**GOVERNMENT OF TAMIL NADU**

# BUSINESS MATHEMATICS AND STATISTICS

## HIGHER SECONDARY FIRST YEAR

### VOLUME - I

## Untouchability is Inhuman and a Crime

## Department of School Education

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# HOW TO USE THE BOOK

**Career Options**

List of Further Studies & Professions.

**Learning Objectives:**

Learning objectives are brief statements that describe what students will be expected to learn by the end of school year, course, unit, lesson or class period.

Additional information about the concept.

Amazing facts, Rhetorical questions to lead students to Mathematical inquiry

**Exercise**

Assess students' critical thinking and their understanding

**ICT**

To enhance digital skills among students

**Web links**

List of digital resources

To motivate the students to further explore the content digitally and take them in to virtual world

**Miscellaneous Problems**

Additional problems for the students

**Glossary**

Tamil translation of Mathematical terms

**References**

List of related books for further studies of the topic

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(iii)

# CAREER OPTIONS IN BUSINESS MATHEMATICS

Higher Secondary students who have taken commerce with Business mathematics can take up careers in BCA, B.Com., and B.Sc. Statistics. Students who have taken up commerce stream, have a good future in banking and financial institutions.

A lot of students choose to do B.Com with a specialization in computers. Higher Secondary Commerce students planning for further studies can take up careers in professional fields such as Company Secretary , Chartered Accountant (CA), ICAI and so on. Others can take up bachelor's degree in commerce (B.Com), followed by M.Com, Ph.D and M.Phil. There are wide range of career opportunities for B.Com graduates.

After graduation in commerce, one can choose MBA, MA Economics, MA Operational and Research Statistics at Postgraduate level. Apart from these, there are several diploma, certificate and vocational courses which provide entry level jobs in the field of commerce.

**Career chart for Higher Secondary students who have taken commerce with Business Mathematics.**

| Courses | Institutions | Scope for further studies |
|---|---|---|
| **B.Com., B.B.A., B.B.M., B.C.A., B.Com (Computer), B.A.** | • Government Arts & Science Colleges, Aided Colleges, Self financing Colleges.<br>• Shri Ram College of Commerce (SRCC), Delhi<br>• Symbiosis Society's College of Arts & Commerce, Pune.<br>• St. Joseph's College, Bangalore | **C.A., I.C.W.A, C.S.** |
| **B.Sc Statistics** | • Presidency College, Chepauk, Chennai.<br>• Madras Christian College, Tambaram<br>• Loyola College, Chennai.<br>• D.R.B.C.C Hindu College, Pattabiram, Chennai. | **M.Sc., Statistics** |
| **B.B.A., LLB, B.A., LLB, B.Com., LL.B. (Five years integrated Course)** | • Government Law College.<br>• School of excellence, Affiliated to Dr.Ambethkar Law University | **M.L.** |
| **M.A. Economics (Integrated Five Year course) – Admission based on All India Entrance Examination** | • Madras School of Economics, Kotturpuram, Chennai. | **Ph.D.,** |
| **B.S.W.** | • School of Social studies, Egmore, Chennai | M.S.W |

# CONTENTS

I7HCTG

E- Book

I7W8V4

Assessment

B184_11_BM_EM

DIGI Links

(v)

| SYLLABUS | |
|---|---|
| **1. MATRICES AND DETERMINANTS** | **(15 periods)** |
| Determinants: Recall - Types of matrices - Determinants Minors – Cofactors - Properties of determinants. Inverse of a matrix: Singular matrix – Non singular matrix - Adjoint of a matrix – Inverse of a matrix - Solution of a system of linear equation. Input - Output analysis: Hawkins Simon conditions. | |
| **2. ALGEBRA** | **(20 periods)** |
| Partial fractions: Denominator contains non - repeated Linear factors - Denominator contains Linear factors, repeated for n times - Denominator contains quadratic factor, which cannot be factorized into linear factors. Permutations: Factorial - Fundamental principle of counting – Permutation - Circular permutation. Combinations - Mathematical Induction - Binomial Theorem. | |
| **3. ANALYTICAL GEOMETRY** | **(20 periods)** |
| Locus: Equation of a locus. System of straight lines: Recall – Various forms of straight lines - Angle between two straight lines – Distance of a point from a line - Concurrence of three lines. Pair of straight lines: Combined equation of pair of straight lines - Pair of straight lines passing through the origin - Angle between pair of straight lines passing through the origin - The condition for general second degree equation to represent the pair of straight lines. Circles: The equation of a circle when the centre and radius are given - Equation of a circle when the end points of a diameter are given - General equation of a circle - Parametric form of a circle – Tangents – Length of tangent to the circle. CONICS: Parabola. | |
| **4. TRIGONOMETRY** | **(25 periods)** |
| Trigonometric ratios: Quadrants - Signs of the trigonometric ratios of an angle $\theta$ as $\theta$ varies from $0^o$ to $360^o$ - Trigonometric ratios of allied angles. Trigonometric ratios of compound angles: Compound angles - Sum and difference formulae of sine, cosine and tangent - Trigonometric ratios of multiple angles. Transformation formulae: Transformation of the products into sum or difference - Transformation of sum or difference into product. Inverse Trigonometric Functions: Properties of Inverse Trigonometric Functions. | |
| **5. DIFFERENTIAL CALCULUS** | **(25 periods)** |
| Functions and their graphs: Some basic concepts – Function - Various types of functions - Graph of a function. Limits and derivatives: Existence of limit - Indeterminate forms and evaluation of limits – Continuous function - Differentiability at a point - Differentiation from first principle. Differentiation techniques: Some standard results - General rules for differentiation - Successive differentiation. | |

# 1

# MATRICES AND DETERMINANTS



**Learning Objectives**

After studying this chapter, the students will be able to understand

- the definition of matrices and determinants
- the properties of determinants
- the concept of inverse matrix
- the concept of adjoint matrix
- the solving simultaneous linear equations
- the input-output analysis

## 1.1 Determinants

### Introduction

The idea of a determinant was believed to be originated from a Japanese Mathematician Seki Kowa (1683) while systematizing the old Chinese method of solving simultaneous equations whose coefficients were represented by calculating bamboos or sticks. Later the German Mathematician Gottfried Wilhelm Von Leibnitz formally developed determinants. The present vertical notation was given in 1841 by Arthur Cayley. Determinant was invented independently by Crammer whose well known rule for solving simultaneous equations was published in 1750.

**Seki Kowa**

In class X, we have studied matrices and algebra of matrices. We have also learnt that a system of algebraic equations can be expressed in the form of matrices. We know that the area of a triangle with vertices $(x_1, y_1)$ $(x_2, y_2)$ and $(x_3, y_3)$ is

$$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

To minimize the difficulty in remembering this type of expression, Mathematicians developed the idea of representing the expression in determinant form and the above expression can be represented in the form

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Thus a determinant is a particular type of expression written in a special concise form. Note that the quantities are arranged in the form of a square between two vertical lines. This arrangement is called a determinant.

In this chapter, we study determinants up to order three only with real entries. Also we shall study various properties of determinant (without proof), minors, cofactors, adjoint, inverse of a square matrix and business applications of determinants.

We have studied matrices in the previous class. Let us recall the basic concepts and operations on matrices.

### 1.1.1 Recall

**Matrix**

> **Definition 1.1**
>
> A matrix is a rectangular arrangement of numbers in horizontal lines (rows) and vertical lines (columns). Numbers are enclosed in square brackets or a open brackets or pair of double bars. It is denoted by A, B, C, …

For example , $A = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 8 & 6 \end{bmatrix}$

**Order of a matrix**

If a matrix A has m rows and n columns, then A is called a matrix of order $m \times n$.

For example, If , $A = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 8 & 6 \end{bmatrix}$ then order of $A$ is $2 \times 3$

**General form of a Matrix**

Matrix of order $m \times n$ is represented as
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1i} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2i} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mi} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

It is shortly written as $[a_{ij}]_{mxn}$  $i = 1, 2, \ldots m; j = 1, 2, \ldots n$ Here $a_{ij}$ is the element in the $i^{th}$ row and $j^{th}$ column of the matrix.

## Types of matrices

### Row matrix

A matrix having only one row is called a row matrix.

For example $A = [\ a_{11}\ a_{12}\ a_{13}\ ............a_{1i}..........a_{1n}]_{1 \times n}$ ; $B = \begin{bmatrix} 1 & 2 \end{bmatrix}_{1 \times 2}$

### Column matrix

A matrix having only one column is called a column matrix

For examples, $A = \begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ . \\ a_{m1} \end{bmatrix}_{m \times 1}$ , $B = \begin{bmatrix} 3 \\ -5 \end{bmatrix}_{2 \times 1}$

### Zero matrix (or) Null matrix

If all the elements of a matrix are zero, then it is called a zero matrix. It is represented by an English alphabet 'O'

> **NOTE**
>
> Zero matrix can be of any order.

For examples, $O = [0]$, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ , $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are all zero matrices.

### Square matrix

If the number of rows and number of columns of a matrix are equal then it is called a square matrix.

For examples, $A = \begin{bmatrix} 8 & 2 \\ 4 & 2 \end{bmatrix}$ is a square matrix of order 2

$B = \begin{bmatrix} 3 & -1 & 0 \\ \sqrt{2} & 0 & 5 \\ 2 & \frac{3}{5} & -4 \end{bmatrix}$ is a square matrix of order 3.

> **NOTE**
>
> The sum of the diagonal elements of a square matrix is called the **trace** of matrix.

### Triangular matrix

A square matrix, whose elements above or below the main diagonal (leading diagonal) are all zero is called a triangular matrix.

For examples, $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ , $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 7 \end{bmatrix}$

## Diagonal matrix

A square matrix in which all the elements other than the main diagonal elements are zero is called the diagonal matrix.

For example, $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a diagonal matrix of order 3

## Scalar matrix

A diagonal matrix with all diagonal elements are equal to K (a scalar) is called a scalar matrix.

For example, $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix of order 3.

## Unit matrix (or) Identity matrix

A scalar matrix having each diagonal element equal to one (unity) is called a Unit matrix.

For examples, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a unit matrix of order 2.

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

## Multiplication of a matrix by a scalar

If $A = [a_{ij}]$ is a matrix of any order and if $k$ is a scalar, then the scalar multiplication of $A$ by the scalar $k$ is defined as $kA = [ka_{ij}]$ for all $i$, $j$.

In other words to multiply a matrix $A$ by a scalar $k$, multiply every element of $A$ by $k$.

For example, if $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & -2 & 8 \end{bmatrix}$ then, $2A = \begin{bmatrix} 2 & 4 & 8 \\ 6 & -4 & 16 \end{bmatrix}$

## Negative of a matrix

The negative of a matrix $A = [a_{ij}]_{m \times n}$ is defined as $-A = [-a_{ij}]_{m \times n}$ for all i, j and is obtained by changing the sign of every element.

For example, if $A = \begin{bmatrix} 2 & -5 & 7 \\ 0 & 5 & 6 \end{bmatrix}$ then $-A = \begin{bmatrix} -2 & 5 & -7 \\ 0 & -5 & -6 \end{bmatrix}$

## Equality of matrices

Two matrices A and B are said to be equal if

(i) they have the same order and   (ii) their corresponding elements are equal.

## Addition and subtraction of matrices

Two matrices $A$ and $B$ can be added, provided both the matrices are of the same order. Their sum $A+B$ is obtained by adding the corresponding entries of both the matrices $A$ and $B$.

Symbolically if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then $A + B = [a_{ij} + b_{ij}]_{m \times n}$

Similarly $A - B = A + (-B) = [a_{ij}]_{m \times n} + [-b_{ij}]_{m \times n} = [a_{ij} - b_{ij}]_{m \times n}$

## Multiplication of matrices

Multiplication of two matrices is possible only when the number of columns of the first matrix is equal to the number of rows of the second matrix. The product of matrices $A$ and $B$ is obtained by multiplying every row of matrix A with the corresponding elements of every column of matrix $B$ element-wise and add the results.

Let $A = [a_{ij}]$ be an $m \times p$ matrix and $B = [b_{ij}]$ be a $p \times n$ matrix, then the product AB is a matrix $C = [c_{ij}]$ of order $m \times n$.

## Transpose of a matrix

Let $A = [a_{ij}]$ be a matrix of order $m \times n$. The transpose of $A$, denoted by $A^T$ of order $n \times m$ is obtained by interchanging either rows into columns or columns into rows of $A$.

For example, if $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix}$

**NOTE**

It is believed that the students might be familiar with the above concepts and our present syllabus continues from the following.

**Definition 1.2**

To every square matrix A of order $n$ with entries as real or complex numbers, we can associate a number called ***determinant*** of matrix $A$ and it is denoted by $|A|$ or det $(A)$ or $\Delta$.

Thus determinant can be formed by the elements of the matrix $A$.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then its $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

**Example 1.1**

Evaluate: $\begin{vmatrix} 2 & 4 \\ -1 & 4 \end{vmatrix}$

*Solution*

$$\begin{vmatrix} 2 & 4 \\ -1 & 4 \end{vmatrix} = (2)(4) - (-1)4$$

$$= 8 + 4 = 12$$

> To evaluate the determinant of order 3 or more, we define minors and cofactors.

## 1.1.2 Minors:

Let $| A | = |[a_{ij}]|$ be a determinant of order $n$. The minor of an arbitrary element $a_{ij}$ is the determinant obtained by deleting the $i^{th}$ row and $j^{th}$ column in which the element $a_{ij}$ stands. The minor of $a_{ij}$ is denoted by $M_{ij}$.

## 1.1.3 Cofactors:

The cofactor is a signed minor. The cofactor of $a_{ij}$ is denoted by $A_{ij}$ and is defined as

$$A_{ij} = (-1)^{i+j} M_{ij}$$

The minors and cofactors of $a_{11}, a_{12}, a_{13}$ in a third order determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ are as follows

(i) Minor of $a_{11}$ is $M_{11}$ $= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

Cofactor of $a_{11}$ is $A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

(ii) Minor of $a_{12}$ is $M_{12}$ $= \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$

Cofactor of $a_{12}$ is $A_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{31}a_{23})$

(iii) Minor of $a_{13}$ is $M_{13}$ $= \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$

Cofactor of is $A_{13}$ $= (-1)^{1+3} M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$

**Example 1.2**

Find the minor and cofactor of all the elements in the determinant $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$

*Solution*

$$\text{Minor of } 1 \quad = \quad M_{11} \quad = \quad 3$$

$$\text{Minor of } -2 \quad = \quad M_{12} \quad = \quad 4$$

$$\text{Minor of } 4 \quad = \quad M_{21} \quad = \quad -2$$

$$\text{Minor of } 3 \quad = \quad M_{22} \quad = \quad 1$$

$$\text{Cofactor of } 1 \quad = \quad A_{11} = (-1)^{1+1} M_{11} = 3$$

$$\text{Cofactor of } -2 \quad = \quad A_{12} = (-1)^{1+2} M_{12} = -4$$

$$\text{Cofactor of } 4 \quad = \quad A_{21} = (-1)^{2+1} M_{21} = 2$$

$$\text{Cofactor of } 3 \quad = \quad A_{22} = (-1)^{2+2} M_{22} = 1$$

## Example 1.3

Find the minor and cofactor of each element of the determinant. $\begin{vmatrix} 3 & 1 & 2 \\ 2 & 2 & 5 \\ 4 & 1 & 0 \end{vmatrix}$

*Solution*

$$\text{Minor of 3 is } M_{11} \quad = \begin{vmatrix} 2 & 5 \\ 1 & 0 \end{vmatrix} = 0 - 5 = -5$$

$$\text{Minor of 1 is } M_{12} \quad = \begin{vmatrix} 2 & 5 \\ 4 & 0 \end{vmatrix} = 0 - 20 = -20$$

$$\text{Minor of 2 is } M_{13} \quad = \begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} = 2 - 8 = -6$$

$$\text{Minor of 2 is } M_{21} \quad = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 0 - 2 = -2$$

$$\text{Minor of 2 is } M_{22} \quad = \begin{vmatrix} 3 & 2 \\ 4 & 0 \end{vmatrix} = 0 - 8 = -8$$

$$\text{Minor of 5 is } M_{23} \quad = \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} = 3 - 4 = -1$$

$$\text{Minor of 4 is } M_{31} \quad = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = 1$$

$$\text{Minor of 1 is } M_{32} \quad = \begin{vmatrix} 3 & 2 \\ 2 & 5 \end{vmatrix} = 15 - 4 = 11$$

$$\text{Minor of 0 is } M_{33} \quad = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} = 6 - 2 = 4$$

$$\text{Cofactor of 3 is } A_{11} = (-1)^{1+1} M_{11} = M_{11} = -5$$

$$\text{Cofactor of 1 is } A_{12} = (-1)^{1+2} M_{12} = -M_{12} = 20$$

Cofactor of 2 is $A_{13} = (-1)^{1+3} M_{13} = M_{13} = -6$

Cofactor of 2 is $A_{21} = (-1)^{2+1} M_{21} = -M_{21} = 2$

Cofactor of 2 is $A_{22} = (-1)^{2+2} M_{22} = M_{22} = -8$

Cofactor of 5 is $A_{23} = (-1)^{2+3} M_{23} = -M_{23} = 1$

Cofactor of 4 is $A_{31} = (-1)^{3+1} M_{31} = M_{31} = 1$

Cofactor of 1 is $A_{32} = (-1)^{3+2} M_{32} = -M_{32} = -11$

Cofactor of 0 is $A_{33} = (-1)^{3+3} M_{33} = M_{33} = 4$

> **NOTE**
>
> Value of a determinant can be obtained by using any row or column

For example

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ (or) $a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$ (expanding along $R_1$)

$\Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$ (or) $a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31}$ (expanding along $C_1$)

**Example 1.4**

Evaluate: $\begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$

*Solution*

$\begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$ = 1 (Minor of 1) –2 (Minor of 2) + 4 (Minor of 4)

$= 1 \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} - 2 \begin{vmatrix} -1 & 0 \\ 4 & 0 \end{vmatrix} + 4 \begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix}$

$= 0 - 0 - 52 = -52.$

## 1.1.4 Properties of determinants (without proof)

1. The value of a determinant is unaltered when its rows and columns are interchanged

2. If any two rows (columns) of a determinant are interchanged then the value of the determinant changes only in sign.

3. If the determinant has two identical rows (columns) then the value of the determinant is zero.

4. If all the elements in a row (column) of a determinant are multiplied by constant $k$, then the value of the determinant is multiplied by $k$.

5. If any two rows (columns) of a determinant are proportional then the value of the determinant is zero.

6. If each element in a row (column) of a determinant is expressed as the sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants of the same order.

7. The value of the determinant is unaltered when a constant multiple of the elements of any row (column) is added to the corresponding elements of a different row (column) in a determinant.

**Example 1.5**

Show that $\begin{vmatrix} x & y & z \\ 2x+2a & 2y+2b & 2z+2c \\ a & b & c \end{vmatrix} = 0$

*Solution*

$$\begin{vmatrix} x & y & z \\ 2x+2a & 2y+2b & 2z+2c \\ a & b & c \end{vmatrix} = \begin{vmatrix} x & y & z \\ 2x & 2y & 2z \\ a & b & c \end{vmatrix} + \begin{vmatrix} x & y & z \\ 2a & 2b & 2c \\ a & b & c \end{vmatrix}$$

$$= 0 + 0$$

$$= 0$$

**Example 1.6**

Evaluate $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$

*Solution*

$$\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix} = x^2 - (x-1)(x+1)$$

$$= x^2 - (x^2 - 1)$$

$$= x^2 - x^2 + 1 = 1$$

**Example 1.7**

Solve $\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$

*Solution*

$$\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$$

**NOTE**

The value of the determinant of triangular matrix is equal to the product of the main diagonal elements.

$$\Rightarrow \ (x-1)(x-2)(x-3) = 0$$

$$x = 1, x = 2, x = 3$$

**Example 1.8**

Evaluate $\begin{vmatrix} 1 & 3 & 4 \\ 102 & 18 & 36 \\ 17 & 3 & 6 \end{vmatrix}$

*Solution*

$$\begin{vmatrix} 1 & 3 & 4 \\ 102 & 18 & 36 \\ 17 & 3 & 6 \end{vmatrix} = 6 \begin{vmatrix} 1 & 3 & 4 \\ 17 & 3 & 6 \\ 17 & 3 & 6 \end{vmatrix}$$

$$= 0 \quad (\text{since } R_2 \equiv R_3)$$

**Example 1.9**

Evaluate $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)\ (b-c)\ (c-a)$

*Solution*

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} \begin{matrix} R_1 \rightarrow R_1 - R_2 \\ R_2 \rightarrow R_2 - R_3 \end{matrix}$$

$$= \begin{vmatrix} 0 & a-b & (a-b)(a+b) \\ 0 & b-c & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)[0 - 0 + \{b + c - (a+b)\}]$$

$$= (a-b)(b-c)(c-a).$$

**Exercise 1.1**

1.  Find the minors and cofactors of all the elements of the following determinants.

    (i) $\begin{vmatrix} 5 & 20 \\ 0 & -1 \end{vmatrix}$   (ii) $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

2.  Evaluate $\begin{vmatrix} 3 & -2 & 4 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

3. Solve: $\begin{vmatrix} 2 & x & 3 \\ 4 & 1 & 6 \\ 1 & 2 & 7 \end{vmatrix} = 0$

4. Find $|AB|$ if $A\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$

5. Solve: $\begin{vmatrix} 7 & 4 & 11 \\ -3 & 5 & x \\ -x & 3 & 1 \end{vmatrix} = 0$

6. Evaluate: $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$

7. Prove that $\begin{vmatrix} \dfrac{1}{a} & bc & b + c \\ \dfrac{1}{b} & ca & c + a \\ \dfrac{1}{c} & ab & a + b \end{vmatrix} = 0$

8. Prove that $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2 b^2 c^2$

## 1.2 Inverse of a matrix

### 1.2.1 Singular matrix :

A Square matrix A is said to be singular, if $|A| = 0$

### 1.2.2 Non – singular matrix:

A square matrix $A$ is said to be non – singular, if $|A| \neq 0$

**Example 1.10**

Show that $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is a singular matrix.

**Solution**

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$

$= 4 - 4 = 0$

$\therefore A$ is a singular matrix.

**Example 1.11**

Show that $\begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}$ is non – singular.

**Solution**

$$\text{Let } A \;=\; \begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}$$

$$|A| \;=\; \begin{vmatrix} 8 & 2 \\ 4 & 3 \end{vmatrix}$$

$$= 24 - 8 \;=\; 16$$

$$\neq 0$$

$\therefore$ $A$ is a non-singular matrix

**DO YOU KNOW?**

If $A$ and $B$ are non – singular matrices of the same order then $AB$ and $BA$ are also non – singular matrices of the same order.

### 1.2.3 Adjoint of a Matrix

The adjoint of a square matrix $A$ is defined as the transpose of a cofactor matrix. Adjoint of the matrix $A$ is denoted by *adj A*

i.e., adj $A = [A_{ij}]^T$ where $[A_{ij}]$ is the cofactor matrix of $A$.

**Example 1.12**

Find adj A for $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

**Solution**

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$\text{adj } A = [A_{ij}]^T$$

$$= \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

**NOTE**

(i) $|adjA| = |A|^{n-1}$ $n$ is the order of the matrix $A$

(ii) $|kA| = k^n |A|$ $n$ is the order of the matrix $A$

(iii) $adj(kA) = k^{n-1}adjA$ $n$ is the order of the matrix $A$

(iv) $A(adj\ A) = (adj\ A)A = |A|I$

(v) $AdjI = I$, $I$ is the unit matrix.

(vi) adj $(AB) = $ (adj $B$)(adj $A$)

(vii) $|AB| = |A||B|$

**Example 1.13**

Find adjoint of $A = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ -4 & 1 & 0 \end{bmatrix}$

**Solution**

$$A_{ij} = (-1)^{i+j} M_{ij}$$

$$A_{11} = (-1)^{1+1} M_{11} \;=\; \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$A_{12} = (-1)^{1+2}M_{12} \quad = -\begin{vmatrix} 0 & 0 \\ -4 & 0 \end{vmatrix} = 0$$

$$A_{13} = (-1)^{1+3}M_{13} \quad = \begin{vmatrix} 0 & 1 \\ -4 & 1 \end{vmatrix} = 0-(-4) = 4$$

$$A_{21} = (-1)^{2+1}M_{21} \quad = -\begin{vmatrix} -2 & -3 \\ 1 & 0 \end{vmatrix} = -(0-(-3)) = -3$$

$$A_{22} = (-1)^{2+2}M_{22} \quad = \begin{vmatrix} 1 & -3 \\ -4 & 0 \end{vmatrix} = 0-12 = -12$$

$$A_{23} = (-1)^{2+3}M_{23} \quad = -\begin{vmatrix} 1 & -2 \\ -4 & 1 \end{vmatrix} = -(1-8) = 7$$

$$A_{31} = (-1)^{3+1}M_{31} \quad = \begin{vmatrix} -2 & -3 \\ 1 & 0 \end{vmatrix} = 0-(-3) = 3$$

$$A_{32} = (-1)^{3+2}M_{32} \quad = -\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix} = 0-0 = 0$$

$$A_{33} = (-1)^{3+3}M_{33} \quad = = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1-0 = 1$$

$$[A_{ij}] = \begin{bmatrix} 0 & 0 & 4 \\ -3 & -12 & 7 \\ 3 & 0 & 1 \end{bmatrix}$$

$$\text{Adj } A = [A_{ij}]^T$$

$$= \begin{bmatrix} 0 & -3 & 3 \\ 0 & -12 & 0 \\ 4 & 7 & 1 \end{bmatrix}$$

### 1.2.4 Inverse of a matrix

Let $A$ be any non-singular matrix of order $n$. If there exists a square matrix B of order $n$ such that $AB = BA = I$ then, $B$ is called the inverse of A and is denoted by $A^{-1}$

We have $A \text{ (adj } A) = \text{(adj } A)A \quad = |A| I$

$\implies A\left(\frac{1}{|A|}adj\ A\right) = \left(\frac{1}{|A|}adj\ A\right)A = I$

$\implies \qquad AB = BA \qquad = I$ where $B = \frac{1}{|A|}adjA$

$$A^{-1} = \frac{1}{|A|}adjA$$

**Example 1.14**

If $A = \begin{bmatrix} 2 & 4 \\ -3 & 2 \end{bmatrix}$ then, find $A^{-1}$.

*Solution*

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 4 \\ -3 & 2 \end{vmatrix}$$

$$= 16 \neq 0$$

Since $A$ is a nonsingular matrix, $A^{-1}$ exists

Now     adj $A = \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} adj A$$

$$= \frac{1}{16} \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}$$

**Example 1.15**

If $A = \begin{bmatrix} -2 & 6 \\ 3 & -9 \end{bmatrix}$ then, find $A^{-1}$

*Solution*

$$|A| = \begin{vmatrix} -2 & 6 \\ 3 & -9 \end{vmatrix} = 0$$

Since A is a singular matrix, $A^{-1}$ does not exist.

## Example 1.16

If A = $\begin{bmatrix} 2 & 4 & 4 \\ 2 & 5 & 4 \\ 2 & 5 & 3 \end{bmatrix}$ then find $A^{-1}$

*Solution*

$$|A| = \begin{vmatrix} 2 & 4 & 4 \\ 2 & 5 & 4 \\ 2 & 5 & 3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 5 & 4 \\ 5 & 3 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} + 4\begin{vmatrix} 2 & 5 \\ 2 & 5 \end{vmatrix}$$

$$= 2(15 - 20) - 4(6 - 8) + 4(10 - 10)$$

$$= 2(-5) - 4(-2) + 4(0)$$

$$= -10 + 8 + 0 = -2 \neq 0$$

Since $A$ is a nonsingular matrix, $A^{-1}$ exists.

$$A_{11} = -5; A_{21} = 8; A_{31} = -4$$

$$A_{12} = 2; A_{22} = -2; A_{32} = 0$$

$$A_{13} = 0; A_{23} = -2; A_{33} = 2$$

$$[A_{ij}] = \begin{bmatrix} -5 & 2 & 0 \\ 8 & -2 & -2 \\ -4 & 0 & 2 \end{bmatrix}$$

$$\text{adj A} = \begin{bmatrix} -5 & 8 & -4 \\ 2 & -2 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{-2}\begin{bmatrix} -5 & 8 & -4 \\ 2 & -2 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} & -4 & 2 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

**Example 1.17**

If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - kA + I_2 = O$ then, find $k$ and also $A^{-1}$.

***Solution***

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$A^2 = A.A$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$$

Given $A^2 - kA + I_2 = O$

$$\Rightarrow \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - k \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 7 - 2k + 1 & 12 - 3k + 0 \\ 4 - k + 0 & 7 - 2k + 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 - 2k & 12 - 3k \\ 4 - k & 8 - 2k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow 4 - k = 0$$

$$\Rightarrow k = 4.$$

Also $\quad |A| = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$

$$= 4 - 3 = 1 \neq 0$$

$$\text{adj } A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$= \frac{1}{1} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

**Example 1.18**

If $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ then, show that $(AB)^{-1} = B^{-1}A^{-1}$.

*Solution*

$$|A| = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 - 2 = -1 \neq 0$$

$\therefore$    $A^{-1}$ exists.

$$|B| = \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = 0 + 1 = 1 \neq 0$$

$B^{-1}$ also exists.

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}$$

$$= -1 \neq 0$$

$(AB)^{-1}$ exists.

$$\text{adj}(AB) = \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{|AB|} adj(AB)$$

$$= \frac{1}{-1} \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \qquad \qquad \dots (1)$$

$$\text{adj } A = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$= \frac{1}{-1} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\text{adj } B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} adjB$$

$$= \frac{1}{1}\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \qquad \dots (2)$$

From (1) and (2), $(AB)^{-1} = B^{-1}A^{-1}$. Hence proved.

## Example 1.19

Show that the matrices $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{-4}{35} & \frac{11}{35} & \frac{-5}{35} \\ \frac{-1}{35} & \frac{-6}{35} & \frac{25}{35} \\ \frac{6}{35} & \frac{1}{35} & \frac{-10}{35} \end{bmatrix}$ are inverses of each other.

*Solution*

$$AB = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} \frac{-4}{35} & \frac{11}{35} & \frac{-5}{35} \\ \frac{-1}{35} & \frac{-6}{35} & \frac{25}{35} \\ \frac{6}{35} & \frac{1}{35} & \frac{-10}{35} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

$$= \frac{1}{35}\begin{bmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{bmatrix}$$

$$= \frac{35}{35}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} \dfrac{-4}{35} & \dfrac{11}{35} & \dfrac{-5}{35} \\ \dfrac{-1}{35} & \dfrac{-6}{35} & \dfrac{25}{35} \\ \dfrac{6}{35} & \dfrac{1}{35} & \dfrac{-10}{35} \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \frac{1}{35} \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \frac{1}{35} \begin{bmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{bmatrix}$$

$$= \frac{35}{35} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Thus $AB = BA = I$

Therefore $A$ and $B$ are inverses of each other.

### Exercise 1.2

1. Find the adjoint of the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

2. If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ then verify that $A\,(adj\,A) = |A|\,I$ and also find $A^{-1}$

3. Find the inverse of each of the following matrices

   (i) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$
   (ii) $\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$
   (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$
   (iv) $\begin{bmatrix} -3 & -5 & 4 \\ -2 & 3 & -1 \\ 1 & -4 & -6 \end{bmatrix}$

4. If $A = \begin{bmatrix} 2 & 3 \\ 1 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}$, then verify $\quad adj\,(AB) = (adj\,B)(adj\,A)$

5. If $A = \begin{bmatrix} 2 & -2 & 2 \\ 2 & 3 & 0 \\ 9 & 1 & 5 \end{bmatrix}$ then, show that $(adj\,A)A = O$

6. If $A = \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix}$ then, show that the inverse of $A$ is $A$ itself.

7. If $A^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ then, find $A$.

8. Show that the matrices $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{4}{5} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$ are inverses of each other.

9. If $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$ then, verify that $(AB)^{-1} = B^{-1}A^{-1}$

10. Find $\lambda$ if the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 2 & \lambda & 4 \\ 9 & 7 & 11 \end{bmatrix}$ has no inverse.

11. If $X = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & p & q \end{bmatrix}$ then, find $p, q$ if $Y = X^{-1}$

### 1.2.5 Solution of a system of linear equations

Consider a system of n linear non – homogeneous equations with n unknowns $x_1, x_2, ......x_n$ as

$$a_{11}x_1 + a_{12}x_2 + .... + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + .... + a_{2n}x_n = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$$

This can be written in matrix form as

$$\begin{bmatrix} a_{11} a_{12} ....a_{1n} \\ a_{21} a_{22} ...a_{2n} \\ ... ... .. ... ... \\ ... ... .. ... ... \\ a_{m1} a_{m2} ...a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}$$

Thus we get the matrix equation $AX = B$ $\qquad$ .( . 1)

$$\text{where } A = \begin{bmatrix} a_{11} a_{12} .....a_{1n} \\ a_{21} a_{22} .....a_{2n} \\ .. ... .... .... ..... \\ .. ... .... .... ..... \\ a_{m1} a_{m2} ....a_{mn} \end{bmatrix} ; X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}$$

From (1) solution is

$$\boxed{X = A^{-1}B}$$

**Example 1.20**

Solve by using matrix inversion method:

$$2x + 5y = 1$$

$$3x + 2y = 7$$

*Solution*

The given system can be written as

$$\begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

i.e.,          $AX = B$

$$X = A^{-1}B$$

where          $A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix}$$

$$= -11 \neq 0$$

$A^{-1}$ exists.

$$adj\ A = \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$= \frac{1}{-11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \frac{1}{-11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$= -\frac{1}{11} \begin{bmatrix} -33 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$x = 3$ and $y = -1$.

## Example 1.21

Solve by using matrix inversion method:

$3x - 2y + 3z = 8; \ 2x + y - z = 1; \ 4x - 3y + 2z = 4$

### Solution

The given system can be written as

$$\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

i.e., $\qquad AX = B$

$$X = A^{-1}B$$

Here $\qquad A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{vmatrix}$$

$$= -17 \neq 0$$

$A^{-1}$ exists

$A_{11} = -1 \qquad A_{12} = -8 \qquad A_{13} = -10$

$A_{21} = -5 \qquad A_{22} = -6 \qquad A_{23} = 1$

$A_{31} = -1 \qquad A_{32} = 9 \qquad A_{33} = 7$

$$[A_{ij}] = \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}$$

$$adj \ A = [A_{ij}]^T$$

$$= \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$= -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$= -\frac{1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$x = 1, y = 2$ and $z = 3$.

### Example 1.22

The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹320. The cost of 2kg onion, 4 kg wheat and 6 kg rice is ₹560. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is ₹380. Find the cost of each item per kg by matrix inversion method.

*Solution*

Let $x$, $y$ and $z$ be the cost of onion, wheat and rice per kg respectively.

Then,    $4x + 3y + 2z = 320$

$2x + 4y + 6z = 560$

$6x + 2y + 3z = 380.$

It can be written as

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 320 \\ 560 \\ 380 \end{bmatrix}$$

$$AX = B$$

where    $A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 320 \\ 560 \\ 380 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix}$$

$$= 50 \neq 0$$

$A^{-1}$ exist

$A_{11} = 0$  $\qquad A_{12} = 30$  $\qquad A_{13} = -20$

$A_{21} = -5$  $\qquad A_{22} = 0$  $\qquad A_{23} = 10$

$A_{31} = 10$  $\qquad A_{32} = -20$  $\qquad A_{33} = 10$

$$[A_{ij}] = \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}$$

$$\text{adj } A = [A_{ij}]^T$$

$$= \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$= \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix} \begin{bmatrix} 320 \\ 560 \\ 380 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix} \begin{bmatrix} 32 \\ 56 \\ 38 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 0 - 280 + 380 \\ 960 + 0 - 760 \\ -640 + 560 + 380 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 40 \\ 60 \end{bmatrix}$$

$x = 20$, $y = 40$ and $z = 60$.

Cost of 1 kg onion is ₹20, cost of 1 kg wheat is ₹40 and cost of 1 kg rice is ₹60 .
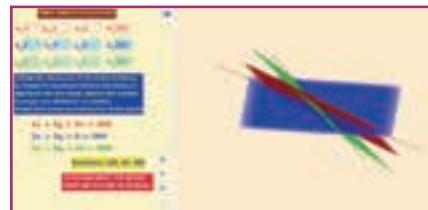
## ICT Corner

**Expected final outcomes**

### Step – 1

Open the Browser and type the URL given (or) Scan the QR Code.

GeoGebra Work book called "**11ᵗʰ BUSINESS MATHS**" will appear. In this several work sheets for Business Maths are given, Open the worksheet named "**Simultaneous Equations**"

### Step - 2

Type the parameters for three simultaneous equations in the respective boxes. You will get the solutions for the equations. Also, you can see the planes for the equations and the intersection point(Solution) on the right-hand side. Solve yourself and check the answer. If the planes do not intersect then there is no solution.

| Step 1 |
| --- |



| Step 2 |
| --- |



Browse in the link

11ᵗʰ Business Maths: https://ggbm.at/qKj9gSTG (or) scan the QR Code

B184_11_BM_EM

1. Solve by matrix inversion method : $2x + 3y - 5 = 0$ ; $x - 2y + 1 = 0$

2. Solve by matrix inversion method :

   (i) $3x - y + 2z = 13$ ; $2x + y - z = 3$ ; $x + 3y - 5z = -8$

   (ii) $x - y + 2z = 3$ ; $2x + z = 1$ ; $3x + 2y + z = 4$.

   (iii) $2x - z = 0$ ; $5x + y = 4$ ; $y + 3z = 5$

3. A sales person Ravi has the following record of sales for the month of January, February and March 2009 for three products $A$, $B$ and $C$  He has been paid a commission at fixed rate per unit but at varying rates for products $A$, $B$ and $C$

| Months | Sales in Units | | | Commission |
|---|---|---|---|---|
| | $A$ | $B$ | $C$ | |
| January | 9 | 10 | 2 | 800 |
| February | 15 | 5 | 4 | 900 |
| March | 6 | 10 | 3 | 850 |

   Find the rate of commission payable on $A$, $B$ and $C$ per unit sold using matrix inversion method.

4. The prices of three commodities $A$, $B$ and $C$ are ₹$x$, ₹$y$ and ₹$z$ per unit respectively. $P$ purchases 4 units of $C$ and sells 3 units of $A$ and 5 units of $B$. $Q$ purchases 3 units of $B$ and sells 2 units of $A$ and 1 unit of $C$. $R$ purchases 1 unit of $A$ and sells 4 units of $B$ and 6 units of $C$. In the process $P$, $Q$ and $R$ earn ₹6,000 , ₹5,000 and ₹13,000 respectively. By using matrix inversion method, find the prices per unit of $A$, $B$ and $C$.

5. The sum of three numbers is 20. If we multiply the first by 2 and add the second number and subtract the third we get 23. If we multiply the first by 3 and add second and third to it, we get 46.  By using matrix inversion method find the numbers.

6. Weekly expenditure in an office for three weeks is given as follows. Assuming that the salary in all the three weeks of different categories of staff did not vary, calculate the salary for each type of staff, using matrix inversion method.

| week | Number of employees | | | Total weekly Salary (in rupees) |
|---|---|---|---|---|
| | A | B | C | |
| 1st week | 4 | 2 | 3 | 4900 |
| 2nd week | 3 | 3 | 2 | 4500 |
| 3rd week | 4 | 3 | 4 | 5800 |

## 1.3 Input – output Analysis

Input –Output analysis is a technique which was invented by Prof. Wassily W.Leontief. Input Output analysis is a form of economic analysis based on the interdependencies between economic sectors. The method is most commonly used for estimating the impacts of positive or negative economic shocks and analyzing the ripple effects throughout an economy.

The foundation of Input - Output analysis involves input – output tables. Such tables include a series of rows and columns of data that quantify the supply chain for sectors of the economy. Industries are listed in the heads of each row and each column. The data in each column corresponds to the level of inputs used in that industry's production function. For example the column for auto manufacturing shows the resources required for building automobiles (ie., requirement of steel, aluminum, plastic, electronic etc.,). Input – Output models typically includes separate tables showing the amount of labour required per rupee unit of investment or production.

Consider a simple economic model consisting of two industries $A_1$ and $A_2$ where each produces only one type of product. Assume that each industry consumes part of its own output and rest from the other industry for its operation. The industries are thus interdependent. Further assume that whatever is produced that is consumed. That is the total output of each industry must be such as to meet its own demand, the demand of the other industry and the external demand (final demand).

Our aim is to determine the output levels of each of the two industries in order to meet a change in final demand, based on knowledge of the current outputs of the two industries, of course under the assumption that the structure of the economy does not change.

Let $a_{ij}$ be the rupee value of the output of $A_i$ consumed by $A_j$, $i, j = 1, 2$

Let $x_1$ and $x_2$ be the rupee value of the current outputs of $A_1$ and $A_2$ respectively.

Let $d_1$ and $d_2$ be the rupee value of the final demands for the outputs of $A_1$ and $A_2$ respectively.

The assumptions lead us to frame the two equations

$$a_{11} + a_{12} + d_1 = x_1; \qquad a_{21} + a_{22} + d_2 = x_2 \qquad \dots (1)$$

Let $\quad b_{ij} = \dfrac{a_{ij}}{x_j} \ i, j = 1, 2$

$$b_{11} = \frac{a_{11}}{x_1}; \quad b_{12} = \frac{a_{12}}{x_2}; \quad b_{21} = \frac{a_{21}}{x_1}; \quad b_{22} = \frac{a_{22}}{x_2}$$

The equations (1) take the form

$$b_{11}x_1 + b_{12}x_2 + d_1 = x_1 \qquad\qquad b_{21}x_1 + b_{22}x_2 + d_2 = x_2$$

The above equations can be rearranged as

$$(1 - b_{11})x_1 - b_{12}x_2 \qquad = d_1$$

$$-b_{21}x_1 + (1 - b_{22})x_2 \qquad = d_2$$

The matrix form of the above equations is

$$\begin{bmatrix} 1-b_{11} & -b_{12} \\ -b_{21} & 1-b_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$(I-B)X = D$

where $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $D = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

By solving we get

$$\boxed{X = (I-B)^{-1}D}$$

The matrix B is known as the *Technology matrix*.

### 1.3.1 The Hawkins – Simon conditions

Hawkins – Simon conditions ensure the viability of the system.

If B is the technology matrix then Hawkins – Simon conditions are

  (i)  the main diagonal elements in $I - B$ must be positive and

  (ii)  $|I - B|$ must be positive.

### Example 1.23

The technology matrix of an economic system of two industries is

$\begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.7 \end{bmatrix}$. Test whether the system is viable as per Hawkins – Simon conditions.

*Solution*

$$\text{B} \quad = \begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.7 \end{bmatrix}$$

$$I - B \quad = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & -0.2 \\ -0.9 & 0.3 \end{bmatrix}$$

$$|I - B| \quad = \begin{vmatrix} 0.2 & -0.2 \\ -0.9 & 0.3 \end{vmatrix}$$

$$= (0.2)(0.3) - (-0.2)(-0.9)$$

$$= 0.06 - 0.18$$

$$= -0.12 < 0$$

Since $|I - B|$ is negative, Hawkins – Simon conditions are not satisfied.

Therefore, the given system is not viable.

## Example 1.24

The following inter – industry transactions table was constructed for an economy of the year 2016.

| Industry | 1 | 2 | Final consumption | Total output |
|----------|------|------|------|------|
| 1 | 500 | 1,600 | 400 | 2,500 |
| 2 | 1,750 | 1,600 | 4,650 | 8,000 |
| Labours | 250 | 4,800 | - | - |

Construct technology co-efficient matrix showing direct requirements. Does a solution exist for this system.

*Solution*

$a_{11} = 500 \qquad a_{12} = 1600 \qquad x_1 = 2500$

$a_{21} = 1750 \qquad a_{22} = 1600 \qquad x_2 = 8000$

$b_{11} = \dfrac{a_{11}}{x_1} = \dfrac{500}{2500} = 0.20; \qquad b_{12} = \dfrac{a_{12}}{x_2} = \dfrac{1600}{8000} = 0.20$

$$b_{21} = \frac{a_{21}}{x_1} = \frac{1750}{2500} = 0.70; \quad b_{22} = \frac{a_{22}}{x_2} = \frac{1600}{8000} = 0.20$$

The technology matrix is $= \begin{bmatrix} 0.2 & 0.2 \\ 0.7 & 0.2 \end{bmatrix}$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.2 \\ 0.7 & 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & -0.2 \\ -0.7 & 0.8 \end{bmatrix}, \text{ elements of main diagonal are positive}$$

Now, $|I - B| = \begin{vmatrix} 0.8 & -0.2 \\ -0.7 & 0.8 \end{vmatrix}$

$$= (0.8)(0.8) - (-0.7)(-0.2)$$

$$= 0.64 - 0.14$$

$$= 0.50 > 0$$

Since diagonal elements of are positive and $|I - B|$ is positive,

Hawkins – Simon conditions are satisfied

Hence this system has a solution.

### Example 1.25

In an economy there are two industries $P_1$ and $P_2$ and the following table gives the supply and the demand position in crores of rupees.

| Production sector | Consumption sector P₁ | Consumption sector P₂ | Final demand | Gross output |
|---|---|---|---|---|
| P₁ | 10 | 25 | 15 | 50 |
| P₂ | 20 | 30 | 10 | 60 |

Determine the outputs when the final demand changes to 35 for $P_1$ and 42 for $P_2$.

### Solution

$a_{11} = 10 \qquad a_{12} = 25 \qquad x_1 = 50$

$a_{21} = 20 \qquad a_{22} = 30 \qquad x_2 = 60$

$$b_{11} = \frac{a_{11}}{x_1} = \frac{10}{50} = \frac{1}{5}; \quad b_{12} = \frac{a_{12}}{x_2} = \frac{25}{60} = \frac{5}{12}$$

$$b_{21} = \frac{a_{21}}{x_1} = \frac{20}{50} = \frac{2}{5}; \quad b_{22} = \frac{a_{22}}{x_2} = \frac{30}{60} = \frac{1}{2}$$

The technology matrix is B $= \begin{bmatrix} \dfrac{1}{5} & \dfrac{5}{12} \\ \dfrac{2}{5} & \dfrac{1}{2} \end{bmatrix}$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \dfrac{1}{5} & \dfrac{5}{12} \\ \dfrac{2}{5} & \dfrac{1}{2} \end{bmatrix}$$

$= \begin{bmatrix} \dfrac{4}{5} & -\dfrac{5}{12} \\ -\dfrac{2}{5} & \dfrac{1}{2} \end{bmatrix}$, elements of main diagonal are positive

Now, $|I - B| = \begin{vmatrix} \dfrac{4}{5} & -\dfrac{5}{12} \\ -\dfrac{2}{5} & \dfrac{1}{2} \end{vmatrix}$

$$= \dfrac{7}{30} > 0$$

Main diagonal elements of $|I - B|$ are positive and $|I - B|$ is positive

∴ The problem has a solution

$adj\,(I{-}B) = \begin{bmatrix} \dfrac{1}{2} & \dfrac{5}{12} \\ \dfrac{2}{5} & \dfrac{4}{5} \end{bmatrix}$

$(I{-}B)^{-1} = \dfrac{1}{|I - B|} adj(I - B)$ (Since $|I - B| \neq 0, (I - B)^{-1}$ exist)

$= \dfrac{1}{\frac{7}{30}}\, adj\,(I - B)$

$= \dfrac{30}{7} \begin{bmatrix} \dfrac{1}{2} & \dfrac{5}{12} \\ \dfrac{2}{5} & \dfrac{4}{5} \end{bmatrix} = \dfrac{1}{7}\begin{bmatrix} 15 & \dfrac{25}{2} \\ 12 & 24 \end{bmatrix}$

$X = (I{-}B)^{-1}D$, where D $= \begin{bmatrix} 35 \\ 42 \end{bmatrix}$

$= \dfrac{1}{7}\begin{bmatrix} 15 & \dfrac{12}{2} \\ 12 & 24 \end{bmatrix}\begin{bmatrix} 35 \\ 42 \end{bmatrix} = \begin{bmatrix} 150 \\ 204 \end{bmatrix}$

The output of industry $P_1$ should be ₹150 crores and $P_2$ should be ₹204 crores.

