1. $P(\text{at least } 1 \text{ correct}) = 1 - \dfrac{D(N)}{N!}$

Derangement's formula,

$$D(N) = N!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots \frac{(-1)^N}{N!}\right)$$

probability of all letters misplaced,

$$\frac{D(N)}{N!} = \sum_{k=0}^{N} \frac{(-1)^k}{k!}$$

as $N \to \infty$, $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1}$

for $N = 50$, error term $\dfrac{(-1)^{51}}{51}$ is negligible,

$$\frac{D(50)}{50!} \approx \frac{1}{e}$$

$$P(\text{at least one correct}) \approx 1 - \frac{1}{e}$$

$$\boxed{P \approx 1 - \frac{1}{e} \approx 0.6321}$$

2. Initially, it is equally likely that \$1000 present in any present.

$$\left.\begin{array}{l} P_1 : 1/3 \\ P_2 : 1/3 \\ P_3 : 1/3 \end{array}\right\} \text{Probability of good gift}$$

Case 1: \$1000 in P_1 (you picked it)

— Probability $= 1/3$.

host opens 2 or 3 with equal probability $\Rightarrow 1/2$.

Given, host opened P_2 with $P = 1/2$.

Case 2: \$1000 in $P_2 \Rightarrow P = 1/3$

. Host cannot open P_2.

must open P_3

since, you picked P1, host must open P3.

This case is ruled out (since 2 is opened).

<u>Case 3:</u> if $ 1000 in P3.

host must open P2.

$P = 1$.

- Conditional prob. (Picked 1 & opened 2)

$P(G1 \cap P_2) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.

$G1 \downarrow$ money in P_1 ; $P_2 \downarrow$ opened P_2

$P(G_3 \cap P_2) = \frac{1}{3} \times 1 = \frac{1}{3}$.

$$P(G_2 \cap P_2) = 0$$

$$P(\text{host opens } P_2) = \frac{1}{2}.$$

So, $$P(G1 \mid \text{host opens } 2) = \frac{1/6}{1/2} = \frac{1}{3}$$

$$P(G_3 \mid \text{host opens } 2) = \frac{1/3}{1/2} = \frac{2}{3}.$$

switching gives you a $2/3$ chance to win \$1000.

Expected winnings after switch

$$= \frac{2}{3} \times \$1000$$

$$\approx \$666.67$$

3.

(a) $P(A \cap B \mid C) = \dfrac{P(A \cap B \cap C)}{P(C)}$

also, $P(A \mid B \cap C) = \dfrac{P(A \cap B \cap C)}{P(B \cap C)}$

$P(B \mid C) = \dfrac{P(B \cap C)}{P(C)}$

$$P(A \mid B \cap C) \cdot P(B \mid C) = \frac{P(A \cap B \cap C)}{P(B \cap C)} \times \frac{P(B \cap C)}{P(C)}$$

$$= \frac{P(A \cap B \cap C)}{P(C)}$$

$$= P(A \cap B \mid C)$$

True

(b) $P(A \cap B | C) = P(A|C)\, P(B|C)$

for independent events A and B.

Sol^n

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A \cap B | C) = P(A|C) \cdot P(B|C)$$

False

(c) Given $\mathbb{P}(A|D \cap B^c) > \mathbb{P}(A|D \cap B)$ and $\mathbb{P}(A|D^c \cap B^c) > \mathbb{P}(A|D^c \cap B)$, $\mathbb{P}(A|B)$ must be greater than $\mathbb{P}(A|B^c)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B \cap D) + P(A \cap B \cap D^c)}{P(B)}$$

$$= \frac{P(A \cap B \cap D)}{P(B \cap D)} \frac{P(B \cap D)}{P(B)} + \frac{P(A \cap B \cap D^c)}{P(B \cap D^c)} \frac{P(B \cap D^c)}{P(B)}$$

$$= P(A|D \cap B)\, P(D|B) + P(A|D^c \cap B)\, P(D^c|B)$$

Similarly,

$$P(A|B^c) = P(A|D \cap B^c) P(D|B^c) + P(A|D^c \cap B^c) P(D^c|B^c)$$

Now, $P(A|D \cap B^c) > P(A|D \cap B)$ &

$P(A|D^c \cap B^c) > P(A|D^c \cap B)$ $\Rightarrow$ False

$\therefore$ Not necessarily True

4.) (a) Let X take values $n = 1, 2, 3, \dots$ with probability $p_n = \frac{c}{n^3}$

Sum, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, so $c = \frac{1}{\zeta(3)}$

$$E[X] = \sum_{n=1}^{\infty} n\, p_n = c \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{converges}$$

$$E[X^2] = \sum_{n=1}^{\infty} n^2 p_n = c \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges (harmonic series)}$$

Therefore exists.

