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SYMBOLS

$=$	equal to
$\neq$	not equal to
$<$	less than
$\leq$	less than or equal to
$>$	greater than
$\geq$	greater than or equal to
$\approx$	equivalent to
$\cup$	union
$\cap$	intersection
$\mathbb{U}$	universal Set
$\in$	belongs to
$\notin$	does not belong to
$\subset$	proper subset of
$\subseteq$	subset of or is contained in
$\not\subset$	not a proper subset of
$\not\subseteq$	not a subset of or is not contained in
A' (or) A^c	complement of A
$\varnothing$ (or) $\{ \}$	empty set or null set or void set
$n(A)$	number of elements in the set A
$P(A)$	power set of A
$\|\|\|^{ly}$	similarly

Δ	symmetric difference
$\mathbb{N}$	natural numbers
$\mathbb{W}$	whole numbers
$\mathbb{Z}$	integers
$\mathbb{R}$	real numbers
$\triangle$	triangle
$\angle$	angle
$\perp$	perpendicular to
$\parallel$	parallel to
$\Longrightarrow$	implies
$\therefore$	therefore
$\because$	since (or) because
$\| \quad \|$	absolute value
$\simeq$	approximately equal to
$\|$ (or) $:$	such that
$\equiv$ (or) $\cong$	congruent
$\equiv$	identically equal to
π	pi
$\pm$	plus or minus

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CONTENTS

Captions used in this Textbook

எண்ணென்ப ஏனை எழுத்தென்ப இவ்விரண்டும்
கண்ணென்ப வாழும் உயிர்க்கு – குறள் 392

Numbers and letters, they are known as
eyes to humans, they are. Kural 392

Learning Outcomes

To transform the classroom processes into learning centric with a set of bench marks

Note

To provide additional inputs for students in the content

Activity

To encourage students to involve in activities to learn mathematics

ICT Corner

To encourage learner's understanding of content through application of technology

Thinking Corner

To kindle the inquisitiveness of students in learning mathematics. To make the students to have a diverse thinking

Points to Remember

To recall the points learnt in the topic

Multiple Choice Questions

To provide additional assessment items on the content

Progress Check

Self evaluation of the learner's progress

Exercise

To evaluate the learners' in understanding the content

Let's use the QR code in the text books! How?

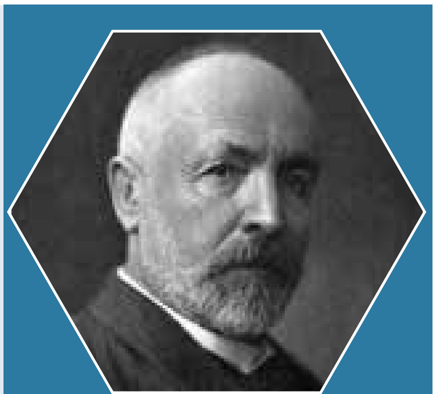
- Download the QR code scanner from the Google PlayStore/ Apple App Store into your smartphone
- Open the QR code scanner application
- Once the scanner button in the application is clicked, camera opens and then bring it closer to the QR code in the text book.
- Once the camera detects the QR code, a url appears in the screen.Click the url and go to the content page.

1 SET LANGUAGE

A set is a many that allows itself to thought of as a one

-Georg Cantor

The theory of sets was developed by German mathematician Georg Cantor. Today it is used in almost every branch of Mathematics. In Mathematics, sets are convenient because all mathematical structures can be regarded as sets.



Georg Cantor
(1845 - 1918)

Learning Outcomes

- To describe a set.
- To represent sets in descriptive form, set builder form and roster form.
- To identify different types of sets.
- To understand and perform set operations.
- To use Venn diagrams to represent sets and set operations.
- To solve life oriented simple word problems.

1.1 Introduction

In our daily life, we often deal with collection of objects like books, stamps, coins, etc. Set language is a mathematical way of representing a collection of objects.

Study the problem: 16 students play only Cricket, 18 students play only Volley ball and 3 students play both Cricket and Volley ball, while 2 students play neither Cricket nor Volley ball. Totally 39 students are there in a class.

We can describe this pictorially as follows:

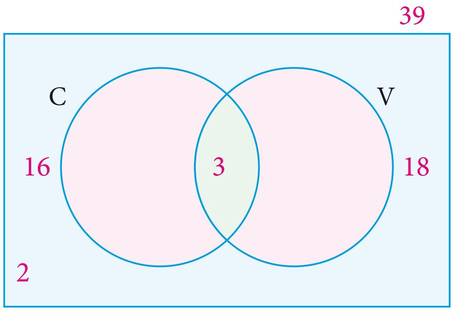


Fig. 1.1

What do the circles in the picture represent? They are **collections** of students who play games. We do not need to draw 39 students, or even 39 symbols, or use different colours to distinguish those who play Cricket from those who play Volleyball etc. Simply calling the collections C and V is enough; we can talk of those in both C and V, in neither and in one but not the other. This is **the language of sets**. A great deal of mathematics is written in this language and hence we are going to study it.

But **why** is this language so important, why should mathematicians want to use this language? One reason is that everyday language is imprecise and can cause confusion. For example, if I write 1, 2, 3,… what do the three dots at the end mean? You say,"Of course, they mean the list of natural numbers". What if I write 1, 2, 4,…? What comes next ? It could be 7, and then 11, and so on. (Can you see why ?) Or it could be continued as 8, 16, and so on. If we explicitly say, "The collection of numbers that are powers of 2", then we know that the latter is meant. So, in general, when we are talking of collections of numbers, we may refer to some collection in some short form, but writing out the collection may be difficult. It is here that the language of sets is of help. We can speak of the powers of 2 as a **set** of numbers.

Note, this is a list that goes on forever, so it is an **infinite** set. By now, we have come across several infinite sets: the set of natural numbers, the set of integers, the set of rational numbers, and many more. We also know the set of prime numbers, again an infinite set. But we know many **finite** sets too. The number of points of intersection of three lines on the plane is an example.

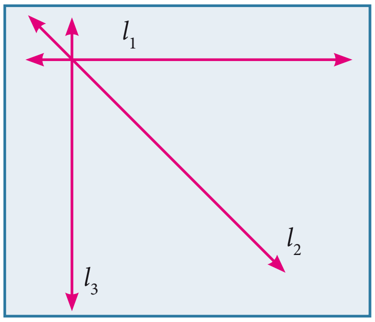


Fig. 1.2

We talk of a point being on a line and we know that a line contains inifinite number of points. For example, five points P,Q,R,S and T which lie on a line can be denoted by a set $A= \{P, Q, R, S, T\}$.

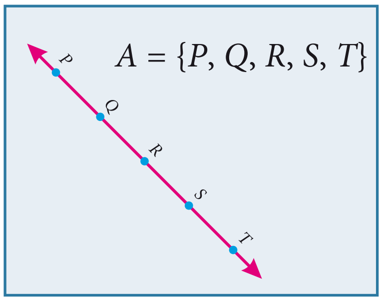


Fig. 1.3

We can already see that many of these sets are important in whatever algebra and geometry we have learnt, and we expect that there will be more important sets coming along as we learn mathematics. That is why we are going to learn the language of sets. For now, we will work with small finite sets and learn its language.

Let us look at the following pictures. What do they represent?

Here, Fig.1.4 represents a collection of fruits and Fig. 1.5 represents a collection of house- hold items.

We observe in the above cases, our attention turns from one individual object to a collection of objects based on their characteristics. Any such collection is called a set.

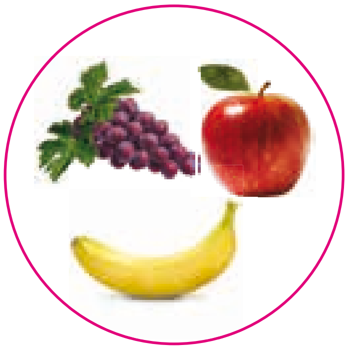


Fig. 1.4

Fig. 1.5

1.2 Set

A set is a well-defined collection of objects.

Here "well-defined collection of objects" means that given a specific object it must be possible for us to decide whether the object is an element of the given collection or not.

The objects of a set are called its members or elements.

For example,

1. The collection of all books in a District Central Library.
2. The collection of all colours in a rainbow.
3. The collection of prime numbers.

We see that in the adjacent box, statements (1), (2), and (4) are well defined and therefore they are sets. Whereas (3) and (5) are not well defined because the words good and beautiful are difficult to agree on. I might consider a student to be good and you may not. I might consider Malligai is beautiful but you may not. So we will consider only those collections to be sets where there is no such ambiguity.

Which of the following are sets ?

1. Collection of Natural numbers.
2. Collection of English alphabets.
3. Collection of good students in a class.
4. Collection of States in our country.
5. Collection of beautiful flowers in a garden.

Therefore (3) and (5) are not sets.

Note

(i) Elements of a set are listed only once.

(ii) The order of listing the elements of the set does not change the set.

Both these conditions are natural. The collection 1,2,3,4,5,6,7,8, … as well as the collection 1, 3, 2, 4, 5, 7, 6, 8, … are the same though listed in different order. Since it is necessary to know whether an object is an element in the set or not, we do not want to list that element many times.

Activity-1

Discuss and give as many examples of collections from your daily life situations, which are sets and which are not sets.

Notation

A set is usually denoted by capital letters of the English Alphabets A, B, P, Q, X, Y, etc.

The elements of a set are denoted by small letters of the English alphabets a, b, p, q, x, y, etc.

The elements of a set is written within curly brackets "{ }"

If x is an element of a set A or x belongs to A, we write $x \in A$.

If x is not an element of a set A or x does not belongs to A, we write $x \notin A$.

For example,

Consider the set $A = \{2,3,5,7\}$ then

2 is an element of A; we write $2 \in A$

5 is an element of A; we write $5 \in A$

6 is not an element of A; we write $6 \notin A$

Example 1.1

Consider the set A = {Ashwin, Muralivijay, Vijay Shankar, Badrinath }.

Fill in the blanks with the appropriate symbol $\in$ or $\notin$.

(i) Muralivijay ____ A. (ii) Ashwin ______ A. (iii) Badrinath ______ A.

(iv) Ganguly _____ A. (v) Tendulkar _____ A

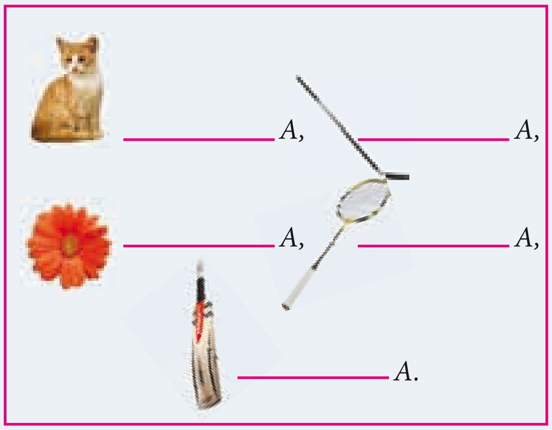
Solution

(i) Muralivijay $\in A$. (ii) Ashwin $\in A$ (iii) Badrinath $\in A$

(iv) Ganguly $\notin A$. (v) Tendulkar $\notin A$.

Activity-2

Insert the appropriate symbol $\in$ (belongs to) or $\notin$ (does not belong to) in the blanks.



1.3 Representation of a Set

The collection of odd numbers can be described in many ways:

(1) "The set of odd numbers" is a fine description, we understand it well.

(2) It can be written as $\{1, 3, 5, \ldots\}$ and you know what I mean.

(3) Also, it can be said as the collection of all numbers x where x is an odd number.

All of them are equivalent and useful. For instance,the two descriptions "The collection of all solutions to the equation $x-5 = 3$" and $\{8\}$ refer to the same set.

A set can be represented in any one of the following three ways or forms:

(i) Descriptive Form.

(ii) Set-Builder Form or Rule Form.

(iii) Roster Form or Tabular Form.

1.3.1 Descriptive Form

In descriptive form, a set is described in words.

For example,

(i) The set of all vowels in English alphabets.

(ii) The set of whole numbers.

1.3.2 Set Builder Form or Rule Form

In set builder form, all the elements are described by a rule.

For example,

(i) $A = \{x : x \text{ is a vowel in English alphabets}\}$

(ii) $B = \{x | x \text{ is a whole number}\}$

Note

The symbol ':' or '|' stands for "such that".

1.3.3 Roster Form or Tabular Form

A set can be described by listing all the elements of the set.

For example,

(i) $A = \{a, e, i, o, u\}$

(ii) $B = \{0,1,2,3,\ldots\}$

Can this form of representation be possible always?

Note

Three dots (...) in the example (ii) is called ellipsis. It indicates that the pattern of the listed elements continues in the same manner.

Activity-3

Write the following sets in respective forms.

S.No.	Descriptive Form	Set Builder Form	Roster Form
1	The set of all natural numbers less than 10		
2		$\{x : x \text{ is a multiple of } 3, x \in \mathbb{N}\}$	
3			$\{2,4,6,8,10\}$
4	The set of all days in a week.		
5			$\{\ldots-3,-2,-1,0,1,2,3\ldots\}$

Example 1.2

Write the set of letters of the following words in Roster form

(i) ASSESSMENT (ii) PRINCIPAL

Solution

(i) ASSESSMENT

$A = \{A, S, E, M, N, T\}$

(ii) PRINCIPAL

$B = \{P, R, I, N, C, A, L\}$

Exercise 1.1

1. Which of the following are sets?

(i) The Collection of prime numbers upto 100.