### Example 1.26

An economy produces only coal and steel. These two commodities serve as intermediate inputs in each other's production. 0.4 tonne of steel and 0.7 tonne of coal are needed to produce a tonne of steel. Similarly 0.1 tonne of steel and 0.6 tonne of coal are required to produce a tonne of coal. No capital inputs are needed. Do you think that

the system is viable? 2 and 5 labour days are required to produce a tonne s of coal and steel respectively. If economy needs 100 tonnes of coal and 50 tonnes of steel, calculate the gross output of the two commodities and the total labour days required.

## Solution

Here the technology matrix is given under

|  | Steel | Coal | Final demand |
|---|---|---|---|
| Steel | 0.4 | 0.1 | 50 |
| Coal | 0.7 | 0.6 | 100 |
| Labour days | 5 | 2 | - |

The technology matrix is $B = \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.1 \\ -0.7 & 0.4 \end{bmatrix}$$

$$|I - B| = \begin{vmatrix} 0.6 & -0.1 \\ -0.7 & 0.4 \end{vmatrix}$$

$$= (0.6)(0.4) - (-0.7)(-0.1)$$

$$= 0.24 - 0.07 = 0.17$$

Since the diagonal elements of $I - B$ are positive and value of $|I - B|$ is positive, the system is viable.

$$\text{adj } (I-B) = \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$$

$$(I-B)^{-1} = \frac{1}{|I - B|} adj(I - B)$$

$$= \frac{1}{0.17} \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$$

$$X = (I-B)^{-1}D, \text{ where } D = \begin{bmatrix} 50 \\ 100 \end{bmatrix}$$

$$= \frac{1}{0.17} \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix} \begin{bmatrix} 50 \\ 100 \end{bmatrix}$$

$$= \frac{1}{0.17} \begin{bmatrix} 30 \\ 95 \end{bmatrix}$$

$$= \begin{bmatrix} 176.5 \\ 558.8 \end{bmatrix}$$

Steel output = 176.5 tonnes

Coal output = 558.8 tonnes

Total labour days required

$$= 5(\text{steel output}) + 2(\text{ coal output})$$

$$= 5(176.5) + 2(558.8)$$

$$= 882.5 + 1117.6 = 2000.1$$

$$\simeq 2000 \text{ labour days.}$$

**Exercise 1.4**

1. The technology matrix of an economic system of two industries is $\begin{bmatrix} 0.50 & 0.30 \\ 0.41 & 0.33 \end{bmatrix}$.
Test whether the system is viable as per Hawkins Simon conditions.

2. The technology matrix of an economic system of two industries is $\begin{bmatrix} 0.6 & 0.9 \\ 0.20 & 0.80 \end{bmatrix}$. Test whether the system is viable as per Hawkins-Simon conditions.

3. The technology matrix of an economic system of two industries is $\begin{bmatrix} 0.50 & 0.25 \\ 0.40 & 0.67 \end{bmatrix}$. Test whether the system is viable as per Hawkins-Simon conditions.

4. Two commodities A and B are produced such that 0.4 tonne of A and 0.7 tonne of B are required to produce a tonne of A. Similarly 0.1 tonne of A and 0.7 tonne of B are needed to produce a tonne of B. Write down the technology matrix. If 6.8 tonnes of A and 10.2 tonnes of B are required, find the gross production of both of them.

5. Suppose the inter-industry flow of the product of two industries are given as under.

| Production sector | Consumption sector | | Domestic demand | Total output |
|---|---|---|---|---|
| | X | Y | | |
| X | 30 | 40 | 50 | 120 |
| Y | 20 | 10 | 30 | 60 |

Determine the technology matrix and test Hawkin's -Simon conditions for the viability of the system. If the domestic demand changes to 80 and 40 units respectively, what should be the gross output of each sector in order to meet the new demands.

6.  You are given the following transaction matrix for a two sector economy.

| Sector | Sales | | Final demand | Gross output |
|---|---|---|---|---|
| | 1 | 2 | | |
| 1 | 4 | 3 | 13 | 20 |
| 2 | 5 | 4 | 3 | 12 |

(i) Write the technology matrix

(ii) Determine the output when the final demand for the output sector 1 alone increases to 23 units.

7.  Suppose the inter-industry flow of the product of two sectors $X$ and $Y$ are given as under.

| Production sector | Consumption Sector | | Domestic demand | Gross output |
|---|---|---|---|---|
| | X | Y | | |
| X | 15 | 10 | 10 | 35 |
| Y | 20 | 30 | 15 | 65 |

Find the gross output when the domestic demand changes to 12 for X and 18 for Y.

## Exercise 1.5

**Choose the correct answer**

1.  The value of $x$ if $\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & x \\ 1 & 3 & x \end{vmatrix} = 0$ is

(a) 0, −1        (b) 0, 1        (c) −1, 1        (d) −1, −1

2.  The value of $\begin{vmatrix} 2x+y & x & y \\ 2y+z & y & z \\ 2z+x & z & x \end{vmatrix}$ is

(a) $x\,y\,z$        (b) $x + y + z$        (c) $2x + 2y + 2z$        (d) 0

3.  The cofactor of −7 in the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ is

(a) −18        (b) 18        (c) −7        (d) 7

4.  If $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$ then $\begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix}$ is

  (a) $\Delta$               (b) $-\Delta$               (c) $3\Delta$               (d) $-3\Delta$

5.  The value of the determinant $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}^2$ is

  (a) $abc$            (b) $0$            (c) $a^2 b^2 c^2$            (d) $-abc$

6.  If $A$ is square matrix of order 3 then $|kA|$ is

  (a) $k|A|$            (b) $-k|A|$            (c) $k^3|A|$            (d) $-k^3|A|$

7.  $adj(AB)$ is equal to

  (a) $adjA \, adjB$     (b) $adjA^T \, adjB^T$     (c) $adjB \, adjA$     (d) $adjB^T \, adjA^T$

8.  The inverse matrix of $\begin{pmatrix} \frac{4}{5} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{1}{2} \end{pmatrix}$ is

  (a) $\frac{7}{30}\begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$     (b) $\frac{7}{30}\begin{pmatrix} \frac{1}{2} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{1}{5} \end{pmatrix}$     (c) $\frac{30}{7}\begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$     (d) $\frac{30}{7}\begin{pmatrix} \frac{1}{2} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{4}{5} \end{pmatrix}$

9.  If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - bc \neq 0$ then $A^{-1}$ is

  (a) $\frac{1}{ad-bc}\begin{pmatrix} d & b \\ -c & a \end{pmatrix}$            (b) $\frac{1}{ad-bc}\begin{pmatrix} d & b \\ c & a \end{pmatrix}$

  (c) $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$            (d) $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ c & a \end{pmatrix}$

10.  The number of Hawkins-Simon conditions for the viability of an input-output analysis is

  (a) 1            (b) 3            (c) 4            (d) 2

11.  The inventor of input - output analysis is

  (a) Sir Francis Galton                      (b) Fisher

  (c) Prof. Wassily W. Leontief           (d) Arthur Caylay

12. Which of the following matrix has no inverse

(a) $\begin{pmatrix} -1 & 1 \\ 1 & -4 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}$

(d) $\begin{pmatrix} \sin a & \sin a \\ -\cos a & \cos a \end{pmatrix}$

13. The inverse matrix of $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ is

(a) $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} -2 & 5 \\ 1 & -3 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -1 \\ -5 & -3 \end{pmatrix}$

(d) $\begin{pmatrix} -3 & 5 \\ 1 & -2 \end{pmatrix}$

14. If $A = \begin{pmatrix} -1 & 2 \\ 1 & -4 \end{pmatrix}$ then $A(adjA)$ is

(a) $\begin{pmatrix} -4 & -2 \\ -1 & -1 \end{pmatrix}$

(b) $\begin{pmatrix} 4 & -2 \\ -1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

15. If $A$ and $B$ non-singular matrix then, which of the following is incorrect?

(a) $A^2 = I$ implies $A^{-1} = A$

(b) $I^{-1} = I$

(c) If $AX = B$ then $X = B^{-1}A$

(d) If $A$ is square matrix of order 3 then $|adj\ A| = |A|^2$

16. The value of $\begin{vmatrix} 5 & 5 & 5 \\ 4x & 4y & 4z \\ -3x & -3y & -3z \end{vmatrix}$ is

(a) 5

(b) 4

(c) 0

(d) 3

17. If $A$ is an invertible matrix of order 2 then $det(A^{-1})$ be equal to

(a) $d\ t(A)$

(b) $\dfrac{1}{det(A)}$

(c) 1

(d) 0

18. If $A$ is $3 \times 3$ matrix and $|A| = 4$ then $|A^{-1}|$ is equal to

(a) $\dfrac{1}{4}$

(b) $\dfrac{1}{16}$

(c) 2

(d) 4

19. If $A$ is a square matrix of order 3 and $|A| = 3$ then $|adjA|$ is equal to

(a) 81

(b) 27

(c) 3

(d) 9

20. The value of $\begin{vmatrix} x & x^2 - yz & 1 \\ y & y^2 - zx & 1 \\ x & z^2 - xy & 1 \end{vmatrix}$ is

(a) 1            (b) 0            (c) ∔            (d) $-xyz$

21. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $|2A|$ is equal to

(a) $4\cos 2\theta$         (b) 4         (c) 2         (d) 1

22. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and $A_{ij}$ is cofactor of $a_{ij}$, then value of $\Delta$ is given by

(a) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$         (b) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$

(c) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$         (d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

23. If $\begin{vmatrix} x & 2 \\ 8 & 5 \end{vmatrix} = 0$ then the value of $x$ is

(a) $\dfrac{-5}{6}$         (b) $\dfrac{5}{6}$         (c) $\dfrac{-16}{5}$         (d) $\dfrac{16}{5}$

24. If $\begin{vmatrix} 4 & 3 \\ 3 & 1 \end{vmatrix} = -5$ then the value of $\begin{vmatrix} 20 & 15 \\ 15 & 5 \end{vmatrix}$ is

(a) –5         (b) –125         (c) –25         (d) 0

25. If any three rows or columns of a determinant are identical then the value of the determinant is

(a) 0         (b) 2         (c) 1         (d) 3

## Miscellaneous Problems

1. Solve: $\begin{vmatrix} x & 2 & -1 \\ 2 & 5 & x \\ -1 & 2 & x \end{vmatrix} = 0$

2. Evaluate $\begin{vmatrix} 10041 & 10042 & 10043 \\ 10045 & 10046 & 10047 \\ 10049 & 10050 & 10051 \end{vmatrix}$

3. Without actual expansion show that the value of the determinant $\begin{vmatrix} 5 & 5^2 & 5^3 \\ 5^2 & 5^3 & 5^4 \\ 5^4 & 5^5 & 5^6 \end{vmatrix}$ is zero.

4. Show that $\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix} = 2a^3 b^3 c^3$.

5. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 7 \\ 1 & -1 & 1 \end{bmatrix}$ verify that $A(adjA) = (adjA)(A) = |A|I_3$.

6. If $A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ then, find the Inverse of $A$.

7. If $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ show that $A^2 - 4A + 5I_2 = O$ and also find $A^{-1}$

8. Solve by using matrix inversion method : $x - y + z = 2, 2x - y = 0, 2y - z = 1$.

9. The cost of 2 Kg of Wheat and 1 Kg of Sugar is ₹70. The cost of 1 Kg of Wheat and 1 Kg of Rice is ₹70. The cost of 3 Kg of Wheat, 2 Kg of Sugar and 1 Kg of rice is ₹170. Find the cost of per kg each item using matrix inversion method.

10. The data are about an economy of two industries $A$ and $B$. The values are in crores of rupees.

| Producer | User | | Final demand | Total output |
|---|---|---|---|---|
| | $A$ | $B$ | | |
| $A$ | 50 | 75 | 75 | 200 |
| $B$ | 100 | 50 | 50 | 200 |

Find the output when the final demand changes to 300 for $A$ and 600 for $B$

## Summary

- Determinant is a number associated to a square matrix.

- If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then its $|A| = a_{11} a_{22} - a_{21} a_{12}$

- Minor of an arbitrary element $a_{ij}$ of the determinant of the matrix is the determinant obtained by deleting the $i^{th}$ row and $j^{th}$ column in which the element $a_{ij}$ stands. The minor of $a_{ij}$ is denoted by $M_{ij}$.

- The cofactor is a signed minor. The cofactor of $a_{ij}$ is denoted by $A_{ij}$ and is defined as $A_{ij} = (-1)^{i+j} M_{ij}$.

- Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

  $\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ or $a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

- adj A $= \left[ A_{ij} \right]^{T}$ where $[A_{ij}]$ is the cofactor matrix of the given matrix.

- $\left| adjA \right| = \left| A \right|^{n-1}$, where n is the order of the matrix $A$.

- $A(adjA) = (adjA)(A) = |A|I$.

- $adjI = I,\ I$ is the unit Matrix.

- adj$(AB) = (adj\ B)(adj\ A)$.

- A square matrix $A$ is said to be singular, if $|A| = 0$.

- A square matrix $A$ is said to be non-singular if $|A| \neq 0$

- Let $A$ be any square matrix of order $n$ and $I$ be the identity matrix of order $n$. If there exists a square matrix $B$ of order n such that $AB = BA = I$ then, $B$ is called the inverse of $A$ and is denoted by $A^{-1}$

- Inverse of $A$ ( if it exists ) is $A^{-1} = \dfrac{1}{|A|} adjA$.

- Hawkins - Simon conditions ensure the viability of the system. If B is the technology matrix then Hawkins-Simon conditions are

  (i) the main diagonal elements in $I - B$ must be positive and

  (ii) $|I - B|$ must be positive.

| GLOSSARY | |
|---|---|
| Adjoint Matrix | சேர்ப்பு அணி |
| Analysis | பகுப்பாய்வு |
| Cofactor | இணைக் காரணி |
| Determinant | அணிக்கோவை |
| Digonal Matrix | மூலை விட்ட அணி |
| Input | உள்ளீடு |
| Inverse Matrix | நேர்மாறு அணி |
| Minors | சிற்றணிக் கோவைகள் |
| Non-Singular Matrix | பூச்சியமற்றக் கோவை அணி |
| Output | வெளியீடு |
| Scalar | திசையிலி |
| Singular Matrix | பூச்சியக் கோவை அணி |
| Transpose of a Matrix | நிரை நிரல் மாற்று அணி |
| Triangular Matrix | முக்கோண அணி |

# ALGEBRA

**Learning Objectives**

After studying this chapter, the students will be able to understand

- definition of partial fractions
- the various techniques to resolve into partial fraction
- different cases of permutation such as word formation
- different cases of combination such as word formation
- difference between permutation and combination
- the concept and principle of mathematical induction
- Binomial expansion techniques

## Introduction

Algebra is a major component of mathematics that is used to unify mathematics concepts. Algebra is built on experiences with numbers and operations along with geometry and data analysis. The word "algebra" is derived from the Arabic word "al-Jabr". The Arabic mathematician Al-Khwarizmi has traditionally been known as the "Father of algebra". Algebra is used to find perimeter, area, volume of any plane figure and solid.

**Al-Khwarizmi**

## 2.1 Partial Fractions

### Rational Expression

An expression of the form $\dfrac{p(x)}{q(x)} = \dfrac{a_0 x^n + a_1 x^{n-1} + \cdots + a_n}{b_0 x^m + b_1 x^{m-1} + \cdots + b_m}$ ,$[q(x) \neq 0]$ is called a rational algebraic expression.

If the degree of the numerator $p(x)$ is less than that of the denominator $q(x)$ then $\dfrac{p(x)}{q(x)}$ is called a proper fraction. If the degree of $p(x)$ is greater than or equal to that of $q(x)$ then $\dfrac{p(x)}{q(x)}$ is called an improper fraction.

An improper fraction can be expressed as the sum of an integral function and a proper fraction.

That is, $\dfrac{p(x)}{q(x)} = f(x) + \dfrac{r(x)}{q(x)}$

For example, $\dfrac{x^2 + x + 1}{x^2 + 2x + 1} = 1 - \dfrac{x}{x^2 + 2x + 1}$

## Partial Fractions

We can express the sum or difference of two fractions $\dfrac{3}{x-1}$ and $\dfrac{2}{x-2}$ as a single fraction.

i.e., $\dfrac{3}{x-1} + \dfrac{2}{x-2} = \dfrac{5x-8}{(x-1)(x-2)}$

$\dfrac{5x-8}{(x-1)(x-2)} = \dfrac{3}{x-1} + \dfrac{2}{x-2}$

Process of writing a single fraction as a sum or difference of two or more simpler fractions is called splitting up into *partial fractions*.

Generally if $p(x)$ and $q(x)$ are two rational integral algebraic functions of $x$ and the fraction $\dfrac{p(x)}{q(x)}$ be expressed as the algebraic sum (or difference) of simpler fractions according to certain specified rules, then the fraction $\dfrac{p(x)}{q(x)}$ is said to be resolved into partial fractions.

### 2.1.1 Denominator contains non-repeated Linear factors

In the rational fraction $\dfrac{p(x)}{q(x)}$, if $q(x)$ is the product of non-repeated linear factors of the form $(ax+b)(cx+d)$, then $\dfrac{p(x)}{q(x)}$ can be resolved into partial fraction of the form $\dfrac{A}{ax+b} + \dfrac{B}{cx+d}$, where $A$ and $B$ are constants to be determined.

For example $\dfrac{3x+7}{(x-1)(x-2)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)}$, the values of $A$ and $B$ are to be determined.

### Example 2.1

Find the values of $A$ and $B$ if $\dfrac{1}{(x^2-1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1}$

*Solution*

Let $\dfrac{1}{(x^2-1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1}$

Multiplying both sides by $(x-1)(x+1)$, we get

$$1 = A(x+1) + B(x-1) \quad \dots (1)$$

Put $x = 1$ in (1) we get, $\quad 1 = A(2)$

$$\therefore A = \frac{1}{2}$$

Put $x = -1$ in (1) we get, $\quad 1 = A(0) + B(-2)$

$$\therefore B = -\frac{1}{2}$$

**Example 2.2**

Resolve into partial fractions : $\quad \dfrac{7x-1}{x^2 - 5x + 6}$

*Solution*

Write the denominator into the product of linear factors.

Here $\quad x^2 - 5x + 6 = (x-2)(x-3)$

Let $\dfrac{7x-1}{x^2 - 5x + 6} = \dfrac{A}{(x-2)} + \dfrac{B}{(x-3)} \quad \dots\dots\dots\dots(1)$

Multiplying both the sides of (1) by *(x-2)(x-3)*, we get

$$7x - 1 = A(x-3) + B(x-2) \quad \dots\dots\dots..(2)$$

Put $\quad x = 3$ in (2), we get

$$21 - 1 = B\,(1)$$

$\Rightarrow \quad\quad\quad\quad B = 20$

Put $\quad x = 2$ in (2), we get

$$14 - 1 = A(-1)$$

$\Rightarrow \quad\quad\quad\quad A = -13$

Substituting the values of $A$ and $B$ in (1), we get

$$\frac{7x-1}{x^2 - 5x + 6} = \frac{-13}{(x-2)} + \frac{20}{(x-3)}$$

$$\frac{7x-1}{x^2-5x+6} = \frac{1}{(x-2)}\left[\frac{(7x-1)}{(x-3)}\right]_{x=2} + \frac{1}{(x-3)}\left[\frac{(7x-1)}{(x-2)}\right]_{x=3}$$

$$= \frac{-13}{(x-2)} + \frac{20}{(x-3)}$$

**Example 2.3**

Resolve into partial fraction : $\dfrac{x+4}{(x^2-4)(x+1)}$

*Solution*

Write the denominator into the product of linear factors

Here $(x^2-4)(x+1)=(x-2)(x+2)(x+1)$

$\therefore \dfrac{x+4}{(x^2-4)(x+1)} = \dfrac{A}{(x-2)}+\dfrac{B}{(x+2)}+\dfrac{C}{(x+1)}$ ...(1)

Multiplying both the sides by $(x-2)(x+2)(x+1)$, we get

$x+4 = A(x+2)(x+1)+B(x-2)(x+1)+C(x-2)(x+2)$ ... (2)

Put $x=-2$ in (2), we get

$-2+4 = A(0)+B(-4)(-1)+C(0)$

$\therefore B = \dfrac{1}{2}$

Put $x=2$ in (2), we get

$2+4 = A(4)(3)+B(0)+C(0)$

$\therefore A = \dfrac{1}{2}$

Put $x=-1$ in (2), we get

$-1+4 = A(0)+B(0)+C(-3)(1)$

$$\therefore \qquad C = -1$$

Substituting the values of $A$, $B$ and $C$ in (1), we get

$$\frac{x+4}{(x^2-4)(x+1)} = \frac{1}{2(x-2)} + \frac{1}{2(x+2)} - \frac{1}{x+1}$$

## 2.1.2 Denominator contains Linear factors, repeated $n$ times

In the rational fraction $\dfrac{p(x)}{q(x)}$ , if $q(x)$ is of the form $(ax+b)^n$ [the linear factor is the product of $(ax+b)$, $n$ times], then $\dfrac{p(x)}{q(x)}$ can be resolved into partial fraction of the form.

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_n}{(ax+b)^n}$$

For example, $\quad \dfrac{9x+7}{(x+4)(x+1)^2} = \dfrac{A}{(x+4)} + \dfrac{B}{(x+1)} + \dfrac{C}{(x+1)^2}$

### Example 2.4

Find the values of $A$, $B$ and C if $\dfrac{x}{(x-1)(x+1)^2} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

### Solution

$$\frac{x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x = A(x+1)^2 + B(x-1)(x+1) + C(x-1) \qquad \text{... (1)}$$

Put $x = 1$ in (1)

$$1 = A(1+1)^2 + B(0) + C(0)$$

$$\therefore A = \frac{1}{4}$$

Put $x = -1$ in (1)

$$-1 = A(0) + B(0) + C(-1-1)$$

$$\therefore C = \frac{1}{2}$$

Equating the constant term on both sides of (1), we get

$$A - B - C = 0$$

$$\Rightarrow B = A - C$$

$$\therefore B = -\frac{1}{4}$$

**Example 2.5**

Resolve into partial fraction $\dfrac{x+1}{(x-2)^2(x+3)}$

*Solution*

Let $\dfrac{x+1}{(x-2)^2(x+3)} = \dfrac{A}{(x-2)} + \dfrac{B}{(x-2)^2} + \dfrac{C}{(x+3)}$ ... (1)

Multiplying both the sides by $(x-2)^2(x+3)$, we get

$$x+1 = A(x-2)(x+3) + B(x+3) + C(x-2)^2 \quad ...(2)$$

Put $x = 2$ in (2), we have

$$2+1 = A(0) + B(5) + C(0)$$

$$\therefore \quad B = \dfrac{3}{5}$$

Put $x = -3$ in (2), we have

$$-3+1 = A(0) + B(0) + C(-3-2)^2$$

$$\therefore \quad C = \dfrac{-2}{25}$$

Equating the coefficient of $x^2$ on both the sides of (2), we get

$$0 = A + C$$

$$A = -C$$

$$\therefore \quad A = \dfrac{2}{25}$$

Substituting the values of $A$, $B$ and $C$ in (1),

$$\dfrac{x+1}{(x-2)^2(x+3)} = \dfrac{2}{25(x-2)} + \dfrac{3}{5(x-2)^2} - \dfrac{2}{25(x+3)}.$$

**Example 2.6**

Resolve into partial fraction $\dfrac{9}{(x-1)(x+2)^2}$.

*Solution*

$$\dfrac{9}{(x-1)(x+2)^2} = \dfrac{A}{(x-1)} + \dfrac{B}{(x+2)} + \dfrac{C}{(x+2)^2} \quad ... (1)$$

Multiplying both sides by $(x-1)(x+2)^2$

$$\therefore \quad 9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1) \quad ... (2)$$

Put $x = -2$ in (2)

$$9 = A(0) + B(0) + C(-2 - 1)$$

$$\therefore \ C = -3$$

$$\text{Put} \ x = 1 \text{ in (2)}$$

$$9 = A(3)^2 + B(0) + C(0)$$

$$\therefore \ A = 1$$

Equating the coefficient of $x^2$ on both the sides of (2), we get

$$0 = A + B$$

$$B = -A$$

$$\therefore \ B = -1$$

Substituting the values of $A$, $B$ and $C$ in (1), we get

$$\frac{9}{(x-1)(x+2)^2} = \frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}$$

## 2.1.3 Denominator contains quadratic factor, which cannot be factorized into linear factors

In the rational fraction $\dfrac{p(x)}{q(x)}$, if $q(x)$ is the quadratic factor $ax^2 + bx + c$ and cannot be factorized into linear factors, then $\dfrac{p(x)}{q(x)}$ can be resolved into partial fraction of the form $\dfrac{Ax + B}{ax^2 + bx + c}$.

### Example 2.7

Resolve into partial fractions: $\dfrac{2x + 1}{(x-1)(x^2 + 1)}$

*Solution*

Here $x^2 + 1$ cannot be factorized into linear factors.

Let $\quad \dfrac{2x + 1}{(x-1)(x^2 + 1)} = \dfrac{A}{x-1} + \dfrac{Bx + c}{x^2 + 1} \qquad \dots (1)$

Multiplying both sides by $(x-1)(x^2 + 1)$

$$2x + 1 = A(x^2 + 1) + (Bx + C)(x - 1) \ \dots (2)$$

$$\text{Put} \quad x = 1 \text{ in (2)}$$

$$2+1 = A(1+1)+0$$

$$\therefore \quad A = \frac{3}{2}$$

Put $x = 0$ in (2)

$$0+1 = A(0+1)+(0+C)(-1)$$

$$1 = A - C$$

$$\therefore \quad C = \frac{1}{2}$$

Equating the coefficient of $x^2$ on both the sides of (2), we get

$$A + B = 0$$

$$B = -A$$

$$\therefore \quad B = \frac{-3}{2}$$

Substituting the values of $A$, $B$ and $C$ in (1), we get

$$\frac{2x+1}{(x-1)(x^2+1)} = \frac{\frac{3}{2}}{x-1} + \frac{-\frac{3}{2}x+\frac{1}{2}}{x^2+1}$$

$$= \frac{3}{2(x-1)} - \frac{3x-1}{2(x^2+1)}$$

### Exercise 2.1

**Resolve into partial fractions for the following:**

1. $\dfrac{3x+7}{x^2-3x+2}$  2. $\dfrac{4x+1}{(x-2)(x+1)}$  3. $\dfrac{1}{(x-1)(x+2)^2}$

4. $\dfrac{1}{x^2-1}$  5. $\dfrac{x-2}{(x+2)(x-1)^2}$  6. $\dfrac{2x^2-5x-7}{(x-2)^3}$

7. $\dfrac{x^2-6x+2}{x^2(x+2)}$  8. $\dfrac{x^2-3}{(x+2)(x^2+1)}$  9. $\dfrac{x+2}{(x-1)(x+3)^2}$

10. $\dfrac{1}{(x^2+4)(x+1)}$

## 2.2 Permutations

### 2.2.1 Factorial

For any natural number n, n factorial is the product of the first n natural numbers and is denoted by $n!$ or $\lfloor n$ .

For any natural number $n$, $n! = n(n-1)(n-2) \ldots 3 \times 2 \times 1$

For examples,

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

$$3! = 3 \times 2 \times 1 = 6$$

$$2! = 2 \times 1 = 2$$

$$1! = 1$$

NOTE

$0! = 1$

For any natural number $n$

$$n! = n(n-1)(n-2) \dots 3 \times 2 \times 1$$

$$= n(n-1)!$$

$$= n(n-1)(n-2)!$$

In particular,    $8! = 8 \times (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)$

$$= 8 \times 7!$$

### Example 2.8

Evaluate the following:

(i) $\dfrac{7!}{6!}$          (ii) $\dfrac{8!}{5!}$          (iii) $\dfrac{9!}{6!3!}$

*Solution*

(i)    $\dfrac{7!}{6!} = \dfrac{7 \times 6!}{6!} = 7.$

(ii)    $\dfrac{8!}{5!} = \dfrac{8 \times 7 \times 6 \times 5!}{5!} = 336.$

(iii)    $\dfrac{9!}{6!3!} = \dfrac{9 \times 8 \times 7 \times 6!}{6! \times 3!} = \dfrac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84.$

### Example 2.9

Rewrite 7! in terms of 5!

*Solution*

$$7! = 7 \times 6!$$

$$= 7 \times 6 \times 5! = 42 \times 5!$$

**Example 2.10**

Find $n$, if $\dfrac{1}{9!} + \dfrac{1}{10!} = \dfrac{n}{11!}$

*Solution*

$$\dfrac{1}{9!} + \dfrac{1}{10!} = \dfrac{n}{11!}$$

$$\dfrac{1}{9!} + \dfrac{1}{10 \times 9!} = \dfrac{n}{11!}$$

$$\dfrac{1}{9!}\left[1 + \dfrac{1}{10}\right] = \dfrac{n}{11!}$$

$$\dfrac{1}{9!} \times \dfrac{11}{10} = \dfrac{n}{11!}$$

$$n = \dfrac{11! \times 11}{9! \times 10}$$

$$= \dfrac{11! \times 11}{10!}$$

$$= \dfrac{11 \times 10! \times 11}{10!}$$

$$= 121.$$

## 2.2.2 Fundamental principle of counting

### Multiplication principle of counting:

Consider the following situation in an auditorium which has three entrance doors and two exit doors. Our objective is to find the number of ways a person can enter the hall and then come out.

Assume that, $P_1$, $P_2$ and $P_3$ are the three entrance doors and $S_1$ and $S_2$ are the two exit doors. A person can enter the hall through any one of the doors $P_1$, $P_2$ or $P_3$ in 3 ways. After entering the hall, the person can come out through any of the two exit doors $S_1$ or $S_2$ in 2 ways.



Hence the total number of ways of entering the hall and coming out is

$3 \times 2 = 6$ ways.

These possibilities are explained in the given flow chart (Fig. 2.1).

Fig. 2.1

The above problem can be solved by applying multiplication principle of counting.
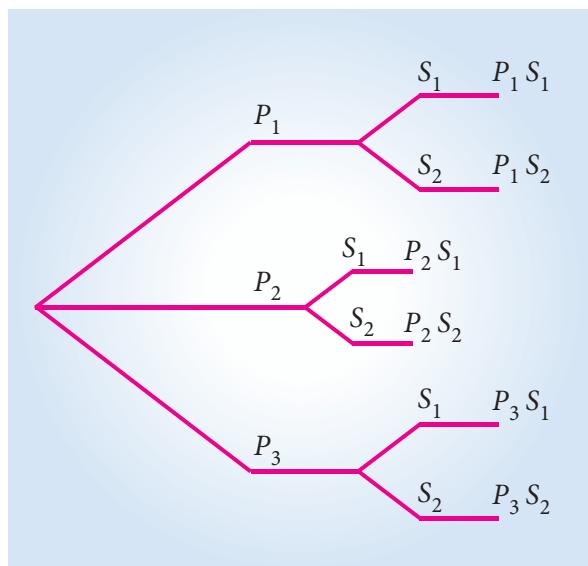
### Definition 2.1

There are two jobs. Of which one job can be completed in m ways, when it has completed in any one of these m ways, second job can be completed in n ways, then the two jobs in succession can be completed in $m \times n$ ways. This is called multiplication principle of counting.

### Example 2.11

Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word " NOTE", where the repetition of the letters is not allowed.

### Solution

Let us allot four vacant places (boxes) to represent the four letters. N,O,T,E of the given word NOTE.



Fig : 2.2

The first place can be filled by any one of the 4 letters in 4 ways. Since repetition is not allowed, the number of ways of filling the second vacant place by any of the remaining 3 letters in 3 ways. In the similar way the third box can be filled in 2 ways and that of last box can be filled in 1 way. Hence by multiplication principle of counting, total number of words formed is $4 \times 3 \times 2 \times 1 = 24$ words.

### Example 2.12

If each objective type questions having 4 choices, then find the total number of ways of answering the 4 questions.

### Solution

Since each question can be answered in 4 ways, the total number of ways of answering 4 questions = $4 \times 4 \times 4 \times 4 = 256$ ways.

### Example 2.13

How many 3 digits numbers can be formed if the repetition of digits is not allowed?

### Solution

Let us allot 3 boxes to represent the digits of the 3 digit number

|  | 100's place | 10's place | unit's place |
|---|---|---|---|

|  |  |  |
|---|---|---|
| 1 | 2 | 3 |
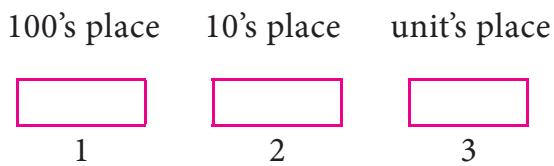
Fig : 2.3

Three digit number never begins with "0". ∴ 100's place can be filled with any one of the digits from 1 to 9 in 9 ways.

Since repetition is not allowed, 10's place can be filled by the remaining 9 numbers (including 0) in 9 ways and the unit place can be filled by the remaining 8 numbers in 8 ways. Then by multiplication principle of counting, number of 3 digit numbers = $9 \times 9 \times 8 = 648$.

## 2.2.3   Addition principle of counting

Let us consider the following situation in a class, there are 10 boys and 8 girls. The class teacher wants to select either a boy or a girl to represent the class in a function. Our objective is to find the number of ways that the teacher can select the student. Here the teacher has to perform either of the following two jobs.

One student can be selected in 10 ways among 10 boys.

One student can be selected in 8 ways among 8 girls

Hence the jobs can be performed in 10 + 8 = 18 ways

This problem can be solved by applying Addition principle of counting.

**Definition 2.2**

If there are two jobs, each of which can be performed independently in *m* and *n* ways respectively, then either of the two jobs can be performed in (m+n) ways. This is called addition principle of counting.

## Example 2.14

There are 6 books on commerce and 5 books on accountancy in a book shop. In how many ways can a student purchase either a book on commerce or a book on accountancy?

*Solution*

Out of 6 commerce books, a book can be purchased in 6 ways.

Out of 5 accountancy books, a book can be purchased in 5 ways.

Then by addition principle counting the total number of ways of purchasing any one book is 5+6=11 ways.

 **Exercise 2.2**

1.  Find $x$ if $\dfrac{1}{6!} + \dfrac{1}{7!} = \dfrac{x}{8!}$ .

2.  Evaluate $\dfrac{n!}{r!\,(n-r)!}$   when $n = 5$ and $r = 2$.

3.  If $(n+2)! = 60[(n-1)!]$, find $n$

4.  How many five digits telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 with no digit appears more than once?

5.  How many numbers lesser than 1000 can be formed using the digits 5, 6, 7, 8 and 9 if no digit is repeated?

### 2.2.4  Permutation

Suppose we have a fruit salad with combination of "APPLES, GRAPES & BANANAS". We don't care what order the fruits are in. They could also be "bananas, grapes and apples" or "grapes, apples and bananas". Here the order of mixing is not important. Any how, we will have the same fruit salad.

But, consider a number lock with the number code 395. If we make more than three attempts wrongly, then it locked permanently. In that situation we do care about the order 395. The lock will not work if we input 3-5-9 or 9-5-3. The number lock will work if it is exactly 3-9-5. We have so many such situations in our practical life. So we should know the order of arrangement, called Permutation.

> **Definition 2.3**
>
> The number of arrangements that can be made out of $n$ things taking $r$ at a time is called the number of permutation of $n$ things taking $r$ at a time.

For example, the number of three digit numbers formed using the digits 1, 2, 3 taking all at a time is 6

6 three digit numbers are 123,132,231,213,312,321

**Notations**: If $n$ and $r$ are two positive integers such that $0 \leq r \leq n$, then the number of all permutations of $n$ distinct things taken $r$ at a time is denoted by

$$P(n,r) \text{ (or) } nP_r$$

We have the following theorem *without proof*.

Theorem : $$nP_r = \frac{n!}{(n-r)!}$$

**Example 2.15**

Evaluate : $5P_3$ and $P(8, 5)$

*Solution*

$$5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 60.$$

$$P(8, 5) = 8P_5$$

$$= \frac{8!}{(8-5)!}$$

$$= \frac{8!}{3!}$$

$$= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3!} = 6720$$

| **Results** |
|---|
| (i) $0! = 1$ |
| (ii) $nP_0 = \dfrac{n!}{(n-0)!} = \dfrac{n!}{n!} = 1$ |
| (iii) $nP_1 = \dfrac{n!}{(n-1)!} = \dfrac{n(n-1)!}{(n-1)!} = n$ |
| (iv) $nP_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = n!$ |
| (v) $nP_r = n(n-1)(n-2) \ldots [n-(r-1)]$ |

**Example 2.16**

Evaluate:  (i) $8P_3$      (ii) $5P_4$

*Solution*

(i) $8P_3 = 8 \times 7 \times 6 = 336.$

(ii) $5P_4 = 5 \times 4 \times 3 \times 2 = 120.$

**Example 2.17**

In how many ways 7 pictures can be hung from 5 picture nails on a wall ?

*Solution*

The number of ways in which 7 pictures can be hung from 5 picture nails on a wall is nothing but the number of permutation of 7 things taken 5 at a time = $7P_5 = \dfrac{7!}{(7-5)!}$ = 2520 ways.

**Example 2.18**

Find how many four letter words can be formed from the letters of the word "LOGARITHMS" ( words are with or without meanings)

*Solution*

There are 10 letters in the word LOGARITHMS

Therefore $\qquad\qquad\qquad\qquad\qquad n = 10$

Since we have to find four letter words, $r = 4$

Hence the required number of four letter words $= np_r = 10\,p_4$

$$= 10 \times 9 \times 8 \times 7$$

$$= 5040.$$

**Example 2.19**

If $nP_r = 360$, find $n$ and $r$.

*Solution*

$$nP_r = 360 = 36 \times 10$$

$$= 3 \times 3 \times 4 \times 5 \times 2$$

$$= 6 \times 5 \times 4 \times 3 = 6P_4$$

Therefore $\quad n = 6$ and $r = 4$.

## Permutation of repeated things:

The number of permutation of $n$ different things taken $r$ at a time, when repetition is allowed is $n^r$.

**Example 2.20**

Using 9 digits from 1,2,3,......9, taking 3 digits at a time, how many 3 digits numbers can be formed when repetition is allowed?

*Solution*

Here, $n = 9$ and $r = 3$

$\therefore$   Number of 3 digit numbers $= n^r$

$$= 9^3 = 729 \text{ numbers}$$

## Permutations when all the objects are not distinct:

The number of permutations of n things taken all at a time, of which p things are of one kind and q things are of another kind, such that $p + q = n$ is $\dfrac{n!}{p! \, q!}$ .

In general, the number of permutation of $n$ objects taken all at a time, $p_1$ are of one kind, $p_2$ are of another kind, $p_3$ are of third kind, ...... $p_k$ are of $k^{th}$ kind such that $p_1 + p_2 + ...+p_k = n$ is $\dfrac{n!}{p_1! p_2! ... p_k!}$ .

**Example 2.21**

How many distinct words can be formed using all the letters of the following words.
(i) MISSISSIPPI      (ii) MATHEMATICS.

*Solution*

(i)   There are 11 letters in the word MISSISSIPPI

In this word  M occurs once

I    occurs 4 times

S    occurs 4 times

P    occurs twice.

Therefore required number of permutation $= \dfrac{11!}{4! \, 4! \, 2!}$

(ii)  In the word  MATHEMATICS

There are 11 letters

Here    M   occurs twice

T    occurs twice

A    occurs twice

and the rest are all different. Therefore number of permutations = $\dfrac{11!}{2!\,2!\,2!}$

## 2.2.5    Circular permutation

In the last section, we have studied permutation of $n$ different things taken all together is $n!$. Each permutation is a different arrangement of n things in a row or on a straight line. These are called Linear permutation. Now we consider the permutation of $n$ things along a circle, called circular permutation.

Consider the four letters A, B, C, D  and the number of row arrangement of these 4 letters can be done in 4! Ways.  Of those 4!  arrangements, the arrangements ABCD, BCDA, CDAB, DABC  are one and the same when they are arranged along a circle.
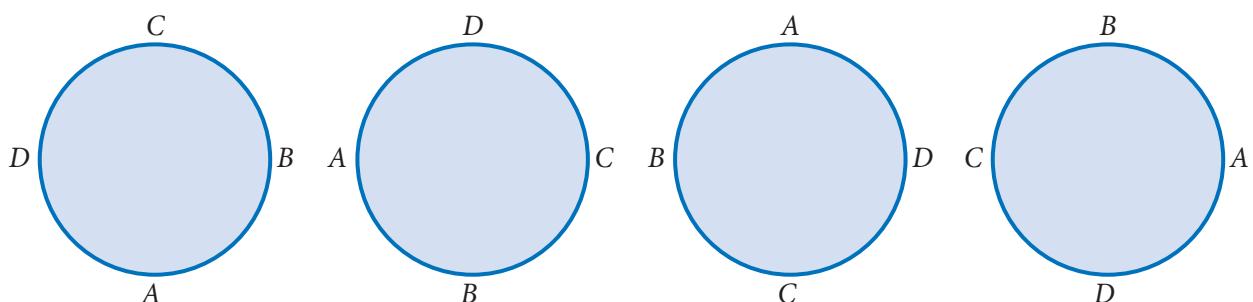


Fig. 2.4

So, the number of permutation of '4' things along a circle is $\dfrac{4!}{4} = 3!$

In general, circular permutation of $n$ different things taken all at the time =  $(n-1)!$

### NOTE

If clock wise and anti-clockwise circular permutations are considered to be same (identical), the number of circular permutation of n objects taken all at a time is $\dfrac{(n-1)!}{2}$

**Example 2.22**

In how many ways 8 students can be arranged

(i) in a line          (ii) along a circle

## Solution

(i) Number of permutations of 8 students along a line = $8P_8$ = 8!

(ii) When the students are arranged along a circle, then the number of permutation is $(8-1)! = 7!$

## Example 2.23

In how many ways 10 identical keys can be arranged in a ring?

## Solution

Since keys are identical, both clock wise and anti-clockwise circular permutation are same.

The number of permutation is

$$\frac{(n-1)!}{2} = \frac{(10-1)!}{2} = \frac{9!}{2}$$

## Example 2.24

Find the rank of the word 'RANK' in dictionary.

## Solutions

Here maximum number of the word in dictionary formed by using the letters of the word 'RANK' is 4!

The letters of the word RANK in alphabetical order are A, K, N, R

Number of words starting with A     = 3! = 6
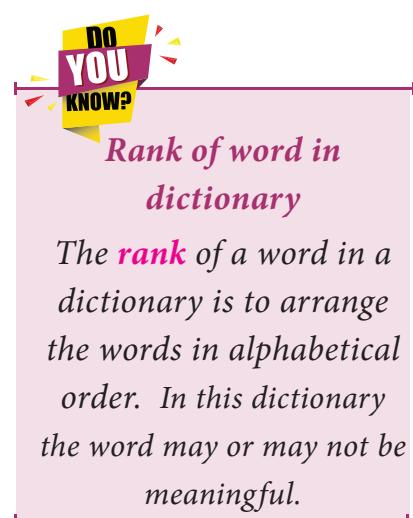
Number of words starting with K     = 3! = 6

Number of words starting with N     = 3! = 6

Number of words starting with RAK = 1! = 1

Number of words starting with RANK = 0! = 1

∴ Rank of the word RANK is 6 + 6 + 6 + 1 + 1 = 20

**DO YOU KNOW?**

*Rank of word in dictionary*

*The **rank** of a word in a dictionary is to arrange the words in alphabetical order. In this dictionary the word may or may not be meaningful.*

1. If $nP_4 = 12(nP_2)$, find $n$.

2. In how many ways 5 boys and 3 girls can be seated in a row, so that no two girls are together?

3. How many 6- digit telephone numbers can be constructed with the digits 0,1,2,3,4,5,6,7,8,9 if each numbers starts with 35 and no digit appear more than once?

4. Find the number of arrangements that can be made out of the letters of the word "ASSASSINATION"

5. (a) In how many ways can 8 identical beads be strung on a necklace?

   (b) In how many ways can 8 boys form a ring?

6. Find the rank of the word 'CHAT' in dictionary.

## 2.3 Combinations

A combination is a selection of items from a collection such that the order of selection does not matter. That is the act of combining the elements irrespective of their order. Let us suppose that in an interview two office assistants are to be selected out of 4 persons namely A, B, C and D. The selection may be of AB, AC, AD, BC, BD, CD (we cannot write BA, since AB and BA are of the same selection). Number of ways of selecting 2 persons out of 4 persons is 6.It is represented as $4C_2 = \frac{4 \times 3}{1 \times 2} = 6$.

The process of different selection without repetition is called Combination.

### Definition 2.4

Combination is the selection of n things taken r at a time without repetition. Out of $n$ things, $r$ things can be selected in $nc_r$ ways.

$$nC_r = \frac{n!}{r!(n-r)!}, 0 \leq r \leq n$$

Here $n \neq 0$ but $r$ may be 0.

For example, out of 5 balls 3 balls can be selected in $5C_3$ ways.

$$5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!\,2!} = \left(\frac{5 \times 4 \times 3}{1 \times 2 \times 3}\right) = 10 \text{ ways.}$$

**Example 2.25**

Find $8C_2$

*Solution*

$$8C_2 = \frac{8 \times 7}{2 \times 1} = 28$$

**Properties**

(i)    $nC_0$    $= nc_n = 1.$

(ii)    $nC_1$    $= n.$

(iii)    $nC_2$    $= \dfrac{n(n-1)}{2!}$

(iv)    $nC_x$    $= nC_y,$
then either $x = y$ or $x + y = n.$

(v)    $nC_r$    $= nC_{n-r}.$

(vi)    $nC_r + nC_{r-1} = (n+1)C_r.$

*(vii)*   $nC_r$     $= \dfrac{np_r}{r!}$

**Example 2.26**

If $nC_4 = nC_6$, find $12C_n$.

*Solution*

If                 $nC_x = nC_y$, then $x + y = n.$

Here              $nC_4 = nC_6$

$\therefore$                   $n = 4 + 6 = 10$

$$12C_n = 12C_{10}$$

$$= 12C_2$$

$$= \frac{12 \times 11}{1 \times 2} = 66$$

**Example 2.27**

If $nP_r = 720$; $nC_r = 120$, find $r$

## Solution

We know that
$$nC_r = \frac{nP_r}{r!}$$

$$120 = \frac{720}{r!}$$

$$r! = \frac{720}{120} = 6 = 3\,!$$

$$\implies \qquad r = 3.$$

## Example 2.28

If $15C_{3r} = 15C_{r+3}$, find $r$

### Solution

$$15C_{3r} = 15C_{r+3}$$

Then by the property, $nC_x = nC_y \implies x + y = n$, we have

$$3r + r + 3 = 15$$

$$\implies \qquad r = 3.$$

## Example 2.29

From a class of 32 students, 4 students are to be chosen for competition. In how many ways can this be done?

### Solution

The number of combination $= 32C_4$

$$= \frac{32!}{4!\,(32-4)!}$$

$$= \frac{32!}{4!\,(28)!}\,.$$

## Example 2.30

A question paper has two parts namely Part *A* and Part *B*. Each part contains 10 questions. If the student has to choose 8 from part *A* and 5 from part *B*, in how many ways can he choose the questions?

### Solution

In part A, out of 10 questions 8 can be selected in $10C_8$ ways. In part B out of 10 questions 5 can be selected in $10C_5$. Therefore by multiplication principle the total number of selection is

$$10C_8 \times 10C_5 = 10C_2 \times 10C_5$$

$$= \frac{10 \times 9}{2 \times 1} \times \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= 11340.$$

## Example 2.31

A Cricket team of 11 players is to be formed from 16 players including 4 bowlers and 2 wicket-keepers. In how many different ways can a team be formed so that the team contains at least 3 bowlers and at least one wicket-keeper?

### *Solution*

A Cricket team of 11 players can be formed in the following ways:

(i) 3 bowlers, 1 wicket keeper and 7 other players can be selected in

$4C_3 \times 2C_1 \times 10C_7$ ways

$$4C_3 \times 2C_1 \times 10C_7 = 4C_1 \times 2C_1 \times 10C_3$$

$$= 960 \text{ ways}$$

(ii) 3 bowlers, 2 wicket keepers and 6 other players can be selected in

$4C_1 \times 2C_2 \times 10C_6$ ways

$$4C_1 \times 2C_2 \times 10C_6 = 4C_1 \times 2C_2 \times 10C_4$$

$$= 840 \text{ ways}$$

(iii) 4 bowlers and 1 wicket keeper and 6 other players $4C_4 \times 2C_1 \times 10C_6$

$$4C_4 \times 2C_1 \times 10C_4 = 420 \text{ ways}$$

4 bowlers, 2 wicket keepers and 5 other players can be selected in

$4C_4 \times 2C_2 \times 10C_5$ ways

$$4C_4 \times 2C_2 \times 10C_5 = 252 \text{ ways}$$

By addition principle of counting.

