Numerical Methods for IVP ODEs

CS/SE 4X03

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Outline

The problem

- Given

$$y' = f(t, y), \qquad y(a) = c$$

  compute $y(t)$ on $[a, b]$
- $y' \equiv y'(t) \equiv \frac{dy}{dt}$
- This is an Initial Value Problem (IVP) in Ordinary Differential Equations (ODEs)
- We approximate $y(t)$ at points t_i in $[a, b]$ using a numerical method

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ODE examples

$$y' = -y + t$$

- Solution is $y(t) = t - 1 + \alpha e^{-t}$:

$$\begin{aligned} y'(t) &= 1 - \alpha e^{-t} \\ -y + t &= -(t - 1 + \alpha e^{-t}) + t = 1 - \alpha e^{-t} \end{aligned}$$

- Given $y(0) = c$, e.g. $c = 5$,

$$y(0) = -1 + \alpha = c = 5, \qquad \alpha = 6$$

$$y(t) = t - 1 + 6e^{-t}$$

is the solution with this initial condition

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Motion of a pendulum

$$\theta'' = -g \sin\theta, \qquad \theta'' = \frac{d^2\theta(t)}{dt^2}$$

- ball of mass 1 attached to the end of a rigid, massless rod of length $r = 1$
- $g \approx 9.81$ is gravity
- t is time
- This is a second-order ODE. To write as a first-order ODE, set $y_1 = \theta$, $y_2 = \theta' = y_1'$:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -g \sin(y_1) \end{aligned}$$

- Needed initial conditions are $y_1(0)$ and $y_2(0)$

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ODEs

System of n first-order equations in n variables

$$y' = f(t, y), \qquad f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

the system is

Nonlinear: if f is nonlinear in y, linear otherwise

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Autonomous ODE

$$y' = f(y), \qquad y(a) = c$$

is an autonomous ODE, does not depend on time explicitly

$$y' = f(t, y), \qquad y(a) = c$$

is non-autonomous

To convert a non-autonomous ODE to an autonomous set $x = t$ and then

$$\begin{aligned} x' &= 1 \\ y' &= f(x, y), \qquad x(a) = a,\ y(a) = c \end{aligned}$$

Set $z = (z_1, z_2)^T = (x, y)^T$. Then $z' = f(z)$:

$$\begin{aligned} z_1' &= 1 \\ z_2' &= f(z) \end{aligned}$$

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High-order ODEs

$$y^{(n)} = f\left(t, y, y', \dots, y^{(n-1)}\right)$$

can be converted to first-order by setting

$$\begin{aligned}
y_1 &= y \\
y_2 &= y' = y_1' \\
y_3 &= y'' = y_2' \\
&\vdots \\
y_n &= y^{(n-1)} = y_{n-1}'
\end{aligned}$$

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Then

$$
\begin{aligned}
y_1' &= y_2 \\
y_2' &= y_3 \\
&\vdots \\
y_n' &= f(t, y_1, y_2, \ldots, y_n)
\end{aligned}
$$

To solve, we need initial values for

$$y_1(a), y_2(a), \ldots, y_n(a)$$

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Euler's method

- Let $h = (b-a)/N$, $N > 1$ is an integer
- h is stepsize
- Let $t_0 = a$, $t_i = a + ih$, $i = 0, 1, \ldots, N$
- From $y'(t_i) = f(t_i, y(t_i))$, we write

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi_i), \quad \xi_i \text{ between } t_i \text{ and} t_{i+1}$$

= y(t_i + h)

$$= y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i)$$

apply Taylor Series

$$\approx y(t_i) + hf(t_i, y(t_i))$$

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- Euler's method:

$$y_0 = c$$
$$y_{i+1} = y_i + hf(t_i, y_i), \qquad i = 0, 1, \ldots, N-1$$

- Example: Euler's method on $y' = -y + t$, $y(0) = y_0 = 5$, with $h = 0.1$:

t_i = a + h*i
y_0 = c

y_i = y(t_i)
is approx. for

$$y_{i+1} = y_i + hf(t_i, y_i) = y_i + h(-y_i + t_i)$$
$$y_1 = y_0 + h(-y_0 + t_0) = 5 + 0.1(-5 + 0) = 4.5$$
$$y_2 = y_1 + h(-y_1 + t_1) = 4.5 + 0.1(-4.5 + 0.1) = 4.06$$
$$y_3 = y_2 + h(-y_2 + t_2) = 4.06 + 0.1(-4.06 + 0.2) = 3.674$$