(ii) The Collection of rich people in India.

(iii) The Collection of all rivers in India.

(iv) The Collection of good Hockey players.

2. List the set of letters of the following words in Roster form.

(i) INDIA (ii) PARALLELOGRAM

(iii) MISSISSIPPI (iv) CZECHOSLOVAKIA

3. Consider the following sets $A = \{0, 3, 5, 8\}$, $B = \{2, 4, 6, 10\}$ and $C = \{12, 14, 18, 20\}$.

(a) State whether True or False:

(i) $18 \in C$ (ii) $6 \notin A$ (iii) $14 \notin C$ (iv) $10 \in B$

(v) $5 \in B$ (vi) $0 \in B$

(b) Fill in the blanks:

(i) $3 \in$ ____ (ii) $14 \in$ ____ (iii) 18 ____ B (iv) 4 ____ B

4. Represent the following sets in Roster form.

(i) A = The set of all even natural numbers less than 20.

(ii) $B = \{y : y = \frac{1}{2n}, n \in \mathbb{N}, n \leq 5\}$

(iii) $C = \{x : x$ is perfect cube, $27 < x < 216\}$

(iv) $D = \{x : x \in \mathbb{Z}, -5 < x \leq 2\}$

5. Represent the following sets in set builder form.

(i) B = The set of all Cricket players in India who scored double centuries in One Day Internationals.

(ii) $C = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$

(iii) D = The set of all tamil months in a year.

(iv) E = The set of odd Whole numbers less than 9.

6. Represent the following sets in descriptive form.

(i) P = { January, June, July}

(ii) $Q = \{7,11,13,17,19,23,29\}$

(iii) $R = \{x : x \in \mathbb{N}, x < 5\}$

(iv) $S = \{x : x$ is a consonant in English alphabets$\}$

1.4 Types of Sets

There is a very special set of great interest: the empty collection ! Why should one care about the empty collection? Consider the set of solutions to the equation $x^2+1 = 0$. It has no elements at all in the set of Real Numbers. Also consider all rectangles with one angle greater than 90 degrees. There is no such rectangle and hence this describes an empty set.

So, the empty set is important, interesting and deserves a special symbol too.

1.4.1 Empty Set or Null Set

A set consisting of no element is called the *empty set* or *null set* or *void set*.

It is denoted by $\varnothing$ or { }.

Thinking Corner

Are the sets $\{0\}$ and $\{\varnothing\}$ empty sets?

For example,

(i) $A=\{x : x$ is an odd integer and divisible by $2\}$

$\therefore A=\{\ \}$ or $\varnothing$

(ii) The set of all integers between 1 and 2.

1.4.2.Singleton Set

A set which has only one element is called a *singleton set*.

For example,

(i) $A = \{\mathbf{x : 3 < x < 5, x} \in \mathbb{N}\}$ (ii) The set of all even prime numbers.

1.4.3 Finite Set

A set with finite number of elements is called a *finite set*.

Note

An empty set has no elements, so $\varnothing$ is a finite set.

For example,

1. The set of family members.
2. The set of indoor/outdoor games you play.
3. The set of curricular subjects you learn in school.
4. $A = \{x : x$ is a factor of $36\}$

1.4.4 Infinite Set

A set which is not finite is called an *infinite set*.

Thinking Corner

Is the set of natural numbers a finite set?

For example,

(i) $\{5,10,15,...\}$ (ii) The set of all points on a line.

To discuss further about the types of sets, we need to know the cardinality of sets.

Cardinal number of a set : When a set is finite, it is very useful to know how many elements it has. The number of elements in a set is called the Cardinal number of the set.

The cardinal number of a set A is denoted by $n(A)$

Example 1.3

If $A = \{1,2,3,4,5,7,9,11\}$, find $n(A)$.

Solution

$A = \{1,2,3,4,5,7,9,11\}$

Since set A contains 8 elements, $n(A) = 8$.

Thinking Corner

If $A = \{1, b, b, \{4, 2\}, \{x, y, z\}, d, \{d\}\}$, then $n(A)$ is____

1.4.5 Equivalent Sets

Two finite sets A and B are said to be equivalent if they contain the same number of elements. It is written as $A \approx B$.

If A and B are equivalent sets, then $n(A) = n(B)$

For example,

Consider $A = \{$ ball, bat$\}$ and $B = \{$history, geography$\}$.

Here A is equivalent to B because $n(A) = n(B) = 2$.

Thinking Corner

Let $A=\{x:x$ is a colour in national flag of India$\}$ and $B=\{$Red, Blue, Green$\}$. Are these two sets equivalent?

Example 1.4

Are $P = \{x : -3 \le x \le 0, x \in \mathbb{Z}\}$ and $Q =$ The set of all prime factors of 210, equivalent sets?

Solution

$P = \{-3, -2, -1, 0\}$, The prime factors of 210 are 2,3,5,and 7 and so, $Q = \{2, 3, 5, 7\}$

$n(P) = 4$ and $n(Q) = 4$. Therefore P and Q are equivalent sets.

1.4.6 Equal Sets

Two sets are said to be equal if they contain exactly the same elements, otherwise they are said to be unequal.

In other words, two sets A and B are said to be equal, if

(i) every element of A is also an element of B

(ii) every element of B is also an element of A

Thinking Corner

Are the sets $\varnothing$, $\{0\}$, $\{\varnothing\}$ equal or equivalent?

For example,

Consider the sets $A = \{1, 2, 3, 4\}$ and $B = \{4, 2, 3, 1\}$

Since A and B contain exactly the same elements, A and B are equal sets.

Note

(i) If A and B are equal sets, we write $A = B$.

(ii) If A and B are unequal sets, we write $A \neq B$.

A set does not change, if one or more elements of the set are repeated.

For example, if we are given

$A=\{a, b, c\}$ and $B=\{a, a, b, b, b, c\}$ then, we write $B = \{a, b, c, \}$. Since, every element of A is also an element of B and every element of B is also an element of A, the sets A and B are equal.

Example 1.5

Are $A = \{x : x \in \mathbb{N}, 4 \leq x \leq 8\}$ and

$B = \{4, 5, 6, 7, 8\}$ equal sets?

Solution

$A = \{4, 5, 6, 7, 8\}$, $B = \{4, 5, 6, 7, 8\}$

A and B are equal sets.

Note

Equal sets are equivalent sets but equivalent sets need not be equal sets. For example, if $A = \{p,q,r,s,t\}$ and $B= \{4,5,6,7,8\}$. Here $n(A)=n(B)$, so A and B are equivalent but not equal.

1.4.7 Subset

Let A and B be two sets. If every element of A is also an element of B, then A is called a subset of B. We write $A \subseteq B$.

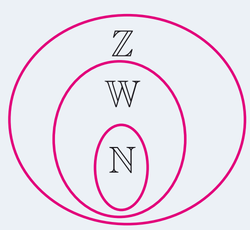
$A \subseteq B$ is read as "A is a subset of B"

Thus $A \subseteq B$, if $a \in A$ implies $a \in B$.

If A is not a subset of B, we write $A \nsubseteq B$

Clearly, if A is a subset of B, then $n(A) \leq n(B)$.

Thinking Corner



Is $\mathbb{W}$ subset of $\mathbb{N}$ or $\mathbb{Z}$?

Since every element of A is also an element of B, the set B must have at least as many elements as A, thus $n(A) \leq n(B)$.

The other way is also true. Suppose that $n(A) > n(B)$, then A has more elements than B, and hence there is at least one element in A that cannot be in B, so A is not a subset of B.

For example,

(i) $\{1\}\subseteq\{1,2,3\}$ (ii) $\{2,4\} \not\subseteq \{1,2,3\}$

Activity-4

Discuss with your friends and give examples of subsets of sets from your daily life situation.

Example1.6

Write all the subsets of $A = \{a, b\}$.

Solution

$A= \{a,b\}$

Subsets of A are $\varnothing$,$\{a\}$, $\{b\}$ and $\{a, b\}$.

Note

(i) If $A\subseteq B$ and $B\subseteq A$, then $A=B$.

In fact this is how we defined equality of sets.

(ii) Empty set is a subset of every set.

This is not easy to see ! Let A be any set. The only way for the empty set to be not a subset of A would be to have an element x in it but with x not in A. But how can x be in the empty set ? That is impossible. So this only way being impossible, the empty set must be a subset of A. (Is your head spinning? Think calmly, explain it to a friend, and you will agree it is alright !)

(iii) Every set is a subset of itself. (Try and argue why.)

1.4.8. Proper Subset

Let A and B be two sets.If A is a subset of B and $A \neq B$, then A is called a proper subset of B and we write $A \subset B$.

For example,

If $A=\{1,2,5\}$ and $B=\{1,2,3,4,5\}$ then A is a proper subset of B ie. $A \subset B$.

Example 1.7

Insert the appropriate symbol $\subseteq$ or $\nsubseteq$ in each blank to make a true statement.

(i) $\{10, 20, 30\}$ ____ $\{10, 20, 30, 40\}$ (ii) $\{p, q, r\}$ _____ $\{w, x, y, z\}$

Solution

(i) $\{10, 20, 30\}$ ____ $\{10, 20, 30, 40\}$

Since every element of $\{10, 20, 30\}$ is also an element of $\{10, 20, 30, 40\}$, we get $\{10, 20, 30\} \subseteq \{10, 20, 30, 40\}$.

(ii) $\{p, q, r\}$ _____ $\{w, x, y, z\}$

Since the element p belongs to $\{p, q, r\}$ but does not belong to $\{w, x, y, z\}$, shows that $\{p, q, r\} \nsubseteq \{w, x, y, z\}$.

1.4.9 Power Set

The fun begins when we realise that elements of sets can themselves be sets !

That is not very difficult to imagine: the people in school form a set, that consists of the set of students, the set of teachers, and the set of other staff. The set of students then has many sets as its elements: the set of students in class 1, the set of class 2 children, and so on. So we can easily talk of sets of sets of sets of …. of sets of elements !

Why bother? There is a particular set of sets that is very interesting.

Let A be any set. Form the set consisting of all subsets of A. Let us call it B. What all sets does B contain? For one thing, A is inside it, since A is a subset of A. The empty set is also a subset of A, so it is in B. If x is in A, then the singleton set $\{x\}$ is in B. (This means that B has at least as many elements as A; so $n(A)$ is equal to $n(B)$). For every pair of distinct elements x, y in A, we have $\{x,y\}$ in B. So yes, B has a lot many sets. It is so rich that it gets a very powerful name !

The set of all subsets of a set A is called the power set of 'A'. It is denoted by $P(A)$.

For example,

(i) If $A=\{2, 3\}$, then find the power set of A.

The subsets of A are $\emptyset$, $\{2\},\{3\},\{2,3\}$.

The power set of A,

$\mathrm{P}(A) = \{\emptyset ,\{2\},\{3\},\{2,3\}\}$

(ii) If $A = \{\emptyset , \{\emptyset\}\}$, then the power set of A is $\{ \emptyset , \{\emptyset , \{\emptyset\}\}, \{\emptyset\} , \{\{\emptyset\}\} \}$.

An important property.

We already noted that $n(A) \leq n[P(A)]$. But how big is $P(A)$? Think about this a bit, and see whether you come to the following conclusion:

(i) If $n(A) = m$, then $n[P(A)] = 2^m$

(ii) The number of proper subsets of a set A is $n[P(A)]-1 = 2^m-1$.

Example 1.8

Find the number of subsets and the number of proper subsets of a set $X=\{a, b, c, x, y, z\}$.

Solution

Given $X=\{a, b, c, x, y, z\}$.Then, $n(X) =6$

The number of subsets $= n[P(X)] \quad = 2^6 \quad = 64$

The number of proper subsets $\quad = n[P(X)]-1 \quad = 2^6-1$

$= 64 - 1 \quad = 63$

Exercise 1.2

1. Find the cardinal number of the following sets.

(i) $M = \{p, q, r, s, t, u\}$

(ii) $P = \{x : x = 3n+2, n\in \mathbb{W} \text{ and } x< 15\}$

(iii) $Q = \{y : y = \frac{4}{3n}, n \in \mathbb{N} \text{ and } 2 < n \leq 5\}$

(iv) $R = \{x : x$ is an integers, $x \in \mathbb{Z}$ and $-5 \leq x < 5\}$

(v) S = The set of all leap years between 1882 and 1906.

2. Identify the following sets as finite or infinite.

(i) X = The set of all districts in Tamilnadu.

(ii) Y = The set of all straight lines passing through a point.

(iii) $A = \{ x : x \in \mathbb{Z}$ and $x < 5\}$

(iv) $B = \{x : x^2 - 5x + 6 = 0, x \in \mathbb{N}\}$

3. Which of the following sets are equivalent or unequal or equal sets?

(i) A = The set of vowels in the English alphabets.

B = The set of all letters in the word "VOWEL"

(ii) $C = \{2,3,4,5\}$

$D = \{ x : x \in \mathbb{W}, 1 < x < 5\}$

(iii) E = The set of A=$\{ x : x$ is a letter in the word "LIFE"$\}$

$F = \{F, I, L, E\}$

(iv) $G = \{x : x$ is a prime number and $3 < x < 23\}$

$H = \{x : x$ is a divisor of $18\}$

4. Identify the following sets as null set or singleton set.

(i) $A = \{x : x \in \mathbb{N}, 1 < x < 2\}$

(ii) B = The set of all even natural numbers which are not divisible by 2

(iii) $C = \{0\}$.

