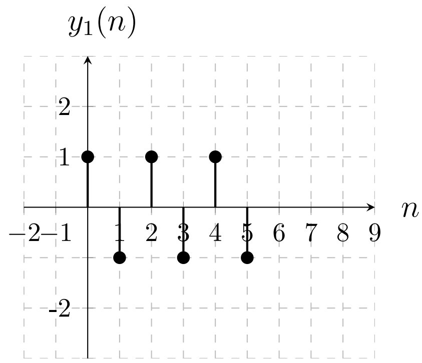
Musterlösung zur Klausur
"Digitale Signalverarbeitung"
vom 12.03.2015

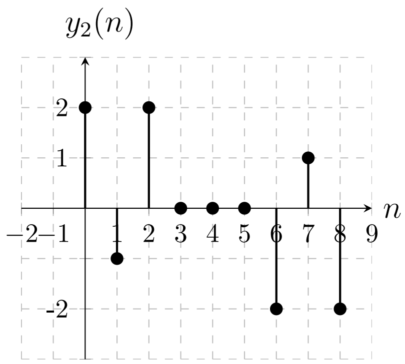
Aufgabe 1: Übertragungsfunktionen, Faltung und Analyse eines LTI-Systems

(14 Punkte gesamt)

a) (2 Punkte)



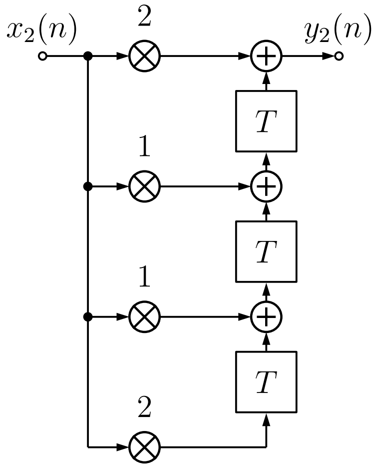
b) (2 Punkte)



c) (1 Punkt)
Ja, Typ II, FIR

d) (1 Punkt)
Nein, $\pi = 0$

e) (1 Punkt)



f) (1 Punkt)

$$
\begin{aligned}
H_2(z) &= \frac{Y_2(z)}{X_2(z)} \\
y_2(n) &= 2x_2(n) + 1x_2(n-1) + 1x_2(n-2) + 2x_2(n-3) \\
Y_2(z) &= 2X_2(z)z^0 + 1X_2(z)z^{-1} + 1X_2(z)z^{-2} + 2X_2(z)z^{-3} \\
&= X_2(z)\left[2z^0 + z^{-1} + z^{-2} + 2z^{-3}\right] \\
H_2(z) &= \frac{Y_2(z)}{X_2(z)} = \left[2z^0 + z^{-1} + z^{-2} + 2z^{-3}\right]
\end{aligned}
$$

g) (1 Punkt)
Typ III, FIR, Bandpass

h) (1 Punkt)
$N_\mathrm{b} = N - 1$
$N_\mathrm{b} = 9 - 1 = 8$

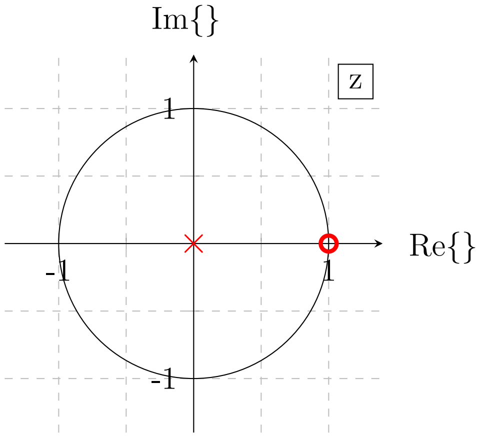
i) (1 Punkt)

$$
\begin{aligned}
H_1(z) &= \frac{Y_1(z)}{X_1(z)} \\
y_1(n) &= x_1(n) - x_1(n-1) \\
Y_1(z) &= X_1(z)z^0 - X_1(z)z^{-1} \\
&= X_1(z)\left[1 - z^{-1}\right] \\
H_2(z) &= \frac{Y_1(z)}{X_1(z)} = 1 - z^{-1}
\end{aligned}
$$

j) (1 Punkt)

