AMC 10 Study Guide

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§1 Introduction

§2 How To Use Guide

Note that the usage of "iff" and "WLOG" throughout this text refers to "if and only if" and "without loss of generality", respectively.

§3 Theorems by Frequency

List of formulas, theorems, lemmas, or useful pieces of knowledge listed in order of frequency.

§3.1 Algebra

§3.1.1 Simon's Favorite Factoring Trick (SFFT)

Theorem 3.1

Given
$$xy + jx + ky = a,$$
where j, k, a are integer constants and the coefficient of xy is 1, this equation can be transformed into
$$(x + k)(y + j) = a + jk.$$

Example 3.2

Let $xy + 3x + 4y = 6$. Express this equation in SFFT form.

Example 3.3

2021 AMC 10B, Problem 11

§3.1.2 Vieta's Formulas

To decomplexify Vieta's formulas, let polynomial $P(x) = ax^n + bx^{n-1} + cx^{n-2} + \ldots$ and let its roots be $r_1, r_2, r_3, \ldots, r_n$.

Vieta's can answer the questions: "What is the sum of the roots?", "What is the product of the roots?", etc.
Given $n = 2$, or a polynomial of the form $ax^2 + bx + c$, the sum of the roots is
$$-\frac{b}{a}$$
and the product of the roots is
$$\frac{c}{a}.$$

Moving to higher values of n, new combinations of roots are introduced. Given a cubic polynomial ($n = 3$), we are able to find the sum $r_1r_2 + r_1r_3 + r_2r_3$ (pairwise sum). This is given by $\frac{c}{a}$, and the product becomes $-\frac{d}{a}$. For higher values of n, there are further combinations[1].
Pattern:
$-\frac{b}{a}, \frac{c}{a}, -\frac{d}{a}, \frac{e}{a}$, etc. Note negative sign repeats.

Example 3.4

2022 AMC 10A, Problem 16

Example 3.5

2021 AMC 12A, Problem 12

[1] k-th symmetric sum

§3.1.3 Discriminant

The discriminant for polynomial $ax^2 + bx + c$ is

$$b^2 - 4ac.$$

This is in the square root of the quadratic formula. Knowing that, it is easy to see that if the discriminant is positive, the equation has two real roots; if the discriminant is negative, the equation has two nonreal roots; and if the discriminant is 0, the equation has a real double root.

§3.1.4 Telescoping

Basically a method for a series that you cancel out most of to leave a few terms that you can work with.

Example 3.6

Find

$$\sum_{n=2}^{\infty} \frac{-2}{(n+1)(n+2)}.$$

Hint. Let's break this bad boy up; write $\frac{-2}{(n+1)(n+2)}$ in the form $\frac{a}{n+1} + \frac{b}{n+2}$. This should allow you to complete the problem, blowing your mind.

§3.1.5 Polynomial Graphs

- The largest possible number of minimum or maximum points is one less than the degree of the polynomial.
- Polynomials of degree greater than 2 can have more than one max or min value.
- Vertex of a quadratic function: $-b/2a$

§3.1.6 Coordinate Definition of a Circle

Given a circle's radius r and center (h, k), the coordinate definition of a circle is

$$(x-h)^2 + (y-k)^2 = r^2.$$

§3.1.7 Rational Root Theorem

Theorem 3.7

Given an integer polynomial $P(x)$ with leading coefficient a_n and the constant term a_0, if $P(x)$ has a rational root $r = p/q$ in lowest terms, then $p|a_0$ and $q|a_n$.

If math is not your native language, then this is what it says: For an integer polynomial with a rational root, the leading coefficient must be divisible by the denominator of the fraction and the constant term must be divisible by the numerator.
This is useful for narrowing down possible roots of a high degree polynomial.

Example 3.8

Find all rational roots of the polynomial $x^6 + x^3 + 2x + 10$.

§3.2 Geometry

Assumes you know certain basic concepts such as the 45-45-90 triangle or 30-60-90 triangle.

