Tính ổn định HTĐKTĐ gián đoạn

14th Lecture, 9th Dec 2023

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Mục đích, Yêu cầu

❑Mục đích

- ✓Determine the input-output stability of a z-transfer function.
- ✓Determine the asymptotic stability of a z-transfer function.
- ✓Determine the internal stability of a digital feedback control system.
- ✓Determine the stability of a z-polynomial using the Routh-Hurwitz criterion.
- ✓Determine the stability of a z-polynomial using the Jury criterion.
- ✓Determine the stable range of a parameter for a z-polynomial.
- ✓Determine the closed-loop stability of a digital system using the Nyquist criterion.
- ✓Determine the gain margin and phase margin of a digital system.

❑Yêu cầu

- ✓Chú ý nghe giảng
- ✓Làm lại các ví dụ và tự thực hiện các bài tập về nhà.
- ✓Sử dụng máy tính và matlab để kiểm chứng lý thuyết

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Nội dung bài học

- ❑ Khái niệm về tính ổn định
- ❑ Mối quan hệ vị trí điểm cực và tính ổn định
- ❑ Các điều kiện ổn định
- ❑ Cách xác định tính ổn định hệ thống

Nội dung bài học

- ❑ **Khái niệm về tính ổn định**
- ❑ Mối quan hệ vị trí điểm cực và tính ổn định
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Definitions of stability

- ❑ **Asymptotic stability**
- ❑ **Marginal stability**
- ❑ **Input-output stability**

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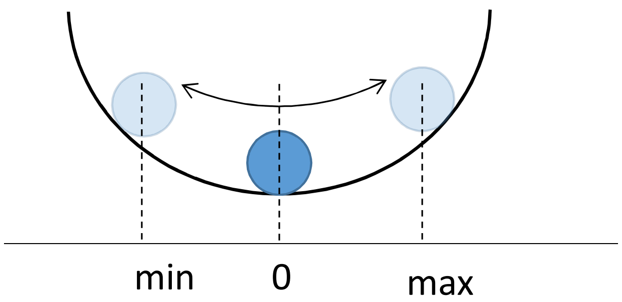
Asymptotic stability

Asymptotic stability: Response due to any initial conditions decays to zero asymptotically in the steady state

$$\lim_{k \to \infty} y(k) = 0$$



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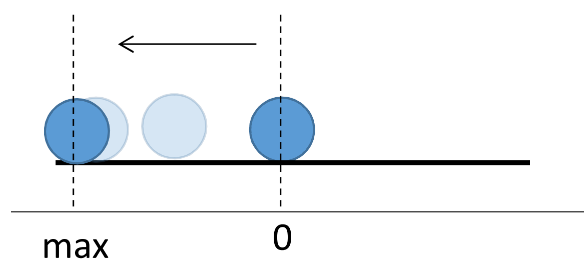
Marginal stability

Marginal stability: response due to any initial conditions remains bounded but does not decay to zero

$$\lim_{k \to \infty} y(k) = const$$



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Bounded-Input-Bounded-Output (BIBO) Stability

BIBO stability: Response due to any bounded input remains bounded

$$u(k) < b_u, k = 0, 1, 2, 3, \ldots$$

$$0 < b_u < \infty$$

$$y(k) < b_y, k = 0, 1, 2, 3, \ldots$$

$$0 < b_y < \infty$$

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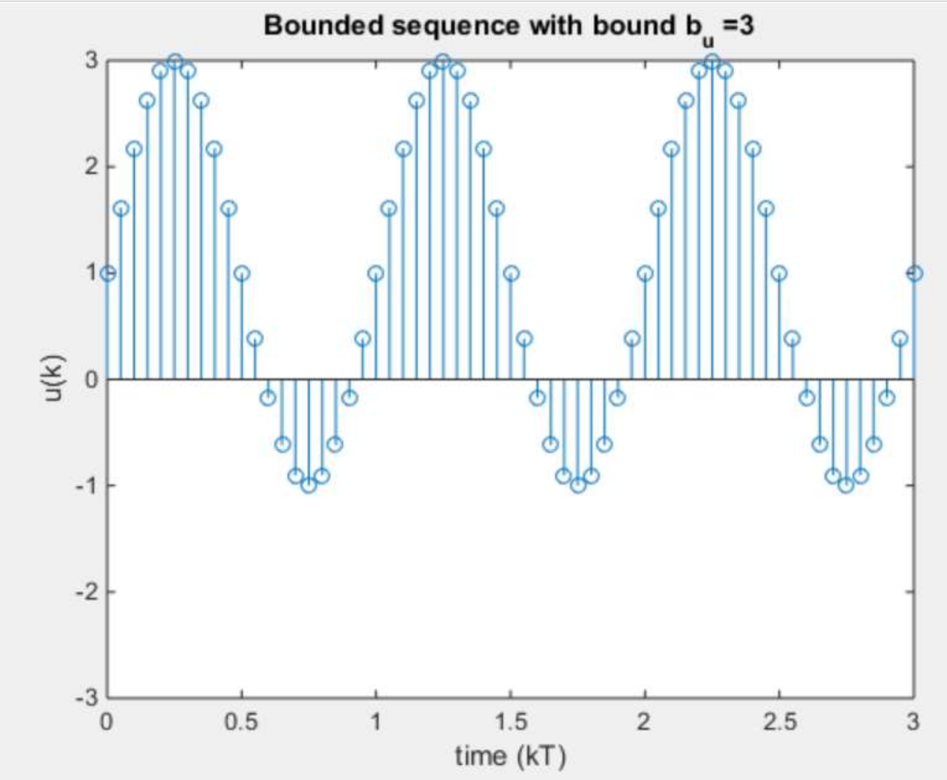
Bounded-Input-Bounded-Output (BIBO) Stability

```
% MATLAB
close all, clear all, clc;
t = 0:3/60:3;
y =2*sin(2*pi*t);
y_add_offset = y+1;
stem(t,y_add_offset)
title('Bounded sequence with bound b_u=3');
xlabel('time (kT)');
ylabel('u(k)');
axis([0 3 -3 3]);
```



Nội dung bài học

- Khái niệm về tính ổn định
- **Mối quan hệ vị trí điểm cực và tính ổn định**
- Các điều kiện ổn định
- Cách xác định tính ổn định hệ thống

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Stable z-domain pole locations (1)

The sampled exponential and its z-transform

$$p^k, k = 0,1,2,\ldots \overset{\mathbb{Z}}{\leftrightarrow} \frac{z}{z-p}$$

Time sequence for large *k*

$$\left|p^k\right| = \begin{cases} 0 & |p| < 1 \\ 1 & |p| = 1 \\ \infty & |p| > 1 \end{cases}$$

Any time sequence

$$f(k) = \sum_{k=0}^{n} A_k p^k, k = 0,1,2,\ldots \overset{\mathbb{Z}}{\leftrightarrow} F(z) = \sum_{k=0}^{n} A_k \frac{z}{z-p_k} \qquad (4.6)$$

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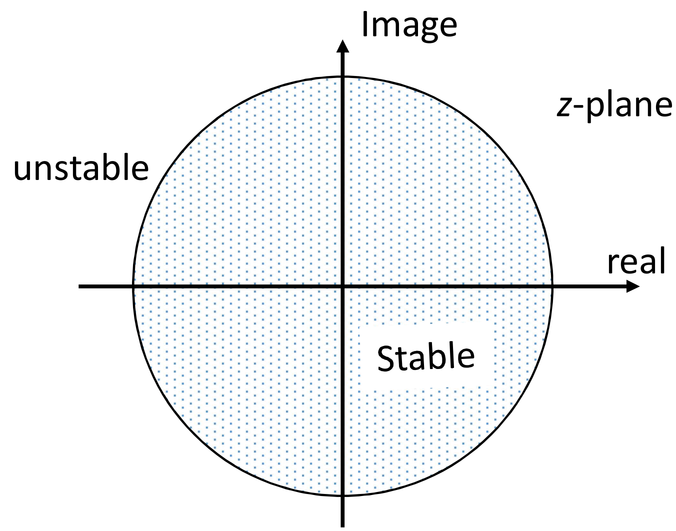
Stable z-domain pole locations (2)

Bounded sequence: if its poles lie in the closed unit disc (i.e., on or inside the unit circle) and decays exponentially if its poles lie in the open unit disc (i.e., inside the unit circle) due to any bounded input remains bounded



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Nội dung bài học

- ❑ Khái niệm về tính ổn định
- ❑ Mối quan hệ vị trí điểm cực và tính ổn định
- ❑ **Các điều kiện ổn định**
- ❑ Cách xác định tính ổn định hệ thống

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Stability conditions

- Asymptotic stability
- BIBO stability
- Internal stability

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Asymptotic stability (1)

Theorem 4.1

Asymptotic stability: In the absence of pole-zero cancellation, an LTI digital system is asymptotically stable if its transfer function poles are in the open unit disc and marginally stable if the poles are in the closed unit disc with no repeated poles on the unit circle

Proof

LTI system $y(k+N)+a_{n-1}y(k+N-1)+\cdots+a_1y(k+1)+a_0y(k)$

$$= b_m u(k+M)+b_{m-1}u(k+M-1)+\cdots+b_1u(k+1)+b_0u(k)$$

$$k = 0,1,2,\ldots$$

Điều kiện ban đầu $y(0),\ y(1),\ \ldots,\ y(N-1)$

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Asymptotic stability (2)