(b) X have density fun, $f(x) = \frac{c}{x^3}$ for $x \geq 1$

$$1 = \int_1^{\infty} \frac{c}{x^3}\, dx = \frac{c}{2} \quad \text{so, } c = 2$$

$$E[X] = \int_1^{\infty} x \cdot 2/x^3 \, dx = 2$$

$$E[X^2] = \int_1^{\infty} x^2 \cdot 2/x^3 \, dx = \infty$$

Therefore exists $\quad f(x) = \begin{cases} 2/x^3 & x \geq 1 \\ 0 & \end{cases}$

(c) $E(e^{-X}) \geq e^{-E[X]} = e^{-1} \approx 0.37$

(By Jensen's inequality).

for any random variable with

$E(X) = 1$ we have $E(e^{-X}) \geq e^{-1} > 1/3$

<u>does not exist</u>

Made with Goodnotes

5. Let $X_1, X_2, \dots X_n$ be the prizes on n tickets drawn, each uniformly dist. on $\{1, 2, \dots N\}$

Prob. max. of these draws is at most k:

$$P(M \leq k) = \left(\frac{k}{N}\right)^n$$

$$P(M = k) = P(M \leq k) - P(M \leq k-1)$$

$$= \left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n$$

$$E(M) = \sum_{k=1}^{N} k \cdot \left[\left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n\right]$$

Let $x = \frac{k}{N}$, so $k = Nx$ and $dx \approx \frac{1}{N}$

$$P(M = k) = \left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n \approx \frac{d}{dk}\left(\frac{k}{N}\right)^n$$

$$= n\left(\frac{k}{N}\right)^{n-1} \frac{1}{N}$$

$$E(M) = \sum_{k=1}^{N} k \cdot n \cdot \left(\frac{k}{N}\right)^{n-1} \frac{1}{N}$$

$$= nN \sum_{k=1}^{N} \left(\frac{k}{N}\right)^n \frac{1}{N}$$

$$E(M) \approx nN \int_0^1 x^n \, dx = nN \left[\frac{x^{n+1}}{n+1} \right]_0^1$$

$$= nN \frac{1}{n+1} = \frac{n}{n+1} N$$

$$\boxed{\begin{array}{l} E(M) = \displaystyle\sum_{k=1}^{N} k \left[\left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n \right] \\[2em] \text{for large } N, \quad E(M) \approx \dfrac{n}{n+1} N \end{array}}$$

6. X and Y be pos^n of two points, dist on $[0, d]$.

$$P\left(|X-Y| < d/3\right)$$

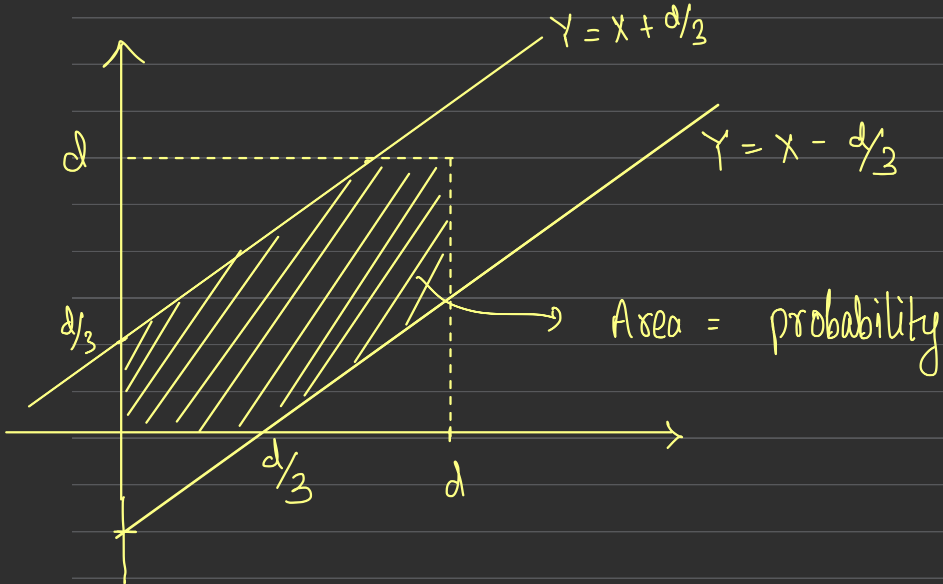
The pair (X, Y) can be repr^n as a point on square $[0,d] \times [0,d]$

