

Errors in Polynomial Interpolation

CS/SE 4X03

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Outline

Polynomial interpolation error

Chebyshev nodes

Polynomial interpolation error



Assume

- Polynomial p_n of degree $\leq n$ interpolates f at $n + 1$ distinct points x_0, x_1, \dots, x_n , where $x_i \in [a, b]$
- $f^{(n+1)}$ is continuous on $[a, b]$

Then, for each $x \in [a, b]$, there is a $\xi = \xi(x) \in (a, b)$ such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n (x - x_i)$$

Polynomial interpolation error cont.

- Let $M = \max_{a \leq t \leq b} |f^{(n+1)}(t)|$

Then

$$|f(x) - p_n(x)| \leq \frac{M}{(n+1)!} \prod_{i=0}^n |x - x_i|$$

- Let $h = (b - a)/n$ and let $x_i = a + ih$ for $i = 0, 1, \dots, n$. It can be shown that

$$\prod_{i=0}^n |x - x_i| \leq \frac{1}{4} h^{n+1} n!$$

Then

$$|f(x) - p_n(x)| \leq \frac{M}{4(n+1)} h^{n+1}$$

for equally spaced points.

$n + 1 = \text{\#points}$

Polynomial interpolation error cont.

**n + 1 points**

Example 1. Consider $\cos(x)$ and assume values $f(x_i) = \cos(x_i)$ are given at 11 equally spaced points in $[a, b] = [-\pi, \pi]$. What is the error in the interpolating polynomial?

Here $n = 10$ and $h = (b - a)/n = 2\pi/10$.

$$M = \max_{-\pi \leq t \leq \pi} |\cos^{(n+1)}(t)| = 1.$$

Then

$$|f(x) - \cos(x)| \leq \frac{M}{4(n+1)} h^{n+1} = \frac{1}{4(11)} (2\pi/10)^{11} \approx 1.3694 \times 10^{-4}$$

Chebyshev nodes

- Suppose $f(x_i)$ is given at $n + 1$ distinct points x_0, x_1, \dots, x_n in $[a, b]$ and $p_n(x)$ of degree $\leq n$ interpolates f at these points
- We have for the error

$$\max_{x \in [a, b]} |f(x) - p_n(x)| \leq \frac{M}{(n+1)!} \max_{s \in [a, b]} \left| \prod_{i=0}^n (s - x_i) \right|$$

where $M = \max_{t \in [a, b]} |f^{(n+1)}(t)|$

- How to choose the x_i so

$$\max_{s \in [a, b]} \left| \prod_{i=0}^n (s - x_i) \right|$$

is minimized?

Chebyshev nodes cont.

- Chebyshev nodes on $[-1, 1]$:

$$x_i = \cos \left(\frac{2i+1}{2n+2} \pi \right), \quad i = 0, 1, \dots, n$$

- Min-max property: over all possible x_i they minimize $\max_{s \in [-1, 1]} |(s - x_0)(s - x_1) \cdots (s - x_n)|$

$$\min_{x_0, x_1, \dots, x_n} \max_{s \in [-1, 1]} |(s - x_0)(s - x_1) \cdots (s - x_n)| = 2^{-n}$$

- Error bound using Chebyshev nodes in $[-1, 1]$:

$$\max_{x \in [-1, 1]} |f(x) - p_n(x)| \leq \frac{M}{2^n (n+1)!}$$

$$M = \max_{t \in [-1, 1]} |f^{(n+1)}(t)|$$

Chebyshev nodes cont.

- For a general $[a, b]$,

$$x_i = \frac{1}{2}(a + b) + \frac{1}{2}(b - a) \cos \left(\frac{2i + 1}{2n + 2} \pi \right), \quad i = 0, 1, \dots, n$$

Example 2. In the previous example, if we chose Chebyshev nodes,

$$|f(x) - \cos(x)| \leq \frac{M}{2^n(n+1)!} = \frac{1}{2^{10}(10+1)!} \approx 2.4465 \times 10^{-11}$$