Total number of ways = 960 + 840 + 420 + 252

$$= 2472$$

## Example 2.32

If $4(nC_2) = (n+2)C_3$, find $n$

*Solution*

$$4(nC_2) = (n+2)C_3$$

$$4\frac{n(n-1)}{1 \times 2} = \frac{(n+2)(n+1)(n)}{1 \times 2 \times 3}$$

$$12(n-1) = (n+2)(n+1)$$

$$12(n-1) = (n^2 + 3n + 2)$$

$$n^2 - 9n + 14 = 0$$

$$(n-2)(n-7) = 0 \Rightarrow n = 2, n = 7$$

## Example 2.33

If $(n+2)C_n = 45$, find $n$

*Solution*

$$(n+2)C_n = 45$$

$$(n+2)C_{n+2-n} = 45$$

$$(n+2)C_2 = 45$$

$$\frac{(n+2)(n+1)}{2} = 45$$

$$n^2 + 3n - 88 = 0$$

$$(n+11)(n-8) = 0$$

$$n = -11, 8$$

$n = -11$ is not possible

$$\therefore \qquad\qquad n = 8$$

**Expected final outcomes**

## Step – 1

Open the Browser and type the URL given (or) Scan the QR Code.

GeoGebra Work book called "**11ᵗʰ BUSINESS MATHS**" will appear. In this several work sheets for Business Maths are given, Open the worksheet named "**Combination Exercise**"

## Step - 2

Combination practice exercise will open. You can generate as many questions you like by clicking "New question" Solve the problem yourself and check your answer by clicking on the "Show solution" box. Also click "Short method" and follow this method.

| Step 1 |
| --- |



| Step 2 |
| --- |



Browse in the link

11ᵗʰ Business Maths: https://ggbm.at/qKj9gSTG (or) scan the QR Code

1. If $nP_r = 1680$ and $nC_r = 70$, find $n$ and $r$.

2. Verify that $8C_4 + 8C_3 = 9C_4$.

3. How many chords can be drawn through 21 points on a circle?

4. How many triangles can be formed by joining the vertices of a hexagon?

5. Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?

6. If four dice are rolled, find the number of possible outcomes in which atleast one die shows 2.

7. There are 18 guests at a dinner party. They have to sit 9 guests on either side of a long table, three particular persons decide to sit on one particular side and two others on the other side. In how many ways can the guests to be seated?

8. If a polygon has 44 diagonals, find the number of its sides.

9. In how many ways can a cricket team of 11 players be chosen out of a batch of 15 players?

    (i) There is no restriction on the selection.

    (ii) A particular player is always chosen.

    (iii) A particular player is never chosen.

10. A Committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways this can be done when

    (i) atleast two ladies are included

    (ii) atmost two ladies are included

## 2.4 Mathematical induction

Mathematical induction is one of the techniques which can be used to prove variety of mathematical statements which are formulated in terms of $n$, where $n$ is a positive integer.

Mathematical Induction is used in the branches of Algebra, Geometry and Analysis where it turns out to be necessary to prove some truths of propositions.

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**The principle of mathematical induction:**

Let $P(n)$ be a given statement for $n \in N$.

  (i) **Initial step:** Let the statement is true for $n = 1$ i.e., $P(1)$ is true and

  (ii) **Inductive step:** If the statement is true for $n = k$ (where $k$ is a particular but arbitrary natural number) then the statement is true for $n = k + 1$. i.e., truth of $P(k)$ implies the truth of $P(k+1)$. Then $P(n)$ is true for all natural numbers $n$.

</div>

### Example 2.34

Using mathematical induction method, Prove that

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}, \, n \in N.$$

### *Solution*

Let the given statement $P(n)$ be defined as

$$1+2+3+\ldots\ldots n = \frac{n(n+1)}{2} \quad \text{for} \quad n \in N.$$

**Step 1:** Put $\qquad\qquad n = 1$

$\qquad$ LHS $\qquad\quad$ P(1) $\;= 1$

$\qquad$ RHS $\qquad\quad$ P(1) $\;= \dfrac{1(1+1)}{2} = 1$

$\qquad$ LHS $=$ RHS for $\quad n = 1$

$\qquad \therefore \qquad$ P(1) is true

**Step 2:** Let us assume that the statement is true for $n = k$.

$\qquad$ i.e., P($k$) is true

$$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2} \text{ is true}$$

**Step 3:** To prove that $P(k + 1)$ is true

$$P(k + 1) = 1+2+3+\ldots\ldots+ k + (k + 1)$$

$$= P(k) + k + 1$$

$$= \frac{k(k+1)}{2} + k + 1$$

$$= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$\therefore$    $P(k+1)$ is true

Thus if $P(k)$ is true, then $P(k+1)$ is also true.

$\therefore$ $P(n)$ is true for all  $n \in N$

Hence $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}, n \in N$

**Example 2.35**

By the principle of Mathematical Induction, prove that $1 + 3 + 5 \ldots + (2n-1) = n^2$, for all  $n \in N$ .

*Solution*

Let $P(n)$  denote the statement

$$1 + 3 + 5 \ldots + (2n-1) = n^2$$

Put                                    $n = 1$

$$\text{LHS} = 1$$

$$\text{RHS} = 1^2$$

$$= 1$$

$$\text{LHS} = \text{RHS}$$

$\therefore$  $P(1)$ is true.

Assume that $P(k)$  is true.

i.e.,     $1 + 3 + 5 \ldots + (2k-1) = k^2$

To prove:  $P(k+1)$  is true.

$\therefore$   $1 + 3 + 5 \ldots + (2k-1) + (2k+1) = P(k) + (2k+1)$

$$= k^2 + 2k + 1$$

$$= (k+1)^2$$

$P(k + 1)$ is true whenever $P(k)$ is true.

$\therefore P(n)$ is true for all $n \in N$.

## Example 2.36

By Mathematical Induction, prove that $1^2+2^2+3^2+\ldots\ldots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$, for all $n \in N$.

### Solution

Let $P(n)$ denote the statement:

$$1^2 + 2^2 + 3^2 +\ldots +n^2 = \dfrac{n(n+1)(2n+1)}{6}$$

Put $n = 1$

$\therefore$ $\quad$ LHS $= 1^2$

$\qquad\qquad = 1$

$\qquad$ RHS $= \dfrac{1(1+1)(2+1)}{6}$

$\qquad\qquad = 1$

$\qquad$ LHS $=$ RHS

$\therefore \quad P(1)$ is true.

Assume that $P(k)$ is true.

$P(k) = 1^2 + 2^2 + 3^2 +\ldots +k^2 = \dfrac{k(k+1)(2k+1)}{6}$

Now,

$1^2 + 2^2+ 3^2+\ldots + k^2 + (k+1)^2 = p(k)+(k+1)^2$

$$= \dfrac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$= \dfrac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \dfrac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6}$$

$$= \dfrac{(k+1)(2k^2+7k+6)}{6}$$

$$= \dfrac{(k+1)(k+2)(2k+3)}{6}$$

$\therefore P(k + 1)$ is true whenever $P(k)$ is true

$\therefore P(n)$ is true for all $n \in N$.

**Example 2.37**

Show by the principle of mathematical induction that $2^{3n}-1$ is a divisible by 7, for all $n \in N$.

*Solution*

Let the given statement $P(n)$ be defined as $P(n) = 2^{3n}-1$

**Step 1:**                  Put   $n = 1$

$$\therefore \quad P(1) = 2^3 - 1$$

$$= 7 \text{ is divisible by 7}$$

i.e.,    $P(1)$ is true.

**Step 2:** Let us assume that the statement is true for $n = k$ i.e., $P(k)$ is true.

We assume $2^{3k}-1$ is divisible by $7 \Rightarrow 2^{3k} - 1 = 7m$

**Step 3**: To prove that $P(k + 1)$ is true

$$P(k+1) = 2^{3(k+1)} - 1$$

$$= 2^{(3k+3)} - 1$$

$$= 2^{3k} \cdot 2^3 - 1$$

$$= 2^{3k} \cdot 8 - 1$$

$$= 2^{3k} \cdot (7+1) - 1$$

$$= 2^{3k} \cdot 7 + 2^{3k} - 1$$

$$= 2^{3k} \cdot 7 + 7m = 7(2^{3k} + m)$$

which is divisible by 7

$P(k+1)$ is true whenever $p(k)$ is true

$\therefore P(n)$ is true for all $n \in N$.

Hence the proof.

**Example 2.38**

By the principle of mathematical induction prove that $n^2 + n$ is an even number, for all $n \in N$.

*Solution*

Let $P(n)$ denote the statement that, "$n^2 + n$ is an even number".

Put $n = 1$

$\therefore$ $1^2 + 1 = 1 + 1 = 2$ , an even number.

Let us assume that $P(k)$ is true.

$\therefore$ $k^2 + k$ is an even number is true

$\therefore$ take $\qquad\qquad k^2 + k = 2m \qquad$ ...(1)

To prove $P(k + 1)$ is true

$$\therefore (k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1$$

$$= k^2 + k + 2k + 2$$

$$= 2m + 2(k+1) \text{ by (1)}$$

$$= 2(m + k + 1) \text{ ( a multiple of 2)}$$

$\therefore$ $(k + 1)^2 + (k + 1)$ is an even number

$\therefore$ $P(k + 1)$ is true whenever $P(k)$ is true.

$P(n)$ is true for $n \in N$.

**Exercise 2.5**

By the principle of mathematical induction, prove the following

1. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$ for all $n \in N$.

2. $1.2 + 2.3 + 3.4 + \ldots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$ , for all $n \in N$.

3. $4 + 8 + 12 + \ldots + 4n = 2n(n+1)$ , for all $n \in N$.

4. $1 + 4 + 7 + \ldots + (3n - 2) = \dfrac{n(3n-1)}{2}$ , for all $n \in N$.

5. $3^{2n} - 1$ is a divisible by 8, for all $n \in N$.

6. $a^n - b^n$ is divisible by $a - b$, for all $n \in N$.

7. $5^{2n} - 1$ is divisible by 24, for all $n \in N$.

8. $n(n + 1)(n+2)$ is divisible by 6, for all $n \in N$.

9. $2^n > n$, for all $n \in N$.

## 2.5 Binomial theorem

An algebraic expression of sum or the difference of two terms is called a binomial. For example $(x+y), (5a - 2b), \left(x + \dfrac{1}{y}\right), \left(p + \dfrac{5}{p}\right), \left(\dfrac{7}{4} + \dfrac{1}{y^2}\right)$ etc., are binomials. The Binomial theorem or Binomial Expression is a result of expanding the powers of binomials. It is possible to expand $(x + y)^n$ into a sum involving terms of the form $ax^b y^c$, exponents $b$ and $c$ are non-negative integers with $b + c = n$, the coefficient 'a' of each term is a positive integer called binomial coefficient.

Expansion of $(x + a)^2$ was given by Greek mathematician Euclid on 4th century and there is an evidence that the binomial theorem for cubes i.e., expansion of $(x+a)^3$ was known by 6th century in India. The term Binomial coefficient was first introduced by Michael Stifle in 1544. Blaise Pascal (19 June 1623 to 19 August 1662) a French Mathematician, Physicist, inventor, writer and catholic theologian. In his Treatise on Arithmetical triangle of 1653 described the convenient tabular presentation for Binomial coefficient now called Pascal's triangle. Sir Issac Newton generalized the Binomial theorem and made it valid for any rational exponent.

Now we study the **Binomial theorem for $(x + a)^n$**

### Theorem(without proof)

If $x$ and 'a' are real numbers, then for all $n \in N$

$$(x + a)^n = nC_0 x^n a^0 + nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 + ... + nC_r x^{n-r} a^r + ...$$

$$+ nC_{n-1} x^1 a^{n-1} + nC_n x^0 a^n = \sum_{r=0}^{n} nC_r x^{n-r} a^r$$

### NOTE

When $n = 0$, $(x+a)^0 = 1$

When $n = 1$, $(x + a) = 1C_0 x + 1C_1 a = x + a$

When $n = 2$, $(x + a)^2 = 2C_0 x^2 + 2C_1 x^a + 2C_2 a^2 = x^2 + 2xa + a^2$

When $n = 3$, $(x + a)^3 = 3C_0 x^3 + 3C_1 x^2 a + 3C_2 xa^2 + 3C_3 a^3$

$$= x^3 + 3x^2 a + 3xa^2 + a^3 \text{ and so on.}$$

Given below is the Pascal's Triangle showing the co-efficient of various terms in Binomial expansion of $(x + a)^n$
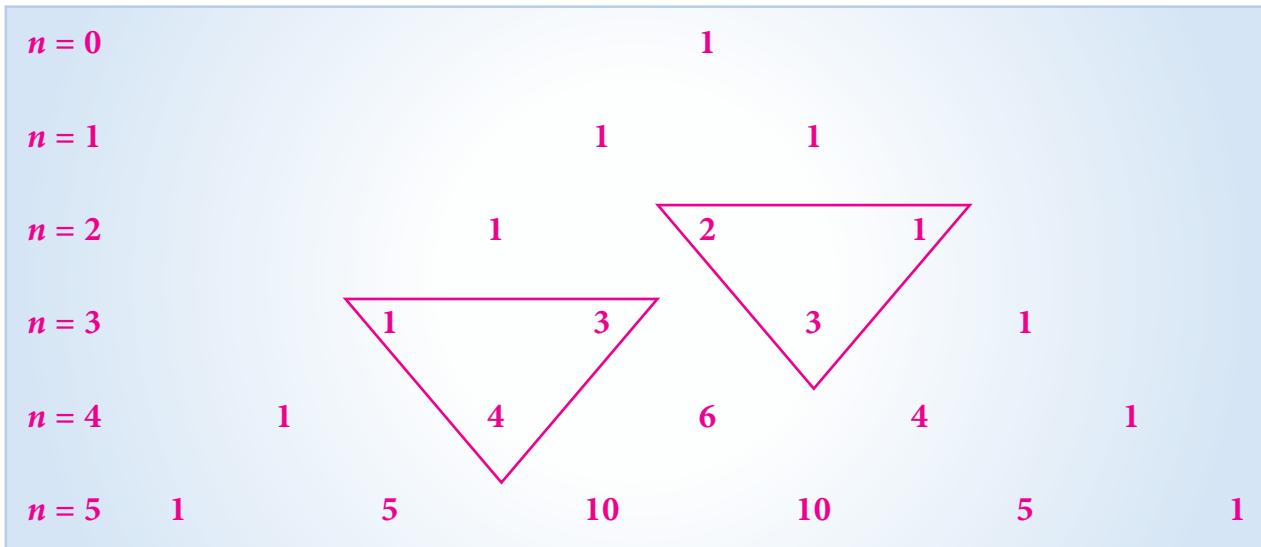


Fig. 2.5

## NOTE

(i) Number of terms in the expansion of $(x + a)^n$ is n+1

(ii) Sum of the indices of $x$ and a in each term in the expansion is n

(iii) $nC_0, nC_1, nC_2, nC_3, \ldots nC_r, \ldots nC_n$ are also represented as $C_0, C_1, C_2, C_3, \ldots, C_r, \ldots, C_n$, are called Binomial co-efficients.

(iv) Since $nC_r = nC_{n-r}$, for $r$ =0, 1, 2, …, $n$ in the expansion of $(x + a)^n$, binomial co-efficients of terms equidistant from the beginning and from the end of the expansion are equal.

(v) Sum of the co-efficients in the expansion of $(1+x)^n$ is equal of $2^n$

(vi) In the expansion of $(1+x)^n$, the sum of the co-efficients of odd terms = the sum of the co-efficients of the even terms $=2^{n-1}$

(vii) General term in the expansion of $(x+a)^n$ is $t_{r+1} = nC_r x^{n-r} a^r$

## Middle term of $(x + a)^n$

**Case (i)** If $n$ is even, then the number of terms $n + 1$ is odd, then there is only one middle term, given by $t_{\frac{n}{2}+1}$

**Case (ii)** If $n$ is odd, then the number of terms $(n+1)$ is even.. Therefore, we have two middle terms given by $t_{\frac{n+1}{2}}$ and $t_{\frac{n+3}{2}}$

Sometimes we need a **particular term** in the expansion of $(x + a)^n$. For this we first write the general term $t_{r+1}$. The value of r can be obtained by taking the term $t_{r+1}$ as the required term. To find the **term independent of $x$** (term without $x$), equate the power of $x$ in $t_{r+1}$ to zero, we get the value of '$r$'. By substituting the value of $r$ in $t_{r+1}$, we get the term independent of $x$.

**NOTE**

$$(x+a)^n = nC_0 x^n a^0 + nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 + ... + nC_r x^{n-r} a^r + ... + a^n \quad ....(1)$$

(i) To find $(x-a)^n$, replace by $-a$

$$(x-a)^n = nC_0 x^n a^0 + nC_1 x^{n-1} (-a) + nC_2 x^{n-2} (-a)^2 + ... + nC_r x^{n-r} (-a)^r + ... + (-a)^n$$

$$= nC_0 x^n a^0 - nC_1 x^{n-1} (a) + nC_2 x^{n-2} (a)^2 + ... + (-1)^r nC_r x^{n-r} (a)^r + ... + (-1)^n (a)^n$$
$$......(2)$$

Note that we have the signs alternatively.

(ii) If we put $a = 1$ in (1), we get

$$(1+x)^n = 1 + nC_1 x + nC_2 x^2 + ... + nC_r x^r + ... + nC_n x^n \qquad ......(3)$$

(iii) If we replace $x$ by $-x$ in (3), we get

$$(1-x)^n = 1 - nC_1 x + nC_2 x^2 + ... + nC_r (-1)^r x^r + ... + nC_n (-1)^n x^n$$

**Example 2.39**

Expand $(2x + 3y)^5$ using binomial theorem.

*Solution*

$$(x+a)^n = nC_0 x^n a^0 + nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 + ... + nC_r x^{n-r} a^r + ...$$

$$+ nC_{n-1} x^1 a^{n-1} + nC_n x^0 a^n$$

$$(2x+3y)^5 = 5C_0 (2x)^5 + 5C_1 (2x)^4 (3y) + 5C_2 (2x)^3 (3y)^2 + 5C_3 (2x)^2 (3y)^3 +$$

$$5C_4 (2x)(3y)^4 + 5C_5 (3y)^5$$

$$= 32x^5 + 5(16)x^4 (3y) + \frac{5 \times 4}{2 \times 1}(8x^3)(9y^2) + \frac{5 \times 4 \times 3}{3 \times 2 \times 1} 4x^2 (27y^3) +$$

$$5(2x)81y^4 + 243y^5$$

$$= 32x^5 + 240x^4 y + 720x^3 y^2 + 1080x^2 y^3 + 810xy^4 + 243y^5$$

## Example 2.40

Using binomial theorem, expand $\left(x^2 + \dfrac{1}{x^2}\right)^4$

*Solution*

$$\left(x^2 + \dfrac{1}{x^2}\right)^4 = (x^2)^4 + 4C_1(x^2)^3\,\dfrac{1}{x^2} + 4C_2(x^2)^2\left(\dfrac{1}{x^2}\right)^2 + 4C_3(x^2)\left(\dfrac{1}{x^2}\right)^3 + 4C_4\left(\dfrac{1}{x^2}\right)^4$$

$$= x^8 + 4x^4 + 6 + \dfrac{4}{x^2} + \dfrac{4}{x^4} + \dfrac{1}{x^8}$$

## Example 2.41

Using binomial theorem, evaluate $(101)^5$

*Solution*

$$(101)^5 = (100 + 1)^5$$

$$= (100)^5 + 5C_1(100)^4 + 5C_2(100)^3 + 5C_3(100)^2 + 5C_4(100) + 5C_5$$

$$= 10000000000 + 5(100000000) + 10(1000000) + 10(10000) + 5(100) + 1$$

$$= 10000000000 + 500000000 + 10000000 + 100000 + 500 + 1$$

$$= 10510100501$$

## Example 2.42

Find the 5$^{\text{th}}$ term in the expansion of $\left(x - \dfrac{3}{x^2}\right)^{10}$

*Solution*

General term in the expansion of $(x + a)^n$ is $t_{r+1} = nC_r x^{n-r} a^r$        ----------(1)

To find the 5$^{\text{th}}$ term of $\left(x - \dfrac{3}{x^2}\right)^{10}$

For this take   $r = 4$

Then,   $t_{4+1} = t_5 = 10C_4(x)^6\left(-\dfrac{3}{x^2}\right)^4$   (Here $n = 10$ , $x = x$ , $a = -\dfrac{3}{x^2}$ )

$$= 10C_4(x)^6\,\dfrac{3^4}{x^8}$$

$$= \dfrac{17010}{x^2}$$

## Example 2.43

Find the middle term in the expansion of $\left(x^2 - \dfrac{2}{x}\right)^{10}$

## Solution

Compare $\left(x^2 - \dfrac{2}{x}\right)^{10}$ with $(x+a)^n$

$$n = 10, \quad x = x^2, \quad a = -\dfrac{2}{x}$$

Since $\quad n = 10$, we have 11 terms (odd)

$\therefore$ $6^{\text{th}}$ term is the middle term.

The general term is $\quad t_{r+1} = nC_r x^{n-r} a^r$ $\qquad$ (1)

To get $t_6$, put $\qquad r = 5$

$$t_{5+1} = t_6 = 10C_5 (x^2)^5 \left(\dfrac{-2}{x}\right)^5$$

$$= 10C_5 x^{10} \dfrac{(-2)^5}{x^5} = -8064 x^5$$

### Example 2.44

Find the middle term in the expansion of $\left(\dfrac{x}{3} + 9y\right)^9$

## Solution

Compare $\left(\dfrac{x}{3} + 9y\right)^9$ with $(x + a)^n$

Since $n = 9$, we have 10 terms (even)

$\therefore$ There are two middle terms namely $\dfrac{t_{n+1}}{2}, \dfrac{t_{n+3}}{2}$ i.e., $\dfrac{t_{9+1}}{2}, \dfrac{t_{9+s}}{2}$

General term in the expansion of $(x + a)^n$ is

$$t_{r+1} = nC_r x^{n-r} a^r \qquad (1)$$

Here $t_5$ and $t_6$ are middle terms.

put $r = 4$ in (1), $\quad t_{4+1} = t_5 = 9C_4 \left(\dfrac{x}{3}\right)^5 \cdot (9y)^4$

$$= 9C_4 \dfrac{x^5}{3^5} \cdot 9^4 y^4$$

$$= \dfrac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \cdot \dfrac{x^5}{3^5} \cdot 9^4 y^4$$

$$= 3402\, x^5 y^4$$

put $r = 5$ in (1), $\quad t_{5+1} = t_6 = 9C_5 \left(\dfrac{x}{3}\right)^4 \cdot (9y)^5$

$$= 9C_5 \dfrac{x^4}{3^4} \cdot 9^5 y^5$$

$$= 91854\, x^4 y^5$$

**Example 2.45**

Find the Coefficient of $x^{10}$ in the binomial expansion of $\left(2x^2 - \dfrac{3}{x}\right)^{11}$

*Solution*

General term of $(x + a)^n$ is   $t_{r+1} = nC_r x^{n-r} a^r$          (1)

Compare $\left(2x^2 - \dfrac{3}{x}\right)^{11}$ with $(x + a)^n$

$$t_{r+1} = 11C_r (2x^2)^{11-r} \left(\dfrac{-3}{x}\right)^r$$

$$= 11C_r 2^{11-r} x^{2(11-r)} (-3)^r \left(\dfrac{1}{x}\right)^r$$

$$= 11C_r 2^{11-r} (-1)^r \cdot 3^r x^{22-3r}$$

To find the co-efficient of $x^{10}$, take $22 - 3r = 10$

$$r = 4$$

Put $r = 4$ in (1),          $t_5 = 11C_4 2^{11-4} 3^4 x^{10} = 11C_4 \cdot 2^7 3^4 x^{10}$

∴   Co-efficient of $x^{10}$ is $11C_4 2^7 3^4$.

**Example 2.46**

Find the term independent of $x$ in the expansion of $\left(2x + \dfrac{1}{2x^2}\right)^9$.

*Solution*

General term in the expansion of $(x + a)^n$ is  $t_{r+1} = nC_r x^{n-r} a^r$

Compare $\left(2x + \dfrac{1}{2x^2}\right)^9$ with $(x + a)^n$

∴                          $t_{r+1} = 9C_r (2x)^{9-r} \left(\dfrac{1}{3x^2}\right)^r$

$$= 9C_r 2^{9-r} x^{9-r} \dfrac{1}{3^r} x^{-2r}$$

$$t_{r+1} = 9C_r \dfrac{2^{9-r}}{3^r} \cdot x^{9-3r} \qquad \qquad \text{... (1)}$$

To get the term independent of $x$, equating the power of $x$ as 0

$$9 - 3r = 0$$

∴                          $r = 3$

Put $r = 3$ in (1)

$$t_{3+1} = 9C_3 \dfrac{2^{9-3}}{3^5} \cdot x^0 = 9C_3 \dfrac{2^6}{3^5} = 1792$$
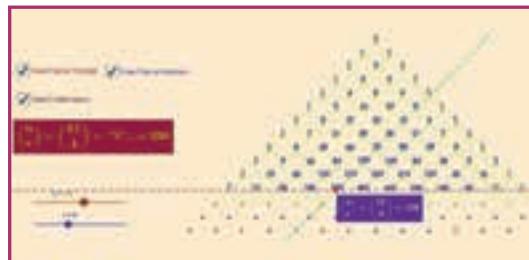
**Expected final outcomes**



**Step – 1**

Open the Browser and type the URL given (or) Scan the QR Code.

GeoGebra Work book called "**11th BUSINESS MATHS**" will appear. In this several work sheets for Business Maths are given, Open the worksheet named "**Pascal's Triangle**"

**Step - 2**

Pascal's triangle page will open. Click on the check boxes "**View Pascal Triangle**" and "**View Pascal's Numbers**" to see the values. Now you can move the sliders n and r to move the red point to respective $^{n}C_{r}$. Now click on view Combination to see the $^{n}C_{r}$ calculation.

| Step 1 |
|---|



| Step 2 |
|---|



Browse in the link

11th Business Maths: https://ggbm.at/qKj9gSTG (or) scan the QR Code

1. Expand the following by using binomial theorem

   (i)   $(2a - 3b)^4$          (ii)   $\left(x + \dfrac{1}{y}\right)^7$          (iii)   $\left(x + \dfrac{1}{x^2}\right)^6$

2. Evaluate the following using binomial theorem:

   (i)   $(101)^4$          (ii)   $(999)^5$

3. Find the $5^{\text{th}}$ term in the expansion of $(x - 2y)^{13}$.

4. Find the middle terms in the expansion of

   (i)   $\left(x + \dfrac{1}{x}\right)^{11}$          (ii)   $\left(3x + \dfrac{x^2}{2}\right)^8$          (iii)   $\left(2x^2 - \dfrac{3}{x^3}\right)^{10}$

5. Find the term independent of $x$ in the expansion of

   (i)   $\left(x^2 - \dfrac{2}{3x}\right)^9$          (ii)   $\left(x - \dfrac{2}{x^2}\right)^{15}$          (iii)   $\left(2x^2 + \dfrac{1}{x}\right)^{12}$

6. Prove that the term independent of $x$ in the expansion of $\left(x + \dfrac{1}{x}\right)^{2n}$ is $\dfrac{1 \cdot 3 \cdot 5 \ldots, (2n - 1)2^n}{n!}$.

7. Show that the middle term in the expansion of $(1 + x)^{2n}$ is $\dfrac{1.3.5 \ldots (2n - 1) 2^n x^n}{n!}$

**Choose the correct answer**

1. If $nC_3 = nC_2$ then the value of $nc_4$ is

   (a) 2          (b) 3          (c) 4          (d) 5

2. The value of $n$, when $np_2 = 20$ is

   (a) 3          (b) 6          (c) 5          (d) 4

3. The number of ways selecting 4 players out of 5 is

   (a) 4!          (b) 20          (c) 25          (d) 5

4. If $nP_r = 720\,(nC_r)$, then $r$ is equal to

   (a) 4          (b) 5          (c) 6          (d) 7

5. The possible out comes when a coin is tossed five times

   (a) $2^5$          (b) $5^2$          (c) 10          (d) $\dfrac{5}{2}$

6. The number of diagonals in a polygon of n sides is equal to

   (a) $nC_2$          (b) $nC_2 - 2$          (c) $nC_2 - n$          (d) $nC_2 - 1$

7. The greatest positive integer which divide $n(n+1)(n+2)(n+3)$ for all $n \in N$ is

   (a) 2          (b) 6          (c) 20          (d) 24

8. If $n$ is a positive integer, then the number of terms in the expansion of $(x+a)^n$ is

   (a) $n$          (b) $n+1$          (c) $n-1$          (d) $2n$

9. For all n > 0, $nC_1 + nC_2 + nC_3 + \dots \dots + nC_n$ is equal to

   (a) $2^n$          (b) $2^n - 1$          (c) $n^2$          (d) $n^2 - 1$

10. The term containing $x^3$ in the expansion of $(x-2y)^7$ is

    (a) $3^{rd}$          (b) $4^{th}$          (c) $5^{th}$          (d) $6^{th}$

11. The middle term in the expansion of $\left(x+\dfrac{1}{x}\right)^{10}$ is

    (a) $10C_4\left(\dfrac{1}{x}\right)$          (b) $10C_5$          (c) $10C_6$          (d) $10C_7 x^4$

12. The constant term in the expansion of $\left(x+\dfrac{2}{x}\right)^6$ is

    (a) 156          (b) 165          (c) 162          (d) 160

13. The last term in the expansion of $(3+\sqrt{2})^8$ is

    (a) 81          (b) 16          (c) $8\sqrt{2}$          (d) $27\sqrt{3}$

14. If $\dfrac{kx}{(x+4)(2x-1)} = \dfrac{4}{x+4} + \dfrac{1}{2x-1}$ then $k$ is equal to

    (a) 9          (b) 11          (c) 5          (d) 7

15. The number of 3 letter words that can be formed from the letters of the word number when the repetition is allowed are

    (a) 206          (b) 133          (c) 216          (d) 300

16. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is

(a) 18  (b) 12  (c) 9  (d) 6

17. There are 10 true or false questions in an examination. Then these questions can be answered in

(a) 240 ways  (b) 120 ways  (c) 1024 ways  (d) 100 ways

18. The value of $(5C_0 + 5C_1) + (5C_1 + 5C_2) + (5C_2 + 5C_3) + (5C_3 + 5C_4) + (5C_4 + 5C_5)$ is

(a) $2^6 - 2$  (b) $2^5 - 1$  (c) $2^8$  (d) $2^7$

19. The total number of 9 digit number which have all different digit is

(a) $10!$  (b) $9!$  (c) $9 \times 9!$  (d) $10 \times 10!$

20. The number of ways to arrange the letters of the word " CHEESE "

(a) 120  (b) 240  (c) 720  (d) 6

21. Thirteen guests has participated in a dinner. The number of handshakes happened in the dinner is

(a) 715  (b) 78  (c) 286  (d) 13

22. Number of words with or without meaning that can be formed using letters of the word "EQUATION" , with no repetition of letters is

(a) $7!$  (b) $3!$  (c) $8!$  (d) $5!$

23. Sum of Binomial co-efficient in a particular expansion is 256, then number of terms in the expansion is

(a) 8  (b) 7  (c) 6  (d) 9

24. The number of permutation of $n$ different things taken $r$ at a time, when the repetition is allowed is

(a) $r^n$  (b) $n^r$  (c) $\dfrac{n!}{(n-r)!}$  (d) $\dfrac{n!}{(n+r)!}$

25. Sum of the binomial coefficients is

(a) $2^n$  (b) $n^2$  (c) $2n$  (d) $n+17$

## Miscellaneous Problems

1. Resolve into Partial Fractions : $\dfrac{5x+7}{(x-1)(x+3)}$

2. Resolve into Partial Fractions: $\dfrac{x-4}{x^2-3x+2}$

3. Decompose into Partial Fractions: $\dfrac{6x^2-14x-27}{(x+2)(x-3)^2}$

4. Decompose into Partial Fractions: $\dfrac{5x^2-8x+5}{(x-2)(x^2-x+1)}$

5. Evaluate the following expression.

   (i) $\dfrac{7!}{6!}$         (ii) $\dfrac{8!}{5!}$         (iii) $\dfrac{9!}{6!\,3!}$

6. How many code symbols can be formed using 5 out of 6 letters $A$, $B$, $C$, $D$, $E$, $F$ so that the letters a) cannot be repeated   b) can be repeated    c) cannot be repeated but must begin with $E$   d) cannot be repeated but end with $CAB$.

7. From 20 raffle tickets in a hat, four tickets are to be selected in order. The holder of the first ticket wins a car, the second a motor cycle, the third a bicycle and the fourth a skateboard. In how many different ways can these prizes be awarded?

8. In how many different ways, 2 Mathematics, 2 Economics and 2 History books can be selected from 9 Mathematics, 8 Economics and 7 History books?

9. Let there be 3 red, 2 yellow and 2 green signal flags. How many different signals are possible if we wish to make signals by arranging all of them vertically on a staff?

10. Find the Co-efficient of $x^{11}$ in the expansion of $\left(x+\dfrac{2}{x^2}\right)^{17}$

## Summary

- For any natural number $n$ , $n$ factorial is the product of the first $n$ natural numbers and is denoted by $n!$ or $\lfloor n$ .

- For any natural number $n$   $n! = n(n-1)\,(n-2)\ldots3\times2\times1$

- $0!=1$

- If there are two jobs, each of which can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $(m+n)$ ways.

- The number of arrangements that can be made out of $n$ things taking $r$ at a time is called the number of permutation of n things taking $r$ at a time.

- $np_r = \dfrac{n!}{(n-r)!}$

- $np_0 = \dfrac{n!}{(n-0)!} = \dfrac{n!}{n!} = 1$

- $np_1 = \dfrac{n!}{(n-1)!} = \dfrac{n(n-1)!}{(n-1)!} = n$

- $np_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = n!$

- $np_r = n(n-1)(n-2)\ldots[n-(r-1)]$

- The number of permutation of n different things taken r at a time, when repetition is allowed is $n^r$.

- The number of permutations of n things taken all at a time, of which $p$ things are of one kind and $q$ things are of another kind, such that $p+q = n$ is $\dfrac{n!}{p!\,q!}$

- Circular permutation of n different things taken all at the time = (n–1)!

- The number of circular permutation of n identical objects taken all at a time is $\dfrac{(n-1)!}{2}$

- Combination is the selection of $n$ things taken $r$ at a time without repetition.

- Out of $n$ things, $r$ things can be selected in $nc_r$ ways.

- $nC_r = \dfrac{n!}{r!(n-r)!}, \ 0 \le r \le n$

- $nC_n = 1.$

- $nC_1 = n.$

- $nC_2 = \dfrac{n(n-1)}{2!}$

- $nC_x = nC_y$, then either $x = y$ or $x + y = n$.

- $nC_r = nC_{n-r}$.

- $nC_r + nC_{r-1} = (n+1)C_r$

- $(x+a)^n = nC_0 x^n a^0 + nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 + \ldots + nC_r x^{n-r} a^r + nC_{n-1} x^1 a^{n-1}$

  $+ nC_n x^0 a^n = \displaystyle\sum_{r=0}^{n} nC_r x^{n-r} a^r$

- Number of terms in the expansion of $(x+a)^n$ is $n+1$

- Sum of the indices of $x$ and $a$ in each term in the expansion of $(x+a)^n$ is $n$

- $nC_0, nC_1, nC_2, nC_3 \ldots\ldots nC_r \ldots nC_n$ are also represented as $C_0, C_1, C_2, C_3 \ldots C_r \ldots C_n$, called Binomial co-efficients.

- Since $nC_r = nC_{n-r}$, for $r = 0, 1, 2,\ldots n$ in the expansion of $(x+a)^n$, Binomial co-efficients which are equidistant from either end are equal

- $nC_0 = nC_n, nC_1 = nC_{n-1}, nC_2 = nC_{n-2}$

- Sum of the co-efficients is equal of $2^n$

- Sum of the co-efficients of odd terms = sum of the co-efficients of even terms = $2^{n-1}$

- General term in the expansion of $(x+a)^n$ is $t_{r+1} = nC_r x^{n-r} a^r$

| GLOSSARY | |
|---|---|
| Binomial | ஈருறுப்பு |
| Circular Permutation | வட்ட வரிசை மாற்றம் |
| Coefficient | கெழு |
| Combination | சேர்வு |
| Factorial | காரணீயப் பெருக்கம் |
| Independent term | சாரா உறுப்பு |
| Linear factor | ஒரு படிக்காரணி |
| Mathematical Induction | கணிதத் தொகுத்தறிதல் |
| Middle term | மைய உறுப்பு |
| Multiplication Principle of counting | எண்ணுதலின் பெருக்கல் கொள்கை |
| Partial fraction | பகுதிப் பின்னங்கள் |
| Pascal's Triangle | பாஸ்கலின் முக்கோணம் |
| Permutations | வரிசை மாற்றங்கள் |
| Principle of counting | எண்ணுதலின் கொள்கை |
| Rational Expression | விகிதமுறு கோவை |

# 3

# ANALYTICAL GEOMETRY

**Learning Objectives**

After studying this chapter, the students will be able to understand

- the locus
- the angle between two lines.
- the concept of concurrent lines.
- the pair of straight lines
- the general equation and parametric equation of a circle.
- the centre and radius of the general equation of a circle.
- the equation of a circle when the extremities of a diameter are given.
- the equation of a tangent to the circle at a given point.
- identification of conics.
- the standard equation of a parabola, its focus, directrix , latus rectum and commercial applications

## Introduction

The word "Geometry" is derived from the word "geo" meaning "earth" and "metron" meaning "measuring". The need of measuring land is the origin of geometry.

"Geometry" is the study of points, lines, curves, surface etc., and their properties. The importance of analytical geometry is that it establishes a correspondence between geometric curves and algebraic equations.

A systematic study of geometry by the use of algebra was first carried out by celebrated French Philosopher and mathematician Rene Descartes (1596-1650), in his book' La Geometry', published in 1637. The resulting combination of analysis and geometry is referred now as analytical geometry. He is known as the father of Analytical geometry

**Rene Descartes (1596-1650)**

Analytical geometry is extremely useful in the aircraft industry, especially when dealing with the shape of an airplane fuselage.

## 3.1 Locus

> **Definition 3.1**
>
> The path traced by a moving point under some specified geometrical condition is called its locus.

### 3.1.1 Equation of a locus

Any relation in $x$ and $y$ which is satisfied by every point on the locus is called the equation of the locus.

For example,

(i) The locus of a point $P(x_1, y_1)$ whose distance from a fixed point $C(h, k)$ is constant, is a circle. The fixed point '$C$' is called the centre.
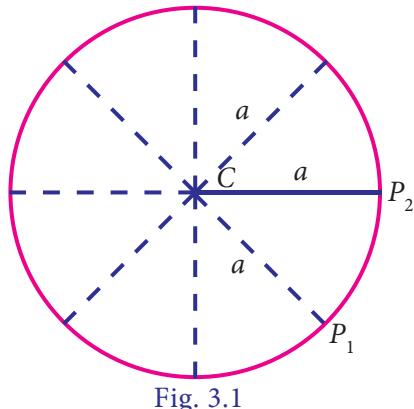


Fig. 3.1

(ii) The locus of a point whose distances from two points $A$ and $B$ are equal is the perpendicular bisector of the line segment $AB$.
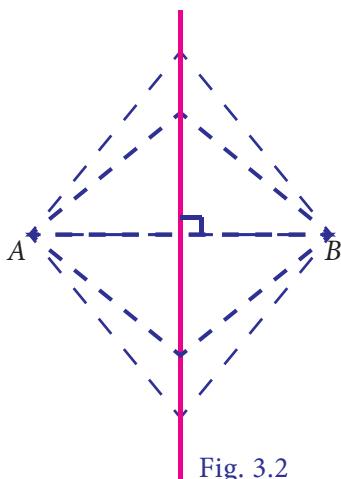


Fig. 3.2

> **DO YOU KNOW?**
>
> Straight line is the locus of a point which moves in the same direction.

### Example 3.1

A point in the plane moves so that its distance from the origin is thrice its distance from the $y$- axis. Find its locus.

### *Solution*

Let $P(x_1, y_1)$ be any point on the locus and $A$ be the foot of the perpendicular from $P(x_1, y_1)$ to the $y$- axis.

Given that OP $= 3$ AP

$$OP^2 = 9AP^2$$

$$(x_1 - 0)^2 + (y_1 - 0)^2 = 9x_1^2$$

$$x_1^2 + y_1^2 = 9x_1^2$$

$$8x_1^2 - y_1^2 = 0$$

$\therefore$ The locus of $P(x_1, y_1)$ is

$$8x^2 - y^2 = 0$$

## Example 3.2

Find the locus of the point which is equidistant from (2, –3) and (3, –4).

### Solution

Let $A(2, -3)$ and $B(3, -4)$ be the given points

Let $P(x_1, y_1)$ be any point on the locus.

Given that $\qquad\qquad PA = PB.$

$$PA^2 = PB^2$$

$$(x_1 - 2)^2 + (y_1 + 3)^2 = (x_1 - 3)^2 + (y_1 + 4)^2$$

$$x_1^2 - 4x_1 + 4 + y_1^2 + 6y_1 + 9 = x_1^2 - 6x_1 + 9 + y_1^2 + 8y_1 + 16$$

i.e., $\qquad\qquad 2x_1 - 2y_1 - 12 = 0$

i.e., $\qquad\qquad x_1 - y_1 - 6 = 0$

The locus of $P(x_1, y_1)$ is $x - y - 6 = 0$.

## Example 3.3

Find the locus of a point, so that the join of (–5, 1) and (3, 2) subtends a right angle at the moving point.

### Solution

Let $A(-5, 1)$ and $B(3, 2)$ be the given points

Let $P(x_1, y_1)$ be any point on the locus.

Given that $\angle APB = 90º$.

Triangle $APB$ is a right angle triangle.

$$BA^2 = PA^2 + PB^2$$

$$(3+5)^2 + (2-1)^2 = (x_1+5)^2 + (y_1-1)^2 + (x_1-3)^2 + (y_1-2)^2$$

$$65 = x_1^2 + 10x_1 + 25 + y_1^2 - 2y_1 + 1 + x_1^2 - 6x_1 + 9 + y_1^2 - 4y_1 + 4$$

i.e., $2x_1^2 + 2y_1^2 + 4x_1 - 6y_1 + 39 - 65 = 0$

i.e., $x_1^2 + y_1^2 + 2x_1 - 3y_1 - 13 = 0$

The locus of $P(x_1, y_1)$ is $x^2 + y^2 - 2x - 3y - 13 = 0$

### Exercise 3.1

1. Find the locus of a point which is equidistant from (1, 3) and $x$ axis.

2. A point moves so that it is always at a distance of 4 units from the point (3, –2)

3. If the distance of a point from the points (2,1) and (1,2) are in the ratio 2 :1, then find the locus of the point.

4. Find a point on $x$ axis which is equidistant from the points (7, –6) and (3, 4).

5. If $A(-1, 1)$ and $B(2, 3)$ are two fixed points, then find the locus of a point P so that the area of triangle $APB$ = 8 sq.units.

## 3.2 System of straight lines

### 3.2.1 Recall

In lower classes, we studied the basic concept of coordinate geometry, like distance formula, section - formula, area of triangle and slope of a straight lines etc.

We also studied various form of equations of lines in X std. Let us recall the equations of straight lines. Which will help us for better understanding the new concept and definitions of XI std co-ordinate geometry.

**Various forms of straight lines:**

**(i) Slope-intercept form**

Equation of straight line having slope $m$ and $y$-intercept 'c' is $y = mx+c$

**(ii) Point- slope form**

Equation of Straight line passing through the given point $P(x_1, y_1)$ and having a slope $m$ is

$$y - y_1 = m(x - x_1)$$

**(iii)    Two Point form**

Equation of a straight line joining the given points $A(x_1,y_1), B(x_2,y_2)$ is $\dfrac{y-y_1}{y_2-y_1} = \dfrac{x-x_1}{x_2-x_1}$.

In determinant form, equation of straight line joining two given points $A(x_1,y_1)$ and $B(x_2,y_2)$ is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

**(iv)    Intercept form**

Equation of a straight line whose $x$ and $y$ intercepts are $a$ and $b$, is $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

**(v)    General form**

Equation of straight line in general form is $ax + by + c = 0$ where $a$, $b$ and $c$ are constants and $a$, $b$ are not simultaneously zero.

### 3.2.2    Angle between two straight lines

Let $l_1$ and $l_2$ be two straight lines represented by the equations $l_1 : y = m_1 x + c_1$ and $l_2 : y = m_2 x + c_2$ intersecting at $P$.

If $\theta_1$ and $\theta_2$ are two angles made by $l_1$ and $l_2$ with $x$- axis then slope of the lines are $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$.

From fig 3.3, if $\theta$ is angle between the lines $l_1$ & $l_2$ then

$$\theta = \theta_1 - \theta_2$$

$$\therefore \qquad \tan\theta = |\tan(\theta_1 - \theta_2)|$$

$$= \left| \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} \right|$$

$$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\theta = \tan^{-1}\left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$
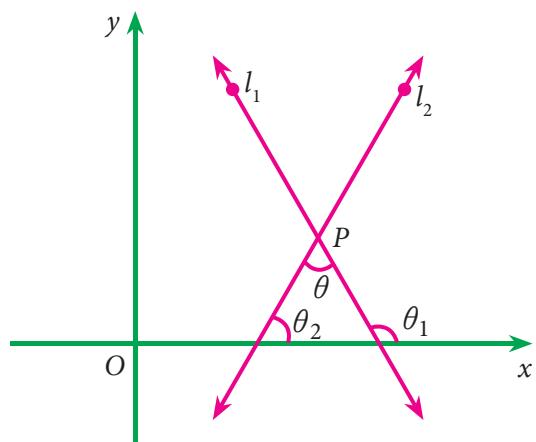
Fig. 3.3

(i) If $\dfrac{m_1 - m_2}{1 + m_1 m_2}$ is positive, then $\theta$, the angle between $l_1$ and $l_2$ is acute and if it is negative, then, $\theta$ is the obtuse.

(ii) We know that two straight lines are parallel if and only if their slopes are equal.

(iii) We know that two lines are perpendicular if and only if the product of their slopes is –1. (Here the slopes $m_1$ and $m_2$ are finite.)

**DO YOU KNOW?**

The straight lines $x$-axis and $y$-axis are perpendicular to each other. But, the condition $m_1 m_2 = -1$ is not true because the slope of the $x$-axis is zero and the slope of the $y$-axis is not defined.

## Example 3.4

Find the acute angle between the lines $2x - y + 3 = 0$ and $x + y + 2 = 0$

### Solution

Let $m_1$ and $m_2$ be the slopes of $2x - y + 3 = 0$ and $x + y + 2 = 0$

Now $m_1 = 2$, $m_2 = -1$

Let $\theta$ be the angle between the given lines

$$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{2 + 1}{1 + 2(-1)} \right| = 3$$

$$\theta = \tan^{-1}(3)$$

### 3.2.3 Distance of a point from a line

(i) The length of the perpendicular from a point $P(l, m)$ to the line $ax + by + c = 0$ is $d = \left| \dfrac{al + bm + c}{\sqrt{a^2 + b^2}} \right|$

(ii) The length of the perpendicular form the origin $(0,0)$ to the line $ax + by + c = 0$ is $d = \left| \dfrac{c}{\sqrt{a^2 + b^2}} \right|$

## Example 3.5

Show that perpendicular distances of the line $x - y + 5 = 0$ from origin and from the point $P(2, 2)$ are equal.

## Solution

Given line is $x - y + 5 = 0$

Perpendicular distance of the given line from $P(2, 2)$ is $= \left| \dfrac{2 - 2 + 5}{\sqrt{1^2 + 1^2}} \right|$

$$= \left| \dfrac{5}{\sqrt{2}} \right|$$

Distance of $(0,0)$ from the given line $= \left| \dfrac{5}{\sqrt{1^2 + 1^2}} \right| = \left| \dfrac{5}{\sqrt{2}} \right|$

The given line is equidistance from origin and $(2, 2)$

### Example 3.6

If the angle between the two lines is $\dfrac{\pi}{4}$ and slope of one of the lines is 3, then find the slope of the other line.

