(\xi,j),1} - \mathcal{E}_k \geq M(w_{i,k}^{\xi,j} - 1) \qquad \xi \in \{s,f,d\}, j \in \{1,2,3\}, \forall i,k \tag{234}$$

$$\mathcal{E}_k - e_i^{(\xi,j),1} \geq M(w_{i,k}^{\xi,j} - 1) \qquad \xi \in \{s,f,d\}, j \in \{1,2,3\}, \forall i,k \tag{235}$$

$$\sum_k w_{i,k}^{\xi,j} = 1 \qquad \forall\ \xi \in \{s,f,d\}, j \in \{1,2,3\}, i \tag{236}$$

$$\sum_{(\xi,j),i} w_{i,k}^{\xi,j} = 1 \qquad \forall\ k \tag{237}$$

$$\sum_k k w_{i+1,k}^{\xi,j} - \sum_k k w_{i,k}^{\xi,j} \geq 1 \qquad \forall\ \xi \in \{s,f,d\}, j \in \{1,2,3\}, i \tag{238}$$

$$e_i^{s,j,1} - e_i^{s,j,0} \geq 0 \qquad j = 1,2,3, \forall\ i \tag{239}$$

$$e_i^{f,j,1} - e_i^{f,j,0} \geq t_i^j \qquad j = 1,2, \forall\ i \tag{240}$$

$$e_i^{d,j,1} - e_i^{d,j,0} \geq 0 \qquad j = 1,2, \forall\ i \tag{241}$$

$$e_i^{d,3,1} - e_i^{d,3,0} \geq t_i^3 \qquad \forall\ i \tag{242}$$

$$e_i^{\xi,j,0} - \mathcal{E}_k \geq M(x_{i,k}^{\xi,j} - 1) \qquad \forall\ \xi \in \{s,f,d\}, j = 1,2,3,k,i \tag{243}$$

$$\mathcal{E}_k - e_i^{\xi,j,0} \geq M(x_{i,k}^{\xi,j} - 1) \qquad \forall\ \xi \in \{s,f,d\}, j = 1,2,3,k,i \tag{244}$$

$$m_k^j = m_{k-1}^j + \sum_{i=1}^{N^j} (w_{i,k}^{s,j} + w_{i,k}^{f,j} - 2w_{i,k}^{d,j}) \qquad j = 1,2, \forall\ k \tag{245}$$

$$m_k^3 = m_{k-1}^3 + \sum_{i=1}^{N^3} (w_{i,k}^{s,3} - w_{i,k}^{d,3}) \qquad \forall\ k \tag{246}$$

$$q_k = q_{k-1} + \sum_{i=1}^{N^3} w_{i,k}^{s,3} - \sum_{i=1}^{N^1} w_{i,k}^{d,1} - \sum_{i=1}^{N^2} w_{i,k}^{d,2} \tag{247}$$

$$m_k^j \geq M(z_k^{s,j} - 1) \qquad j = 1,2,3, \forall\ k \tag{248}$$

$$1 - m_k^j \geq M(z_k^{f,j} - 1) \qquad j = 1,2, \forall\ k \tag{249}$$

$$m_k^j - 1 \geq M(z_k^{f,j} - 1) \qquad j = 1,2, \forall\ k \tag{250}$$

$$m_k^j - 2 \geq M(z_k^{d,j} - 1) \qquad j = 1,2, \forall\ k \tag{251}$$

$$q_k - 1 \geq M(z_k^{d,j} - 1) \qquad j = 1,2, \forall\ k \tag{252}$$

$$1 - m_k^1 \geq M(z_k^{d,2} - 1) \qquad \forall\ k \tag{253}$$

$$m_k^3 - 1 \geq M(z_k^{d,3} - 1) \qquad \forall\ k \tag{254}$$

$$(Q - 1) - q_k \geq M(z_k^{s,3} - 1) \qquad \forall\ k \tag{255}$$

$$\sum_k x_{i,k}^{\xi,j} = 1 \qquad \forall\ \xi \in \{s,f,d\}, j = 1,2,3, \forall\ i,k \tag{256}$$

$$\sum_{i=1}^{N^j} x_{i,k}^{\xi,j} \leq z_k^{\xi,j} \qquad \forall\ \xi \in \{s,f,d\}, j = 1,2,3,k \tag{257}$$

$$\sum_k k x_{i+1,k}^{\xi,j} - \sum_k k x_{i,k}^{\xi,j} \geq 0 \qquad \forall\ \xi \in \{s,f,d\}, j = 1,2,3,i \tag{258}$$

5.5. Merge - 2 machines in station 3

Variable	Value	Initialization	Description
e^1	0,1,2	2	number of empty machines in station 1
f^1	0,1,2	0	number of finished jobs in station 1
e^2	0,1	1	number of empty machines in station 2
f^2	0,1	0	number of finished jobs in station 2
e^3	0,1	1	number of empty machines in station 3
q	0,...,Q	Q	number of available spaces in queue

Table 8. State variables: Merge-S3M211

Variable	Event	Condition to schedule	Delay	# executions	State change
ai $e^{f,1}$	Finish m1	$1 \leq e^1 \leq 2$	t^1	2	$f^1 + +$
$e^{d,1}$	Depart m1	$1 \leq f^1 \leq 2\ AND$ $1 \leq q \leq Q$	0	1	$e^1 + +,\ f^1 - -,\ q - -$
$e^{s,2}$	Start m2	$1 \leq e^2 \leq 1$	0	1	$e^2 - -$
$e^{f,2}$	Finish m2	$1 \leq e^2 \leq 1$	t^2	1	$f^2 + +$
$e^{d,2}$	Depart m2	$1 \leq f^2 \leq 1\ AND$ $1 \leq q \leq Q\ AND$ $0 \leq f^1 \leq 0$	0	1	$f^2 - -, e^2 + +,\ q - -$
$e^{s,3}$	Start m3	$1 \leq e^3 \leq 1\ AND$ $0 \leq q \leq Q - 1$	0	1	$e^3 - -,\ q + +$
$e^{d,3}$	Depart m3	$1 \leq e^3 \leq 1\ AND$	t^3	1	$e^3 + +$
		$0 \leq q \leq Q - 1$			