area of square $= d^2$

Cond^n defines region bet^n lines

$$Y = X + d/3 \quad \& \quad Y = X - d/3$$

in square.



I^{st}. $X \in [0, d/3]$

$$I_1 = \int_0^{d/3} \left(X + d/3\right) dX = \int_0^{d/3} X \, dX + \int_0^{d/3} \frac{d}{3} \, dX$$

$$= \left[\frac{X^2}{2}\right]_0^{d/3} + \left[\frac{d}{3} X\right]_0^{d/3}$$

$$I_1 = \frac{d^2}{6}$$

II^{nd} : $\left[d/3 , d - d/3\right]$

$$I_2 = \int_{d/3}^{2d/3} \frac{2d}{3} \, dX = \frac{2d^2}{9}$$

III^{rd}. $\left[d - d/3 , d\right]$

$$I_3 = \int_{2d/3}^{d} (d - X) \, dX = \left[dX - \frac{X^2}{2}\right]_{2d/3}^{d}$$

$$I_3 = \frac{d^2}{6}$$

$$\text{Area} = \frac{d^2}{6} + \frac{2d^2}{9} + \frac{d^2}{6} = \frac{5d^2}{9}$$

$$\boxed{\text{Probability} = \frac{\left(\frac{5d^2}{9}\right)}{d^2} = \frac{5}{9}}$$

7) a) At each step, rumor is passed to any one of other n people.

$\therefore P$(original person is not chosen) $= 1 - \frac{1}{n}$

P(original person is not chosen r times)

$$= \left(1 - \frac{1}{n}\right)^{r}$$

b) At each step, rumor must be passed to a new person.

At k^{th} step : $n-k+1$ persons left

$\therefore$ probability of rumor being told r times

$$= \frac{n}{n} \times \frac{n-1}{n} \times \frac{n-2}{n} \cdots \frac{n-(r-1)}{n}$$

$$= \prod_{k=0}^{r-1} \left(1 - \frac{k}{n}\right)$$

Now, Rumor is told to N people at each step

$\therefore$ Total people $= rN$

a) P(original person is not chosen rN times)

$$= \left(1 - \frac{1}{n}\right)^{rN}$$

b) Now $rN \leq n$ $\because$ Every time we choose from $n-k$ people

Probability of no repetition $=$

$$\prod_{k=0}^{rN-1} \left(1 - \frac{k}{n}\right)$$

8) $\mathbb{P}(\cap A_i^c) = \mathbb{P}(A_1^c \cap A_2^c \cap A_3^c \ldots)$

$\because$ A_i's are independent

$\therefore$ $\mathbb{P}(A_1^c \cap A_2^c \cap A_3^c \ldots A_n^c) = \prod_{i=1}^{n} \mathbb{P}(A_i^c)$

$= \prod_{i=1}^{n} (1 - \mathbb{P}(A_i))$

$e^{-P(A_1) - P(A_2) \ldots - P(A_n)}$

$= e^{-\sum_{i=1}^{n} P(A_i)}$

$\because$ $\boxed{1 - x \leq e^{-x}}$ $\forall x > 0$ & $\mathbb{P}(A_i) \geq 0$

→ On applying this inequality to each term

$$\prod_{i=1}^{n} (1 - \mathbb{P}(A_i)) \leq \prod_{i=1}^{n} e^{-\mathbb{P}(A_i)} = e^{-\sum \mathbb{P}(A_i)}$$

→ Proved.