## Solution

We know that the acute angle $\theta$ between two lines with slopes $m_1$ and $m_2$ is given by

$$\tan \theta = \left| \dfrac{m_1 - m_2}{1 + m_1 m_2} \right|$$

Given $m_1 = 3$ and $\theta = \dfrac{\pi}{4}$

$\therefore \qquad \tan \dfrac{\pi}{4} = \left| \dfrac{3 - m_2}{1 + 3m_2} \right|$

$$1 = \left( \dfrac{3 - m_2}{1 + 3m_2} \right)$$

$$1 + 3m_2 = 3 - m_2$$

$\Rightarrow \qquad m_2 = \dfrac{1}{2}$

Hence the slope of the other line is $\dfrac{1}{2}$.

### 3.2.4  Concurrence of three lines

If two lines $l_1$ and $l_2$ meet at a common point $P$, then that $P$ is called point of intersection of $l_1$ and $l_2$. This point of intersection is obtained by solving the equations of $l_1$ and $l_2$.

If three or more straight lines will have a point in common then they are said to be concurrent.

The lines passing through the common point are called concurrent lines and the common point is called concurrent point.

## Conditions for three given straight lines to be concurrent

Let $a_1x + b_1y + c_1 = 0$ → (1)

$a_2x + b_2y + c_2 = 0$ → (2)

$a_3x + b_3y + c_3 = 0$ → (3)

be the equations of three straight lines, then the condition that these lines to be concurrent is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

## Example 3.7

Show that the given lines $3x - 4y - 13 = 0, 8x - 11y = 33$ and $2x - 3y - 7 = 0$ are concurrent and find the concurrent point.

### *Solution*

Given lines $\quad 3x - 4y - 13 = 0 \quad$ ... (1)

$8x - 11y = 33 \quad$ ... (2)

$2x - 3y - 7 = 0 \quad$ ... (3)

Conditon for concurrent lines is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

i.e., $\begin{vmatrix} 3 & -4 & -13 \\ 8 & -11 & -33 \\ 2 & -3 & -7 \end{vmatrix} = 3(77-99) + 4(-56+66) - 13(-24+22)$

$= -66 + 40 + 26 = 0$

⇒ Given lines are concurrent.

To get the point of concurrency solve the equations (1) and (3)

Equation (1) ✕2 ⇒ $6x - 8y = 26$

Equation (3) ✕3 ⇒ $6x - 9y = 21$

_____

$y = 5$

_____

When $y=5$ from (2) $8x = 88$

$$x = 11$$

Point of concurrency is $(11, 5)$

## Example 3.8

If the lines $3x - 5y - 11 = 0, 5x + 3y - 7 = 0$ and $x + ky = 0$ are concurrent, find the value of k.

### Solution

Given the lines are concurrent.

Therefore
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 3 & -5 & -11 \\ 5 & 3 & -7 \\ 1 & k & 0 \end{vmatrix} = 0$$

$$1(35+33) - k(-21+55) = 0$$

$$\Rightarrow \quad 34k = 68. \quad \therefore \ k = 2.$$

## Example 3.9

A private company appointed a clerk in the year 2012, his salary was fixed as ₹20,000. In 2017 his salary raised to ₹25,000.

(i) Express the above information as a linear function in $x$ and $y$ where $y$ represent the salary of the clerk and $x$-represent the year

(ii) What will be his salary in 2020?

### Solution

Let $y$ represent the salary (in Rs) and $x$ represent the year

| year $(x)$ | salary $(y)$ |
|---|---|
| 2012$(x_1)$ | 20,000$(y_1)$ |
| 2017$(x_2)$ | 25,000$(y_2)$ |
| 2020 | ? |

The equation of straight line expressing the given information as a linear equation in $x$ and $y$ is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 20,000}{25,000 - 20,000} = \frac{x - 2012}{2017 - 2012}$$

$$\frac{y - 20,000}{5000} = \frac{x - 2012}{5}$$

$$y = 1000x - 2012000 + 20,000$$

$$y = 1000x - 19,92,000$$

In 2020 his salary will be

$$y = 1000(2020) - 19,92,000$$

$$y = ₹28000$$

**Exercise 3.2**

1. Find the angle between the lines whose slopes are $\frac{1}{2}$ and 3

2. Find the distance of the point (4,1) from the line $3x - 4y + 12 = 0$

3. Show that the straight lines $x + y - 4 = 0, 3x + 2 = 0$ and $3x - 3y + 16 = 0$ are concurrent.

4. Find the value of 'a' for which the straight lines $3x + 4y = 13; 2x - 7y = -1$ and $ax - y - 14 = 0$ are concurrent.

5. A manufacturer produces 80 TV sets at a cost of ₹2,20,000 and 125 TV sets at a cost of ₹2,87,500. Assuming the cost curve to be linear, find the linear expression of the given information. Also estimate the cost of 95 TV sets.

## 3.3 Pair of straight lines

### 3.3.1 Combined equation of the pair of straight lines

Let us consider the two individual equations of straight lines

$l_1 x + m_1 y + n_1 = 0$ and $l_2 x + m_2 y + n_2 = 0$

Then their combined equation is

$\left(l_1 x + m_1 y + n_1\right)\left(l_2 x + m_2 y + n_2\right) = 0$

$$l_1 l_2 x^2 + (l_1 m_2 + l_2 m_1)xy + m_1 m_2 y^2 + (l_1 n_2 + l_2 n_1)x + (m_1 n_2 + m_2 n_1)y + n_1 n_2 = 0$$

Hence the general equation of pair of straight lines can be taken as

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

where $a, b, c, f, g$ and $h$ are all constants.

### 3.3.2 Pair of straight lines passing through the origin

The homogeneous equation $ax^2 + 2hxy + by^2 = 0$     ... (1)

of second degree in $x$ and $y$ represents a pair of straight lines passing through the origin.

Let $y = m_1 x$ and $y = m_2 x$ be two straight lines passing through the origin.

Then their combined equation is

$$(y - m_1 x)(y - m_2 x) = 0$$

$$\Rightarrow m_1 m_2 x^2 - (m_1 + m_2)xy + y^2 = 0 \qquad \text{... (2)}$$

(1) and (2) represent the same pair of straight lines

$$\therefore \quad \frac{a}{m_1 m_2} = \frac{2h}{-(m_1 + m_2)} = \frac{b}{1}$$

$$\Rightarrow \quad m_1 m_2 = \frac{a}{b} \text{ and } m_1 + m_2 = -\frac{2h}{b}.$$

i.e., product of the slopes $= \dfrac{a}{b}$ and sum of the slopes $= -\dfrac{2h}{b}$

### 3.3.3 Angle between pair of straight lines passing through the origin

The equation of the pair of straight lines passing through the origin is

$$ax^2 + 2hxy + by^2 = 0$$

Let $m_1$ and $m_2$ be the slopes of above lines.

Here $m_1 + m_2 = -\dfrac{2h}{b}$ and $m_1 m_2 = \dfrac{a}{b}$.

Let $\theta$ be the angle between the pair of straight lines.

Then

$$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{\pm 2\sqrt{h^2 - ab}}{a + b} \right|$$

Let us take '$\theta$' as acute angle

$$\therefore \qquad \theta = \tan^{-1} \left| \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| \right|$$

## NOTE

(i) If $\theta$ is the angle between the pair of straight lines

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then $\theta = \tan^{-1} \left| \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| \right|$

(ii) If the straight lines are parallel, then $h^2 = ab$ .

(iii) If the straight lines are perpendicular, then $a + b = 0$

i.e., *coefficient of* $x^2$ + *coefficient of* $y^2 = 0$

### 3.3.4 The condition for general second degree equation to represent the pair of straight lines

The condition for a general second degree equation in $x$, $y$ namely

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of straight lines is

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

## NOTE

The condition in determinant form is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

### Example 3.10

Find the combined equation of the given straight lines whose separate equations are $2x + y - 1 = 0$ and $x + 2y - 5 = 0$.

### Solution

The combined equation of the given straight lines is

$$(2x + y - 1)(x + 2y - 5) = 0$$

i.e., $2x^2 + xy - x + 4xy + 2y^2 - 2y - 10x - 5y + 5 = 0$

i.e., $2x^2 + 5xy + 2y^2 - 11x - 7y + 5 = 0$

**Example 3.11**

Show that the equation $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ represents a pair of straight lines. Also find the angle between them.

*Solution*

Compare the equation

$$2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0 \text{ with}$$

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \text{ we get}$$

$$a = 2, b = 3, h = \frac{5}{2}, g = 3, f = \frac{7}{2} \text{ and } c = 4$$

Condition for the given equation to represent a pair of straight lines is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 24 + \frac{105}{2} - \frac{49}{2} - 27 - 25 = 0$$

Hence the given equation represents a pair of straight lines.

Let $\theta$ be the angle between the lines.

Then $\theta = \tan^{-1} \left| \left| \dfrac{2\sqrt{h^2 - ab}}{a+b} \right| \right| = \tan^{-1} \left[ \dfrac{2\sqrt{\frac{25}{4} - 6}}{5} \right]$

$$\therefore \ \theta = \tan^{-1}\left(\frac{1}{5}\right)$$

**Example 3.12**

The slope of one of the straight lines $ax^2 + 2hxy + by^2 = 0$ is twice that of the other, show that $8h^2 = 9ab$.

*Solution*

Let $m_1$ and $m_2$ be the slopes of the pair of straight lines

$$ax^2 + 2hxy + by^2 = 0$$

$$\therefore \qquad m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b}$$

It is given that one slope is twice the other, so let $m_2 = 2m_1$

$$\therefore \qquad m_1 + 2m_1 = -\frac{2h}{b} \text{ and } m_1 \cdot 2m_1 = \frac{a}{b}$$

$$\therefore \qquad m_1 = -\frac{2h}{3b} \text{ and } 2m_1^2 = \frac{a}{b}$$

$$\Rightarrow \qquad 2\left(-\frac{2h}{3b}\right)^2 = \frac{a}{b}$$

$$\Rightarrow \qquad \frac{8h^2}{9b^2} = \frac{a}{b}$$

$$\Rightarrow \qquad 8h^2 = 9ab$$

## Example 3.13

Show that the equation $2x^2 + 7xy + 3y^2 + 5x + 5y + 2 = 0$ represent two straight lines and find their separate equations.

### Solution

Compare the equation $2x^2 + 7xy + 3y^2 + 5x + 5y + 2 = 0$ with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,

$a = 2,\ b = 3,\ h = \dfrac{7}{2},\ g = \dfrac{5}{2},\ f = \dfrac{5}{2},\ c = 2$

Now
$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} a & \frac{7}{2} & \frac{5}{2} \\ \frac{7}{2} & 3 & \frac{5}{2} \\ \frac{5}{2} & \frac{5}{2} & 2 \end{vmatrix}$$

$$= 2\left(6 - \frac{25}{4}\right) - \frac{7}{2}\left(9 - \frac{25}{4}\right) + \frac{5}{2}\left(\frac{35}{4} - \frac{15}{2}\right)$$

$$= 2\left(-\frac{1}{4}\right) - \frac{7}{2}\cdot\frac{3}{4} + \frac{5}{2}\cdot\frac{5}{4}$$

$$= -\frac{1}{2} - \frac{21}{8} + \frac{25}{8} = 0$$

Hence the given equation represents a pair of straight lines.

Now consider,

$$2x^2 + 7xy + 3y^2 = 2x^2 + 6xy + xy + 3y^2$$

$$= 2x(x + 3y) + y(x + 3y)$$

$$= (x + 3y)(2x + y)$$

Let $2x^2 + 7xy + 3y^2 + 5x + 5y + 2 = (x + 3y + l)(2x + y + m)$

Comparing the coefficient of $x$, $\qquad 2l + m = 5 \qquad$ (1)

Comparing the coefficient of $y$, $\qquad l + 3m = 5 \qquad$ (2)

Solving (1) and (2), we get $\quad m = 1$ and $l = 2$

$\therefore \qquad$ The separate equations are $x + 3y + 2 = 0$ and $2x + y + 1 = 0$.

**Example 3.14**

Show that the pair of straight lines $4x^2 - 12xy + 9y^2 + 18x - 27y + 8 = 0$ represents a pair of parallel straight lines and find their separate equations.

*Solution*

The given equation is $4x^2 - 12xy + 9y^2 + 18x - 27y + 8 = 0$

Here $a = 4$, $b = 9$ and $h = -6$

$$h^2 - ab = 36 - 36 = 0$$

Hence the given equation represents a pair of parallel straight lines.

Now $4x^2 - 12xy + 9y^2 = (2x - 3y)^2$

Consider $4x^2 - 12xy + 9y^2 + 18x - 27y + 8 = 0$

$\Rightarrow \quad (2x - 3y)^2 + 9(2x - 3y) + 8 = 0$

Put $\quad 2x - 3y = z$

$$z^2 + 9z + 8 = 0$$

$$(z+1)(z+8) = 0$$

$$z+1 = 0 \qquad\qquad z+8 = 0$$

$$2x - 3y + 1 = 0 \qquad 2x - 3y + 8 = 0$$

Hence the separate equations are

$$2x - 3y + 1 = 0 \quad \text{and} \quad 2x - 3y + 8 = 0$$

**Example 3.15**

Find the angle between the straight lines $x^2 + 4xy + y^2 = 0$

*Solution*

The given equation is $x^2 + 4xy + y^2 = 0$

Here $a = 1$, $b = 1$ and $h = 2$.

If $\theta$ is the angle between the given straight lines, then

$$\theta = \tan^{-1}\left|\left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|\right|$$

$$= \tan^{-1}\left|\left|\frac{2\sqrt{4-1}}{2}\right|\right|$$

$$= \tan^{-1}\left(\sqrt{3}\right)$$

$$\theta = \frac{\pi}{3}$$

### Example 3.16

For what value of $k$ does $2x^2 + 5xy + 2y^2 + 15x + 18y + k = 0$ represent a pair of straight lines.

### Solution

Here $a = 2$, $b = 2$, $h = \frac{5}{2}$, $g = \frac{15}{2}$, $f = 9$, $c = k$.

The given line represents a pair of straight lines if,

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

i.e., $\quad 4k + \dfrac{675}{2} - 162 - \dfrac{225}{2} - \dfrac{25}{4}k = 0$

$\Rightarrow \quad 16k + 1350 - 648 - 450 - 25k = 0$

$\Rightarrow \quad 9k = 252 \quad \therefore k = 28$

**Exercise 3.3**

1. If the equation $ax^2 + 5xy - 6y^2 + 12x + 5y + c = 0$ represents a pair of perpendicular straight lines, find $a$ and $c$.

2. Show that the equation $12x^2 - 10xy + 2y^2 + 14x - 5y + 2 = 0$ represents a pair of straight lines and also find the separate equations of the straight lines.

3. Show that the pair of straight lines $4x^2 + 12xy + 9y^2 - 6x - 9y + 2 = 0$ represents two parallel straight lines and also find the separate equations of the straight lines.

4. Find the angle between the pair of straight lines $3x^2 - 5xy - 2y^2 + 17x + y + 10 = 0$

## 3.4 Circles

### Definition 3.2

A circle is the locus of a point which moves in such a way that its distance from a fixed point is always constant. The fixed point is called the centre of the circle and the constant distance is the radius of the circle.

### 3.4.1 The equation of a circle when the centre and radius are given.

Let $C(h, k)$ be the centre and '$r$' be the radius of the circle



Fig. 3.4

Let P$(x, y)$ be any point on the circle

$$CP = r$$

$$CP^2 = r^2$$

$(x–h)^2 + (y–k)^2 = r^2$ is the equation of the circle.
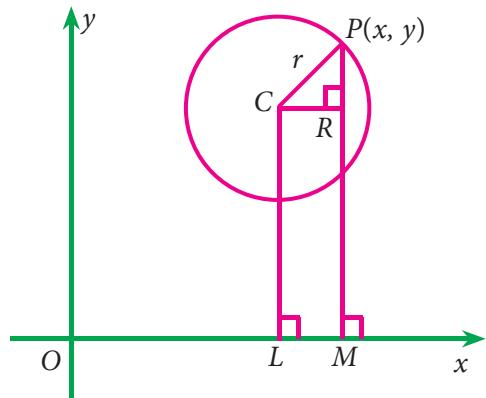
In particular, if the centre is at the origin, the equation of circle is $x^2 + y^2 = r^2$

### Example 3.17

Find the equation of the circle with centre at (3, –1) and radius is 4 units.

### *Solution*

Equation of circle is $(x-h)^2 + (y-k)^2 = r^2$

Here $(h, k) = (3, –1)$ and $r = 4$

Equation of circle is

$$(x-3)^2 + (y+1)^2 = 16$$

$$x^2 - 6x + 9 + y^2 + 2y + 1 = 16$$

$$x^2 + y^2 - 6x + 2y - 6 = 0$$

### Example 3.18

Find the equation of the circle with centre at origin and radius is 3 units.

*Solution*

Equation of circle is $x^2 + y^2 = r^2$

Here   $r = 3$

i.e equation of circle is  $x^2 + y^2 = 9$

## 3.4.2 Equation of a circle when the end points of a diameter are given

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter of a circle and

$P(x, y)$ be any point on the circle.

We know that angle in the semi circle is 90°

$\therefore$   $\underline{|APB} = 90°$

$\therefore$     (Slope of AP) (Slope of BP)   $=$   $-1$

$$\left(\frac{y - y_1}{x - x_1}\right) \times \left(\frac{y - y_2}{x - x_2}\right) = -1$$

$$(y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$$
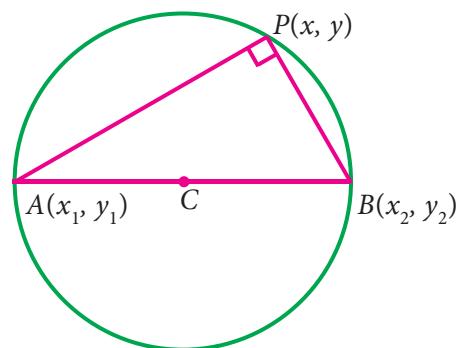
Fig. 3.5

$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ is the required equation of circle.

## Example 3.19

Find the equation of the circle when the end points of the diameter are (2, 4) and (3, –2).

*Solution*

Equation of a circle when the end points of the diameter are given is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Here  $(x_1, y_1) = (2, 4)$ and $(x_2, y_2) = (3, -2)$

$\therefore (x - 2)(x - 3) + (y - 4)(y + 2) = 0$

$$x^2 + y^2 - 5x - 2y - 2 = 0$$

## 3.4.3 General equation of a circle

The general equation of circle is  $x^2 + y^2 + 2gx + 2fy + c = 0$ where $g, f$ and $c$ are constants.

$$\text{i.e } x^2 + y^2 + 2gx + 2fy = -c$$

$$x^2 + 2gx + g^2 - g^2 + y^2 + 2fy + f^2 - f^2 = -c$$

$$(x+g)^2 - g^2 + (y+f)^2 - f^2 = -c$$

$$(x+g)^2 + (y+f)^2 = g^2 + f^2 - c$$

$$[x-(-g)]^2 + [y-(-f)]^2 = \left[\sqrt{g^2+f^2-c}\right]^2$$

Comparing this with the circle $(x-h)^2 + (y-k)^2 = r^2$

We get, centre is $(-g, -f)$ and radius is $\sqrt{g^2+f^2-c}$

## NOTE

The general second degree equation $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if

(i)  $a = b$  i.e., co-efficient of $x^2$ = co-efficient of $y^2$.

(ii)  $h = 0$  i.e., no $xy$ term.

## Example 3.20

Find the centre and radius of the circle $x^2 + y^2 - 8x + 6y - 24 = 0$

### Solution

Equation of circle is $x^2 + y^2 - 8x + 6y - 24 = 0$

Here  $g = -4$ , $f = 3$ and $c = -24$

Centre $= C(-g, -f) = C(4, -3)$ and

Radius:  $r = \sqrt{g^2 + f^2 - c}$

$$= \sqrt{16 + 9 + 24} = 7 \text{ unit.}$$

## Example 3.21

For what values of  $a$ and  $b$ does the equation

$(a-2)x^2 + by^2 + (b-2)xy + 4x + 4y - 1 = 0$ represents a circle? Write down the resulting equation of the circle.

*Solution*

The given equation is $(a-2)x^2 + by^2 + (b-2)xy + 4x + 4y - 1 = 0$

As per conditions noted above,

(i) coefficient of $xy = 0 \Rightarrow b - 2 = 0$

$$\therefore b = 2$$

(ii) coefficient of $x^2$ = Coefficient of $y^2 \Rightarrow a - 2 = b$

$$a - 2 = 2 \Rightarrow a = 4$$

$\therefore$ Resulting equation of circle is $2x^2 + 2y^2 + 4x + 4y - 1 = 0$

**Example 3.22**

If the equation of a circle $x^2 + y^2 + ax + by = 0$ passing through the points (1, 2) and (1, 1), find the values of $a$ and $b$

*Solution*

The circle $x^2 + y^2 + ax + by = 0$ passing through (1, 2) and (1, 1)

$\therefore$ We have $1 + 4 + a + 2b = 0$ and $1 + 1 + a + b = 0$

$$\Rightarrow \qquad a + 2b = -5 \qquad (1)$$

and $\qquad a + b = -2 \qquad (2)$

solving (1) and (2), we get $a = 1, b, = -3$ .

**Example 3.23**

If the centre of the circle $x^2 + y^2 + 2x - 6y + 1 = 0$ lies on a straight line

$ax + 2y + 2 = 0$, then find the value of '$a$'

*Solution*

Centre C(−1, 3)

It lies on $ax + 2y + 2 = 0$

$$-a + 6 + 2 = 0$$

$$a = 8$$

**Example 3.24**

Show that the point (7, –5) lies on the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ and find the coordinates of the other end of the diameter through this point .

*Solution*

Let   $A(7, -5)$

Equation of circle is   $x^2 + y^2 - 6x + 4y - 12 = 0$

Substitute $(7, -5)$ for $(x, y)$ , we get

$$x^2 + y^2 - 6x + 4y - 12 = 7^2 + (-5)^2 - 6(7) + 4(-5) - 12$$

$$= 49 + 25 - 42 - 20 - 12 = 0$$

$\therefore$ $(7, -5)$ lies on the circle

Here   $g = -3$ and $f = 2$

$\therefore$     Centre = $C(3, -2)$

Let the other end of the diameter be $B\ (x, y)$

Midpoint of $AB = \left(\dfrac{x+7}{2}, \dfrac{y-5}{2}\right) = C(3, -2)$

$$\dfrac{x+7}{2} = 3 \qquad\qquad \dfrac{y-5}{2} = -2$$

$$x = -1 \qquad\qquad\qquad y = 1$$

Other end of the diameter is $(-1 , 1)$.

**Example 3.25**

Find the equation of the circle passing through the points (0,0), (1, 2) and (2,0).

*Solution*

Let the equation of the circle be   $x^2 + y^2 + 2gx + 2fy + c = 0$

The circle passes through the point (0, 0)

$$c = 0 \qquad\qquad \text{... (1)}$$

The circle passes through the point (1, 2)

$$1^2 + 2^2 + 2g(1) + 2f(2) + c = 0$$

$$2g + 4f + c = -5 \qquad \text{... (2)}$$

The circle passes through the point $(2, 0)$

$$2^2 + 0 + 2g(2) + 2f(0) + c = 0$$

$$4g + c = -4 \qquad \text{... (3)}$$

Solving (1), (2) and (3), we get

$$g = -1, \; f = -\frac{3}{4} \text{ and } c = 0$$

∴ The equation of the circle is

$$x^2 + y^2 + 2(-1)x + 2\left(\frac{-3}{4}\right)y + 0 = 0$$

ie., $\qquad 2x^2 + 2y^2 - 4x - 3y = 0$

### 3.4.4 Parametric form of a circle

Consider a circle with radius r and centre at the origin. Let $P(x, y)$ be any point on the circle. Assume that $OP$ makes an angle $\theta$ with the positive direction of $x$-axis.

Draw PM perpendicular to $x$-axis.

From the figure, $\qquad \cos\theta = \dfrac{x}{r} \Rightarrow x = r\cos\theta$

$$\sin\theta = \dfrac{x}{r} \Rightarrow y = r\sin\theta$$



Fig. 3.6

The equations $x = r\cos\theta$, $y = r\sin\theta$ are called the parametric equations of the circle $x^2 + y^2 = r^2$. Here '$\theta$' is called the parameter and $0 \le \theta \le 2\pi$.
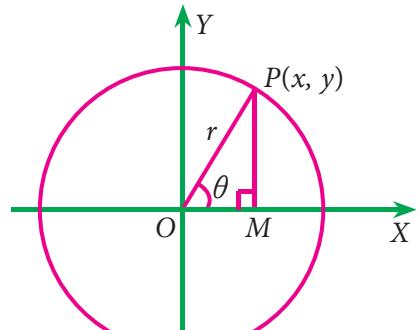
### Example 3.26

Find the parametric equations of the circle $x^2 + y^2 = 25$

### Solution

Here $r^2 = 25 \Rightarrow r = 5$

Parametric equations are $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow x = 5\cos\theta$, $y = 5\sin\theta$, $0 \le \theta \le 2\pi$

1. Find the equation of the following circles having

    (i) the centre (3,5) and radius 5 units

    (ii) the centre (0,0) and radius 2 units

2. Find the centre and radius of the circle

    (i) $x^2 + y^2 = 16$

    (ii) $x^2 + y^2 - 22x - 4y + 25 = 0$

    (iii) $5x^2 + 5y^2 + 4x - 8y - 16 = 0$

    (iv) $(x+2)(x-5) + (y-2)(y-1) = 0$

3. Find the equation of the circle whose centre is $(-3, -2)$ and having circumference $16\pi$

4. Find the equation of the circle whose centre is (2,3) and which passes through (1,4)

5. Find the equation of the circle passing through the points (0, 1),(4, 3) and (1, –1).

6. Find the equation of the circle on the line joining the points (1,0), (0,1) and having its centre on the line $x + y = 1$

7. If the lines $x + y = 6$ and $x + 2y = 4$ are diameters of the circle, and the circle passes through the point (2, 6) then find its equation.

8. Find the equation of the circle having (4, 7) and (–2, 5) as the extremities of a diameter.

9. Find the Cartesian equation of the circle whose parametric equations are $x = 3\cos\theta$, $y = 3\sin\theta$, $0 \le \theta \le 2\pi$ .

### 3.4.5 Tangents

The equation of the tangent to the circle
$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ at } (x_1, y_1) \text{ is}$$
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ .

**Corollary:**

The equation of the tangent at $(x_1, y_1)$ to the circle
$$x^2 + y^2 = a^2 \text{ is } xx_1 + yy_1 = a^2$$

C(-g, -f)
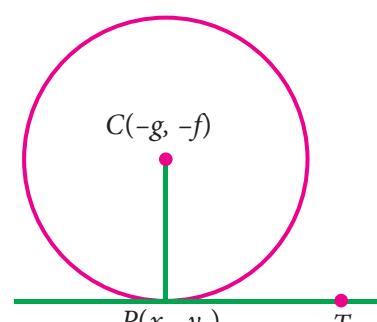
P($x_1$, $y_1$)     T

Fig. 3.7

To get the equation of the tangent at $(x_1, y_1)$ to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots (1)$$

replace $x^2$ by $xx_1$, $y^2$ by $yy_1$, $x$ by $\dfrac{x + x_1}{2}$ and $y$ by $\dfrac{y + y_1}{2}$ in equation (1).

### Example 3.27

Find the equation of tangent at the point $(-2, 5)$ on the circle

$$x^2 + y^2 + 3x - 8y + 17 = 0.$$

*Solution*

The equation of the tangent at $(x_1, y_1)$ to the given circle $x^2 + y^2 + 3x - 8y + 17 = 0$ is

$$xx_1 + yy_1 + 3 \times \frac{1}{2}(x + x_1) - 8 \times \frac{1}{2}(y + y_1) + 17 = 0$$

Here $(x_1, y_1) = (-2, 5)$

$$-2x + 5y + \frac{3}{2}(x - 2) - 4(y + 5) + 17 = 0$$

$$-2x + 5y + \frac{3}{2}x - 3 - 4y - 20 + 17 = 0$$

$$-4x + 10y + 3x - 6 - 8y - 40 + 34 = 0$$

$$x - 2y + 12 = 0 \text{ is the required equation.}$$

### Length of the tangent to the circle

Length of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from a point P $(x_1, y_1)$

is PT $= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$
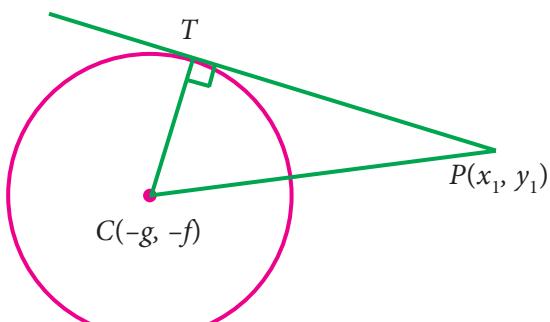


Fig. 3.8

(i)   If the point P is on the circle then $PT^2 = 0$

(ii)   If the point P is outside the circle then $PT^2 > 0$

(iii)   If the point P is inside the circle then $PT^2 < 0$

## Example 3.28

Find the length of the tangent from the point (2 ,3) to the circle

$$x^2 + y^2 + 8x + 4y + 8 = 0$$

### Solution

The length of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from a point $(x_1, y_1)$ is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

$$\begin{aligned}
\text{Length of the tangent} &= \sqrt{x_1^2 + y_1^2 + 8x_1 + 4y_1 + 8} \\
&= \sqrt{2^2 + 3^2 + 8(2) + 4(3) + 8} \quad [\text{Here} \, (x_1, y_1) = (2,3)] \\
&= \sqrt{49}
\end{aligned}$$

Length of the tangent  = 7 units

## Example 3.29

Determine whether the points $P(0,1)$, $Q(5,9)$, $R(-2, 3)$ and $S(2, 2)$ lie outside the circle, on the circle or inside the circle  $x^2 + y^2 - 4x + 4y - 8 = 0$

### Solution

The equation of the circle is  $x^2 + y^2 - 4x + 4y - 8 = 0$

$$PT^2 = x_1^2 + y_1^2 - 4x_1 + 4y_1 - 8$$

At $P(0, 1)$ $PT^2 = 0 + 1 + 0 + 4 - 8 = -3 < 0$

At $Q(5, 9)$ $QT^2 = 25 + 81 - 20 + 36 - 8 = 114 > 0$

At $R(-2, 3)$ $RT^2 = 4 + 9 + 8 + 12 - 8 = 25 > 0$

At $S(2, 2)$ $ST^2 = 4 + 4 - 8 + 8 - 8 = 0$

$\therefore$   The point $P$ lies inside the circle.  The points $Q$ and $R$ lie outside the circle and the point  $S$  lies on the circle.

| Result | Condition for the straight line $y = mx + c$ to be a tangent to the circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1 + m^2)$ |
|---|---|

**Example 3.30**

Find the value of $k$ so that the line $3x + 4y - k = 0$ is a tangent to $x^2 + y^2 - 64 = 0$

*Solution*

The given equations are $x^2 + y^2 - 64 = 0$ and $3x + 4y - k = 0$

The condition for the tangency is $c^2 = a^2(1 + m^2)$

Here $a^2 = 64$ , $m = \dfrac{-3}{4}$ and $c = \dfrac{k}{4}$

$c^2 = a^2(1 + m^2) \Rightarrow \dfrac{k^2}{16} = 64\left(1 + \dfrac{9}{16}\right)$

$$k^2 = 64 \times 25$$

$$k = \pm 40$$

### Exercise 3.5

1. Find the equation of the tangent to the circle $x^2 + y^2 - 4x + 4y - 8 = 0$ at $(-2, -2)$

2. Determine whether the points $P(1, 0)$, $Q(2, 1)$ and $R(2, 3)$ lie outside the circle, on the circle or inside the circle $x^2 + y^2 - 4x - 6y + 9 = 0$

3. Find the length of the tangent from $(1, 2)$ to the circle $x^2 + y^2 - 2x + 4y + 9 = 0$

4. Find the value of $P$ if the line $3x + 4y - P = 0$ is a tangent to the circle $x^2 + y^2 = 16$

## 3.5 Conics

**Definition 3.3**

If a point moves in a plane such that its distance from a fixed point bears a constant ratio to its perpendicular distance from a fixed straight line, then the path described by the moving point is called a conic.



Fig. 3.9

In figure, the fixed point F is called focus, the fixed straight line $l$ is called directrix and $P$ is the moving point such that $\dfrac{FP}{PM} = e$, a constant. Here the locus of $P$ is called a conic and the constant '$e$' is called the eccentricity of the conic.
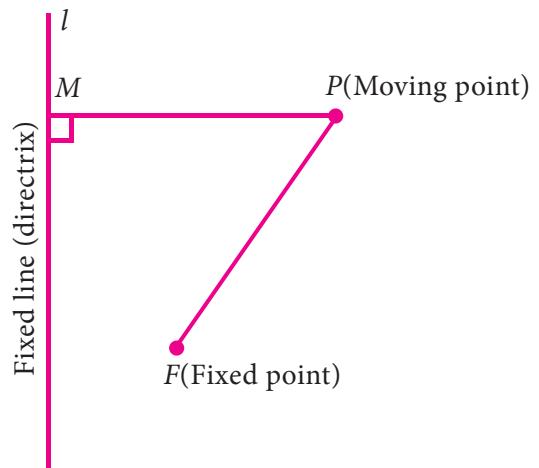
Based on the value of eccentricity we can classify the conics namely,

a)   If $e = 1$, then, the conic is called a parabola

b)   If $e < 1$, then, the conic is called an ellipse

c)   If $e > 1$, then, the conic is called a hyperbola.

The general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents,

 (i)   a pair of straight lines if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

(ii)   a circle if $a = b$ and $h = 0$

If the above two conditions are not satisfied, then $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents,

 (i)   a parabola if $h^2 - ab = 0$

(ii)   an ellipse if $h^2 - ab < 0$

(iii)   a hyperbola if $h^2 - ab > 0$

In this chapter, we study about parabola only.

## 3.5.1 Parabola

**Definition 3.4**

The locus of a point whose distance from a fixed point is equal to its distance from a fixed line is called a parabola.

$y^2 = 4ax$ is th stad rd q tirf    th p rab a. It is p rr i ward

## 3.5.2 Definitions regarding a parabola: $y^2 = 4ax$
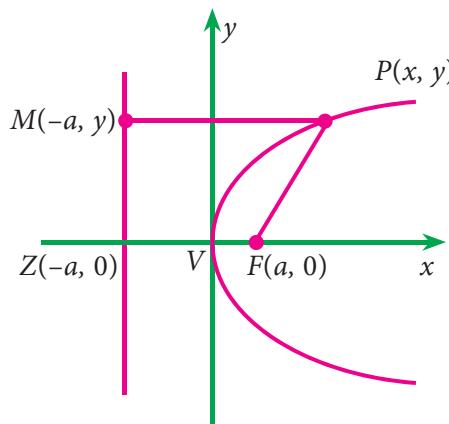


Fig. 3.10

| | |
|---|---|
| **Focus** | The fixed point used to draw the parabola is called the focus ($F$). Here, the focus is $F(a, 0)$. |
| **Directrix** | The fixed line used to draw a parabola is called the directrix of the parabola. Here, the equation of the directrix is $x = -a$. |
| **Axis** | The axis of the parabola is the axis of symmetry. The curve $y^2 = 4ax$ is symmetrical about x-axis and hence $x$-axis or $y = 0$ is the axis of the parabola $y^2 = 4ax$. Note that the axis of the parabola passes through the focus and perpendicular to the directrix. |
| **Vertex** | The point of intersection of the parabola with its axis is called its vertex. Here, the vertex is $V(0, 0)$. |
| **Focal distance** | The distance between a point on the parabola and its focus is called a focal distance |
| **Focal chord** | A chord which passes through the focus of the parabola is called the focal chord of the parabola. |
| **Latus rectum** | It is a focal chord perpendicular to the axis of the parabola. Here, the equation of the latus rectum is $x = a$. Length of the latus rectum is $4a$. |

### 3.5.3 Other standard parabolas :

1. **Open leftward : $y^2 = -4ax[a > 0]$**

   If $x > 0$, then y become imaginary. i.e., the curve exist for $x \leq 0$.
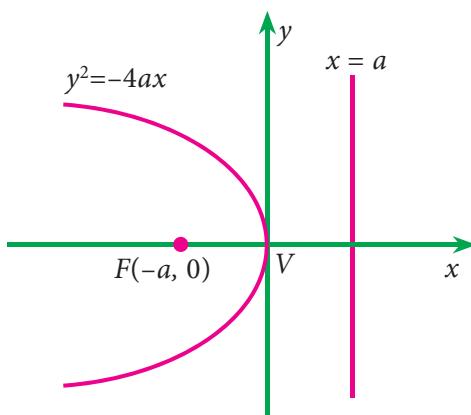


Fig. 3.11

2. **Open upward : $x^2 = 4ay[a > 0]$**

If $y < 0$, then x becomes imaginary. i.e., the curve exist for $y \geq 0$.
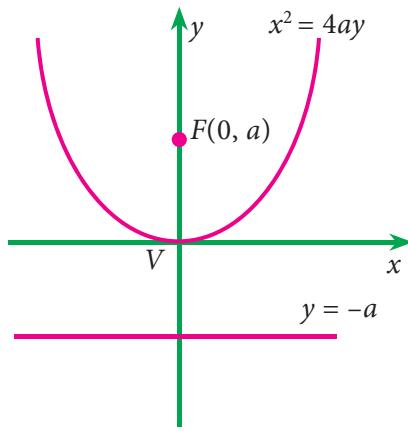


Fig. 3.12

3.  **Open downward :** $x^2 = -4ay[a > 0]$

If $y > 0$, then $x$ becomes imaginary. i.e., the curve exist for $y \leq 0$.
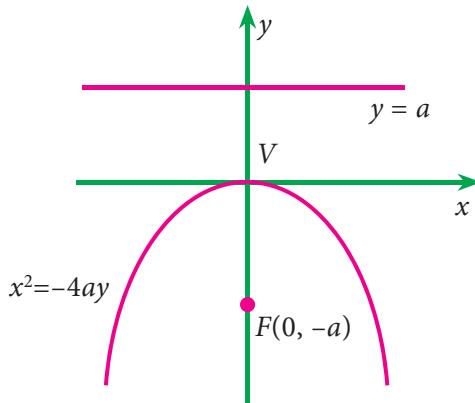


Fig. 3.13

| Equations | $y^2 = 4ax$ | $y^2 = -4ax$ | $x^2 = 4ay$ | $x^2 = -4ay$ |
|---|---|---|---|---|
| Axis | $y = 0$ | $y = 0$ | $x = 0$ | $x = 0$ |
| Vertex | $V(0, 0)$ | $V(0, 0)$ | $V(0, 0)$ | $V(0, 0)$ |
| Focus | $F(a, 0)$ | $F(-a, 0)$ | $F(0, a)$ | $F(0, -a)$ |
| Equation of directrix | $x = -a$ | $x = a$ | $y = -a$ | $y = a$ |
| Length of Latus rectum | $4a$ | $4a$ | $4a$ | $4a$ |
| Equation of Latus rectum | $x = a$ | $x = -a$ | $y = a$ | $y = -a$ |

## The process of shifting the origin or translation of axes.

Consider the *xoy* system. Draw a line parallel to *x* -axis (say X axis) and draw a line parallel to *y* – axis (say Y axis). Let $P(x,y)$ be a point with respect to *xoy* system and $P(X, Y)$ be the same point with respect to $XO'Y$ system.

*Analytical Geometry* | 111
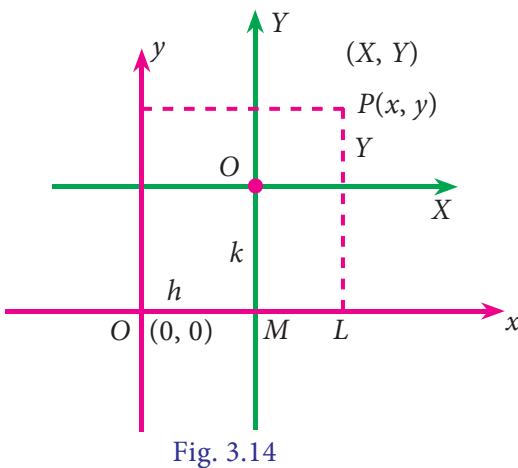
Fig. 3.14

Let the co-ordinates of $O'$ with respect to xoy system be $(h, k)$

The co-ordinate of P with respect to xoy system:

$OL$   $= OM + ML$

   $= h + X$

i.e) $x = X + h$

similarly   $y = Y + k$

$\therefore$ The new co-ordinates of $P$ with respect to $XO'Y$ system.

$X = x - h$ and $Y = y - k$

### 3.5.4 General form of the standard equation of a parabola, which is open rightward (i.e., the vertex other than origin) :

Consider a parabola with vertex $V$ whose co-ordinates with respect to $XOY$ system is $(0, 0)$ and with respect to xoy system is $(h, k)$. Since it is open rightward, the equation of the parabola w.r.t. $XOY$ system is $Y^2 = 4aX$.

By shifting the origin $X = x - h$ and $Y = y - k$, the equation of the parabola

with respect to old xoy system is $(y - k)^2 = 4a(x - h)$.

This is the general form of the standard equation of the parabola, which is open rightward.

Similarly the other general forms are

$(y - k)^2 = - 4a(x - h)$ (open leftwards)

$$(x - h)^2 = 4a(y - k) \text{ (open upwards)}$$

$$(x - h)^2 = -4a(y - k) \text{ (open downwards)}$$

### Example 3.31

Find the equation of the parabola whose focus is (1,3) and whose directrix is $x - y + 2 = 0$.

### *Solution*

$F$ is (1, 3) and directrix is $x - y + 2 = 0$

$$x - y + 2 = 0$$

let $P(x,y)$ be any point on the parabola.

For parabola $\dfrac{FP}{PM} = 1$

$$FP = PM$$

$$FP^2 = (x - 1)^2 + (y - 3)^2$$

$$= x^2 - 2x + 1 + y^2 - 6y + 9$$

$$= x^2 + y^2 - 2x - 6y + 10$$

$$PM = \pm \dfrac{(x - y + 2)}{\sqrt{1 + 1}}$$

$$= \pm \dfrac{(x - y + 2)}{\sqrt{2}}$$

$$PM^2 = \dfrac{(x - y + 2)^2}{2} = \dfrac{x^2 + y^2 + 4 - 2xy - 4y + 4x}{2}$$

$$FP^2 = PM^2$$

$$2x^2 + 2y^2 - 4x - 12y + 20 = x^2 + y^2 - 2xy - 4y + 4x + 4$$

The required equation of the parabola is

$$\Rightarrow x^2 + y^2 + 2xy - 8x - 8y + 16 = 0$$

*Fig. 3.15*

### Example 3.32

Find the focus, the vertex, the equation of the directrix, the axis and the length of the latus rectum of the parabola $y^2 = -12x$
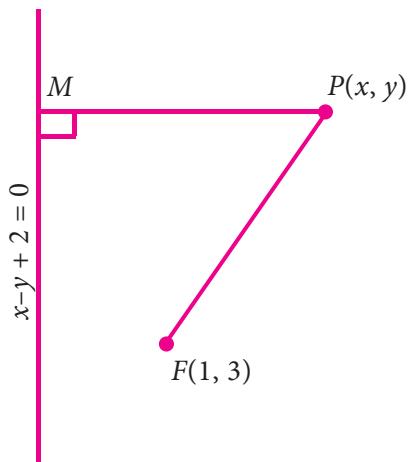
*Solution*

The given equation is of the form $y^2 = -4ax$

where $4a = 12,$     $a = 3.$

The parabola is open left, its focus is

$$F(-a, o), F(-3, 0)$$

Its vertex is    $V(h, k) = V(0, 0)$

The equation of the directrix is $x = a$

      i.e.,    $x = 3$



Fig. 3.16

Its axis is x – axis whose equation is $y = 0$.

Length of the latus rectum = $4a = 12$

### Example 3.33

Show that the demand function $x = 10p - 20 - p^2$ is a parabola and price is maximum at its vertex.

*Solution*

$$x = 10p - 20 - p^2$$

$$= -p^2 - 20 + 10p + 5 - 5$$

$$x - 5 = -p^2 + 10p - 25$$

$$= -\left(p^2 - 10p + 25\right)$$

$$= -(p - 5)^2$$

Put    $X = x - 5$ and $P = p - 5$

$X = -P^2$   i.e $P^2 = -X$

It is a parabola open downward.

$\therefore$ At the vertex $p = 5$ when $x = 5$.   i.e.,   price is maximum at the vertex.

### Example 3.34

Find the axis, vertex, focus, equation of directrix and length of latus rectum for the parabola $x^2 + 6x - 4y + 21 = 0$
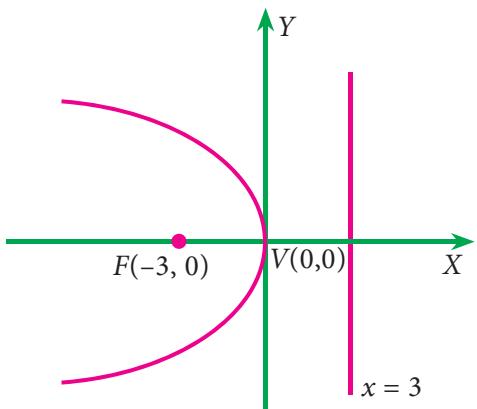
## Solution

$$4y = x^2 + 6x + 21$$

$$4y = (x^2 + 6x + 9) + 12$$

$$4y - 12 = (x + 3)^2$$

$$(x + 3)^2 = 4(y-3)$$

$$X = x + 3, Y = y - 3$$

$$x = X - 3 \text{ and } y = Y + 3$$

$$X^2 = 4Y$$

Comparing with $X^2 = 4aY, 4a = 4$

$$a = 1$$

| | Referred to $(X, Y)$ | Referred to $(x, y)$ $x = X - 3, y = Y + 3$ |
|---|---|---|
| Axis $y = 0$ | $Y = 0$ | $y = 3$ |
| Vertex $V(0,0)$ | $V(0,0)$ | $V(-3,3)$ |
| Focus $F(0, a)$ | $F(0, 1)$ | $F(-3, 4)$ |
| Equation of directrix $(y=-a)$ | $Y = -1$ | $y = 2$ |
| Length of Latus rectum $(4a)$ | $4(1) = 4$ | $4$ |

### Example 3.35

The supply of a commodity is related to the price by the relation $x = \sqrt{5P - 15}$ . Show that the supply curve is a parabola.

### Solution

The supply price relation is given by $x^2 = 5p - 15$

$$= 5(p - 3)$$

$\Rightarrow \quad x^2 = 4aP$ Where $X = x$ and $P = p - 3$

$\therefore$ the supply curve is a parabola whose vertex is $(X = 0, P = 0)$

i.e., The supply curve is a parabola whose vertex is $(0, 3)$
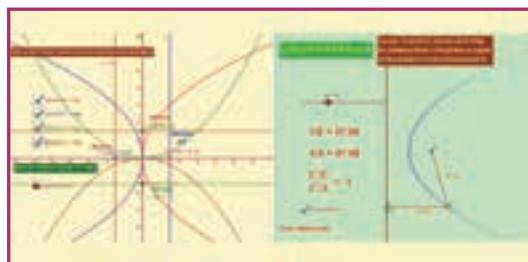
**Expected final outcomes**

### Step – 1

Open the Browser and type the URL given (or) Scan the QR Code.

GeoGebra Work book called "**11th BUSINESS MATHS**" will appear. In this several work sheets for Business Maths are given, Open the worksheet named "**Locus-Parabola**"

### Step - 2

Locus-Parabola page will open. Click on the check boxes on the Left-hand side to see the respective parabola types with Directrix and Focus. On right-hand side Locus for parabola is given. You can play/pause for the path of the locus and observe the condition for the locus.

| Step 1 |
| --- |

| Step 2 |
| --- |

Browse in the link

11th Business Maths: https://ggbm.at/qKj9gSTG (or) scan the QR Code

B184_11_BM_EM

1. Find the equation of the parabola whose focus is the point $F(-1, -2)$ and the directrix is the line $4x-3y+2 = 0$.

2. The parabola $y^2 = kx$ passes through the point $(4, -2)$. Find its latus rectum and focus.

3. Find the vertex, focus, axis, directrix and the length of latus rectum of the parabola $y^2 - 8y - 8x + 24 = 0$

4. Find the co-ordinates of the focus, vertex, equation of the directrix, axis and the length of latus rectum of the parabola

   (a) $y^2 = 20x$     (b) $x^2 = 8y$     (c) $x^2 = -16y$

5. The average variable cost of a monthly output of x tonnes of a firm producing a valuable metal is ₹ $\frac{1}{5}x^2 - 6x + 100$. Show that the average variable cost curve is a parabola. Also find the output and the average cost at the vertex of the parabola.

