Problem 1 [3 points] Without pivoting. Multiply the first row by $1/3 = 3.3333 \times 10^{-1}$ and subtract from second and multiply by 2 and subtract from third:

$$\begin{bmatrix} 3 & 4 & 3 \\ & 3.6667 & -2.0000 \\ & -5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 3.6667 \\ -5 \end{bmatrix}$$

Multiply second row by $-5/3.6667 = -1.3636$ and subtract from third:

$$\begin{bmatrix} 3 & 4 & 3 \\ & 3.6667 & -2.0000 \\ & & -1.7272 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 3.6667 \\ -8.7880 \times 10^{-5} \end{bmatrix}$$

Then

$$\begin{aligned} x_3 &= -8.7880 \times 10^{-5}/(-1.7272) \\ &= 5.0880 \times 10^{-5} \\ x_2 &= (b_2 - a_{23}x_3)/a_{22} \\ &= (3.6667 + (-2.0000) \times 5.0880 \times 10^{-5})/3.6667 \\ &= (3.6667 - 1.0176 \times 10^{-4})/3.6667 \\ &= 3.6666/3.6667 \\ &= 9.9997 \times 10^{-1} \\ x_1 &= (b_1 - a_{12}x_2 - a_{13}x_3)/a_{11} \\ &= (10 - 4 \times 9.9997 \times 10^{-1} - 3 \times 5.0880 \times 10^{-5})/3 \\ &= (10 - 3.9999 - 1.5264 \times 10^{-4})/3 \\ &= (6.0001 - 1.5264 \times 10^{-4})/3 \\ &= 5.9999/3 \\ &= 2.0000. \end{aligned}$$

With partial pivoting. Multiply third row by 0.5 and subtract from first, and multiply by $1/6 = 1.6667 \times 10^{-1}$ and subtract from second:

$$\begin{bmatrix} & 2.5 & -5.0000 \times 10^{-1} \\ & 4.5 & -2.1667 \\ 6 & 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 4.5 \\ 15 \end{bmatrix} \tag{1}$$

Multiply second row by $2.5/4.5 = 5.5556 \times 10^{-1}$ and subtract from first:

$$\begin{bmatrix} & & 7.0373 \times 10^{-1} \\ & 4.5 & -2.1667 \\ 6 & 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2.0000 \times 10^{-5} \\ 4.5 \\ 15 \end{bmatrix} \tag{2}$$

Then

$$
\begin{aligned}
x_3 &= -2.0000 \times 10^{-5}/7.0373 \times 10^{-1} \\
&= -2.8420 \times 10^{-5} \\
x_2 &= (4.5 - 4.5 \times -2.8420 \times 10^{-5})/4.5 \\
&= (4.5 + 1.2789 \times 10^{-4})/4.5 \\
&= 4.5001/4.5 \\
&= 1 \\
x_1 &= (15 - 3 \times 1 - 7 \times -2.8420 \times 10^{-5})/6 \\
&= (15 - 3 + 1.9894 \times 10^{-4})/6 \\
&= (12 + 1.9894 \times 10^{-4})/6 \\
&= 1.2000 \times 10^{1}/6 \\
&= 2
\end{aligned}
$$

Problem 2 [3 points] The inverse of A is

$$A^{-1} = \frac{1}{\epsilon^2}\begin{bmatrix} 1 & -1-\epsilon \\ -1+\epsilon & 1 \end{bmatrix}.$$

In the infinity norm, for small ϵ,

$$\text{cond}(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty = (2+\epsilon)\frac{(2+\epsilon)}{\epsilon^2} = \frac{(2+\epsilon)^2}{\epsilon^2} \approx \frac{4}{\epsilon^2}.$$

Hence for $\epsilon \approx \sqrt{\epsilon_{\text{mach}}}$,

$$\text{cond}(A) \approx \frac{4}{\epsilon_{\text{mach}}}.$$

In double precision the machine epsilon is $\approx 2.2204 \times 10^{-16}$ and and hence

$$\text{cond}(A) \approx 1.8014 \times 10^{16}.$$

The script eps_system.m produces

```
 eps            |x1-1|    |x2-eps|/eps    cond(A)
0.9*sqrt(eps)  1.00e+00   7.46e+07     5.81e+16
1.0*sqrt(eps)  0.00e+00   0.00e+00     1.80e+16
1.5*sqrt(eps)  1.11e-01   4.97e+06     8.01e+15
2.0*sqrt(eps)  0.00e+00   0.00e+00     4.50e+15
```