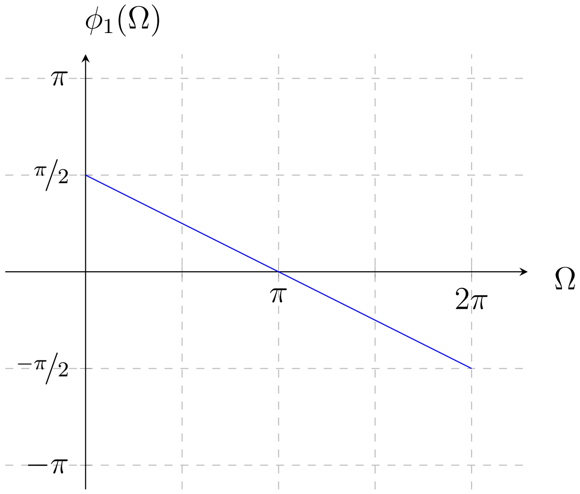
Pol-/Nullstellen:

$$\begin{aligned}
&\frac{1-z^{-1}}{1} \quad | \cdot \frac{z}{z} \\
\Rightarrow\ &\frac{z-1}{z} \\
&z_\infty = 0 \\
&z_0 = 1
\end{aligned}$$



k) (2 Punkte)

$$\begin{aligned}
H_1(e^{j\Omega}) &= (1-e^{-j\Omega}) \quad | \cdot \left[e^{-j\frac{\Omega}{2}} \cdot e^{j\frac{\Omega}{2}}\right] \\
&= (1-e^{-j\Omega}) \cdot \left[e^{-j\frac{\Omega}{2}} \cdot e^{j\frac{\Omega}{2}}\right] \\
&= \left[e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}\right] \cdot e^{-j\frac{\Omega}{2}} \\
&= \sin\frac{\Omega}{2} \cdot 2j \cdot e^{-j\frac{\Omega}{2}} \\
&= \sin\frac{\Omega}{2} \cdot 2e^{j\frac{\pi}{2}} \cdot e^{-j\frac{\Omega}{2}} \\
&= 2\sin\frac{\Omega}{2} \cdot e^{j(\frac{\pi-\Omega}{2})} \\
\Rightarrow \phi_1(\Omega) &= \frac{\pi-\Omega}{2}
\end{aligned}$$


$\phi_1(\Omega)$
π
$\pi/2$
$-\pi/2$
$-\pi$
π
2π
Ω

Aufgabe 2: Zerlegung eines LTI-Systems

(11 Punkte gesamt)

a) (3 Punkte)
Nullstellen:

$$\begin{aligned} 6 - 2z^{-2} &\stackrel{!}{=} 0 \\ \Rightarrow 6z^2 &= 2 \\ \Leftrightarrow z^2 &= \frac{1}{3} \\ \Rightarrow z_{0_{1,2}} &= \pm\frac{1}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned} 3 - 27z^{-2} &\stackrel{!}{=} 0 \\ \Rightarrow 3z^2 &= 27 \\ \Leftrightarrow z^2 &= 9 \\ \Rightarrow z_{0_{3,4}} &= \pm 3 \end{aligned}$$

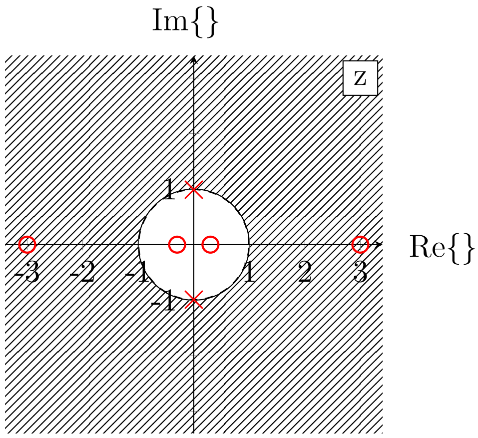
Polstellen:

$$\begin{aligned} 1 + 0{,}9801z^{-2} &\stackrel{!}{=} 0 \\ \Rightarrow z^2 &= -0{,}9801 \\ \Leftrightarrow z &= \sqrt{j^2 \cdot 0{,}9801} \\ \Rightarrow z_{\infty_{1,2}} &= \pm j \cdot 0{,}99 \\ z_{\infty_{3,4}} &= 0 \end{aligned}$$