- Exact solution is $y(t) = t - 1 + 6e^{-t}$
- The corresponding exact values are $y(0.1) \approx 4.5290$, $y(0.2) \approx 4.1124$, $y(0.3) \approx 3.7449$

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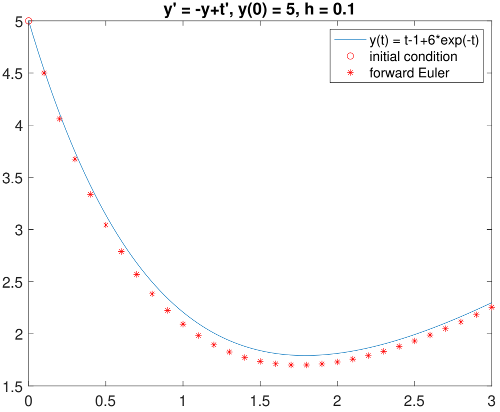
Example: Forward Euler on $y' = -y + t$



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Backward Euler

- We can write

= y(t_(i+1) - h)

$$
\begin{aligned}
y(t_i) &= y(t_{i+1}) - hy'(t_{i+1}) + \frac{h^2}{2}y''(\eta_i) \\
&\approx y(t_{i+1}) - hf(t_{i+1}, y(t_{i+1}))
\end{aligned}
$$

apply Taylor Series $y(t_{i+1}) \approx y(t_i) + hf(t_{i+1}, y(t_{i+1}))$

- Backward Euler

y_0 = c

$$y_{i+1} = y_i + hf(t_{i+1}, y_{i+1})$$

- This is an implicit method; forward Euler is explicit

solve for y_(i + 1)

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- Example: Backward Euler method on $y' = -y + t$, $y(0) = y_0 = 5$, with $h = 0.1$:

$$\begin{aligned} y_{i+1} &= y_i + hf(t_{i+1}, y_{i+1}) \\ &= y_i + h(-y_{i+1} + t_{i+1}) \end{aligned}$$

- We need to solve for y_{i+1}:

$$\begin{aligned} y_{i+1} &= y_i - hy_{i+1} + ht_{i+1} \\ y_{i+1} + hy_{i+1} &= y_i + ht_{i+1} \\ y_{i+1} &= \frac{y_i + ht_{i+1}}{1 + h} \end{aligned}$$

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- We compute

$$
\begin{aligned}
y_1 &= \frac{y_0 + ht_1}{1+h} = \frac{5 + 0.1 \cdot 0.1}{1 + 0.1} \approx 4.5545 \\
y_2 &= \frac{y_1 + ht_2}{1+h} \approx \frac{4.5545 + 0.1 \cdot 0.2}{1 + 0.1} \approx 4.1586 \\
y_3 &= \frac{y_2 + ht_3}{1+h} \approx \frac{4.1586 + 0.1 \cdot 0.3}{1 + 0.1} \approx 3.7987
\end{aligned}
$$

- The corresponding exact values are $y(0.1) \approx 4.5290$, $y(0.2) \approx 4.1124$, $y(0.3) \approx 3.7449$
- Here it was easy to solve for y_{i+1}: $f(t, y) = -y + t$ is linear in y
- In general, it is non-linear: apply Newton's method

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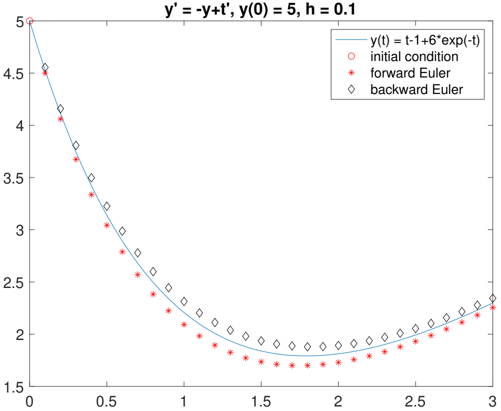
Example: FE and BE on $y' = -y + t$



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Stability

Forward Euler

- Consider $y' = \lambda y$, $y(0) = y_0$
- The exact solution is $y(t) = e^{\lambda t} y_0$
- Forward Euler with constant stepsize h is

$$
\begin{aligned}
y_{i+1} &= y_i + hf(t_i, y_i) = y_i + h\lambda y_i \\
&= (1 + h\lambda) y_i \\
&= (1 + h\lambda)^2 y_{i-1} \\
&\;\;\vdots \\
&= (1 + h\lambda)^{i+1} y_0
\end{aligned}
$$

- If $\lambda < 0$, $y(t)$ is decaying. Since $|y(t_{i+1})| < |y(t_i)|$, we want $|y_{i+1}| \leq |y_i|$

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Stability cont.