§3.2.1 Similarity

Two objects are similar if they are similar in every aspect except possibly size or orientation.
Triangle similarity rules: AA, SSS, SAS. Essential topic in geometry.

§3.2.2 Pythagorean Triples

- (3, 4, 5)
- (5, 12, 13)
- (7, 24, 25)
- (8, 15, 17)
- (9, 40, 41)

There are infinitely many but I would only take the time to memorize these.

§3.2.3 Congruent Triangles

Triangle congruency rules: SSS, SAS, ASA, AAS, HL.

§3.2.4 Angle Bisector Theorem

If a problem has the words "angle bisector", immediately think about this.

Theorem 3.9

Given triangle $\triangle ABC$ and angle bisector AD, where D is on side BC, then $\frac{c}{m} = \frac{b}{n}$ where m is BD, n is DC, c is AB, and b is AC. It follows that $\frac{c}{b} = \frac{m}{n}$.

This works for the converse.

Example 3.10

2018 AMC 10A, Problem 24

Example 3.11

2012 AIME I, Problem 12

§3.2.5 Polygon Angles

The sum of the inscribed angles in a polygon with n sides in degrees is

$$180(n-2).$$

In every polygon, the exterior angles always add up to $360°$. The measure of one exterior angle for a regular polygon is thus $360/n$.

§3.2.6 Triangle Area

Along with $\frac{bh}{2}$ and Heron's Formula, there are a couple of other ways to find a triangle's area, denoted by K.

- $K = \frac{1}{2}ab\sin\theta$, where a, b are adjacent sides and θ is the angle between them.
- $K = rs$, where r is the radius of the incircle and s is the semiperimeter. Note: this works for any polygon with an incircle!
- $K = \frac{abc}{4R}$, where a, b c, are lengths of the sides of the triangle and R is the circumradius.

  In addition, the area of an equilateral triangle with side length n is

  $$K = \frac{n^2\sqrt{3}}{4}.$$

  Therefore, the height of an equilateral triangle is

  $$h = \frac{n\sqrt{3}}{2}.$$

> **Example 3.12**
> What is the area of a regular hexagon in terms of n?

§3.2.7 Other Polygonal Area

- For a regular polygon,
  $$K = \frac{ap}{2}$$
  where a is the apothem, or the line segment from the center to the midpoint of one of its sides, and p is the perimeter.
- The area of a trapezoid is
  $$K = \frac{(b_1 + b_2)h}{2}$$
  where b_1 and b_2 are bases and h is the height.
- The area of a kite or rhombus is
  $$K = bh = \frac{d_1 \cdot d_2}{2}$$
  where b is the base, h is the height, and d_1 and d_2 are diagonals.
- Others you should hopefully know already: Square, rectangle, circle (both area and perimeter), volumes and surface area of: cubes, rectangular prisms, spheres ($V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$), cylinders, cones.

§3.2.8 Thales' Theorem

> **Theorem 3.13**
> If there are three points on a circle, A, B, C with AC being a diameter, $\angle ABC = 90°$.

§3.2.9 Right Triangle Altitude Theorem

In any right triangle, the altitude squared is equal to the product of the two lengths that it divides the base into.[2]

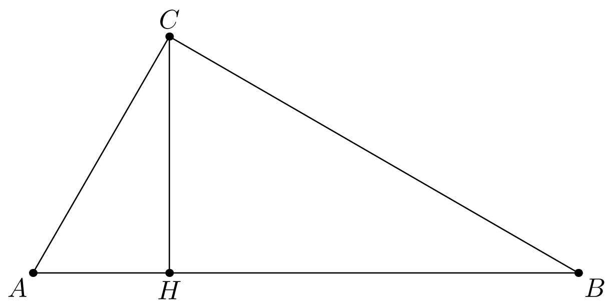
In other words, given right triangle $\triangle ABC$, altitude h, $AH = d$ and $HB = e$, we have

$$h^2 = de$$

Hence,

$$a^2 = dc$$

where a is the length across point A and c is the length across point C. This result comes from Pythagoras' theorem (notice $c = d + e$).



