Trigonometry

SMO Senior Training 2022

1 Formulas

Definitions

$$\sin x = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos x = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan x = \frac{\text{opposite}}{\text{adjacent}}$$

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

Addition Angle Formulas

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Double Angle Formulas

$$\begin{aligned}\sin 2x &= 2 \sin x \cos x \\ \cos 2x &= 2\cos^2 x - 1 \\ &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ \tan 2x &= \frac{\tan 2x}{1 - \tan^2 x}\end{aligned}$$

Half Angle Formulas

$$\begin{aligned}\sin \frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{2}} \\ \cos \frac{x}{2} &= \pm\sqrt{\frac{1+\cos x}{2}} \\ \tan \frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}\end{aligned}$$

Pythagorean Identities

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1 \\ \tan^2 x + 1 &= \csc^2 x \\ 1 + \cot^2 x &= \sec^2 x\end{aligned}$$

Sum to Product

$$\begin{aligned}\sin A + \sin B &= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \\ \sin A - \sin B &= 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \\ \cos A + \cos B &= 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \\ \cos A - \cos B &= -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}\end{aligned}$$

Product to Sum

$$\begin{aligned}2 \sin A \cos B &= \sin(A+B) + \sin(A-B) \\ 2 \cos A \sin B &= \sin(A+B) - \sin(A-B) \\ 2 \cos A \cos B &= \cos(A-B) + \cos(A+B) \\ 2 \sin A \sin B &= \cos(A-B) - \cos(A+B)\end{aligned}$$

R-Formula

$$A \sin x \pm B \cos x = R \sin(x \pm \alpha)$$
$$A \cos x \pm B \sin x = R \cos(x \mp \alpha)$$

where $R = \sqrt{A^2 + B^2}$, $\tan \alpha = \dfrac{B}{A}$.
$\implies |A \sin x \pm B \cos x| \leq \sqrt{A^2 + B^2}$.

Others

$$\begin{aligned}\sin(90^\circ - x) &= \cos x & \sin(-x) &= -\sin x \\ \cos(90^\circ - x) &= \sin x & \cos(-x) &= \cos x \\ \tan(90^\circ - x) &= \cot x & \tan(-x) &= -\tan x \\ \sin(90^\circ + x) &= \cos x \\ \cos(90^\circ + x) &= -\sin x \\ \tan(90^\circ + x) &= -\cot x\end{aligned}$$

2 Trigonometry Calculations

1. Find the value of $100(\sin 253^\circ \sin 313^\circ + \sin 163^\circ \sin 223^\circ)$.

   Answer. 50

   Proof.

$$\begin{aligned}
&100(\sin 253^\circ \sin 313^\circ + \sin 163^\circ \sin 223^\circ)\\
&= 100((-\sin 73^\circ)(-\sin 47^\circ) + \sin 17^\circ(-\sin 43^\circ))\\
&= 100(\cos 17^\circ \cos 43^\circ - \sin 17 \sin 43^\circ)\\
&= 100\cos(17^\circ + 43^\circ)\\
&= 100\cos 60^\circ\\
&= 100\left(\frac{1}{2}\right)\\
&= 50.
\end{aligned}$$

   $\square$

2. Given that $(1+\tan 1^\circ)(1+\tan 2^\circ)...(1+\tan 45^\circ) = 2^n$, find n.

   Answer. 23

   Proof. We first note that

$$\begin{aligned}
&(1+\tan x)(1+\tan(45^\circ - x))\\
&= (1+\tan x)\left(1+\frac{\tan 45^\circ - \tan x}{1+\tan 45^\circ \cdot \tan x}\right)\\
&= (1+\tan x)\left(1+\frac{1-\tan x}{1+\tan x}\right)\\
&= (1+\tan x) + (1-\tan x)\\
&= 2.
\end{aligned}$$

   Thus

$$\begin{aligned}
&(1+\tan 1^\circ)(1+\tan 2^\circ)...(1+\tan 45^\circ)\\
&= (1+\tan 1^\circ)(1+\tan 44^\circ)...(1+\tan 22^\circ)(1+\tan 23^\circ)(1+\tan 45^\circ)\\
&= 2^{23},
\end{aligned}$$

   which means $n = 23$. $\square$

   Remark. Whenever you are asked to calculate a long chain of trigonometric expressions, it is usually helpful to think about pairing some terms together.