(iv) D = The set of all triangles having four sides.

5. If S = {square, rectangle, circle, rhombus, triangle}, list the elements of the following subset of S.

(i) The set of shapes which have 4 equal sides.

(ii) The set of shapes which have no sides.

(iii) The set of shapes in which the sum of all interior angles is 180°

(iv) The set of shapes which have 5 sides.

6. If $A = \{a, \{a, b\}\}$, write all the subsets of A.

7. Write down the power set of the following sets.

   (i) $A = \{a, b\}$ (ii) $B = \{1, 2, 3\}$ (iii) $D = \{p, q, r, s\}$ (iv) $E = \varnothing$

8. Find the number of subsets and the number of proper subsets of the following sets.

   (i) $W = \{\text{red, blue, yellow}\}$ (ii) $X = \{ x^2 : x \in \mathbb{N}, x^2 \leq 100\}$.

9. (i) If $n(A) = 4$, find $n[P(A)]$.

   (ii) If $n(A)=0$, find $n[P(A)]$.

   (iii) If $n[P(A)] = 256$, find $n(A)$.

1.5 Set Operations

We started with numbers and very soon we learned arithmetical operations on them. In algebra we learnt expressions and soon started adding and multiplying them as well, writing (x^2+2) $(x-3)$ etc. Now that we know sets, the natural question is, what can we do with sets, what are natural operations on them ?

What can we do with sets ? We can pick an element. But then which element ? There are many in general, and hence "picking an element" is not an operation on a set. But like we did with addition, subtraction etc, we can try and think of operations that combine two given sets to get a new set. How can we do this ?

A simple way is to put the two sets together. This gives us a new set, and exactly one set, so it is an operation. We could pick out exactly the common elements of the two given sets. That's an interesting operation too. We could talk of all the elements not in the given set. But this is problematic: we can list the elements in a set, but which are the elements not in it ? Almost anything. We know that 5734 is in the set of natural numbers, but we also know that my chair is not in it, an elephant is not in it, and so on. How could we ever hope to describe all these elements ? But then when we speak of sets of numbers we are clearly not talking of elephants ! So we should really speak of numbers not in the set of natural numbers. For instance we could implicitly fix the integers and talk of integers not in the set of natural numbers. In general, we call this "fixed" set the universal set (relative to which we speak of what is or not in a given set).

When two or more sets combine together to form one set under the given conditions, then operations on sets can be carried out. We can visualize the relationship between sets and set operations using Venn diagram.

1.5.1 Universal Set

A Universal set is a set which contains all the elements of all the sets under consideration and is usually denoted by U.

For example,

(i) If we discuss about elements in Natural numbers, then the universal set U is the set of all Natural numbers. U=$\{x : x \in \mathbb{N}\}$.

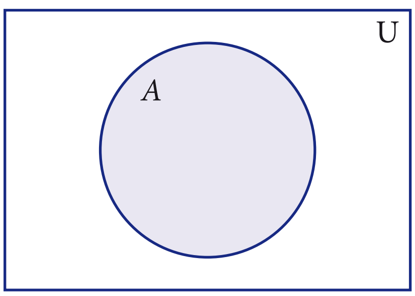
(ii) If A={earth, mars, jupiter}, then the universal set U is the planets of solar system.

1.5.2 Complement of a Set

The Complement of a set A is the set of all elements of U (the universal set) that are not in A.

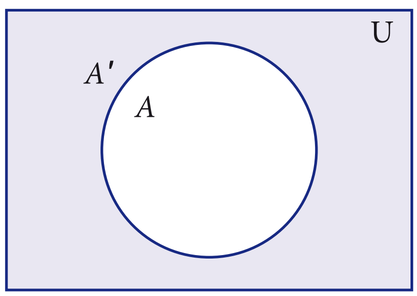
It is denoted by A' or A^c. In symbols $A' = \{x : x \in U, x \notin A\}$

Venn diagram for complement of a set



A (shaded region)

Fig. 1.6



A' (shaded region)

Fig. 1.7

For example,

If U = {all boys in a class} and A= {boys who play Cricket}, then complement of the set A is A'= {boys who do not play Cricket}.

Example 1.9

If U = $\{c, d, e, f, g, h, i, j\}$ and $A = \{c, d, g, j\}$, find A'.

Solution

$U = \{c, d, e, f, g, h, i, j\}$, $A = \{c, d, g, j\}$

$A' = \{e, f, h, i\}$

Note

(i) $(A')' = A$

(ii) $U' = \varnothing$

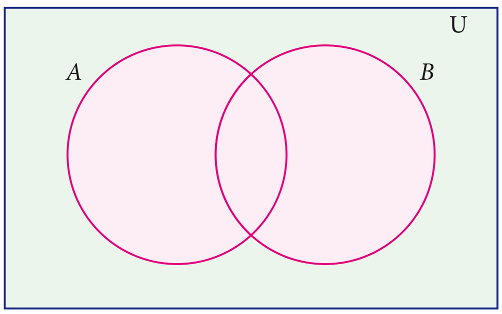
(iii) $\varnothing' = U$

1.5.3 Union of Two Sets

The union of two sets A and B is the set of all elements which are either in A or in B or in both. It is denoted by $A \cup B$ and read as A union B.

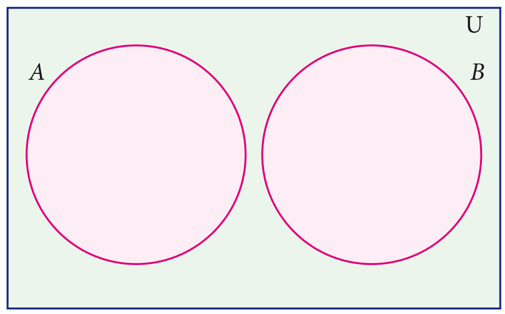
In symbol, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

The union of two sets can be represented by Venn diagram as given below



Sets A and B have common elements

Fig. 1.8



Sets A and B are disjoint

Fig. 1.9

For example,

If P ={Asia,Africa, Antarctica, Australia} and Q = {Europe, North America, South America}, then the union set of P and Q is $P \cup Q$ = {Asia, Africa, Antartica, Australia, Europe, North America, South America}.

Note

(i) $A \cup A = A$

(ii) $A \cup \varnothing = A$

(iii) $A \cup U = U$ where A is any subset of universal set U

(iv) $A \subseteq A \cup B$ and $B \subseteq A \cup B$

(v) $A \cup B = B \cup A$ (union of two sets is commutative)

Example 1.10

If $A=\{1, 2, 6\}$ and $B=\{2, 3, 4\}$, find $A\cup B$.

Solution

Given $A=\{1, 2, 6\}$, $B=\{2, 3, 4\}$

$A\cup B=\{1, 2, 3, 4,6\}$.

Example 1.11

If $P=\{m, n\}$ and $Q= \{m, i, j\}$, represent P and Q in Venn diagram and find $P\cup Q$.

Solution

Given $P=\{m, n\}$ and $Q= \{m, i, j\}$

From the diagram,

$P\cup Q=\{n, m, i, j\}$.

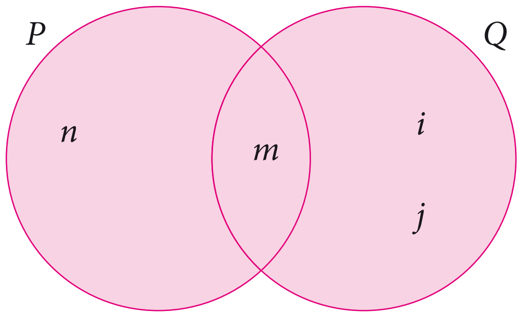


Fig. 1.10

1.5.4 Intersection of TwoSets

The intersection of two sets A and B is the set of all elements common to both A and B. It is denoted by $A\cap B$ and read as A intersection B.

In symbol , $A\cap B=\{x : x\in A \text{ and } x\in B\}$

Intersection of two sets can be represented by a Venn diagram as given below

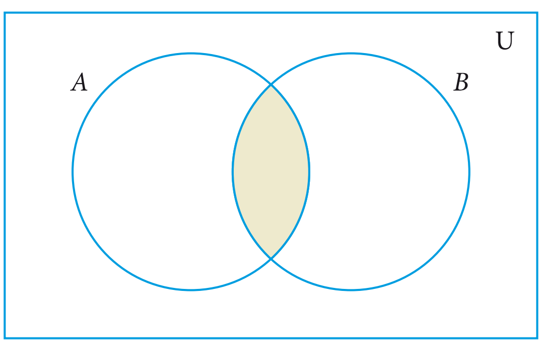


Fig. 1.11

For example,

If $A = \{1, 2, 6\}$; $B = \{2, 3, 4\}$, then $A\cap B = \{2\}$ because 2 is common element of the sets A and B.

Note

(i) $A\cap A = A$ (ii) $A\cap\varnothing = \varnothing$

(iii) $A\cap U = A$ where A is any subset of universal set U

(iv) $A\cap B \subseteq$ A and $A\cap B \subseteq B$

(v) $A\cap B = B\cap A$ (Intersection of two sets is commutative)

Can we determine $n(A \cap B)$ in terms of $n(A)$ and $n(B)$? This seems to be difficult, but what about $n(A \cup B)$ in terms of $n(A)$ and $n(B)$?

Notice that all the elements in A are in $(A \cup B)$, and all the elements in B also in $A \cup B$

Can we say that $n(A \cup B) = n(A) + n(B)$?

Unfortunately not. Consider an element is common to both A and B? Of course it is in A union B, but is counted both in A and in B and we don't want to count the same element twice in A union B !

So indeed, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. But then it is easy to see that $n(A \cap B) = n(A) + n(B) - n(A \cup B)$. So between the union and the intersection we need to know one to determine the other, given $n(A)$ and $n(B)$.

Example 1.12

Let $A = \{x : x$ is an even natural number and $1 < x \leq 12\}$ and

$B = \{ x : x$ is a multiple of 3, $x \in \mathbb{N}$ and $x \leq 12\}$ be two sets. Find $A\cap B$.

Solution

Here $A = \{2, 4, 6, 8, 10, 12\}$ and $B = \{3, 6, 9, 12\}$

$A\cap B = \{6, 12\}$

Example 1.13

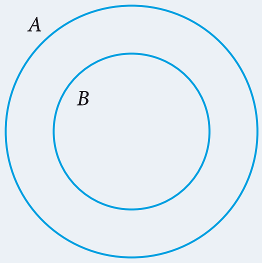
If $A = \{2, 3\}$ and $C = \{\ \}$, find $A\cap C$.

Solution

There is no common element and hence $A\cap C = \{\ \}$

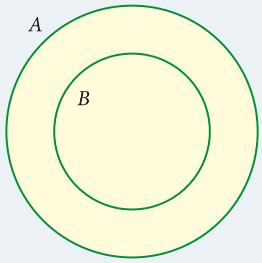
Note

(i) When $B \subseteq A$, the union and intersection of A and B are represented in Venn diagram as follows



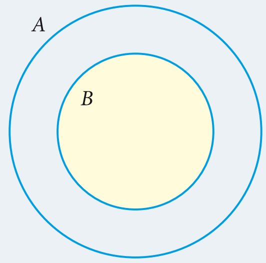
$B \subseteq A$

Fig. 1.12



shaded region is
$A \cup B = A$

Fig. 1.13



shaded region is
$A \cap B = B$

Fig. 1.14

(ii) For any two sets A and B, $A \cup B = A \cap B$. $\therefore A = B$

(iii) Let $n(A) = p$ and $n(B) = q$ then

(a) Minimum of $n(A \cup B) = \max\{p, q\}$

(b) Maximum of $n(A \cup B) = p + q$

(c) Minimum of $n(A \cap B) = 0$

(d) Maximum of $n(A \cap B) = \min\{p, q\}$

1.5.5 Difference of Two Sets

Let A and B be two sets, the difference of sets A and B is the set of all elements which are in A, but not in B. It is denoted by $A-B$ or $A \backslash B$ and read as A difference B.

In symbol, $A-B = \{x : x \in A \text{ and } x \notin B\}$

$$B-A = \{y : y \in B \text{ and } y \notin A\}.$$

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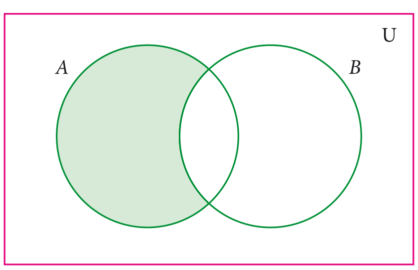
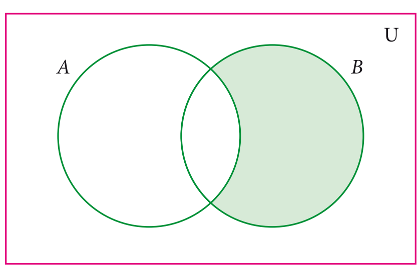
Venn diagram for set difference



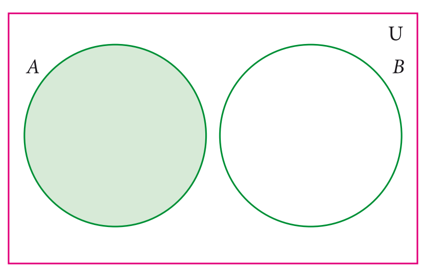
$A-B$

Fig. 1.15



$B-A$

Fig. 1.16



$A-B$

Fig. 1.17

Example 1.14

If $A=\{-3, -2, 1, 4\}$ and $B= \{0, 1, 2, 4\}$, find (i) $A-B$ (ii) $B-A$.