Proof

Using the z-transform of the output, we observe that the response of the system due to the initial conditions with the input zero

$$Y(z) = \frac{N(z)}{z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0}$$

$N(z)$ is a polynomial dependent on the initial conditions

Hàm ảnh *Y*(*z*) có các đặc điểm sau:

+ transfer function zeros arise from transforming the input terms, they have no influence on the response due to the initial conditions

+ The denominator of the output z-transform is the same as the denominator of the z-transfer function in the absence of pole-zero cancellation

+ the poles of the function Y(z) are the poles of the system transfer function

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Asymptotic stability (3)

The output due to the initial conditions is bounded for system poles in the closed unit disc with no repeated poles on the unit circle. It decays exponentially for system poles in the open unit disc (i.e., inside the unit circle)

❑ Examples

$$a)H(z)=\frac{4(z-2)}{(z-2)(z-0.1)}$$

$$b)H(z)=\frac{4(z-0.2)}{(z-0.2)(z-0.1)}$$

$$c)H(z)=\frac{5(z-0.3)}{(z-0.2)(z-0.1)}$$

$$d)H(z)=\frac{8(z-0.2)}{(z-0.1)(z-1)}$$

```
% Asymtotic Stablily
syms z;
%% Ha
Ha = 4*(z-2)/((z-2)*(z-0.1));
y = iztrans(Ha)
%% Hb
Hb = 4*(z-0.2)/((z-0.2)*(z-0.1));
y = iztrans(Hb)
%% Hc
Hc = 5*(z-0.3)/((z-0.2)*(z-0.1));
y = iztrans(Hc)
%% Hd
Hd = 8*(z-0.2)/((z-0.1)*(z-1));
y = iztrans(Hd)
```

BIBO stability

- BIBO stability concerns the response of a system to a bounded input
- The response of the system to any input is given by the convolution summation

$$y(k) = \sum_{i=0}^{k} h(k-i)u(i), \qquad k = 0, 1, 2, ... \quad (4.7)$$

h(*k*) - the impulse response sequence

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BIBO stability

- Let the impulse response of a linear system be bounded and strictly positive with lower bound bh_1 and upper bound bh_2

$$0 < bh_1 < h(k) < bh_2 < \infty$$

- Reapply to (4.7)

$$\left|y(k)\right| = \sum_{i=0}^{k} h(k-i)u(i) > \sum_{i=0}^{k} bh_1 u(i)$$

which is unbounded as k tiến đến vô cùng nếu tác động đầu vào là tín hiệu bị chặn được xác định như sau:

$$u(k) = 1, \qquad k = 0, 1, 2, 3, \ldots$$

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BIBO stability

Theorem 4.2

A discrete-time linear system is BIBO stable if and only if its impulse response sequence is absolutely summabled, that is,

$$\sum_{i=0}^{\infty} |h(i)| < \infty \qquad (4.10)$$

Proof

1. Necessity (only if) To prove necessity by contradiction

2. Sufficiency (if) To prove sufficiency, we assume that (4.10) is satisfied and then show that the system is BIBO stable

BIBO stability

Proof

1. Necessity (only if) Ta chứng minh bằng phản chứng.

Giả sử hệ thống ổn định BIBO, song không thoả mãn (4.10)

Chọn tín hiệu đầu vào có dạng

$$u(k) = \begin{cases} 1 & h(k) \geq 0 \\ -1 & h(k) < 0 \end{cases}$$

The corresponding output is

$$y(k) = \sum_{i=0}^{k} |h(i)|$$

which is unbounded as $k \to \infty$ Mâu thuẫn với giả thiết là hệ ổn định BIBO

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BIBO stability

Proof

2. Sufficiency (if)

To prove sufficiency, we assume that (4.10) is satisfied and then show that the system is BIBO stable

The inequality

$$|y(k)| = \sum_{i=0}^{k} |h(i)||u(k-i)| < b_u \sum_{i=0}^{k} |h(i)|$$

$$k = 0, 1, 2, 3, \ldots$$

Rõ ràng ta thấy $|y(k)|$ bị chặn.

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BIBO stability

Theorem 4.3

A discrete-time linear system is BIBO stable if and only if the poles of its transfer function lie inside the unit circle

Proof

1. **Necessity (only if):**

+ The impulse response and transfer function shows that the impulse response is bounded if the poles of the transfer function are in the closed unit disc and decays exponentially if the poles are in the open unit disc.

+ It has already been established that systems with a bounded impulse response that does not decay exponentially are not BIBO stable. Thus, for BIBO stability, the system poles must lie inside the unit circle

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BIBO stability

Proof

2. Sufficiency (if): assume an exponentially decaying impulse response (i.e., poles inside the unit circle). Let A_r be the coefficient of largest magnitude and $|p_s|<1$ be the system pole of largest magnitude in (4.6). Then the impulse response (assuming no repeated poles for simplicity) is bounded by

$$|h(k)| = \left|\sum_{i=1}^{n} A_i p_i^k\right| \le \sum_{i=1}^{n} |A_i||p_i|^k \le n|A_r||p_s|^k$$

$$\sum_{i=0}^{\infty} |h(k)| \le n|A_r| \sum_{i=0}^{\infty} |p_s|^k = n|A_r| \frac{1}{1-|p_s|} < \infty$$

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BIBO stability

Ví dụ 4.2: FIR

Investigate the BIBO stability of the class of systems with the impulse response

$$h(k) = \begin{cases} K & 0 \le k \le m < \infty \\ 0 & k > m \end{cases}$$

where *K* is a finite constant.

Solution

The impulse response satisfies

$$\sum_{i=0}^{\infty} |h(i)| = \sum_{i=0}^{m} |h(i)| = (m+1)|K| < \infty$$ **FIR systems are all BIBO stable**

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BIBO stability

Ví dụ 4.3

Investigate the BIBO stability

$$a) H(z) = \frac{4(z-2)}{(z-2)(z-0.1)}$$

$$b) H(z) = \frac{4(z-0.2)}{(z-0.2)(z-0.1)}$$

$$c) H(z) = \frac{5(z-0.3)}{(z-0.2)(z-0.1)}$$

$$d) H(z) = \frac{8(z-0.2)}{(z-0.1)(z-1)}$$

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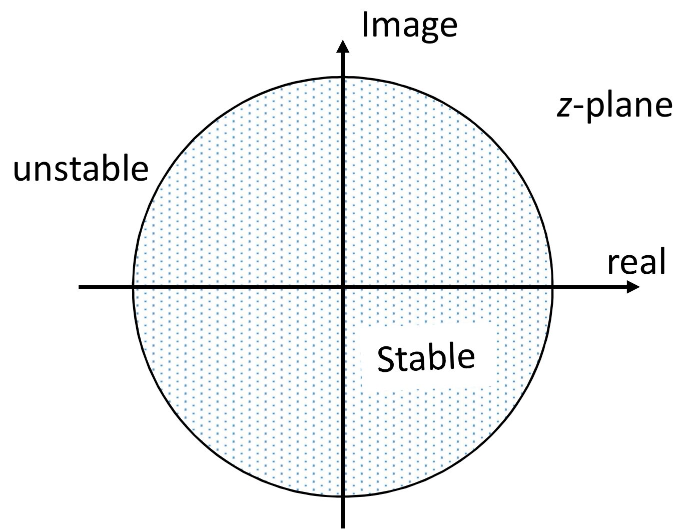
BIBO stability

Kết luận

+ BIBO and asymptotic stability are **equivalent** and can be investigated using the **same tests** (no zero-pole cancelation)



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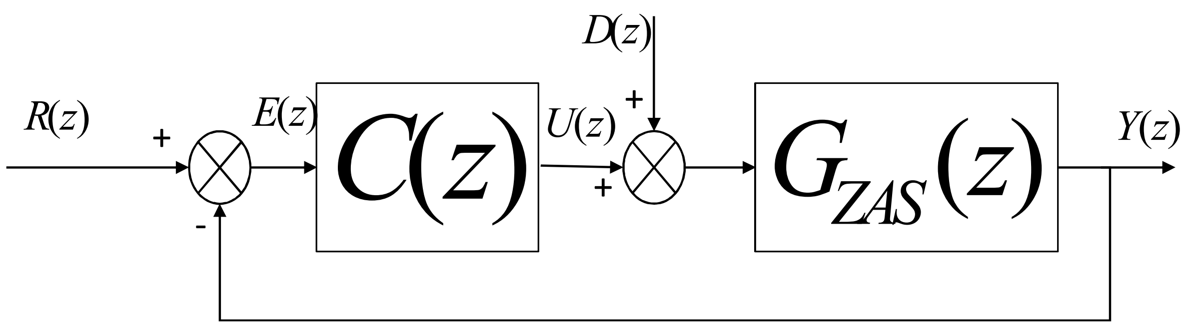
Internal stability

- the stability of the closed-loop transfer function is not always sufficient for proper system operation because some of the internal variables may be unbounded
- In a feedback control system, it is essential that all the signals in the loop be bounded when bounded exogenous inputs are applied to the system