3. Find the value of $\dfrac{\sin 7^\circ + \sin 8^\circ \cos 15^\circ}{\cos 7^\circ - \sin 8^\circ \sin 15^\circ}$.

   Answer. $2-\sqrt{3}$

Proof.

$$\begin{aligned}
&\frac{\sin 7^\circ + \sin 8^\circ \cos 15^\circ}{\cos 7^\circ - \sin 8^\circ \sin 15^\circ} \\
&= \frac{\sin 7^\circ + \frac{1}{2}(\sin 23^\circ + \sin(-7^\circ))}{\cos 7^\circ - \frac{1}{2}(\cos(-7^\circ) - \cos 23^\circ)} \text{ (product to sum)} \\
&= \frac{\sin 7^\circ + \sin 23^\circ}{\cos 7^\circ + \cos 23^\circ} \\
&= \frac{2\sin 15^\circ \cos 8^\circ}{2\cos 15^\circ \cos 8^\circ} \text{ (sum to product)} \\
&= \tan 15^\circ \\
&= \frac{1-\cos 30^\circ}{\sin 30^\circ} \\
&= 2-\sqrt{3}.
\end{aligned}$$

□

3 Trigonometric Equations

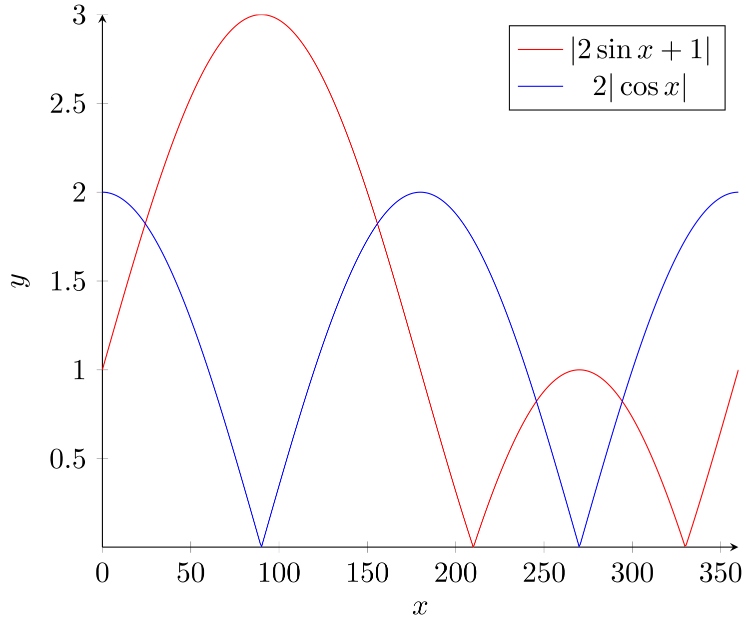
1. Find the number of solutions of the equation

$$|2\sin x + 1| = 2|\cos x|,$$

where $0^\circ \le x \le 360^\circ$.

Answer. 4

Proof 1 (Graph plotting). We can plot the following graph of the two functions.



Clearly, there are 4 intersections, thus there are 4 solutions. □

Proof 2 (Algebra). We have $|2\sin x + 1| = 2|\cos x| \iff (2\sin x + 1)^2 = (2\cos x)^2$.

$$\begin{aligned}
(2\sin x + 1)^2 &= (2\cos x)^2 \\
4\sin^2 x + 4\sin x + 1 &= 4\cos^2 x \\
4\sin^2 x + 4\sin x + 1 &= 4 - 4\sin^2 x \\
8\sin^2 x + 4\sin x - 3 &= 0
\end{aligned}$$

This means $\sin x = \dfrac{-1 \pm \sqrt{7}}{4}$. Since each value of $\sin x$ corresponds to 2 values of x in $[0°, 360°]$, there are 4 solutions in total. □

2. Given that $\sin\theta - \cos\theta = \dfrac{1}{2}$, find the value of $\sin^3\theta - \cos^3\theta$.

   Answer. $\dfrac{11}{16}$

   Proof. Firstly, we have

$$\begin{aligned}
(\sin\theta - \cos\theta)^2 &= \frac{1}{4} \\
\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta &= \frac{1}{4} \\
1 - 2\sin\theta\cos\theta &= \frac{1}{4}
\end{aligned}$$

   so $\sin\theta\cos\theta = \dfrac{3}{8}$. Now

$$\begin{aligned}
&\sin^3\theta - \cos^3\theta \\
&= (\sin\theta - \cos\theta)(\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta) \\
&= \left(\frac{1}{2}\right)\left(1 + \frac{3}{8}\right) \\
&= \frac{11}{16}.
\end{aligned}$$

   □

3. Given that x and y are acute angles such that $\cos(x+y) = \dfrac{4}{5}$ and $\cos(2x+y) = \dfrac{5}{13}$, find the value of $130\cos x$.