6. The profit ₹ $y$ accumulated in thousand in $x$ months is given by $y = -x^2 + 10x - 15$. Find the best time to end the project.

Exercise 3.7

**Choose the correct answer**

1. If $m_1$ and $m_2$ are the slopes of the pair of lines given by $ax^2 + 2hxy + by^2 = 0$, then the value of $m_1 + m_2$ is

   (a) $\frac{2h}{b}$       (b) $-\frac{2h}{b}$       (c) $\frac{2h}{a}$       (d) $-\frac{2h}{a}$

2. The angle between the pair of straight lines $x^2 - 7xy + 4y^2 = 0$ is

   (a) $\tan^{-1}\left(\frac{1}{3}\right)$       (b) $\tan^{-1}\left(\frac{1}{2}\right)$       (c) $\tan^{-1}\left(\frac{\sqrt{33}}{5}\right)$       (d) $\tan^{-1}\left(\frac{5}{\sqrt{33}}\right)$

3. If the lines $2x - 3y - 5 = 0$ and $3x - 4y - 7 = 0$   $= 0$ are the diameters of a circle, then its centre is

   (a) $(-1, 1)$       (b) $(1, 1)$       (c) $(1, -1)$       (d) $(-1, -1)$

4. The $x$-intercept of the straight line $3x + 2y - 1 = 0$ is

   (a) 3       (b) 2       (c) $\frac{1}{3}$       (d) $\frac{1}{2}$

5. The slope of the line $7x + 5y - 8 = 0$ is

(a) $\frac{7}{5}$        (b) $-\frac{7}{5}$        (c) $\frac{5}{7}$        (d) $-\frac{5}{7}$

6. The locus of the point $P$ which moves such that $P$ is at equidistance from their coordinate axes is

(a) $y = \frac{1}{x}$        (b) $y = -x$        (c) $y = x$        (d) $y = \frac{-1}{x}$

7. The locus of the point $P$ which moves such that $P$ is always at equidistance from the line $x + 2y + 7 = 0$ is

(a) $x + 2y + 2 = 0$   (b) $x - 2y + 1 = 0$   (c) $2x - y + 2 = 0$   (d) $3x + y + 1 = 0$

8. If $kx^2 + 3xy - 2y^2 = 0$ represent a pair of lines which are perpendicular then $k$ is equal to

(a) $\frac{1}{2}$        (b) $-\frac{1}{2}$        (c) 2        (d) –2

9. (1, –2) is the centre of the circle $x^2 + y^2 + ax + by - 4 = 0$, then its radius

(a) 3        (b) 2        (c) 4        (d) 1

10. The length of the tangent from (4,5) to the circle $x^2 + y^2 = 16$ is

(a) 4        (b) 5        (c) 16        (d) 25

11. The focus of the parabola $x^2 = 16y$ is

(a) (4 ,0)        (b) (–4 , 0)        (c) (0, 4)        (d) (0, –4)

12. Length of the latus rectum of the parabola $y^2 = -25x$.

(a) 25        (b) –5        (c) 5        (d) –25

13. The centre of the circle $x^2 + y^2 - 2x + 2y - 9 = 0$ is

(a) (1 ,1)        (b) (–1, –1)        (c) (–1,1)        (d) (1, –1)

14. The equation of the circle with centre on the $x$ axis and passing through the origin is

(a) $x^2 - 2ax + y^2 = 0$             (b) $y^2 - 2ay + x^2 = 0$

(c) $x^2 + y^2 = a^2$             (d) $x^2 - 2ay + y^2 = 0$

15. If the centre of the circle is $(-a, -b)$ and radius is $\sqrt{a^2 - b^2}$, then the equation of circle is

    (a) $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$
    (b) $x^2 + y^2 + 2ax + 2by - 2b^2 = 0$
    (c) $x^2 + y^2 - 2ax - 2by - 2b^2 = 0$
    (d) $x^2 + y^2 - 2ax - 2by + 2b^2 = 0$

16. Combined equation of co-ordinate axes is

    (a) $x^2 - y^2 = 0$
    (b) $x^2 + y^2 = 0$
    (c) $xy = c$
    (d) $xy = 0$

17. $ax^2 + 4xy + 2y^2 = 0$ represents a pair of parallel lines then '$a$' is

    (a) 2
    (b) –2
    (c) 4
    (d) –4

18. In the equation of the circle $x^2 + y^2 = 16$ then $y$ intercept is (are)

    (a) 4
    (b) 16
    (c) ±4
    (d) ±16

19. If the perimeter of the circle is $8\pi$ units and centre is (2,2) then the equation of the circle is

    (a) $(x - 2)^2 + (y - 2)^2 = 4$
    (b) $(x - 2)^2 + (y - 2)^2 = 16$
    (c) $(x - 4)^2 + (y - 4)^2 = 2$
    (d) $x^2 + y^2 = 4$

20. The equation of the circle with centre (3, –4) and touches the $x$ – axis is

    (a) $(x - 3)^2 + (y - 4)^2 = 4$
    (b) $(x - 3)^2 + (y + 4)^2 = 16$
    (c) $(x - 3)^2 + (y - 4)^2 = 16$
    (d) $x^2 + y^2 = 16$

21. If the circle touches $x$ axis, $y$ axis and the line $x = 6$ then the length of the diameter of the circle is

    (a) 6
    (b) 3
    (c) 12
    (d) 4

22. The eccentricity of the parabola is

    (a) 3
    (b) 2
    (c) 0
    (d) 1

23. The double ordinate passing through the focus is

    (a) focal chord
    (b) latus rectum
    (c) directrix
    (d) axis

24. The distance between directrix and focus of a parabola $y^2 = 4ax$ is

    (a) $a$          (b) $2a$          (c) $4a$          (d) $3a$

25. The equation of directrix of the parabola $y^2 = -x$ is

    (a) $4x + 1 = 0$     (b) $4x - 1 = 0$     (c) $x - 4 = 0$     (d) $x + 4 = 0$

## Miscellaneous Problems

1. A point $P$ moves so that $P$ and the points $(2, 2)$ and $(1, 5)$ are always collinear. Find the locus of $P$.

2. As the number of units produced increases from 500 to 1000 and the total cost of production increases from ₹ 6000 to ₹9000. Find the relationship between the cost $(y)$ and the number of units produced $(x)$ if the relationship is linear.

3. Prove that the lines $4x + 3y = 10, 3x–4y = -5$ and $5x + y = 7$ are concurrent.

4. Find the value of $p$ for which the straight lines $8px + (2 - 3p)y + 1 = 0$ and $px + py - 7 = 0$ are perpendicular to each other.

5. If the slope of one of the straight lines $ax^2 + 2hxy + by^2 = 0$ is thrice that of the other, then show that $3h^2 = 4ab$.

6. Find the values of $a$ and $b$ if the equation $(a - 1)x^2 + by^2 + (b - 8)xy + 4x + 4y - 1 = 0$ represents a circle.

7. Find whether the points $(-1, -2), (1, 0)$ and $(-3, -4)$ lie above, below or on the line $3x + 2y + 7 = 0$.

8. If $(4, 1)$ is one extremity of a diameter of the circle $x^2 + y^2 - 2x + 6y - 15 = 0$, find the other extremity.

9. Find the equation of the parabola which is symmetrical about x-axis and passing through $(-2, -3)$.

10. Find the axis, vertex, focus, equation of directrix length of latus rectum of the parabola $(y - 2)^2 = 4(x - 1)$

# Summary

- Angle between the two intersecting lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$ is $\theta = \tan^{-1}\left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

- Condition for the three straight lines $a_1 x + b_1 y + c_1 = 0$, $a_2 x + b_2 y + c_2 = 0$, and $a_3 x + b_3 y + c_3 = 0$ to be concurrent is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

- The condition for a general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

- Pair of straight lines passing through the origin is $ax^2 + 2hxy + by^2 = 0$.

- Let $ax^2 + 2hxy + by^2 = 0$ be the straight lines passing through the origin then the product of the slopes is $\dfrac{a}{b}$ and sum of the slopes is $\dfrac{-2h}{b}$.

- If $\theta$ is the angle between the pair of straight lines

- $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then $\theta = \tan^{-1}\left|\left|\dfrac{\pm 2\sqrt{h^2 - ab}}{a + b}\right|\right|$.

- The equation of a circle with centre $(h, k)$ and the radius r is $(x - h)^2 + (y - k)^2 = r^2$

- The equation of a circle with centre at origin and the radius $r$ is $x^2 + y^2 = r^2$.

- The equation of the circle with $(x_1, y_1)$ and $(x_2, y_2)$ as end points of a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

- The parametric equations of the circle $x^2 + y^2 = r^2$ are $x = r\cos\theta$, $y = r\sin\theta$ $0 \le \theta \le 2\pi$

- The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at $(x_1, y_1)$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

- The equation of the tangent at $(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

- Length of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from a point $(x_1, y_1)$ is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.

- Condition for the straight line $y = mx + c$ to be a tangent to the circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1 + m^2)$.

- The standard equation of the parabola is $y^2 = 4ax$

| GLOSSARY | |
|---|---|
| Centre | மையம் |
| Chord | நாண் |
| Circle | வட்டம் |
| Concurrent Line | ஒரு புள்ளி வழிக் கோடு |
| Conics | கூம்பு வெட்டிகள் |
| Diameter | விட்டம் |
| Directix | இயக்குவரை |
| Equation | சமன்பாடு |
| Focal distance | குவியத்தூரம் |
| Focus | குவியம் |
| Latus rectum | செவ்வகலம் |
| Length of the tangent | தொடுகோட்டின் நீளம் |
| Locus | நியமப்பாதை அல்லது இயங்குவரை |
| Origin | ஆதி |
| Pair of straight line | இரட்டை நேர்கோடு |
| Parabola | பரவளையம் |
| Parallel line | இணை கோடு |
| Parameter | துணையலகு |
| Perpendicular line | செங்குத்து கோடு |
| Point of concurrency | ஒருங்கிணைவுப் புள்ளி |
| Point of intersection | வெட்டும் புள்ளி |
| Radius | ஆரம் |
| Straight line | நேர்கோடு |
| Tangent | தொடுகோடு |
| Vertex | முனை |

# TRIGONOMETRY

## Learning Objectives
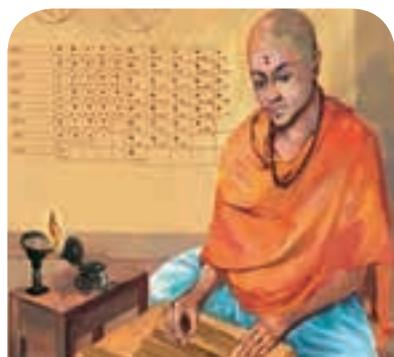
After studying this chapter, the students will be able to understand

- trigonometric ratio of angles
- addition formulae, multiple and sub-multiple angles
- transformation of sums into products and vice versa
- basic concepts of inverse trigonometric functions
- properties of inverse trigonometric functions

## Introduction

The word trigonometry is derived from the Greek word 'tri' (meaning three), 'gon' (meaning sides) and 'metron' (meaning measure). In fact, trigonometry is the study of relationships between the sides and angles of a triangle. Around second century A.D. George Rheticus was the first to define the trigonometric functions in terms of right angles. The study of trigonometry was first started in India. The ancient Indian Mathematician, Aryabhatta, Brahmagupta, Bhaskara I and Bhaskara II obtained important results. Bhaskara I gave formulae to find the values of sine functions for angles more than 90 degrees. The earliest applications of trigonometry were in the fields of navigation, surveying and astronomy.



**Brahmagupta**

**DO YOU KNOW?**

*Currently, trigonometry is used in many areas such as electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analyzing a musical tone.*

### Recall

1.  $\sin\theta = \dfrac{\textit{side opposite to angle } \theta}{\textit{Hypotenuse}}$

2.  $\cos\theta = \dfrac{\textit{Adjacent side to angle } \theta}{\textit{Hypotenuse}}$

3.  $\tan\theta = \dfrac{\textit{side opposite to angle } \theta}{\textit{Adjacent side to angle } \theta}$

4.  $\cot\theta = \dfrac{\textit{Adjacent side to angle } \theta}{\textit{side opposite to angel } \theta}$

5.  $\sec\theta = \dfrac{\textit{Hypotenuse}}{\textit{Adjacent side to angle } \theta}$

6.  $\csc\theta = \dfrac{\textit{Hypotenuse}}{\textit{side opposite to angle } \theta}$

### Relations between trigonometric ratios

1.  $\sin\theta = \dfrac{1}{\csc\theta}$      or      $\csc\theta = \dfrac{1}{\sin\theta}$

2.  $\cos\theta = \dfrac{1}{\sec\theta}$      or      $\sec\theta = \dfrac{1}{\cos\theta}$

3.  $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$      or      $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

4.  $\tan\theta = \dfrac{1}{\cot\theta}$      or      $\cot\theta = \dfrac{1}{\tan\theta}$

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### Trigonometric Identities

1.  $\sin^2\theta + \cos^2\theta = 1$

2.  $1 + \tan^2\theta = \sec^2\theta$

3.  $1 + \cot^2\theta = \csc^2\theta.$

---

### Angle

Angle is a measure of rotation of a given ray about its initial point. The ray 'OA' is called the initial side and the final position of the ray 'OB' after rotation is called the terminal side of the angle. The point 'O' of rotation is called the vertex.



Fig. 4.1

If the direction of rotation is anticlockwise, then angle is said to be positive and if the direction of rotation is clockwise, then angle is said to be negative.
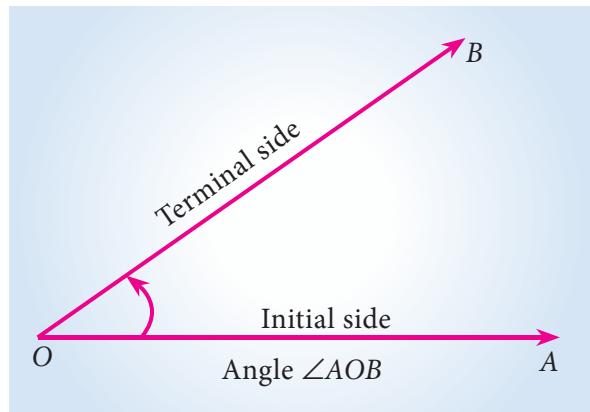
### Measurement of an angle

Two types of units of measurement of an angle which are most commonly used namely degree measure and radian measure.

## Degree measure

If a rotation from the initial position to the terminal position $\left(\frac{1}{360}\right)^{th}$ of the revolution, the angle is said to have a measure of one degree and written as $1^\circ$. A degree is divided into minutes and minute is divided into seconds.

One degree = 60 minutes $60'$
One minute = 60 seconds ($60''$)

## Radian measure

The angle subtended at the centre of the circle by an arc equal to the length of the radius of the circle is called a radian, and it is denoted by $1^c$
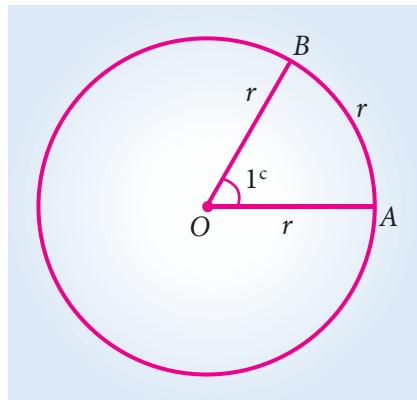


Fig. 4.2

**NOTE**

The number of radians in an angle subtended by an arc of a circle at the centre of a circle is

$$= \frac{length\ of\ the\ arc}{radius}$$

i.e., $\theta = \frac{s}{r}$, where $\theta$ is the angle subtended at the centre of a circle of radius r by an arc of length $s$.

## Relation between degrees and radians

We know that the circumference of a circle of radius 1unit is $2\pi$. One complete revolution of the radius of unit circle subtends $2\pi$ radians. A circle subtends at the centre an angle whose degree measure is $360^\circ$.

$2\pi$ radians = $360^\circ$

$\pi$ radians = $180^\circ$

1radian = $\frac{180^\circ}{\pi}$

## Example 4.1

Convert (i) $160^\circ$ into radians (ii) $\frac{4\pi}{5}$ radians into degree (iii) $\frac{1}{4}$ radians into degree

### Solution

*i)* $160° = 160 \times \dfrac{\pi}{180} \, radians = \dfrac{8}{9}\pi$

*ii)* $\dfrac{4\pi}{5} \, radians = \dfrac{4}{5}\pi \times \dfrac{180°}{\pi} = 144°$

*iii)* $\dfrac{1}{4} \, radian = \dfrac{1}{4}\dfrac{180}{\pi} = \dfrac{1}{4} \times 180 \times \dfrac{7}{22} = 14°19'5''$

## 4.1 Trigonometric ratios

### 4.1.1 Quadrants:

Let $X'OX$ and $Y'OY$ be two lines at right angles to each other as in the figure. We call $X'OX$ and $Y'OY$ as $X$ axis and $Y$ axis respectively. Clearly these axes divided the entire plane into four equal parts called "Quadrants" .

The parts $XOY$, $YOX'$, $X'OY'$ and $Y'OX$ are known as the first, second, third and the fourth quadrant respectively.
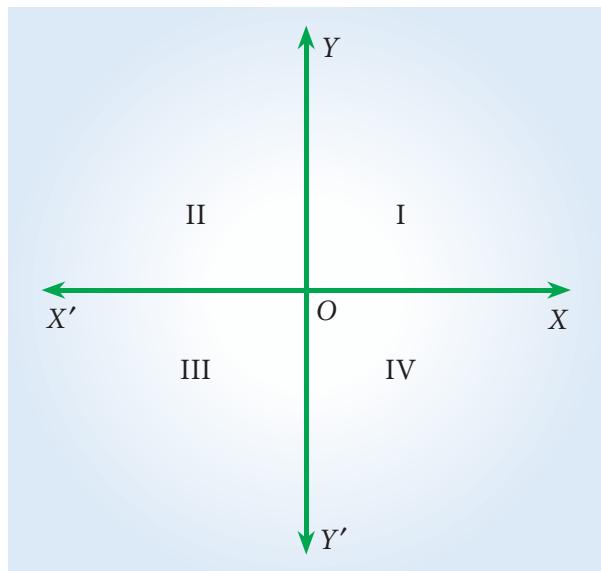


Fig. 4.3

### Example 4.2

Find the quadrants in which the terminal sides of the following angles lie.

(i) $-70°$  (ii) $-320°$   (iii) $1325°$

### Solution (i)

1.  (i)   The terminal side of $-70°$ lies in IV quadrant.
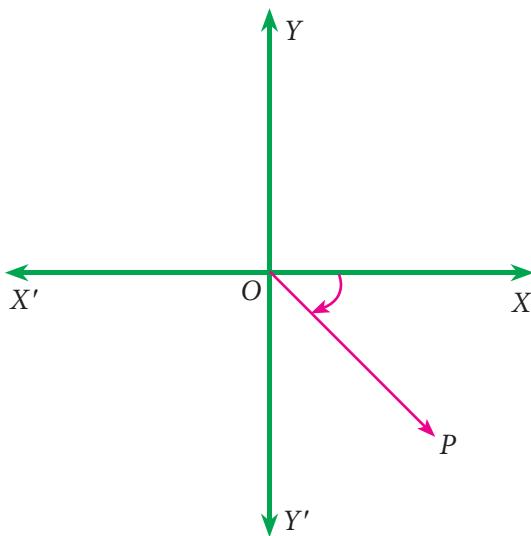


Fig. 4.4

(ii)   The terminal side $-320°$ of lies in I quadrant.
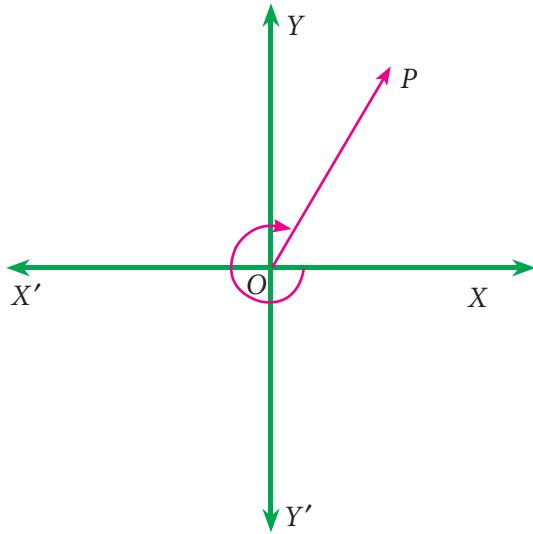


Fig. 4.5

(iii)   $1325° = (3 \times 360) + 180° + 65°$   the terminal side lies in III quadrant.
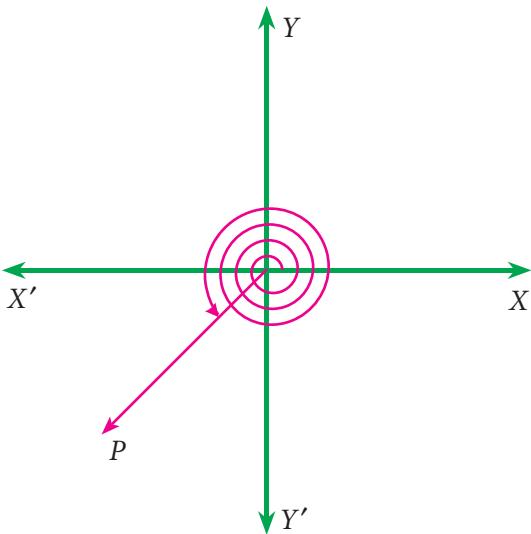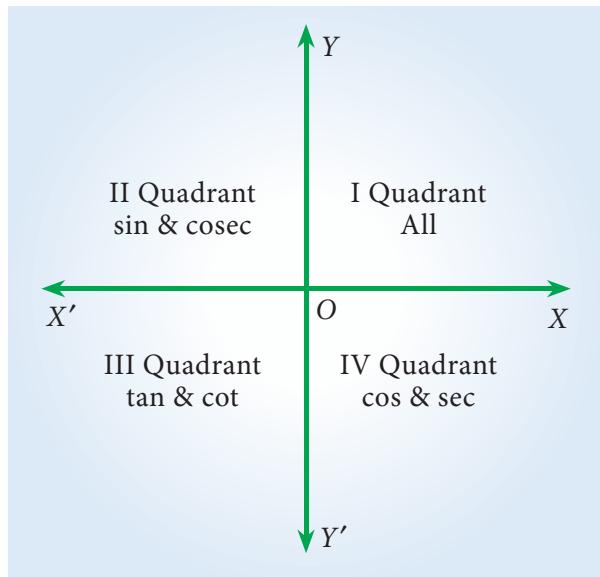


Fig. 4.6

## 4.1.2   Signs of the trigonometric ratios of an angle θ as it varies from 0° to 360°

In the first quadrant both $x$ and $y$ are positive. So all trigonometric ratios are positive. In the second quadrant $(90° < \theta < 180°)$ $x$ is negative and $y$ is positive. So trigonometric ratios sin θ and cosec θ are positive. In the third quadrant $(180° < \theta < 270°)$ both $x$ and $y$ are negative. So trigonometric ratios tan θ and cot θ are positive. In the fourth quadrant $(270° < \theta < 360°)$ $x$ is positive and $y$ is negative. So trigonometric ratios cos θ and sec θ are positive.

A function $f(x)$ is said to be odd function if $f(-x) = -f(x)$. sin θ, tan θ, cot θ and cosec θ are all odd function.  A function $f(x)$ is said to be even function if $f(-x) = f(x)$. cos θ and sec θ are even function.

ASTC : All Sin Tan Cos

Fig. 4.7

## 4.1.3   Trigonometric ratios of allied angles:

Two angles are said to be allied angles when their sum or difference is either zero or a multiple of 90°.  The angles $-\theta$, $90° \pm \theta$, $180° \pm \theta$, $360° \pm \theta$ etc .,are angles allied to the angle θ. Using trigonometric  ratios of the allied angles we can find the trigonometric ratios of any angle.

| Angle/ Function | $-\theta$ | $90° - \theta$ or $\frac{\pi}{2} - \theta$ | $90° + \theta$ or $\frac{\pi}{2} + \theta$ | $180° - \theta$ or $\pi - \theta$ | $180° + \theta$ or $\pi + \theta$ | $270° - \theta$ or $\frac{3\pi}{2} - \theta$ | $270° + \theta$ or $\frac{3\pi}{2} + \theta$ | $360° - \theta$ or $2\pi - \theta$ | $360° + \theta$ or $2\pi + \theta$ |
|---|---|---|---|---|---|---|---|---|---|
| Sine | $-\sin\theta$ | $\cos\theta$ | $\cos\theta$ | $\sin\theta$ | $-\sin\theta$ | $-\cos\theta$ | $-\cos\theta$ | $-\sin\theta$ | $\sin\theta$ |
| Cosine | $\cos\theta$ | $\sin\theta$ | $-\sin\theta$ | $-\cos\theta$ | $-\cos\theta$ | $-\sin\theta$ | $\sin\theta$ | $\cos\theta$ | $\cos\theta$ |
| tangent | $-\tan\theta$ | $\cot\theta$ | $-\cot\theta$ | $-\tan\theta$ | $\tan\theta$ | $\cot\theta$ | $-\cot\theta$ | $-\tan\theta$ | $\tan\theta$ |
| cotangent | $-\cot\theta$ | $\tan\theta$ | $-\tan\theta$ | $-\cot\theta$ | $\cot\theta$ | $\tan\theta$ | $-\tan\theta$ | $-\cot\theta$ | $\cot\theta$ |
| secant | $\sec\theta$ | $\csc\theta$ | $-\csc\theta$ | $-\sec\theta$ | $-\sec\theta$ | $-\csc\theta$ | $\csc\theta$ | $\sec\theta$ | $\sec\theta$ |
| cosecant | $-\csc\theta$ | $\sec\theta$ | $\sec\theta$ | $\csc\theta$ | $-\csc\theta$ | $-\sec\theta$ | $-\sec\theta$ | $-\csc\theta$ | $\csc\theta$ |

Table : 4.1

### Example 4.3

Find the values of each of the following trigonometric ratios.

(i)   $\sin 150°$   (ii) $\cos(-210°)$   (iii)   $\csc 390°$   (iv) $\tan(-1215°)$

(v) $\sec 1485°$

*Solution*

(i)    $\sin 150° = \sin(1 \times 90° + 60°)$

Since $150°$ lies in the second quadrant, we have

$\sin 150° = \sin(1 \times 90° + 60°)$

$= \cos 60° = \dfrac{1}{2}$

(ii)  we have $\cos(-210°) = \cos 210°$

Since $210°$ lies in the third quadrant, we have

$\cos 210° = \cos(180° + 30°) = -\cos 30° = -\dfrac{\sqrt{3}}{2}$

(iii)  $\operatorname{cosec} 390° = \operatorname{cosec}(360° + 30°) = \operatorname{cosec} 30° = 2$

(iv)  $\tan(-1215°) = -\tan(1215°) = -\tan(3 \times 360° + 135°)$

$= -\tan 135° = -\tan(90° + 45°) = -(-\cot 45°) = 1$

(v)   $\sec 1485° = \sec(4 \times 360° + 45°) = \sec 45° = \sqrt{2}$

## Example 4.4

Prove that $\sin 600° \cos 390° + \cos 480° \sin 150° = -1$

*Solution*

$\sin 600° = \sin(360° + 240°) = \sin 240° = \sin(180° + 60°) = -\sin 60° = -\dfrac{\sqrt{3}}{2}$

$\cos 390° = \cos(360° + 30°) = \cos 30° = \dfrac{\sqrt{3}}{2}$

$\cos 480° = \cos(360° + 120) = \cos 120°$

$= \cos(180° - 60°) = -\cos 60° = -\dfrac{1}{2}$

$\sin 150° = \sin(180° - 30°) = \sin 30° = \dfrac{1}{2}$

Now $\sin 600° \cos 390° + \cos 480° \sin 150°$

$= \left(-\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{\sqrt{3}}{2}\right) + \left(-\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right) = -\dfrac{3}{4} - \dfrac{1}{4} = \dfrac{-3-1}{4} = -1$

**Example 4.5**

Prove that $\dfrac{\sin(-\theta)\tan(90°-\theta)\sec(180°-\theta)}{\sin(180+\theta)\cot(360-\theta)\csc(90°-\theta)} = 1$

*Solution*

$$\text{L.H.S} = \frac{\sin(-\theta)\tan(90°-\theta)\sec(180°-\theta)}{\sin(180+\theta)\cot(360-\theta)\csc(90°-\theta)}$$

$$= \frac{(-\sin\theta)\cot\theta(-\sec\theta)}{(-\sin\theta)(-\cot\theta)\sec\theta} = 1$$

## Exercise 4.1

1. Convert the following degree measure into radian measure

   (i) 60°  (ii) 150°  (iii) 240°  (iv) –320°

2. Find the degree measure corresponding to the following radian measure.

   (i) $\dfrac{\pi}{8}$  (ii) $\dfrac{9\pi}{5}$  (iii) –3  (iv) $\dfrac{11\pi}{18}$

3. Determine the quadrants in which the following degree lie.

   (i) 380°  (ii) –140°  (iii) 1195°

4. Find the values of each of the following trigonometric ratios.

   (i) sin300°  (ii) cos(–210°)  (iii) sec390°  (iv) tan(–855°)

   (v) cosec1125°

5. Prove that: (i) $\tan(-225°)\cot(-405°) - \tan(-765°)\cot(675°) = 0$

   (ii) $2\sin^2\dfrac{\pi}{6} + \csc^2\dfrac{7\pi}{6}\cos^2\dfrac{\pi}{3} = \dfrac{3}{2}$

   (iii) $\sec\left(\dfrac{3\pi}{2} - \theta\right)\sec\left(\theta - \dfrac{5\pi}{2}\right) + \tan\left(\dfrac{5\pi}{2} + \theta\right)\tan\left(\theta - \dfrac{5\pi}{2}\right) = -1$

6. If *A, B, C, D* are angles of a cyclic quadrilateral, prove that :

   $\cos A + \cos B + \cos C + \cos D = 0$

7. Prove that : (i) $\dfrac{\sin(180°-\theta)\cos(90°+\theta)\tan(270-\theta)\cot(360-\theta)}{\sin(360-\theta)\cos(360+\theta)\sin(270-\theta)\csc(-\theta)} = -1$

   (ii) $\sin\theta \cdot \cos\theta\left\{\sin\left(\dfrac{\pi}{2}-\theta\right)\cdot\csc\theta + \cos\left(\dfrac{\pi}{2}-\theta\right)\cdot\sec\theta\right\} = 1$

8. Prove that : $\cos 510^\circ \cos 330^\circ + \sin 390^\circ \cos 120^\circ = -1$

9. Prove that: (i) $\tan(\pi + x)\cot(x - \pi) - \cos(2\pi - x)\cos(2\pi + x) = \sin^2 x$

   (ii) $\dfrac{\sin(180^\circ + A)\cos(90^\circ - A)\tan(270^\circ - A)}{\sin(540^\circ - A)\cos(360^\circ + A)\text{cosec}(270^\circ + A)} = -\sin A \cos^2 A$

10. If $\sin\theta = \dfrac{3}{5}$, $\tan\varphi = \dfrac{1}{2}$ and $\dfrac{\pi}{2} < \theta < \pi < \varphi < \dfrac{3\pi}{2}$ , then find the value of $8\tan\theta - \sqrt{5}\sec\varphi$

## 4.2 Trigonometric ratios of compound angles

### 4.2.1 Compound angles

When we add or subtract angles, the result is called a compound angle. i.e.,the algebraic sum of two or more angles are called compound angles

For example, If *A, B, C* are three angles then $A \pm B$, $A + B + C$, $A - B + C$ etc., are compound angles.

### 4.2.2 Sum and difference formulae of sine, cosine and tangent

(i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(iii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(iv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

**Example 4.6**

If $\cos A = \dfrac{4}{5}$ and $\cos B = \dfrac{12}{13}$, $\dfrac{3\pi}{2} < A, B < 2\pi$, find the value of

(i) $\sin(A-B)$    (ii) $\cos(A+B)$.

*Solution*

Since $\dfrac{3\pi}{2} < A, B < 2\pi$, both *A* and *B* lie in the fourth quadrant ,

$\therefore \sin A$ and $\sin B$ are negative.

Given $\cos A = \dfrac{4}{5}$ and $\cos B = \dfrac{12}{13}$,

Therefore, $\sin A = -\sqrt{1 - \cos^2 A}$ $\qquad$ Sin B $= -\sqrt{1 - \cos^2 B}$

$$= -\sqrt{1 - \frac{16}{25}} \qquad\qquad = -\sqrt{1 - \frac{144}{169}}$$

$$= -\sqrt{\frac{25 - 16}{25}} \qquad\qquad = -\sqrt{\frac{169 - 144}{169}}$$

$$= -\frac{3}{5} \qquad\qquad\qquad\quad = -\frac{5}{13}$$

(i) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$= \frac{4}{5} \times \frac{12}{13} - \left(\frac{-3}{5}\right) \times \left(\frac{-5}{13}\right)$$

$$= \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$$

(ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$$= \left(\frac{-3}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{-5}{13}\right)$$

$$= \frac{-36}{65} + \frac{20}{65} = \frac{-16}{65}$$

**Example 4.7**

Find the values of each of the following trigonometric ratios.

(i) sin15°   (ii) cos(–105°)   (iii) tan75°   (iv) –sec165°

*Solution*

(i) $\quad \sin 15° = \sin(45° - 30°)$

$$= \sin 45° \cos 30° - \cos 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

(ii) $\cos(-105°) = \cos 105°$

$$= \cos(60° + 45°)$$

$$= \cos 60° \cos 45° - \sin 60° \sin 45°$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} = \frac{1 - \sqrt{3}}{2\sqrt{2}}$$

(iii) $\quad \tan 75° = \tan(45° + 30°)$

$$= \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$$

$$= \frac{1 + \dfrac{1}{\sqrt{3}}}{1 - \dfrac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + \sqrt{3}$$

(iv) $\cos 165° = \cos(180° - 15°)$

$$= -\cos 15°$$

$$= -\cos(60° - 45°)$$

$$= -(\cos 60° \cos 45° + \sin 60° \sin 45°)$$

$$= -\left( \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} \right)$$

$$= -\left( \frac{1 + \sqrt{3}}{2\sqrt{2}} \right)$$

$$\therefore \quad -\sec 165° = \frac{2\sqrt{2}}{1 + \sqrt{3}}$$

$$= \frac{2\sqrt{2}}{1 + \sqrt{3}} \times \frac{1 - \sqrt{3}}{1 - \sqrt{3}}$$

$$= \sqrt{2}(\sqrt{3} - 1)$$

**Example 4.8**

If $\tan A = m \tan B,$ prove that $\dfrac{\sin(A + B)}{\sin(A - B)} = \dfrac{m + 1}{m - 1}$

***Solution***

Given $\tan A = m \tan B$

$$\frac{\sin A}{\cos A} = m \frac{\sin B}{\cos B}$$

$$\frac{\sin A \cos B}{\cos A \sin B} = m$$

Applying componendo and dividendo rule, we get

$$\frac{\sin A \cos B + \cos A \sin B}{\sin A \cos B - \cos A \sin B} = \frac{m + 1}{m - 1}$$

$$\frac{\sin(A + B)}{\sin(A - B)} = \frac{m + 1}{m - 1}, \quad \text{which completes the proof.}$$

### 4.2.3 Trigonometric ratios of multiple angles

Identities involving sin2A, cos2A, tan2A, sin3A etc., are called multiple angle identities.

(i) $\sin 2A = \sin(A+A) = \sin A \cos A + \cos A \sin A = 2 \sin A \cos A.$

(ii) $\cos 2A = \cos(A+A) = \cos A \cos A - \sin A \sin A = \cos^2 A - \sin^2 A$

(iii) $\sin 3A = \sin(2A+A)$

$$= \sin 2A \cos A + \cos 2A \sin A$$

$$= (2 \sin A \cos A) \cos A + (1 - 2 \sin^2 A) \sin A$$

$$= 2 \sin A \cos^2 A + \sin A - 2 \sin^3 A$$

$$= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A$$

$$= 3 \sin A - 4 \sin^3 A$$

Thus we have the following multiple angles formulae

1.  (i)   $\sin 2A = 2 \sin A \cos A$

    (ii)   $\sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$

2.  (i)   $\cos 2A = \cos^2 A - \sin^2 A$

    (ii)   $\cos 2A = 2 \cos^2 A - 1$

    (iii)   $\cos 2A = 1 - 2 \sin^2 A$

    (iv)   $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

3.   $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

4.   $\sin 3A = 3 \sin A - 4 \sin^3 A$

5.   $\cos 3A = 4 \cos^3 A - 3 \cos A$

6.   $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

### NOTE

(i)   $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$   (or)   $\cos A = \pm \sqrt{\dfrac{1 + \cos 2A}{2}}$

(ii)   $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$   (or)   $\sin A = \pm \sqrt{\dfrac{1 - \cos 2A}{2}}$

**Example 4.9**

Show that $\dfrac{\sin 2\theta}{1 + \cos 2\theta} = \tan\theta$

*Solution*

$$\frac{\sin 2\theta}{1 + \cos 2\theta} = \frac{2\sin\theta\cos\theta}{2\cos^2\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

**Example 4.10**

Using multiple angle identity, find $\tan 60^\circ$

*Solution*

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Put $A = 30^\circ$ in the above identity, we get

$$\tan 60^\circ = \frac{2\tan 30^\circ}{1 - \tan^2 30^\circ}$$

$$= \frac{2\dfrac{1}{\sqrt{3}}}{1 - \dfrac{1}{3}} \qquad = \frac{\dfrac{2}{\sqrt{3}}}{\dfrac{2}{3}} = \frac{2}{\sqrt{3}} \times \frac{3}{2}$$

$$= \sqrt{3}$$

**Example 4.11**

If $\tan A = \dfrac{1}{7}$ and $\tan B = \dfrac{1}{3}$, show that $\cos 2A = \sin 4B$

*Solution*

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{1 - \dfrac{1}{49}}{1 + \dfrac{1}{49}} = \frac{48}{49} \times \frac{49}{50} = \frac{24}{25} \quad \cdots (1)$$

Now $\quad \sin 4B = 2\sin 2B \cos 2B$

$$= 2\frac{2\tan B}{1 + \tan^2 B} \times \frac{1 - \tan^2 B}{1 + \tan^2 B} = \frac{24}{25} \qquad \cdots (2)$$

From (1) and (2) we get,

$$\cos 2A = \sin 4B.$$

**Example 4.12**

If $\tan A = \dfrac{1 - \cos B}{\sin B}$ then prove that $\tan 2A = \tan B$

*Solution*

Consider $\dfrac{1-\cos B}{\sin B} = \dfrac{2\sin^2 \dfrac{B}{2}}{2\sin\dfrac{B}{2}\cos\dfrac{B}{2}} = \dfrac{\sin\dfrac{B}{2}}{\cos\dfrac{B}{2}} = \tan\dfrac{B}{2}$

$$\tan A = \dfrac{1-\cos B}{\sin B}$$

$$= \tan\dfrac{B}{2}$$

$$A = \dfrac{B}{2}$$

$$2A = B$$

$\therefore$    $\tan 2A = \tan B$

## Example 4.13

If $\tan\alpha = \dfrac{1}{3}$ and $\tan\beta = \dfrac{1}{7}$ then prove that $(2\alpha + \beta) = \dfrac{\pi}{4}$.

*Solution*

$$\tan(2\alpha + \beta) = \dfrac{\tan 2\alpha + \tan\beta}{1 - \tan 2\alpha \cdot \tan\beta}$$

$$\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha} = \dfrac{2\times\dfrac{1}{3}}{1-\dfrac{1}{9}} = \dfrac{3}{4}$$

$$\tan(2\alpha + \beta) = \dfrac{\dfrac{3}{4}+\dfrac{1}{7}}{1-\dfrac{3}{4}\times\dfrac{1}{7}} = \dfrac{\dfrac{25}{28}}{\dfrac{25}{28}}$$

$$= 1 = \tan\dfrac{\pi}{4}$$

$$2\alpha + \beta = \dfrac{\pi}{4}$$

### Exercise 4.2

1. Find the values of the following : (i) cosec15° (ii) sin(-105°) (iii) cot75°

2. Find the values of the following

   (i) sin76° cos16° – cos76° sin16°
   
   (ii) $\sin\dfrac{\pi}{4}\cos\dfrac{\pi}{12} + \cos\dfrac{\pi}{4}\sin\dfrac{\pi}{12}$

   (iii) cos70° cos10° – sin70° sin10°
   
   (iv) $\cos^2 15° - \sin^2 15°$

3. If $\sin A = \dfrac{3}{5}, 0 < A < \dfrac{\pi}{2}$ and $\cos B = \dfrac{-12}{13}, \pi < A < \dfrac{3\pi}{2}$ find the values of the following :

   (i) cos(A+B)          (ii) sin(A−B)          (iii) tan(A−B)

4. If $\cos A = \dfrac{13}{4}$ and $\cos B = \dfrac{1}{7}$ where $A$, $B$ are acute angles prove that $A-B = \dfrac{\pi}{3}$

5. Prove that $2\tan 80° = \tan 85° - \tan 5°$

6. If $\cot \alpha = \dfrac{1}{2}$ , $\sec \beta = \dfrac{-5}{3}$ , where $\pi < \alpha < \dfrac{3\pi}{2}$ and $\dfrac{\pi}{2} < \beta < \pi$, find the value of $\tan(\alpha + \beta)$.  State the quadrant in which $\alpha + \beta$ terminates.

7. If $A+B = 45°$, prove that $(1+\tan A)(1+\tan B) = 2$ and hence deduce the value of $\tan 22\dfrac{1}{2}^{\circ}$

8. Prove that (i) $\sin(A + 60°) + \sin(A - 60°) = \sin A$

   (ii) $\tan 4A \tan 3A \tan A + \tan 3A + \tan A - \tan 4A = 0$

9. (i) If $\tan\theta = 3$ find $\tan 3\theta$

   (ii) If $\sin A = \dfrac{12}{13}$, find $\sin 3A$

10. If $\sin A = \dfrac{3}{5}$, find the values of $\cos 3A$ and $\tan 3A$

11. Prove that $\dfrac{\sin(B-C)}{\cos B \cos C} + \dfrac{\sin(C-A)}{\cos C \cos A} + \dfrac{\sin(A-B)}{\cos A \cos B} = 0.$

12. If $\tan A - \tan B = x$ and $\cot B - \cot A = y$ prove that $\cot(A - B) = \dfrac{1}{x} + \dfrac{1}{y}.$

13. If $\sin \alpha + \sin \beta = a$ and $\cos \alpha + \cos \beta = b$, then prove that $\cos(\alpha-\beta) = \dfrac{a^2 + b^2 - 2}{2}.$

14. Find the value of $\tan \dfrac{\pi}{8}$ .

15. If $\tan \alpha = \dfrac{1}{7}$, $\sin \beta = \dfrac{1}{\sqrt{10}}$  Prove that  $\alpha + 2\beta = \dfrac{\pi}{4}$ where $0 < \alpha < \dfrac{\pi}{2}$ and $0 < \beta < \dfrac{\pi}{2}$.

## 4.3 Transformation formulae

The two sets of transformation formulae are described below.

### 4.3.1 Transformation of the products into sum or difference

$\sin A \cos B + \cos A \sin B = \sin(A + B)$    ... (1)

$$\sin A \cos B - \cos A \sin B = \sin(A - B) \quad \dots (2)$$

$$\cos A \cos B - \sin A \sin B = \cos(A + B) \quad \dots (3)$$

$$\cos A \cos B + \sin A \sin B = \cos(A - B) \quad \dots (4)$$

Adding (1) and (2), we get

$$2\sin A \cos B = \sin(A + B) + \sin(A - B)$$

Subtracting (1) from (2), we get

$$2\cos A \sin B = \sin(A + B) - \sin(A - B)$$

Adding (3) and (4), we get

$$2\cos A \cos B = \cos(A + B) + \cos(A - B)$$

Subtracting (3) from (4), we get

$$2\sin A \sin B = \cos(A - B) - \cos(A + B)$$

Thus, we have the following formulae;

$$2\sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$2\cos A \sin B = \sin(A + B) - \sin(A - B)$$

$$2\cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$2\sin A \sin B = \cos(A - B) - \cos(A + B)$$

Also $\quad \sin^2 A - \sin^2 B = \sin(A + B)\sin(A - B)$

$$\cos^2 A - \sin^2 B = \cos(A + B)\cos(A - B)$$

### 4.3.2 Transformation of sum or difference into product

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

**Example 4.14**

Express the following as sum or difference

(i) $2 \sin 2\theta \cos \theta$        (ii) $2 \cos 3\theta \cos \theta$        (iii) $2 \sin 4\theta \sin 2\theta$

(iv) $\cos 7\theta \ \cos 5\theta$        (v) $\cos \dfrac{3A}{2} \cos \dfrac{5A}{2}$        (vi) $\cos 9\theta \sin 6\theta$

(vii) $2 \cos 13A \sin 15A$

**Solution**

(i)      $2 \sin 2\theta \cos \theta = \sin(2\theta + \theta) + \sin(2\theta - \theta)$

                        $= \sin 3\theta + \sin \theta$

(ii)      $2 \cos 3\theta \cos \theta = \cos(3\theta + \theta) + \cos(3\theta - \theta)$

                        $= \cos 4\theta + \cos 2\theta$

(iii)      $2 \sin 4\theta \sin 2\theta = \cos(4\theta - 2\theta) - \cos(4\theta + 2\theta)$

                        $= \cos 2\theta - \cos 6\theta$

(iv)      $\cos 7\theta \ \cos 5\theta = \dfrac{1}{2}\left[\cos(7\theta + 5\theta) + \cos(7\theta - 5\theta)\right]$

                        $= \dfrac{1}{2}\left[\cos 12\theta + \cos 2\theta\right]$

(v)      $\cos \dfrac{3A}{2} \cos \dfrac{5A}{2} = \dfrac{1}{2}\cos\left(\dfrac{3A}{2} + \dfrac{5A}{2}\right) + \cos\left(\dfrac{3A}{2} - \dfrac{5A}{2}\right)$

                        $= \dfrac{1}{2}\left[\cos \dfrac{8A}{2} + \cos\left(\dfrac{-2A}{2}\right)\right]$

                        $= \dfrac{1}{2}\left[\cos 4A + \cos(-A)\right]$

                        $= \dfrac{1}{2}\left[\cos 4A + \cos A\right]$

(vi)      $\cos 9\theta \sin 6\theta = \dfrac{1}{2}\left[\sin(9\theta + 6\theta) - \sin(9\theta - 6\theta)\right]$

                        $= \dfrac{1}{2}\left[\sin(15\theta) - \sin 3\theta\right]$

(vii)      $2 \cos 13A \sin 15A = \sin(13A + 15A) - \sin(13A - 15A)$

                        $= \sin 28A + \sin 2A$

**Example 4.15**

Show that $\sin 20° \sin 40° \sin 80° = \dfrac{\sqrt{3}}{8}$

*Solution*

$$\text{LHS} = \sin 20° \sin 40° \sin 80°$$

$$= \sin 20° \sin(60° - 20°)\sin(60° + 20°)$$

$$= \sin 20°[\sin^2 60° - \sin^2 20°]$$

$$= \sin 20°\left[\frac{3}{4} - \sin^2 20°\right]$$

$$= \sin 20°\left[\frac{3 - 4\sin^2 20°}{4}\right]$$

$$= \frac{3\sin 20° - 4\sin^3 20°}{4}$$

$$= \frac{\sin 60°}{4} = \frac{\sqrt{3}/2}{4} = \frac{\sqrt{3}}{8}.$$

## Example 4.16

Show that $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

*Solution*

$$\text{L.H.S} = \sin 20° \sin 40° \sin 60° \sin 80°$$

$$= \sin 60° \sin 20° \sin(60° - 20°)\sin(60° + 20°)$$

$$= \frac{\sqrt{3}}{2}\sin 20°(\sin^2 60° - \sin^2 20°)$$

$$= \frac{\sqrt{3}}{2}\sin 20°\left(\frac{3}{4} - \sin^2 20°\right)$$

$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{4}\right)(3\sin 20° - 4\sin^3 20°)$$

$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{4}\right)\sin 60° = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{16} = \text{R.H.S.}$$

## Example 4.17

Prove that $\cos^2 A + \cos^2(A + 120°) + \cos^2(A - 120°) = \frac{3}{2}$

*Solution*

$$\cos^2 A + \cos^2(A + 120°) + \cos^2(A - 120°)$$

$$= \cos^2 A + \cos^2(A + 120°) + (1 - \sin^2(A - 120°)) \quad (\text{Since } \cos^2 A = 1 - \sin^2 A)$$

$$= \cos^2 A + 1 + (\cos^2(A + 120°)) - \sin^2(A - 120°)$$

$$= \cos^2 A + 1 + \cos[(A + 120°) + (A - 120°)]\cos[(A + 120°) - (A - 120°)]$$

$$= \cos^2 A + 1 + \cos 2A \cos 240° \qquad (\text{Since } \cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)\ )$$

$$= \cos^2 A + 1 + (2\cos^2 A - 1)\cos(180° + 60°) \qquad (\text{since } 2\cos^2 A = 1 + \cos 2A)$$

$$= \cos^2 A + 1 + (2\cos^2 A - 1)(-\cos 60°)$$

$$= \cos^2 A + 1 + (2\cos^2 A - 1)\left(-\frac{1}{2}\right)$$

$$= \cos^2 A + 1 - \cos^2 A + \frac{1}{2} = \frac{3}{2}$$

## Example 4.18

Convert the following into the product of trigonometric functions.