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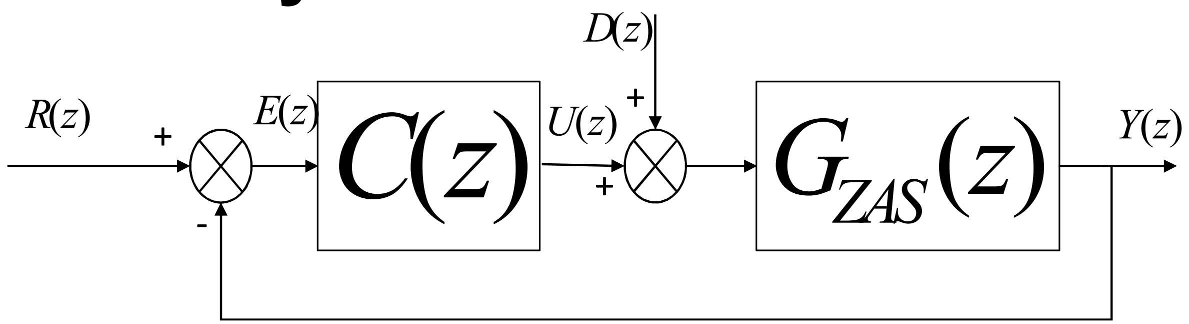
Internal stability



- ❑It is not sufficient to prove that the output of the controlled system Y is bounded for bounded reference input R because the controller output U can be unbounded.
- ❑The system output must be bounded when a different input is applied to the system namely, in the presence of a disturbance

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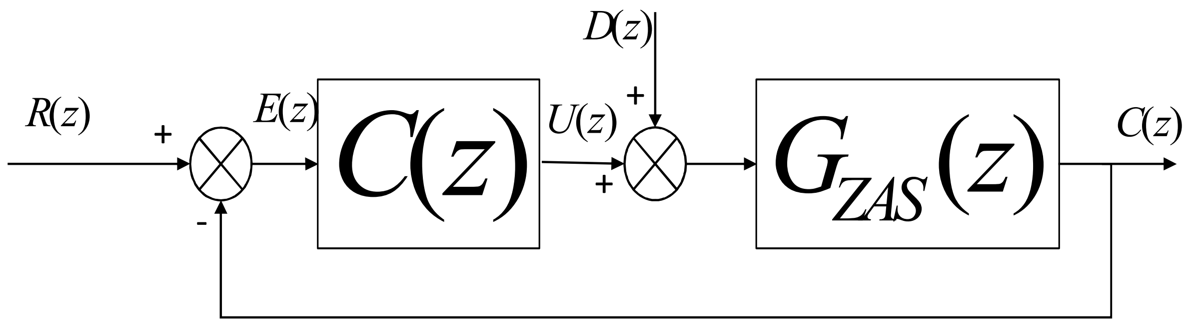
Internal stability



$$\begin{bmatrix} Y(z) \\ U(z) \end{bmatrix} = \begin{bmatrix} \dfrac{C(z)G_{ZAS}(z)}{1+C(z)G_{ZAS}(z)} & \dfrac{G_{ZAS}(z)}{1+C(z)G_{ZAS}(z)} \\ \dfrac{C(z)}{1+C(z)G_{ZAS}(s)} & \dfrac{C(z)G_{ZAS}(z)}{1+C(z)G_{ZAS}(z)} \end{bmatrix} \begin{bmatrix} R(z) \\ D(z) \end{bmatrix}$$

Internal stability

Internal stability: If all the transfer functions that relate system inputs (R and D) to the possible system outputs (Y and U) are BIBO stable, then the system is said to be internally stable.

- The stability of the closed-loop transfer function is not always sufficient for proper system operation because some of the internal variables may be unbounded
- In a feedback control system, it is essential that all the signals in the loop be bounded when bounded exogenous inputs are applied to the system

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Internal stability

Theorem 4.3

The system is internally stable if and only if all its closed-loop poles are in the open unit disc.

Proof

1. **Necessity (only if)** we write $C(z)$ and $G_{ZAS}(z)$

$$C(z) = \frac{N_C(z)}{D_C(z)}; \qquad G_{ZAS}(z) = \frac{N_G(z)}{D_G(z)}$$

$$\begin{bmatrix} Y \\ G \end{bmatrix} = \frac{1}{D_C D_G + N_C N_G} \begin{bmatrix} N_C N_G & D_C D_G \\ N_C N_G & -N_C N_G \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

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Internal stability

Theorem 4.4

The system is internally stable if and only if all its closed-loop poles are in the open unit disc.

Proof

2. Sufficiency (if)

It is evident that if the characteristic polynomial $D_C D_G + N_C N_G$ has no zeros on or outside the unit circle, then all the transfer functions are asymptotically stable and the system is internally stable

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Internal stability

Theorem 4.5

The system is internally stable if and only if the following two conditions hold:

1. The characteristic polynomial $1 + C(z)G_{ZAS}(z)$ has no zeros on or outside the unit circle.

2. The loop gain $C(z)G_{ZAS}(z)$ has no pole-zero cancellation on or outside the unit circle

Proof

1. Necessity (only if)

2. Sufficiency (if)

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Internal stability

Proof

❑Necessity (only if)

+ Condition 1 is clearly necessary by Theorem 4.4.
+ Necessity of condition 2 (chứng minh phản chứng)

$$C(z) = \frac{N_C(z)}{D_C(z)}; \qquad G_{ZAS}(z) = \frac{N_G(z)}{D_G(z)}$$

$$\begin{bmatrix} Y \\ G \end{bmatrix} = \frac{1}{D_C D_G + N_C N_G} \begin{bmatrix} N_C N_G & D_C D_G \\ N_C N_G & -N_C N_G \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

the characteristic polynomial $D_C D_G + N_C N_G$

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Internal stability

Proof

❑Necessity (only if)

the characteristic polynomial $D_C D_G + N_C N_G$

We assume that there exists which is a zero of $D_C D_G$ as well as a zero of $N_C N_G$

$$Z_0, \quad |Z_0| \geq 1$$

Then clearly Z_0 is also a zero of the characteristic polynomial $D_C D_G + N_C N_G$, and the system is unstable

This establishes the necessity of condition 2

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Internal stability

Proof

❑ **Sufficiency (if)**

+ By Theorem 4.4, condition 1 implies internal stability unless unstable pole-zero cancellation occurs in the characteristic polynomial $1 + C(z)G_{ZAS}(z)$. We therefore have internal stability if condition 2 implies the absence of unstable pole-zero cancellation.

+ If the loop gain $C(z)G_{ZAS}(z) = N_C N_G / D_C D_G$ has no unstable pole-zero cancellation, then $1+C(z)G_{ZAS}(z)=[D_C D_G + N_C N_G]/D_C D_G$ does not have unstable pole-zero cancellation, and the system is internally stable.

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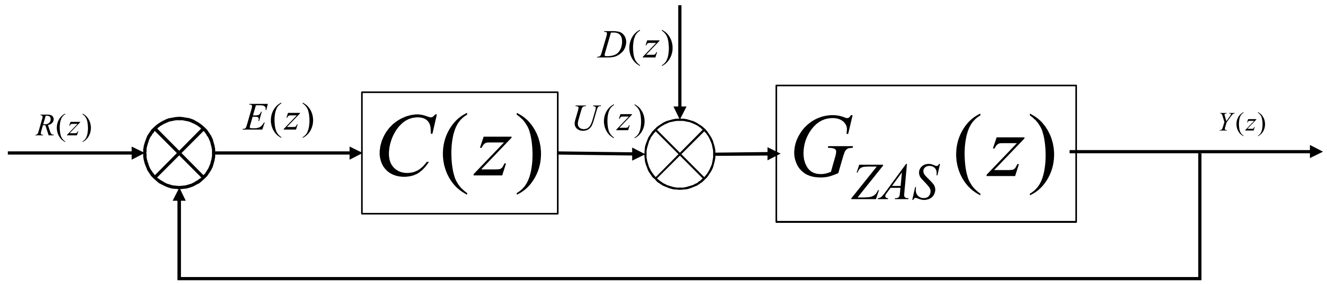
Stability Condition

❑Example 4.3



$$C(z) = \frac{-10(z-0.8149)(z-0.7655)}{(z-1)(z-1.334)} \qquad G(s) = \frac{0.5848(-0.3549s+1)}{0.1828s^2+0.8627+1}$$

1. Xác định $G_{ZAS}(z)$ với T = 0.1

2. Kiểm tra tính ổn định internal stability của hệ thống

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Stability Condition

❑Example 4.3

1. Xác định $G_{ZAS}(z)$ với T = 0.1 $\quad G(s) = \dfrac{0.5848(-0.3549s+1)}{0.1828s^2+0.8627+1}$

$$G_{ZAS}(z) = (1-z^{-1})\mathbb{Z}\left\{L^{-1}\left[\frac{G(s)}{s}\right]\right\} = \frac{-0.075997(z-1.334)}{(z-0.8149)(z-0.7655)}$$

$$\frac{Y(z)}{R(z)} = \frac{C(z)G_{ZAS}(z)}{1+C(z)G_{ZAS}(z)} = \frac{0.75997}{z-0.24}$$

$$\frac{U(z)}{R(z)} = \frac{C(z)}{1+C(z)G_{ZAS}(z)} = \frac{-10(z-0.8149)(z-0.7655)}{(z-0.24)(z-1.334)}$$

Condition 2 of Theory 4.5

$$C(z)G_{ZAS}(z) = \frac{-10(z-0.8149)(z-0.7655)}{(z-1)(z-1.334)} \times \frac{-0.075997(z-1.334)}{(z-0.8149)((z-0.7655)}$$

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- ❑ **Cách xác định tính ổn định hệ thống**