If the condition number is $\approx 10^k$, we expect to lose about k digits of accuracy. However, how to explain the second and fourth lines? With partial pivoting, we have

$$\begin{bmatrix} 1 & 1+\epsilon \\ 0 & 1-(1+\epsilon)(1-\epsilon) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1+(1+\epsilon)\epsilon \\ 1-\big(1+(1+\epsilon)\epsilon\big)(1-\epsilon) \end{bmatrix}.$$

Consider the (2,2) entry of the matrix:

$$1 - (1+\epsilon)(1-\epsilon) = \epsilon^2.$$

When ϵ is small, $(1+\epsilon)(1-\epsilon) \approx 1$. If in the evaluation of $(1+\epsilon)(1-\epsilon)$ there are roundoff errors, after the cancellations the result can be very inaccurate. However, for $\epsilon = \sqrt{\epsilon_{\text{mach}}}$ and $2\sqrt{\epsilon_{\text{mach}}}$, $1 - (1+\epsilon)(1-\epsilon)$ is the same as ϵ^2, so no errors. Try computing for various values the error

$$\frac{1 - (1+\epsilon)(1-\epsilon)}{\epsilon^2}.$$

Problem 3 [12 points] The relative error in the solution is

$$\frac{\|x - \widetilde{x}\|}{\|\widetilde{x}\|} \le \text{cond}(A)\frac{\|r\|}{\|b\|},$$

where $r = b - A\widetilde{x}$. Gauss elimination with partial pivoting produces relative residual which is about the machine epsilon, in double precision of the order of 10^{-16}. For $\text{cond}(A) \approx 10^k$, the error is at most 10^{k-16}, so we expect about $k - 16$ accurate digits.

With no pivoting, the residual can be larger, so the larger errors in the third column.

```
Random matrices of size 2000
              n A\b     no pivot. pivoting cond(A)
              -----------------------------------
              1 1.4e-13 1.9e-12 1.2e-13 4.4e+03
              2 8.9e-14 2.1e-12 9.5e-14 3.9e+03
              3 2.0e-13 1.6e-12 1.5e-13 4.0e+03
              4 1.4e-12 9.8e-10 3.3e-13 1.8e+05
              5 4.9e-13 9.6e-12 5.3e-13 2.2e+04
```

Problem 4 [4 points] To have degree $n = 5$, we need $(n+1)$ points. Let $h = 1/5 = 0.2$. Since e^x is bounded by e on $[0,1]$, the error is

$$\frac{M}{4(n+1)}h^{n+1} \le \frac{e}{4(5+1)}0.2^{5+1} \approx 7.2488 \times 10^{-6}.$$

Let $h = 1/n$. Then we want

$$\frac{M}{4(n+1)}h^{n+1} \le \frac{e}{4(n+1)}\left(1/n\right)^{n+1} \le 10^{-8}.$$

By trial and error, one can easily find that $n \ge 8$, so we need at least 9 equally spaced points.

Problem 5 [5 points]

a. The script interp_sqrt.m produces

```
sqrt(0.05) ~ 1.024700: error 4.923e-06
sqrt(0.15) ~ 1.072350: error 3.053e-05
```

b. We have a polynomial of degree $n = 3$ and spacing $h = 0.1$. The error bound is

$$|\sqrt{x+1} - p_3(x)| \leq \frac{M}{4(n+1)} h^{n+1}, \tag{3}$$

where

$$|f^{(4)}(x)| \leq M \quad \text{for all } x \in [a, b].$$

The 4th derivative of $f(x) = \sqrt{x+1}$ is

$$\sqrt{x+1}^{(4)} = \frac{-15}{16(x+1)^{7/2}}$$

On $[0, 0.3]$

$$|\sqrt{x+1}^{(4)}| = \frac{15}{16(x+1)^{7/2}} \leq \frac{15}{16(0+1)^{7/2}} = 0.9375 = M.$$

The error bound is

$$|\sqrt{x+1} - p_3(x)| \leq \frac{M}{4(n+1)} h^{n+1} = \frac{0.9375}{4 \times 4} 0.1^4 \approx 5.8594 \times 10^{-6}.$$

c. The first error is within the error bound, but not the second. The reason is the y values are approximations of $\sqrt{x+1}$ and accurate up to 4 digits. Replace in this script `y` by `y = sqrt(x+1)` and compare the errors.

Problem 6 [4 points] Run main_interp_absx.m.

Problem 7 [3 points] Run main_interp_sinx.m.

In the bound for the error, we have the $(n+1)$st derivative of the function $f(x)$ being interpolated, and this derivative should be continuous for the error bound to hold. When $f(x) = \sin(x)$, $|f^{(n+1)}(x)| \leq 1$ for any x, so the small errors. When $f(x) = |x|$, the first derivative if discontinuous.

Problem 8 [4 points] Run main_interp.m