- For the method to be numerically stable, we require

$$|y_{i+1}| = |1 + h\lambda| \cdot |y_i| \le |y_i|$$

- That is $|1 + h\lambda| \le 1$, or

$$-1 \le 1 + h\lambda \le 1$$

$$-2 \le h\lambda \le 0$$

forward Euler Stability condition

$$h \le \frac{2}{|\lambda|}$$

- If $|\lambda|$ is large, we can have a severe restriction on the stepsize
  If e.g. $y' = \underbrace{-10^6}_{\lambda} y$, $h \le 2 \cdot 10^{-6}$

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Stability cont.

Example 1.

- Consider $y' = -10y$, $y(0) = y_0$
- Euler's method is

$$y_{i+1} = y_i + h\lambda y_i = (1 - 10h)y_i$$

- For stability $h \le 0.2$
- If e.g. $h = 0.21$ then

$$\begin{aligned} y_1 &= (1 - 10 \cdot 0.21)y_0 = -1.1y_0 \\ y_2 &= -1.1y_1 = 1.21y_0 \\ y_3 &= -1.1y_2 = -1.331y_0 \\ &\vdots \\ y_i &= (-1.1)^i y_0 \end{aligned}$$

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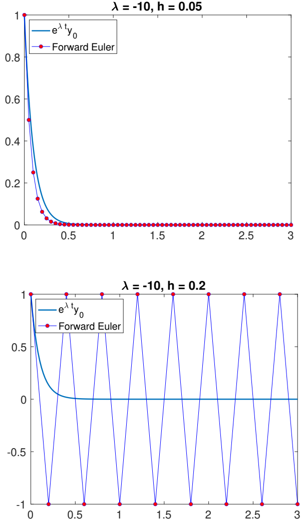
Stability

Example 1. cont.



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Stability

Backward Euler

- Consider the backward Euler on $y' = \lambda y$, where $\lambda < 0$

$$\begin{aligned}
y_{i+1} &= y_i + h\lambda y_{i+1} \\
y_{i+1} &= \frac{1}{1-h\lambda} y_i \\
|y_{i+1}| &= \frac{1}{|1-h\lambda|}|y_i| \\
&\le |y_i| \qquad \text{for any } h > 0
\end{aligned}$$

backward Euler
Stability condition

Stability

Example 2.

- $y' = -10y$
- Backward Euler is

$$y_{i+1} = \frac{1}{1 + 10h} y_i$$

- Stable for any $h > 0$
- Backward Euler is absolutely (for any $h > 0$) stable

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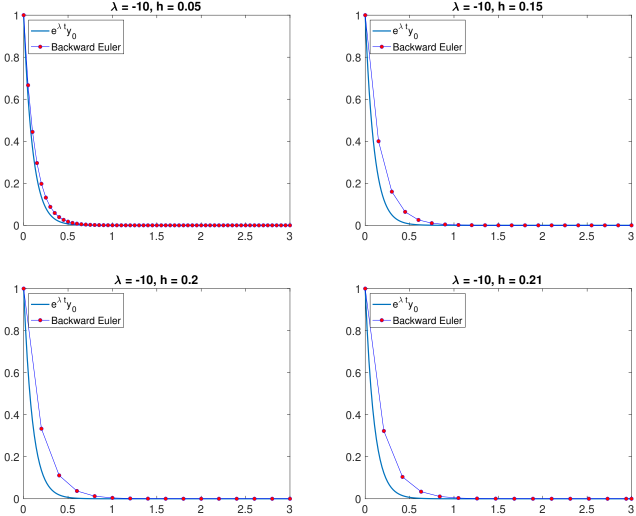
Stability

Example 2. cont.



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