LU Decomposition

CS/SE 4X03

Ned Nedialkov

McMaster University

September 30, 2021

Outline

LU decomposition

- Decompose A as $A = LU$, where
  - L is unit lower-triangular
    1's on the main diagonal, 0's above it
  - U is upper-triangular
    0's below the main diagonal
- Consider solving $Ax = b$. From

$$Ax = LUx = b$$
$$L\underbrace{(Ux)}_{y} = b$$

we can solve first $Ly = b$ for y and then $Ux = y$ for x

LU decomposition cont.

A is $n \times n$

- Gauss elimination takes $O(n^3)$ arithmetic operations
- LU decomposition takes $O(n^3)$ arithmetic operations
- Solving each of $Ly = b$ and $Ux = y$ takes $O(n^2)$ arithmetic operations
- Suppose we need to solve m systems $Ax = b^{(i)}$, $i = 1, \dots, m$
  A is the same, the right-hand side changes
- If we solve them with GE $O(mn^3)$
- Do LU decomposition first $O(n^3)$
- Solve $Ly = b^{(i)}$, $Ux = y$, for $i = 1 : m$ $O(mn^2)$
  Total LU+triangular solves $O(n^3 + mn^2)$

Example of LU decomposition

$$A = \begin{array}{c c} \begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} & \begin{array}{cc} \times 1 & \times 3 \\ \downarrow & \\ & \downarrow \end{array} \end{array}$$

- multipliers $l_{2,1} = 1$, $l_{3,1} = 3$

$$M_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -3 \\ 0 & 1 & -8 \end{bmatrix} = A^{(1)}$$

- multiplier $l_{3,2} = \frac{1}{2}$

$$M_2A^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -3 \\ 0 & 1 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & -6.5 \end{bmatrix} = A^{(2)} = U$$

We have

$$M_2A^{(1)} = (M_2M_1)A = U$$
$$A = \underbrace{\left(M_1^{-1}M_2^{-1}\right)}_{L} U$$

To compute M_1^{-1}, M_2^{-1} flip the signs of nonzero entries below the main diagonal

Then

$$L = M_1^{-1} M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & \frac{1}{2} & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & \frac{1}{2} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & -6.5 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}}_{A}$$

General derivation

- Let $l_{i,1} = a_{i,1}/a_{1,1}$ for $l = 2, \dots n$

$$
\underbrace{\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
-l_{2,1} & 1 & 0 & \cdots & 0 \\
-l_{3,1} & 0 & 1 & \cdots & 0 \\
 & \vdots & & & \\
-l_{n,1} & 0 & & \cdots & 1
\end{bmatrix}}_{M_1}
\underbrace{\begin{bmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\
a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\
a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\
 & \vdots & & & \\
a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n}
\end{bmatrix}}_{A}
$$

$$
= \underbrace{\begin{bmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\
0 & a_{2,2}^{(1)} & a_{2,3}^{(1)} & \cdots & a_{2,n}^{(1)} \\
0 & a_{3,2}^{(1)} & a_{3,3}^{(1)} & \cdots & a_{3,n}^{(1)} \\
 & \vdots & & & \\
0 & a_{n,2}^{(1)} & a_{n,3}^{(1)} & \cdots & a_{n,n}^{(1)}
\end{bmatrix}}_{A^{(1)}}
$$

- Let $l_{i,2} = a_{i,2}^{(1)}/a_{2,2}^{(1)}$ for $l = 3, \dots n$

$$
\underbrace{\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & -l_{3,2} & 1 & \cdots & 0 \\
 & \vdots & & & \\
0 & -l_{n,2} & & \cdots & 1
\end{bmatrix}}_{M_2}
\underbrace{\begin{bmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\
0 & a_{2,2}^{(1)} & a_{2,3}^{(1)} & \cdots & a_{2,n}^{(1)} \\
0 & a_{3,2}^{(1)} & a_{3,3}^{(1)} & \cdots & a_{3,n}^{(1)} \\
 & \vdots & & & \\
0 & a_{n,2}^{(1)} & a_{n,3}^{(1)} & \cdots & a_{n,n}^{(1)}
\end{bmatrix}}_{A^{(1)}}
$$

$$
= \underbrace{\begin{bmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\
0 & a_{2,2}^{(1)} & a_{2,3}^{(1)} & \cdots & a_{2,n}^{(1)} \\
0 & a_{3,2}^{(2)} & a_{3,3}^{(2)} & \cdots & a_{3,n}^{(2)} \\
 & \vdots & & & \\
0 & a_{n,2}^{(2)} & a_{n,3}^{(2)} & \cdots & a_{n,n}^{(2)}
\end{bmatrix}}_{A^{(2)}}
$$

- $M_2 A^{(1)} = M_2 M_1 A = A^{(2)}$
- Continuing in this way

$$M_{n-1} \cdots M_2 M_1 A = A^{(n-1)} = U$$

  which is upper triangular
- Multiplying by M_i^{-1} both sides

$$A = \underbrace{\left(M_1^{-1} M_2^{-1} \cdots M_{n-1}^{-1}\right)}_{L} U$$

- M_i is of the form

$$M_i = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & -l_{i+1,i} & 1 & & \\ & & -l_{i+2,i} & & \ddots & \\ & & \vdots & & & \ddots \\ & & -l_{n,i} & & & & 1 \end{bmatrix}$$

$$M_i^{-1} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & l_{i+1,i} & 1 & & \\ & & l_{i+2,i} & & \ddots & \\ & & \vdots & & & \ddots \\ & & l_{n,i} & & & & 1 \end{bmatrix}$$