Errors, Convergence, Stiffness

CS/SE 4X03

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Outline

Local truncation error and order

- *Local truncation error* is the amount by which the exact solution fails to satisfy the numerical method
- Forward Euler $y_{i+1} = y_i + hf(t_i, y_i)$
  Using the exact solution $y(t)$ in this formula

$$d_i = \frac{y(t_{i+1}) - y(t_i)}{h} - f(t_i, y(t_i)) = \frac{h}{2} y''(\eta_i)$$

- Backward Euler $d_i = -\frac{h}{2} y''(\xi_i)$
- A method is of *order* q, if q is the lowest positive integer such that for any sufficiently smooth exact solution $y(t)$

$$\max_i |d_i| = O(h^q)$$

- Forward and backward Euler are of order $q = 1$

Local and global error

- Global error is

$$\boxed{e_i = y(t_i) - y_i, \quad i = 0, 1, \ldots, N,}$$

  where $y(t_i)$ is the exact solution at t_i and y_i is the computed approximation

- Consider

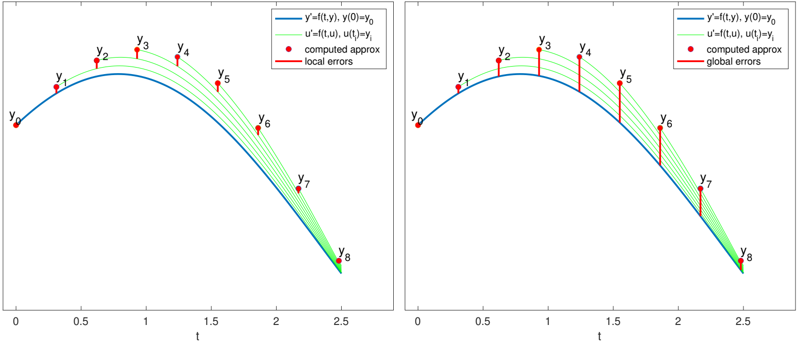
$$u' = f(t, u), \quad u(t_{i-1}) = y_{i-1}$$

  The local error is

$$l_i = u(t_i) - y_i$$

  where $u(t_i)$ is the exact solution to $u' = f(t, u)$ with initial condition u_i at t_i

Local vs global error



- Numerical methods control the local error
- That is, select a stepsize such that the local error is within a given tolerance
- Typically the global error is proportional to the tolerance

Convergence

- A methods is said to *converge* if the maximum global error goes to 0 as $h \to 0$
- That is

$$\max_i e_i = \max_i [y(t_i) - y_i] \to 0 \quad \text{as } h \to 0$$

Stiffness

- When the stepsize is restricted by stability rather than accuracy
- When an explicit solver takes very small steps
- Matlab: nonstiff solvers `ode45`, `ode113`,...
  stiff solvers: `ode15s`, `ode23s`

Stiffness cont.

Van der Pol

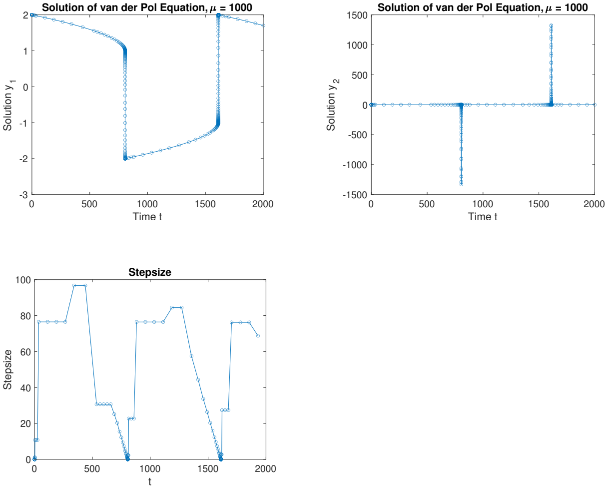
$$\begin{aligned} y_1' &= y_2 \\ y_2' &= \mu(1 - y_1^2)y_2 - y_1 \end{aligned}$$

μ is a constant

$y(0) = (2, 0)^T$, $t \in [0, 2000]$

Stiff vs Nonstiff

ode15s on Van der Pol, $\mu = 1000$: integrated in ≈ 0.2 seconds, 408 steps



Stiff vs Nonstiff

ode45 on Van der Pol, $\mu = 1000$: integrated in ≈ 15 seconds, 4,624,409 steps