ROC: $|z| > 0{,}99$

Faktorisierte Form: $H(z) = \frac{(1-\frac{1}{\sqrt{3}}z^{-1})(1+\frac{1}{\sqrt{3}}z^{-1})(1-3z^{-1})(1+3z^{-1})}{(1-j\cdot 0{,}99z^{-1})(1+j\cdot 0{,}99z^{-1})}$

b) (2 Punkte)



c) (2 Punkte)
Ja, da alle Polstellen innerhalb des Einheitskreises!

d) (3 Punkte)

$$
\begin{aligned}
P_{\mathrm{OUT}}(z) &= (1 - 3z^{-1})(1 + 3z^{-1}) \\
P_{\mathrm{REST}}(z) &= (1 - \frac{1}{\sqrt{3}} z^{-1})(1 + \frac{1}{\sqrt{3}} z^{-1}) \\
P'_{\mathrm{OUT}}(z) &= P_{\mathrm{OUT}}(z = \frac{1}{z^*_{0\nu}}) = (1 - \frac{1}{3} z^{-1})(1 + \frac{1}{3} z^{-1}) \\
H_{\mathrm{AP}}(z) &= \frac{1}{b_0} \frac{(1 - 3z^{-1})(1 + 3z^{-1})}{(1 - \frac{1}{3} z^{-1})(1 + \frac{1}{3} z^{-1})} \\
H_{\min}(z) &= b_0 \cdot \frac{(1 - \frac{1}{\sqrt{3}} z^{-1})(1 + \frac{1}{\sqrt{3}} z^{-1}) \cdot (1 - \frac{1}{3} z^{-1})(1 + \frac{1}{3} z^{-1})}{(1 - j0{,}99 z^{-1})(1 + j0{,}99 z^{-1})} \\
b_0 &= \prod_{\nu=1}^{N} (-\frac{1}{z^*_{0\nu}}) = -(-\frac{1}{3}) \cdot (-\frac{1}{3}) = -\frac{1}{9}
\end{aligned}
$$

e) (1 Punkt)
Ja, weil $G(z) = \frac{1}{H_{\min}(z)}$ wieder ein stabiles System ist.

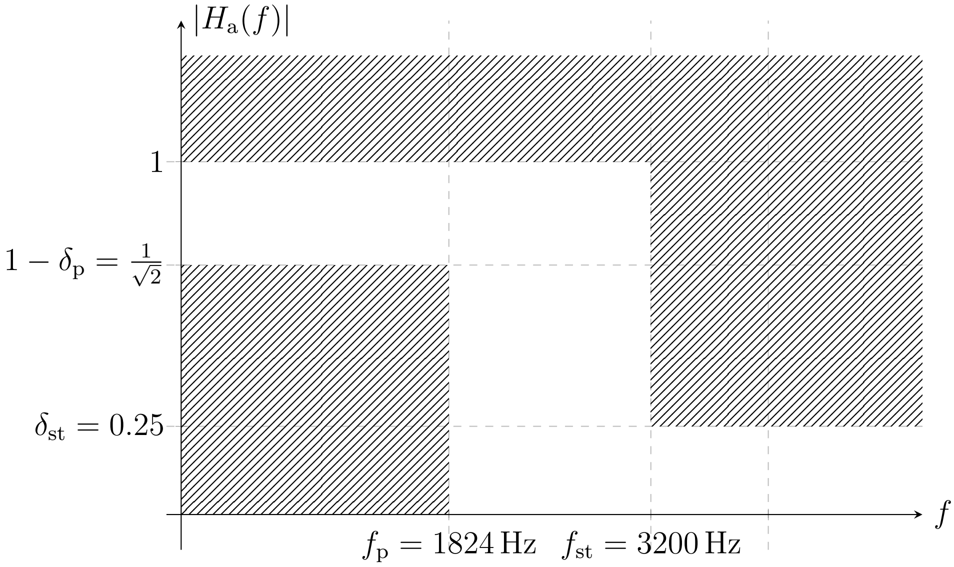
Aufgabe 3: Filterentwurf

(14 Punkte gesamt)

