# Newton's Method for Nonlinear Equations CS/SE 4X03

Ned Nedialkov

McMaster University

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#### Outline

Scalar case

Examples

Convergence

Subtleties

Newton for systems of equations

#### Scalar case

- $\bullet$  Given a scalar function f find a zero/root of f, i.e. an r such that f(r)=0
- ullet f may have no zeros, one, or many
- Let r be a root of f and let  $x_n \approx r$ From

$$0 = f(r) = f(x_n) + f'(x_n)(r - x_n) + O(|r - x_n|^2)$$
  

$$0 = f(r) \approx f(x_n) + f'(x_n)(r - x_n)$$

we find  $x_{n+1}$  by solving

$$f(x_n) + f'(x_n)(x_{n+1} - x_n) = 0 (1)$$

Scalar case Examples Convergence Subtleties N for systems Scalar case cont.

That is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

- We start with an initial guess  $x_0$  and compute  $x_1, x_2, \dots$
- How to choose  $x_0$ , does it converge to a root, when to stop iterating...?

# Interpretation

Given  $x_0$ , we compute

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The tangent line at  $(x, f(x_0))$  is

$$l(x) = f(x_0) + f'(x_0)(x - x_0)$$

We find  $x_1$  such that l(x) crosses the x axis,  $l(x_1) = 0$ :

$$0 = l(x_1) = f(x_0) + f'(x_0)(x_1 - x_0)$$

Similarly for  $x_2$ ,  $x_3$ , ...

# Examples

#### Square root

## Newton's method for nonlinear equation



- Given a > 0, compute  $\sqrt{a}$
- Write  $x = \sqrt{a}$ ,  $f(x) = x^2 a$
- Apply (2):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n}$$
$$= x_n - \frac{x_n}{2} + \frac{a}{2x_n}$$
$$= 0.5 \left(x_n + \frac{a}{x_n}\right)$$

- Let a = 2 and  $x_0 = 3$
- We compute

$$i x_i |x_i - \sqrt{2}|$$

- 1 1.83333333333333 4.19e-01
- 2 1.462121212121212 4.79e-02
- 3 1.4149984298948031 7.85e-04
- 4 1.4142137800471977 2.18e-07
- 5 1.4142135623731118 1.67e-14
- 6 1.4142135623730949 2.22e-16

# Examples cont.

Dividing without division operation

- How to obtain a/b without division?
- a/b = a \* (1/b)
- Find 1/b. Write f(x) = 1/x b and apply (2)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{1/x_n - b}{-1/x_n^2}$$
$$= x_n + x_n - bx_n^2$$
$$= x_n(2 - bx_n)$$

# Scalar case Examples Convergence Subtleties N for systems Examples cont.

• With b=3 and  $x_0=0.3$ , we compute i  $x_i$   $|x_i-1/3|$  1 0.3300000000000000 3.33e-03 2 0.333330000000000 3.33e-05 3 0.333333333333333333 5.55e-17

### Convergence

Theorem 1. If f, f', and f'' are continuous in a neighbourhood of a root r of f and  $f'(r) \neq 0$ , then  $\exists \delta > 0$  such that if  $|r - x_0| \leq \delta$ , then all  $x_n$  satisfy

$$|r - x_n| \le \delta,\tag{3}$$

$$|r - x_{n+1}| \le c(\delta)|r - x_n|^2,$$
 (4)

where  $c(\delta)$  is defined in (6), and  $x_n$  converges to r

Let  $e_n = r - x_n$ . (4) is

$$|e_{n+1}| \le c(\delta)|e_n|^2 \tag{5}$$

If e.g.  $|e_n| \approx 10^{-4}$ ,  $|e_{n+1}| \lesssim c(\delta) 10^{-8}$ 

If sufficiently close to r, each iteration  $\approx$  doubles the number of accurate digits

Quadratic convergence  $|e_{n+1}| \le \text{constant} \cdot |e_n|^2$ 

Order of convergence is 2

# Convergence cont.

Proof. From Taylor series,

$$\begin{split} 0 &= f(r) = f(x_n) + f'(x_n)(r - x_n) + \frac{f''(\xi)}{2}(r - x_n)^2 \\ &= f(x_n) + f'(x_n)e_n + \frac{f''(\xi)}{2}e_n^2 \\ f(x_n) + f'(x_n)e_n &= -\frac{f''(\xi)}{2}e_n^2, \quad \xi \text{ is between } r \text{ and } x_n \end{split}$$