   Answer. 112

   Proof. Firstly, since x and y are both acute angles, and $\cos(x+y)$ and $\cos(2x+y)$ are both positive, $x+y$ and $2x+y$ are also both acute angles. Now

$$\begin{aligned}
\cos x &= \cos((2x+y) - (x+y)) \\
&= \cos(2x+y)\cos(x+y) + \sin(2x+y)\sin(x+y) \\
&= \left(\frac{5}{13}\right)\left(\frac{4}{5}\right) + \left(\frac{12}{13}\right)\left(\frac{3}{5}\right) \\
&= \frac{56}{65},
\end{aligned}$$

   hence $130\cos x = 112$. □

4 Trigonometric Inequalities

1. Find the minimum value of $13\sec\theta - 9\sin\theta\tan\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

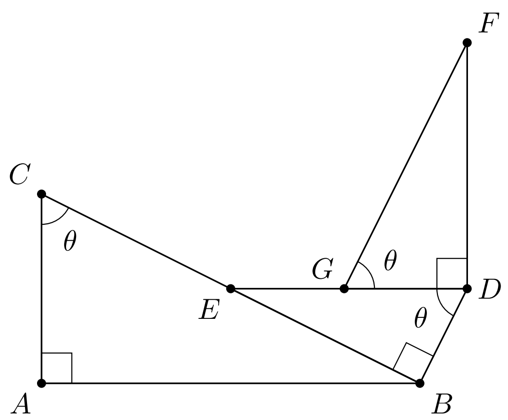
   Answer. 12

   Proof.

$$
\begin{aligned}
&13\sec\theta - 9\sin\theta\tan\theta \\
&= \frac{13 - 9\sin^2\theta}{\cos\theta} \\
&= \frac{13 - (9 - 9\cos^2\theta)}{\cos\theta} \\
&= \frac{4}{\cos\theta} + 9\cos\theta \\
&\geq 2\sqrt{\frac{4}{\cos\theta}\cdot 9\cos\theta} \text{ (AMGM)} \\
&= 12,
\end{aligned}
$$

   where equality holds if and only if $\frac{4}{\cos\theta} = 9\cos\theta \implies \cos\theta = \pm\frac{2}{3}$. $\square$

2. The diagram below shows three right-angled triangles, where $BC = 14$, $GF = 10$, $DE = 7$ and $\angle BCA = \angle BDE = \angle FGD = \theta$. Find the maximum possible value of $AB+BD+DF$.



   Answer. 25

   Proof.

$$
\begin{aligned}
&AB + BD + DF \\
&= 14\sin\theta + 7\cos\theta + 10\sin\theta \\
&= 24\sin\theta + 7\cos\theta \\
&= \sqrt{24^2 + 7^2}\sin\left(\theta + \tan^{-1}\left(\frac{7}{24}\right)\right) \\
&\leq 25,
\end{aligned}
$$

   where equality holds if and only if $\theta = -\tan^{-1}\left(\frac{7}{24}\right)$. $\square$

3. Find the maximum value of $\dfrac{\sqrt{3}\sin x}{2-\cos x}$ for $x \in \mathbb{R}$.

Answer. 1

Proof. Let $k = \dfrac{\sqrt{3}\sin x}{2-\cos x}$. Then $\sqrt{3}\sin x = 2k - k\cos x$, or $\sqrt{3}\sin x + k\cos x = 2k$. But by R-formula,

$$\sqrt{3}\sin x + k\cos x = \sqrt{3+k^2}\sin\left(x + \tan^{-1}\left(\frac{k}{3}\right)\right),$$

so

$$\sqrt{3+k^2}\sin\left(x + \tan^{-1}\left(\frac{k}{3}\right)\right) = 2k.$$

Since $\sin\left(x + \tan^{-1}\left(\frac{k}{3}\right)\right) \leq 1$ and it attains the maximum value of 1 at $x = -\tan^{-1}\left(\frac{k}{3}\right)$, we just need

$$\begin{aligned}
\sqrt{3+k^2} &\geq 2k \\
3+k^2 &\geq 4k^2 \\
3 &\geq 3k^2
\end{aligned}$$

which implies $-1 \leq k \leq 1$. Thus the maximum value of k is 1. □

4. Given that α and β are acute angles such that $\dfrac{\sin\alpha}{\sin\beta} = \sin(\alpha+\beta)$, let m be the maximum value of $\tan\alpha$. Find $9m$.

