Computer Arithmetic

CS/SE 4X03

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What is the output

```
a(1) =  (1/cos(100*pi+pi/4))^2; % (1/cos(100π + π/4))^2 = 2
a(2) = 3*tan(atan(1e7))/1e7;    % 3 tan(arctan(10^7))/10^7 = 3
x = 4;
for i=1:20 x = sqrt(x); end
for i=1:20 x = x*x; end
a(3) = x;                       % = 4
a(4) = 5*(1+exp(-100)-1)/(1+exp(-100)-1); % 5 (1+e^-100 -1)/(1+e^-100 -1) = 5
a(5) = log(exp(6e3))/1e3;                 % ln(e^6000)/1000 = 6
for i = 1:5
    fprintf('%d: %.16f\n', i+1, a(i));
end
```

Outline

Useful links

- IEEE 754 double precision visualization
- C. Moler. Floating Point Numbers
- IEEE 754
- N. Higham. Half Precision Arithmetic: fp16 Versus bfloat16
- GNU Multiple Precision Arithmetic Library
- The GNU Multiple Precision (GMP) Library, in Maple
- Quadruple-precision floating-point format

Fixed-point vs. floating-point

- Fixed-point: the position of the radix (e.g. decimal, binary) point is fixed
  - Consider unsigned, 6 decimal digit representation with 4 digits for the integer part and 2 digits for the fractional part; e.g. 1234.56, 12.34
  - Largest number 9999.99, smallest 0.01
  - Can be interpreted as integers with implicit scaling factor, here $1/100$
- Floating-point: the radix point can "float" between the digits
  - E.g. with 6 digits we can represent 0.123456, 0.00123456, 12345600000
  - Position of radix point is determined from an exponent
    E.g. $0.123456 = 1.23456 \times 10^{-1} = 1234.56 \times 10^{-4}$
  - To ensure uniqueness of representation, assume first digit nonzero, followed by "." followed by 5 digits

Floating-point number system

A floating-point (FP) system is characterized by four integers (β, t, L, U), where

- β base of the system or radix
- t number of digits or precision
- $[L, U]$ exponent range

A number x in the system is represented as

$$x = \pm \left(d_0 + \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots + \frac{d_{t-1}}{\beta^{t-1}} \right) \times \beta^e$$

where

- $0 \leq d_i \leq \beta - 1$, $i = 0, \ldots, t-1$
- $e \in [L, U]$

Floating-point number system cont.

- The string of base β digits $d_0 d_1 \cdots d_{t-1}$ is called mantissa or significand
- $d_1 d_2 \cdots d_{t-1}$ is called fraction
- A common way of expressing x is

$$\pm\, d_0.d_1 \cdots d_{t-1} \times \beta^e$$

- A FP number is normalized if d_0 is nonzero
  denormalized otherwise

Floating-point number system cont.

Example 1. Consider the FP $(10, 3, -2, 2)$.

- Numbers are of the form

$$d_0.d_1d_2 \times 10^e, \quad d_0 \neq 0,\ e \in [-2, 2]$$

- largest positive number 9.99×10^2
- smallest positive normalized number 1.00×10^{-2}
- smallest positive denormalized number 0.01×10^{-2}
- denormalized numbers are e.g. 0.23×10^{-2}, 0.11×10^{-2}
- 0 is represented as 0.00×10^0

Rounding

How to store a real number

$$x = \pm\, d_0.d_1 \cdots d_{t-1} d_t d_{t+1} \cdots \times \beta^e$$

in t digits?

Denote by $\mathrm{fl}\,(x)$ the FP representation of x

- Rounding by chopping (also called rounding towards zero)
- Rounding to nearest. $\mathrm{fl}\,(x)$ is the nearest FP to x
  If a tie, round to the even FP
- Rounding towards $+\infty$. $\mathrm{fl}\,(x)$ is the smallest FP $\geq x$
- Rounding towards $-\infty$. $\mathrm{fl}\,(x)$ is the largest FP $\leq x$

Rounding cont.

Example 2. Consider the FP $(10, 3, -2, 2)$.
Let $x = 1.2789 \times 10^1$

- chopping: $\mathsf{fl}(x) = 1.27 \times 10^1$
- nearest: $\mathsf{fl}(x) = 1.28 \times 10^1$
- $+\infty$: $\mathsf{fl}(x) = 1.28 \times 10^1$
- $-\infty$: $\mathsf{fl}(x) = 1.27 \times 10^1$

Let $x = 1.275000 \times 10^1$.