Solution

$A-B = \{-3, -2, 1, 4\} - \{0, 1, 2, 4\} = \{-3, -2\}$

$B-A = \{0, 1, 2, 4\} - \{-3, -2, 1, 4\} = \{0, 2\}$

Note

(i) $A' = U - A$

(ii) $A-B = A \cap B'$

(iii) $A-A = \varnothing$

(iv) $A-\varnothing = A$

(v) $A-B = B-A \Leftrightarrow A=B$

(vi) $A-B = A$ if $A \cap B = \varnothing$

1.5.6 Symmetric Difference of Sets

The symmetric difference of two sets A and B is the set $(A-B)\cup(B-A)$. It is denoted by $A\Delta B$.

$A\Delta B=\{x : x \in A-B \text{ or } x \in B-A\}$

Example 1.15

If $A = \{6, 7, 8, 9\}$ and $B=\{8, 10, 12\}$, find $A\Delta B$.

Solution

$A-B = \{6, 7, 9\}$

$B-A = \{10, 12\}$

$A\Delta B = (A-B)\cup(B-A) = \{6, 7, 9\}\cup\{10,12\}$

$A\Delta B = \{6, 7, 9, 10, 12\}$.

Thinking Corner

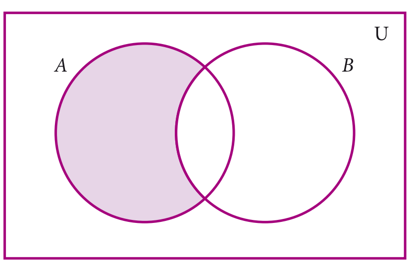
What is $(A-B) \cap (B-A)$?

Example 1.16

Represent $A\Delta B$ through Venn diagram.

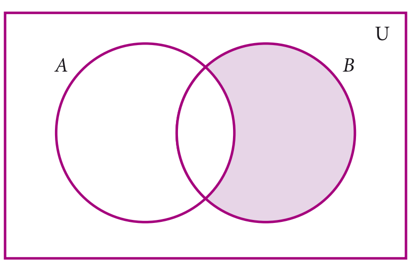
Solution

$A\Delta B= (A-B) \cup (B-A)$



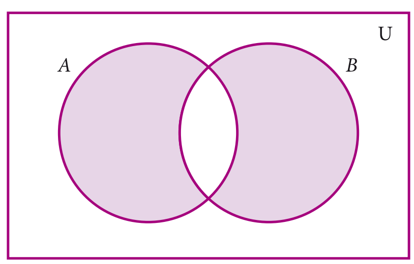
$A-B$

Fig. 1.18



$B-A$

Fig. 1.19



$(A-B) \cup (B-A)$

Fig. 1.20

Note

(i) $A\Delta A=\varnothing$ (ii) $A\Delta B = B\Delta A$

(iii) $A\Delta B=\{x : x\in A\cup B \text{ and } x\notin A\cap B\}$ (iv) $A\Delta B= (A\cup B) - (A\cap B)$

1.5.7 Disjoint Sets

Two sets A and B are said to be disjoint if they do not have common elements.

In other words, if $A\cap B=\varnothing$, then A and B are said to be disjoint sets.

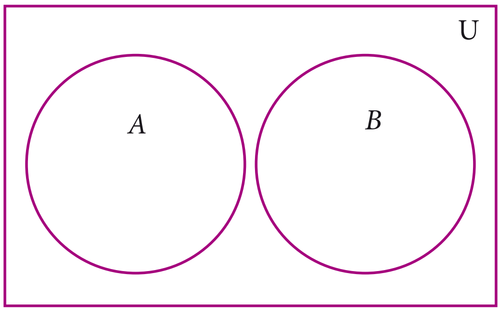


Fig. 1.21

Example 1.17

Verify whether $A=\{20, 22, 23, 24\}$ and $B=\{25, 30, 40, 45\}$ are disjoint sets.

Solution

$$A = \{20,22, 23, 24\}\ ,\ B=\{25, 30, 40, 45\}$$

$$A\cap B = \{20,22, 23, 24\} \cap \{25, 30, 40, 45\}$$

$$= \{\ \}$$

Since $A\cap B = \varnothing$, A and B are disjoint sets.

Note

If $A\cap B\neq \varnothing$, then A and B are said to be overlapping sets .Thus if two sets have atleast one common element, they are called overlapping sets.

Example 1.18

From the given Venn diagram, write the elements of

(i) A (ii) B (iii) $A-B$ (iv) $B-A$

(v) A' (vi) B' (vii) U

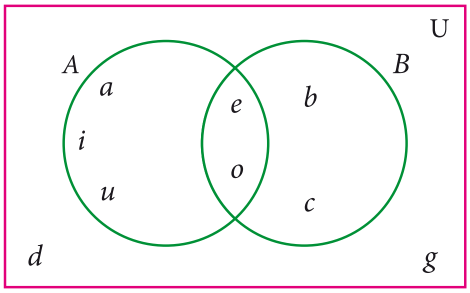


Fig. 1.22

Solution

(i) $A = \{a, e, i, o, u\}$

(ii) $B = \{b, c, e, o\}$

(iii) $A - B = \{a, i, u\}$

(iv) $B - A = \{b, c\}$

(v) $A' = \{b, c, d, g\}$

(vi) $B' = \{a, d, g, i, u\}$

(vii) $\text{U} = \{a, b, c, d, e, g, i, o, u\}$

Example 1.19

Draw a Venn diagram similar to one at the side and shade the region representing the following sets

(i) A' (ii) $(A-B)'$ (iii) $(A \cup B)'$

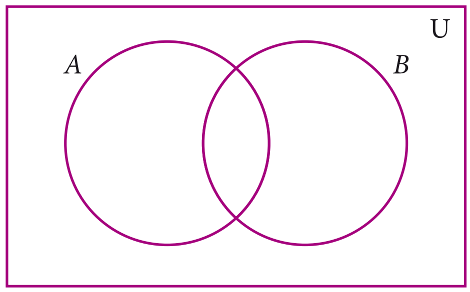


Fig. 1.23

Solution

(i) A'

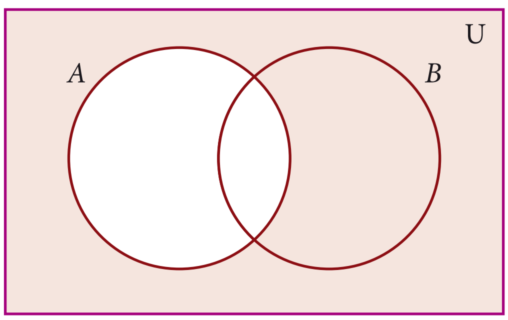


Fig. 1.24 A'

(ii) $(A-B)'$

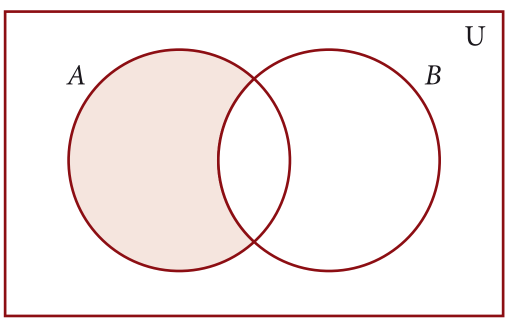


Fig. 1.25 $A-B$

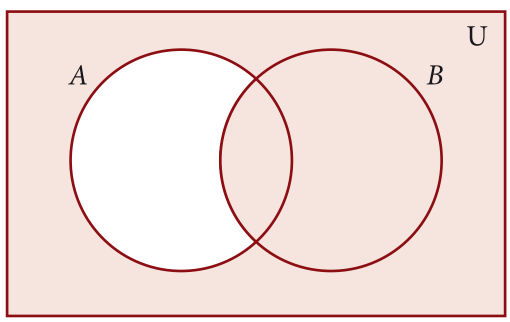


Fig. 1.26 $(A-B)'$

(iii) $(A \cup B)'$

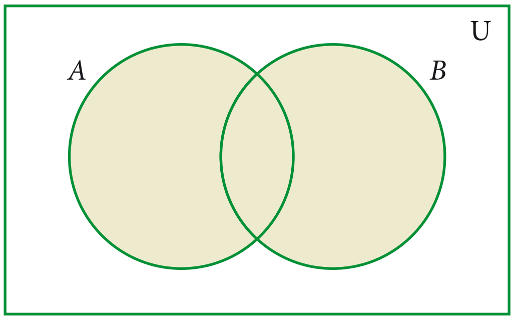


Fig. 1.27 $A \cup B$

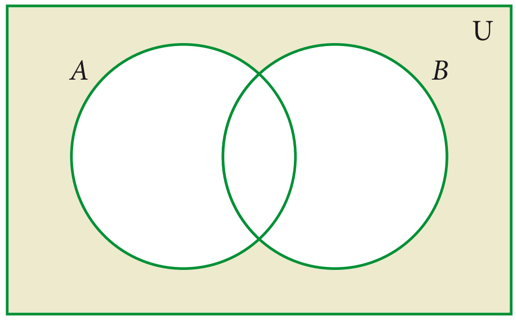


Fig. 1.28 $(A \cup B)'$

Exercise 1.3

1. Using the given Venn diagram, write the elements of

   (i) A (ii) B (iii) $A \cup B$ (iv) $A \cap B$

   (v) $A-B$ (vi) $B-A$ (vii) A' (viii) B'

   (ix) U

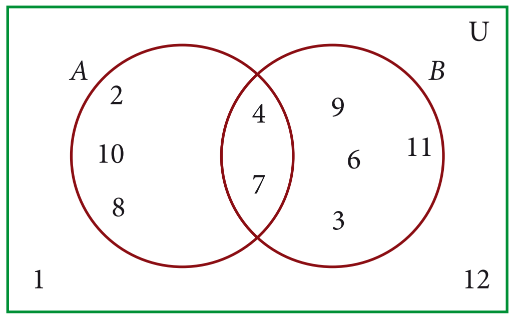


Fig. 1.29

2. Find $A \cup B$, $A \cap B$, $A-B$ and $B-A$ for the following sets.

   (i) $A = \{2, 6, 10, 14\}$ and $B=\{2, 5, 14, 16\}$

   (ii) $A = \{a, b, c, e, u\}$ and $B=\{a, e, i, o, u\}$

   (iii) $A = \{x : x \in N, x \leq 10\}$ and $B=\{x : x \in W, x < 6\}$

   (iv) A = Set of all letters in the word "mathematics" and

   B = Set of all letters in the word "geometry"

3. If U=$\{a, b, c, d, e, f, g, h\}$, $A=\{b, d, f, h\}$ and $B=\{a, d, e, h\}$, find the following sets.

   (i) A' (ii) B' (iii) $A' \cup B'$ (iv) $A' \cap B'$ (v) $(A \cup B)'$

   (vi) $(A \cap B)'$ (vii) $(A')'$ (viii) $(B')'$

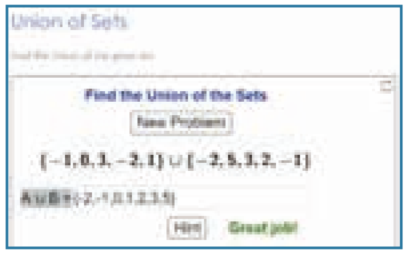
4. Let U=$\{0, 1, 2, 3, 4, 5, 6, 7\}$, $A=\{1, 3, 5, 7\}$ and $B=\{0, 2, 3, 5, 7\}$, find the following sets.

   (i) A' (ii) B' (iii) $A' \cup B'$ (iv) $A' \cap B'$ (v) $(A \cup B)'$

   (vi) $(A \cap B)'$ (vii) $(A')'$ (viii) $(B')'$

ICT Corner

Expected Result is shown in this picture



Step – 1

Open the Browser and copy and paste the Link given below (or) by typing the URL given (or) Scan the QR Code.

Step - 2

GeoGebra worksheet “Union of Sets” will appear. You can create new problems by clicking on the box
“NEW PROBLEM”

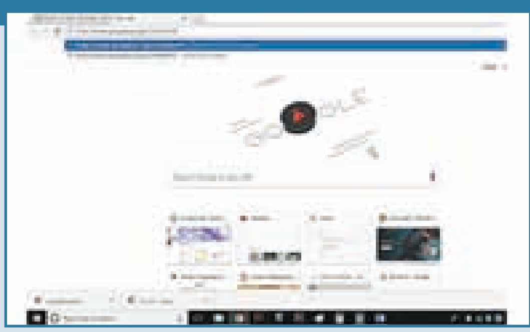
Step-3

Enter your answer by typing the correct numbers in the Question Box and then hit enter. If you have any doubt, you can hit the “HINT” button

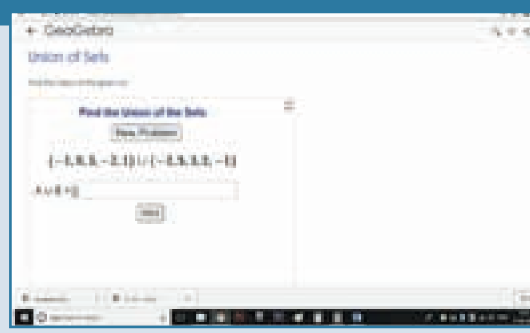
Step-4

If your answer is correct “GREAT JOB” menu will appear. And if your answer is Wrong “Try Again!” menu will appear.
Keep on working new problems until you get 5 consecutive trials as correct.