Answer. 12

Proof 1. Firstly,

$$\begin{aligned}
\frac{\sin\alpha}{\sin\beta} &= \sin(\alpha+\beta) \\
\frac{\sin\alpha}{\cos\alpha} &= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)\left(\frac{\sin\beta}{\cos\alpha}\right) \\
\tan\alpha &= \tan\alpha\sin\beta\cos\beta + \sin^2\beta \\
(1-\sin\beta\cos\beta)\tan\alpha &= \sin^2\beta \\
\tan\alpha &= \frac{\sin^2\beta}{1-\sin\beta\cos\beta} \\
\tan\alpha &= \frac{1-\cos 2\beta}{2-\sin 2\beta}
\end{aligned}$$

so we just need to maximise $k = \dfrac{1-\cos 2\beta}{2-\sin 2\beta}$. Cross-multiplying,

$$\begin{aligned}
1-\cos 2\beta &= 2k - k\sin 2\beta \\
k\sin 2\beta - \cos 2\beta &= 2k-1
\end{aligned}$$

But by R-formula,

$$k\sin 2\beta - \cos 2\beta = \sqrt{k^2+1}\sin\left(2\beta - \tan^{-1}\left(\frac{1}{k}\right)\right),$$

so

$$\sqrt{k^2+1}\sin\left(2\beta - \tan^{-1}\left(\frac{1}{k}\right)\right) = 2k-1.$$

Since $\sin\left(2\beta - \tan^{-1}\left(\frac{1}{k}\right)\right) \leq 1$ and it attains the maximum value of 1 at $2\beta = \tan^{-1}\left(\frac{1}{k}\right)$, we just need

$$\begin{aligned}
\sqrt{k^2+1} &\geq 2k-1 \\
k^2+1 &\geq 4k^2-4k+1 \\
0 &\geq 3k^2-4k
\end{aligned}$$

which implies $0 \leq k \leq \frac{4}{3}$. Thus the maximum value of k is $\frac{4}{3}$, so $9m = 12$. □

Proof 2. Similar to Proof 1, we get $\tan\alpha = \dfrac{\sin^2\beta}{1-\sin\beta\cos\beta}$. Now consider

$$\begin{aligned}
\cot\alpha &= \frac{1-\sin\beta\cos\beta}{\sin^2\beta} \\
&= \frac{1}{\sin^2\beta} - \frac{\cos\beta}{\sin\beta} \\
&= \csc^2\beta - \cot\beta \\
&= \cot^2\beta - \cot\beta + 1 \\
&= \left(\cot^2\beta - \frac{1}{2}\right)^2 + \frac{3}{4} \\
&\geq \frac{3}{4},
\end{aligned}$$

hence $\tan\alpha \leq \frac{4}{3}$, so $9m = 12$. □

5 Exercises

1. Given that $\alpha, \beta \in \left(\frac{3\pi}{4}, \pi\right)$ such that $\cos(\alpha+\beta) = \frac{4}{5}$ and $\sin\left(\alpha - \frac{\pi}{4}\right) = \frac{12}{13}$, find the value of $-130\cos\left(\beta + \frac{\pi}{4}\right)$.

   Answer. 32

   Proof. First, note that $\alpha+\beta$ is in quadrant IV and $\alpha - \frac{\pi}{4}$ is in quadrant I. Thus

$$\begin{aligned}
\cos\left(\beta + \frac{\pi}{4}\right) &= \cos\left((\alpha+\beta) - \left(\alpha - \frac{\pi}{4}\right)\right) \\
&= \cos(\alpha+\beta)\cos\left(\alpha - \frac{\pi}{4}\right) + \sin(\alpha+\beta)\sin\left(\alpha - \frac{\pi}{4}\right) \\
&= \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right) \\
&= -\frac{16}{65},
\end{aligned}$$

   so $-130\cos x = 32$. □

2. Evaluate $256 \sin 10° \sin 30° \sin 50° \sin 70°$.

Answer. 16

Proof.

$$\begin{aligned}
&256 \sin 10° \sin 30° \sin 50° \sin 70° \\
&= 128 \sin 10° \sin 50° \sin 70° \\
&= 64 \sin 10° (\cos 20° - \cos 120°) \text{ (product to sum)} \\
&= 64 \sin 10° \cos 20° + 32 \sin 10° \\
&= 32(\sin 30° - \sin 10°) + 32 \sin 10° \text{ (product to sum)} \\
&= 32 \sin 30° \\
&= 16.
\end{aligned}$$

□

3. Find the value of $\sin^2 1° + \sin^2 2° + \sin^2 3° + ... + \sin^2 360°$.

Answer. 180

Proof.

$$\begin{aligned}
&\sin^2 1° + \sin^2 2° + \sin^2 3° + ... + \sin^2 360° \\
&= 4(\sin^2 1° + \sin^2 2° + \sin^2 3° + ... + \sin^2 89°) + 2(\sin^2 90° + \sin^2 180°) \\
&= 4(\sin^2 1° + \sin^2 2° + ... + \sin^2 45° + \cos^2 44° + ... + \cos^2 1°) + 2 \\
&= 4((\sin^2 1° + \cos^2 1°) + (\sin^2 2° + \cos^2 2°) + ... + (\sin^2 44° + \cos^2 44°) + \sin^2 45°) + 2 \\
&= 4(44 + \frac{1}{2}) + 2 \\
&= 180.
\end{aligned}$$

□

4. Prove the identity $\cos x \cos 2x \cos 4x = \dfrac{\sin 8x}{8 \sin x}$.

Proof.

$$\begin{aligned}
\cos x \cos 2x \cos 4x &= \frac{8 \sin x \cos x \cos 2x \cos 4x}{8 \sin x} \\
&= \frac{4 \sin 2x \cos 2x \cos 4x}{8 \sin x} \\
&= \frac{2 \sin 4x \cos 4x}{8 \sin x} \\
&= \frac{\sin 8x}{8 \sin x}
\end{aligned}$$

□

(a) Find the value of $\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7}$.