- nearest: $\mathsf{fl}(x) = 1.28 \times 10^1$

Machine epsilon

Machine epsilon: the distance from 1 to the next larger FP number
E.g. in $(10, 3, -2, 2)$, $\epsilon_{\text{mach}} = 1.01 - 1 = 0.01 = 10^{-2}$

Another definition: smallest $\epsilon > 0$ such that $\mathrm{fl}\,(1 + \epsilon) > 1$
These two definitions are not equivalent

Unit roundoff: $u = \epsilon_{\text{mach}}/2$
When rounding to the nearest

how do we prove this?

$$\mathrm{fl}\,(x) = x(1 + \epsilon), \quad \text{where } |\epsilon| \leq u$$

i.e.

$$\frac{\mathrm{fl}\,(x) - x}{x} = \epsilon, \quad \left| \frac{\mathrm{fl}\,(x) - x}{x} \right| \leq u$$

IEEE 754

- IEEE 754 standard for FP arithmetic (1985)
- IEEE 754-2008, IEEE 754-2019
- Most common (binary) single and double precision
  since 2008 half precision

	bits	t	L	U	ϵ_{mach}
single	32	24	-126	127	1.2×10^{-7}
double	64	53	-1022	1023	2.2×10^{-16}

	range	smallest	
		normalized	denormalized
single	$\pm 3.4 \times 10^{38}$	$\pm 1.2 \times 10^{-38}$	$\pm 1.4 \times 10^{-45}$
double	$\pm 1.8 \times 10^{308}$	$\pm 2.2 \times 10^{-308}$	$\pm 4.9 \times 10^{-324}$

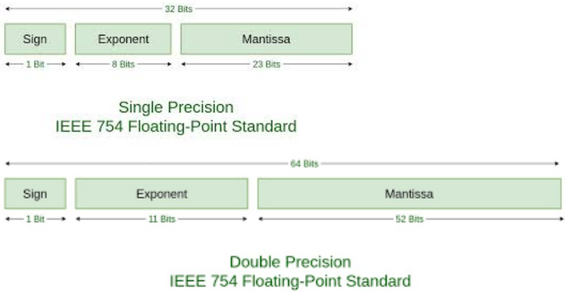
(These are $\approx$ values)

IEEE 754 cont.

Exceptional values

- `Inf`, `-Inf` when the result overfl~~ows~~, e.g. `1/0.0`
- `NaN` "Not a Number" results from undefined operations e.g. `0/0`, `0*Inf`, `Inf/Inf`
  `NaNs` propagate through computations

IEEE 754 cont.



(From `https://www.geeksforgeeks.org/ieee-standard-754-floating-point-numbers/`)

- sign 0 positive, 1 negative
- exponent is biased
- first bit of mantissa is not stored, sticky bit, assumed 1

IEEE 754 cont.

Single precision

- `Inf`
  - sign: 0 for +`Inf`, 1 for −`Inf`
  - biased exponent: all 1's, 255
  - fraction: all 0's
- `NaN`
  - sign: 0 or 1
  - biased exponent: all 1's, 255
  - fraction: at least one 1
- 0
  - sign: 0 for $+0$, 1 for -0
  - biased exponent: all 0's
  - mantissa: all 0's
- FP numbers
  - biased exponent: from 1 to 254, bias: 127
  - actual exponent: $1 - 127 = -126$ to $254 - 127 = 127$

IEEE 754 cont.

Double precision

- bias 1023
- biased exponent: from 1 to 2046
- actual exponent: from -1022 to 1023
- rest similar to single

IEEE 754 cont.

FP arithmetic

For a real x and rounding to nearest

$$\mathsf{fl}\,(x) = x(1+\epsilon), \quad |\epsilon| \le u$$

u is the unit roundoff of the precision

The arithmetic operations are correctly rounded, i.e. for x and y IEEE numbers (e.g. in double or single precision) and rounding to the nearest

$$\mathsf{fl}\,(x \circ y) = (x \circ y)(1+\epsilon), \quad \circ \in \{+,-,*,/\}, \quad |\epsilon| \le u$$

IEEE 754 cont.