§3.2.10 Cevians, Medians, Altitudes, Perpendicular Bisectors

Cevians: a line segment or ray that extends from one vertex of a polygon (usually a triangle) to the opposite side (or the extension of that side). Medians, altitudes, and perpendicular bisectors are special cevians.

Medians: Cevian that divides the opposite side into two congruent lengths. See centroid.

Altitudes: A line segment that passes through a vertex of a triangle and is perpendicular to the line opposite to the vertex. See orthocenter.

Perpendicular Bisector: In a plane, the perpendicular bisector of a line segment AB is a line l such that AB and l are perpendicular and l passes through the midpoint of AB. See circumcircle.

Read the last few entries in the Geometry section of this guide for theorems relevant to cevians.

§3.2.11 Centroid, Orthocenter, Incircle, Circumcircle

Centroid: Point of intersection of the medians of a triangle; center of mass.
The centroid divides the medians in a $2:1$ ratio. Also, the three medians of a triangle divide the triangle into six equal regions area $(1/6)$. Note that the coordinates of the centroid of a triangle with points (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

[2] See Wikipedia article for a nice proof

Example 3.14

2018 AMC 10B, Problem 12

Orthocenter: Point of intersection of the altitudes of a triangle.
If the orthocenter's triangle is acute, then the orthocenter is in the triangle; if the triangle is right, then it is on the vertex opposite the hypotenuse; and if it is obtuse, then the orthocenter is outside the triangle.

Incircle: An incircle of a convex polygon is a circle which is inside the figure and tangent to each side.
Every triangle and regular polygon has a unique incircle, but general polygons with 4 or more sides (such as non- square rectangles) do not have an incircle. The incenter (center of incircle) can be constructed by drawing the intersection of **angle bisectors**.

The radius of an incircle of a triangle with perimeter p and area A is

$$r = \frac{2A}{p}.$$

Notice this comes from the $A = rs$ formula previously mentioned in Triangle Area.

Circumcircle: The circumcircle of a triangle or other polygon is the circle which passes through all of its vertices.
Every triangle and regular polygon has a circumcircle, but most other polygons do not. The circumcenter (center of circumcircle) is the intersection of the perpendicular bisectors of the edges of the polygon.

The radius of a circumcircle is

$$R = \frac{abc}{4A}.$$

Also previously mentioned in Triangle Area.

§3.2.12 Inscribed Angle Theorem

Inscribed angle: In a given circle, an inscribed angle is an angle whose vertex lies on the circle and each of whose sides either intersects the circle in a second point (and so includes a chord of the circle) or is tangent to the circle.

Theorem 3.15

The measure of an inscribed angle is equal to **half** of the measure of the arc it intercepts or subtends. Thus, in particular it does not depend on the location of the vertex on the circle.

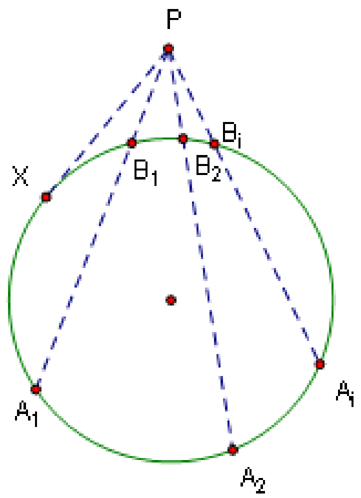
§3.2.13 Power of a Point

This is a mindblowing theorem once you really understand what it's talking about.

Theorem 3.16

Consider a circle O and a point P in the plane where P is not on the circle (it is either outside or inside the circle). Now draw a line through P that intersects the circle in two places (or one place). By the Power of a Point Theorem, the product of the length from P to the first point of intersection and the length from P to the second point of intersection is constant for any choice of a line through P that intersects the circle.

$$PX^2 = PA_1 \cdot PB_1 = PA_2 \cdot PB_2 = \cdots = PA_i \cdot PB_i$$



If you do not understand after reading through this explanation a few times, check out the full AoPS article on Power of a Point.