Answer. $-\dfrac{1}{8}$

Proof.

$$\begin{aligned}
\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} &= \frac{\sin\frac{8\pi}{7}}{8\sin\frac{\pi}{7}} \\
&= \frac{-\sin\frac{\pi}{7}}{8\sin\frac{\pi}{7}} \\
&= -\frac{1}{8}.
\end{aligned}$$

□

(b) Find the value of $\cos 20^\circ \cos 40^\circ \cos 80^\circ$.

Answer. $\frac{1}{8}$

Proof.

$$\begin{aligned}
\cos 20^\circ \cos 40^\circ \cos 80^\circ &= \frac{\sin 160^\circ}{8\sin 20^\circ} \\
&= \frac{\sin 20^\circ}{8\sin 20^\circ} \\
&= \frac{1}{8}.
\end{aligned}$$

□

5. Given that $\cos\alpha$ and $\cos\beta$ are roots of the equation $5x^2 - 3x - 1 = 0$, where α and β are acute angles, find the value of $10\sqrt{7}\sin\alpha\sin\beta$.

Answer. 14

Proof. From Vieta's formula, we know that $\cos\alpha + \cos\beta = \frac{3}{5}$ and $\cos\alpha\cos\beta = -\frac{1}{5}$. Now

$$\begin{aligned}
\sin^2\alpha\sin^2\beta &= (1-\cos^2\alpha)(1-\cos^2\beta) \\
&= 1 - \cos^2\alpha - \cos^2\beta + \cos^2\alpha\cos^2\beta \\
&= (1+\cos\alpha\cos\beta)^2 - (\cos\alpha+\cos\beta)^2 \\
&= \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 \\
&= \frac{7}{25}.
\end{aligned}$$

Since α and β are acute angles, $\sin\alpha\sin\beta = \frac{\sqrt{7}}{5}$, so the answer is 14. □

6. Given that $\frac{\cos 3x}{\cos x} = \frac{1}{3}$, find the value of $\frac{\sin 3x}{\sin x}$.

Answer. $\frac{7}{3}$

Proof. Note that

$$\begin{aligned}
\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} &= \frac{\sin 3x\cos x - \cos 3x\sin x}{\sin x\cos x} \\
&= \frac{\sin 2x}{\frac{1}{2}\sin 2x} \\
&= 2,
\end{aligned}$$

so $\frac{\sin 3x}{\sin x} = \frac{\cos 3x}{\cos x} + 2 = \frac{7}{3}$. □

7. Find the value of $\csc 10^\circ - 4\sin 70^\circ$.

Answer. 2

Proof.

$$\begin{aligned}
\csc 10^\circ - 4\sin 70^\circ &= \frac{1}{\sin 10^\circ} - 4\cos 20^\circ \\
&= \frac{1 - 4\sin 10^\circ \cos 20^\circ}{\sin 10^\circ} \\
&= \frac{1 - 2(\sin 30^\circ - \sin 10^\circ)}{\sin 10^\circ} \\
&= \frac{2\sin 10^\circ}{\sin 10^\circ} \\
&= 2.
\end{aligned}$$

□

8. Find the value of $\sin 20^\circ \cos^2 25^\circ - \sin 20^\circ \sin^2 25^\circ + \cos^2 50^\circ + \sin^2 20^\circ$.

Answer. $\frac{3}{4}$

Proof.

$$\begin{aligned}
&\sin 20^\circ \cos^2 25^\circ - \sin 20^\circ \sin^2 25^\circ + \cos^2 50^\circ + \sin^2 20^\circ \\
&= \sin 20^\circ (\cos^2 25^\circ - \sin^2 25^\circ) + \cos^2 50^\circ + \sin^2 20^\circ \\
&= \sin 20^\circ \cos 50^\circ + \cos^2 50^\circ + \sin^2 20^\circ \\
&= \cos 50^\circ (\sin 20^\circ + \sin 40^\circ) + \sin^2 20^\circ \\
&= \cos 50^\circ (2\sin 30^\circ \cos 10^\circ) + \sin^2 20^\circ \\
&= \cos 50^\circ \cos 10^\circ + \sin^2 20^\circ \\
&= \frac{\cos 60^\circ + \cos 40^\circ}{2} + \frac{1 - \cos 40^\circ}{2} \\
&= \frac{3}{4}.
\end{aligned}$$