Example 3. Consider a decimal floating-point system with $t = 5$ and rounding to nearest

- The machine epsilon is $1.0001 - 1.0000 = 0.0001 = 10^{-4}$
- Unit roundoff is $u = 10^{-4}/2 = 5 \times 10^{-5}$
- Let $x = 1.162611735194631$
  With rounding to nearest, $\mathsf{fl}(x) = 1.1626$

$$\mathsf{fl}(x) = x(1 + \epsilon)$$
$$\epsilon = \frac{\mathsf{fl}(x) - x}{x} = \frac{1.1626 - 1.162611735194631}{1.162611735194631} \approx -1.0094 \times 10^{-5}$$
$$|\epsilon| \approx 1.0094 \times 10^{-5} < 5 \times 10^{-5}$$

IEEE 754 cont.

Example 3. cont.

Let $x = 1.162611735194631 \times 10^8$. Then $\text{fl}(x) = 1.1626 \times 10^8$

$$\epsilon = \frac{\text{fl}(x) - x}{x} = \frac{1.1626 \times 10^8 - 1.162611735194631 \times 10^8}{1.162611735194631 \times 10^8}$$

$$\approx -1.0094 \times 10^{-5}$$

$$|\epsilon| \approx 1.0094 \times 10^{-5} < 5 \times 10^{-5}$$

Let $x = 1.00005000000000001$. Then $|\epsilon| \approx 4.9998 \times 10^{-5} < 5 \times 10^{-5}$

IEEE 754 cont.

Example 4. Assume x, y, z machine numbers. Find the error in $\text{fl}\,(z(x+y))$

$$\begin{aligned}
\text{fl}\,(z(x+y)) &= \text{fl}\,(z)\,\text{fl}\,(x+y)\,(1+\delta_1) \\
&= z(\text{fl}\,(x) + \text{fl}\,(y))(1+\delta_1)(1+\delta_2) \\
&= z(x+y)(1+\delta_1)(1+\delta_2) \\
&= z(x+y)(1+\delta_1+\delta_2+\delta_1\delta_2) \\
&= z(x+y)(1+\delta_1+\delta_2),
\end{aligned}$$

where $|\delta_{1,2}| \le u$. $|\delta_1\delta_2| \ll u$ is very small, so we neglect it

Denoting $\delta = \delta_1 + \delta_2$

$$\text{fl}\,(z(x+y)) = z(x+y)(1+\delta), \quad \text{where} |\delta| \le 2u$$

IEEE 754 cont.

Example 5. Assume x, y real. What is the error in $\mathrm{fl}\,(xy)$?

$$\begin{aligned}
\mathrm{fl}\,(xy) &= \mathrm{fl}\,(x)\,\mathrm{fl}\,(y)\,(1+\delta_1) \\
&= x(1+\delta_2)y(1+\delta_3)(1+\delta_1) \\
&= xy(1+\delta_1+\delta_2+\delta_3+\underbrace{\delta_1\delta_2+\delta_1\delta_3+\delta_2\delta_3+\delta_1\delta_2\delta_3}_{\text{very small}}) \\
&\approx xy(1+\delta_1+\delta_2+\delta_3),
\end{aligned}$$

where $\mathrm{fl}\,(x) = x(1+\delta_2)$, $\mathrm{fl}\,(y) = x(1+\delta_3)$, $|\delta_{1,2,3}| \le u$

Denoting $\delta = \delta_1 + \delta_2 + \delta_3$,

$$|\delta| \le |\delta_1| + |\delta_2| + |\delta_3| \le 3u$$

and

$$\mathrm{fl}\,(xy) = xy(1+\delta), \quad \text{where } |\delta| \le 3u$$

Example 6 (Computing $\sqrt{x^2 + y^2}$).

- One can do `sqrt(x*x+y*y)`
- Assume double precision and suppose `x = 1e200` and `y= 1e100`
- `x*x` will overflow and the result is `Inf`
- `sqrt(Inf+1e200)` gives `Inf`
- Let $M = \max\{|x|, |y|\}$ and assume $M = |x|$. Then

$$\sqrt{x^2 + y^2} = M\sqrt{1 + (y/M)^2}$$

- Setting `M=1e200`, `y1=y/M`, compute `M*sqrt(1+y1*y1)`, which gives `1e200`

IEEE 754 cont.