(i)  $\sin 9A + \sin 7A$        (ii)  $\sin 7\theta - \sin 4\theta$        (iii)  $\cos 8A + \cos 12A$

(iv)  $\cos 4\alpha - \cos 8\alpha$        (v)  $\cos 20° - \cos 30°$        (vi)  $\cos 75° + \cos 45°$

(vii)  $\cos 55° + \sin 55°$

## Solution

(i)      $\sin 9A + \sin 7A$
$$= 2\sin\left(\frac{9A+7A}{2}\right)\cos\left(\frac{9A-7A}{2}\right)$$
$$= 2\sin\left(\frac{16A}{2}\right)\cos\left(\frac{2A}{2}\right)$$
$$= 2\sin 8A \cos A$$

(ii)      $\sin 7\theta - \sin 4\theta$
$$= 2\cos\left(\frac{7\theta+4\theta}{2}\right)\sin\left(\frac{7\theta-4\theta}{2}\right)$$
$$= 2\cos\left(\frac{11\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)$$

(iii)      $\cos 8A + \cos 12A$
$$= 2\cos\left(\frac{8A+12A}{2}\right)\cos\left(\frac{8A-12A}{2}\right)$$
$$= 2\cos\left(\frac{20A}{2}\right)\cos\left(\frac{-4A}{2}\right)$$
$$= 2\cos 10A \cos(-2A)$$
$$= 2\cos 10A \cos 2A$$

(iv)      $\cos 4\alpha - \cos 8\alpha$
$$= -2\sin\left(\frac{4\alpha+8\alpha}{2}\right)\sin\left(\frac{4\alpha-8\alpha}{2}\right)$$
$$= 2\sin\left(\frac{12\alpha}{2}\right)\sin\left(\frac{4\alpha}{2}\right)$$
$$= 2\sin 6\alpha \sin 2\alpha$$

(v)      $\cos 20° - \cos 30°$
$$= -2\sin\left(\frac{20°+30°}{2}\right)\sin\left(\frac{20°-30°}{2}\right)$$
$$= 2\sin\left(\frac{50°}{2}\right)\sin\left(\frac{10°}{2}\right)$$
$$= 2\sin 25° \sin 5°$$

(vi) $\quad \cos 75° + \cos 45° = 2\cos\left(\dfrac{75° + 45°}{2}\right)\cos\left(\dfrac{75° - 45°}{2}\right)$

$$= 2\cos\left(\dfrac{120°}{2}\right)\cos\left(\dfrac{30°}{2}\right)$$

$$= 2\cos 60° \cos 15°$$

(vii) $\quad \cos 55° + \sin 55° = \cos 55° + \cos(90° - 55°)$

$$= \cos 55° + \cos 35°$$

$$= 2\cos\left(\dfrac{55° + 35°}{2}\right)\cos\left(\dfrac{55° - 35°}{2}\right)$$

$$= 2\cos 45° \cos 10° = 2\left(\dfrac{1}{\sqrt{2}}\right) \cdot \cos 10°$$

$$= \sqrt{2}\cos 10°$$

### Example 4.19

If three angles $A$, $B$ and $C$ are in arithmetic progression, Prove that

$$\cot B = \dfrac{\sin A - \sin C}{\cos C - \cos A}$$

### Solution

$$\dfrac{\sin A - \sin C}{\cos C - \cos A} = \dfrac{2\sin\left(\dfrac{A - C}{2}\right)\cos\left(\dfrac{A + C}{2}\right)}{2\sin\left(\dfrac{A + C}{2}\right)\sin\left(\dfrac{A - C}{2}\right)}$$

$$= \dfrac{\cos\left(\dfrac{A + C}{2}\right)}{\sin\left(\dfrac{A + C}{2}\right)} = \cot\left(\dfrac{A + C}{2}\right)$$

$$= \cot B \quad \text{(since } A, B, C \text{ are in A.P., } B = \dfrac{A + C}{2}\text{)}$$

### Example 4.20

Prove that $\dfrac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \tan x$

### Solution

$$\dfrac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \dfrac{\sin 5x + \sin x - 2\sin 3x}{\cos 5x - \cos x}$$

$$= \dfrac{2\sin 3x \cos 2x - 2\sin 3x}{\cos 5x - \cos x}$$

$$= \dfrac{2\sin 3x (\cos 2x - 1)}{-2\sin 3x \sin 2x} = \dfrac{1 - \cos 2x}{\sin 2x}$$

$$= \dfrac{2\sin^2 x}{2\sin x \cos x} = \tan x$$

**Example 4.21**

Find $\sin 105° + \cos 105°$

*Solution*

$\sin 105° + \cos 105°$

$$= \sin(90° + 15°) + \cos 105°$$

$$= \cos 15° + \cos 105°$$

$$= \cos 105° + \cos 15° \ = \ 2\cos\left(\frac{105° + 15°}{2}\right)\cos\left(\frac{105° - 15°}{2}\right)$$

$$= 2\cos\left(\frac{120°}{2}\right)\cos\left(\frac{90°}{2}\right)$$

$$= 2\cos 60° \ \cos 45° \ = \ \frac{1}{\sqrt{2}}$$

**Example 4.22**

Prove that $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\left(\frac{\alpha - \beta}{2}\right)$

*Solution*

$$\cos\alpha + \cos\beta \ = \ 2\cos\left(\frac{\alpha + \beta}{2}\right)\cdot\cos\left(\frac{\alpha - \beta}{2}\right) \qquad \text{..........(1)}$$

$$\sin\alpha + \sin\beta \ = \ 2\sin\left(\frac{\alpha + \beta}{2}\right)\cdot\cos\left(\frac{\alpha - \beta}{2}\right) \qquad \text{..........(2)}$$

Squaring and adding (1) and (2)

$(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$

$$= 4\cos^2\left(\frac{\alpha + \beta}{2}\right)\cos^2\left(\frac{\alpha - \beta}{2}\right) + \ 4\sin^2\left(\frac{\alpha + \beta}{2}\right)\cos^2\left(\frac{\alpha - \beta}{2}\right)$$

$$= 4\cos^2\left(\frac{\alpha - \beta}{2}\right)\left[\cos^2\left(\frac{\alpha + \beta}{2}\right) + \sin^2\left(\frac{\alpha + \beta}{2}\right)\right]$$

$$= 4\cos^2\left(\frac{\alpha - \beta}{2}\right)$$

**Exercise 4.3**

1.  Express each of the following as the sum or difference of sine or cosine:

   (i)  $\sin\dfrac{A}{8}\sin\dfrac{3A}{8}$

   (ii)  $\cos(60° + A)\sin(120° + A)$

   (iii)  $\cos\dfrac{7A}{3}\sin\dfrac{5A}{3}$

   (iv)  $\cos 7\theta \sin 3\theta$

2. Express each of the following as the product of sine and cosine

    (i) $\sin A + \sin 2A$                   (ii) $\cos 2A + \cos 4A$

    (iii) $\sin 6\theta - \sin 2\theta$               (iv) $\cos 2\theta - \cos \theta$

3. Prove that

    (i) $\cos 20° \cos 40° \cos 80° = \dfrac{1}{8}$     (ii) $\tan 20° \tan 40° \tan 80° = \sqrt{3}$

4. Prove that

    (i) $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\left(\dfrac{\alpha - \beta}{2}\right)$

    (ii) $\sin A \sin(60° + A)\sin(60° - A) = \dfrac{1}{4}\sin 3A$

5. Prove that

    (i) $\sin(A - B)\sin C + \sin(B - C)\sin A + \sin(C - A)\sin B = 0$

    (ii) $2\cos\dfrac{\pi}{13}\cos\dfrac{9\pi}{13} + \cos\dfrac{3\pi}{13} + \cos\dfrac{5\pi}{13} = 0$

6. Prove that

    (i) $\dfrac{\cos 2A - \cos 3A}{\sin 2A - \sin 3A} = \tan\dfrac{A}{2}$

    (ii) $\dfrac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \cot A$

7. Prove that $\cos 20° \cos 40° \cos 60° \cos 80° = \dfrac{1}{16}$

8. Evaluate

    (i) $\cos 20° + \cos 100° + \cos 140°$

    (ii) $\sin 50° - \sin 70° + \sin 10°$

9. If $\cos A + \cos B = \dfrac{1}{2}$ and $\sin A + \sin B = \dfrac{1}{4}$, prove that $\tan\left(\dfrac{A + B}{2}\right) = \dfrac{1}{2}$

10. If $\sin(y + z - x), \sin(z + x - y), \sin(x + y - z)$    are   in   A.P,    then   prove   that $\tan x$, $\tan y$ and $\tan z$ are in A.P

11. If $\operatorname{cosec} A + \sec A = \operatorname{cosec} B + \sec B$ prove that $\cot\left(\dfrac{A + B}{2}\right) = \tan A \tan B$

## 4.4 Inverse Trigonometric Functions

### 4.4.1 Inverse Trigonometric Functions

The quantities such as $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc., are known as inverse trigonometric functions.

i.e., if $\sin\theta = x$, then $\theta = \sin^{-1}x$. The domains and ranges (Principal value branches) of trigonometric functions are given below

| Function | Domain $x$ | Range $y$ |
|---|---|---|
| $\sin^{-1}x$ | $[-1, 1]$ | $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$ |
| $\cos^{-1}x$ | $[-1, 1]$ | $[0, \pi]$ |
| $\tan^{-1}x$ | $R = (-\infty, \infty)$ | $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ |
| $\csc^{-1}x$ | $R-(-1, 1)$ | $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right], y \neq 0$ |
| $\sec^{-1}x$ | $R-(-1, 1)$ | $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right], y \neq \dfrac{\pi}{2}$ |
| $\cot^{-1}x$ | $R$ | $(0, \pi)$ |

### 4.4.2 Properties of Inverse Trigonometric Functions

**Property (1)**

(i) $\sin^{-1}(\sin x) = x$         (ii) $\cos^{-1}(\cos x) = x$

(iii) $\tan^{-1}(\tan x) = x$        (iv) $\cot^{-1}(\cot x) = x$

(v) $\sec^{-1}(\sec x) = x$        (vi) $\csc^{-1}(\csc x) = x$

**Property (2)**

(i) $\sin^{-1}\left(\dfrac{1}{x}\right) = \csc^{-1}(x)$        (ii) $\cos^{-1}\left(\dfrac{1}{x}\right) = \sec^{-1}(x)$

(iii) $\tan^{-1}\left(\dfrac{1}{x}\right) = \cot^{-1}(x)$        (iv) $\csc^{-1}\left(\dfrac{1}{x}\right) = \sin^{-1}(x)$

(v) $\sec^{-1}\left(\dfrac{1}{x}\right) = \cos^{-1}(x)$        (vi) $\cot^{-1}\left(\dfrac{1}{x}\right) = \tan^{-1}(x)$

## Property (3)

(i) $\sin^{-1}(-x) = -\sin^{-1}(x)$  (ii) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

(iii) $\tan^{-1}(-x) = -\tan^{-1}(x)$  (iv) $\cot^{-1}(-x) = -\cot^{-1}(x)$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$  (vi) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}(x)$

## Property (4)

(i) $\sin^{-1}(x) + \cos^{-1}(x) = \dfrac{\pi}{2}$  (ii) $\tan^{-1}(x) + \cot^{-1}(x) = \dfrac{\pi}{2}$

(iii) $\sec^{-1}(x) + \operatorname{cosec}^{-1}(x) = \dfrac{\pi}{2}$

## Property (5)

If $xy < 1$, then

(i) $\tan^{-1}(x) + \tan^{-1}(y) = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

(ii) $\tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

## Property (6)

$$\sin^{-1}(x) + \sin^{-1}(y) = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

### Example 4.23

Find the principal value of (i) $\sin^{-1}(1/2)$ (ii) $\tan^{-1}(-\sqrt{3})$

*Solution*

(i)  Let  $\sin^{-1}(1/2) = y$  where $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$

$\therefore$  $\sin y = (1/2) = \sin\left(\dfrac{\pi}{6}\right)$

$y = \dfrac{\pi}{6}$

The principal value of $\sin^{-1}(1/2)$ is $\dfrac{\pi}{6}$

(ii)  Let $\tan^{-1}(-\sqrt{3}) = y$  where $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$

$\therefore$  $\tan y = -\sqrt{3} = \tan\left(-\dfrac{\pi}{3}\right)$

$y = -\dfrac{\pi}{3}$

The principal value of $\tan^{-1}(-\sqrt{3})$ is $-\dfrac{\pi}{3}$

**Example 4.24**

Evaluate the following  (i) $\cos\left(\sin^{-1}\dfrac{5}{13}\right)$  (ii) $\tan\left(\cos^{-1}\dfrac{8}{17}\right)$

*Solution*

(i) Let $\left(\sin^{-1}\dfrac{5}{13}\right) = \theta$       ... (1)

$$\sin\theta = \dfrac{5}{13}$$

$$\cos\theta = \sqrt{1-\sin^2\theta}$$

$$= \sqrt{1-\dfrac{25}{169}}$$

$$= \dfrac{12}{13} \quad\quad ... (2)$$

From (1) and (2), we get

Now $\cos\left(\sin^{-1}\dfrac{5}{13}\right) = \cos\theta = \dfrac{12}{13}$

(ii)Let $\left(\cos^{-1}\dfrac{8}{17}\right) = \theta$

$$\cos\theta = \dfrac{8}{17}$$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-\dfrac{64}{289}}$$

$$= \dfrac{15}{17} \quad\quad ... (2)$$

From (1) and (2), we get

Now $\tan\left(\cos^{-1}\dfrac{8}{17}\right) = \tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$= \dfrac{\frac{15}{17}}{\frac{8}{17}} = \dfrac{15}{8}$$

**Example 4.25**

Prove that  (i) $\tan^{-1}\left(\dfrac{1}{7}\right) + \tan^{-1}\left(\dfrac{1}{13}\right) = \tan^{-1}\left(\dfrac{2}{9}\right)$

(ii) $\cos^{-1}\left(\dfrac{4}{5}\right) + \tan^{-1}\left(\dfrac{3}{5}\right) = \tan^{-1}\left(\dfrac{27}{11}\right)$

*Solution*

(i) $\tan^{-1}\left(\dfrac{1}{7}\right) + \tan^{-1}\left(\dfrac{1}{13}\right) = \tan^{-1}\left[\dfrac{\frac{1}{7}+\frac{1}{13}}{1-\left(\frac{1}{7}\right)\left(\frac{1}{13}\right)}\right]$

$$= \tan^{-1}\left(\frac{20}{90}\right)$$

$$= \tan^{-1}\left(\frac{2}{9}\right)$$

(ii) Let $\cos^{-1}\left(\frac{4}{5}\right) = \theta$. Then $\cos\theta = \frac{4}{5} \Rightarrow \sin\theta = \frac{3}{5}$

$$\therefore \quad \tan\theta = \frac{3}{4} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\therefore \cos^{-1}\left(\frac{4}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$$

Now $\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{3}{5}\right)$

$$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{4} + \frac{3}{5}}{1 - \frac{3}{4} \times \frac{3}{5}}\right)$$

$$= \tan^{-1}\left(\frac{27}{11}\right)$$

**Example 4.26**

Find the value of $\tan\left[\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{8}\right)\right]$

*Solution*

$$\text{Let} \qquad \tan^{-1}\left(\frac{1}{8}\right) = \theta$$

$$\text{Then} \quad \tan\theta = \frac{1}{8}$$

$$\therefore \tan\left[\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{8}\right)\right] = \tan\left(\frac{\pi}{4} - \theta\right)$$

$$= \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}$$

$$= \frac{1 - \frac{1}{8}}{1 + \frac{1}{8}} = \frac{7}{9}$$

**Example 4.27**

Solve $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

*Solution*

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \tan^{-1}\left[\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right] = \tan^{-1}\left(\frac{2x^2 - 4}{-3}\right)$$

Given that

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

$$\therefore \qquad \tan^{-1}\left(\frac{2x^2-4}{-3}\right) = \frac{\pi}{4}$$

$$\frac{2x^2-4}{-3} = \tan\frac{\pi}{4} = 1$$

$$2x^2 - 4 = -3$$

$$\Rightarrow \qquad 2x^2 - 1 = 0$$

$$\Rightarrow \qquad x = \pm\frac{1}{\sqrt{2}}$$

## Example 4.28

Simplify: $\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2}{3}\right)$

### *Solution*

We know that

$$\sin^{-1}(x) + \sin^{-1}(y) = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

$$\therefore \qquad \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2}{3}\right) = \sin^{-1}\left[\frac{1}{3}\sqrt{1-\frac{4}{9}} + \frac{2}{3}\sqrt{1-\frac{1}{9}}\right]$$

$$= \sin^{-1}\left[\frac{1}{3}\sqrt{\frac{5}{9}} + \frac{2}{3}\sqrt{\frac{8}{9}}\right]$$

$$= \sin^{-1}\left(\frac{\sqrt{5}+4\sqrt{2}}{9}\right)$$

## Example 4.29

Solve $\tan^{-1}(x+2) + \tan^{-1}(2-x) = \tan^{-1}\left(\frac{2}{3}\right)$

### *Solution*

$$\tan^{-1}\left[\frac{(x+2)+(2-x)}{1-(x+2)(2-x)}\right] = \tan^{-1}\left(\frac{2}{3}\right)$$

$$\Rightarrow \qquad \tan^{-1}\left(\frac{4}{1-(4-x^2)}\right) = \tan^{-1}\left(\frac{2}{3}\right)$$

$$\Rightarrow \qquad 2x^2 - 6 = 12$$

$$\Rightarrow \qquad x^2 = 9$$

$$\therefore \qquad x = \pm 3$$

## Example 4.30

If $\tan(x+y) = 42$ and $x = \tan^{-1}(2)$, then find $y$

*Solution*

$$\tan(x+y) = 42$$

$$x + y = \tan^{-1}(42)$$

$$\tan^{-1}(2)+y = \tan^{-1}(42)$$

$$y = \tan^{-1}(42) - \tan^{-1}(2)$$

$$= \tan^{-1}\left[\frac{42-2}{1+(42 \times 2)}\right]$$

$$= \tan^{-1}\left(\frac{40}{85}\right)$$

$$y = \tan^{-1}\left[\frac{8}{17}\right]$$

**Exercise 4.4**

1. Find the pricipal value of the following

   (i) $\sin^{-1}\left(-\frac{1}{2}\right)$    (ii) $\tan^{-1}(-1)$    (iii) $\operatorname{cosec}^{-1}(2)$    (iv) $\sec^{-1}(-\sqrt{2})$

2. Prove that

   (i) $2\tan^{-1}(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

   (ii) $\tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

3. Show that $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{2}{11}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

4. Solve $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

5. Solve $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{4}{7}\right)$

6. Evaluate (i) $\cos\left[\tan^{-1}\left(\frac{3}{4}\right)\right]$    (ii) $\sin\left[\frac{1}{2}\cos^{-1}\left(\frac{4}{5}\right)\right]$

7. Evaluate: $\cos\left(\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{12}{13}\right)\right)$

8. Prove that $\tan^{-1}\left(\frac{m}{n}\right) - \tan^{-1}\left(\frac{m-n}{m+n}\right) = \frac{\pi}{4}$

9. Show that $\sin^{-1}\left(-\frac{3}{5}\right) - \sin^{-1}\left(-\frac{8}{7}\right) = \cos^{-1}\frac{84}{85}$

10. Express $\tan^{-1}\left[\frac{\cos x}{1-\sin x}\right], -\frac{\pi}{2} < x < \frac{3\pi}{2}$, in the simplest form.
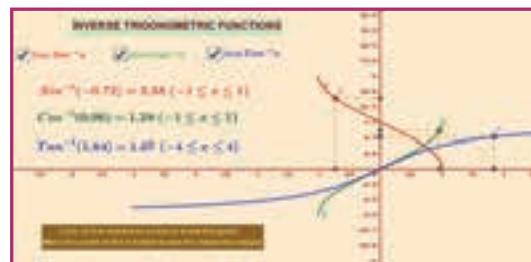
**Expected final outcomes**



### Step – 1

Open the Browser and type the URL given (or) Scan the QR Code.

GeoGebra Work book called "**11<sup>th</sup> BUSINESS MATHS**" will appear. In this several work sheets for Business Maths are given, Open the worksheet named "**Inverse Trigonometric Functions**"

### Step - 2

Inverse Trigonometric Functions page will open. Click on the check boxes on the Left-hand side to see the respective Inverse trigonometric functions. You can move and observe the point on the curve to see the x-value and y-value along the axes. Sine and cosine inverse are up to the max limit. But Tangent inverse is having customised limit.

| Step 1 |
| --- |



| Step 2 |
| --- |



Browse in the link

11<sup>th</sup> Business Maths: https://ggbm.at/qKj9gSTG (or) scan the QR Code



B184_11_BM_EM

**Choose the correct answer**

1. The degree measure of $\frac{\pi}{8}$ is

   (a) $20°60'$      (b) $22°30'$      (c) $22°60'$      (d) $20°30'$

2. The radian measure of $37°30'$ is

   (a) $\frac{5\pi}{24}$      (b) $\frac{3\pi}{24}$      (c) $\frac{7\pi}{24}$      (d) $\frac{9\pi}{24}$

3. If $\tan\theta = \frac{1}{\sqrt{5}}$ and $\theta$ lies in the first quadrant then $\cos\theta$ is

   (a) $\frac{1}{\sqrt{6}}$      (b) $\frac{-1}{\sqrt{6}}$      (c) $\frac{\sqrt{5}}{\sqrt{6}}$      (d) $\frac{-\sqrt{5}}{\sqrt{6}}$

4. The value of $\sin 15°$ is

   (a) $\frac{\sqrt{3}+1}{2\sqrt{2}}$      (b) $\frac{\sqrt{3}-1}{2\sqrt{2}}$      (c) $\frac{\sqrt{3}}{\sqrt{2}}$      (d) $\frac{\sqrt{3}}{2\sqrt{2}}$

5. The value of $\sin(-420°)$ is

   (a) $\frac{\sqrt{3}}{2}$      (b) $-\frac{\sqrt{3}}{2}$      (c) $\frac{1}{2}$      (d) $\frac{-1}{2}$

6. The value of $\cos(-480°)$ is

   (a) $\sqrt{3}$      (b) $-\frac{\sqrt{3}}{2}$      (c) $\frac{1}{2}$      (d) $\frac{-1}{2}$

7. The value of $\sin 28° \cos 17° + \cos 28° \sin 17°$ is

   (a) $\frac{1}{\sqrt{2}}$      (b) $1$      (c) $\frac{-1}{\sqrt{2}}$      (d) $0$

8. The value of $\sin 15° \cos 15°$ is

   (a) $1$      (b) $\frac{1}{2}$      (c) $\frac{\sqrt{3}}{2}$      (d) $\frac{1}{4}$

9. The value of $\sec A \sin(270° + A)$ is

   (a) $-1$      (b) $\cos^2 A$      (c) $\sec^2 A$      (d) $1$

10. If $\sin A + \cos A = 1$ then $\sin 2A$ is equal to

    (a) $1$      (b) $2$      (c) $0$      (d) $\frac{1}{2}$

11. The value of $\cos^2 45° - \sin^2 45°$ is

(a) $\dfrac{\sqrt{3}}{2}$    (b) $\dfrac{1}{2}$    (c) 0    (d) $\dfrac{1}{\sqrt{2}}$

12. The value of $1-2\sin^2 45°$ is

(a) 1    (b) $\dfrac{1}{2}$    (c) $\dfrac{1}{4}$    (d) 0

13. The value of $4\cos^3 40° - 3\cos 40°$ is

(a) $\dfrac{\sqrt{3}}{2}$    (b) $-\dfrac{1}{2}$    (c) $\dfrac{1}{2}$    (d) $\dfrac{1}{\sqrt{2}}$

14. The value of $\dfrac{2\tan 30°}{1+\tan^2 30°}$ is

(a) $\dfrac{1}{2}$    (b) $\dfrac{1}{\sqrt{3}}$    (c) $\dfrac{\sqrt{3}}{2}$    (d) $\sqrt{3}$

15. If $\sin A = \dfrac{1}{2}$ then $4\cos^3 A - 3\cos A$ is

(a) 1    (b) 0    (c) $\dfrac{\sqrt{3}}{2}$    (d) $\dfrac{1}{\sqrt{2}}$

16. The value of $\dfrac{3\tan 10° - \tan^3 10°}{1 - 3\tan^2 10°}$ is

(a) $\dfrac{1}{\sqrt{3}}$    (b) $\dfrac{1}{2}$    (c) $\dfrac{\sqrt{3}}{2}$    (d) $\dfrac{1}{\sqrt{2}}$

17. The value of $\operatorname{cosec}^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$ is

(a) $\dfrac{\pi}{4}$    (b) $\dfrac{\pi}{2}$    (c) $\dfrac{\pi}{3}$    (d) $\dfrac{\pi}{6}$

18. $\sec^{-1}\dfrac{2}{3} + \operatorname{cosec}^{-1}\dfrac{2}{3} =$

(a) $\dfrac{-\pi}{2}$    (b) $\dfrac{\pi}{2}$    (c) $\pi$    (d) $-\pi$

19. If $\alpha$ and $\beta$ be between 0 and $\dfrac{\pi}{2}$ and if $\cos(\alpha+\beta) = \dfrac{12}{13}$ and $\sin(\alpha-\beta) = \dfrac{3}{5}$ then $\sin 2\alpha$ is

(a) $\dfrac{16}{15}$    (b) 0    (c) $\dfrac{56}{65}$    (d) $\dfrac{64}{65}$

20. If $\tan A = \dfrac{1}{2}$ and $\tan B = \dfrac{1}{3}$ then $\tan(2A+B)$ is equal to

(a) 1    (b) 2    (c) 3    (d) 4

21. $\tan\left(\dfrac{\pi}{4} - x\right)$ is

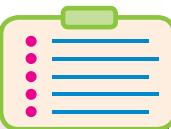(a) $\left(\dfrac{1+\tan x}{1-\tan x}\right)$    (b) $\left(\dfrac{1-\tan x}{1+\tan x}\right)$    (c) $1-\tan x$    (d) $1+\tan x$

22. $\sin\left(\cos^{-1}\frac{3}{5}\right)$ is

  (a) $\frac{3}{5}$          (b) $\frac{5}{3}$          (c) $\frac{4}{5}$          (d) $\frac{5}{4}$

23. The value of $\dfrac{1}{\operatorname{cosec}(-45°)}$ is

  (a) $\dfrac{-1}{\sqrt{2}}$          (b) $\dfrac{1}{\sqrt{2}}$          (c) $\sqrt{2}$          (d) $-\sqrt{2}$

24. If $p\sec 50° = \tan 50°$ then $p$ is

  (a) $\cos 50°$        (b) $\sin 50°$        (c) $\tan 50°$        (d) $\sec 50°$

25. $\left(\dfrac{\cos x}{\operatorname{cosec} x}\right) - \sqrt{1 - \sin^2 x}\sqrt{1 - \cos^2 x}$ is

  (a) $\cos^2 x - \sin^2 x$     (b) $\sin^2 x - \cos^2 x$     (c) 1          (d) 0

## Miscellaneous Problems

1. Prove that $\dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$

2. Prove that $\sqrt{3}\operatorname{cosec} 20° - \sin 20° = 4$

3. Prove that $\cot 4x(\sin 5x + \sin 3x) = \cot x(\sin 5x - \sin 3x)$.

4. If $\tan x = \frac{3}{4}$, $\pi < x < \frac{3\pi}{2}$ then find the value of $\sin\frac{x}{2}$ and $\cos\frac{x}{2}$

5. Prove that $2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{3\pi}{4} = 10$

6. Find the value of (i) $\sin 75°$ (ii) $\tan 15°$

7. If $\sin A = \frac{1}{3}$, $\sin B = \frac{1}{4}$ then find the value of $\sin(A+B)$, where $A$ and $B$ are acute angles.

8. Show that $\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$

9. If $\cos(\alpha + \beta) = \frac{4}{5}$ and $\sin(\alpha - \beta) = \frac{5}{13}$ where $\alpha + \beta$ and $\alpha - \beta$ are acute, then find $\tan 2\alpha$

10. Express $\tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right)$, $0 < x < \pi$ in the simplest form.

**Summary**

- If in a circle of radius r, an arc of length $l$ subtends an angle of $\theta$ radians, then $l = r\theta$

- $\sin^2 x + \cos^2 x = 1$

- $\tan^2 x + 1 = \sec^2 x$

- $\cot^2 x + 1 = \text{cosec}^2 x$

- $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

- $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

- $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

- $\cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \sin\alpha\sin\beta$

- $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

- $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$
- $\sin 2\alpha = 2\sin\alpha\cos\alpha$

- $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$

- $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$

- $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

- $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

- $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$

- $\sin x + \sin y = 2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$

- $\sin x - \sin y = 2\cos\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$

- $\cos x + \cos y = 2\cos\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$

- $\cos x - \cos y = -2\sin\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$

- $\sin x\cos y = \dfrac{1}{2}[\sin(x+y) + \sin(x-y)]$

- $\cos x\sin y = \dfrac{1}{2}[\sin(x+y) - \sin(x-y)]$

- $\cos x\cos y = \dfrac{1}{2}[\cos(x+y) + \cos(x-y)]$

- $\sin x\sin y = \dfrac{1}{2}[\cos(x-y) - \cos(x+y)]$

| GLOSSARY | |
|---|---|
| Allied angles | துணைக் கோணங்கள் |
| Angle | கோணம் |
| Componendo and dividendo. | கூட்டல் மற்றும் கழித்தல் விகித சமம் |
| Compound angle | கூட்டுக் கோணம் |
| Degree measure | பாகை அளவை |
| Inverse function | நேர்மாறு சார்பு |
| Length of the arc. | வில்லின் நீளம் |
| Multiple angle | மடங்கு கோணம் |
| Quadrants | கால் பகுதிகள் |
| Radian measure | ரேடியன் அளவு / ஆரையன் அளவு |
| Transformation formulae | உருமாற்று சூத்திரங்கள் |
| Trignometric identities | திரிகோணமிதி முற்றொருமைகள் |
| Trignometry Ratios | திரிகோணமிதி விகிதங்கள் |

**Learning Objectives**

After studying this chapter, the students will be able to understand

- the concept of limit and continuity
- formula of limit and definition of continuity
- basic concept of derivative
- how to find derivative of a function by first principle
- derivative of a function by certain direct formula and its respective application
- how to find higher order derivatives



## Introduction

Calculus is a Latin word which means that *Pebble or a small* stone used for calculation. The word calculation is also derived from the same Latin word. Calculus is primary mathematical tool for dealing with change. The concept of derivative is the basic tool in science of calculus. Calculus is essentially concerned with the rate of change of dependent variable with respect to an independent variable. Sir Issac Newton (1642 – 1727 CE) and the German mathematician G.W. Leibnitz (1646 – 1716 CE) invented and developed the subject independently and almost simultaneously.



**Isaac Newton**

In this chapter we will study about functions and their graphs, limits, derivatives and differentiation technique.

## 5.1 Functions and their graphs

**Some basic concepts**

### 5.1.1    Quantity

Anything which can be performed on basic mathematical operations like addition, subtraction, multiplication and division is called a quantity.



**G.W. Leibnitz**

### 5.1.2 Constant

A quantity which retains the same value throughout a mathematical investigation is called a constant.

Basically constant quantities are of two types

(i) **Absolute constants** are those which do not change their values in any mathematical investigation. In other words, they are fixed for ever. **Examples:** $3$, $\sqrt{3}$, $\pi$, ...

(ii) **Arbitrary constants** are those which retain the same value throughout a problem, but we may assign different values to get different solutions. The arbitrary constants are usually denoted by the letters *a, b, c,* ...

**Example:** In an equation *y=mx+4*, *m* is called arbitrary constant.

### 5.1.3 Variable

A variable is a quantity which can assume different values in a particular problem. Variables are generally denoted by the letters *x, y, z,* ...

**Example:** In an equation of the straight line $\dfrac{x}{a} + \dfrac{y}{b} = 1$,

*x* and *y* are variables because they assumes the co-ordinates of a moving point in a straight line and thus changes its value from point to point. *a* and *b* are intercept values on the axes which are arbitrary.

There are two kinds of variables

(i) A variable is said to be an **independent variable** when it can have any arbitrary value.

(ii) A variable is said to be a **dependent variable** when its value depend on the value assumed by some other variable.

**Example:** In the equation y = $5x^2 - 2x + 3$,

"*x*" is the independent variable,

"*y*" is the dependent variable and

"3" is the constant.

### 5.1.4 Intervals

The real numbers can be represented geometrically as points on a number line called real line. The symbol *R* denotes either the real number system or the real line. A subset of the real line is an interval. It contains atleast two numbers and all the real numbers lying between them.



Fig. 5.1

## (i) Open interval

The set $\{x : a < x < b\}$ is an open interval, denoted by $(a, b)$.

In this interval, the boundary points $a$ and $b$ are not included.



Fig. 5.2

For example, in an open interval $(4, 7)$, 4 and 7 are not elements of this interval.
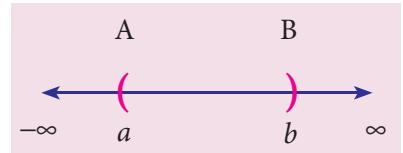
But, 4.001 and 6.99 are elements of this interval.

## (ii) Closed interval

The set $\{x : a \leq x \leq b\}$ is a closed interval and is denoted by $[a, b]$.

In the interval $[a, b]$, the boundary points $a$ and $b$ are included.



Fig. 5.3

For example, in an interval $[4, 7]$, 4 and 7 are also elements of this interval.
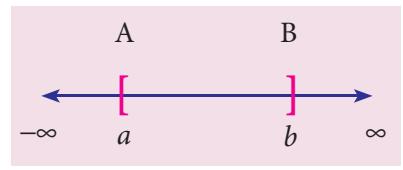
Also we can make a mention about semi closed or semi open intervals.

$$(a, b] = \{x : a < x \leq b\} \text{ is called left open interval}$$

and $\qquad [a, b) = \{x : a \leq x < b\}$ is called right open interval.

In all the above cases, $b - a = h$ is called the length of the interval.

## 5.1.5 Neighbourhood of a point

Let '$a$' be any real number. Let $\varepsilon > 0$ be arbitrarily small real number.

Then $(a - \varepsilon, a + \varepsilon)$ is called an $\varepsilon$ - neighbourhood of the point '$a$' and denoted by $N_{a, \varepsilon}$

For examples,

$$N_{5, \frac{1}{4}} = \left(5 - \frac{1}{4}, 5 + \frac{1}{4}\right)$$

$$= \left\{x : \frac{19}{4} < x < \frac{21}{4}\right\}$$

$$N_{2, \frac{1}{7}} = \left(2 - \frac{1}{7}, 2 + \frac{1}{7}\right) = \left\{x : \frac{13}{7} < x < \frac{15}{7}\right\}$$

### 5.1.6 Function

Let $X$ and $Y$ be two non-empty sets of real numbers. If there exists a rule $f$ which associates to every element $x \in X$, a unique element $y \in Y$, then such a rule $f$ is called a function (or mapping) from the set $X$ to the set $Y$. We write $f : X \longrightarrow Y$.

The set $X$ is called the domain of $f$, $Y$ is called the co-domain of $f$ and the range of $f$ is defined as $f(X) = \{f(x) \,/\, x \in X\}$.   Clearly $f(X) \subseteq Y$.

A function of $x$ is generally denoted by the symbol $f(x)$, and read as *"function of x"* or *"f of x"*.

### 5.1.7 Classification of functions

Functions can be classified into two groups.

#### (i) Algebraic functions

Algebraic functions are algebraic expressions using a finite number of terms, involving only the algebraic operations addition, subtraction, multiplication, division and raising to a fractional power.

**Examples** : $y = 3x^4 - x^3 + 5x^2 - 7$, $y = \dfrac{2x^3 + 7x - 3}{x^3 + x^2 + 1}$,  $y = \sqrt{3x^2 + 6x - 1}$ are algebraic functions.

($a$) $y = 3x^4 - x^3 + 5x^2 - 7$ is a polynomial or rational integral function.

($b$) $y = \dfrac{2x^3 + 7x - 3}{x^3 + x^2 + 1}$ is a rational function.

($c$) $y = \sqrt{3x^2 + 6x - 1}$ is an irrational function.

#### (ii) Transcendental functions

A function which is not algebraic is called transcendental function.

**Examples:** $\sin x, \sin^{-1} x, e^x, \log_a x$  are transcendental functions.

($a$)  $\sin x$, $\tan 2x$, ... are trigonometric functions.

($b$)  $\sin^{-1} x$, $\cos^{-1} x$, ... are inverse trigonometric functions.

($c$)  $e^x$, $2^x$, $x^x$, ... are exponential functions.

($d$)  $\log_a x$, $\log_e (\sin x)$, ... are logarithmic functions.

## 5.1.8 Even and odd functions

A function $f(x)$ is said to be an even function of $x$, if $f(-x) = f(x)$.

A function $f(x)$ is said to be an odd function of $x$, if $f(-x) = -f(x)$.

**Examples:** $f(x) = x^2$ and $f(x) = \cos x$ are even functions.

$f(x) = x^3$ and $f(x) = \sin x$ are odd functions.

## 5.1.9 Explicit and implicit functions

A function in which the dependent variable is expressed explicitly in terms of some independent variables is known as **explicit function**.

**Examples:** $y = x^2 + 3$ and $y = e^x + e^{-x}$ are explicit functions of $x$.

If two variables $x$ and $y$ are connected by the relation or function $f(x, y) = 0$ and none of the variable is directly expressed in terms of the other, then the function is called an **implicit function**.

**Example:** $x^3 + y^3 - xy = 0$ is an implicit function.

## 5.1.10 Constant function

If $k$ is a fixed real number then the function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function.

**Examples:** $y = 3$, $f(x) = -5$ are constant functions.

## 5.1.11 Identity function

A function that associates each real number to itself is called the identity function and is denoted by I.

i.e. A function defined on $R$ by $f(x) = x$ for all $x \in R$ is an identity function.

**Example:** The set of ordered pairs {(1, 1), (2, 2), (3, 3)} defined by $f : A \to A$ where $A = \{1, 2, 3\}$ is an identity function.

## 5.1.12 Modulus function

A function $f(x)$ defined by $f(x) = |x|$, where $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ is called the modulus function.

It is also called an absolute value function.

| Remark | Domain is $R$ and range set is $[0, \infty)$ |

NOTE

$|5| = 5$, $|-5| = -(-5) = 5$

## 5.1.13 Signum function

A function $f(x)$ is defined by $f(x) = \begin{cases} \dfrac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is called the Signum function.

| Remark | Domain is $R$ and range set is $\{-1, 0, 1\}$ |

## 5.1.14 Step function

### (i) Greatest integer function

The function whose value at any real number $x$ is the greatest integer less than or equal to $x$ is called the greatest integer function. It is denoted by $\lfloor x \rfloor$.

i.e. $f : R \longrightarrow R$ defined by $f(x) = \lfloor x \rfloor$ is called the greatest integer function.

NOTE

$\lfloor 2.5 \rfloor = 2$, $\lfloor -2.1 \rfloor = -3$,
$\lfloor 0.74 \rfloor = 0$, $\lfloor -0.3 \rfloor = -1$,
$\lfloor 4 \rfloor = 4$.

### (ii) Least integer function

The function whose value at any real number $x$ is the smallest integer greater than or equal to $x$ is called the least integer function. It is denoted by $\lceil x \rceil$.

i.e. $f : R \longrightarrow R$ defined by $f(x) = \lceil x \rceil$ is called the least integer function

NOTE

$\lceil 4.7 \rceil = 5$, $\lceil -7.2 \rceil = -7$,
$\lceil 5 \rceil = 5$, $\lceil 0.75 \rceil = 1$.

| Remark | Function's domain is $R$ and range is $Z$ [set of integers] |

## 5.1.15 Rational function

A function $f(x)$ is defined by the rule $f(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$ is called rational function.

**Example:** $f(x) = \dfrac{x^2 + 1}{x - 3}$, $x \neq 3$ is a rational function.

### 5.1.16 Polynomial function

For the real numbers $a_0, a_1, a_2, ..., a_n$; $a_0 \neq 0$ and $n$ is a non-negative integer, a function $f(x)$ given by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n$ is called as a polynomial function of degree $n$.

**Example:** $f(x) = 2x^3 + 3x^2 + 2x - 7$ is a polynomial function of degree 3.

### 5.1.17 Linear function

For the real numbers $a$ and $b$ with a $\neq 0$, a function $f(x) = ax + b$ is called a linear function.

**Example:** $y = 2x + 3$ is a linear function.

### 5.1.18 Quadratic function

For the real numbers $a$, $b$ and $c$ with $a \neq 0$, a function $f(x) = ax^2 + bx + c$ is called a quadratic function.

**Example:** $f(x) = 3x^2 + 2x - 7$ is a quadratic function.

### 5.1.19 Exponential function

A function $f(x) = a^x$, $a \neq 1$ and $a > 0$, for all $x \in R$ is called an exponential function

**Remark**   Domain is $R$ and range is $(0, \infty)$ and $(0, 1)$ is a point on the graph

**Examples:** $e^{2x}$, $e^{x^2+1}$ and $2^x$ are exponential functions.

### 5.1.20 Logarithmic function

For $x > 0$, $a > 0$ and $a \neq 1$, a function $f(x)$ defined by $f(x) = \log_a x$ is called the logarithmic function.

**Remark**   Domain is $(0, \infty)$, range is $R$ and $(1, 0)$ is a point on the graph

**Example:** $f(x) = \log_e(x+2)$, $f(x) = \log_e(\sin x)$ are logarithmic functions.

### 5.1.21 Sum, difference, product and quotient of two functions

If $f(x)$ and $g(x)$ are two functions having same domain and co-domain, then

(i)  $(f \pm g)(x) = f(x) \pm g(x)$

(ii)  $(fg)(x) = f(x)\, g(x)$

(iii) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ , $g(x) \neq 0$

(iv) $(cf)(x) = cf(x)$, $c$ is a constant.

## 5.1.22 Graph of a function

The graph of a function is the set of points $(x, f(x))$ where $x$ belongs to the domain of the function and $f(x)$ is the value of the function at $x$.

To draw the graph of a function, we find a sufficient number of ordered pairs $(x, f(x))$ belonging to the function and join them by a smooth curve.

**Example 5.1**

Find the domain for which the functions $f(x) = 2x^2-1$ and $g(x) = 1-3x$ are equal.

*Solution*

Given that $\quad f(x) = g(x)$

$\Rightarrow \qquad 2x^2 - 1 = 1-3x$

$\qquad 2x^2 +3x - 2 = 0$

$\qquad (x+2)(2x-1) = 0$

$\therefore \qquad\qquad x = -2, x = \dfrac{1}{2}$

Domain is $\left\{-2, \dfrac{1}{2}\right\}$

**Example 5.2**

$f = \{(1, 1), (2, 3)\}$ be a function described by the formula $f(x) = ax+b$. Determine $a$ and $b$ ?

*Solution*

Given that $\quad f(x) = ax + b$ , $f(1) = 1$ and $f(2) = 3$

$\therefore \qquad\qquad a + b = 1$ and $2a + b = 3$

$\Rightarrow \qquad\qquad a = 2$ and $b = -1$

**Example 5.3**

If $f(x) = x + \dfrac{1}{x}$, then show that $[f(x)]^3 = f(x^3) + 3f\left(\dfrac{1}{x}\right)$

*Solution*

$$f(x) = x + \frac{1}{x}, \quad f(x^3) = x^3 + \frac{1}{x^3} \quad \text{and} \quad f\!\left(\frac{1}{x}\right) = f(x)$$

Now, $\quad LHS = [f(x)]^3$

$$= \left[x + \frac{1}{x}\right]^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)$$

$$= f(x^3) + 3f(x) = f(x^3) + 3f\!\left(\frac{1}{x}\right)$$

$$= RHS$$

**Example 5.4**

Let $f$ be defined by $f(x) = x - 5$ and $g$ be defined by

$$g(x) = \begin{cases} \dfrac{x^2 - 25}{x + 5} & \text{if } x \ne -5 \\ \lambda & \text{if } x = -5 \end{cases} \quad \text{Find } \lambda \text{ such that } f(x) = g(x) \text{ for all } x \in R$$

*Solution*

Given that $\quad f(x) = g(x)$ for all $x \in R$

$\therefore \qquad\qquad f(-5) = g(-5)$

$\qquad\qquad\qquad -5 - 5 = \lambda$

$\therefore \qquad\qquad\quad \lambda = -10$

**Example 5.5**

If $f(x) = 2^x$, show that $f(x) \cdot f(y) = f(x + y)$

*Solution*

$$f(x) = 2^x \ (\text{Given})$$

$\therefore \qquad\quad f(x + y) = 2^{x+y}$

$$= 2^x \cdot 2^y$$

$$= f(x) \cdot f(y)$$

**Example 5.6**

If $f(x) = \dfrac{x-1}{x+1}$, $0 < x < 1$, then show that $f[f(x)] = -\dfrac{1}{x}$

**Solution**

$$f(x) = \frac{x-1}{x+1} \quad \text{(Given)}$$

$$\therefore \qquad f[f(x)] = \frac{f(x)-1}{f(x)+1} = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1}$$

$$= \frac{x-1-x-1}{x-1+x+1} = -\frac{2}{2x} = -\frac{1}{x}$$

**Example 5.7**

If $f(x) = \log\dfrac{1+x}{1-x}$, show that $f\left(\dfrac{2x}{1+x^2}\right) = 2f(x)$.

**Solution**

$$f(x) = \log\frac{1+x}{1-x} \quad \text{(Given)}$$

$$\therefore \qquad f\left(\frac{2x}{1+x^2}\right) = \log\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}$$

$$= \log\frac{1+x^2+2x}{1+x^2-2x} = \log\frac{(1+x)^2}{(1-x)^2}$$

$$= 2\log\frac{1+x}{1-x} = 2f(x).$$

**Example 5.8**

If $f(x) = x$ and $g(x) = |x|$, then find

(i) $(f+g)(x)$        (ii) $(f-g)(x)$        (iii) $(fg)(x)$

**Solution**

(i)      $(f+g)(x) = f(x) + g(x) = x + |x|$

$$= \begin{cases} x+x & \text{if } x \geq 0 \\ x-x & \text{if } x < 0 \end{cases} = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

(ii)      $(f-g)(x) = f(x) - g(x) = x - |x|$

$$= \begin{cases} x-x & \text{if } x \geq 0 \\ x-(-x) & \text{if } x < 0 \end{cases} = \begin{cases} 0 & \text{if } x \geq 0 \\ 2x & \text{if } x < 0 \end{cases}$$

(iii)    $(f\,g)\,(x)\;=f(x)\,g(x)\;=\;x\,|x|$

$$= \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

## Example 5.9

A group of students wish to charter a bus for an educational tour which can accommodate atmost 50 students. The bus company requires at least 35 students for that trip.  The bus company charges ₹ 200 per student up to the strength of 45. For more than 45 students, company charges each student ₹ 200 less $\frac{1}{5}$ times the number more than 45. Consider the number of students who participates the tour as a function, find the total cost and its domain.