a) (3 Punkte)

$$\begin{aligned}
v &= \frac{\omega'}{\tan\frac{\Omega'}{2}} = \frac{\Omega_\mathrm{st}\cdot f_\mathrm{s}}{\tan\frac{\Omega_\mathrm{st}}{2}} = \frac{0{,}4\pi\cdot 16\mathrm{kHz}}{\tan 0{,}2\pi} = 27673{,}80\mathrm{s}^{-1}\\
\omega_\mathrm{st} &= \Omega_\mathrm{st}\cdot f_\mathrm{s} = 20106{,}19\mathrm{s}^{-1}\\
f_\mathrm{st} &= \frac{\omega_\mathrm{st}}{2\pi} = 3200\mathrm{Hz}\\
\omega_\mathrm{p} &= v\cdot\tan\frac{\Omega_\mathrm{p}}{2} = 11462{,}86\mathrm{s}^{-1}\\
f_\mathrm{p} &= \frac{\omega_\mathrm{p}}{2\pi} = 1824{,}37\mathrm{Hz}\\
\delta_\mathrm{st} &= 10^{\frac{-12\mathrm{dB}}{20\mathrm{dB}}} = 0{,}25
\end{aligned}$$

b) (1 Punkt)



c) (2 Punkte)

$$|H_a(j\omega_\mathrm{p})|^2 = \frac{1}{1+(\frac{\omega_\mathrm{p}}{\omega_\mathrm{c}})^{2N}} = (1-\delta_\mathrm{p})^2 = (\frac{1}{\sqrt{2}})^2 = \frac{1}{2}$$
$$\Rightarrow (\frac{\omega_\mathrm{p}}{\omega_\mathrm{c}})^{2N} = 1 \qquad \text{(I)}$$
$$|H_a(j\omega_\mathrm{st})|^2 = \frac{1}{1+(\frac{\omega_\mathrm{st}}{\omega_\mathrm{c}})^{2N}} = \delta_\mathrm{st}^2 = 0{,}0631$$
$$\Rightarrow (\frac{\omega_\mathrm{st}}{\omega_\mathrm{c}})^{N} = \sqrt{\frac{1}{\delta_\mathrm{st}^2} - 1} = 3{,}8534 \qquad \text{(II)}$$
$$\frac{\text{II}}{\text{I}}:$$
$$(\frac{\omega_\mathrm{st}}{\omega_\mathrm{p}})^{N} \geq \frac{3{,}8534}{1}$$
$$\frac{\omega_\mathrm{st}}{\omega_\mathrm{p}} = 1{,}75$$
$$N \cdot \log 1{,}75 \geq \log 3{,}8534$$
$$N \geq \frac{\log 3{,}8534}{\log 1{,}75} = 2{,}41$$
$$N = 3$$

d) (2 Punkte)
II:

$$(\frac{\omega_\mathrm{st}}{\omega_\mathrm{c}})^{N} = 3{,}8534 \qquad | \cdot \omega_\mathrm{c}^N$$
$$\omega_\mathrm{st}^N = \omega_\mathrm{c}^N \cdot 3{,}8534 \qquad | \div 3{,}8534$$
$$\omega_\mathrm{c}^N = \frac{\omega_\mathrm{st}^N}{3{,}8534} \qquad | \sqrt[N]{}$$
$$\omega_\mathrm{c} = (\frac{\omega_\mathrm{st}^N}{3{,}8534})^{\frac{1}{N}} = \frac{\omega_\mathrm{st}}{1{,}5678} = 12825\mathrm{s}^{-1}$$
$$f_\mathrm{c} = \frac{\omega_\mathrm{c}}{2\pi} = 2041\mathrm{Hz}$$
$$\Omega_\mathrm{c} = 2 \cdot \arctan\frac{\omega_\mathrm{c}}{v} = 0{,}868 = 0{,}276\pi$$