□

9. Find the value of $\dfrac{\sin 80^\circ}{\sin 20^\circ} - \dfrac{\sqrt{3}}{2\sin 80^\circ}$.

Answer. 2

Proof.

$$\begin{aligned}
\frac{\sin 80^\circ}{\sin 20^\circ} - \frac{\sqrt{3}}{2\sin 80^\circ} &= \frac{\sin 80^\circ}{\sin 20^\circ} - \frac{\sin 60^\circ}{\sin 80^\circ} \\
&= \frac{\sin^2 80^\circ - \sin 60^\circ \sin 20^\circ}{\sin 20^\circ \sin 80^\circ} \\
&= \frac{\cos^2 10^\circ - \frac{\cos 40^\circ - \cos 80^\circ}{2}}{\frac{\cos 60^\circ - \cos 100^\circ}{2}} \\
&= \frac{2\cos^2 10^\circ - \cos 40^\circ + \cos 80^\circ}{\cos 60^\circ + \cos 80^\circ} \\
&= \frac{1 + \cos 20^\circ - \cos 40^\circ + \cos 80^\circ}{\cos 60^\circ + \cos 80^\circ} \\
&= \frac{1 + 2\sin 10^\circ \sin 30^\circ + \cos 80^\circ}{\cos 60^\circ + \cos 80^\circ} \\
&= \frac{1 + 2\cos 80^\circ}{\frac{1}{2} + \cos 80^\circ} \\
&= 2.
\end{aligned}$$

□

10. Find the smallest positive integer n such that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + ... + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}.$$

Answer. 1

Proof. Observe that

$$\begin{aligned}
\frac{1}{\sin n \sin(n+1^\circ)} &= \frac{1}{\sin 1^\circ} \cdot \frac{\sin((n+1^\circ)-n)}{\sin n \sin(n+1^\circ)} \\
&= \frac{1}{\sin 1^\circ} \cdot \frac{\sin(n+1^\circ)\cos n - \cos(n+1^\circ)\sin n}{\sin n \sin(n+1^\circ)} \\
&= \frac{1}{\sin 1^\circ}\left(\frac{\cos n}{\sin n} - \frac{\cos(n+1^\circ)}{\sin(n+1^\circ)}\right) \\
&= \frac{1}{\sin 1^\circ}(\cot n - \cot(n+1^\circ)),
\end{aligned}$$

so

$$\begin{aligned}
&\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + ... + \frac{1}{\sin 133^\circ \sin 134^\circ} \\
&= \frac{1}{\sin 1^\circ}(\cot 45^\circ - \cot 46^\circ + \cot 47^\circ - \cot 48^\circ + ... + \cot 133^\circ - \cot 134^\circ).
\end{aligned}$$

Since $\cot(180^\circ - x) = -\cot x$, the expression reduces to

$$\frac{1}{\sin 1^\circ}(\cot 45^\circ - \cot 90^\circ) = \frac{1}{\sin 1^\circ}.$$

Therefore the smallest positive value of n is 1. □

11. Find the value of $100\left|\cos\frac{\pi}{15} - \cos\frac{2\pi}{15} - \cos\frac{4\pi}{15} + \cos\frac{7\pi}{15}\right|$.

Answer. 50

Proof.

$$\begin{aligned}
&\cos\frac{\pi}{15}-\cos\frac{2\pi}{15}-\cos\frac{4\pi}{15}+\cos\frac{7\pi}{15}\\
&=\left(\cos\frac{\pi}{15}+\cos\frac{7\pi}{15}\right)-\left(\cos\frac{2\pi}{15}+\cos\frac{4\pi}{15}\right)\\
&=2\cos\frac{4\pi}{15}\cos\frac{\pi}{5}-2\cos\frac{\pi}{5}\cos\frac{\pi}{15}\\
&=2\cos\frac{\pi}{5}\left(\cos\frac{4\pi}{15}-\cos\frac{\pi}{15}\right)\\
&=2\cos\frac{\pi}{5}\left(-2\sin\frac{\pi}{6}\sin\frac{\pi}{10}\right)\\
&=-2\cos\frac{\pi}{5}\sin\frac{\pi}{10}
\end{aligned}$$

But since $\cos\frac{\pi}{10}=\sin\frac{2\pi}{5}=2\sin\frac{\pi}{5}\cos\frac{\pi}{5}=4\sin\frac{\pi}{10}\cos\frac{\pi}{10}\cos\frac{\pi}{5}$ which implies $4\cos\frac{\pi}{5}\sin\frac{\pi}{5}=1$, we have $2\cos\frac{\pi}{5}\sin\frac{\pi}{10}=\frac{1}{2}$, thus the answer is $100\left|-\frac{1}{2}\right|=50$. ☐