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Stability determination

- ❑Using Numerical Method (MATLAB)
- ❑Tiêu chuẩn Routh-Hurwitz
- ❑Tiêu chuẩn Jury
- ❑Tiêu chuẩn Nyquist

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Stability determination

- ❑ **Using Numerical Method (MATLAB)**
- ❑ Tiêu chuẩn Routh-Hurwitz
- ❑ Tiêu chuẩn Jury
- ❑ Tiêu chuẩn Nyquist

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Using Numerical Method (MATLAB)

```
clear all, close all, clc
pole(gd);      % if using transfer function
roots(gd.den); % using polynomial
abs(pole(gd)   % Calculate magnitude of all roots
abs(roots(gd)) % Calculate magnitude of all roots
```

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Stability determination

- ❑Using Numerical Method (MATLAB)
- ❑**Tiêu chuẩn Routh-Hurwitz**
- ❑Tiêu chuẩn Jury
- ❑Tiêu chuẩn Nyquist

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Tiêu chuẩn Routh-Hurwitz

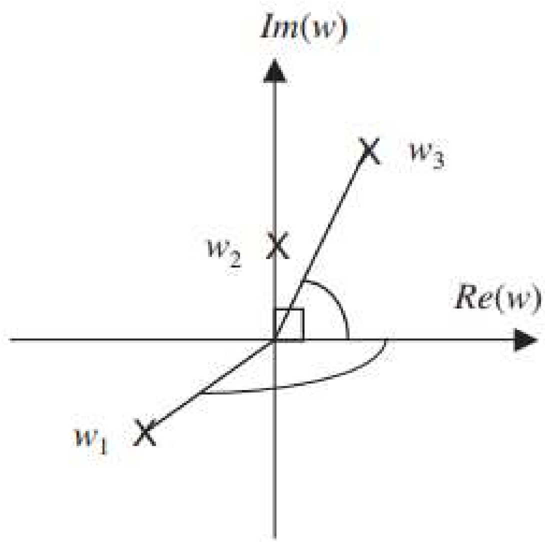
- ❑Phép biến đổi song tuyến tính
- ❑Tiêu chuẩn ổn định đại số

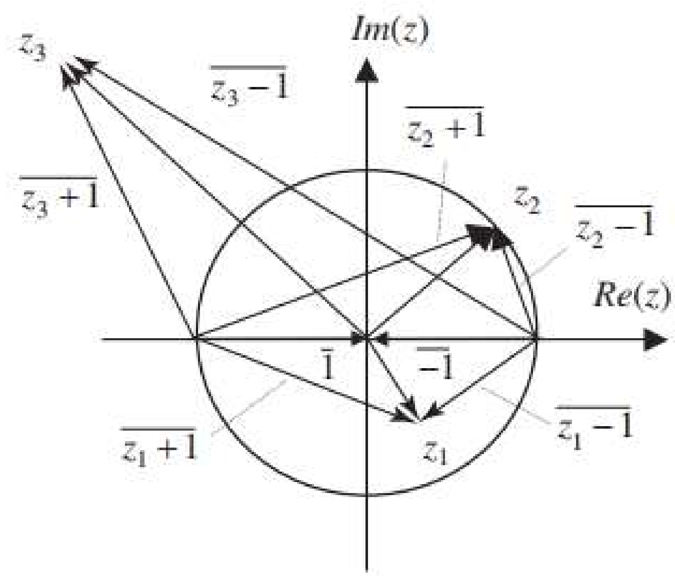
Phép biến đổi song tuyến tính

$$z = \frac{1+\mathrm{w}}{1-\mathrm{w}} \Leftrightarrow \mathrm{w} = \frac{z-1}{z+1}$$

The angle of w after bilinear transformation is

$$\angle(\mathrm{w}) = \angle(z-1) - \angle(z+1)$$





Tiêu chuẩn ổn định Routh-Hurwitz

- For the general z-polynomial, we have the transform pair

$$\boxed{F(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}$$

$$z = \frac{1+\mathrm{w}}{1-\mathrm{w}} \qquad \mathrm{w} = \frac{z-1}{z+1}$$

$$\boxed{F(\mathrm{w}) = a_0\left(\frac{1+\mathrm{w}}{1-\mathrm{w}}\right)^n + a_1\left(\frac{1+\mathrm{w}}{1-\mathrm{w}}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{1+\mathrm{w}}{1-\mathrm{w}}\right) + a_n}$$

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Tiêu chuẩn ổn định Routh-Hurwitz

- The Routh-Hurwitz approach becomes progressively more difficult as the order of the z-polynomial increases
- For high-order polynomials, a symbolic manipulation package can be used to perform the necessary algebraic manipulations

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Tiêu chuẩn ổn định Routh-Hurwitz

❑Ví dụ 4.5: Xác định tính ổn định

1. Đa thức bậc nhất $f(z) = a_1 z + a_0, \quad a_1 > 0$
2. Đa thức bậc hai $f(z) = a_2 z^2 + a_1 z + a_0, \quad a_2 > 0$

Lời giải: Tìm nghiệm của phương trình và xác định điều kiện để biên độ <1

1. Đa thức bậc nhất $z_1 = \dfrac{a_0}{a_2}, \qquad |z| < 1 \Leftrightarrow \left|\dfrac{a_0}{a_1}\right| < 1$
2. Đa thức bậc hai $z_{1,2} = \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_1}$

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Tiêu chuẩn ổn định Routh-Hurwitz

Using bilinear transformation: for 2nd order equation

$$a_2\left(\frac{1+w}{1-w}\right)^2 + a_1\left(\frac{1+w}{1-w}\right) + a_0$$

$$\frac{a_2(1+w)^2 + a_1(1+w)(1-w) + a_0(1-w)^2}{(1-w)^2}$$

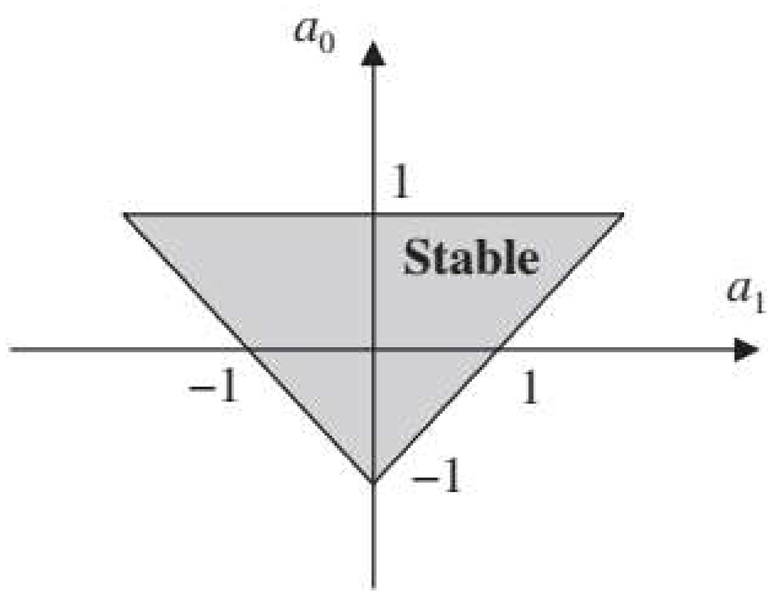
$$\frac{(a_2 - a_1 + a_0)w^2 + (2a_2 - 2a_0)w + (a_2 + a_1 + a_0)}{(1-w)^2}$$

Apply Routh- Hurwitz for w

$$a_2 - a_1 + a_0 > 0 \qquad a_2 + a_0 > 0 \Leftrightarrow a_0 > -a_2$$

$$a_2 - a_0 > 0 \qquad -a_0 < a_2$$

$$a_2 + a_1 + a_0 > 0$$



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Stability determination

- Using Numerical Method (MATLAB)
- Tiêu chuẩn Routh-Hurwitz
- Tiêu chuẩn Jury
- Tiêu chuẩn Nyquist

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Tiêu chuẩn ổn định Jury

- It is possible to investigate the stability of z-domain polynomials directly
  - Jury test for real coefficients
  - Schur-Cohn test for complex coefficients
- The Jury test Procedure

Theorem 4.6

The Jury test Procedure

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Jury Test Procedure

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

the roots of the polynomial are inside the unit circle if and only if

(1) F(1)>0

(2) $(-1)^n F(-1) > 0$

(3) $|a_0| < |a_n|$

(4) $|b_0| > |b_{n-1}|$

(5) $|c_0| > |c_{n-2}|$

•

•

$(n+1)$ $|r_0| > |r_2|$

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Jury Test Procedure

❑Jury table

Row	z^0	z^1	z^2	…	z^{n-k}	…	z^{n-1}	z^n
1	a_0	a_1	a_2	…	a_{n-k}	…	a_{n-1}	a_n
2	a_n	a_{n-1}	a_{n-2}	…	a_k	…	a_1	a_0
3	b_0	b_1	b_2	…	b_{n-k}	…	b_{n-1}	
4	b_{n-1}	b_{n-2}	b_{n-3}	…	b_k	…	b_0	
5	c_0	c_1	c_2	…	…	c_{n-2}		
6	c_{n-2}	c_{n-3}	c_{n-4}	…	…	c_0		
.	.	.	.	…	…			
.	.	.	.	…	…			
.	.	.	.	…	…			
$2n-5$	s_0	s_1	s_2	s_3				
$2n-4$	s_3	s_2	s_1	s_0				
$2n-3$	r_0	r_1	r_2					

Jury Test Procedure

- The entries of the table are calculated as follows:

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \qquad k = 0,1,2,\cdots n-1$$

$$c_k = \begin{vmatrix} b_0 & b_{n-k-1} \\ b_{n-1} & b_k \end{vmatrix}, \qquad k = 0,1,2,\cdots n-2$$

.

