Errors in Polynomial Interpolation

CS/SE 4X03

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Outline

Polynomial interpolation error

Chebyshev nodes

Polynomial interpolation error

$n = \#\text{points} - 1$

Assume

- Polynomial p_n of degree $\leq n$ interpolates f at $n+1$ distinct points $x_0, x_1, \ldots, x_n$, where $x_i \in [a, b]$
- $f^{(n+1)}$ is continuous on $[a, b]$

Then, for each $x \in [a, b]$, there is a $\xi = \xi(x) \in (a, b)$ such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

Polynomial interpolation error cont.

- Let $M = \max_{a \le t \le b} |f^{(n+1)}(t)|$
  Then
  $$|f(x) - p_n(x)| \le \frac{M}{(n+1)!} \prod_{i=0}^{n} |x - x_i|$$
- Let $h = (b-a)/n$ and let $x_i = a + ih$ for $i = 0, 1, \ldots, n$. It can be shown that
  $$\prod_{i=0}^{n} |x - x_i| \le \frac{1}{4} h^{n+1} n!$$

  Then
  $$|f(x) - p_n(x)| \le \frac{M}{4(n+1)} h^{n+1}$$

for equally spaced points.
n + 1 = #points

Polynomial interpolation error cont.

n + 1 points

Example 1. Consider $\cos(x)$ and assume values $f(x_i) = \cos(x_i)$ are given at 11 equally spaced points in $[a, b] = [-\pi, \pi]$. What is the error in the interpolating polynomial?

Here $n = 10$ and $h = (b - a)/n = 2\pi/10$.
$M = \max_{-\pi \le t \le \pi} |\cos^{(n+1)}(t)| = 1$.

Then

$$|f(x) - \cos(x)| \le \frac{M}{4(n+1)} h^{n+1} = \frac{1}{4(11)} (2\pi/10)^{11} \approx 1.3694 \times 10^{-4}$$

Chebyshev nodes

- Suppose $f(x_i)$ is given at $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in $[a,b]$ and $p_n(x)$ of degree $\leq n$ interpolates f at these points
- We have for the error

$$\max_{x\in[a,b]} |f(x) - p_n(x)| \leq \frac{M}{(n+1)!} \max_{s\in[a,b]} \left| \prod_{i=0}^{n} (s - x_i) \right|$$

where $M = \max_{t\in[a,b]} |f^{(n+1)}(t)|$

- How to chose the x_i so

$$\max_{s\in[a,b]} \left| \prod_{i=0}^{n} (s - x_i) \right|$$

is minimized?

Chebyshev nodes cont.

- Chebyshev nodes on $[-1, 1]$:

$$x_i = \cos\left(\frac{2i+1}{2n+2}\pi\right), \quad i = 0, 1, \ldots, n$$

- Min-max property: over all possible x_i they minimize $\max_{s\in[-1,1]} |(s-x_0)(s-x_1)\cdots(s-x_n)|$

$$\min_{x_0,x_1,\ldots,x_n} \max_{s\in[-1,1]} |(s-x_0)(s-x_1)\cdots(s-x_n)| = 2^{-n}$$

- Error bound using Chebyshev nodes in $[-1, 1]$:

$$\max_{x\in[-1,1]} |f(x) - p_n(x)| \le \frac{M}{2^n(n+1)!}$$

$M = \max_{t\in[-1,1]} |f^{(n+1)}(t)|$

$n = \#\text{points} - 1$

Chebyshev nodes cont.

- For a general $[a, b]$,

$$x_i = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos\left(\frac{2i+1}{2n+2}\pi\right), \quad i = 0, 1, \ldots, n$$

Example 2. In the previous example, if we chose Chebyshev nodes,

$$|f(x) - \cos(x)| \leq \frac{M}{2^n(n+1)!} = \frac{1}{2^{10}(10+1)!} \approx 2.4465 \times 10^{-11}$$