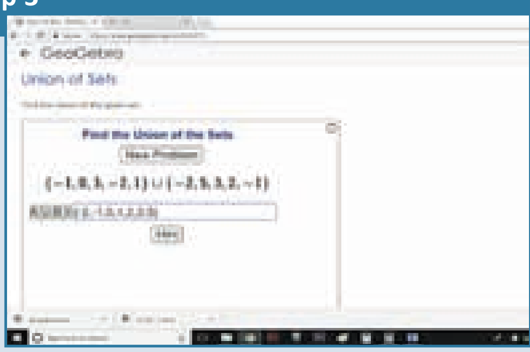
Step 1



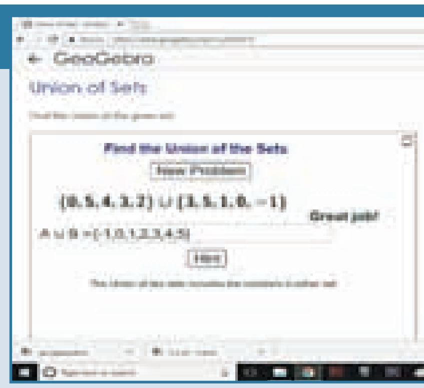
Step 2



Step 3



Step 4



Browse in the link

Union of Sets: https://www.geogebra.org/m/ufxdh47G

5. State which pairs of sets are disjoint or overlapping?

(i) $A=\{f, i, a, s\}$ and $B=\{a, n, f, h, s\}$

(ii) $C=\{x : x$ is a prime number, $x>2\}$ and $D=\{x : x$ is an even prime number$\}$

(iii) $E=\{x : x$ is a factor of $24\}$ and $F=\{x : x$ is a multiple of $3, x<30\}$

6. Find the symmetric difference between the following sets.

(i) $P=\{2, 3, 5, 7, 11\}$ and $Q=\{1, 3, 5, 11\}$

(ii) $R=\{l, m, n, o, p\}$ and $S=\{j, l, n, q\}$

(iii) $X=\{5, 6, 7\}$ and $Y=\{5, 7, 9, 10\}$

7. Using the set symbols, write down the expressions for the shaded region in the following

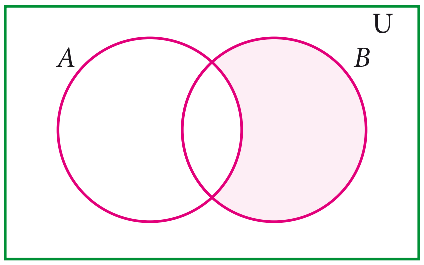


Fig. 1.30

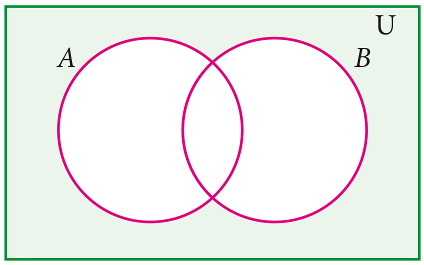


Fig. 1.31

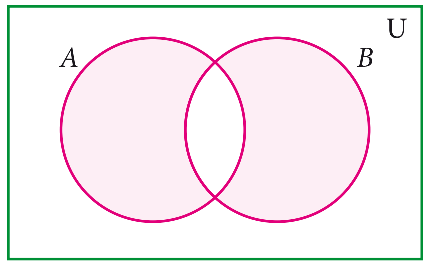


Fig. 1.32

8. Let A and B be two overlapping sets and the universal set be U. Draw appropriate Venn diagram for each of the following,

(i) $A\cup B$ (ii) $A\cap B$ (iii) $(A\cap B)'$ (iv) $(B-A)'$ (v) $A'\cup B'$ (vi) $A'\cap B'$

(vii) What do you observe from the Venn diagram (iii) and (v)?

1.6 Cardinality and Practical Problems on Set Operations

We have learnt about the union, intersection, complement and difference of sets.

Now we will go through some practical problems on sets related to everyday life.

Results :

If A and B are two finite sets, then

(i) $n(A\cup B) = n(A)+n(B) - n(A\cap B)$

(ii) $n(A-B) = n(A) - n(A\cap B)$

(iii) $n(B-A) = n(B) - n(A\cap B)$

(iv) $n(A') = n(\mathrm{U}) - n(A)$

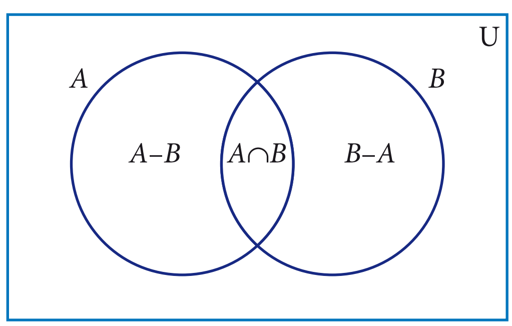


Fig. 1.33

Note

From the above results we may get,

(i) $n(A \cap B) = n(A) + n(B) - n(A \cup B)$ (ii) $n(\mathrm{U}) = n(A) + n(A')$

(iii) If A and B are disjoint sets, $n(A \cup B) = n(A) + n(B)$.

Example1.20

From the Venn diagram, verify that

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

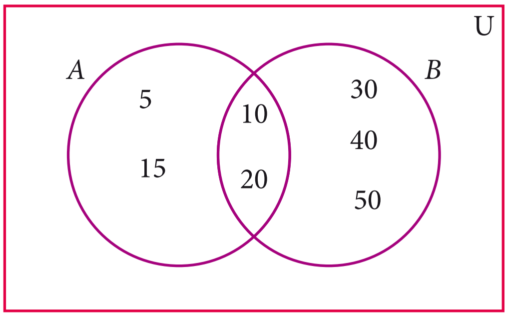


Fig. 1.34

Solution

From the venn diagram,

$$A = \{5, 10, 15, 20\}$$

$$B = \{10, 20, 30, 40, 50,\}$$

Then $\quad A \cup B = \{5, 10, 15, 20, 30, 40, 50\}$

$$A \cap B = \{10, 20\}$$

$n(A) = 4, \quad n(B) = 5, n(A \cup B) = 7, \quad n(A \cap B) = 2$

$$n(A \cup B) = 7 \qquad \rightarrow (1)$$

$$n(A) + n(B) - n(A \cap B) = 4 + 5 - 2$$

$$= 7 \qquad \rightarrow (2)$$

From (1) and (2), $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Example 1.21

If $n(A) = 36$, $n(B) = 10$, $n(A \cup B) = 40$, and $n(A') = 27$ find $n(\mathrm{U})$ and $n(A \cap B)$.

Solution

$n(A) = 36, \quad n(B) = 10, \quad n(A \cup B) = 40, \quad n(A') = 27$

(i) $n(\mathrm{U}) = n(A) + n(A') = 36 + 27 = 63$

(ii) $n(A \cap B) = n(A) + n(B) - n(A \cup B) = 36 + 10 - 40 = 46 - 40 = 6$

Activity-5

Fill in the blanks with appropriate cardinal numbers.

S.No.	$n(A)$	$n(B)$	$n(A\cup B)$	$n(A\cap B)$	$n(A-B)$	$n(B-A)$
1	30	45	65			
2	20		55	10		
3	50	65		25		
4	30	43	70			

Example 1.22

Let $A=\{b, d, e, g, h\}$ and $B = \{a, e, c, h\}$. Verify that $n(A-B) = n(A)-n(A\cap B)$.

Solution

$A = \{b, d, e, g, h\}$, $B = \{a, e, c, h\}$

$$A - B = \{b, d, g\}$$

$$n(A-B) = 3 \qquad \ldots (1)$$

$$A \cap B = \{e, h\}$$

$$n(A \cap B) = 2, \quad n(A) = 5$$

$$n(A) - n(A\cap B) = 5\text{-}2$$

$$= 3 \qquad \ldots (2)$$

Form (1) and (2) we get $n(A-B) = n(A)-n(A\cap B)$.

Example 1.23

In a school, all students play either Hockey or Cricket or both. 300 play Hockey, 250 play Cricket and 110 play both games. Find

(i) the number of students who play only Hockey.

(ii) the number of students who play only Cricket.

(iii) the total number of students in the School.

Solution:

Let H be the set of all students play Hockey and C be the set of all students play Cricket.

Then $n(H) = 300$, $n(C) = 250$ and $n(H \cap C) = 110$

Using Venn diagram,

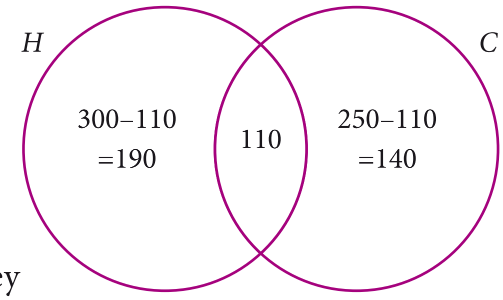


Fig. 1.35

From the Venn diagram,

(i) The number of students who play only Hockey $= 190$

(ii) The number of students who play only Cricket $= 140$

(iii) The total number of students in the school $= 190+110+140 = 440$

Aliter

(i) The number of students who play only Hockey

$$n(H-C) = n(H) - n(H \cap C)$$
$$= 300 - 110 = 190$$

(ii) The number of students who play only Cricket

$$n(C-H) = n(C) - n(H \cap C)$$
$$= 250 - 110 = 140$$

(iii) The total number of students in the school

$$n(H \cup C) = n(H) + n(C) - n(H \cap C)$$
$$= 300+250 - 110 = 440$$

Example1.24

In a party of 60 people, 35 had Vanilla ice cream, 30 had Chocolate ice cream. All the people had at least one ice cream. Then how many of them had,

(i) both Vanilla and Chocolate ice cream.

(ii) only Vanilla ice cream.

(iii) only Chocolate ice cream.

Solution :

Let V be the set of people who had Vanilla ice cream and C be the set of people who had Chocolate ice cream.

Then $n(V) = 35$, $n(C) = 30$, $n(V \cup C) = 60$,

Let x be the number of people who had both ice creams.

From the Venn diagram

$$35 - x + x + 30 - x = 60$$

$$65 - x = 60$$

$$x = 5$$

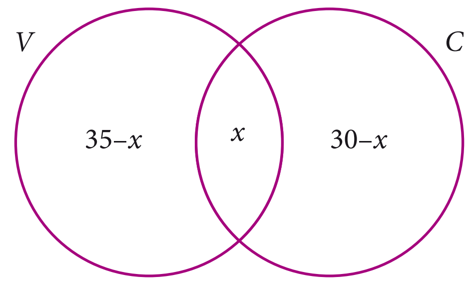


Fig. 1.36

Hence 5 people had both ice creams.

(i) Number of people who had only Vanilla ice cream $= 35 - x$

$= 35 - 5 = 30$

(ii) Number of people who had only Chocolate ice cream $= 30 - x$

$= 30 - 5 = 25$

Aliter

Number of people had both Vanilla and Chocolate ice creams

$$n(V \cap C) = n(V) + n(C) - n(V \cup C)$$

$$= 35 + 30 - 60 = 5$$

Number of people had only Vanilla ice creams

$$n(V - C) = n(V) - n(V \cap C)$$

$$= 35 - 5 = 30$$

Number of people who had only Chocolate ice creams

$$n(C - V) = n(C) - n(V \cap C)$$

$$= 30 - 5 = 25$$

E4AHV7

Exercise 1.4

1. (i) If $n(A) = 25$, $n(B) = 40$, $n(A \cup B) = 50$ and $n(B') = 25$, find $n(A \cap B)$ and $n(U)$.

   (ii) If $n(A) = 300$, $n(A \cup B) = 500$, $n(A \cap B) = 50$ and $n(B') = 350$, find $n(B)$ and $n(U)$.

2. If $U = \{x : x \in \mathbb{N}, x \leq 10\}$, $A = \{2,3,4,8,10\}$ and $B = \{1,2,5,8,10\}$, then verify that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

3. If $U = \{1, 2, 3, \ldots., 10\}$, $P = \{3, 4, 5, 6\}$ and $Q = \{x : x \in \mathbb{N}, x < 5\}$, then verify that $n(Q - P) = n(Q) - n(P \cap Q)$.

4. In a class, all students take part in either music or drama or both. 25 students take part in music, 30 students take part in drama and 8 students take part in both music and drama. Find

   (i) The number of students who take part in only music.

   (ii) The number of students who take part in only drama.

   (iii) The total number of students in the class.

5. In a Mathematics class, 20 children forgot to bring their rulers, 17 children forgot to bring their pencils and 5 children forgot to bring both ruler and pencil. Then find the number of children

   (i) who forgot to bring only pencil. (ii) who forgot to bring only ruler.

   (iii) in the class.

6. In a village of 100 families, 65 families buy Tamil newspapers and 55 families buy English newspapers. Find the number of families who buy

   (i) both Tamil and English newspapers. (ii) Tamil newspapers only.

   (iii) English newspapers only.

7. In a party of 45 people, each one likes tea or coffee or both. 35 people like tea and 20 people like coffee. Find the number of people who

   (i) like both tea and coffee. (ii) do not like Tea.

   (iii) do not like coffee.

