

# Numerical Integration

## Composite Rules

CS/SE 4X03

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# Outline

Composite trapezoidal rule

Error of composite trapezoidal rule

Composite Simpson & midpoint rules

## How to increase the accuracy of a rule

- We can increase the degree of the polynomial, but the error might be large
- Apply a basic rule over small subintervals
  - subdivide  $[a, b]$  into  $r$  subintervals
  - $h = \frac{b-a}{r}$  length of each subinterval
  - $t_i \equiv a + ih, i = 0, 1, \dots, r$  not the same  $i$  with below  
 $t_0 = a, t_r = b$

$$\int_a^b f(x)dx = \sum_{i=1}^r \int_{t_{i-1}}^{t_i} f(x)dx$$

## Composite trapezoidal rule

From the basic rule on  $[t_{i-1}, t_i]$ ,  $i = 1, \dots, r$

$$\int_{t_{i-1}}^{t_i} f(x) dx \approx \frac{t_i - t_{i-1}}{2} [f(t_{i-1}) + f(t_i)] = \frac{h}{2} [f(t_{i-1}) + f(t_i)]$$

we derive

$$\begin{aligned} \int_a^b f(x) dx &= \sum_{i=1}^r \int_{t_{i-1}}^{t_i} f(x) dx \approx \frac{h}{2} \sum_{i=1}^r [f(t_{i-1}) + f(t_i)] \\ &= \frac{h}{2} \left( \sum_{i=1}^r f(t_{i-1}) + \sum_{i=1}^r f(t_i) \right) \\ &= \frac{h}{2} (f(t_0) + f(t_1) + \dots + f(t_{r-1})) \\ &\quad + \frac{h}{2} (f(t_1) + \dots + f(t_{r-1}) + f(t_r)) \\ &= \frac{h}{2} [f(a) + f(b)] + h \sum_{i=1}^{r-1} f(t_i) \end{aligned}$$

## Error of composite trapezoidal rule

From

$$\int_{t_{i-1}}^{t_i} f(x)dx = \frac{h}{2} [f(t_{i-1}) + f(t_i)] - \frac{f''(\eta_i)}{12} h^3$$

we have

$$\int_a^b f(x)dx = \underbrace{\sum_{i=1}^r \frac{h}{2} [f(t_{i-1}) + f(t_i)]}_{\text{composite}} - \underbrace{\sum_{i=1}^r \frac{f''(\eta_i)}{12} h^3}_{\text{error}}$$

Assuming  $f''(x)$  continuous on  $[a, b]$ ,

$$\min_{x \in [a, b]} f''(x) \leq f''(\eta_i) \leq \max_{x \in [a, b]} f''(x)$$

Then

$$\min_{x \in [a, b]} f''(x) \leq \frac{1}{r} \sum_{i=1}^r f''(\eta_i) \leq \max_{x \in [a, b]} f''(x)$$

## Error of composite trapezoidal rule cont.

From the Intermediate Value Theorem, there exists  $\mu$ , such that

$$f''(\mu) = \frac{1}{r} \sum_{i=1}^r f''(\eta_i)$$

Then the error is

$$\begin{aligned}
 -\sum_{i=1}^r \frac{f''(\eta_i)}{12} h^3 &= -\frac{1}{12} \left[ \frac{1}{r} \sum_{i=1}^r f''(\eta_i) \right] r \cdot h \cdot h^2 \\
 &= -\frac{f''(\mu)}{12} (b-a) h^2,
 \end{aligned}$$

$$h = (b-a)/r, \text{ and } r \cdot h = b-a$$

# Composite Simpson & midpoint rules

Simpson:

$$\int_a^b f(x) dx \approx \frac{h}{3} \left[ f(a) + 2 \sum_{i=1}^{r/2-1} f(t_{2i}) + 4 \sum_{i=1}^{r/2} f(t_{2i-1}) + f(b) \right]$$

Error

$$\left| -\frac{f^{(4)}(\zeta)}{180} (b-a) h^4 \right|$$

Midpoint:

$$\int_a^b f(x) dx \approx h \sum_{i=1}^r f(a + (i-1/2)h)$$

Error

$$\left| \frac{f''(\xi)}{24} (b-a) h^2 \right|$$