Partial Pivoting

CS/SE 4X03

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September 30, 2021

Outline

Examples

- The matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

  is nonsingular, but does not have LU factorization
  Gauss elimination breaks down on this matrix since the multiplier is $1/0$

-

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

  is singular and has the LU factorization

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = LU$$

Small pivots

$$A = \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix}$$

- Multiply the first row by $1/\epsilon$ and subtract from the second

$$L = \begin{bmatrix} 1 & 0 \\ \frac{1}{\epsilon} & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} \epsilon & 1 \\ 0 & 1 - \frac{1}{\epsilon} \end{bmatrix}$$

- When ϵ small, in floating-point arithmetic,

$$U \approx \begin{bmatrix} \epsilon & 1 \\ 0 & -\frac{1}{\epsilon} \end{bmatrix}$$

as $1 - \frac{1}{\epsilon} \approx -\frac{1}{\epsilon}$. Take e.g. $\epsilon = 10^{-16}$ in double precision

$$LU \approx \begin{bmatrix} 1 & 0 \\ \frac{1}{\epsilon} & 1 \end{bmatrix} \begin{bmatrix} \epsilon & 1 \\ 0 & -\frac{1}{\epsilon} \end{bmatrix} = \begin{bmatrix} \epsilon & 1 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix} = A$$

- Loss of accuracy

$$A = \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix}$$

- Permute the rows

$$\overline{A} = \begin{bmatrix} 1 & 1 \\ \epsilon & 1 \end{bmatrix}$$

- Multiple first row by ϵ and subtract from second row

$$\begin{bmatrix} 1 & 1 \\ 0 & 1-\epsilon \end{bmatrix}$$

$$\overline{L} = \begin{bmatrix} 1 & 0 \\ \epsilon & 1 \end{bmatrix}, \qquad \overline{U} = \begin{bmatrix} 1 & 1 \\ 0 & 1-\epsilon \end{bmatrix}$$

- Permuting the rows of A is PA, where P is permutation matrix

$$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \epsilon & 1 \end{bmatrix}$$

Example

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 1 \\
x_1 + x_2 + 2x_3 &= 2 \\
x_1 + 2x_2 - 16x_3 &= 1
\end{aligned}
$$

Eliminating with the first row:

$$
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & -16 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -17 & 0 \end{array}\right]
$$

GE breaks down, as the next multiplier is 1/0

Swap second and third rows and eliminate:

$$
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -17 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -17 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]
$$

Example: Replace $a_{22} = 1$ by 1.0001 and consider

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + 1.0001x_2 + 2x_3 &= 2 \\ x_1 + 2x_2 - 16x_3 &= 1 \end{aligned}$$

Gauss elimination gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1.0001 & 2 & 2 \\ 1 & 2 & -16 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0.0001 & 1 & 1 \\ 0 & 1 & -17 & 0 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0.0001 & 1 & 1 \\ 0 & 0 & -10017 & -10000 \end{array}\right]$$

Applying backward substitution in 5-digit decimal arithmetic:

$$
\begin{aligned}
\widetilde{x}_3 &= -10000/(-10017) \\
&= 0.9983, \quad \text{roundoff} \approx 2.8851 \times 10^{-6} \\
0.0001x_2 + x_3 &= 1 : \\
\widetilde{x}_2 &= (1 - \widetilde{x}_3)/0.0001 = 0.0017/0.0001 \\
&= 17 \\
x_1 + x_2 + x_3 &= 1 : \\
\widetilde{x}_1 &= 1 - \widetilde{x}_2 - \widetilde{x}_3 = 1 - 17 - 0.9983 \\
&= -16.998
\end{aligned}
$$

Exact solution to 5 digits is $(-16.969, 16.971, 0.9983)^T$

Why $\widetilde{x}_2$ is not accurate?

- $\widetilde{x}_3 = x_3 + \epsilon$, ϵ is the roundoff error
- Then

$$
\begin{aligned}
\widetilde{x}_2 &= (1 - \widetilde{x}_3)/0.0001 \\
&= (1 - x_3 - \epsilon)/0.0001 \\
&= (1 - x_3)/0.0001 - \epsilon/0.0001 \\
&= x_2 - \epsilon \times 10^4
\end{aligned}
$$

- The roundoff error in $\widetilde{x}_3$ is multiplied by 10^4: $\epsilon \times 10^4 \approx 0.029$
- $\widetilde{x}_2 - x_2 = 17 - 16.971 = 0.029$

Before eliminating with second row, swap second and third row and then eliminate

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -17 & 0 \\ 0 & 0.0001 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1.0017 & 1 \end{array}\right]$$

We have

$$\widehat{x}_3 = 1/1.0017 = 0.9983, \quad \text{roundoff} \approx 2.8851 \times 10^{-6}$$

$$\widehat{x}_2 = 0 + 17\widehat{x}_3 = 16.971, \qquad \text{roundoff multiplied by 17}$$

$$\widehat{x}_1 = 1 - \widehat{x}_2 - \widehat{x}_3 = -16.969$$

Exact solution to 5 digits is $(-16.969, 16.971, 0.9983)^T$

Partial pivoting (PP)

- If a pivot is small, then 1/(pivot) is large
- Roundoff errors are multiplied

Partial pivoting

- at step $k = 1 : n - 1$ chose the row q for which $|a_{qk}|$ is the largest
- eliminate with row q
  now we divide by the largest element in column k