.

$$r_0 = \begin{vmatrix} s_0 & s_3 \\ s_3 & s_0 \end{vmatrix}, r_1 = \begin{vmatrix} s_0 & s_2 \\ s_2 & s_1 \end{vmatrix}, r_2 = \begin{vmatrix} s_0 & s_1 \\ s_3 & s_2 \end{vmatrix}$$

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Jury Test Procedure

- The first row of the Jury table is a listing of the coefficients of the polynomial F(z) in order of increasing power of z.
- The number of rows of table 2 n3 is always odd, and the coefficients of each even row are the same as the odd row directly above it with the order of the coefficients reversed
- There are n + 1 conditions in (4.22) that correspond to the n+1 coefficients of F(z)
- Condition 3 through n + 1 of (4.22) are calculated using the coefficient of the first column of the Jury table together with the last coefficient of the preceding row. The middle coefficient of the last row is never used and need not be calculated
- Conditions 1 and 2 of (4.22) are calculated from F(z) directly. If one of the first two conditions is violated, we conclude that F(z) has roots on or outside the unit circle without the need to construct the Jury table or test the remaining conditions

Jury Test Procedure

- Condition 3 of (4.22), with $a_n = 1$, requires the constant term of the polynomial to be less than unity in magnitude. The constant term is simply the product of the roots and must be smaller than unity for all the roots to be inside the unit circle.
- Condition (4.22) reduce to conditions (4.19) and (4.20) for first and second-order systems, respectively, where the Jury table is simply one row
- For higher-order systems, applying the Jury test by hand is laborious, and it is preferable to test the stability of a polynomial F(z) using a computer-aided design (CAD) package
- If the coefficients of the polynomial are functions of system parameters, the Jury test can be used to obtain the stable ranges of the system parameters

Jury Test Procedure

❑Example 4.6

$$F(z) = z^5 + 2.6z^4 - 0.56z^3 - 2.05z^2 + 0,0775z + 0.35 = 0$$

Row	z^0	z^1	z^2	z^3	z^4	z^5
1	0.35	0.0775	−2.05	−0.56	2.6	1
2	1	2.6	−0.56	−2.05	0.0775	0.35
3	−0.8775	−2.5729	−0.1575	1.854	0.8325	
4	0.8325	1.854	−0.1575	−2.5729	−0.8775	
5	0.0770	0.7143	0.2693	0.5151		
6	0.5151	0.2693	0.7143	0.0770		
7	−0.2593	−0.0837	−0.3472			

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Jury Test Procedure

❑Example 4.6

The first two conditions require the evaluation of $F(z)$ at $z = \pm 1$.

1. $F(1) = 1 + 2.6 - 0.56 - 2.05 + 0.0775 - 0.35 = 1.4175 > 0$
2. $(-1)^5 F(-1) = (-1)\,(-1 + 2.6 + 0.56 - 2.05 - 0.0775 + 0.35) = -0.3825 < 0$
   Conditions 3 through 6 can be checked quickly using the entries of the first column of the Jury table.
3. $|0.35| < 1$
4. $|-0.8775| > |0.8325|$
5. $|0.0770| < |0.5151|$
6. $|-0.2593| < |-0.3472|$

$$F(z) = (z-0.7)(z-0.5)(z+0.5)(z+0.8)(z+2.5)$$

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Jury Test Procedure

❑Example 4.7

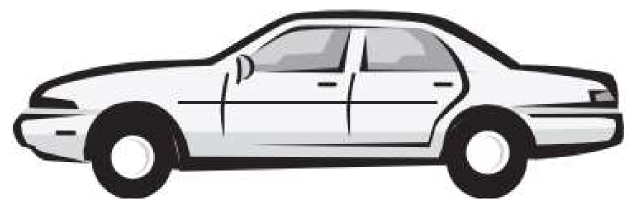
Find the stable range of the gain *K* for the unity feedback digital cruise control system

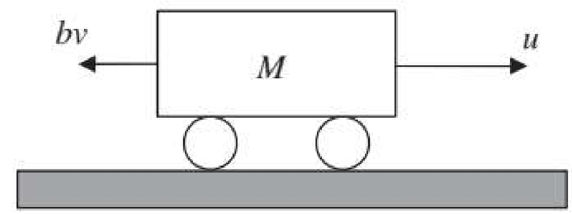
$$G(s) = \frac{K}{s+3}$$

DAC and ADC if the sampling period is 0.02s

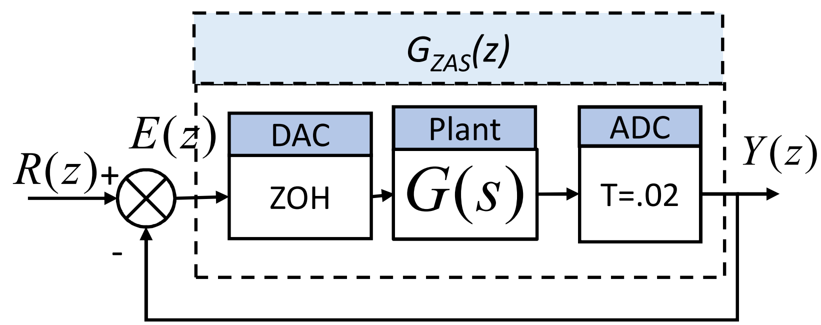
a. Find the transfer function for analog subsystem ADC and DAC

b. Find the stable range of the gain K





$$M\dot{v}(t) + bv(t) = u(t)$$



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Jury Test Procedure

❑Example 4.7

a. The transfer function for analog subsystem ADC and DAC is

$$G_{ZAS}(s) = (1-z^{-1})\mathbb{Z}\left\{L^{-1}\left[\frac{G(s)}{s}\right]\right\} = (1-z^{-1})\mathbb{Z}\left\{L^{-1}\left[\frac{K}{s(s+3)}\right]\right\}$$

$$\frac{K}{s(s+3)} = \frac{K}{3}\left(\frac{1}{s} - \frac{1}{s+3}\right)$$

$$G_{ZAS}(z) = \frac{1.9412\times10^{-2}K}{z-0.9418}$$

b. Find stable Range of *K*

$$p_{cl}(z) = 1 + G_{ZAS}(z) = z - 0.9418 + 1.9412\times10^{-2}K = 0$$

$$-3 < K < 100.3$$

```
clear all, close all, clc;
s = tf('s');
sys = K/(s+3);
T = 0.02;
sys_d  = c2d(sys,T,'zoh')
```

Jury Test Procedure

❑Example 4.8

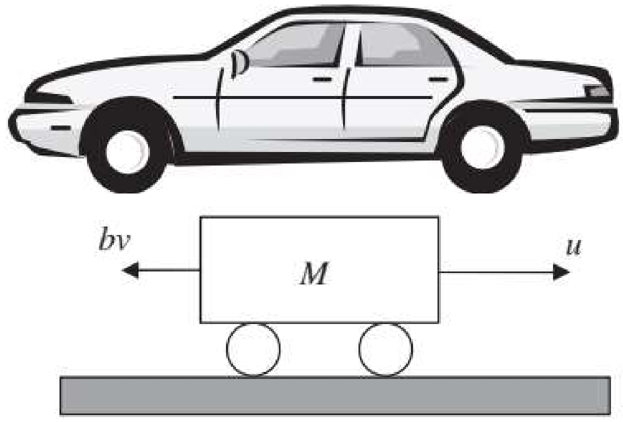
Find the stable range of the gain K for the unity feedback digital cruise control system

$$G(s) = \frac{10K}{s(s+10)}$$

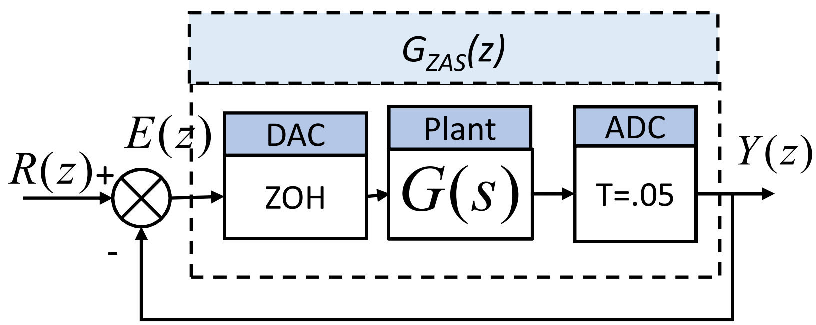
DAC and ADC if the sampling period is 0.05s

a. Find the transfer function for analog subsystem ADC and DAC

b. Find the stable range of the gain K



$$M\ddot{y}(t) + b\dot{y}(t) = u(t)$$



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Jury Test Procedure

Example 4.4

a. The transfer function for analog subsystem ADC and DAC is

$$G_{ZAS}(s) = (1-z^{-1})\mathbb{Z}\left\{L^{-1}\left[\frac{G(s)}{s}\right]\right\} = (1-z^{-1})\mathbb{Z}\left\{L^{-1}\left[\frac{10K}{s^2(s+10)}\right]\right\}$$

$$\frac{10K}{s^2(s+10)} = \frac{K}{10}\left(\frac{10}{s^2} - \frac{1}{s} + \frac{1}{s+10}\right)$$

$$G_{ZAS}(z) = \frac{1.0653\times 10^{-2} K(z+0.8467)}{(z-1)(z-0.6065)}$$

```
clear all, close all, clc;
s = tf('s');
K=1
sys = 10*K/(s*(s+3));
T = 0.02;
sys_d  = c2d(sys,T,'zoh')
```

