MIG Telegram Problem 2024/1 — 19 Jul 2024

Let $s_n(k)$ denote the sum of digits of a positive integer k in its base-n representation. For every positive integer n, let $f(n)$ be the number of positive integers m such that for every positive integer k, k is divisible by m if and only $s_n(k)$ divisible by m. For example, $f(10) = 3$ as $m = 1, 3, 9$ satisfies the condition. Evaluate

$$\frac{f(4^2)f(8^2)f(12^2)\cdots f(2024^2)}{f(2^2)f(6^2)f(10^2)\cdots f(2022^2)}.$$

Authored by Ding Chenghao

Answer. 15

Solution. Let $d(n)$ denote the number of positive integer divisors of n, and denote the condition "for every positive integer k, k is divisible by m if and only $s_n(k)$ divisible by m" by (*). The main claim is as follows.

Claim. $f(n) = d(n-1)$.

Proof. In fact, we will prove an integer m satisfies the condition (*) if and only if it is a divisor of $n-1$.

Suppose an integer m satisfies the condition (*). First, we will prove $m < n$. By considering $k = m$ in (*), we obtain $m \mid m \iff m \mid s_n(m)$, so $m \mid s_n(m)$ for all such m. But if $m \geq n$, then $m > s_n(m) > 0$, so m cannot divide $s_n(m)$, a contradiction.

Now we prove $m \mid n-1$. By considering $k = m(n-1)$ in (*), we obtain $m \mid m(n-1) \iff m \mid s_n(m(n-1))$, so $m \mid s_n(m(n-1))$ for all such m. But since $m < n$, we have

$$m(n-1) = (m-1)n + (n-m) \implies m(n-1) = \overline{[m-1][n-m]}_n,$$

i.e. $s_n(m(n-1)) = (m-1) + (n-m) = n-1$. Thus $m \mid n-1$.

On the other hand, we can prove that all $m \mid n-1$ works. Consider any base-n number

$$A = \overline{b_k b_{k-1} \dots b_1 b_0}_n = n^k b_k + n^{k-1} b_{k-1} + \cdots + n b_1 + b_0.$$

We have

$$\begin{aligned} A &= n^k b_k + n^{k-1} b_{k-1} + \cdots + n b_1 + b_0 \\ &\equiv 1^k b_k + 1^{k-1} b_{k-1} + \cdots + 1 b_1 + b_0 \pmod m \text{ (since } m \mid n-1) \\ &\equiv b_k + b_{k-1} + \cdots + b_1 + b_0 \pmod m \\ &\equiv s_n(A) \pmod m, \end{aligned}$$

which implies $m \mid A \iff m \mid s_n(A)$ as desired. □

Now it remains to evaluate

$$\frac{d(4^2-1)d(8^2-1)d(12^2-1)\cdots d(2024^2-1)}{d(2^2-1)d(6^2-1)d(10^2-1)\cdots d(2022^2-1)}.$$

At this point, the following property of $d(n)$ is useful.

Property. $d(ab) = d(a)d(b)$ if a and b are coprime. (This can be proven by considering their prime factorisations.)

In particular, for any **even** n, we have $d(n^2-1) = d((n-1)(n+1)) = d(n-1)d(n+1)$ since $\gcd(n-1, n+1) = \gcd(n-1, 2) = 1$ (because $n-1$ is odd).

With this, the required expression finally reduces to

$$\frac{d(3)d(5)d(7)d(9)\cdots d(2023)d(2025)}{d(1)d(3)d(5)d(7)\cdots d(2021)d(2023)} = \frac{d(2025)}{d(1)} = \boxed{15}.$$

$\square$