8. In an examination 50% of the students passed in Mathematics and 70% of students passed in Science while 10% students failed in both subjects. 300 students passed in atleast one subject. Find the total number of students who appeared in the examination, if they took examination in only two subjects.

9. A and B are two sets such that $n(A-B) = 32 + x$, $n(B-A) = 5x$ and $n(A\cap B) = x$. Illustrate the information by means of a Venn diagram. Given that $n(A) = n(B)$, calculate the value of x.

10. Out of 500 car owners investigated, 400 owned car A and 200 owned car B, 50 owned both A and B cars. Is this data correct?

Exercise 1.5

Multiple Choice Questions

1. Which of the following is correct?

   (*a*) $\{7\} \in \{1,2,3,4,5,6,7,8,9,10\}$ (*b*) $7 \in \{1,2,3,4,5,6,7,8,9,10\}$

   (*c*) $7 \notin \{1,2,3,4,5,6,7,8,9,10\}$ (*d*) $\{7\} \not\subseteq \{1,2,3,4,5,6,7,8,9,10\}$

2. The set $P = \{x \mid x \in \mathbb{Z}, -1 < x < 1\}$ is a

   (*a*) Singleton set (*b*) Power set (*c*) Null set (*d*) Subset

3. If $U = \{x \mid x \in \mathbb{N}, x < 10\}$ and $A = \{x \mid x \in \mathbb{N}, 2 \le x < 6\}$ then $(A')'$ is

   (*a*) $\{1, 6, 7, 8, 9\}$ (*b*) $\{1, 2, 3, 4\}$ (*c*) $\{2, 3, 4, 5\}$ (*d*) $\{\ \}$

4. If $B \subseteq A$ then $n(A\cap B)$ is

   (*a*) $n(A-B)$ (*b*) $n(B)$ (*c*) $n(B-A)$ (*d*) $n(A)$

5. If $A = \{x, y, z\}$ then the number of non- empty subsets of A is

   (*a*) 8 (*b*) 5 (*c*) 6 (*d*) 7

6. Which of the following is correct?

   (*a*) $\varnothing \subseteq \{a, b\}$ (*b*) $\varnothing \in \{a, b\}$ (*c*) $\{a\} \in \{a, b\}$ (*d*) $a \subseteq \{a, b\}$

7. If $A\cup B = A\cap B$, then

   (*a*) $A \neq B$ (*b*) $A = B$ (*c*) $A \subset B$ (*d*) $B \subset A$

8. If $B - A$ is B, then $A\cap B$ is

   (*a*) A (*b*) B (*c*) U (d) $\varnothing$

9. The shaded region in the adjacent diagram represents

   (*a*) $(A\cup B)'$ (*b*) $(A\cap B)'$

   (*c*) $A'\cap B'$ (*d*) $A\cap B$

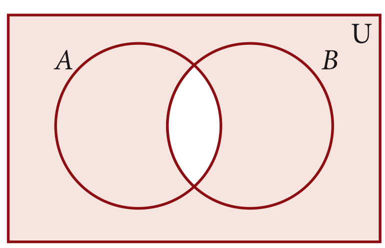


Fig. 1.37

10. From the adjacent diagram $n[P(A \Delta B)]$ is

(*a*) 8 (*b*) 16

(*c*) 32 (*d*) 64

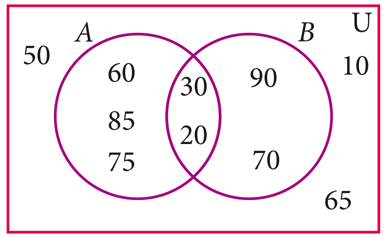


Fig. 1.38

11. If $n(A) = 10$ and $n(B) = 15$, then the minimum and maximum number of elements in $A \cap B$ is

(*a*) (10,15) (*b*) (15,10) (*c*) (10,0) (*d*) (0,10)

12. If $X = \{x : x = 4(n-1),\ n \in \mathbb{N}\}$ and $Y = \{y : y = 3^n - 2n - 1,\ n \in \mathbb{N}\}$, then $X \cup Y$ is

(*a*) $\mathbb{W}$ (*b*) X (*c*) Y (*d*) $\mathbb{N}$

13. Let $A = \{\varnothing\}$ and $B = P(A)$, then $A \cap B$ is

(*a*) $\{\varnothing, \{\varnothing\}\}$ (*b*) $\{\varnothing\}$ (*c*) $\varnothing$ (*d*) $\{0\}$

14. In a class of 50 boys, 35 boys play Carrom and 20 boys play Chess then the number of boys play both games is

(*a*) 5 (*b*) 30 (*c*) 15 (*d*) 10.

Points to remember

1. A set is a well defined collection of objects.
2. Sets are represented in three forms (i) Descriptive form (ii) Set – builder form (iii) Roster form.
3. The number of elements in a set is called the cardinal number of the set.
4. A set consisting of no element is called an empty set .
5. If the number of elements in a set is zero or finite, it is a finite set. Otherwise it is an infinite set.
6. Two finite sets are said to be equivalent if they contain the same number of elements.
7. Two sets are said to be equal when they contain the same elements.

8. If every element of A is also an element of B, then A is called a subset of B.

9. If $A \subseteq B$ and $A \neq B$, then A is a proper subset of B.

10. The power set of the set A is the set of all the subsets of A and it is denoted by $P(A)$.

11. The number of subsets of a set with $\boldsymbol{m}$ elements is $\mathbf{2^m}$.

12. The number of proper subsets of a set with $\boldsymbol{m}$ elements is $\mathbf{2^m - 1}$.

13. The complement A′ of the set A contains all the elements of the universal set except that of A.

14. The union of two sets A and B is the set of all elements which are either in A or in *B or in both*.

15. The set of all common elements of the sets A and B is called the intersection of the sets A and B.

16. If $A \cap B = \varnothing$ then A and B are disjoint sets. If $A \cap B \neq \varnothing$ then A and B are overlapping.

17. The difference of two sets A and B is the set of all elements in A but not in B.

18. The symmetric difference of two sets A and B is the union of A-B and B-A.

19. For any two finite sets A and B, we have

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

(iii) $n(A - B) = n(A) - n(A \cap B)$

(iv) $n(B - A) = n(B) - n(A \cap B)$

(v) $n(U) = n(A) + n(A')$

Answers

Exercise 1.1

1. (i) set (ii) not a set (iii) Set (iv) not a set

2. (i) {I, N, D, A} (ii) {P, A, R, L, E, O, G, M} (iii) {M, I, S, P}
   (iv) {C, Z, E, H, O, S, L, V, A, K, I}

3. (a) (i) True (ii) True (iii) False (iv) True (v) False (vi) False
   (b) (i) A (ii) C (iii) $\notin$ (iv) $\in$

4. (i) $A = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$ (ii) $B = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}\right\}$
   (iii) $C = \{64, 125\}$ (iv) $D = \{-4, -3, -2, -1, 0, 1, 2\}$

5. (i) $B = \{x : x$ is an Indian player who scored double centuries in One Day International$\}$
   (ii) $C = \left\{x : x = \frac{n}{n+1}, n \in \mathbb{N}\right\}$ (iii) $D = \{x : x$ is a tamil month in a year$\}$
   (iv) $E = \{x : x$ is an odd whole number less than $9\}$

6. (i) P = The set of English months starting with letter '*J*'
   (ii) Q = The set of Prime numbers between 5 and 31
   (iii) R = The set of natural numbers less than 5
   (iv) S = The set of English consonants

Exercise 1.2

1. (i) $n(M) = 6$ (ii) $n(P) = 5$ (iii) $n(Q) = 3$ (iv) $n(R) = 10$ (v) $n(S) = 5$

2. (i) finite (ii) infinite (iii) infinite (iv) finite

3. (i) Equivalent sets (ii) Unequal sets (iii) Equal sets (iv) Equivalent sets

4. (i) null set (ii) null set (iii) singleton set (iv) null set

5. (i) {square, rhombus} (ii) {circle} (iii) {triangle} (iv) { }

6. (i) $\{\ \}, \{a\}, \{a, b\}\ \{a, \{a, b\}\}$

7. (i) $\{\{\ \}, \{a\},\ \{b\}\ \{a, b\}\}$

(ii) $\{\{\ \}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

(iii) $\{\{\ \}, \{p\}, \{q\}\ \{r\}, \{s\}, \{p, q\}, \{p, r\}, \{p, s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, \{p, q, r, s\}\}$ (iv) $P(E)= \{\{\ \}\}$

8. (i) 8, 7 (ii) 1024, 1023

9. (i) 16 (ii) 1 (iii) 8

Exercise 1.3

1. (i) $\{2, 4, 7, 8, 10\}$ (ii) $\{3, 4, 6, 7, 9, 11\}$ (iii) $\{2, 3, 4, 6, 7, 8, 9, 10, 11\}$

(iv) $\{4, 7\}$ (v) $\{2, 8, 10\}$ (vi) $\{3, 6, 9, 11\}$

(vii) $\{1, 3, 6, 9, 11, 12\}$ (viii) $\{1, 2, 8, 10, 12\}$

(ix) $\{1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12\}$

2. (i) $\{2, 5, 6, 10, 14, 16\}, \{2, 14\}, \{6, 10\}, \{5, 16\}$

(ii) $\{a, b, c, e, i, o, u\}, \{a, e, u\}, \{b, c\}, \{i, o\}$

(iii) $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, \{1, 2, 3, 4, 5\}, \{6, 7, 8, 9, 10,\}, \{0\}$

(iv) $\{m, a, t, h, e, i, c, s, g, o, r, y\}, \{e, m, t,\}, \{a, h, i, c, s\}, \{g, o, r, y\}$

3. (i) $\{a, c, e, g\}$ (ii) $\{b, c, f, g\}$ (iii) $\{a, b, c, e, f, g\}$ (iv) $\{c, g\}$ (v) $\{c, g\}$

(vi) $\{a, b, c, e, f, g\}$ (vii) $\{b, d, f, h\}$ (viii) $\{a, d, e, h\}$

4. (i) $\{0, 2, 4, 6\}$ (ii) $\{1, 4, 6\}$ (iii) $\{0, 1, 2, 4, 6\}$ (iv) $\{4, 6\}$ (v) $\{4, 6\}$

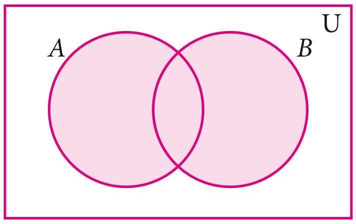
(vi) $\{0, 1, 2, 4, 6\}$ (vii) $\{1, 3, 5, 7\}$ (viii) $\{0, 2, 3, 5, 7\}$

5. (i) overlapping (ii) disjoint (iii) overlapping

6. (i) $\{1, 2, 7\}$ (ii) $\{m, o, p, q, j\}$ (iii) $\{6, 9, 10\}$

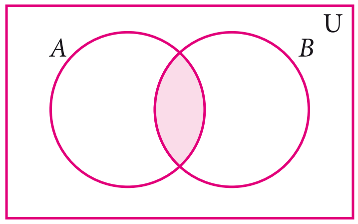
7. (i) $B-A$ (ii) $(A \cup B)'$ (iii) $(A - B) \cup (B - A)$

8. (i)



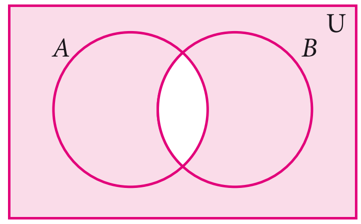
$A \cup B$

(ii)



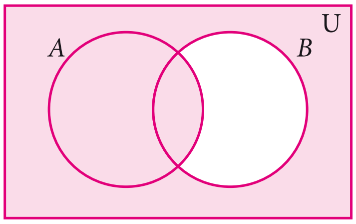
$A \cap B$

(iii)



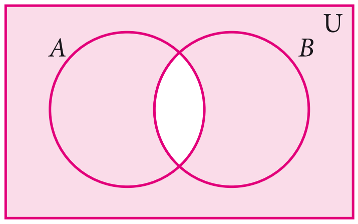
$(A \cap B)'$

(iv)



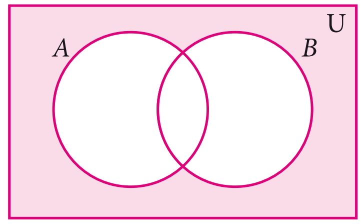
$(B - A)'$

(v)



$A' \cup B'$

(vi)



$A' \cap B'$

(vii) $(A \cap B)' = A' \cup B'$

Exercise 1.4

1. (i) 15, 65 (ii) 250, 600 4. (1) 17 (ii) 22 (iii) 47

5. (i) 12 (ii) 15 (iii) 32

6. (i) 20 (ii) 45 (iii) 35

7. (i) 10 (ii) 10 (iii) 25 8. 1000 9. 8

10. Not correct

Exercise 1.5

1. *b* 2. *a* 3. *c* 4. *b* 5. *d* 6. *a* 7. *b* 8. *d* 9. *b*

10. *c* 11. *d* 12. *b* 13. *b* 14. *a*

2 REAL NUMBERS

E4JDHU

" Numbers form Universe"

" Pythagoras"

A primitive digital computer was first designed in the year 1926 by American Mathematician and number theorist Norman Lehmer using bicycle chain and rods. It is used to find the primes and in factoring numbers of the form $2^{93} + 1$ in three seconds. Later in 1936, it was modified with 16mm films instead of chains and rods.