Jury Test Procedure

❑**Example 4.8**

b. Find stable Range of *K*

$$p_{cl}(z) = 1 + G_{ZAS}(z)$$

$$p_{cl}(z) = (z-1)(z-0.6065) + 1.0653\times10^{-2}K(z+0.8467)$$

$$p_{cl}(z) = z^2 + (1.0653\times10^{-2}K - 1.6065)z + 9.02\times10^{-2}K$$

1. $F(1) = 1 + (1.0653\times10^{-2}K - 1.6065) + 0.6065 + 9.02\times10^{-3}K > 0 \Leftrightarrow K > 0$
2. $F(-1) = 1 - (1.0653\times10^{-2}K - 1.6065) + 0.6065 + 9.02\times10^{-3}K > 0 \Leftrightarrow K < 1967.582$
3. $|0.6065 + 0.0902\,K| < 1 \Leftrightarrow +(0.6065 + 0.0902\,K) < 1 - (0.6065 + 0.0902\,K) < 1$

$$-178.104 < K < 43.6199$$

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Stability determination

- ❑Using Numerical Method (MATLAB)
- ❑Tiêu chuẩn Routh-Hurwitz
- ❑Tiêu chuẩn Jury
- ❑**Tiêu chuẩn Nyquist**

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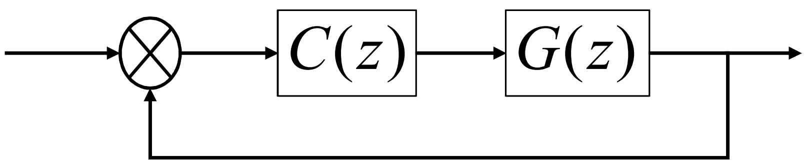
Tiêu chuẩn Nyquist

The Nyquist criterion allows us to answer two questions:
1. Does the system have closed-loop poles outside the unit circle?
2. If the answer to the first question is yes, how many closed-loop poles are outside the unit circle?

The Nyquist criterion is particularly useful in situations where the frequency response can be obtained ***experimentally*** and used to obtain a polar or Nyquist plot.



The closed-loop characteristic polynomial

$$p_{cl}(z) = 1 + C(z)G(z) = 1 + L(z)$$

$$p_{cl}(z) = 1 + \frac{N_L(z)}{D_L(z)} = \frac{N_L(z) + D_L(z)}{D_L(z)}$$

1. zeros of the $p_{cl}(z)$ are the closed-loop poles
2. poles of the $p_{cl}(z)$ are the open-loop poles

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Tiêu chuẩn Nyquist

- **Problem Statement:**

We assume that we are given the number of open-loop poles outside the unit circle (P). We need to find the number of closed-loop poles outside the unit circle (Z)

- **Nguyên lý góc quay**
- **Tiêu chuẩn Nyquist**
- **Phase and magnitude margin**

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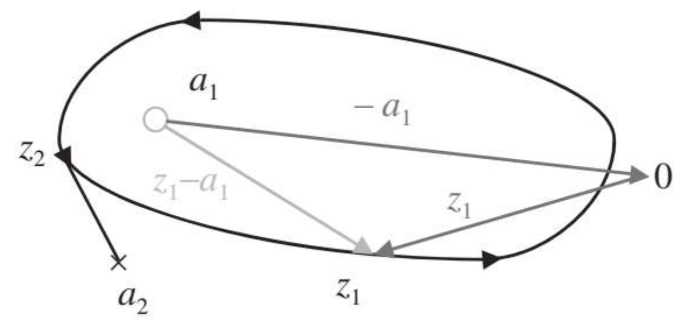
Nguyên lý góc quay

Contour: A contour is a closed directed simple (does not cross itself) curve

1. In the complex plane the vector connecting any point *a* to a point *z* is the vector (*z*-*a*)
2. The net angle change for the vector (*z*-*a*) as the point *z* traverses the contour in the shown (counterclockwise) direction by determining the net number of rotations of the corresponding vector
3. The point a_1 - which is inside the contour- that the net rotation is one full turn (2π radians)
4. The point a_2 - which is outside the contour- The net rotation is zero (0 radians)
5. If the point in question corresponds to a zero, then the rotation gives a numerator angle; if it is a pole, we have a denominator angle



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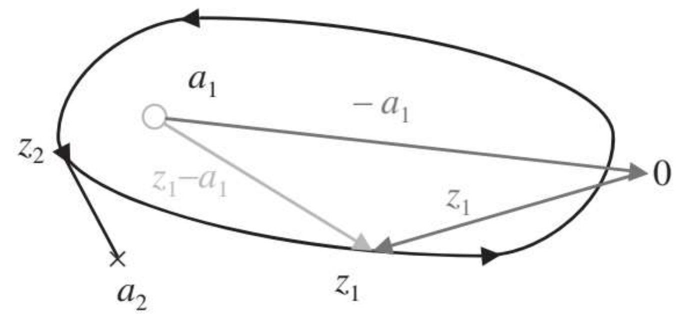
Nguyên lý góc quay

- The net angle change for a rational function is the change in the angle of the numerator minus the change in the angle of the denominator

we have one counterclockwise rotation because of a_1 and no rotation as a result of a_2 for a net angle change of one counterclockwise rotation. Angles are typically measured in the counterclockwise direction and we count clockwise rotations as negative



Re-statements: how to determine the number of zeros of a rational function in a specific region; given the number of poles

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Nguyên lý góc quay

- Select a closed contour surrounding the region.
- Compute the net angle change for the function as we traverse the contour once.
- The net angle change or number of counterclockwise rotations N is equal to the number of closed-loop poles inside the contour Z_{in} minus the angle of open-loop poles inside the contour P_{in}, that is, the number of counterclockwise encirclements of the origin is given by

$$N = Z_{in} - P_{in}$$

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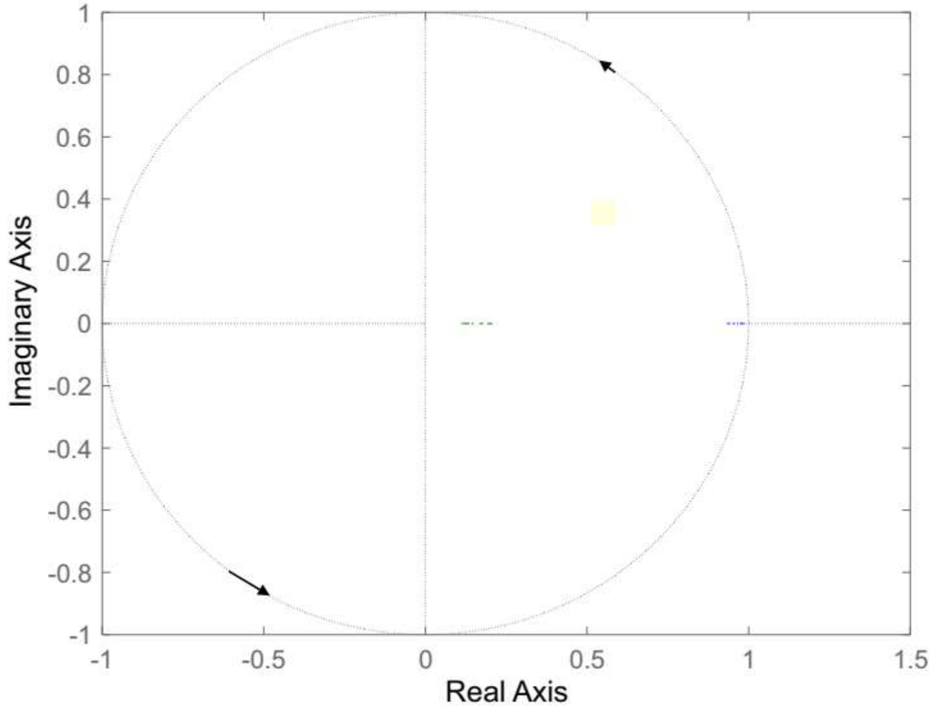
Tiêu chuẩn ổn định Nyquist

- To use this to determine closed-loop stability, we need to count the closed-loop poles outside the unit circle.(an nth order system)

$$N = Z_{in} - P_{in} = (n - Z) - (n - P)$$
$$Z = -N + P$$

- The stability contour is the unit circle and is traversed in the counterclockwise direction



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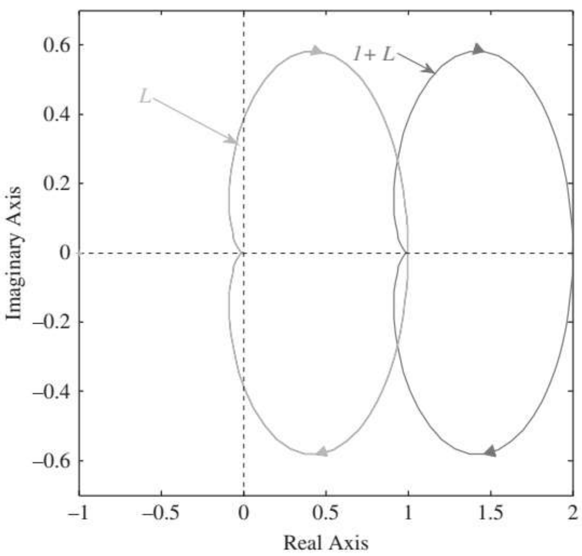
Tiêu chuẩn ổn định Nyquist