Table 9. Events: Merge-S3M211

5.6. Failure

5.7. Jobshop

5.8. Identifying Resource-type variables

6. Gradient-based approximate cut

6.1. Gradient estimation

6.2. Gradient-based feasibility cut

7. Combinatorial cut generation

7.1. Combinatorial cut

7.2. Heuristic for tightening Exact combinatorial cut

8. Feasibility-cut-based algorithm

The complete algorithm for solving RAP–PC is summarized in Algorithm 1. The resource capacities are initialized to the lower bound. The searching region of RAP–PC–MIP is initialized to $\mathbb{X}$, and the lower and upper bounds of the objective function, C^L and C^U, respectively, are set considering the upper bound and lower bound of the capacity of each resource. Lines 7 to 11 show that approximate cuts are generated and used in the model when infeasible solutions are found. Once a feasible solution is found, the upper bound C^U, which is also the incumbent solution, can be updated after comparing the value of the found feasible solution and that of the current incumbent. Then, all the currently used approximate cuts are replaced by exact cuts of the DIS. If there are only exact cuts in RAP–PC–MIP, the solution is the new lower bound C^L. The algorithm terminates when the gap between the upper bound and lower bound is within a tolerance or the time limit is exceeded.

9. Numerical analysis

10. Conclusion

References

Alfieri, A., & Matta, A. (2012). Mathematical programming formulations for approximate simulation of multistage production systems. *European Journal of Operational Research*, *219*(3), 773–783.

Alfieri, A., Matta, A., & Pastore, E. (2020). The time buffer approximated buffer allocation problem: A row–column generation approach. *Computers & Operations Research*, *115*, 104835.

Chan, W. K., & Schruben, L. (2008). Optimization models of discrete-event system dynamics. *Operations Research*, *56*(5), 1218–1237.

Law, A. M. (2014). *Simulation modeling and analysis* (Vol. 5). McGraw-Hill New York.

Matta, A. (2008). Simulation optimization with mathematical programming representation of discrete event systems. In *2008 winter simulation conference* (pp. 1393–1400).

Pedrielli, G., Alfieri, A., & Matta, A. (2015). Integrated simulation–optimisation of pull control systems. *International Journal of Production Research*, *53*(14), 4317–4336.

Tan, B. (2015). Mathematical programming representations of the dynamics of continuous-flow production systems. *IIE Transactions*, *47*(2), 173–189.

Algorithm 1 MIP–based algorithm.

Input:

Lower bound $\mathbf{a} = [a_1, ..., a_J]$ and upper bound $\mathbf{b} = [b_1, ..., b_J]$ of resource capacity $\mathbf{x}$, such that $a_j \leq x_j \leq b_j \ \forall \ j = 1, ..., J$.

Tolerance of optimality gap ε_{opt}.

Optional input: time limit of the algorithm T_{lim}.

Ensure:

Sample-path global optimal $\mathbf{x}^*$.

```
Initialize system with lower bound x ← a
Initialize incumbent with upper bound x* ← b.
Initialize lower bound of the objective C^L ← c^T a.
Initialize upper bound of the objective C^U ← c^T b.
Add initial constraints which defines 𝕏 to the RAP–PC–MIP.
while C^U − C^L > ε_opt and T_lim is not exceeded. do
    while There exists at least one violated performance constraint do
        Generate one approximate cut CA(x̄, l) for each violated constraints l and
        add all the generated cuts to the RAP–PC–MIP.
        x̄ ← solution of the RAP–PC–MIP.
        Simulate the system of x̄.
    end while
    Update upper bound and incumbent C^U ← c^T x̄, x* ← x̄ if c^T x̄ < C^U.
    if There exist approximate cuts in RAP–PC–MIP then
        For all the currently used approximate cuts CA(x̄^r, l), find dominating infea-
        sible solution x̄_d(x̄^r) and replace approximate cuts CA(x̄^r, l) by exact cuts
        CE(x̄_d(x̄^r), l) of the DIS.
        x̄ ← solution of the RAP–PC–MIP.
        Simulate the system of x̄.
        Update lower bound C^L ← max{c^T x̄, C^L}.
    end if
end while
```

Weiss, S., & Stolletz, R. (2015). Buffer allocation in stochastic flow lines via sample-based optimization with initial bounds. *OR spectrum*, *37*(4), 869–902.

Zhang, M., & Matta, A. (2018). Data-driven downtime bottleneck detection in open flow lines. In *2018 ieee 14th international conference on automation science and engineering (case)* (pp. 1513–1518).

Zhang, M., & Matta, A. (2020). Models and algorithms for throughput improvement problem of serial production lines via downtime reduction. *IISE Transactions*, 1–15.

Zhang, M., Matta, A., & Alfieri, A. (2020). Simulation optimization with mathematical programming representation of discrete event systems. In *2020 winter simulation conference* (p. accepted).

Zhang, M., Pastore, E., Matta, A., & Alfieri, A. (n.d.). Buffer allocation problem in production flow lines: a new benders decomposition based exact solution approach. *under review*.