12. If m is the maximum value of $\cos^3 x+\sin^2 x-\cos x$ for $x\in\mathbb{R}$, find the value of $108m$.

Answer. 128

Proof. We have

$$\begin{aligned}
\cos^3 x+\sin^2 x-\cos x&=\cos x(\cos^2 x-1)+\sin^2 x\\
&=\sin^2 x-\sin^2 x\cos x\\
&=\sin^2 x(1-\cos x)\\
&=2\sin^2 x\sin^2\frac{x}{2}\\
&=8\sin^4\frac{x}{2}\cos^2\frac{x}{2}\\
&=4\sin^2\frac{x}{2}\cdot\sin^2\frac{x}{2}\cdot 2\cos^2\frac{x}{2}\\
&\le 4\left(\frac{\sin^2\frac{x}{2}+\sin^2\frac{x}{2}+2\cos^2\frac{x}{2}}{3}\right)^3 \quad \text{(AMGM)}\\
&=\frac{32}{27},
\end{aligned}$$

thus $108m=128$. ☐

13. Let $0<\theta<\pi$. Find the minimum value of $\dfrac{100}{\sqrt{3}\left(\sin\frac{\theta}{2}\right)(1+\cos\theta)}$.

Answer. 75

Proof. We have

$$\begin{aligned}
\sin\frac{\theta}{2}(1+\cos\theta) &= 2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\\
&= 2\cdot 2\left(\sin^2\frac{\theta}{2}\cdot\frac{1}{2}\cos^2\frac{\theta}{2}\cdot\frac{1}{2}\cos^2\frac{\theta}{2}\right)^{\frac{1}{2}}\\
&\leq 4\left(\frac{\sin^2\frac{\theta}{2}+\frac{1}{2}\cos^2\frac{\theta}{2}+\frac{1}{2}\cos^2\frac{\theta}{2}}{3}\right)^{\frac{3}{2}}\\
&= \frac{4}{3\sqrt{3}},
\end{aligned}$$

so $\dfrac{100}{\sqrt{3}\left(\sin\frac{\theta}{2}\right)(1+\cos\theta)} \geq 75.$ □

14. Find the maximum value of $3\sin\left(x+\frac{\pi}{9}\right)+5\sin\left(x+\frac{4\pi}{9}\right)$, where x ranges over all real numbers.

Answer. 7

Proof. By letting $y = x+\frac{\pi}{9}$, the expression becomes $3\sin y+5\sin\left(y+\frac{\pi}{3}\right)$, where the second term can be written as

$$5\sin\left(y+\frac{\pi}{3}\right) = 5\sin y\cos\frac{\pi}{3}+5\cos y\sin\frac{\pi}{3} = \frac{5}{2}\sin y+\frac{5\sqrt{3}}{2}\cos y.$$

Thus we can write the given expression as

$$\begin{aligned}
\frac{11}{2}\sin y+\frac{5\sqrt{3}}{2}\cos y &= \sqrt{\left(\frac{11}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2}\sin\left(y+\tan^{-1}\frac{5\sqrt{3}}{11}\right)\\
&\leq 7.
\end{aligned}$$

□

15. Determine the minimum value of $\dfrac{\sec^4\alpha}{\tan^2\beta}+\dfrac{\sec^4\beta}{\tan^2\alpha}$ over all $\alpha,\beta\in\mathbb{R}$.

Answer. 8

Proof. We have

$$\begin{aligned}
\frac{\sec^4\alpha}{\tan^2\beta}+\frac{\sec^4\beta}{\tan^2\alpha} &= \frac{\cos^2\beta}{\sin^2\beta\cos^4\alpha}+\frac{\cos^2\alpha}{\sin^2\alpha\cos^4\beta}\\
&\geq \frac{2}{\sin\alpha\cos\alpha\sin\beta\cos\beta} \quad \text{(AMGM)}\\
&\geq \frac{2}{\left(\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{2}\right)^2} \quad \text{(AMGM)}\\
&= \frac{8}{(\sin(\alpha+\beta))^2}\\
&\geq 8.
\end{aligned}$$

□

16. Evaluate

$$\cos a \cos 2a \cos 3a \ldots \cos 999a,$$

where $a = \dfrac{2\pi}{1999}$.