Made with Goodnotes

9) let X & Y be two independent R.V. with pdf $f_X(x)$ & $f_Y(y)$

$$f_Z(z) = f_X * f_Y = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy$$

let $Z = X+Y$

$$f_Z(z) = \frac{d}{dz} P(X+Y \le z)$$

& $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ [Independent]

$\therefore x+y = z$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\, dx = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy \quad \text{[Symmetric]}$$

$$\therefore f_{X+Y}(z) = (f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy$$

→ convolution

$\therefore$ The convolution of two distribution results in a distribution.

10) $\int_\Omega \int_0^\infty I_{[0, X(\omega)]}(x)\, dx\, dP(\omega)$

$\int_0^\infty I_{[0, X(\omega)]}(x)\, dx = X(\omega)$

$\therefore \int_\Omega X(\omega)\, dP(\omega) = EX$

$\because$ Integrand is non-negative, we can switch Integrals

$\therefore \int_\Omega \int_0^\infty I_{[0, X(\omega)]}(x)\, dx\, dP(\omega) = \int_0^\infty \int_\Omega I_{[0, X(\omega)]}(x)\, dP(\omega)\, dx$

$\int_\Omega I_{[0, X(\omega)]}\, dP(\omega)$ $\qquad \because I_{[0, X(\omega)]}(x) = \begin{cases} 1 & x < X(\omega) \\ 0 & o/w \end{cases}$

$\therefore$ Inner Intergral $= P(x < X(\omega))$
$= P(X(\omega) > x) = 1 - F(x)$

$\therefore \int_\Omega \int_0^\infty I_{[0, X(\omega)]}(x)\, dx\, dP(\omega) = \int_0^\infty (1 - F(x))\, dx$

11) $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

(i) $\mathbb{E}\, e^{ux} = \displaystyle\int e^{ux} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$

$$= \frac{1}{\sqrt{\sigma^2 2\pi}} \int e^{ux - \frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

Now,

exponent $= ux - \dfrac{(x-\mu)^2}{2\sigma^2} = \dfrac{-1}{2\sigma^2}\left[(x-\mu)^2 - 2\sigma^2 ux\right]$

$$= \frac{-1}{2\sigma^2}\left[(x-\mu)^2 - 2\sigma^2 u(x-\mu)\right] + u\mu$$

$$= \frac{-1}{2\sigma^2}\left[\left((x-\mu) - \sigma^2 u\right)^2 - \sigma^4 u^2\right] + u\mu$$

$$= -\frac{(x-\mu-\sigma^2 u)^2}{2\sigma^2} + \frac{\sigma^2 u^2}{2} + u\mu$$

$\therefore \mathbb{E}\, e^{ux} = \dfrac{1}{\sqrt{2\pi\sigma^2}} \displaystyle\int e^{-\frac{(x-\mu-\sigma^2 u)^2}{2\sigma^2}} e^{\mu u + \frac{\sigma^2 u^2}{2}}\, dx$

$$= \frac{e^{\mu u + \frac{\sigma^2 u^2}{2}}}{\sqrt{2\pi\sigma^2}} \int e^{-\frac{(x-\mu-\sigma^2 u)^2}{2\sigma^2}}\, dx$$

$$= e^{\mu u + \frac{\sigma^2 u^2}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-(\mu+\sigma^2 u))^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dx$$

$\hookrightarrow$ Pdf with with $\mu + \sigma^2 u$ $\therefore$ Integral $= 1$

$\therefore \mathbb{E}\, e^{ux} = e^{\mu u + \frac{\sigma^2 u^2}{2}}$

(ii) $EX = \mu$ $\qquad \therefore \varphi(EX) = e^{u\mu}$

$\varphi(x) = e^{ux}$

$\therefore E[\varphi(X)] = Ee^{uX} = e^{u\mu + \frac{1}{2}u^2\sigma^2}$

$\because \frac{1}{2}u^2\sigma^2 \geq 0$

$\therefore e^{u\mu + \frac{1}{2}u^2\sigma^2} \geq e^{u\mu}$