- The value of the loop gain on the unit circle is $L(e^{j\omega T})$ $(\omega T = [-\pi,\pi])$
- Because the order of the numerator is equal to that of the denominator or less, points on the large circle map to zero or to a single constant value
- Note that **traversing the contour counterclockwise implies increasing ω** and following the direction typically shown on Nyquist plots



We can simplify the test by plotting $L(e^{j\omega T})$ as we traverse the contour and then counting its encirclements of the point (-1, j0)

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Tiêu chuẩn ổn định Nyquist

Theorem 4.7

Let the number of counterclockwise encirclements of the point (1, j0) for a loop gain *L(z)* when traversing the stability contour be *N* (i.e., *N* for clockwise encirclements), where *L(z)* has *P* open-loop poles inside the contour. Then the system has *Z* closed-loop poles outside the unit circle with *Z* given by

$$Z = (-N) + P$$

Hệ quả

An open-loop stable system is closed-loop stable if and only if its Nyquist plot does not encircle the point (1, j0) (i.e., if *N = 0*)

Tiêu chuẩn ổn định Nyquist

Theorem 4.7

- Although counting encirclements appears complicated, it is actually quite simple using the following recipe:
  - Starting at a distant point, move toward the point (1, 0).
  - Count all lines of the stability contour crossed. Count each line with an arrow pointing from your left to your right as negative and every line with an arrow pointing from your right to your left as positive.
  - The net number of lines counted is equal to the number of clockwise encirclements of the point (1, 0).

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Tiêu chuẩn ổn định Nyquist

- **Special Case:** Open-loop system has poles on unit Circle
  => the contour passes through poles and the test fails
  We modify the contour to avoid these open-loop poles

The contour includes an additional circular arc of infinitesimal radius

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Tiêu chuẩn ổn định Nyquist

The most common case is a pole at unity

$$L(z) = \frac{N_L(z)}{(z-1)^m D(z)}$$

The value of the transfer function on the circular arc is approximately given by

$$L(z)\big|_{z\to 1+\varepsilon e^{j\theta}} = \frac{K}{\varepsilon e^{jm\phi}}, \qquad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad K = \frac{N_L(1)}{D(1)}, \qquad (z-1)^m = e^{jm\theta}$$

A net denominator angle change of mπ radians (counterclockwise) (i.e., m half circles). The net angle change for the quotient on traversing the small circular arc is thus mπ radians (clockwise). For a type *m* system, the Nyquist contour will include *m* large clockwise semicircles

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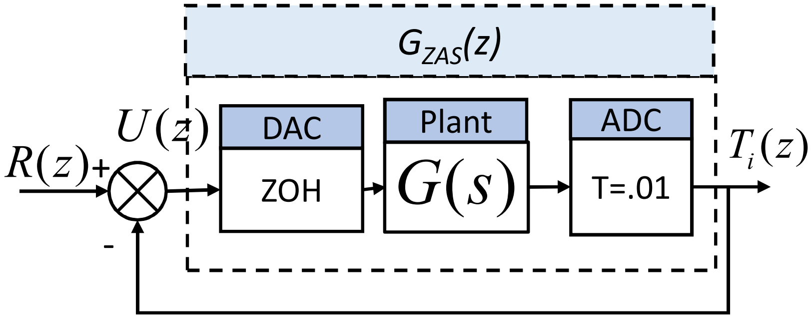
Tiêu chuẩn ổn định Nyquist

❑Example 4.9

A linearized model of a furnace

In heating phase $$G(s) = \frac{1}{s^2 + 3s + 1}$$

$$G(s) = \frac{T_i(s)}{U(s)} = \frac{K g_{rw} g_{iw}}{s^2 + (2g_{iw} + g_{rw})s + g_{rw} g_{iw}}$$



$$G_{ZAS}(z) = 10^{-5} \frac{4.95z + 4.901}{z^2 - 1.97z + 0.9704}$$

```
clear all, close all, clc;
s = tf('s');
sys = 1/(s^2+3*s+1);
T = 0.01;
sys_d = c2d(sys,T,'zoh')
```

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Tiêu chuẩn ổn định Nyquist

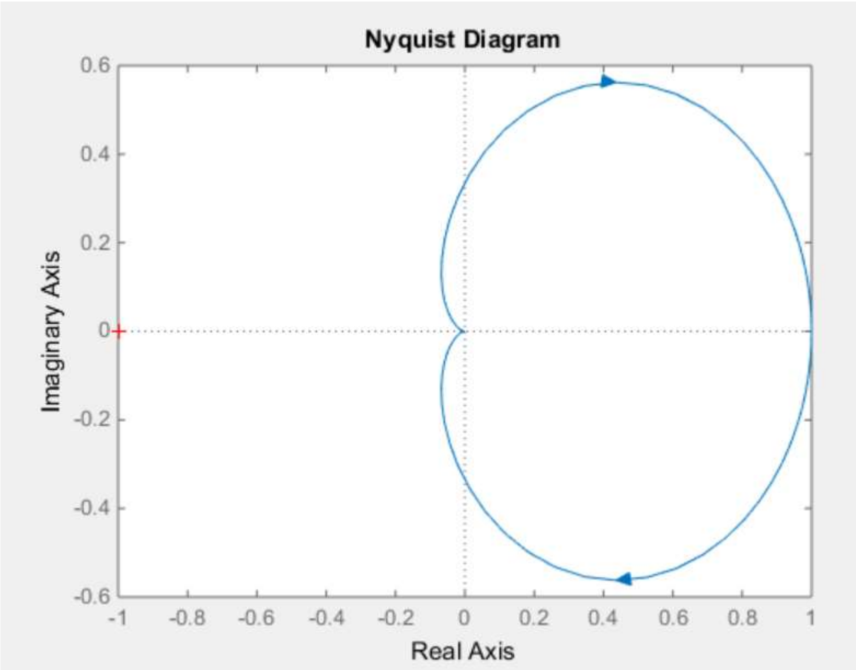
❑Example 4.9

+ N = 0 - the Nyquist plot does not encircle the point (1, j0)

+ P = 0 - the open-loop transfer function has no unstable poles

Z = 0 - the system is closed-loop **stable**

```
nyquist(sys_d)
```



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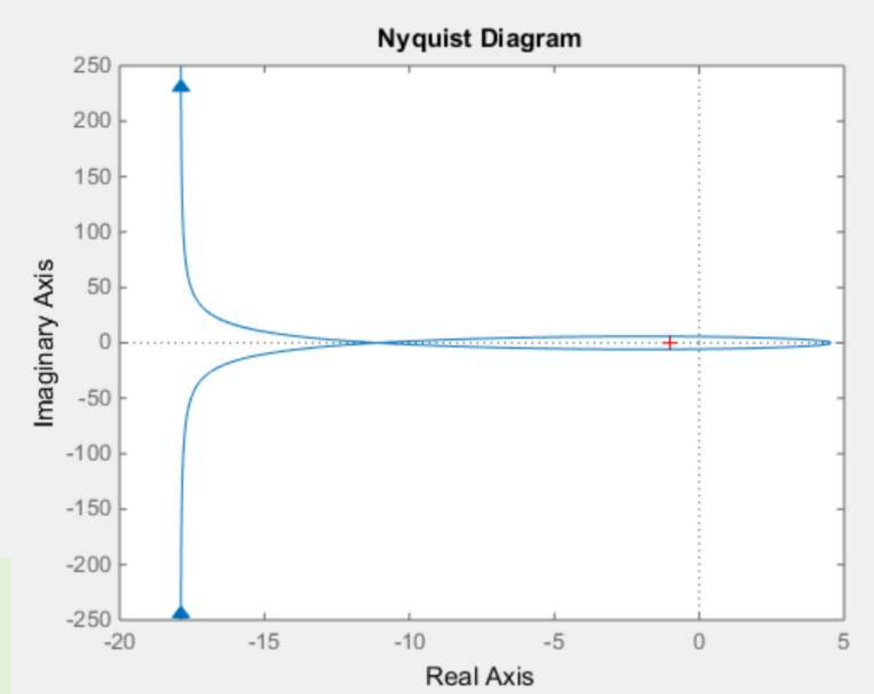
Tiêu chuẩn ổn định Nyquist

Example 4.10: The stability of the type-1 transfer function

$$G(z)=\frac{10}{(z-1)(z-0.1)}$$

```
clear all, close all, clc;
T = 0.01;
sys_d = tf(10,[1 -1.1 1],T)
nyquist(sys_d)
```



+ The Nyquist contour has two clockwise encirclements (N = -2)

Z = 2 - the system is closed-loop **unstable**

23

Phase and Gain Margin

- the stability of a mathematical model is not sufficient to guarantee acceptable system performance

  ➡ We therefore need to determine how far the system is from instability. This degree of stability is known as **relative stability**

- we restrict it to open-loop stable systems where zero encirclements guarantee stability. For open-loop stable systems that are nominally closed-loop stable, the distance from instability can be measured by the distance between the set of points of the Nyquist plot and the point (1, j0)
- The distance = magnitude distance + angular distance