Example 3.17

2013 AMC 10A, Problem 23 (There are numerous ways to approach this. Do whatever you want, but definitely try Power of a Point)

§3.2.14 Mass Points

Mass point geometry involves systematically assigning 'weights' to points using **ratios**[3] of lengths relating vertices, which can then be used to deduce other lengths using a few key facts:

- The lengths must be inversely proportional to their weight (just like a balanced lever).
- The point dividing the line has a mass equal to the sum of the weights on either end of the line (like the fulcrum of a lever).

First, choose a point for the entire figure to balance around. From there, WLOG a first weight can be assigned.

- Any line passing this central point will balance the figure.

[3]If you are given only ratios in a problem, you know what to do :D (other methods are available, however, depending on the problem

- If two points balance, the product of the mass and distance from a line of balance of one point will equal the product of the mass and distance from the same line of balance of the other point.
- If two points balance, the point on the balancing line used to balance them has a mass of the sum of the masses of the two points; meaning if two points A and B have masses m and n, respectively, a third point between A and B which divides AB into the ratio $\frac{m}{n}$ will have mass $m+n$.

Example 3.18

2004 AMC 10B, Problem 20

§3.2.15 Triangle Inequality

Theorem 3.19

The Triangle Inequality says that in nondegenerate (a normal) $\triangle ABC$: $AB+BC > AC$, $BC+AC > AB$, $AC+AB > BC$.

Example 3.20

2018 AMC 10B, Problem 22

§3.2.16 Unnamed Hypothenuse Theorem

Theorem 3.21

The midpoint of the hypotenuse is equidistant from the vertices of the right triangle.

§3.2.17 Common Trig Values

This assumes you are familar with basic trigonometery, such as the right triangle definitions of trigonometric functions. Useful additional knowledge includes the unit circle definition.

Recall $\tan x = \sin x / \cos x$. Therefore you could just memorize sin and cos.

rad	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
deg	0	$30°$	$45°$	$60°$	$90°$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	-

§3.2.18 Law of Cosines

Theorem 3.22

For a triangle with sides a, b, c and opposite angles A, B, C respectively, the Law of Cosines states $c^2 = a^2 + b^2 - 2ab\cos C$.

This is extremely useful for finding various values in a triangle.

Example 3.23

2017 AMC 12B, Problem 15

§3.2.19 Law of Sines

Theorem 3.24

In triangle $\triangle ABC$, where a is the side opposite to A, b opposite to B, c opposite to C, and where R is the circumradius:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

.

§3.2.20 Heron's Formula

See: Triangle Area

Theorem 3.25

For any triangle with side lengths a, b, c, the area A can be found using the following formula:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where the semi-perimeter $s = \frac{a+b+c}{2}$.

§3.2.21 Cyclic Quadrilaterals

A quadrilateral is cyclic iff it can be inscribed in a circle. Properties include:

- Opposite angles add up to $180°$.
- A convex quadrilateral is cyclic iff the four perpendicular bisectors to the sides are concurrent. This common point is the circumcenter. (Remember the converse!)
- In cyclic quadrilateral $ABCD$, $\angle ABD = \angle ACD$, $\angle BCA = \angle BDA$, $\angle BAC = \angle BDC$, $\angle CAD = \angle CBD$

Here are some nice theorems regarding cyclic quadrilaterals.

Ptolemy's Theorem:

Theorem 3.26

Given a cyclic quadrilateral $ABCD$ with side lengths a, b, c, d and diagonals e, f:

$$ac + bd = ef$$

Example 3.27

2022 AMC 10A Problem 23 Hint: Try using Stewart's

Example 3.28

2023 AIME I, Problem 5

Brahmagupta's Formula: You could use this but more likely there is a more efficient method using Ptolemy's and other theorems. Worst case scenario,

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

where K denotes area, s denotes semiperimeter, and a, b, c and d are side lengths.