Note

expression	produces
`y1*y1`	`1e-100`
`1+y1*y1`	`1`
`sqrt(1+y1*y1)`	`1`

Cancellations

Example 7. Assume a decimal FP system with $t = 5$ digits and rounding to nearest. Let $x = 1.234567$ and $y = 1.234512$

- Then

$$\begin{aligned} \mathrm{fl}(x) &= \mathrm{fl}(1.234567) = 1.2346 && \text{roundoff error} \\ \mathrm{fl}(y) &= \mathrm{fl}(1.234512) = 1.2345 && \text{roundoff error} \\ \mathrm{fl}(x) - \mathrm{fl}(y) &= 0.0001 = 1.0000 \times 10^{-4} && \text{NO roundoff error} \end{aligned}$$

- $\mathrm{fl}(x) - \mathrm{fl}(y) = 1.0000 \times 10^{-4}$ has no correct diggits: catastrophic cancellation
- $x - y = 5.5000 \times 10^{-5}$
- error in $\mathrm{fl}(x) - \mathrm{fl}(y)$ is $\frac{[\mathrm{fl}(x) - \mathrm{fl}(y)] - (x-y)}{x-y} \approx 8.1818 \times 10^{-1}$
- Repeat the same computation with $x = 4.894080$ and $y = 5.384576$. Is the error $\leq$ unit roundoff?

Cancellations cont.

Both of the FP systems has 8 decimal digits, while relative errors are 7.3*10^-5 and 10^-7, so digits lost are 8-5=3 and 8-7=1

Example 8.

- Consider FP arithmetic with 8 decimal digits
- Let $x = 1$, $h = 10^{-4}$
- Compute $\mathsf{fl}\big[\sin(x+h) - \sin(x)\big]$

$$
\begin{aligned}
\mathsf{fl}\big[\sin(x+h)\big] &= 8.4152501 \times 10^{-1} \\
\mathsf{fl}\big[\sin(x)\big] &= 8.4147098 \times 10^{-1} \\
\mathsf{fl}\big[\sin(x+h)\big] - \mathsf{fl}\big[\sin(x)\big] &= 0.0005403 \times 10^{-1} \\
&= 5.4030000 \times 10^{-5} \\
\sin(x+h) - \sin(x) &= 5.402602314186211 \times 10^{-5}
\end{aligned}
$$

- We lose 3 digits: cancellations. Relative error $\approx 7.3 \times 10^{-5}$
- Note there are no roundoff errors in the subtraction

Cancellations cont.

Example 9.

- Consider FP arithmetic with 8 decimal digits
- Let $x = 1$, $h = 10^{-1}$
- Compute $\sin(x+h) - \sin(x)$

$$\begin{aligned}
\mathsf{fl}\big[\sin(x+h)\big] &= 8.9120736 \times 10^{-1} \\
\mathsf{fl}\big[\sin(x)\big] &= 8.4147098 \times 10^{-1} \\
\mathsf{fl}\big[\sin(x+h)\big] - \mathsf{fl}\big[\sin(x)\big] &= 0.4973638 \times 10^{-1} \\
&= 4.9736380 \times 10^{-2} \\
\sin(x+h) - \sin(x) &= 4.9736375 \times 10^{-2}
\end{aligned}$$

- We lose 1 digit. Relative error $\approx 9.5 \times 10^{-8} \approx 10^{-7}$
- Note there are no roundoff errors in the subtraction

Cancellations cont.

Example 10. Consider the equivalent expressions $x^2 - y^2$ and $(x-y)(x+y)$. Suppose $|x| \approx |y|$. Which one is better to evaluate? Assume $x, y > 0$; the case $x, y < 0$ is similar

- $x - y$ may have cancellations; $x + y$ does not
- x^2 and y^2 would have (in general) roundoff errors from the multiplications
- due to them, the cancellations in $x^2 - y^2$ can be worse than in $(x-y)(x+y)$

Try

```
x = 10000 * rand; y = x * (1 + 1e-14);
eval1 = (x - y) * (x + y); eval2 = x * x - y * y;
%compute more accurate result using
xv = vpa(x); yv = vpa(y); acc = (xv - yv) * (xv + yv);
fprintf('error in (x-y)*(x+y) = % .2e\n', double(acc - eval1));
fprintf('error in x*x - y*y    = % .2e\n', double(acc - eval2));
```