e) (4 Punkte)

$$s_{\infty,\nu} = \omega_\mathrm{c} \cdot e^{j(\pi/2N + \pi/2 + \nu\cdot\pi/N)}, \text{ mit } \nu = 0{,}1{,}2$$
$$s_{\infty,0} = \omega_\mathrm{c} \cdot e^{j\frac{\pi}{6} + \frac{3\pi}{6} + 0} = \omega_\mathrm{c} \cdot e^{j\frac{2\pi}{3}}$$
$$s_{\infty,1} = \omega_\mathrm{c} \cdot e^{j\frac{2\pi}{3} + \frac{\pi}{3}} = \omega_\mathrm{c} \cdot e^{j\pi} = -\omega_\mathrm{c}$$
$$s_{\infty,2} = \omega_\mathrm{c} \cdot e^{j\frac{2\pi}{3} + \frac{2\pi}{3}} = \omega_\mathrm{c} \cdot e^{j\frac{4\pi}{3}}$$

$$z_{\infty,0} = \frac{v + s_{\infty,0}}{v - s_{\infty,0}} = 0{,}4679 + j \cdot 0{,}4783$$
$$z_{\infty,1} = \frac{v - \omega_c}{v + \omega_c} = 0{,}3667$$
$$z_{\infty,2} = z_{\infty,0}^{*} = 0{,}4679 - j \cdot 0{,}4783$$

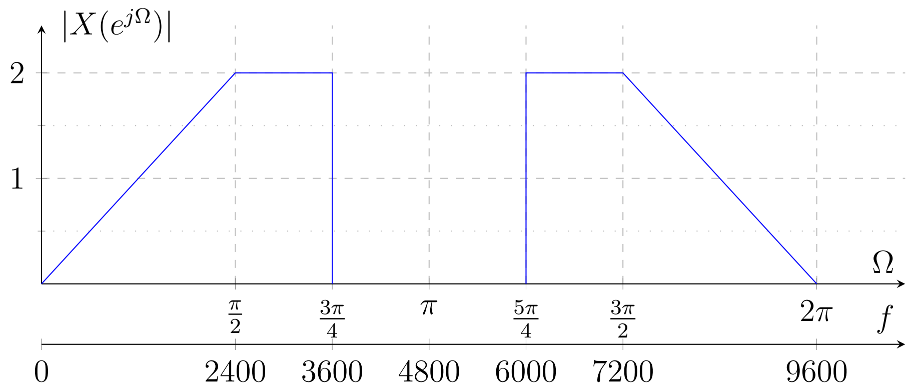
f) (2 Punkte)

$$N = 3$$
$$\Rightarrow \text{3-fache Nullstelle bei } z = -1$$

Aufgabe 4: Abtastratenwandlung

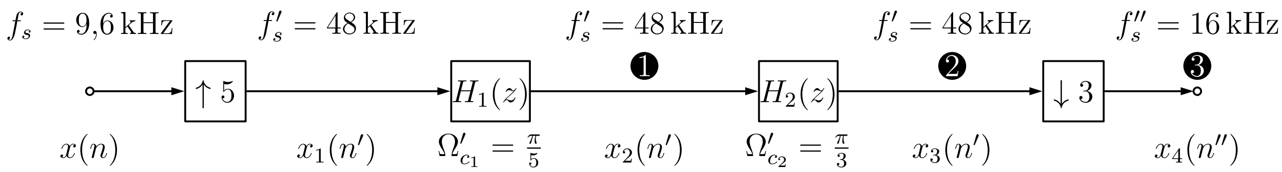
(11 Punkte gesamt)

a) (2 Punkte)