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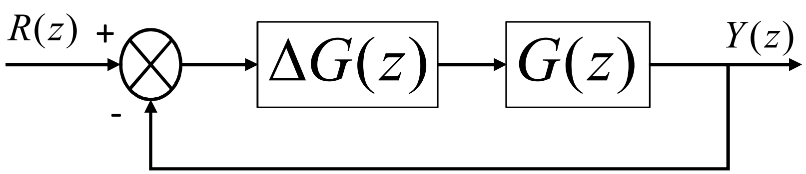
Phase and Gain margin

Gain margin: **The gain margin is the gain perturbation that makes the system marginally stable.**

Phase margin: **The phase margin is the negative phase perturbation that makes the system marginally stable**



Gain disturbance $\Delta G(z) = \Delta K$

Phase disturbance $\Delta G(z) = e^{-j\Delta\theta}$

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Phase and Gain Margin

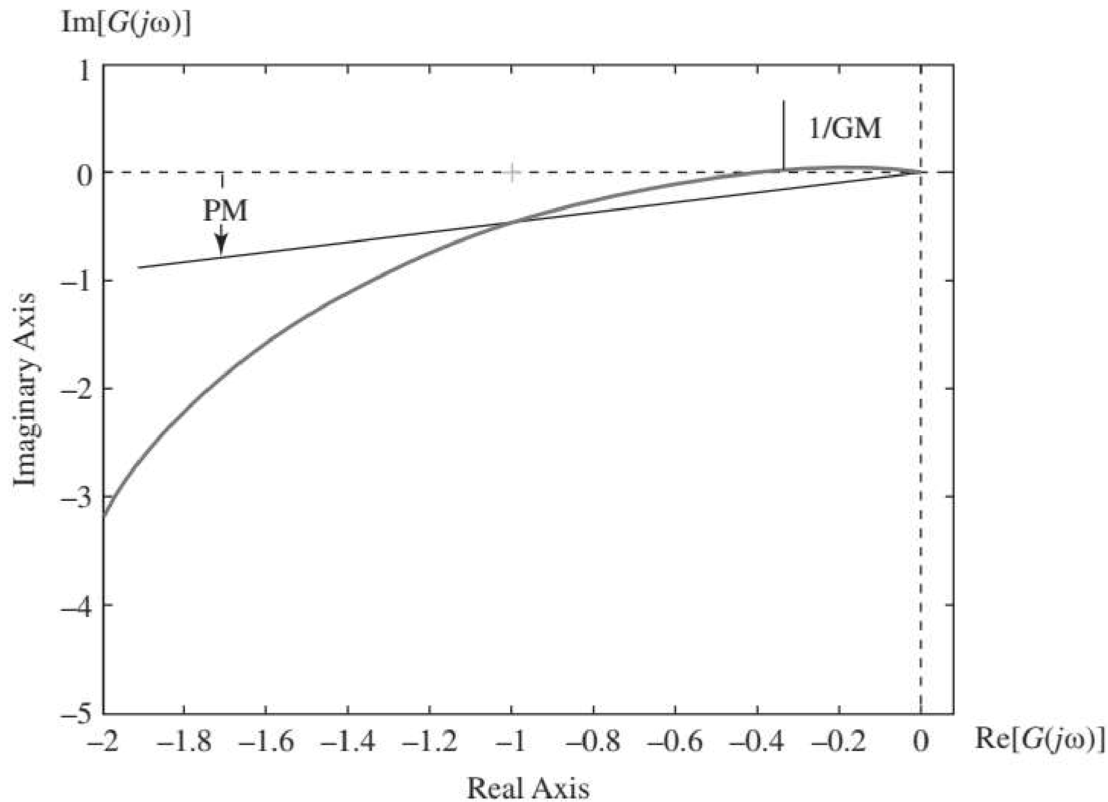
The MATLAB commands for obtaining frequency response plots

```
nyquist(gd)  % Nyquist Plot
bode(gd)     % Bode Plot
```

Gain and Phase margins

```
[GM, PM] = margin(gd); % find Margin
```

On Nyquist and Bode plot, can select option



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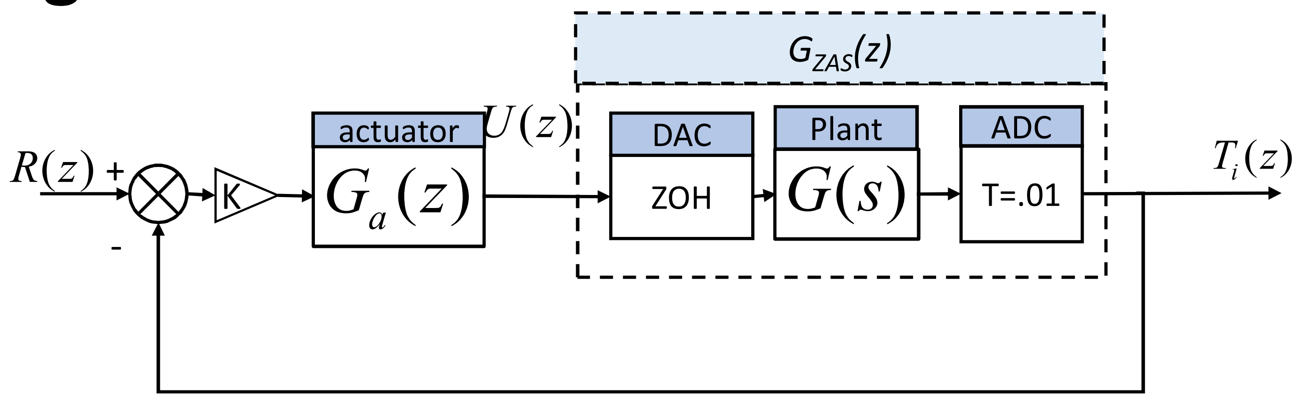
Phase and Gain margin

❑ **Example 4.11**

$$G(s) = \frac{1}{s^2 + 3s + 1}$$



$$G_{ZAS}(z) = 10^{-5}\frac{4.95z + 4.901}{z^2 - 1.97z + 0.9704} \qquad G_a(z) = \frac{0.9516}{z - 0.9048} \qquad K = 1 \div 5$$

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Phase and Gain margin

❑Example 4.11

```
clear all; close all; clc;
T = 0.01;
z = tf('z',T);
K=1;
Ga_d  = 0.9516/(z-0.9048);
Gzas_d = 10^-5*(4.95*z+4.901)/(z^2-1.97*z+0.9704)
L_d = K*Ga_d*Gzas_d;
[GM,PM] = margin(L_d)
```

```
GM =

   3.5699


PM =

  46.3442
```

$$L(z) = G_a(z)G_{ZAS}(z) = 10^{-5}\frac{4.711z + 4.644}{z^3 - 2.875z^2 + 2.753z - 0.87810.9704}$$

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Phase and Gain margin

❑Example 4.11

```
clear all; close all; clc;
T = 0.01;
z = tf('z',T);
K=5;
Ga_d  = 0.9516/(z-0.9048);
Gzas_d = 10^-5*(4.95*z+4.901)/(z^2-1.97*z+0.9704)
L_d = K*Ga_d*Gzas_d;
[GM,PM] = margin(L_d)
```

```
GM =

   0.7140


PM =

  -9.0842
```

$$L(z)=G_a(z)G_{ZAS}(z)=10^{-5}\frac{5(4.711z+4.644)}{z^3-2.875z^2+2.753z-0.87810.9704}$$

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Phase and Gain Margin

❑Example 4.12

Determine the closed-loop stability of the digital control system for the position control system with analog transfer function (T=0.01)

$$G(s) = \frac{10}{s(s+3)}$$

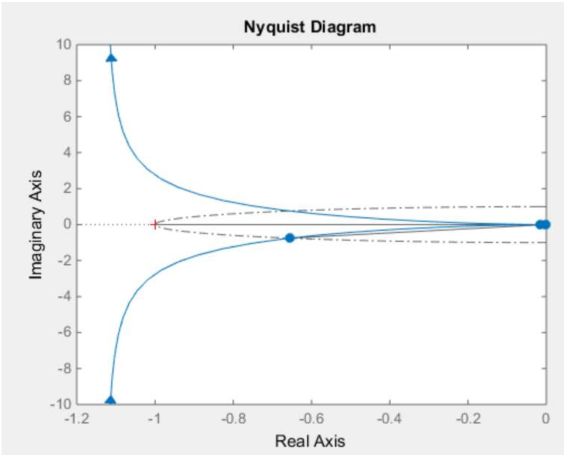
```
close all; clear all; clc
G = tf(10,[1 3 0]);
T = 0.01;
G_d = c2d(G,T,'zoh');
[GM, PM] = margin(G_d)
nyquist(G_d)
```

```
GM =

  60.3028


PM =

  48.9872
```



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Summary

- ❑Determine the input-output stability of a z-transfer function.
- ❑Determine the asymptotic stability of a z-transfer function.
- ❑Determine the internal stability of a digital feedback control system.
- ❑Determine the stability of a z-polynomial using the Routh-Hurwitz criterion.
- ❑Determine the stability of a z-polynomial using the Jury criterion.
- ❑Determine the stable range of a parameter for a z-polynomial.
- ❑Determine the closed-loop stability of a digital system using the Nyquist criterion.
- ❑Determine the gain margin and phase margin of a digital system.

Template cho khais nieemj

Internal stability: **If all the transfer functions that relate system inputs (R and D) to the possible system outputs (Y and U) are BIBO stable, then the system is said to be internally stable.C**

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