§3.2.22 Stewart's Theorem

Theorem 3.29

Let ABC be a triangle, with side lengths a, b, c and opposite vertices A, B, and C respectively. If cevian AD is drawn so that $BD = m$, $DC = n$, and $AD = d$, then $b^2m + c^2n = amn + d^2a$.

Example 3.30

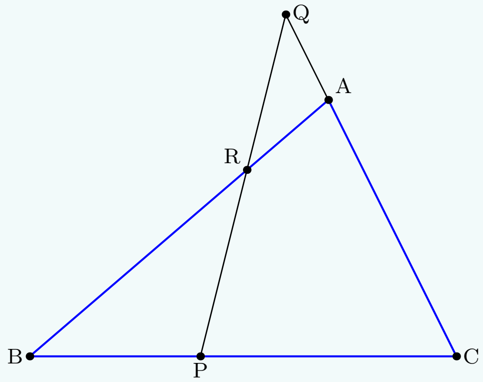
2013 AIME II, Problem 13

§3.2.23 Menelaus' Theorem

Theorem 3.31

If line PQ intersecting AB on $\triangle ABC$, where P is on BC, Q is on the extension of AC, and R on the intersection of PQ and AB, then

$$\frac{PB}{CP} \cdot \frac{QC}{QA} \cdot \frac{AR}{RB} = 1.$$



Example 3.32

2011 AIME II, Problem 4

§3.2.24 Shoelace Theorem

Theorem 3.33

Given any polygon[a] with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots (x_n, y_n)$, listed in clockwise (or counterclockwise) order, the area A of the polygon is

$$A = \frac{1}{2}\left|\sum_{i=1}^{n}(x_{i_1} + x_i)(y_{i+1} - y_i)\right|$$

or equivalently,

$$A = \tfrac{1}{2}\left|\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{array} \quad - \quad \begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{array}\right|$$

[a]There is also a 3D version, analogous to 3D Pythagoras'

Example 3.34

Vaishnavi go solve this problem. 2017 AIME II, Problem 3

§3.2.25 Ceva's Theorem

Theorem 3.35

Let ABC be a triangle, and let D, E, and F be points on lines BC, CA, AB, respectively. Lines AD, BE, CF are **concurrent** iff

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

This works for the reciprocal of the ratios. The cevians do not necessarily have to lie within the triangle.

§3.3 Combinatorics

§3.3.1 Stars and Bars

The number of ways to put k indistinguishable objects into n distinguishable bins is

$$\binom{n+k-1}{n-1}.$$

Beware: what if there are restrictions involved? If there must be at least a objects in each bin, then there are

$$\binom{k-na+n-1}{n-1}.$$

Notice this dissolves to $\binom{k-1}{n-1}$ when $a = 1$, and of course the regular stars and bars technique when $a = 0$.
Note we are subtracting na because since each bin has at least a objects, we are getting rid of what does not need to be counted.

Example 3.36

If a, b, c are positive integers less than 1000, how many ordered pairs (a, b, c) are there such that $a + b + c = 2023$?

§3.3.2 Binomial Theorem

Theorem 3.37

For real or complex a, b, and non-negative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Example 3.38

2011 AMC 10B, Problem 23

Hint. Use a modulo because we only want the hundreds digit. Then, try and use the binomial theorem.

t

§3.3.3 Rearrangements

- Notice the number of rectangles in a rectangular grid with size $m \times n$ is
$$\binom{m+1}{2}\binom{n+1}{2},$$
since a rectangle is just two horizontal lines and two vertical lines.

- There are
$$\binom{n}{4}$$
ways to find an intersection point of two diagonals in an n-gon. We just choose 4 points from the vertices of an n-gon.[4]

- How to count the number of rearrangements of the word "SHINING"? It is $\frac{7!}{2!2!} = 1260$. This makes sense as 7 is the number of characters we are rearranging and there are 2 "I"s and 2 "N"s, which we take the factorial of and divide off.