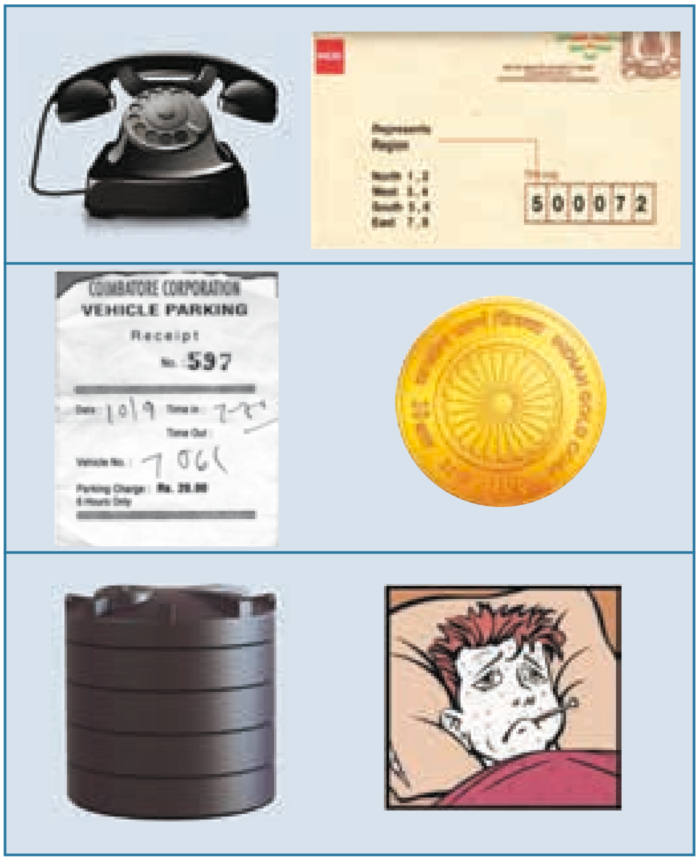
Even today his works on number theory is one of the pioneers in the basic ideas for designing Sieves in Integrated circuits or softwares in computers.

Derrick Norman Lehmer
1867-1930

Learning Outcomes

- To recall the concept of rational numbers.
- To know there exist of infinitely many rational numbers between two given rational numbers.
- To denote rational numbers on a number line.
- To represent rational numbers as terminating decimals.
- To express an irrational number in decimal form.
- To locate irrational numbers on the number line.
- To visualize the real numbers on the number line.

2.1 Introduction



Numbers, numbers, everywhere!

- Do you have a phone at home? How many digits does its dial have?
- What is the Pin code of your locality? How is it useful?
- When you park a vehicle, do you get a 'token'? What is its purpose?
- Have you handled 24 'carat' gold? How do you decide its purity?
- How high is the 'power' of your spectacles?
- How much water does the overhead tank in your house can hold?
- Does your friend have fever? What is his body temperature?

You have learnt about many types of numbers so far. Now is the time to extend the ideas further.

2.2 Rational Numbers

When you want to count the number of books in your cupboard, you start with 1, 2, 3, … and so on. These counting numbers 1, 2, 3, … , are called Natural numbers. You know to show these numbers on a line (see Fig. 2.1).

Fig. 2.1

We use $\mathbb{N}$ to denote the set of all natural numbers.

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

Suppose there are 5 books in your cupboard and you remove them one by one; the number of books diminish step by step. You remove one, it becomes 4, remove one more, it becomes 3, again one more is removed leaving 2, once again remove one and you are left with 1. If this last one is also taken out, the cupboard is empty (since no books are there). To denote such a situation we use the symbol 0. It denotes absence of any quantity. Thus to say “there are no books”, you can write “the number of books is zero”. Including zero as a digit you can now consider the numbers 0, 1, 2, 3, … and call them **Whole numbers**. With this additional entity, the number line will look as shown below

Fig. 2.2

We use $\mathbb{W}$ to denote the set of all Whole numbers.

$$\mathbb{W} = \{ 0, 1, 2, 3, \dots \}$$

Certain conventions lead to more varieties of numbers. Let us agree that certain conventions may be thought of as “**positive**” denoted by a ‘+’ sign. A thing that is ‘up’ or ‘forward’ or ‘more’ or ‘increasing’ is positive; and anything that is ‘down’ or ‘backward’ or ‘less’ or ‘decreasing’ is “**negative**” denoted by a ‘–’ sign.

For example, if I make a profit of Rs.1000 in my business, I would call that +1000, and if I lose Rs. 5000, that would be -5000. Why? Similarly, if a mountain’s base is 2 km below sea level and its peak is 3 km above sea level, then the altitude of its base is –2 and the altitude of its peak is +3. (What is its total height? Is it 5 km?).

With this understanding, you can treat natural numbers as positive numbers and rename them as **positive integers**; thereby you have enabled the entry of negative integers –1, –2, –3, … .

Note that –2 is “more negative” than –1. Therefore, among –1 and –2, you find that –2 is smaller and –1 is bigger. Are –2 and –1 smaller or greater than –3? Think about it.

The number line at this stage may be given as follows:

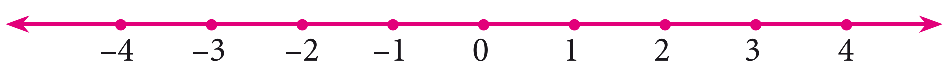


Fig. 2.3

We use $\mathbb{Z}$ to denote the set of all **Integers**.

$$\mathbb{Z} = \{ \dots, -3, -2, -1, 0, 1, 2, 3, \dots \}.$$

Draw a copy of Fig 2.2. Hold your whole number line up to a mirror on zero. You will see the natural numbers reflected in the mirror. The reflected numbers attached with a minus sign are negative integers. So the numbers to the left of 0 are negative, and the numbers to the right of 0 are positive. But 0 is neither negative nor positive; 0 is just 0. It’s non-committal!

When you look at the figures (Fig. 2.2 and 2.3) above, you are sure to get amused by the gap between any pair of consecutive integers. Could there be some numbers in between?

How did you actually draw the number line $\mathbb{N}$ (Fig. 2.1) initially? Draw any line, mark a point 1 on it. From 1, choose another point on its right side at a preferred 'unit' distance and call it 2. Repeat this as many times as you desire. To get $\mathbb{W}$ (Fig. 2.2), from 1, go one unit on the left to get 0. Now $\mathbb{Z}$ is easier; just repeat the exercise on the left side.

You have come across fractions already. How will you mark the point that shows $\frac{1}{2}$ on $\mathbb{Z}$? It is just midway between 0 and 1. In the same way, you can plot $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, 2\frac{3}{4}$ ... etc. You may find that many different fractions are shown by the same point. Can you say 'why'? Will $\frac{5}{4}$ and $\frac{10}{8}$ be represented by the same point? Do you think $\frac{7}{9}$ and $\frac{35}{55}$ represent the same point? You will now easily visualize similar fractions on the left side of zero. These are all fractions of the form $\frac{a}{b}$ where a and b are integers with one restriction that $b \neq 0$. (Why?) If a fraction is in decimal form, even then the setting is same.

Because of the connection between fractions and ratios of lengths, we name them as **Rational numbers**. Here is a rough picture of the situation:

Fig. 2.4

> A **rational number** is a fraction indicating the quotient of two integers, excluding division by zero.

Since a fraction can have many equivalent fractions , there are many possible names for the same rational number. Thus $\frac{1}{3}, \frac{2}{6}, \frac{8}{24}$ all these denote the same rational number.

Progress check

(a) Of the two numbers in each pair given, which is to the right of the other on the number line?

(i) 4, –5 (ii) –4, –5 (iii) $\frac{16}{3}, \frac{21}{4}$

(b) Consider $\frac{12}{48}$. What type of number is it? Natural? Whole? Integer? Rational?

(c) Discuss similarly the nature of (i) the number $\frac{0}{11}$ and (ii) the number $\frac{-6}{-3}$.

(d) Draw a number line and plot the numbers 5, $-1\frac{2}{3}$, 2.5, $\frac{5}{2}$ and 2^2.

2.2.1 Denseness Property of Rational Numbers

Consider a, b where $a > b$ and their AM(Arithmetic Mean) given by $\frac{a+b}{2}$. Is this AM a rational number? Let us see.

If $a = \frac{p}{q}$ (p, q integers and $q \neq 0$); $b = \frac{r}{s}$ (r, s integers and $s \neq 0$), then

$$\frac{a+b}{2} = \frac{\frac{p}{q}+\frac{r}{s}}{2} = \frac{ps+qr}{2qs}$$ which is a rational number.

We have to show that this rational number lies between a and b.

$$a - \left(\frac{a+b}{2}\right) = \frac{2a-a-b}{2} = \frac{a-b}{2}$$ which is > 0 since $a>b$.

Therefore, $a > \left(\frac{a+b}{2}\right)$ ... (1)

$$\left(\frac{a+b}{2}\right) - b = \frac{a+b-2b}{2} = \frac{a-b}{2}$$ which is > 0 since $a>b$.

Therefore, $\left(\frac{a+b}{2}\right) > b$ ... (2)

From (1) and (2) we see that $a > \left(\frac{a+b}{2}\right) > b$, which can be visualized as follows:

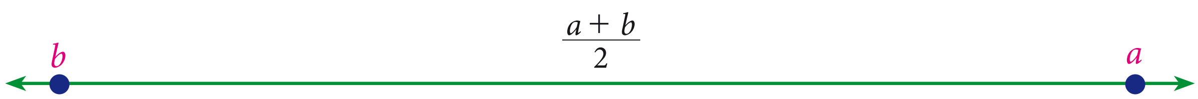


Fig. 2.5

Thus, for any two rational numbers, their average/midpoint is rational. We can repeat this process indefinitely to produce infinitely many rational numbers.

Example 2.1

Find any two rational numbers between $\frac{1}{2}$ and $\frac{2}{3}$.

Solution 1

A rational number between $\frac{1}{2}$ and $\frac{2}{3} = \frac{1}{2}\left(\frac{1}{2}+\frac{2}{3}\right) = \frac{1}{2}\left(\frac{3+4}{6}\right) = \frac{1}{2}\left(\frac{7}{6}\right) = \frac{7}{12}$

A rational number between $\frac{1}{2}$ and $\frac{7}{12} = \frac{1}{2}\left(\frac{1}{2}+\frac{7}{12}\right) = \frac{1}{2}\left(\frac{6+7}{12}\right) = \frac{1}{2}\left(\frac{13}{12}\right) = \frac{13}{24}$

Hence two rational numbers between $\frac{1}{2}$ and $\frac{2}{3}$ are $\frac{7}{12}$ and $\frac{13}{24}$ (of course, there are many more!)

There is an interesting result that could help you to write instantly rational numbers between any two given rational numbers.

Result

If $\frac{p}{q}$ and $\frac{r}{s}$ are any two rational numbers such that $\frac{p}{q} < \frac{r}{s}$, then $\frac{p+q}{r+s}$ is a rational number, such that $\frac{p}{q} < \frac{p+q}{r+s} < \frac{r}{s}$.

Let us take the same example: Find any two rational numbers between $\frac{1}{2}$ and $\frac{2}{3}$

Solution 2

$\frac{1}{2}<\frac{2}{3}$ gives $\frac{1}{2}<\frac{1+2}{2+3}<\frac{2}{3}$ *or* $\frac{1}{2}<\frac{3}{5}<\frac{2}{3}$ gives $\frac{1}{2}<\frac{1+3}{2+5}<\frac{3}{5}<\frac{3+2}{5+3}<\frac{2}{3}$ *or* $\frac{1}{2}<\frac{4}{7}<\frac{3}{5}<\frac{5}{8}<\frac{2}{3}$

Solution 3

Any more new methods to solve? Yes, if decimals are your favourites, then the above example can be given an alternate solution as follows:

$$\frac{1}{2} = 0.5 \text{ and } \frac{2}{3} = 0.66...$$

Hence rational numbers between $\frac{1}{2}$ and $\frac{2}{3}$ can be listed as $0.51, 0.57, 0.58, \ldots$

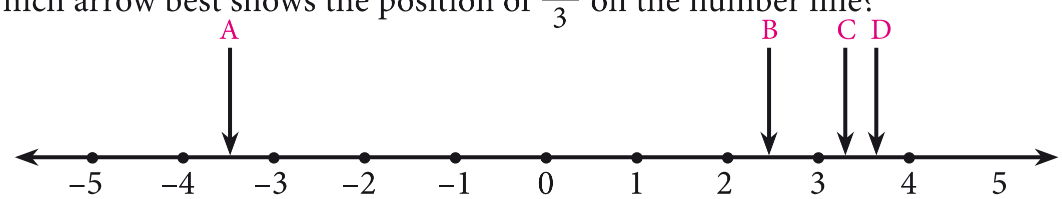
Solution 4

There is one more way to solve some problems. For example, to find four rational numbers between $\frac{4}{9}$ and $\frac{3}{5}$, note that the LCM of 9 and 5 is 45; so we can write $\frac{4}{9}=\frac{20}{45}$ and $\frac{3}{5}=\frac{27}{45}$.

Therefore, four rational numbers between $\frac{4}{9}$ and $\frac{3}{5}$ are $\frac{21}{45}, \frac{22}{45}, \frac{23}{45}, \frac{24}{45}, \ldots$

Exercise 2.1

1. Which arrow best shows the position of $\frac{11}{3}$ on the number line?



2. Find any three rational numbers between $\frac{-7}{11}$ and $\frac{2}{11}$.

3. Find any five rational numbers between (i) $\frac{1}{4}$ and $\frac{1}{5}$ (ii) 0.1 and 0.11 (iii) –1 and –2

2.3 Irrational Numbers

You saw that each rational number is assigned to a point on the number line and learnt about the denseness property of the rational numbers. Does that mean the line is entirely filled with the rational numbers and there are no more numbers on the number line? Let us explore.