### Solution

Let '*x*' be the number of students who participate the tour.

Then $35 \leq x \leq 50$ and  *x* is a positive integer.

> **Formula :**   *Total cost = (cost per student) × (number of students)*

 (i)  If the number of students are between 35 and 45, then

the cost per student is ₹ 200

∴  The total cost is *y = 200x.*

 (ii)  If the number of students are between 46 and 50, then

the cost per student is  ₹ $\left\{200 - \frac{1}{5}(x - 45)\right\} = 209 - \frac{x}{5}$

∴    The total cost is $y = \left(209 - \frac{x}{5}\right)x = 209x - \frac{x^2}{5}$

∴    $y = \begin{cases} 200x & ;\ 35 \leq x \leq 45 \\ 209x - \dfrac{x^2}{5} & ;\ 46 \leq x \leq 50 \end{cases}$  and

the domain is {35, 36, 37, ..., 50}.

## Example 5.10

Draw the graph of the function $f(x) = |x|$

### Solution

$$y = f(x) = |x|$$

$$= \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Choose suitable values for $x$ and determine $y$.

Thus we get the following table.

| $x$ | −2 | −1 | 0 | 1 | 2 |
|-----|-----|-----|-----|-----|-----|
| $y$ | 2 | 1 | 0 | 1 | 2 |

Table : 5.1

Plot the points $(-2, 2)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, $(2, 2)$ and draw a smooth curve.

The graph is as shown in the figure
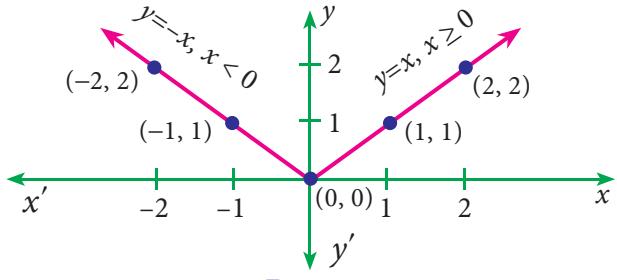


Fig. 5.4

## Example 5.11

Draw the graph of the function $f(x) = x^2 - 5$

### Solution

Let $y = f(x) = x^2 - 5$

Choose suitable values for $x$ and determine $y$.

Thus we get the following table.

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $y$ | 4 | −1 | −4 | −5 | −4 | −1 | 4 |

Table : 5.2

Plot the points $(-3, 4)$, $(-2, -1)$, $(-1, -4)$, $(0, -5)$, $(1, -4)$, $(2, -1)$, $(3, 4)$ and draw a smooth curve.
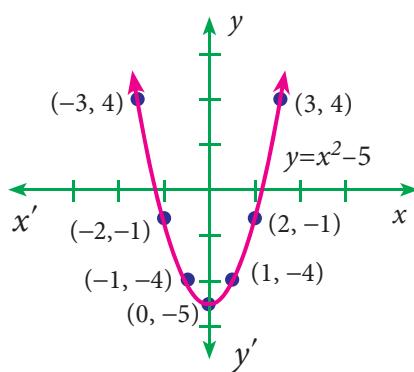
The graph is as shown in the figure



Fig 5.5

## Example 5.12

Draw the graph of $f(x) = a^x$, $a \neq 1$ and $a > 0$

### Solution

We know that, domain set is $R$, range set is $(0, \infty)$ and the curve passing through the point $(0, 1)$

**Case (i)** when $a > 1$

$$y = f(x) = a^x = \begin{cases} < 1 & \text{if } x < 0 \\ = 1 & \text{if } x = 0 \\ > 1 & \text{if } x > 0 \end{cases}$$

We noticed that as $x$ increases, $y$ is also increases and $y > 0$.

∴    The graph is as shown in the figure.



Fig 5.6

**Case (ii)** when $0 < a < 1$

$$y = f(x) = a^x = \begin{cases} > 1 & \text{if } x < 0 \\ = 1 & \text{if } x = 0 \\ < 1 & \text{if } x > 0 \end{cases}$$

We noticed that as $x$ increases, $y$ decreases and $y > 0$.

∴    The graph is as shown in the figure.
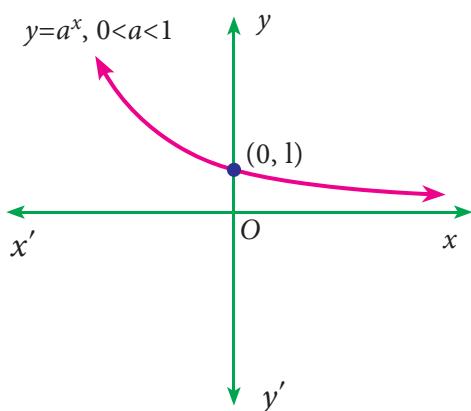


Fig 5.7

**NOTE**

(i)   The value of e lies between 2 and 3. i.e. $2 < e < 3$

(ii)  The graph of $f(x) = e^x$ is identical to that of $f(x) = a^x$ if $a > 1$ and the graph of $f(x) = e^{-x}$ is identical to that of $f(x) = a^x$ if $0 < a < 1$

**Example 5.13**

Draw the graph of $f(x) = \log_a x$ ; $x > 0$, $a > 0$ and $a \neq 1$.

## Solution
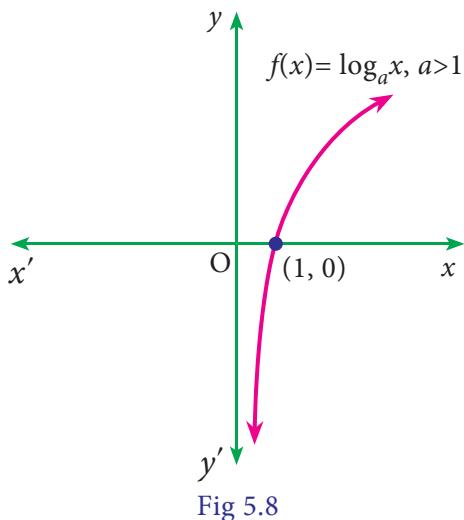
We know that, domain set is $(0, \infty)$, range set is R and the curve passing the point $(1, 0)$

**Case (i)** when $a > 1$ and $x > 0$

$$f(x) = \log_a x = \begin{cases} < 0 & \text{if } 0 < x < 1 \\ = 0 & \text{if } x = 1 \\ > 0 & \text{if } x > 1 \end{cases}$$

We noticed that as $x$ increases, $f(x)$ is also increases.
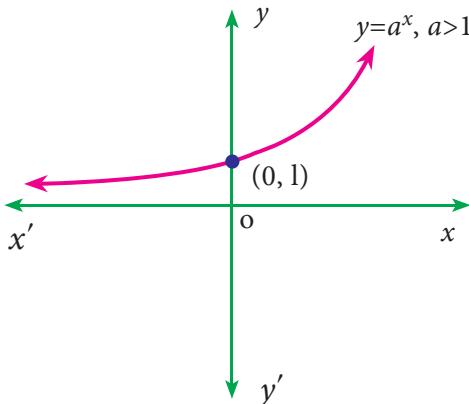
∴    The graph is as shown in the figure.



Fig 5.8
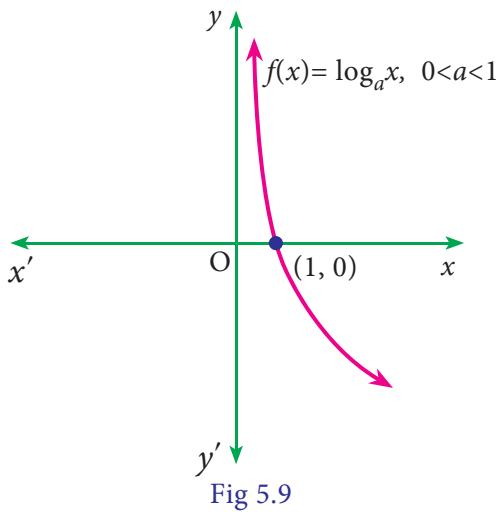
**Case (ii)** when $0 < a < 1$ and $x > 0$

$$f(x) = \log_a x = \begin{cases} > 0 & \text{if } 0 < x < 1 \\ = 0 & \text{if } x = 1 \\ < 0 & \text{if } x > 1 \end{cases}$$

We noticed that $x$ increases, the value of $f(x)$ decreases.

∴    The graph is as shown in the figure.



Fig 5.9

**Expected final outcomes**



### Step – 1

Open the Browser and type the URL given (or) Scan the QR Code.

GeoGebra Work book called "**11ᵗʰ BUSINESS MATHS**" will appear. In this several work sheets for Business Maths are given, Open the worksheet named "**Types of Functions**"

### Step - 2

Types of Functions page will open. Click on the check boxes of the functions on the Right-hand side to see the respective functions. You can move the slider and change the function. Observe each function.

| Step 1 |
| --- |



| Step 2 |
| --- |



Browse in the link

11ᵗʰ Business Maths: https://ggbm.at/qKj9gSTG  (or) scan the QR Code



B184_11_BM_EM

## Exercise 5.1

1. Determine whether the following functions are odd or even?

    (i) $f(x) = \left( \dfrac{a^x - 1}{a^x + 1} \right)$        (ii) $f(x) = \log\left( x^2 + \sqrt{x^2 + 1} \right)$

    (iii) $f(x) = \sin x + \cos x$        (iv) $f(x) = x^2 - |x|$        (v) $f(x) = x + x^2$

2. Let $f$ be defined by $f(x) = x^3 - kx^2 + 2x$, $x \in R$. Find $k$, if $'f'$ is an odd function.

3. If $f(x) = x^3 - \dfrac{1}{x^3}$, then show that $f(x) + f\left(\dfrac{1}{x}\right) = 0$

4. If $f(x) = \dfrac{x + 1}{x - 1}$, then prove that $f(f(x)) = x$

5. For $f(x) = \dfrac{x - 1}{3x + 1}$, write the expressions of $f\left(\dfrac{1}{x}\right)$ and $\dfrac{1}{f(x)}$

6. If $f(x) = e^x$ and $g(x) = \log_e x$, then find

    (i) $(f + g)(1)$        (ii) $(fg)(1)$        (iii) $(3f)(1)$        (iv) $(5g)(1)$

7. Draw the graph of the following functions :

    (i) $f(x) = 16 - x^2$        (ii) $f(x) = |x - 2|$        (iii) $f(x) = x|x|$

    (iv) $f(x) = e^{2x}$        (v) $f(x) = e^{-2x}$        (vi) $f(x) = \dfrac{|x|}{x}$

## 5.2 Limits and derivatives

In this section, we will discuss about limits, continuity of a function, differentiability and differentiation from first principle. Limits are used when we have to find the value of a function near to some value.

### Definition 5.1

Let $f$ be a real valued function of $x$. Let $l$ and $a$ be two fixed numbers. If $f(x)$ approaches the value $l$ as $x$ approaches to $a$, then we say that $l$ is the limit of the function $f(x)$ as $x$ tends to $a$, and written as

$$\lim_{x \to a} f(x) = l$$

**Notations**: (i)   $L[f(x)]_{x=a} = \lim_{x \to a^-} f(x)$ is known as left hand limit of $f(x)$ at $x=a$.

   (ii)   $R[f(x)]_{x=a} = \lim_{x \to a^+} f(x)$ is known as right hand limit of $f(x)$ at $x=a$.

## 5.2.1 Existence of limit

$\lim_{x \to a} f(x)$ exist if and only if both $L[f(x)]_{x=a}$ and $R[f(x)]_{x=a}$ exist and are equal.

i.e., $\lim_{x \to a} f(x)$ exist if and only if $L[f(x)]_{x=a} = R[f(x)]_{x=a}$.

> **NOTE**
>
> $L[f(x)]_{x=a}$ and $R[f(x)]_{x=a}$ are shortly written as $L[f(a)]$ and $R[f(a)]$

## 5.2.2 Algorithm of left hand limit : $L[f(x)]_{x=a}$

(i) Write $\lim_{x \to a^-} f(x)$.

(ii) Put $x=a-h$, $h > 0$ and replace $x \to a^-$ by $h \to 0$.

(iii) Obtain $\lim_{h \to 0} f(a-h)$.

(iv) The value obtained in step (iii) is called as the value of left hand limit of the function $f(x)$ at $x=a$.

## 5.2.3 Algorithm of right hand limit : $R[f(x)]_{x=a}$

(i) Write $\lim_{x \to a^+} f(x)$.

(ii) Put $x = a+h$, $h>0$ and replace $x \to a^+$ by $h \to 0$.

(iii) Obtain $\lim_{h \to 0} f(a+h)$.

(iv) The value obtained in step (iii) is called as the value of right hand limit of the function $f(x)$ at $x=a$.

### Example 5.14

Evaluate the left hand and right hand limits of the function

$$f(x) = \begin{cases} \dfrac{|x-3|}{x-3} & \text{if } x \neq 3 \\ 0 & \text{if } x = 3 \end{cases} \quad \text{at } x = 3.$$

### Solution

$$\begin{aligned} L[f(x)]_{x=3} &= \lim_{x \to 3^-} f(x) \\ &= \lim_{h \to 0} f(3-h), \ x = 3-h \\ &= \lim_{h \to 0} \frac{|(3-h)-3|}{(3-h)-3} \end{aligned}$$

$$= \lim_{h \to 0} \frac{|-h|}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h}$$

$$= \lim_{h \to 0} -1$$

$$= -1$$

$$R[f(x)]_{x=3} = \lim_{x \to 3^+} f(x)$$

$$= \lim_{h \to 0} f(3 + h)$$

$$= \lim_{h \to 0} \frac{|(3 + h) - 3|}{(3 + h) - 3}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

$$= \lim_{h \to 0} 1$$

$$= 1$$

### Example 5.15

Verify the existence of the function $f(x) = \begin{cases} 5x - 4 & \text{if } 0 < x \le 1 \\ 4x^3 - 3 & \text{if } 1 < x < 2 \end{cases}$ at $x = 1$.

*Solution*

$$L[f(x)]_{x=1} = \lim_{x \to 1^-} f(x)$$

$$= \lim_{h \to 0} f(1 - h), \; x = 1 - h$$

$$= \lim_{h \to 0} [5(1-h)-4]$$

$$= \lim_{h \to 0} (1-5h) = 1$$

$$R[f(x)]_{x=1} = \lim_{x \to 1^+} f(x)$$

$$= \lim_{h \to 0} f(1 + h), \; x = 1 + h$$

$$= \lim_{h \to 0} [4(1 + h)^3 - 3(1 + h)]$$

$$= 4(1)^3 - 3(1)$$

Clearly, $L[f(1)] = R[f(1)]$

$\therefore \quad \lim_{x \to 1} f(x)$ exists and equal to 1.

*Let a be a point and f(x) be a function, then the following stages may happen*

(i)   $\lim\limits_{x \to a} f(x)$ exists but $f(a)$ does not exist.

(ii)  The value of $f(a)$ exists but $\lim\limits_{x \to a} f(x)$ does not exist.

(iii) Both $\lim\limits_{x \to a} f(x)$ and $f(a)$ exist but are unequal.

(iv)  Both $\lim\limits_{x \to a} f(x)$ and $f(a)$ exist and are equal.

### 5.2.4   Some results of limits

If $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exists, then

(i)   $\lim\limits_{x \to a} (f \pm g)(x) \quad = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$

(ii)  $\lim\limits_{x \to a} (fg)(x) \quad = \lim\limits_{x \to a} f(x) \lim\limits_{x \to a} g(x)$

(iii) $\lim\limits_{x \to a} \left( \dfrac{f}{g} \right)(x) \quad = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, whenever $\lim\limits_{x \to a} g(x) \neq 0$

(iv)  $\lim\limits_{x \to a} k\, f(x) \quad = k \lim\limits_{x \to a} f(x)$, where $k$ is a constant.

### 5.2.5   Indeterminate forms and evaluation of limits

Let $f(x)$ and $g(x)$ are two functions in which the limits exist.

If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$, then $\dfrac{f(a)}{g(a)}$ takes $\dfrac{0}{0}$ form which is meaningless but does not imply that $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ is meaningless.

In many cases, this limit exists and have a finite value. Finding the solution of such limits are called evaluation of the indeterminate form.

The indeterminate forms are $\dfrac{0}{0}, \dfrac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, \infty^0$ *and* $1^\infty$.

Among all these indeterminate forms, $\dfrac{0}{0}$ is the fundamental one.

### 5.2.6   Methods of evaluation of algebraic limits

(i)   Direct substitution

(ii)  Factorization

(iii) Rationalisation

(iv)  By using some standard limits

## 5.2.7 Some standard limits

(i) $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} \qquad = na^{n-1}$ if $n \in Q$

(ii) $\lim\limits_{\theta \to 0} \dfrac{\sin \theta}{\theta} \qquad = \lim\limits_{\theta \to 0} \dfrac{\tan \theta}{\theta} = 1$, $\theta$ is in radian.

(iii) $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} \qquad = \log_e a$ if $a > 0$

(iv) $\lim\limits_{x \to 0} \dfrac{\log(1 + x)}{x} \qquad = 1$

(v) $\lim\limits_{x \to 0} (1 + x)^{\frac{1}{x}} \qquad = e$

(vi) $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x \qquad = e$

(vii) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} \qquad = 1$

## Example 5.16

Evaluate: $\lim\limits_{x \to 1} (3x^2 + 4x - 5)$.

### Solution

$\lim\limits_{x \to 1} (3x^2 + 4x - 5)$

$= 3(1)^2 + 4(1) - 5 = 2$

## Example 5.17

Evaluate: $\lim\limits_{x \to 2} \dfrac{x^2 - 4x + 6}{x + 2}$.

### Solution

$\lim\limits_{x \to 2} \dfrac{x^2 - 4x + 6}{x + 2} = \dfrac{\lim\limits_{x \to 2} (x^2 - 4x + 6)}{\lim\limits_{x \to 2} (x + 2)}$

$\qquad = \dfrac{(2)^2 - 4(2) + 6}{2 + 2} = \dfrac{1}{2}$

## Example 5.18

Evaluate : $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{5 \sin 2x - 2 \cos 2x}{3 \cos 2x + 2 \sin 2x}$.

### Solution

$\lim\limits_{x \to \frac{\pi}{4}} \dfrac{5 \sin 2x - 2 \cos 2x}{3 \cos 2x + 2 \sin 2x} = \dfrac{\lim\limits_{x \to \frac{\pi}{4}} (5 \sin 2x - 2 \cos 2x)}{\lim\limits_{x \to \frac{\pi}{4}} (3 \cos 2x + 2 \sin 2x)}$

$$= \frac{5 \sin \frac{\pi}{2} - 2 \cos \frac{\pi}{2}}{3 \cos \frac{\pi}{2} + 2 \sin \frac{\pi}{2}}$$

$$= \frac{5(1) - 2(0)}{3(0) + 2(1)} = \frac{5}{2}.$$

### Example 5.19

Evaluate: $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1}$.

*Solution*

$\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1}$ is of the type $\dfrac{0}{0}$

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{(x - 1)}$$

$$= \lim_{x \to 1} (x^2 + x + 1)$$

$$= (1)^2 + 1 + 1 = 3$$

### Example 5.20

Evaluate: $\lim\limits_{x \to 0} \dfrac{\sqrt{2 + x} - \sqrt{2}}{x}$.

*Solution*

$$\lim_{x \to 0} \frac{\sqrt{2 + x} - \sqrt{2}}{x} = \lim_{x \to 0} \frac{(\sqrt{2 + x} - \sqrt{2})(\sqrt{2 + x} + \sqrt{2})}{x(\sqrt{2 + x} + \sqrt{2})}$$

$$= \lim_{x \to 0} \frac{2 + x - 2}{x(\sqrt{2 + x} + \sqrt{2})}$$

$$= \lim_{x \to 0} \frac{1}{(\sqrt{2 + x} + \sqrt{2})}$$

$$= \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}.$$

### Example 5.21

Evaluate: $\lim\limits_{x \to a} \dfrac{x^{\frac{3}{5}} - a^{\frac{3}{5}}}{x^{\frac{1}{5}} - a^{\frac{1}{5}}}$.

*Solution*

$$\lim_{x \to a} \frac{x^{\frac{3}{5}} - a^{\frac{3}{5}}}{x^{\frac{1}{5}} - a^{\frac{1}{5}}} = \lim_{x \to a} \frac{\dfrac{x^{\frac{3}{5}} - a^{\frac{3}{5}}}{x - a}}{\dfrac{x^{\frac{1}{5}} - a^{\frac{1}{5}}}{x - a}}$$

$$= \frac{\dfrac{3}{5}(a)^{-\frac{2}{5}}}{\dfrac{1}{5}(a)^{-\frac{4}{5}}} = 3a^{\frac{-2}{5} + \frac{4}{5}} = 3a^{\frac{2}{5}}.$$

**Example 5.22**

Evaluate: $\lim\limits_{x \to 0} \dfrac{\sin 3x}{\sin 5x}$.

*Solution*

$$\lim_{x \to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x \to 0} \left\{ \frac{3x \times \dfrac{\sin 3x}{3x}}{5x \times \dfrac{\sin 5x}{5x}} \right\}$$

$$= \frac{3}{5} \frac{\lim\limits_{x \to 0} \dfrac{\sin 3x}{3x}}{\lim\limits_{x \to 0} \dfrac{\sin 5x}{5x}}$$

$$= \frac{3}{5} \left( \frac{1}{1} \right) = \frac{3}{5}$$

**Example 5.23**

Evaluate: $\lim\limits_{x \to \infty} \dfrac{6 - 5x^2}{4x + 15x^2}$.

*Solution*

$$\lim_{x \to \infty} \frac{6 - 5x^2}{4x + 15x^2} = \lim_{x \to \infty} \frac{\dfrac{6}{x^2} - 5}{\dfrac{4}{x} + 15}$$

$$= \frac{0 - 5}{0 + 15} = -\frac{1}{3}.$$

**Example 5.24**

Evaluate: $\lim\limits_{x \to \infty} x \tan\left(\dfrac{1}{x}\right)$.

*Solution*

Let $\qquad y = \dfrac{1}{x}$ and $y \to 0$ as $x \to \infty$

$$\lim_{x \to \infty} x \tan\left(\frac{1}{x}\right) = \lim_{y \to 0} \frac{\tan y}{y} = 1.$$

**Example 5.25**

Evaluate: $\lim\limits_{n \to \infty} \dfrac{\sum n^2}{n^3}$.

*Solution*

$$\lim_{n \to \infty} \frac{\sum n^2}{n^3} = \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3}$$

$$= \lim_{n \to \infty} \frac{1}{6} \left\{ \left( \frac{n}{n} \right) \left( \frac{n+1}{n} \right) \left( \frac{2n+1}{n} \right) \right\}$$

$$= \frac{1}{6} \lim_{n \to \infty} \left\{ \left( 1 + \frac{1}{n} \right) \left( 2 + \frac{1}{n} \right) \right\}$$

$$= \frac{1}{6}(1)(2)$$

$$= \frac{1}{3}.$$

**Example 5.26**

Show that $\lim\limits_{x \to 0} \dfrac{\log(1 + x^3)}{\sin^3 x} = 1.$

*Solution*

$$\lim_{x \to 0} \frac{\log(1 + x^3)}{\sin^3 x} = \lim_{x \to 0} \left\{ \frac{\log(1 + x^3)}{x^3} \times \frac{x^3}{\sin^3 x} \right\}$$

$$= \lim_{x \to 0} \frac{\log(1 + x^3)}{x^3} \times \frac{1}{\lim\limits_{x \to 0} \left( \frac{\sin x}{x} \right)^3}$$

$$= 1 \times \frac{1}{1} = 1$$

**Exercise 5.2**

1. Evaluate the following

   (i) $\lim\limits_{x \to 2} \dfrac{x^3 + 2}{x + 1}$ 

   (ii) $\lim\limits_{x \to \infty} \dfrac{2x + 5}{x^2 + 3x + 9}$ 

   (iii) $\lim\limits_{x \to \infty} \dfrac{\sum n}{n^2}$

   (iv) $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + x} - \sqrt{1 - x}}{5x}$ 

   (v) $\lim\limits_{x \to a} \dfrac{x^{\frac{5}{8}} - a^{\frac{5}{8}}}{x^{\frac{2}{3}} - a^{\frac{2}{3}}}$ 

   (vi) $\lim\limits_{x \to 0} \dfrac{\sin^2 3x}{x^2}$

2. If $\lim\limits_{x \to a} \dfrac{x^9 + a^9}{x + a} = \lim\limits_{x \to 3}(x + 6)$, find the values of $a$.

3. If $\lim\limits_{x \to 2} \dfrac{x^n - 2^n}{x - 2} = 448$, then find the least positive integer $n$.

4. If $f(x) = \dfrac{x^7 - 128}{x^5 - 32}$, then find $\lim\limits_{x \to 2} f(x)$

5. Let $f(x) = \dfrac{ax + b}{x + 1}$, If $\lim\limits_{x \to 0} f(x) = 2$ and $\lim\limits_{x \to \infty} f(x) = 1$, then show that $f(-2) = 0$.

## Derivative

Before going into the topic, let us first discuss about the continuity of a function. A function $f(x)$ is continues at $x = a$ if its graph has no break at $x = a$

If there is any break at the point $x = a$ then we say that the function is not continuous at the point $x = a$.

If a function is continuous at all the point in an interval, then it is said to be a continuous in that interval.

### 5.2.8 Continuous function

A function $f(x)$ is continuous at $x = a$ if

(i)  *f(a)* exists

(ii) $\lim\limits_{x \to a} f(x)$ exists

(iii) $\lim\limits_{x \to a} f(x) = f(a)$

**Observation**

In the above statement, if atleast any one condition is not satisfied at a point $x = a$ by the function *f(x)*, then it is said to be a discontinuous function at $x = a$

### 5.2.9 Some properties of continuous functions

Let $f(x)$ and $g(x)$ are two real valued continuous functions at $x = a$ , then

(i)  $f(x) \pm g(x)$ is continuous at $x = a$.

(ii) $f(x) \, g(x)$ is continuous at $x = a$.

(iii) $kf(x)$ is continuous at $x = a$, $k$ be a real number.

(iv) $\dfrac{1}{f(x)}$ is continuous at $x = a$, if $f(a) \neq 0$.

(v) $\dfrac{f(x)}{g(x)}$ is continuous at $x = a$, if $g(a) \neq 0$.

(vi) $|f(x)|$ is continuous at $x = a$.

**Example 5.27**

Show that $f(x) = \begin{cases} 5x - 4, & \text{if } 0 < x \le 1 \\ 4x^3 - 3x, & \text{if } 1 < x < 2 \end{cases}$ is continuous at $x = 1$

*Solution*

$$L[f(x)]_{x=1} = \lim_{x \to 1^-} f(x)$$

$$= \lim_{h \to 0} f(1 - h), \quad x = 1 - h$$

$$= \lim_{h \to 0} [5(1 - h) - 4]$$

$$= 5(1) - 4 = 1$$

$$R[f(x)]_{x=1} = \lim_{x \to 1^+} f(x)$$

$$= \lim_{h \to 0} f(1 + h), \quad x = 1 + h$$

$$= \lim_{h \to 0} [4(1 + h)^3 - 3(1 - h)]$$

$$= 4(1)^3 - 3(1)$$

$$= 4 - 3 = 1$$

Now, $\quad f(1) = 5(1) - 4 = 5 - 4 = 1$

$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1),$

$f(x)$ is continuous at $x = 1$.

**Example 5.28**

Verify the continuity of the function $f(x)$ given by

$$f(x) = \begin{cases} 2 - x \text{ if } x < 2 \\ 2 + x \text{ if } x \ge 2 \end{cases} \text{ at } x = 2.$$

*Solution*

$$L[f(x)]_{x=2} = \lim_{x \to 2^-} f(x)$$

$$= \lim_{h \to 0} f(2 - h), \quad x = 2 - h$$

$$= \lim_{h \to 0} \{2 - (2 - h)\}$$

$$= \lim_{h \to 0} h = 0$$

$$R\big[f(x)\big]_{x=2} = \lim_{x+2^+} f(x)$$

$$= \lim_{h \to 0} f(2+h), \; x = 2 + h$$

$$= \lim_{h \to 0}\{2 + (2+h)\}$$

$$= \lim_{h \to 0}(4+h) = 4$$

Now, $\quad f(2) = 2+2 = 4.$

Here $\displaystyle\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

Hence $f(x)$ is not continuous at $x = 2$.

### Observations

(i) A constant function is everywhere continuous.

(ii) The identity function is everywhere continuous.

(iii) A polynomial function is everywhere continuous.

(iv) The modulus function is everywhere continuous.

(v) The exponential function $a^x$, $a>0$ is everywhere continuous.

(vi) The logarithmic function is continuous in its domain.

(vii) Every rational function is continuous at every point in its domain.

**Exercise 5.3**

1. Examine the following functions for continuity at indicated points

(a) $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & if \;\; x \neq 2 \\ 0, & if \;\; x = 2 \end{cases}$ at $x = 2$

(b) $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, & if \;\; x \neq 3 \\ 6, & if \;\; x = 3 \end{cases}$ at $x = 3$

2. Show that $f(x) = |x|$ is continuous at $x = 0$

### 5.2.10 Differentiability at a point

Let $f(x)$ be a real valued function defined on an open interval $(a, b)$ and let

$c \in (a, b)$. $f(x)$ is said to be differentiable or derivable at $x = c$ if and only if $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists finitely.

This limit is called the derivative (or) differential co-efficient of the function $f(x)$ at $x = c$ and is denoted by $f'(c)$ (or) $D f(c)$ (or) $\left[\dfrac{d}{dx}(f(x))\right]_{x = c}$.

Thus $f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$.

### 5.2.11  Left hand derivative and right hand derivative

(i) $\lim\limits_{x \to c^-} \dfrac{f(x) - f(c)}{x - c}$    or    $\lim\limits_{h \to 0} \dfrac{f(c - h) - f(c)}{-h}$   is called the left hand derivative of $f(x)$ at $x = c$ and it denoted by $f'(c^-)$ or $L[f'(c)]$.

(ii) $\lim\limits_{x \to c^+} \dfrac{f(x) - f(c)}{x - c}$    or    $\lim\limits_{h \to 0} \dfrac{f(c + h) - f(c)}{h}$   is called the right hand derivative of $f(x)$ at $x = c$ and it denoted by $f'(c^+)$ or $R[f'(c)]$.

| **Result** | $f(x)$ is differentiable at $x = c \Longleftrightarrow L\big[f'(c)\big] = R\big[f'(c)\big]$ |
|---|---|

**Remarks:**

(i) If $L\big[f'(c)\big] \neq R\big[f'(c)\big]$, then we say that $f(x)$ is not differentiable at $x = c$.

(ii) $f(x)$ is differentiable at $x = c \Rightarrow f(x)$ is continuous at $x = c$.

**DO YOU KNOW?**

A function may be continuous at a point but may not be differentiable at that point.

### Example 5.29

Show that the function $f(x) = |x|$ is not differentiable at $x = 0$.

### *Solution*

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$L[f'(0)] = \lim\limits_{x \to 0^-} \dfrac{f(x) - f(0)}{x - 0},$$

$$= \lim\limits_{h \to 0} \dfrac{f(0 - h) - f(0)}{0 - h - 0}, \quad x = 0 - h$$

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{|-h| - |0|}{-h}$$

$$= \lim_{h \to 0} \frac{|-h|}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h}$$

$$= \lim_{h \to 0} (-1)$$

$$= -1$$

$$R[\, f'(0)] = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{h \to 0} \frac{f(0 + h) - f(0)}{0 + h - 0} \;,\; x = 0 + h$$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|h| - |0|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= \lim_{h \to 0} 1$$

$$= 1$$

Here, $L[\, f'(0)] \neq R[\, f'(0)]$

$\therefore f(x)$ is not differentiable at $x=0$

**Example 5.30**

Show that $f(x) = x^2$ differentiable at $x=1$ and find $f'(1)$.

*Solution*

$$f(x) = x^2$$

$$L[\, f'(1)\,] \;=\; \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{h \to 0} \frac{f(1 - h) - f(1)}{1 - h - 1}$$

$$= \lim_{h \to 0} \frac{(1 - h)^2 - 1}{-h}$$

$$= \lim_{h \to 0} \frac{1 + h^2 - 2h - 1}{-h}$$

$$= \lim_{h \to 0}(-h + 2)$$

$$= 2$$

$$R[\, f'(1)\,] \;=\; \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{h \to 0} \frac{f(1 + h) - f(1)}{1 + h - 1}$$

$$= \lim_{h \to 0} \frac{(1 + h)^2 - 1}{h}$$

$$= \lim_{h \to 0} \frac{1 + h^2 + 2h - 1}{h}$$

$$= \lim_{h \to 0}(h + 2)$$

$$= 2$$

Here, $L[\, f'(1)\,] = R[\, f'(1)\,]$

$\therefore$ $f(x)$ is differentiable at $x = 1$

and  $f'(1) = 2$

## 5.2.12 Differentiation from first principle

The process of finding the derivative of a function by using the definition that

$f'(x) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$  is called the differentiation from first principle.  It is convenient to write $f'(x)$ as $\dfrac{dy}{dx}$ .

### 5.2.13 Derivatives of some standard functions using first principle

1. For $x \in R, \dfrac{d}{dx}(x^n) = n\, x^{n-1}$

   **Proof:**

   Let $\qquad f(x) = x^n$

   then $\qquad f(x+h) = (x+h)^n$

   $$\dfrac{d}{dx}\big(f(x)\big) = \lim_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$$

   $$= \lim_{h \to 0} \dfrac{(x+h)^n - (x)^n}{h}$$

   $$= \lim_{h \to 0} \dfrac{(x+h)^n - (x)^n}{(x+h)-x}$$

   $$= \lim_{z \to x} \dfrac{z^n - x^n}{z - x}, \quad \text{where } z = x + h \text{ and } z \to x \text{ as } h \to 0$$

   $$= nx^{n-1} \qquad \left[\because \lim_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}\right]$$

   i.e., $\quad \dfrac{d}{dx}(x^n) = nx^{n-1}$

2. $\dfrac{d}{dx}(e^x) = e^x$

   **proof:**

   Let $f(x) = e^x$

   then $\qquad f(x+h) = e^{x+h}$

   $$\dfrac{d}{dx}\big(f(x)\big) = \lim_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$$

   $$= \lim_{h \to 0} \dfrac{e^{x+h} - e^x}{h}$$

   $$= \lim_{h \to 0} \dfrac{e^x e^h - e^x}{h}$$

   $$= \lim_{h \to 0} e^x \left[\dfrac{e^h - 1}{h}\right]$$

$$= e^x \lim_{h \to 0} \left[ \frac{e^h - 1}{h} \right]$$

$$= e^x \times 1 \qquad\qquad \left[ \because \lim_{h \to 0} \left[ \frac{e^h - 1}{h} \right] = 1 \right]$$

$$= e^x$$

i.e., $\dfrac{d}{dx}(e^x) = e^x$.

3.  $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$

**Proof:**

Let $\qquad f(x) = \log_e x$

Then, $\quad f(x+h) = \log_e(x+h)$

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\log_e(x+h) - \log_e x}{h}$$

$$= \lim_{h \to 0} \frac{\log_e\left(1 + \dfrac{h}{x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\log_e\left(1 + \dfrac{h}{x}\right)}{\dfrac{h}{x}} \; \frac{1}{x}$$

$$= \frac{1}{x} \lim_{h \to 0} \frac{\log_e\left(1 + \dfrac{h}{x}\right)}{\dfrac{h}{x}}$$

$$= \frac{1}{x} \times 1 \qquad\qquad \left[ \because \lim_{x \to 0} \frac{\log_e(1+x)}{x} = 1 \right]$$

$$= \frac{1}{x}$$

**Example 5.31**

Find $\dfrac{d}{dx}(x^3)$ from first principle.

*Solution*

Let $f(x) = x^3$ then

$$f(x+h) = (x+h)^3$$

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$$

$$= 3x^2 + 3x(0) + (0)^2$$

$$= 3x^2$$

$$\frac{d}{dx}(x^3) = 3x^2$$

### Example 5.32

Find $\frac{d}{dx}(e^{3x})$ from first principle.

*Solution*

Let $f(x) = e^{3x}$

then    $f(x + h) = e^{3(x+h)}$

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{3(x+h)} - e^{3x}}{h}$$

$$= \lim_{h \to 0} \frac{e^{3x} \cdot e^{3h} - e^{3x}}{h}$$

$$= \lim_{h \to 0} \frac{e^{3x}(e^{3h} - 1)}{h}$$

$$= 3e^{3x} \lim_{h \to 0} \frac{e^{3h} - 1}{3h}$$

$$= 3e^{3x} \times 1 \qquad\qquad [\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1]$$

$$= 3e^{3x}$$

**Exercise 5.4**

1.   Find the derivative of the following functions from first principle.

(i) $x^2$      (ii) $e^{-x}$      (iii) $\log(x+1)$

## 5.3  Differentiation techniques

In this section we will discuss about different techniques to obtain the derivatives of the given functions.

### 5.3.1  Some standard results [formulae]

1.   $\dfrac{d}{dx}(x^n) = nx^{n-1}$

2.   $\dfrac{d}{dx}(x) = 1$

3.   $\dfrac{d}{dx}(k) = 0$, $k$ is a constant

4.   $\dfrac{d}{dx}(kx) = k$, $k$ is a constant

5.   $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$

6.   $\dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$

7.   $\dfrac{d}{dx}(e^x) = e^x$

8.   $\dfrac{d}{dx}(e^{ax}) = ae^{ax}$

9.   $\dfrac{d}{dx}(e^{ax+b}) = ae^{ax+b}$

10.   $\dfrac{d}{dx}(e^{-x^2}) = -2x\,e^{-x^2}$

11.   $\dfrac{d}{dx}(a^x) = a^x \log_e a$

12. $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$

13. $\dfrac{d}{dx}\log_e(x+a) = \dfrac{1}{x+a}$

14. $\dfrac{d}{dx}(\log_e(ax+b)) = \dfrac{a}{(ax+b)}$

15. $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x\log_e a}$

16. $\dfrac{d}{dx}(\sin x) = \cos x$

17. $\dfrac{d}{dx}(\cos x) = -\sin x$

18. $\dfrac{d}{dx}(\tan x) = \sec^2 x$

19. $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

20. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

21. $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

## 5.3.2   General rules for differentiation

**(i)   Addition rule**

$$\dfrac{d}{dx}\left[f(x)+g(x)\right] = \dfrac{d}{dx}\left[f(x)\right] + \dfrac{d}{dx}\left[g(x)\right]$$

**(ii)   Subtraction rule (or) Difference rule**

$$\dfrac{d}{dx}\left[f(x)-g(x)\right] = \dfrac{d}{dx}\left[f(x)\right] - \dfrac{d}{dx}\left[g(x)\right]$$

**(iii)   Product rule**

$$\dfrac{d}{dx}\left[f(x)\cdot g(x)\right] = \dfrac{d}{dx}\left[f(x)\right]g(x) + f(x)\dfrac{d}{dx}\left[g(x)\right]$$

(or)  If $u = f(x)$ and $v = g(x)$ then, $\dfrac{d}{dx}(uv) = u\dfrac{d}{dx}(v) + v\dfrac{d}{dx}(u)$

**(iv)   Quotient rule**

$$\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\dfrac{d}{dx}\left[f(x)\right] - f(x)\dfrac{d}{dx}\left[g(x)\right]}{\left[g(x)\right]^2}$$

(or) If $u = f(x)$ and $v = g(x)$ then, $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{d}{dx}(u) - u\dfrac{d}{dx}(v)}{v^2}$

**(v) Scalar product**

$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$, where $c$ is a constant.

**(vi) Chain rule:**

$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

(or) If $y = f(t)$ and $t = g(x)$ then, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

Here we discuss about the differentiation for the explicit functions using standard results and general rules for differentiation.

## Example 5.33

Differentiate the following functions with respect to $x$.

(i) $x^{\frac{3}{2}}$ 

(ii) $7e^x$ 

(iii) $\frac{1-3x}{1+3x}$

(iv) $x^2 \sin x$ 

(v) $\sin^3 x$ 

(vi) $\sqrt{x^2 + x + 1}$

*Solution*

(i) $\quad \frac{d}{dx}\left(x^{\frac{3}{2}}\right) = \frac{3}{2}x^{\frac{3}{2}-1}$

$\qquad\qquad = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

(ii) $\quad \frac{d}{dx}(7e^x) = 7\frac{d}{dx}(e^x) = 7e^x$

(iii) Differentiating $y = \frac{1-3x}{1+3x}$ with respect to $x$

$\quad \frac{dy}{dx} = \frac{(1+3x)\frac{d}{dx}(1-3x)-(1-3x)\frac{d}{dx}(1+3x)}{(1+3x)^2}$

$\qquad = \frac{(1+3x)(-3)-(1-3x)(3)}{(1+3x)^2} = \frac{-6}{(1+3x)^2}$

(iv) Differentiating $y = x^2 \sin x$ with respect to $x$

$\quad \frac{dy}{dx} = x^2\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(x^2)$

$\qquad = x^2 \cos x + 2x \sin x$

$\qquad = x(x\cos x + 2\sin x)$

(v)  $y = \sin^3 x$  (or) $(\sin x)^3$

Let $u = \sin x$ then $y = u^3$

Differentiating $u = \sin x$ with respect to $x$

We get, $\dfrac{du}{dx} = \cos x$

Differentiating $y = u^3$ with respect to $u$

We get, $\dfrac{dy}{du} = 3u^2 = 3\sin^2 x$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = 3\sin^2 x \cdot \cos x$

(vi)  $y = \sqrt{x^2 + x + 1}$

Let $u = x^2 + x + 1$ then $y = \sqrt{u}$

Differentiating $u = x^2 + x + 1$ with respect to $x$

We get, $\dfrac{du}{dx} = 2x + 1$

Differentiating $y = \sqrt{u}$ with respect to $u$.

We get, $\dfrac{dy}{du} = \dfrac{1}{2\sqrt{u}}$

$= \dfrac{1}{2\sqrt{x^2 + x + 1}}$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \dfrac{1}{2\sqrt{x^2 + x + 1}} \cdot (2x + 1)$

$= \dfrac{2x + 1}{2\sqrt{x^2 + x + 1}}$

**DO YOU KNOW?**

★ $\dfrac{d}{dx}(\sin(\log x)) = \dfrac{d}{d(\log x)}(\sin(\log x))\dfrac{d}{dx}(\log x)$

★ $\dfrac{d}{dx}(x^2 + 3x + 1)^5 = \dfrac{d}{d(x^2 + 3x + 1)}(x^2 + 3x + 1)^5 \dfrac{d}{dx}(x^2 + 3x + 1)$

★ $\dfrac{d}{dx}(e^{(x^3 + 3)}) = \dfrac{d}{d(x^3 + 3)}(e^{(x^3 + 3)})\dfrac{d}{dx}(x^3 + 3)$

**Example 5.34**

If $f(x) = x^n$ and $f'(1) = 5$, then find the value of $n$.

*Solution*

$$f(x) = x^n$$

$$\therefore \qquad f'(x) = nx^{n-1}$$

$$f'(1) = n(1)^{n-1}$$

$$f'(1) = n$$

$$f'(1) = 5 \text{ (given)}$$

$$\implies \qquad n = 5$$

**Example 5.35**

If $y = \dfrac{1}{u^2}$ and $u = x^2 - 9$, find $\dfrac{dy}{dx}$.

*Solution*

$$y = \frac{1}{u^2} = u^{-2}$$

$$\frac{dy}{du} = -\frac{2}{u^3}$$

$$\frac{dy}{du} = -\frac{2}{(x^2 - 9)^3} \qquad\qquad (\because u = x^2 - 9)$$

$$u = x^2 - 9$$

$$\frac{du}{dx} = 2x$$

Now, $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= -\frac{2}{(x^2 - 9)^3} \cdot 2x$$

$$= -\frac{4x}{(x^2 - 9)^3}.$$
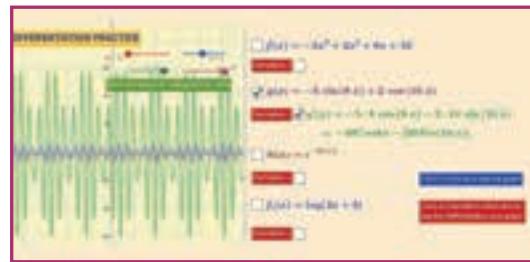
**Expected final outcomes**



**Step – 1**

Open the Browser and type the URL given (or) Scan the QR Code.

GeoGebra Work book called "**11<sup>th</sup> BUSINESS MATHS**" will appear. In this several work sheets for Business Maths are given, Open the worksheet named "**Elementary Differentiation**"

**Step - 2**

Elementary Differentiation and Functions page will open. Click on the function check boxes on the Right-hand side to see the respective graph. You can move the sliders a, b, c and d to change the co-efficient and work out the differentiation. Then click the Derivatives check box to see the answer and respective graphs

| Step 1 |
| --- |



| Step 2 |
| --- |



Browse in the link

11<sup>th</sup> Business Maths: https://ggbm.at/qKj9gSTG  (or) scan the QR Code



B184_11_BM_EM

1. Differentiate the following with respect to $x$.

   (i) $3x^4 - 2x^3 + x + 8$

   (ii) $\dfrac{5}{x^4} - \dfrac{2}{x^3} + \dfrac{5}{x}$

   (iii) $\sqrt{x} + \dfrac{1}{\sqrt[3]{x}} + e^x$

   (iv) $\dfrac{3 + 2x - x^2}{x}$

   (v) $x^3 e^x$

   (vi) $(x^2 - 3x + 2)(x + 1)$

   (vii) $x^4 - 3\sin x + \cos x$

   (viii) $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2$

2. Differentiate the following with respect to $x$.

   (i) $\dfrac{e^x}{1 + x}$

   (ii) $\dfrac{x^2 + x + 1}{x^2 - x + 1}$

   (iii) $\dfrac{e^x}{1 + e^x}$

3. Differentiate the following with respect to $x$.

   (i) $x \sin x$

   (ii) $e^x \sin x$

   (iii) $e^x(x + \log x)$

   (iv) $\sin x \cos x$

   (v) $x^3 e^x$

4. Differentiate the following with respect to $x$.

   (i) $\sin^2 x$

   (ii) $\cos^2 x$

   (iii) $\cos^3 x$

   (iv) $\sqrt{1 + x^2}$

   (v) $(ax^2 + bx + c)^n$

   (vi) $\sin(x^2)$

   (vii) $\dfrac{1}{\sqrt{1 + x^2}}$

## 5.3.3 Derivative of implicit functions

For the implicit function $f(x,y) = 0$, differentiate each term with respect to $x$ treating $y$ as a function of $x$ and then collect the terms of $\dfrac{dy}{dx}$ together on left hand side and remaining terms on the right hand side and then find $\dfrac{dy}{dx}$.

**NOTE**

If the function $f(x, y) = 0$ is an implicit function, then $\dfrac{dy}{dx}$ contains both $x$ and $y$.

**Example 5.36**

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ then find $\dfrac{dy}{dx}$.

*Solution*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

IBFYV9

Differentiating both side with respect to $x$,

$$2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$(2ax + 2hy + 2g) + (2hx + 2by + 2f)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{(ax + hy + g)}{(hx + by + f)}.$$

**Example 5.37**

If $x^3 + y^3 = 3axy$, then find $\dfrac{dy}{dx}$ .

*Solution*

$$x^3 + y^3 = 3axy$$

Differentiating both sides with respect to $x$,

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(3axy)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$$

$$x^2 + y^2\frac{dy}{dx} = ax\frac{dy}{dx} + ay$$

$$(y^2 - ax)\frac{dy}{dx} = (ay - x^2)$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

**Example 5.38**

Find $\dfrac{dy}{dx}$ at $(1,1)$ to the curve $2x^2 + 3xy + 5y^2 = 10$

*Solution*

$$2x^2 + 3xy + 5y^2 = 10$$

Differentiating both sides with respect to $x$,

$$\frac{d}{dx}\left[2x^2 + 3xy + 5y^2\right] = \frac{d}{dx}[10]$$

$$4x + 3x\frac{dy}{dx} + 3y + 10y\frac{dy}{dx} = 0$$

$$(3x + 10y)\frac{dy}{dx} = -3y - 4x$$

$$\frac{dy}{dx} = -\frac{(3y+4x)}{(3x+10y)}$$

Now,  $\quad \frac{dy}{dx}$ at $(1, 1) = -\frac{3+4}{3+10}$

$$= -\frac{7}{13}$$

## Example 5.39

If $\sin y = x \sin (a+y)$, then prove that $\dfrac{dy}{dx} = \dfrac{\sin^2 (a+y)}{\sin a}$.