- On a similar note, how many ways to get from $(0, 0)$ to $(4, 5)$ by just moving up or right? In a path, there is guaranteed to be 4 rights and 5 ups, just the order differs. So we can use the above method and get $\frac{9!}{4!5!}$.

§3.3.4 Pigeonhole Principle

If we distribute n balls into k boxes such that $n > k$ then at least one box must have multiple balls.

Example 3.39
2011 AMC 10B, Problem 11

§3.3.5 Combinatorial Identities

There are three main ones, namely Pascal's Identity, Vandermonde's Identity, and the Hockey-Stick Identity. Read the AoPS page for this one, honestly, as these will not show up often. However, I would like to add[5]

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

§3.3.6 Chicken McNuggies

Theorem 3.40
The Chicken McNuggets theorem states that for any two relatively prime positive integers m, n, the greatest integer that cannot be written in the form $am + bn$ for nonnegative integers a, b, is $mn - m - n$.

Example 3.41
2015 AMC 10B, Problem 15

[4] How many diagonals are there in an n-gon?
[5] See committee forming if you would like to know why

§3.4 Number Theory

§3.4.1 Arithmetic and Geometric Sequences

Let a_n be the nth term, d be the common difference, and r be the common ratio.

Arithmetic sequence: $a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \ldots$
Geometric sequence: $a_1, a_1 \cdot r, a_1 \cdot r^2, a_1 \cdot r^3, \ldots$

Know how to find the common difference and the sum of a finite arithmetic sequence. The most important is the sum of an infinite geometric series (valid iff $|r| < 1$; in other words, when the geometric sequence is mostly fractions).

$$S = \frac{a_1}{1 - r}$$

§3.4.2 Special Series

The sum of odd numbers $1 + 3 + \ldots + (2n - 1)$ is n^2. Note n is the position of the last number in the sequence. The sum of $1 + 3 + \ldots + 2023$ is therefore 1012^2.

The sum of even numbers $2 + 4 + \ldots + 2n$ is $n(n + 1)$. Again, n is the position.

§3.4.3 Prime Factorization (Cool Stuff)

For a prime factorization $n = p^a q^b r^c \ldots$ where $p, q, r, \ldots$ are primes, the number of factors (divisors) are

$$(a + 1)(b + 1)(c + 1) \ldots.$$

Prime factorization is extremely common in NT problems and you should know how to split a number into it quickly.

Example 3.42
2017 AMC 10B, Problem 20

§3.4.4 GCD and LCM

- GCD is found by taking the lowest exponents of the prime factorizations of m and n.
- LCM is found by taking the highest exponents of the prime factorizations of m and n.
- $ab = \text{lcm}(a, b) \cdot \gcd(a, b)$, for any positive integers a, b

§3.4.5 Euclidean Algorithm

This is a nice way to find GCD without prime factorization.

Start with nonnegative integers a, b.

- If $b = 0$, then $\gcd(a, b) = a$.
- Otherwise, take the remainder when a is divided by b. Find $\gcd(a, a \pmod b)$.

- Repeat until $a \pmod b = 0$.

Example 3.43
Given that a is an odd multiple of 1183, find the greatest common divisor of $2a^2 + 29a + 65$ and $a + 13$. (Source: Alcumus)

Example 3.44
1959 IMO, Problem 1

§3.4.6 Modular Arithmetic

Let a, b, c, d be integers and m be a positive integer. If $a \equiv b \pmod m$ and $c \equiv d \pmod m$, the following identities hold:

- $$a + c \equiv b + d \pmod m$$
- $$a - c \equiv b - d \pmod m$$
- $$ac \equiv bd \pmod m$$
- If e is a common divisor of a and b,
$$\frac{a}{e} \equiv \frac{b}{e} \quad \left(\bmod \frac{m}{\gcd(m, e)}\right)$$
- If e is a positive integer,
$$a^e \equiv b^e \pmod m$$

Example 3.45
Note that any perfect square n^2 is congruent to $0 \pmod 4$ when n is even and $1 \pmod 4$ when n is odd.

