Gauss Elimination with Partial Pivoting

CS/SE 4X03

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Outline

Example

Example 1. Consider

$$
\begin{aligned}
10^{-5}x_1 + x_2 &= 1 \\
x_1 + x_2 &= 2
\end{aligned}
$$

Solution is

$$
\begin{aligned}
x_1 &= \frac{1}{1-10^{-5}} \approx 1 + 10^{-5} = 1.00001 \\
x_2 &= 2 - x_1 \approx 1 - 10^{-5} = 0.99999
\end{aligned}
$$

Solve by Gauss elimination (GE) in 4-digit decimal floating-point arithmetic

Example cont.

Example 1. cont.

- Multiply first row by $1/10^{-5} = 10^5$ and subtract from second

$$10^{-5}x_1 + x_2 = 1$$
$$(1 - 10^5)x_2 = 2 - 10^5$$

- Pivot 10^{-5}
- In this arithmetic $1 - 10^5$ and $2 - 10^5$ round to -10^5
- Second equation becomes $-10^5 x_2 = -10^5$
- Backward substitution gives

$$\widetilde{x}_2 = 1, \qquad \widetilde{x}_1 = \frac{1 - \widetilde{x}_2}{10^{-5}} = 0$$

Solution is $x_1 \approx 1.00001$, $x_2 \approx 0.99999$

Example cont.

Example 1. cont.

- Error in $\widetilde{x}_2$ is

$$\delta = \widetilde{x}_2 - x_2 \approx 1 - 0.99999 = 10^{-5}$$

- From $10^{-5}x_1 + x_2 = 1$,

$$\begin{aligned} \widetilde{x}_1 &= \frac{1 - \widetilde{x}_2}{10^{-5}} = \frac{1 - (x_2 + \delta)}{10^{-5}} \\ &= \frac{1 - x_2}{10^{-5}} - \frac{\delta}{10^{-5}} \\ &= x_1 - 10^5\delta \end{aligned}$$

- Error in $\widetilde{x}_1$ is $-10^5\delta \approx -1$
- Error in $\widetilde{x}_2$ is multiplied by -10^5 in $\widetilde{x}_1$

Example cont.

Example 1. cont.

- Swap rows

$$x_1 + x_2 = 2$$
$$10^{-5}x_1 + x_2 = 1$$

- Multiply first row by $10^{-5}/1$ and subtract from second

$$x_1 + x_2 = 2$$
$$x_2(1 - 10^{-5}) = 1 - 2 \times 10^{-5}$$

- Pivot 1
- $1 - 10^{-5}$ and $1 - 2 \times 10^{-5}$ round to 1
- Second equation is $x_2 = 1$

Example cont.

pattern
~x_1 = x_1 - 1/pivot * delta
error in x_1 = - 1/pivot * delta

Example 1. cont.

- Backward substitution gives

$$\widehat{x}_2 = 1$$
$$\widehat{x}_1 = 2 - \widehat{x}_2 = 1$$

- Error in $\widehat{x}_2 = 1$ same as before
- From $x_1 + x_2 = 2$ and $\widehat{x}_2 = x_2 + \delta$,

$$\begin{aligned}\widehat{x}_1 &= 2 - \widehat{x}_2 = 2 - x_2 - \delta \\ &= x_1 - \delta\end{aligned}$$

- Error δ in $\widehat{x}_2$ is not magnified in $\widehat{x}_1$

GE with partial pivoting

GE with partial pivoting

- Choose the row with the largest (in magnitude) entry

Scaled partial pivoting

Example 2. Consider

$$2x_1 + 2cx_2 = 2c$$
$$x_1 + x_2 = 2$$

$c > 1$ is a constant

- Partial pivoting: first row as pivot row $(2 > 1)$
- GE gives

$$2x_1 + 2cx_2 = 2c$$
$$(1 - c)x_2 = 2 - c$$

- For c sufficiently large, $1 - c \approx -c$, $2 - c \approx -c$

Scaled partial pivoting cont.

Example 2. cont.

