Problem 1 [5 points] Suppose you enter two numbers x and y from the keyboard on your computer, store them in double precision variables, and compute `x*y*y`. Assuming that this expression is evaluated in double precision, calculate a bound for the error in the computed result.

Problem 2 [4 points] For this problem, do not use a calculator or a computer.
Consider $f(x) = (e^{2x} - 1)/(2x)$. Let $x = 1\text{e}{-10}$ and assume double precision.

(a) When evaluated in double precision, **exp**`(2*x)` is `1.000000000200000`. Without using the **exp** function, how would you obtain this value?

(b) Describe an approach for computing $f(x) = (e^{2x} - 1)/(2x)$ such that loss of significance is avoided when x is near zero.

(c) Using your approach, what would you obtain with $x = 1\text{e}{-10}$?

Problem 3 [4 points] Suppose $\cos x$ is approximated by an interpolating polynomial of degree n using $(n+1)$ equally spaced points in the interval $[0,1]$.

(a) How accurate is this approximation in terms of n.

(b) What is the minimum number of points needed to achieve error less than 10^{-6}.

Problem 4 [3 points] Given an $a > 0$, you wish to compute $a^{1/3}$, that is, the cubic root of a. You have available only the operations addition, subtraction, multiplication and division.

(a) (2 points) Describe how you can compute it.

(b) (1 points) Then compute $3^{1/3}$ up to 4 accurate digits after the decimal point. Show all the steps in your calculation.

Problem 5 [5 points]
Suppose that r is a double root of $f(x)$, $f \in \mathbb{R} \to \mathbb{R}$. That is $f(r) = f'(r) = 0$ and $f''(r) \neq 0$. For example $f(x) = (x-2)^2$ has a double root $x = 2$.
Suppose f, f', f'' are continuous in a neighborhood of r.
Assume that you apply Newton's method to find this root of f. Denote $e_n = r - x_n$ and assume x_n is near r. Show that

$$e_{n+1} \approx \frac{1}{2} e_n$$

Problem 6 [3 points] You are given the data points

x_i	1	2	3
y_i	2	3	5

Suppose we want to find the coefficients a and b in the function $f(x) = ax + be^x$ that fits these data in a least squares sense.
Describe how you would setup a least squares problem in Matlab and how you can compute these coefficients. You don't have to compute them.

Problem 7 [4 points]

(a) (2 points) Let A be nonsingular, $n \times n$ lower-triangular matrix. Write an algorithm in pseudo-code for solving the system $Ax = b$, where b is an n column vector. For example, the following is a lower-triangular matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

(b) (2 points) Derive a formula for the number of arithmetic operations to solve this system.

Problem 8 [5 points]
Consider the ODE $y' = -5y$ with $y(0) = 1$. Suppose you solve this ODE with constant stepsize $h = 0.5$. Provide sufficient detail when answering the following questions.

(a) Is the solution to this ODE stable?

(b) Is the forward Euler method stable for this ODE using this stepsize?

(c) Is the backward Euler method stable for this ODE using this stepsize?

(d) Compute the numerical value for the approximate solution at $t = 0.5$ by the forward Euler method.

(e) Compute the numerical value for the approximate solution at $t = 0.5$ by the backward Euler method.

Problem 9 [3 points]
What is the smallest number of points that are needed to compute $\int_0^1 e^x\,dx$ with accuracy 10^{-8} using Simpson's composite rule with equally spaced points.