Knowing this, are there solutions to $x^2 + y^2 = 2023$, where x and y are integers?

Example 3.46
2022 AMC 10A, Problem 19

Example 3.47
2015 AIME I, Problem 3

§3.4.7 Chinese Remainder Theorem

This theorem is notoriously quite difficult to understand.

Theorem 3.48
Let's find the least integer x which satisfies

$$y_1 \pmod{d_1},\ y_2 \pmod{d_2},\ \ldots,\ y_n \pmod{d_n},$$

such that d_1, d_2, …, d_n are all relatively prime.

Let M equal the product of all the divisors $M = d_1 d_2 \cdots d_n$, and let $b_i = \frac{M}{d_i}$.

If the integers a_i and b_i satisfy

$$a_i b_i \equiv 1 \pmod{d_i}$$

for every i in $1 \le i \le n$ (n being the number of congruences), then a solution for x is

$$x = \sum_{i=1}^{n} a_i b_i y_i \mod M.$$

Try out an example to get a better understanding. If you are lost, check out this video.

§3.4.8 Fermat's Little Theorem

Theorem 3.49
If a is an integer, p is a prime number and a is not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$.

Useful corollary:

Corollary 3.50

$$a^p \equiv a \pmod{p}$$

§3.4.9 Bases

Base n, where n is a positive integer $n \ge 2$ refers to a number system in which the set of digits used are limited to 0 through $(n-1)$.

For example, binary, base 2, only uses 0 and 1. Bases above 10 will use letters (i.e. hexadecimal, base 16, uses letters A-F to represent 10-15). Bases are indicated with subscripts.

$$432_7 = (4 \cdot 7^2 + 3 \cdot 7^1 + 2 \cdot 7^0)_{10}$$

This should give you an idea of how to convert bases. Starting from the units digit representing n^0, the "tens" digit now becoming the "n" digit, "hundreds" digit becoming "n^2" digit, etc.

Example 3.51
2021 AMC 10A, Problem 11

Example 3.52
2013 AMC 10A, Problem 19

§4 Strategies by Subject

§4.0.1 General

- Create variables
- Create patterns
- Generalize
- Modify the problem - try smaller cases!
- Draw diagram
- Look for symmetery
- Sometimes, work backwards
- Consider parity (evenness/oddness)

§4.0.2 Algebra

- Recall trivialities such as the zero product property.
- For problems involving units digits, you can just work with the units digits. Ex. what is the units digit of 2023^{2023}?

§4.0.3 Geometry

- Identify similar triangles
- Use ratios (not just mass points or similar triangles but in general)
- Create auxillary lines
- Angle chase
- Coordinate bashing
- In a 3D problem, perhaps find a 2D simplification

§4.0.4 Combinatorics

- Complementary counting
- Casework
- PIE

§4.0.5 Number Theory

- When a problem asks for the last digit, last 2 digits, or last digits, take modulos 10, 100, or 1000.
- It may help to rewrite $a \equiv b \pmod{p}$ as $a = b + nk$ for some integer n instead.
- Sometimes you will encounter numbers in the form $\overline{abc\ldots}$ and $0.\overline{abc\ldots}$. It's helpful to write these out with their expanded form: i.e. $\overline{abc} = 100a + 10b + c$.
- Let $s(n)$ denote the sum of the digits of n; note that $n \equiv s(n) \pmod{9}$
- Understand recursion

§5 Other Tips

- Make sure you're not stuck on one problem for over 2-3 minutes (you're just staring at it, making no progress). Move on!
- In modern AMC 10s, you only need 13-15 questions correct (and the rest blank) to qualify for AIME. This means you should prioritize checking over trying to solve new problems in the last 15 or so minutes.
- When you're solving practice problems and you are stuck, read the solution one line at a time, using the line as a hint. (If you're weird and impatient like me, you could look at the solution for under 1 second, giving you maybe a couple ideas you could use)[6]
- Simulate a testing environment to your best extent (maybe don't listen to music while mocking an AMC)
- Lock in

[6] The first part of what I wrote is what many people who are good at math (and competitive programming) recommend

§6 AMC 12 and Beyond

You should only move on to this section once you feel like you have a solid grasp on the fundamentals of competitive math, including all of the concepts in the above sections.