Answer. $\dfrac{1}{2^{999}}$

Proof. Let

$$\begin{aligned} P &= \cos a \cos 2a \cos 3a \ldots \cos 999a, \\ Q &= \sin a \sin 2a \sin 3a \ldots \sin 999a. \end{aligned}$$

Then

$$\begin{aligned} 2^{999} PQ &= (2\sin a \cos a)(2\sin 2a \cos 2a) \ldots (2\sin 999a \cos 999a) \\ &= \sin 2a \sin 4a \ldots \sin 1998a \\ &= (\sin 2a \sin 4a \ldots \sin 998a)(-\sin(2\pi - 1000a))(-\sin(2\pi - 1002a)) \ldots (-2\sin(2\pi - 1998a)) \\ &= \sin 2a \sin 4a \ldots \sin 998a \sin 999a \sin 997a \ldots \sin a \\ &= Q. \end{aligned}$$

Since clearly $Q \neq 0$, we must have $P = \dfrac{1}{2^{999}}$. □

17. In $\triangle ABC$, a, b, c are sides of the triangle opposite to $\angle A, \angle B, \angle C$ respectively. Given that $1 + \dfrac{\tan B}{\tan A} = \dfrac{2c}{\sqrt{3}a}$, find $\angle B$ in degrees.

Answer. 30

Proof. From the given conditions, we have

$$\begin{aligned} 1 + \frac{\frac{\sin B}{\cos B}}{\frac{\sin A}{\cos A}} &= \frac{2\sin C}{\sqrt{3}\sin A} \text{ (sine rule)} \\ \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} &= \frac{2}{\sqrt{3}} \frac{\sin C}{\cos A} \\ \sin A \cos B + \sin B \cos A &= \frac{2}{\sqrt{3}} \sin C \cos B \end{aligned}$$

But since $\sin A \cos B + \sin B \cos A = \sin(A + B) = \sin(180^\circ - A - B) = \sin C$,

$$\begin{aligned} \sin C &= \frac{2}{\sqrt{3}} \sin C \cos B \\ \cos B &= \frac{\sqrt{3}}{2} \end{aligned}$$

which implies $B = 30^\circ$. □

18. In $\triangle ABC$, if $AC = \sqrt{2}$, $AB = 2$ and $\dfrac{\sqrt{3}\sin A + \cos A}{\sqrt{3}\cos A - \sin A} = \tan \dfrac{5\pi}{12}$, find the value of BC^2.

Answer. 2

Proof. Notice that

$$\frac{\sqrt{3}\sin A+\cos A}{\sqrt{3}\cos A-\sin A}=\frac{2\sin\left(A+\frac{\pi}{6}\right)}{2\cos\left(A+\frac{\pi}{6}\right)}=\tan\left(A+\frac{\pi}{6}\right),$$

so we have $\tan\left(A+\frac{\pi}{6}\right)=\tan\frac{5\pi}{12}$, which implies $A=\frac{\pi}{4}$. Now by cosine rule,

$$\begin{aligned}BC^2&=AB^2+AC^2-2\cdot AB\cdot AC\cdot\cos A\\&=4+2-2\cdot 2\cdot\sqrt{2}\cdot\frac{\sqrt{2}}{2}\\&=2.\end{aligned}$$

□

19. In $\triangle ABC$, let $a=BC$, $b=CA$, $c=AB$, $A=\angle A$, $B=\angle B$, $C=\angle C$. Given that $\tan C=\dfrac{\sin A+\sin B}{\cos A+\cos B}$, $\sin(B-A)=\cos C$ and the area of $\triangle ABC$ is $3+\sqrt{3}$, find the value of a^2.

Answer. 8

Proof. From $\tan C=\dfrac{\sin A+\sin B}{\cos A+\cos B}$, we have

$$\begin{aligned}\frac{\sin C}{\cos C}&=\frac{\sin A+\sin B}{\cos A+\cos B}\\\sin C\cos A+\sin C\cos B&=\cos C\sin A+\cos C\sin B\\\sin C\cos A-\cos C\sin A&=\cos C\sin B-\sin C\cos B\\\sin(C-A)&=\sin(B-C)\end{aligned}$$

Since it is impossible to have $C-A=180^\circ-(B-C)\implies B-A=180^\circ$, we must have $C-A=B-C\implies 2C=A+B$. This means $C=60^\circ$ and $A+B=120^\circ$. But we also have $\sin(B-A)=\cos C=\dfrac{1}{2}$, so $B-A=30^\circ$. Therefore we have $A=45^\circ$, $B=75^\circ$ and $C=60^\circ$. Now $3+\sqrt{3}=[ABC]=\dfrac{1}{2}ac\sin B=\frac{\sqrt{6}+\sqrt{2}}{8}ac$. On the other hand, by sine rule, $\dfrac{a}{\sin A}=\dfrac{c}{\sin C}$, or $\dfrac{a}{\frac{\sqrt{2}}{2}}=\dfrac{c}{\frac{\sqrt{3}}{2}}$. Solving, we get $a=2\sqrt{2}$, so $a^2=8$. □

20. Find the number of solutions of the equation $\tan 645x-\tan 670x+\tan 695x=0$ for $x\in[0,\pi]$.

Answer. 666

Proof. Left as an exercise to the reader. □