Notes on Utility-Based Newsvendor Models

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Abstract

We aim to understand the working details behind variations of utility-based newsvendor models, and how we could potentially apply these mechanisms to decision-making in financial markets.

1 Background

The aim of the newsvendor model is to maximize the expected utility of cash flow at the end of a single period by choosing the optimal order quantity of a perishable good. In the financial hedging context, the key decisions at the end of a period are:

- Choosing an optimal ordering policy
- Selecting the optimal portfolio by finding a portfolio of financial securities that minimizes the variance of the hedged cash flow for any possible order quantity

2 Utility-Based Newsvendor with Random Demand

2.1 Notation

$$\begin{aligned}
D &= \text{single-period demand, with CDF } G_D(x) = P(D \le x) \text{ and PDF } g_D(\cdot) \\
y &= \text{order quantity chosen under random demand } D,\ y^* = \text{optimal order quantity} \\
z_0 &= \text{initial wealth of newsvendor} \\
c &= \text{unit purchase cost, } c_h = \text{unit purchase cost when } D \ge y \\
s &= \text{unit sale price} \\
v &= \text{unit salvage value} \\
p &= \text{shortage penalty when } D \ge y \\
u &= \text{utility function capturing the risk sensitivity of newsvendor}
\end{aligned}$$

2.2 Model Assumptions

$$\begin{aligned}
&s > c > v \ge 0 \\
&s \ge c_h \ge c \\
&p \ge c_h - s \ge c - s \text{ i.e. negative unit shortage penalty} \\
&u \text{ is a strictly increasing, concave function s.t. } u' > 0, u'' \le 0
\end{aligned}$$

2.3 Solving for the Optimal Order Quantity

We aim to solve the constrained optimization problem:

$$\max_{y \geq 0} E[u(CF(D,y))] \tag{1}$$

where $CF(D,y)$ is the random cash flow given by the formula:

$$CF(D,y) = z_0 - (c-v)y + (s+p-v)\min\{D,y\} - pD$$

To characterize the optimal solution y^*, it will be helpful to define random cash flow with respect to a threshold x. Thus we can analyze the maximal cash flow position with respect to whether or not our threshold (or demand) has exceeded order quantity:

$$CF(D,y) = \begin{cases} CF_-(x,y) = z_0 - (c-v)y + (s-v)x & x \leq y \\ CF_+(x,y) = z_0 + (s+p-c)y - px & x > y \end{cases} \tag{2}$$

Note that $CF(y,y) = CF_-(y,y) = CF_+(y,y) = z_0 + (s-c)y$. In other words, when we have perfectly matched order quantity to our random demand, the overall cash flow will be equal to net gains proportional to demand centred by vendor's initial wealth.

Using the partition defined in (2), we derive the formula for expectation as follows:

$$E[u(CF(D,y))] = \int_0^y u(CF_-(x,y))g_D(x)dx + \int_y^\infty u(CF_+(x,y))g_D(x)dx$$

Thus, to solve the optimization problem (1) we solve for the first order condition as follows:

$$\begin{aligned} \frac{d}{dy}E[u(CF(D,y))] &= 0 \\ \implies -(c-v)E[u'(CF(D,y^*))1\{D \leq y^*\}] + (s+p-c)E[u'(CF(D,y^*))1\{D > y^*\}] &= 0 \\ \implies -(c-v)E[u'(CF(D,y^*))1\{D \leq y^*\}] + (s+p-c)(E[u'(CF(D,y^*))] - E[u'(CF(D,y^*))1\{D \leq y^*\}]) &= 0 \\ \implies -(s+p-v)E[u'(CF(D,y^*))1\{D \leq y^*\}] + (s+p-c)E[u'(CF(D,y^*))] &= 0 \\ \implies \frac{E[u'(CF(D,y^*))1\{D \leq y^*\}]}{E[u'(CF(D,y^*))]} &= \frac{s+p-c}{s+p-v} \end{aligned}$$

In other words, the optimal order quantity y^* satisfies the following characterization of the critical ratio:

$$\frac{E[u'(CF(D,y^*))1\{D \leq y^*\}]}{E[u'(CF(D,y^*))]} = \frac{s+p-c}{s+p-v} =: \hat{p}$$

2.3.1 Example: Linear Utility

Suppose that the decision maker is risk-neutral, so the utility function can be represented by $u(x) = a + bx$ for some fixed $a, b \in \mathbb{R}$. Then:

$$\begin{aligned} u(CF(D,y)) &= a + b(z_0 - (c-v)y + (s+p-v)\min\{D,y\} - pD) \\ \implies u'(CF(D,y)) &= \begin{cases} -b(c-v) + b(s+p-v) = b(s+p-c) & \text{if } D > y \\ -b(c-v) + b(s+p-v) - bp = b(s-c) & \text{if } D = y \\ -b(c-v) & \text{if } D < y \end{cases} \end{aligned}$$

Due to independence and linearity, the optimality condition reduces to $P(D \leq y^*) = \hat{p}$.

3 Utility-Based Newsvendor with Random Demand and Supply

3.1 Additional Notation

$W \in [0, 1]$ = random yield

$K \geq 0$ = random capacity of the supplier, with CDF $G_K(x) = P(K \leq x)$ and PDF $g_K(\cdot)$

We can extend the model in Section 2 by including random supply, modelled by:

$$Q(y) = W \min\{K, y\}$$

We can consider this to mean that once y units have been ordered, the supplier ships at most K and the proportion W will be received in good shape.

3.2 Additional Model Assumptions

D, K, W are not necessarily independent and have:

- Joint distribution function $F_{DKW}(x, z, w) = P(D \leq x, K \leq z, W \leq w)$
- Conditional density functions $g_{K|W=w}$ and $g_{D|K=z,W=w}$

3.3 Solving for the Optimal Order Quantity

We now aim to solve the constrained optimization problem:

$$\max_{y \geq 0} E[u(CF(D, K, W, y))] \tag{3}$$

where $CF(D, K, W, y)$ is the random cash flow given by the formula as in Section 2, except we replace y with $Q(y)$:

$$CF(D, y) = z_0 - (c - v)W \min\{K, y\} + (s + p - v) \min\{D, W \min\{K, y\}\} - pD$$

4 References

http://home.ku.edu.tr/fkaraesmen/public_html/pdfpapers/SKO_Utility_Based_Hedging_2014.pdf