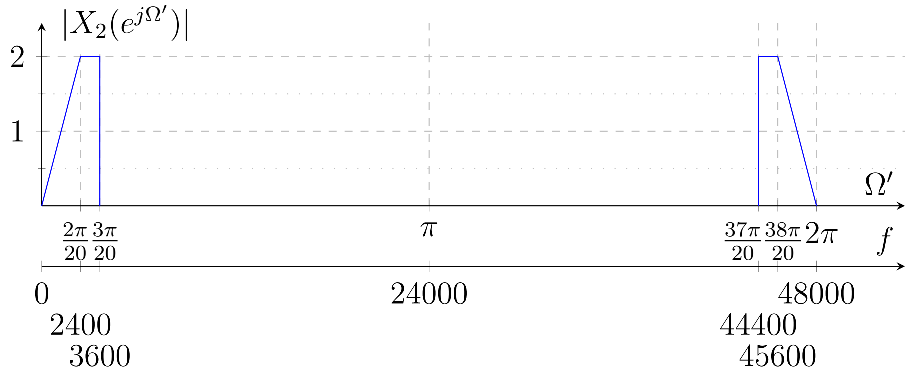
b) (1 Punkt)
$r = \frac{p}{q} = \frac{5}{3}$

c) (2 Punkte)



d) (3 Punkte)

❶



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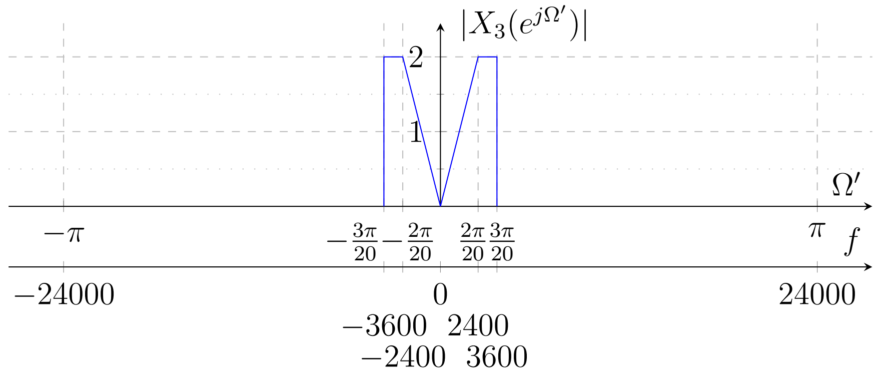

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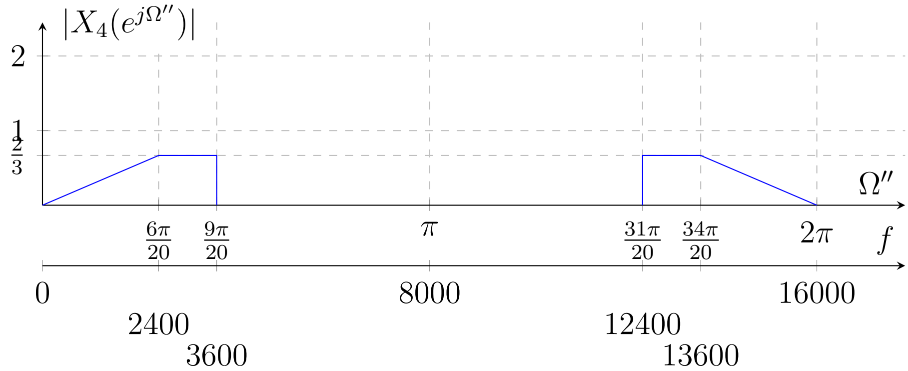
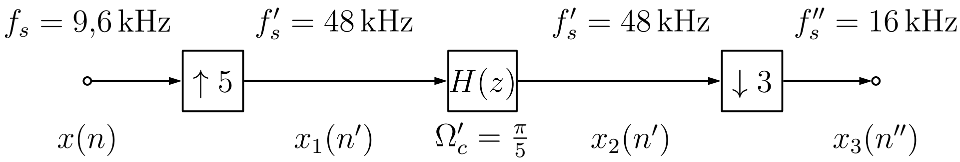


e) (1 Punkt)



$H(z)$ stellt einen Tiefpass mit Grenzfrequenz $\Omega_c' = \frac{\pi}{5}$ dar.

f) (1 Punkt)
$\Omega_c' = \frac{3\pi}{20}$

g) (1 Punkt)
$\frac{N}{15}$