- Backward substitution gives

$$\widetilde{x}_2 \approx 1, \qquad \widetilde{x}_1 = \frac{2c - 2c\widetilde{x}_2}{2} \approx 0$$

If $\delta = \widetilde{x}_2 - x_2$,

$$\widetilde{x}_1 = \frac{2c - 2c\widetilde{x}_2}{2} = c - c(x_2 + \delta) = c - cx_2 - c\delta = x_1 - c\delta$$

- Error is multiplied by c
- When c is sufficiently large,

$$x_2 = \frac{c-2}{c-1} \approx 1, \qquad x_1 = \frac{c}{c-1} \approx 1$$

Scaled partial pivoting cont.

Example 2. cont.

Chose the row with the largest entry with respect to the entries in this row

$$2x_1 + 2cx_2 = 2c$$
$$1x_1 + 1x_2 = 1$$

- Scale vector $s = (2c, 1)$, $2c$ largest in first row, 1 largest in second row
- Ratio vector
$$r = \left(\frac{2}{2c}, \frac{1}{1}\right)$$
- Chose row with largest ratio as pivot row
- Eliminate with second row

Scaled partial pivoting cont.

Example 2. cont.

$$
\begin{aligned}
x_1 + x_2 &= 2 \\
2x_1 + 2cx_2 &= 2c
\end{aligned}
$$

- GE gives

$$
\begin{aligned}
x_1 + x_2 &= 2 \\
(2c-2)x_2 &= 2c-4
\end{aligned}
$$

- Backward substitution (when c sufficiently large)

$$
\begin{aligned}
\widehat{x}_2 &\approx 1 \\
\widehat{x}_1 &\approx 1
\end{aligned}
$$

Scaled partial pivoting cont.

Example 3.

$$Ax = \begin{bmatrix} 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \\ 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -19 \\ 34 \\ 16 \\ 26 \end{bmatrix} = b$$

- Scale vector $s = (13, 18, 6, 12)$

  $s_i = \max\{|a_{ij}| \mid j = 1, 2, 3, 4\}$

- Ratio vector

$$r = \left(\frac{3}{13}, \frac{6}{18}, \frac{6}{6}, \frac{12}{12} \right)$$

- Select index in r with largest ratio: 3 or 4
- Pick 3 and eliminate with row 3

Scaled partial pivoting cont.

Example 3. cont.

$$\begin{bmatrix} 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \\ 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -27 \\ 50 \\ 16 \\ -6 \end{bmatrix}$$

- Scale vector $s = (13, 18, 6, 12)$
- Ratio vector

$$r = \left(\frac{12}{13}, \frac{2}{18}, -, \frac{4}{12} \right)$$

  – means entry does not matter
- Select index from 1,2,4 with largest ratio: 1
- Eliminate with row 1

Scaled partial pivoting cont.

Example 3. cont.

With rounding to 4 decimal places

$$\begin{bmatrix} 0 & -12 & 8 & 1 \\ 0 & 0 & 4.3333 & -13.8333 \\ 6 & -2 & 2 & 4 \\ 0 & 0 & -0.6667 & 1.6667 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -27 \\ 45.5 \\ 16 \\ 3 \end{bmatrix}$$

- Scale vector $s = (13, 18, 6, 12)$
- Ratio vector
$$r = \left(-, \frac{4.3333}{18}, -, \frac{0.6667}{12} \right)$$
- Select index from 2,4 with largest ratio: 2
- Eliminate with row 2

Example 3. cont.

$$\begin{bmatrix} 0 & -12 & 8 & 1 \\ 0 & 0 & 4.3333 & -13.8333 \\ 6 & -2 & 2 & 4 \\ 0 & 0 & -0 & -0.4615 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -27 \\ 45.5 \\ 16 \\ 10 \end{bmatrix}$$

$$\begin{aligned} x_4 &= -21.6667 \\ x_3 &= (45.5 - (-13.8333) * (-21.6667))/(4.3333) \\ &= -58.6671 \\ x_2 &= (-27 - 8 * (-58.6671) - 1 * (-21.6667))/(-12) \\ &= -38.6667 \\ x_1 &= (16 - (-2) * (-38.6667) - 2 * (-58.6671) - 4 * (-21.6667))/6 \\ &= 23.7779 \end{aligned}$$