The error in  $x_{n+1}$  is

$$e_{n+1} = r - x_{n+1} = r - \left(x_n - \frac{f(x_n)}{f'(x_n)}\right) = r - x_n + \frac{f(x_n)}{f'(x_n)}$$

$$= e_n + \frac{f(x_n)}{f'(x_n)} = \frac{f(x_n) + e_n f'(x_n)}{f'(x_n)}$$

$$= -\frac{1}{2} \frac{f''(\xi)}{f'(x_n)} e_n^2$$

# Convergence cont.

For a  $\delta > 0$ , let

$$c(\delta) = \frac{1}{2} \frac{\max_{|r-x| \le \delta} |f''(x)|}{\min_{|r-x| \le \delta} |f'(x)|}$$

$$\tag{6}$$

Then (4) follows from

$$|e_{n+1}| = \frac{1}{2} \frac{|f''(\xi)|}{|f'(x_n)|} e_n^2 \le \frac{1}{2} \frac{\max_{|r-x| \le \delta} |f''(x)|}{\min_{|r-x| \le \delta} |f'(x)|} e_n^2$$
  
 
$$\le c(\delta)e_n^2$$

Choose  $\delta$  such that  $c(\delta)\delta < 1$ . This is possible since

$$c(\delta) o rac{1}{2} \left| rac{f''(r)}{f'(r)} 
ight| \quad \text{as } \delta o 0$$

and  $f'(r) \neq 0$  by assumption

# Convergence cont.

If 
$$|e_n| = |r - x_n| \le \delta$$
, then

$$\begin{split} |e_{n+1}| & \leq c(\delta)e_n^2 = c(\delta) \cdot e_n \cdot e_n \leq c(\delta)\delta \cdot e_n \\ & < \rho e_n, \quad \text{where } \rho = \delta c(\delta) < 1 \end{split}$$

and (3) follows

Hence

$$|e_n| \le \rho |e_{n-1}| \le \rho^2 |e_{n-2}| \le \dots \le \rho^n |e_0|$$

Since 
$$\rho < 1$$
,  $|e_n| \to r$  as  $n \to \infty$ 

#### **Subtleties**

We require  $f'(r) \neq 0$ 

If 
$$f'(r) = 0$$
 and  $f''(r) \neq 0$ ,  $r$  is a double root, e.g.  $f(x) = (x-1)^2$ 

A root r is of multiplicity m if  $f^{(k)}(r)=0$  for all  $k=1,2,\ldots m-1$  and  $f^{(m)}(r)\neq 0$ . In this case

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)}$$

is quadratically convergent

If  $f'(x_n)$  is not available, we can approximate  $f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$  Then

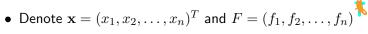
$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

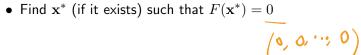
This is the secant method. Order of convergence is  $(1+\sqrt{5})/2\approx 1.618$  (golden ratio)

# Newton for systems of equations

ullet Consider a system of n equations in n variables

$$f_1(x_1, x_2, \dots, x_n) = 0$$
  
 $f_2(x_1, x_2, \dots, x_n) = 0$   
 $\vdots$   
 $f_n(x_1, x_2, \dots, x_n) = 0$ 





# Newton for systems of equations cont.

- Assume  $\mathbf{x}^*$  is such that  $F(\mathbf{x}^*) = 0$  and  $\mathbf{x}^{(k)} \approx \mathbf{x}^*$
- From

$$0 = F(\mathbf{x}^*) \approx F(\mathbf{x}^{(k)}) + F'(\mathbf{x}^{(k)})(\mathbf{x}^* - \mathbf{x}^{(k)})$$

find  $\mathbf{x}^{(k+1)}$  by solving (cf. (1))

$$F(\mathbf{x}^{(k)}) + F'(\mathbf{x}^{(k)})(\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}) = 0$$
 (7)

•  $F'(\mathbf{x}^{(k)})$  is the Jacobian of F at  $\mathbf{x}^{(k)}$ , an  $n \times n$  matrix

# Newton for systems of equations cont.

- Let  $s = \mathbf{x}^{(k+1)} \mathbf{x}^{(k)}$
- ullet Solve (assuming  $F'(\mathbf{x}^{(k)})$  nonsingular) linear system

$$F'(\mathbf{x}^{(k)})s = -F(\mathbf{x}^{(k)}) \tag{8}$$

and set

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + s \tag{9}$$

• (8,9) is basic Newton for systems of equations

# Example

Consider

$$0 = F(\mathbf{x}) = \begin{cases} x_1^2 + x_2^2 - 25 \\ x_1^2 - x_2 - 1 \end{cases}$$

Jacobian is

$$F'(\mathbf{x}) = \begin{pmatrix} 2x_1 & 2x_2 \\ 2x_1 & -1 \end{pmatrix}$$

• Let  $x_0 = (5,1)^T$ 

Then

$$F(\mathbf{x}^{(0)}) = (1, 23)^T$$
$$J(\mathbf{x}^{(0)}) = \begin{pmatrix} 10 & 2\\ 10 & -1 \end{pmatrix}$$

- Solve  $J(\mathbf{x}^{(0)})s = -F(\mathbf{x}^{(0)})$  for s
- $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + s$  and so on
- We compute

i	$x_1$	$x_2$	$  F(\mathbf{x})  $
1	3.433333333333334	8.33333333333333	5.63e+01
2	2.632585333089088	5.289308176100628	9.93e+00
3	2.358810087435537	4.489032143454986	7.19e-01
4	2.329316858408983	4.424847176309882	5.06e-03
5	2.329040359270796	4.424428918660463	2.63e-07
6	2.329040339044829	4.424428900898053	7.11e-15