Despite that, some of the theorems do show up in AMC 10. There is an alternate solution, usually, but these may be helpful.

§6.1 Algebra

§6.2 Geometry

§6.3 Combinatorics

§6.4 Number Theory

§6.4.1 Euler's Totient Theorem

Theorem 6.1

Let $\phi(n)$ be Euler's totient function. If n is a positive integer, $\phi(n)$ is the number of integers in the range $\{1, 2, 3 \cdots, n\}$ which are relatively prime to n. If a is an integer and m is a positive integer relatively prime to a, then

$$a^{\phi(m)} \equiv 1 \pmod{m}.$$

Euler's totient function: Given the general prime factorization of $n = p_1^{e_1} p_2^{e_2} \dots p_m^{e_m}$:

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_m}\right).$$

§6.4.2 Wilson's Theorem

Theorem 6.2

Given an integer $p > 1$, $(p-1)! + 1$ is divisible by p if and only if p is prime.

This theorem is useful for simplifying the modulos of large factorials.

- **AMC 12+**
- trig identities
- log rules
- polynomial division
- Roots of unity
- Piecewise functions (observe function behavior on borders)
- ellipses
- generalized triangle inequality

- Functional equations (cyclic functions)
- RMS-AM-GM-HM
- Cauchy Schwartz ineq
- Ratio lemma
- de moivre's theorem
- uvw method
- Complex conjugate root theorem (see rational root theorem)
- Legendre's formula
- Newton's sums
- Recursion
- Homothety
- Descartes' Rule of Signs
- Bezout's theorem
- Bezout's identity
- Inversive distance
- Set theory
- Invertibility/inverse modulos
- **AIME+**
- Advanced: Calculus (1, 2, 3)
- Advanced: Invariants
- Advanced: Generating funcs
- Advanced: Incenter-Excenter formula
- Advanced: pilot's theorem
- Advanced: Lagrange interpolation
- Advanced: Hook length

§7 All Examples:

1. Grandma has just finished baking a large rectangular pan of brownies. She is planning to make rectangular pieces of equal size and shape, with straight cuts parallel to the sides of the pan. Each cut must be made entirely across the pan. Grandma wants to make the same number of interior pieces as pieces along the perimeter of the pan. What is the greatest possible number of brownies she can produce?
   (A) 24 **(B)** 30 **(C)** 48 **(D)** 60 **(E)** 64

2. The roots of the polynomial $10x^3 - 39x^2 + 29x - 6$ are the height, length, and width of a rectangular box (right rectangular prism). A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?
(A) $\frac{24}{5}$ **(B)** $\frac{42}{5}$ **(C)** $\frac{81}{5}$ **(D)** 30 **(E)** 48

3. All the roots of the polynomial $z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers, possibly repeated. What is the value of B?
(A) 88 **(B)** -80 **(C)** -64 **(D)** -41 **(E)** -40

4. Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?
(A) 88 **(B)** -80 **(C)** -64 **(D)** -41 **(E)** -40

5. Let $\triangle ABC$ be a right triangle with right angle at C. Let D and E be points on $\overline{AB}$ with D between A and E such that $\overline{CD}$ and $\overline{CE}$ trisect $\angle C$. If $\frac{DE}{BE} = \frac{8}{15}$, then $\tan B$ can be written as $\frac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers, and p is a positive integer not divisible by the square of any prime. Find $m + n + p$.

6. Line segment $\overline{AB}$ is a diameter of a circle with $AB = 24$. Point C, not equal to A or B, lies on the circle. As point C moves around the circle, the centroid (center of mass) of $\triangle ABC$ traces out a closed curve missing two points. To the nearest positive integer, what is the area of the region bounded by this curve?
(A) 25 **(B)** 38 **(C)** 50 **(D)** 63 **(E)** 75