

# Polynomial Interpolation

## Newton's Form

CS/SE 4X03

Ned Nedialkov

McMaster University

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# Outline

Basis

Computing coefficients

Divided differences

Example

# Basis

- Basis functions are

$$\phi_j(x) = \prod_{i=0}^{j-1} (x - x_i) = (x - x_0)(x - x_1) \cdots (x - x_{j-1}), \quad j = 0 : n$$

- Example: for a cubic interpolant, we have

$$\phi_0(x) = 1$$

$$\phi_1(x) = x - x_0$$

$$\phi_2(x) = (x - x_0)(x - x_1)$$

$$\phi_3(x) = (x - x_0)(x - x_1)(x - x_2)$$

# Computing coefficients

Let  $y_i = f(x_i)$ . From

$$p_n(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \cdots \\ + c_n(x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

$$p_n(x_i) = c_0 + c_1(x_i - x_0) + c_2(x_i - x_0)(x_i - x_1) + \cdots \\ + c_n(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{n-1}) = f(x_i)$$

at  $x = x_0$ , we have

$$p_n(x_0) = c_0 + c_1(x_0 - x_0) + c_2(x_0 - x_0)(x_0 - x_1) + \cdots \\ + c_n(x_0 - x_0)(x_0 - x_1) \cdots (x_0 - x_{n-1}) = f(x_0)$$

$$c_0 = f(x_0)$$

# Computing coefficients

At  $x_1$ ,

$$\begin{aligned} p_n(x_1) &= c_0 + c_1(x_1 - x_0) + c_2(x_1 - x_0)(x_1 - x_1) + \cdots \\ &\quad + c_n(x_1 - x_0)(x_1 - x_1) \cdots (x_1 - x_{n-1}) = f(x_1) \end{aligned}$$

$$c_0 + c_1(x_1 - x_0) = f(x_1)$$

$$\begin{aligned} c_1 &= \frac{f(x_1) - c_0}{x_1 - x_0} \\ &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \end{aligned}$$

## Computing coefficients

At  $x_2$ ,

$$\begin{aligned} p_n(x_2) &= c_0 + c_1(x_2 - x_0) + c_2(x_2 - x_0)(x_2 - x_1) \\ &\quad + c_3(x_2 - x_0)(x_2 - x_1)(x_2 - x_2) + \cdots \\ &\quad + c_n(x_1 - x_0)(x_1 - x_1) \cdots (x_1 - x_{n-1}) = f(x_1) \end{aligned}$$

Then

$$c_0 + c_1(x_2 - x_0) + c_2(x_2 - x_0)(x_2 - x_1) = f(x_2)$$

$$c_2 = \frac{f(x_2) - c_0 - c_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Exercise: verify the last equality

# Divided differences

**n + 1 points**

Given  $x_0, x_1, \dots, x_n$ , where  $0 \leq i < j \leq n$ , define

$$f[x_i] = f(x_i)$$

$$f[x_i, \dots, x_j] = \frac{f[x_{i+1}, \dots, x_j] - f[x_i, \dots, x_{j-1}]}{x_j - x_i}$$

$f[x_i, \dots, x_j]$  are divided differences over  $x_i, \dots, x_j$

## Divided differences

$$c_0 = f(x_0) = f[x_0]$$

$$c_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1]$$

$$c_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = f[x_0, x_1, x_2]$$

$$\vdots$$

$$c_n = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0} = f[x_0, x_1, \dots, x_n]$$

$$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ + f[x_0, x_1, \dots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1})$$



## Example

$i$	$x_i$	$f[x_i]$	$f[\cdot, \cdot]$	$f[\cdot, \cdot, \cdot]$
0	1	1 $c_0$		
1	2	3	2 $c_1$	
2	4	3	0	$-\frac{2}{3}$ $c_2$

$$\begin{aligned}
 \underline{p_2}(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\
 &= 1 + 2(x - 1) - \frac{2}{3}(x - 1)(x - 2)
 \end{aligned}$$

# Example

Suppose we add a new point  $(3, 5)$

Then

$i$	$x_i$	$f[x_i]$	$f[\cdot, \cdot]$	$f[\cdot, \cdot, \cdot]$	$f[\cdot, \cdot, \cdot, \cdot]$
0	1	1			
1	2	3	2		
2	4	3	0	$-\frac{2}{3}$	
3	3	5	-2	-2	$-\frac{2}{3}$

$$p_3(x) = 1 + 2(x - 1) - \frac{2}{3}(x - 1)(x - 2) - \frac{2}{3}(x - 1)(x - 2)(x - 4)$$