*Solution*

$$\sin y = x \sin (a+y)$$

$$x = \frac{\sin y}{\sin(a+y)}$$

Differentiating with respect to $y$,

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y \cdot \cos(a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\sin^2(a+y)}$$

$$= \frac{\sin a}{\sin^2(a+y)}$$

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

## Exercise 5.6

1. Find $\dfrac{dy}{dx}$ for the following functions

(i) $xy = \tan(xy)$  (ii) $x^2 - xy + y^2 = 7$  (iii) $x^3 + y^3 + 3axy = 1$

2. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ and $x \neq y$, then prove that $\dfrac{dy}{dx} = -\dfrac{1}{(x+1)^2}$

3. If $4x + 3y = \log(4x - 3y)$, then find $\dfrac{dy}{dx}$

### 5.3.4 Logarithmic differentiation

Some times, the function whose derivative is required involves products, quotients, and powers. For such cases, differentiation can be carried out more conveniently if we take logarithms and simplify before differentiation.

## Example 5.40

Differentiate the following with respect to $x$.

(i) $x^x$        (ii) $(\log x)^{\cos x}$

*Solution*

(i)    Let     $y = x^x$

Taking logarithm on both sides

$$\log y = x \log x$$

Differentiating with respect to $x$,

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\frac{dy}{dx} = y[1 + \log x]$$

$$\frac{dy}{dx} = x^x[1 + \log x]$$

(ii)   Let     $y = (\log x)^{\cos x}$

Taking logarithm on both sides

$$\therefore \quad \log y = \cos x \ \log(\log x)$$

Differentiating with respect to $x$,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \cos x \frac{1}{\log x} \cdot \frac{1}{x} + \left[\log(\log x)\right](-\sin x)$$

$$\frac{dy}{dx} = y\left[\frac{\cos x}{x \log x} - \sin x \ \log(\log x)\right]$$

$$= (\log x)^{\cos x}\left[\frac{\cos x}{x \log x} - \sin x \ \log(\log x)\right]$$

## Example 5.41

If $x^y = e^{x-y}$, prove that $\dfrac{dy}{dx} = \dfrac{\log x}{(1 + \log x)^2}$

*Solution*

$$x^y = e^{x-y}$$

Taking logarithm on both sides,

$$y \log x = (x-y)$$

$$y(1+\log x) = x$$

$$y = \frac{x}{1+\log x}$$

Differentiating with respect to $x$

$$\frac{dy}{dx} = \frac{(1+\log x)(1) - x\left(0 + \frac{1}{x}\right)}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

**Example 5.42**

Differentiate: $\sqrt{\dfrac{(x-3)(x^2+4)}{3x^2+4x+5}}$

*Solution*

Let $\quad y = \sqrt{\dfrac{(x-3)(x^2+4)}{3x^2+4x+5}} = \left[\dfrac{(x-3)(x^2+4)}{3x^2+4x+5}\right]^{\frac{1}{2}}$

Taking logarithm on both sides,

$$\log y = \frac{1}{2}\left[\log(x-3) + \log(x^2+4) - \log(3x^2+4x+5)\right]$$

$$\left[\because \log ab = \log a + \log b \quad \text{and} \quad \log \frac{a}{b} = \log a - \log b\right]$$

Differentiating with respect to $x$,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

**Exercise 5.7**

1. Differentiate the following with respect to $x$,

   (i) $x^{\sin x}$      (ii) $(\sin x)^x$      (iii) $(\sin x)^{\tan x}$      (iv) $\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x^2+x+1)}}$

2. If $x^m \cdot y^n = (x+y)^{m+n}$, then show that $\dfrac{dy}{dx} = \dfrac{y}{x}$

## 5.3.5 Differentiation of parametric functions

If the variables $x$ and $y$ are functions of another variable namely $t$, then the functions are called a parametric functions. The variable $t$ is called the parameter of the function.

If $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

## 5.3.6 Differentiation of a function with respect to another function

Let $u = f(x)$ and $v = g(x)$ be two functions of $x$. The derivative of $f(x)$ with respect to $g(x)$ is given by the formula,

$$\frac{d(f(x))}{d(g(x))} = \frac{du/dx}{dv/dx}$$

**Example 5.43**

Find $\dfrac{dy}{dx}$  if  (i) $x = at^2$, $y = 2at$        (ii) $x = a\cos\theta$, $y = a\sin\theta$

*Solution*

(i)     $x = at^2$          $y = 2at$

$$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

$$\therefore \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{2a}{2at}$$

$$= \frac{1}{t}$$

(ii)     $x = a\cos\theta$, $y = a\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = a\cos\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{a\cos\theta}{-a\sin\theta}$$

$$= -\cot\theta$$

**Example 5.44**

If $x = a\theta$ and $y = \dfrac{a}{\theta}$, then prove that $\dfrac{dy}{dx} + \dfrac{y}{x} = 0$

*Solution*

$$x = a\theta \qquad\qquad\qquad y = \frac{a}{\theta}$$

$$\frac{dx}{d\theta} = a \qquad\qquad\qquad \frac{dy}{d\theta} = \frac{-a}{\theta^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{\left(\frac{-a}{\theta^2}\right)}{a}$$

$$= -\frac{1}{\theta^2}$$

$$= -\frac{y}{x}$$

i.e. $\quad \dfrac{dy}{dx} + \dfrac{y}{x} = 0$

**Aliter :**

Take $\quad xy = a\theta \cdot \dfrac{a}{\theta}$

$$xy = a^2$$

Differentiating with respect to $x$,

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} + \frac{y}{x} = 0$$

**Example 5.45**

Differentiate $\dfrac{x^2}{1 + x^2}$ with respect to $x^2$

## Solution

Let    $u = \dfrac{x^2}{1+x^2}$

$\dfrac{du}{dx} = \dfrac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2}$

$= \dfrac{2x}{(1+x^2)^2}$

$v = x^2$

$\dfrac{dv}{dx} = 2x$

$\dfrac{du}{dv} = \dfrac{\left(\dfrac{du}{dx}\right)}{\left(\dfrac{dv}{dx}\right)}$

$= \dfrac{\left[\dfrac{2x}{(1+x^2)^2}\right]}{2x}$

$= \dfrac{1}{(1+x^2)^2}$

**Exercise 5.8**

1. Find $\dfrac{dy}{dx}$ of the following functions

   (i) $x = ct, \ y = \dfrac{c}{t}$

   (ii) $x = \log t, \ y = \sin t$

   (iii) $x = a\cos^3\theta, \ y = a\sin^3\theta$

   (iv) $x = a(\theta - \sin\theta), \ y = a(1 - \cos\theta)$

2. Differentiate $\sin^3 x$ with respect to $\cos^3 x$

3. Differentiate $\sin^2 x$ with respect to $x^2$

### 5.3.7   Successive differentiation

The process of differentiating the same function again and again is called successive differentiation.

(i) The derivative of $y$ with respect to $x$ is called the first order derivative and is denoted by $\dfrac{dy}{dx}$ (or) $y_1$ (or) $f'(x)$

(ii) If $f'(x)$ is differentiable, then the derivative of $f'(x)$ with respect to $x$ is called the second order derivative and is denoted by $\dfrac{d^2y}{dx^2}$ (or) $y_2$ (or) $f''(x)$

(iii) Further $\dfrac{d^n y}{dx^n}$ (or) $y_n$ (or) $f^{(n)}(x)$ is called $n^{\text{th}}$ order derivative of the function

$y = f(x)$

**Example 5.46**

Find the second order derivative of the following functions with respect to $x$,

(i) $3\cos x + 4\sin x$      (ii) $x = at^2,\ \ y = 2at$      (iii) $x\sin x$

*Solution*

(i)    Let     $y = 3\cos x + 4\sin x$

$\qquad\qquad y_1 = -3\sin x + 4\cos x$

$\qquad\qquad y_2 = -3\cos x - 4\sin x$

$\qquad\qquad y_2 = -3(\cos x + 4\sin x)$

$\qquad\qquad y_2 = -y\ (\text{or})\ y_2 + y = 0$

(ii)    $x = at^2,\qquad\qquad y = 2at$

$\qquad \dfrac{dx}{dt} = 2at \qquad\qquad \dfrac{dy}{dt} = 2a$

$\qquad\qquad \dfrac{dy}{dx} = \dfrac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

$\qquad\qquad\quad = \dfrac{1}{t}$

Now, $\qquad \dfrac{d^2 y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$

$\qquad\qquad\qquad = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\dfrac{dt}{dx}$

$\qquad\qquad\qquad = \dfrac{d}{dt}\left(\dfrac{1}{t}\right)\cdot\dfrac{1}{2at}$

$\qquad\qquad\qquad = -\dfrac{1}{t^2}\cdot\dfrac{1}{2at}$

$\qquad\qquad\qquad = -\dfrac{1}{2at^3}$

(iii) $\qquad y = x \sin x$

$$\frac{dy}{dx} = x \cos x + \sin x$$

$$\frac{d^2 y}{dx^2} = -x \sin x + \cos x + \cos x$$

$$= 2 \cos x - x \sin x$$

## Example 5.47

If $y = A \sin x + B \cos x$, then prove that $y_2 + y = 0$

### Solution

$$y = A \sin x + B \cos x$$

$$y_1 = A \cos x - B \sin x$$

$$y_2 = -A \sin x - B \cos x$$

$$y_2 = -y$$

$$y_2 + y = 0$$

**Exercise 5.9**

1. Find $y_2$ for the following functions

   (i) $y = e^{3x+2}$ \qquad (ii) $y = \log x + a^x$ \qquad (iii) $x = a \cos \theta, \; y = a \sin \theta$

2. If $y = 500 e^{7x} + 600 e^{-7x}$, then show that $y_2 - 49y = 0$

3. If $y = 2 + \log x$, then show that $xy_2 + y_1 = 0$

4. If $y = a \cos mx + b \sin mx$, then show that $y_2 + m^2 y = 0$

5. If $y = (x + \sqrt{1+x^2})^m$, then show that $(1 + x^2)y_2 + xy_1 - m^2 y = 0$

6. If $y = \sin(\log x)$, then show that $x^2 y_2 + xy_1 + y = 0$.

**Choose the correct answer**
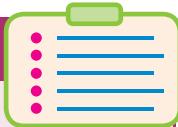
1.  If $f(x) = x^2 - x + 1$ then $f(x+1)$ is

    (a) $x^2$      (b) $x$      (c) 1      (d) $x^2 + x + 1$

2.  Let $f(x) = \begin{cases} x^2 - 4x & \text{if } x \geq 2 \\ x + 2 & \text{if } x < 2 \end{cases}$ then $f(5)$ is

    (a) −1      (b) 2      (c) 5      (d) 7

3.  For $f(x) = \begin{cases} x^2 - 4x & \text{if } x \geq 2 \\ x + 2 & \text{if } x < 2 \end{cases}$ then, $f(0)$ is

    (a) 2      (b) 5      (c) −1      (d) 0

4.  If $f(x) = \dfrac{1-x}{1+x}$ then $f(-x)$ is equal to

    (a) $-f(x)$      (b) $\dfrac{1}{f(x)}$      (c) $-\dfrac{1}{f(x)}$      (d) $f(x)$

5.  The graph of the line $y = 3$ is

    (a) Parallel to $x$-axis      (b) Parallel to $y$-axis

    (c) Passing through the origin      (d) Perpendicular to $x$-axis

6.  The graph of $y = 2x^2$ is passing through

    (a) (0,0)      (b) (2,1)      (c) (2,0)      (d) (0,2)

7.  The graph of $y = e^x$ intersect the $y$-axis at

    (a) (0, 0)      (b) (1, 0)      (c) (0, 1)      (d) (1, 1)

8.  The minimum value of the function $f(x) = |x|$ is

    (a) 0      (b) −1      (c) +1      (d) $-\infty$

9.  Which one of the following functions has the property $f(x) = f\left(\dfrac{1}{x}\right)$

    (a) $f(x) = \dfrac{x^2 - 1}{x}$    (b) $f(x) = \dfrac{1 - x^2}{x}$    (c) $f(x) = x$    (d) $f(x) = \dfrac{x^2 + 1}{x}$

10. If $f(x) = 2^x$ and $g(x) = \dfrac{1}{2^x}$ then $(fg)(x)$ is

    (a) 1      (b) 0      (c) $4^x$      (d) $\dfrac{1}{4^x}$

11. Which of the following function is neither even nor odd?

(a) $f(x) = x^3 + 5$    (b) $f(x) = x^5$      (c) $f(x) = x^{10}$      (d) $f(x) = x^2$

12. $f(x) = -5$, for all $x \in R$ is a

(a) an identity function          (b) modulus function

(c) exponential function         (d) constant function

13. The range of $f(x) = |x|$, for all $x \in R$ is

(a) $(0, \infty)$      (b) $[0, \infty)$      (c) $(-\infty, \infty)$      (d) $[1, \infty)$

14. The graph of $f(x) = e^x$ is identical to that of

(a) $f(x) = a^x, a > 1$          (b) $f(x) = a^x, a < 1$

(c) $f(x) = a^x, 0 < a < 1$       (d) $y = ax + b, a \neq 0$

15. If $f(x) = x^2$ and $g(x) = 2x+1$ then $(fg)(0)$ is

(a) 0      (b) 2      (c) 1      (d) 4

16. $\lim\limits_{\theta \to 0} \dfrac{\tan \theta}{\theta} =$

(a) 1      (b) $\infty$      (c) $-\infty$      (d) $\theta$

17. $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} =$

(a) $e$      (b) $nx^{n-1}$      (c) 1      (d) 0

18. For what value of $x$, $f(x) = \dfrac{x+2}{x-1}$ is not continuous?

(a) –2      (b) 1      (c) 2      (d) –1

19. A function $f(x)$ is continuous at $x = a$ if $\lim\limits_{x \to a} f(x)$ is equal to

(a) $f(-a)$      (b) $f\left(\dfrac{1}{a}\right)$      (c) $2f(a)$      (d) $f(a)$

20. $\dfrac{d}{dx}\left(\dfrac{1}{x}\right)$ is equal to

(a) $-\dfrac{1}{x^2}$      (b) $-\dfrac{1}{x}$      (c) $\log x$      (d) $\dfrac{1}{x^2}$

21. $\dfrac{d}{dx}(5e^x - 2\log x)$ is equal to

(a) $5e^x - \dfrac{2}{x}$  (b) $5e^x - 2x$  (c) $5e^x - \dfrac{1}{x}$  (d) $2\log x$

22. If $y = x$ and $z = \dfrac{1}{x}$ then $\dfrac{dy}{dz} =$

(a) $x^2$  (b) $1$  (c) $-x^2$  (d) $-\dfrac{1}{x^2}$

23. If $y = e^{2x}$ then $\dfrac{d^2y}{dx^2}$ at $x = 0$ is

(a) $4$  (b) $9$  (c) $2$  (d) $0$

24. If $y = \log x$ then $y_2 =$

(a) $\dfrac{1}{x}$  (b) $-\dfrac{1}{x^2}$  (c) $-\dfrac{2}{x^2}$  (d) $e^2$

25. $\dfrac{d}{dx}(a^x) =$

(a) $\dfrac{1}{x\log_e a}$  (b) $a^a$  (c) $x\log_e a$  (d) $a^x\log_e a$

## Miscellaneous Problems

1. If $f(x) = \dfrac{1}{2x+1}$, $x \neq -\dfrac{1}{2}$, then show that $f(f(x)) = \dfrac{2x+1}{2x+3}$, provided $x \neq -\dfrac{3}{2}$

2. Draw the graph of $y = 9 - x^2$

3. If $f(x) = \begin{cases} \dfrac{x-|x|}{x} & if\ x \neq 0 \\ 2 & if\ x = 0 \end{cases}$ then show that $\lim\limits_{x\to 0} f(x)$ does not exist.

4. Evaluate: $\lim\limits_{x\to 1} \dfrac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$

5. Show that the function $f(x) = 2x - |x|$ is continuous at $x = 0$

6. Verify the continuity and differentiability of $f(x) = \begin{cases} 1-x & if\ x < 1 \\ (1-x)(2-x) & if\ 1 \le x \le 2 \\ 3-x & if\ x > 2 \end{cases}$ at $x = 1$ and $x = 2$

7. If $x^y = y^x$, then prove that $\dfrac{dy}{dx} = \dfrac{y}{x}\left(\dfrac{x\log y - y}{y\log x - x}\right)$

8. If $xy^2 = 1$, then prove that $2\dfrac{dy}{dx} + y^3 = 0$.

9. If $y = \tan x$, then prove that $y_2 - 2yy_1 = 0$.

10. If $y = 2\sin x + 3\cos x$, then show that $y_2 + y = 0$.

# Summary

- Let $A$ and $B$ be two non empty sets, then a function $f$ from $A$ to $B$, associates every element of $A$ to an unique element of $B$.

- $\lim\limits_{x \to a} f(x)$ exists $\Leftrightarrow$ $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$.

- For the function $f(x)$ and a real number $a$, $\lim\limits_{x \to a} f(x)$ and $f(a)$ may not be the same.

- A function $f(x)$ is a continuous function at $x = a$ if only if $\lim\limits_{x \to a} f(x) = f(a)$.

- A function $f(x)$ is continuous, if it is continuous at every point of its domain.

- If $f(x)$ and $g(x)$ are continuous on their common domain, then $f \pm g$, $f \cdot g$, $kf$ ($k$ is a constant) are continuous and if $g \neq 0$ then $\dfrac{f}{g}$ is also continuous.

- A function $f(x)$ is differentiable at $x = c$ if and only if $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists finitely and denoted by $f'(c)$.

- $L[f'(c)] = \lim\limits_{h \to 0} \dfrac{f(c - h) - f(c)}{-h}$ and $R[f'(c)] = \lim\limits_{h \to 0} \dfrac{f(c + h) - f(c)}{h}$

- A function $f(x)$ is said to be differentiable at $x = c$ if and only if $L[f'(c)] = R[f'(c)]$.

- Every differentiable function is continuous but, the converse is not necessarily true.

- If $y = f(x)$ then $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ is called second order derivative of y with respect to $x$.

- If $x = f(t)$ and $y = g(t)$ then $\dfrac{d^2 y}{dx^2} = \dfrac{d}{dx}\left\{\dfrac{g'(t)}{f'(t)}\right\}$ (or) $\dfrac{d^2 y}{dx^2} = \dfrac{d}{dt}\left\{\dfrac{g'(t)}{f'(t)}\right\} \cdot \dfrac{dt}{dx}$

| GLOSSARY | |
|---|---|
| Absolute constants | முழுமையான மாறிலிகள் |
| Algebric functions | இயற்கணித சார்புகள் |
| Arbitrary constants | தன்னிச்சை மாறிலிகள் |
| Chain rule | சங்கிலி விதி |
| Closed interval | மூடிய இடைவெளி |
| Constant | மாறிலி |
| Continuous | தொடர்ச்சி |
| Dependent variable | சார்ந்த மாறி |
| Derivative | வகையீடு |
| Domain | சார்பகம் / அரங்கம் |
| Explict | வெளிபடு |
| Exponential | அடுக்கு |
| Function | சார்பு |
| Identity | சமனி |
| Implict | உட்படு |
| Independent variable | சாரா மாறி |
| Interval | இடைவெளி |
| Left limit | இடக்கை எல்லை |
| Limit | எல்லை |
| Logarithmic | மடக்கை |
| Modulus | மட்டு |
| Neighbourhood | அண்மையகம் |
| Open interval | திறந்த இடைவெளி |
| Parametric functions | துணையலகு சார்புகள் |
| Range | வீச்சகம் |
| Right limit | வலக்கை எல்லை |
| Signum function | குறிச் சார்பு |
| Successive differentiation | தொடர் வகையிடல் |
| Transcendental functions | விஞ்சிய சார்புகள் |
| Variable | மாறி |

# ANSWERS

## 1. MATRICES AND DETERMINANTS

### Exercise 1.1

1.(i)  $M_{11} =- 1$   $M_{12} = 0$   $M_{21} = 20$   $M_{22} = 5$

$A_{11} =- 1$   $A_{12} = 0$   $A_{21} =- 20$   $A_{22} = 5$

(ii)  $M_{11} =- 12$   $M_{12} = 2$   $M_{13} = 23$   $M_{21} =- 16$   $M_{22} =- 4$   $M_{23} = 14$

$M_{31} =- 4$   $M_{32} =- 6$   $M_{33} = 11$

$A_{11} =- 12$   $A_{12} =- 2$   $A_{13} = 23$   $A_{21} = 16$   $A_{22} =- 4$   $A_{23} =- 14$

$A_{31} =- 4$   $A_{32} = 6$   $A_{33} = 11$

2.  20          3.  $\dfrac{1}{2}$          4.  −30       5.  $\dfrac{13}{2}, 2$     6. 0

### Exercise 1.2

1.  $\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$     2.  $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$   3.(i)  $\dfrac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$     (ii)  $\dfrac{1}{10}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

(iii)  $\dfrac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$          (iv)  $\dfrac{1}{151}\begin{bmatrix} -22 & -46 & -7 \\ -13 & 14 & -11 \\ 5 & -17 & -19 \end{bmatrix}$

7.  $\dfrac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$   10.  $\lambda = \dfrac{7}{4}$          11.  $p = 2, q =- 3$

### Exercise 1.3

1.  $x = 1, y = 1$          2.(i)  $x = 3, y =- 2, z = 1$

(ii)  $x =- 1, y = 2, z = 3$   (iii)  $x = 1, y =- 1, z = 2$     3. 17.95, 43.08, 103.85

4.  3000, 1000, 2000                5.  13,2,5                6.  700, 600, 300

## Exercise 1.4

1. It is viable     2. It is not viable     3. It is viable

4. $A = 250.36$ tonnes  $B = 62.44$ tonnes   5. 181.62, 84.32

6. 34.16, 17.31,     7. 42 and 78

## Exercise 1.5

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|----|----|----|----|
| (b) | (d) | (b) | (b) | (c) | (c) | (c) | (c) | (c) | (d) | (c) | (b) | (a) |

| 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|----|----|----|----|----|----|----|----|----|----|----|----|
| (c) | (c) | (c) | (b) | (a) | (d) | (b) | (b) | (d) | (d) | (b) | (a) |

## Miscellaneous Problems

1. $x = 3. x = -1$     2. 0     6. $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

7. $\dfrac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$     8. $x = 1, y = 2, z = 3$     9. $x = 20, y = 30, z = 50$

10. ₹1200 Crores, ₹1600 Crores

# 2. ALGEBRA

## Exercise 2.1

1. $\dfrac{13}{x-2} - \dfrac{10}{x-1}$

2. $\dfrac{3}{x-2} + \dfrac{1}{x+1}$

3. $\dfrac{1}{9(x-1)} - \dfrac{1}{9(x+2)} - \dfrac{1}{3(x+2)^2}$

4. $\dfrac{1}{2(x-1)} - \dfrac{1}{2(x+1)}$

5. $\dfrac{-4}{9(x+2)} + \dfrac{4}{9(x-1)} - \dfrac{1}{3(x-1)^2}$

6. $\dfrac{2}{x-2} + \dfrac{3}{(x-2)^2} - \dfrac{9}{(x-2)^2}$

7. $\dfrac{-7}{2x} + \dfrac{1}{x^2} + \dfrac{9}{2(x+2)}$

8. $\dfrac{1}{5(x+2)} + \dfrac{4(x-2)}{5(x^2+1)}$

9. $\dfrac{3}{16(x-1)} - \dfrac{3}{16(x+3)} + \dfrac{1}{4(x+3)^2}$

10. $\dfrac{1-x}{5(x^2+4)} + \dfrac{1}{5(x+1)}$

## Exercise 2.2

1. 64     2. 10     3. 3     4. 336     5. 85

## Exercise 2.3

1. 6     2. 14400     3. 1680     4. $\dfrac{13!}{4!\,3!\,2!\,2!}$

5. (a) $\dfrac{7!}{2}$     (b) 7!     6. Rank of the word CHAT = 9.

# Exercise 2.4

1. 4

2. $8C_4 + 8C_3 = 9C_4 = 126$

3. 210 cards

4. 20

5. 25200

6. 15

7. $\dfrac{13!}{7!\,6!} \times 9! \times 9!$

8. 11

9.(i) 1365

(ii) 1001

(iii) 364

10.(i) 186

(ii) 186

# Exercise 2.6

1.(i) $16a^4 - 96a^3 b + 216a^2 b^2 - 216ab^3 + 81b^4$

(ii) $x^7 + \dfrac{7x^6}{y} + \dfrac{21x^5}{y^2} + \dfrac{35x^4}{y^3} + \dfrac{35x^3}{y^4} + \dfrac{21x^2}{y^5} + \dfrac{7x}{y^6} + \dfrac{1}{y^7}$

(iii) $x^6 + 6x^3 + 15 + \dfrac{20}{x^3} + \dfrac{15}{x^6} + \dfrac{6}{x^9} + \dfrac{1}{x^{12}}$

2.(i) 10, 40,60,401

(ii) 995009990004999

3. 11440 $x^9 y^4$

4.(i) $11C_5 x$, $\dfrac{11C_5}{x}$

(ii) $\dfrac{8C_4(81)}{16} x^{12}$

(iii) $\dfrac{-10C_5(6)^5}{x^5}$

5.(i) $9C_6 \dfrac{2^6}{3^6}$

(ii) $-32(15C_5)$

(iii) 7920.

# Exercise 2.7

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|----|----|----|----|
| (d) | (c) | (d) | (c) | (a) | (c) | (d) | (b) | (b) | (c) | (b) | (d) | (b) |

| 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|----|----|----|----|----|----|----|----|----|----|----|----|
| (a) | (c) | (a) | (c) | (a) | (c) | (a) | (b) | (c) | (d) | (b) | (a) |

# Miscellaneous Problems

1. $\dfrac{3}{x-1} + \dfrac{2}{x+3}$

2. $\dfrac{3}{x-1} - \dfrac{2}{x-2}$

3. $\dfrac{1}{x+2} + \dfrac{5}{x-3} - \dfrac{3}{(x-3)^2}$

4. $\dfrac{3}{x-2} + \dfrac{2x-1}{x^2-x+1}$

5.(i) 7

(ii) 336

(iii) 84

6.(a) 720

(b) 7776

(c) 120

(d) 6

7. 74 ways

8. 85 ways

9. 210

10. 544

# 3. ANALYTICAL GEOMETRY

## Exercise 3.1

1. $x^2 - 2x - 6y + 10 = 0$
2. $x^2 + y^2 - 6x + 4y - 3 = 0$
3. $3x^2 + 3y^2 - 4x - 14y + 15 = 0$
4. $\left(\frac{15}{2},\ 0\right)$
5. $2x - 3y + 21 = 0$

## Exercise 3.2

1. Angle between the line is $45°$
2. 4 units
4. $a = 5$
5. $y = 1500x + 100000$ , cost of 95 TV sets ₹ 2,42,500

## Exercise 3.3

1. $a = 6,\ c = 6$
2. $2x - y + 2 = 0$ and $6x - 2y + 1 = 0$
3. $2x + 3y - 1 = 0$ and $2x + 3y - 2 = 0$
4. $\theta = \tan^{-1}(7)$

## Exercise 3.4

1.(i) $x^2 + y^2 - 6x - 10y + 9 = 0$
(ii) $x^2 + y^2 - 4 = 0$

2.(i) $C(0, 0),\quad r = 4$
(ii) $C(11, 2),\quad r = 10$
(iii) $C\left(\frac{-2}{5}, \frac{4}{5}\right),\ r = 2$
(iv) $C\left(\frac{3}{2}, \frac{3}{2}\right),\ r = \dfrac{5}{\sqrt{2}}$

3. $x^2 + y^2 + 6x + 4y - 3 = 0$
4. $x^2 + y^2 - 4x - 6y + 13 = 0$
5. $x^2 + y^2 - 5x - 2y + 1 = 0$
6. $x^2 + y^2 - x - y = 0$
7. $x^2 + y^2 - 16x + 4y - 32 = 0$
8. $x^2 + y^2 - 2x - 12y + 27 = 0$
9. $x^2 + y^2 = 9$

## Exercise 3.5

1. $x + 2 = 0$
2. $R$ lies inside, $P$ lies outside and $Q$ lies on the circle.
3. $\sqrt{20}$ units
4. $p = \pm\, 20$.

## Exercise 3.6

1. $9x^2 + 16y^2 + 24xy + 34x + 112y + 121 = 0$
2. $k = \frac{1}{4}$, Length of latus rectum $= 1$, Focus $F\left(\frac{1}{4},\ 0\right)$

**3.**

| Axis | Vertex | Focus | Equation of directrix | Length of latus rectum |
|------|--------|-------|-----------------------|------------------------|
| $y = 4$ | $V(1, 4)$ | $F(3, 4)$ | $x + 1 = 0$ | 4 units |

**4.**

| Problem | Axis | Vertex | Focus | Equation of directrix | Length of latusrectum |
|---------|------|--------|-------|-----------------------|------------------------|
| (a) $y^2 = 20x$ | $y = 0$ | $V(0, 0)$ | $F(5, 0)$ | $x = -5$ | 20 units |
| (b) $x^2 = 8y$ | $x = 0$ | $V(0, 0)$ | $F(0, 2)$ | $y = -2$ | 8 units |
| (c) $x^2 = -16y$ | $x = 0$ | $V(0, 0)$ | $F(0, -4)$ | $y = 4$ | 16 units |

5.  $(x - 15)^2 = 5(y - 55)$. The output and the average cost at the vertex are 15 kg. and ₹ 55.

6.  when $x = 5$ months

### Exercise 3.7

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|----|----|----|----|
| (b) | (c) | (c) | (c) | (b) | (c) | (a) | (c) | (a) | (b) | (c) | (a) | (d) |

| 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|----|----|----|----|----|----|----|----|----|----|----|----|
| (a) | (a) | (d) | (a) | (c) | (b) | (b) | (a) | (d) | (b) | (b) | (b) |

### Miscellaneous problems

1.  $2x + y - 7 = 0$     2. $y = 6x + 3000$     4.  $p = 1, p = 2$    6.  $a = 9, b = 8$

7.  $(-1, -2)$ is on the line, $(1, 0)$ is above the line and $(-3, -4)$ is below the line

8.  $(-2, -7)$          9.  $y^2 = -\dfrac{9}{2}x$

10.  Axis : $y = 2$, Vertex : $(1, 2)$, Focus : $(2, 2)$,
     Equation of directrix : $x = 0$ and Length of latus rectum : 4 units

## 4. TRIGONOMETRY

### Exercise 4.1

1.(i)  $\dfrac{\pi}{3}$          (ii)  $\dfrac{5\pi}{6}$          (iii)  $\dfrac{4\pi}{3}$          (iv)  $\dfrac{-16\pi}{9}$

2.(i) 22°30′ (ii) 405°30′ (iii) −171°48′ (iv) 110°

3.(i) 1$^{st}$ quadrant (ii) 3$^{rd}$ quadrant (iii) 2$^{nd}$ quadrant.

4.(i) $\dfrac{-\sqrt{3}}{2}$ (ii) $\dfrac{-1}{2}$ (iii) $\dfrac{2}{\sqrt{3}}$ (iv) 1

(v) $\sqrt{2}$ 10. $\dfrac{-7}{2}$

## Exercise 4.2

1.(i) $\dfrac{2\sqrt{2}}{\sqrt{3}-1}$ (ii) $-\left(\dfrac{\sqrt{3}+1}{2\sqrt{2}}\right)$ (iii) $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

2.(i) $\sin 92°$ (ii) $\dfrac{\sqrt{3}}{2}$ (iii) $\cos 80°$ (iv) $\dfrac{\sqrt{3}}{2}$

3.(i) $\dfrac{-33}{65}$ (ii) $\dfrac{-16}{65}$ (iii) $\dfrac{16}{33}$

6. $\dfrac{2}{11}$, $\alpha+\beta$ lies in 1$^{st}$ quadrant. 7. $\pm\sqrt{2}-1$

9.(i) $\dfrac{9}{13}$ (ii) $-\dfrac{828}{2197}$ 10. $-\dfrac{44}{125}$, $-\dfrac{117}{44}$ 14. $\sqrt{2}-1$

## Exercise 4.3

1.(i) $\dfrac{1}{2}\left(\cos\dfrac{A}{4}-\cos\dfrac{A}{2}\right)$ (ii) $\dfrac{1}{2}\left(-\sin 2A+\dfrac{\sqrt{3}}{2}\right)$

(iii) $\dfrac{1}{2}\left(\sin 4A-\sin\dfrac{2A}{3}\right)$ (iv) $\dfrac{1}{2}(\cos 7\theta+\cos\theta)$

2.(i) $2\sin\dfrac{3A}{2}\cos\dfrac{A}{2}$ (ii) $2\cos 3A\cos A$

(iii) $2\sin 2\theta\cos 4\theta$ (iv) $2\sin\dfrac{3\theta}{2}\sin\dfrac{\theta}{2}$

## Exercise 4.4

1.(i) $\dfrac{-\pi}{6}$ (ii) $\dfrac{-\pi}{4}$ (iii) $\dfrac{\pi}{6}$ (iv) $\dfrac{\pi}{4}$

4. $x=\dfrac{1}{6}$ 5. $x=\dfrac{1}{2}$

6.(i) $\dfrac{4}{5}$ (ii) $\dfrac{1}{\sqrt{10}}$ 7. $\dfrac{-27}{65}$ 10. $\dfrac{\pi}{4}+\dfrac{x}{2}$

## Exercise 4.5

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|----|----|----|----|
| (b) | (a) | (c) | (b) | (b) | (d) | (a) | (d) | (a) | (c) | (c) | (d) | (b) |

| 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|----|----|----|----|----|----|----|----|----|----|----|----|
| (c) | (b) | (a) | (c) | (b) | (c) | (c) | (b) | (c) | (a) | (b) | (d) |

4. $\dfrac{3}{\sqrt{10}}$ and $\dfrac{-1}{\sqrt{10}}$

6.(i) $\dfrac{\sqrt{6}+\sqrt{2}}{4}$    (ii) $2-\sqrt{3}$

7. $\dfrac{\sqrt{15}+2\sqrt{2}}{12}$    9. $\dfrac{56}{33}$

10. $\dfrac{\pi}{4}-x$

## 5. DIFFERENTIAL CALCULUS

### Exercise 5.1

1.(i) Odd function    (ii)   Even function

(iii) Neither even nor odd function    (iv) Even function

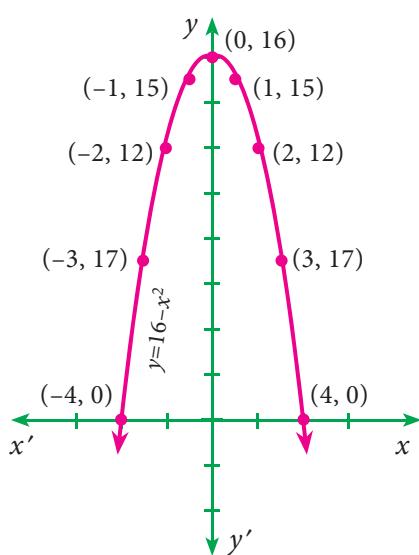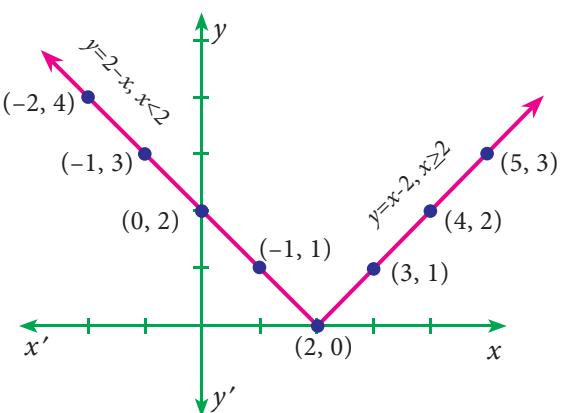(v) Neither even nor odd function    2. $k=0$    5. $\dfrac{1-x}{3+x}$, $\dfrac{3x+1}{x-1}$

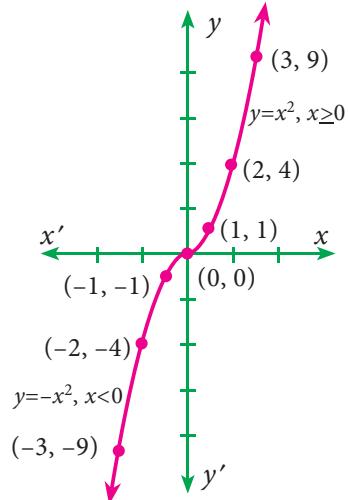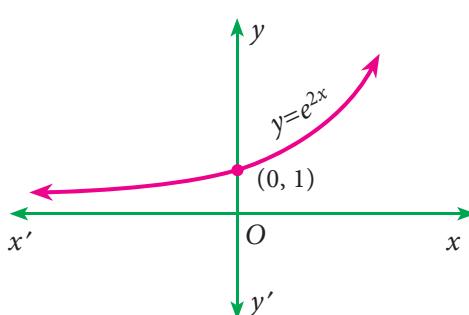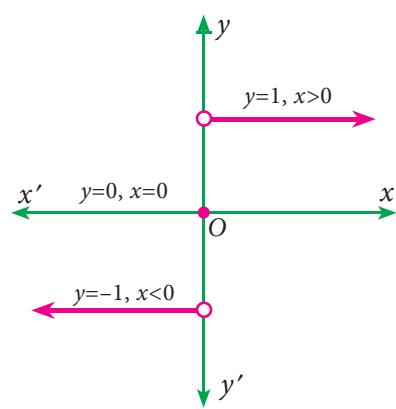6.(i) $e$      (ii) 0      (iii) $3e$      (iv) 0

7.(i)



(ii)



(iii)



(iv)

(v)

$y=e^{-2x}$

(0, 1)

$y$

$x'$   $O$   $x$   $y'$

(vi)

$y$

$y=1, x>0$

$x'$   $y=0, x=0$   $x$
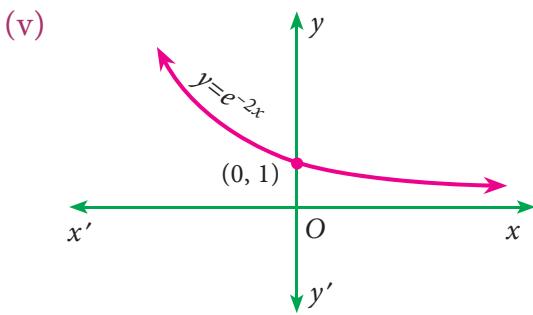
$O$

$y=-1, x<0$

$y'$

## Exercise 5.2

1.(i) $\dfrac{10}{3}$     (ii) 0     (iii) $\dfrac{1}{2}$     (iv) 1

(v) $\dfrac{15}{16}a^{-\frac{1}{24}}$     (vi) 9

2. $a = \pm 1$     3. $n = 7$     4. $\dfrac{28}{5}$     5. $f(-2) = 0$

## Exercise 5.3

1.(a) Not a continuous function at $x = 2$     (b) continuous function at $x = 3$

## Exercise 5.4

(i) $2x$     (ii) $-e^{-x}$     (iii) $\dfrac{1}{x+1}$

## Exercise 5.5

1.(i) $12x^3 - 6x^2 + 1$ (ii) $\dfrac{-20}{x^5} + \dfrac{6}{x^4} - \dfrac{5}{x^2}$ (iii) $\dfrac{1}{2\sqrt{x}} - \dfrac{1}{3\sqrt[3]{x^4}} + e^x$

(iv) $-\dfrac{3+x^2}{x^2}$     (v) $x^2 e^x (x+3)$     (vi) $3x^2 - 4x - 1$

(vii) $4x^3 - 3\cos x - \sin x$     (viii) $1 - \dfrac{1}{x^2}$

2.(i) $\dfrac{xe^x}{(1+x)^2}$     (ii) $\dfrac{2(x^2+1)}{(x^2-x+1)^2}$     (iii) $\dfrac{e^x}{(1+e^x)^2}$

3.(i) $x\cos x + \sin x$   (ii) $e^x (\sin x + \cos x)$

(iii) $e^x \left(1 + \dfrac{1}{x} + x + \log x\right)$     (iv) $\cos 2x$     (v) $e^x x^2 (x+3)$

4.(i) $\sin 2x$     (ii) $-\sin 2x$     (iii) $\dfrac{-3}{2}\cos x \sin 2x$ (iv) $\dfrac{x}{\sqrt{1+x^2}}$

(v) $n(ax^2 + bx + c)^{n-1}(2ax + b)$     (vi) $2x(\cos(x^2))$   (vii) $\dfrac{-x}{(1+x^2)\sqrt{1+x^2}}$

1.(i) $-\dfrac{y}{x}$  (ii) $\dfrac{y-2x}{2y-x}$  (iii) $-\dfrac{(x^2+ay)}{(y^2+ay)}$  3. $\dfrac{4}{3}\left(\dfrac{1-4x+3y}{1+4x-3x}\right)$

## Exercise 5.7

1.(i) $x^{\sin x}\left[\dfrac{\sin x}{x}+\cos x\log x\right]$  (ii) $(\sin x)^x\left[x\cot x+\log(\sin x)\right]$

(iii) $(\sin x)^{\tan x}\left[1+\sec^2 x\log(\sin x)\right]$

(iv) $\dfrac{1}{2}\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x^2+x+1)}}\left\{\dfrac{1}{x-1}+\dfrac{1}{x-2}-\dfrac{1}{x-3}-\dfrac{2x+1}{x^2+x+1}\right\}$

## Exercise 5.8

1.(i) $-\dfrac{1}{t^2}$  (ii) $t\cos t$  (iii) $-\tan\theta$  (iv) $\cot\dfrac{\theta}{2}$

2. $-\tan x$  3. $\dfrac{\sin 2x}{2x}$

## Exercise 5.9

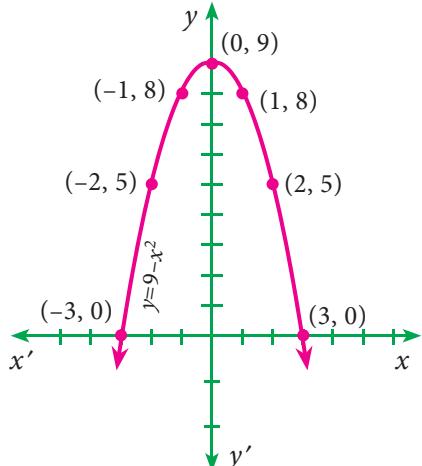1.(i) $9y$  (ii) $-\dfrac{1}{x^2}+a^x(\log a)^2$  (iii) $-\dfrac{1}{a}\operatorname{cosec}^3\theta$

## Exercise: 5.10

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|----|----|----|----|
| (d) | (c) | (a) | (b) | (a) | (a) | (c) | (a) | (d) | (a) | (a) | (d) | (b) |

| 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|----|----|----|----|----|----|----|----|----|----|----|----|
| (a) | (a) | (a) | (c) | (b) | (d) | (a) | (a) | (c) | (a) | (b) | (d) |

## Miscellaneous Problems

2.



4. $-\dfrac{1}{10}$

6.(i) continuous at $x=1$; differentiable at $x=1$

(ii) continuous at $x=2$; not differentiable at $x=2$

| GLOSSARY | |
|---|---|
| Absolute constants | முழுமையான மாறிலிகள் |
| Adjoint Matrix | சேர்ப்பு அணி |
| Algebric functions | இயற்கணித சார்புகள் |
| Allied angle | துணைக் கோணங்கள் |
| Analysis | பகுப்பாய்வு |
| Angle | கோணம் |
| Arbitrary constants | தன்னிச்சை மாறிலிகள் |
| Binomial | ஈருறுப்பு |
| Centre | மையம் |
| Chain rule | சங்கிலி விதி |
| Chord | நாண் |
| Circle | வட்டம் |
| Circular Permutation | வட்ட வரிசை மாற்றம் |
| Closed interval | மூடிய இடைவெளி |
| Coefficient | கெழு |
| Cofactor | இணைக் காரணி |
| Combination | சேர்வு |
| Componendo and dividendo. | கூட்டல் மற்றும் கழித்தல் விகித சமம் |
| Compound angle | கூட்டுக் கோணம் |
| Concurrent Line | ஒரு புள்ளி வழிக் கோடு |
| Conics | கூம்பு வெட்டிகள் |
| Constant | மாறிலி |
| Continuous | தொடர்ச்சி |
| Degree measure | பாகை அளவை |
| Dependent variable | சார்ந்த மாறி |
| Derivative | வகையீடு |
| Determinant | அணிக்கோவை |
| Diameter | விட்டம் |
| Digonal Matrix | மூலை விட்ட அணி |
| Directix | இயக்குவரை |
| Domain | சார்பகம் / அரங்கம் |
| Equation | சமன்பாடு |
| Explict | வெளிபடு |
| Exponential | அடுக்கு |
| Factorial | காரணீயப் பெருக்கம் |
| Focal distance | குவியத்தூரம் |
| Focus | குவியம் |
| Function | சார்பு |
| Identity | சமனி (அல்லது) முற்றொருமை |
| Implict | உட்படு |
| Independent term | சாரா உறுப்பு |
| Independent variable | சாரா மாறி |
| Input | உள்ளீடு |
| Interval | இடைவெளி |
| Inverse function | நேர்மாறு சார்பு |
| Inverse Matrix | நேர்மாறு அணி |
| Latus rectum | செவ்வகலம் |
| Left limit | இடக்கை எல்லை |
| Length of the arc. | வில்லின் நீளம் |

| Length of the tangent | தொடுகோட்டின் நீளம் |
|---|---|
| Limit | எல்லை |
| Linear factor | ஒரு படிக்காரணி |
| Locus | நியமப்பாதை அல்லது இயங்குவரை |
| Logarithmic | மடக்கை |
| Mathematical Induction | கணிதத் தொகுத்தறிதல் |
| Middle term | மைய உறுப்பு |
| Minors | சிற்றணிக் கோவைகள் |
| Modulus | மட்டு |
| Multiple angle | மடங்கு கோணம் |
| Multiplication Principle of counting | எண்ணுதலின் பெருக்கல் கொள்கை |
| Neighborhood | அண்மையகம் |
| Non-Singular Matrix | பூச்சியமற்றக் கோவை அணி |
| Open interval | திறந்த இடைவெளி |
| Origin | ஆதி |
| Output | வெளியீடு |
| Pair of straight line | இரட்டை நேர்கோடு |
| Parabola | பரவளையம் |
| Parallel line | இணை கோடு |
| Parameter | துணையலகு |
| Parametric functions | துணையலகு சார்புகள் |
| Partial fraction | பகுதிப் பின்னங்கள் |
| Pascals's Triangle | பாஸ்கலின் முக்கோணம் |
| Permutations | வரிசை மாற்றங்கள் |
| Perpendicular line | செங்குத்து கோடு |
| Point of concurrency | ஒருங்கிணைவுப் புள்ளி |
| Point of intersection | வெட்டும் புள்ளி |
| Principle of counting | எண்ணுதலின் கொள்கை |
| Quadrants | கால் பகுதிகள் |
| Radian measure | ரேடியன் அளவு / ஆரையன் அளவு |
| Radius | ஆரம் |
| Range | வீச்சகம் / வீச்சு |
| Rational Expression | விகித முறு கோவை |
| Right limit | வலக்கை எல்லை |
| Scalar | திசையிலி |
| Signum function | குறிச் சார்பு |
| Singular Matrix | பூச்சியக் கோவை அணி |
| Straight line | நேர்க் கோடு |
| Successive differentiation | தொடர் வகையிடல் |
| Tangent | தொடுகோடு |
| Transcendental functions | விஞ்சிய சார்புகள் |
| Transformation formulae | உருமாற்று சூத்திரங்கள் |
| Transpose of a Matrix | நிரை நிரல் மாற்று அணி |
| Triangular Matrix | முக்கோண அணி |
| Trignometric identities | திரிகோணமிதி முற்றொருமைகள் |
| Trignometry Ratios | திரிகோணமிதி விகிதங்கள் |
| Variable | மாறி |
| Vertex | முனை |

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