MIG Telegram Problem 2024/2 — 29 Jul 2024

Everyone knows that Tails the Fox has two really, really fluffy tails with a lot of really, really fluffy tail fur. As such, when Tails was brushing his really, really fluffy tails one day, he naturally wondered just how much really, really fluffy tail fur he had. This would have been impossible to count, had he not had his best friend Sonic who could do things really, really fast, including counting individual tail furs. And even then, it took him 5 full days just to count all of Tails' really, really fluffy tail furs. Maybe it would have been faster if Sonic hadn't been taking the opportunity to brush and fluff with Tails' tails, which would only serve to make them even fluffier, and occasionally put Sonic to sleep due to just how really, really fluffy they are. And by the end, Sonic had already lost count! However, he remembers that:

1. On the first day, he could only brush Tails' tails once and count 1 tail fur before he lost focus, fluffed with Tails' tails and immediately fell asleep.
2. Before the end of every subsequent day, Sonic would have fallen asleep a number of times equal to the total number of days so far including that day.
3. Before every subsequent time he falls asleep, Sonic would have fluffed with Tails' tails a number of times equal to the number of times he had fluffed with Tails' tails before the previous time he had fallen asleep, plus the total number of times he had fallen asleep including this time.
4. Before each time he fluffs with Tails' tails, Sonic would have brushed them a number of times equal to the number of times he had brushed them before the previous time he had fluffed with them, plus the total number of times he had fluffed with them including this time.
5. After brushing Tails' tails but before next fluffing with them, Sonic would have counted as many tail furs as the total number of times he had brushed them.

Note that all 5 events (days passing, Sonic sleeping, fluffing with and brushing Tails' tails and counting tail furs) only start being counted from and including the first day, and the cumulative numbers of times each event occurs never reset throughout the 5 days. We can also assume that even though Sonic lost count of the number of furs, he remembers which tail furs he has already counted, even across days, so he does not miss any fur out or count any fur multiple times across the whole 5 days. Given this information, can you help Tails calculate exactly how many really, really fluffy tail furs he has on his really, really fluffy tails?

As a sanity check, Sonic would have counted a total of 715 really, really fluffy tail furs by the end of the second day.

Bonus question: Guess who made this problem?

(Tan Yong Xin, Jeryoan)

Answer. 8987726770

Solution. The number of times Sonic falls asleep is the sum of the first 5 integers, or the 5th triangular number, calculated as $\frac{5\times6}{2} = 15$.

The number of times Sonic fluffs with Tails' tails is the sum of the first 15 triangular number, or the 15th tetrahedral number.

If you know/googled the formula, this is $\frac{17\times16\times15}{1\times2\times3} = 680$. If you don't know the formula, you can notice that the first diagonal of Pascal's triangle is all 1s, the second diagonal is the positive integers, the third diagonal is the triangular numbers and the fourth diagonal is the tetrahedral

numbers.

If you don't see why this must be true, try to find the number of ways we can split a group of n identical objects into $k+1$ groups, where groups can contain no objects, but obviously not a negative number of objects. This can be done by placing the objects in a row and inserting k identical partitions between objects, where the $k+1$ groups are the objects between partitions and at the edges of the row. This is equivalent to choosing k spots to fill with partitions in a row containing $n+k$ spots, and filling the rest of the spots with objects. It is now trivial to see that there are $\binom{n+k}{k}$ ways to do so. At the same time, consider a possible algorithm to sequentially generate all possible ways as follows:

- If there are no partitions, then it is trivial to see that there is only 1 way, with all objects in the same group. Similarly, $\binom{n+0}{0}$ forms the first diagonal in Pascal's triangle, which is all 1s.
- If there is one partition, select the spot to contain the partition. There are $n+1$ ways to choose this spot, and as such there are $n+1$ ways. Similarly, $\binom{n+1}{1}$ forms the second diagonal, which is the positive integers.
- If there are 2 partitions, first select the spot to contain the left partition. There are still $n+1$ ways to choose this spot, even though there are $n+2$ spots now, since it cannot be at the last spot, otherwise the right partition would have to be on the left of it. If it is at the first spot, there are $n+1$ ways to choose the position of the right partition, and for every subsequent spot, there is one less way. Thus, it forms the triangular numbers, just like $\binom{n+2}{2}$.

It should now be trivial to see why the third diagonal forms the tetrahedral numbers, and thus the 15th tetrahedral number is $\binom{17}{3} = 680$.

Finally, the number of furs Sonic counts is the sum of the first 680 tetrahedral numbers, or the 680th 5-cellular number, or the 680th number in the fourth diagonal of Pascal's Triangle, or $\binom{680+3}{4}$, or simply $\frac{680\times681\times682\times683}{1\times2\times3\times4} = 8987726770$. Guess you could say... that escalated really quickly. In conclusion, Tails has a total of 8987726770 really, really fluffy tail furs, which is why it comes as no surprise that Tails has two really, really fluffy tails and as such is really, really fluffy.

Remark. Alternatively, you may also list out the first few terms to spot the pattern. □