Consider an isosceles right-angled triangle whose legs are each 1 unit long. Using Pythagoras theorem, the hypotenuse can be seen having a length $\sqrt{1^2+1^2}=\sqrt{2}$ (see Fig. 2.6). Greeks found that this

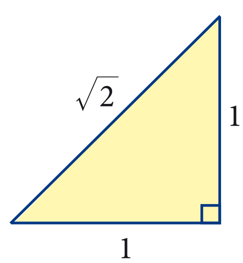


Fig.2.6

$\sqrt{2}$ is neither a whole number nor an ordinary fraction. The belief of relationship between points on the number line and all numbers was shattered! $\sqrt{2}$ was called an irrational number.

An irrational number is a number that cannot be expressed as an ordinary ratio of two integers.

A natural question is how one knows that $\sqrt{2}$ is irrational. It is not difficult to justify it.

If $\sqrt{2}$ is really rational, let it be equal to $\frac{p}{q}$ where p, q are integers without any common factors (so that $\frac{p}{q}$ will be in its simplest form) and $q \neq 0$.

Now, $\frac{p^2}{q^2} = \left(\frac{p}{q}\right)^2 = (\sqrt{2})^2 = 2$

So, $p^2 = 2q^2$ ... (1)

which means p^2 is even ... (2)

As a result, p is even. .... (3)

(Can you prove this?)

Let $p = 2m$ (How?); you get $p^2 = 4m^2$;

This, because of (1)

$\Rightarrow 2q^2 = 4m^2$ or $q^2 = 2m^2$.

As a result q is even ... (4).

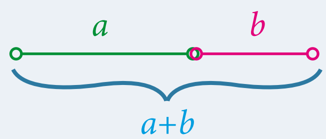
(3) and (4) show that p and q have a common factor 2.

This contradicts our assumption that p and q have no common factors and hence our assumption that $\sqrt{2}$ can be written as $\frac{p}{q}$ is wrong. That is, $\sqrt{2}$ is not rational.

GOLDEN RATIO

Φ The Golden Ratio has been heralded as the most beautiful ratio in art and architecture.

Take a line segment and divide it into two smaller segments such that the ratio of the whole line segment (a+b) to segment a is the same as the ratio of segment a to the segment b.



$a+b : a = a : b$

This gives the proportion $\frac{a+b}{a} = \frac{a}{b}$

Notice that a is the geometric mean of a+b and b.

Examples

1. Apart from $\sqrt{2}$, one can produce a number of examples for such irrational numbers. Here are a few: $\sqrt{5}, \sqrt{7}, 2\sqrt{3}, \ldots$
2. π , the ratio of the circumference of a circle to the diameter of that same circle, is another example for an irrational number.
3. e, also known as Euler's number, is another common irrational number.

4. Φ , the golden ratio, also known as golden mean, or golden section, is a number often stumbled upon when taking the ratios of distances in simple geometric figures such as the pentagon, the pentagram, decagon and dodecahedron, etc., it is an irrational number.

2.3.1 Irrational Numbers on the Number Line

Where are the points on the number line that correspond to the irrational numbers? As an example, let us locate $\sqrt{2}$ on the number line. This is easy.

Remember that $\sqrt{2}$ is the length of the diagonal of the square whose side is 1 unit (How?)Simply construct a square and transfer the length of one of its diagonals to our number line. (see Fig.2.7).

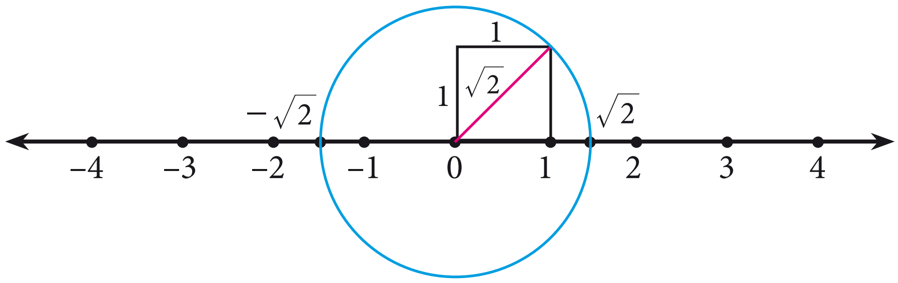
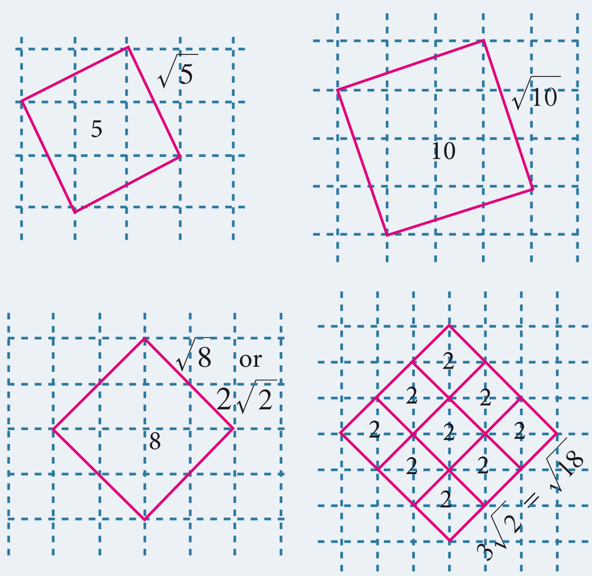


Fig.2.7

Squares on grid sheets can be used to produce irrational lengths.

Here are a few examples :



We draw a circle with centre at 0 on the number line,with a radius equal to that of diagonal of the square. This circle cuts the number line in two points, locating $\sqrt{2}$ on the right of 0 and $-\sqrt{2}$ on its left. (You wanted to locate $\sqrt{2}$; you have also got a bonus in $-\sqrt{2}$)

You started with Natural numbers and extended it to rational numbers and then irrational numbers. You may wonder if further extension on the number line waits for us. Fortunately it stops and you can learn about it in higher classes.

Representation of a Rational number as terminating and non terminating decimal helps us to understand irrational numbers. Let us see the decimal expansion of rational numbers.

2.3.2 Decimal Representation of a Rational Number

If you have a rational number written as a fraction, you get the decimal representation by long division. Study the following examples where the remainder is 0 always:

Consider the examples,

$$\begin{array}{r|l} & 0.875 \\ \hline 8 & 7.000 \\ & 64 \\ \hline & \;\;60 \\ & \;\;56 \\ \hline & \;\;\;\;40 \\ & \;\;\;\;40 \\ \hline & \;\;\;\;\;\;0 \end{array}$$

$$\frac{7}{8} = 0.875$$

$$\begin{array}{r|l} & 0.84 \\ \hline 25 & 21.00 \\ & 200 \\ \hline & \;\;100 \\ & \;\;100 \\ \hline & \;\;\;\;0 \end{array}$$

$$\frac{21}{25} = 0.84$$

$$\begin{array}{r|l} & 2.71875 \\ \hline 32 & 87.00000 \\ & 64 \\ \hline & 230 \\ & 224 \\ \hline & \;\;60 \\ & \;\;32 \\ \hline & \;\;280 \\ & \;\;256 \\ \hline & \;\;\;\;240 \\ & \;\;\;\;224 \\ \hline & \;\;\;\;\;\;160 \\ & \;\;\;\;\;\;160 \\ \hline & \;\;\;\;\;\;\;\;0 \end{array}$$

$$\frac{-87}{32} = -2.71875$$

Note

These show that the process could lead to a decimal with finite number of decimal places. They are called terminating decimals.

Can the decimal representation of a rational number lead to forms of decimals that do not terminate? The following examples (with non-zero remainder) throw some light on this point.

Example 2.2

Represent the following as decimal form (i) $\frac{-4}{11}$ (ii) $\frac{11}{75}$

Solution

$$\begin{array}{r|l} & 0.3636.... \\ \hline 11 & 4.0000 \\ & 33 \\ \hline & \;\;70 \\ & \;\;66 \\ \hline & \;\;\;\;40 \\ & \;\;\;\;33 \\ \hline & \;\;\;\;\;\;70 \\ & \;\;\;\;\;\;66 \\ \hline & \;\;\;\;\;\;\;\;4 \\ & \;\;\;\;\;\;\;\;\vdots \end{array}$$

$$\begin{array}{r|l} & 0.1466... \\ \hline 75 & 11.0000 \\ & 7\,5 \\ \hline & \;\;350 \\ & \;\;300 \\ \hline & \;\;\;\;500 \\ & \;\;\;\;450 \\ \hline & \;\;\;\;\;\;500 \\ & \;\;\;\;\;\;450 \\ \hline & \;\;\;\;\;\;\;\;50 \\ & \;\;\;\;\;\;\;\;\vdots \end{array}$$

Note

In this case the decimal expansion does not terminate! The remainders repeat again and again! We get non-terminating but recurring block of digits.

Thus we see that, $\frac{-4}{11} = -0.\overline{36}$ $\qquad \frac{11}{75} = 0.14\overline{6}$

A rational number can be expressed by

(i) **either** a terminating

(ii) **or** a non-terminating and recurring (repeating) decimal expansion.

The converse of this statement is also true.

That is, if the decimal expansion of a number is terminating or non-terminating and recurring, then the number is a rational number.

The reciprocals of Natural Numbers are Rational numbers. It is interesting to note their decimal forms. See the first ten.

S.No.	Reciprocal	Decimal Representation
1	$\frac{1}{1} = 1.0$	Terminating
2	$\frac{1}{2} = 0.5$	Terminating
3	$\frac{1}{3} = 0.\overline{3}$	Non-terminating Recurring
4	$\frac{1}{4} = 0.25$	Terminating
5	$\frac{1}{5} = 0.2$	Terminating
6	$\frac{1}{6} = 0.1\overline{6}$	Non-terminating Recurring
7	$\frac{1}{7} = 0.\overline{142857}$	Non-terminating Recurring
8	$\frac{1}{8} = 0.125$	Recurring
9	$\frac{1}{9} = 0.\overline{1}$	Non-terminating Recurring
10	$\frac{1}{10} = 0.1$	Recurring

Progress check

Convert the following rational numbers into decimals:

1. a) $\frac{3}{4}$ b) $\frac{5}{8}$ c) $\frac{9}{25}$
2. a) $\frac{2}{3}$ b) $\frac{47}{99}$ c) $-\frac{16}{45}$
3. Can we write 0.25 as 0.250000… ? Can a terminating decimal be written as a recurring decimal?

2.3.3 Period of Decimal

In the decimal expansion of the rational numbers, the number of repeating decimals is called the length of the period of decimals.

For example,

(i) $\frac{25}{7} = 3.\overline{571428}$ has the length of the period of decimal = 6

(ii) $\frac{27}{110} = 0.2\overline{45}$ has the length of the period of decimal = 2

Example 2.3

Express the rational number $\frac{1}{27}$ in recurring decimal form by using the recurring decimal expansion of $\frac{1}{3}$. Hence write $\frac{59}{27}$ in recurring decimal form.

Solution

We know that $\frac{1}{3} = 0.\overline{3}$

Therefore, $\frac{1}{27} = \frac{1}{9} \times \frac{1}{3} = \frac{1}{9} \times 0.333... = 0.037037... = 0.\overline{037}$

Also, $\frac{59}{27} = 2\frac{5}{27} = 2 + \frac{5}{27}$

$= 2 + \left(5 \times \frac{1}{27}\right)$

$= 2 + (5 \times 0.\overline{037}) = 2 + (5 \times 0.037037037...) = 2 + 0.185185... = 2.185185... = 2.\overline{185}$

2.3.4 Conversion of Terminating Decimals into Rational Numbers

Let us now try to convert a terminating decimal, say 2.945 as rational number in the fraction form.

You know that $0.1 = \frac{1}{10}$; $0.01 = \frac{1}{100}$; $0.001 = \frac{1}{1000}$ etc.

That is, in any decimal number, each digit after the decimal point is a fraction with a denominator in increasing powers of 10. Thus,

$$\begin{aligned} 2.945 &= 2 + 0.945 \\ &= 2 + \frac{9}{10} + \frac{4}{100} + \frac{5}{1000} \\ &= 2 + \frac{900}{1000} + \frac{40}{1000} + \frac{5}{1000} \text{ (making denominators common)} \\ &= 2 + \frac{945}{1000} \\ &= \frac{2945}{1000} \text{ or } \frac{589}{200} \text{ which is required} \end{aligned}$$

(In the above, is it possible to write directly $2.945 = \frac{2945}{1000}$?)

Example 2.4

Convert the following decimal numbers in the form of $\frac{p}{q}$, where p and q are integers and $q \neq 0$: (i) 0.35 (ii) 2.176 (iii) – 0.0028

Solution

(i) $0.35 = \frac{35}{100} = \frac{7}{20}$

(ii) $2.176 = \frac{2176}{1000} = \frac{272}{125}$

(iii) $-0.0028 = \frac{-28}{10000} = \frac{-7}{2500}$

2.3.5 Conversion of Non-terminating and recurring decimals into Rational Numbers

It was very easy to handle a terminating decimal. When we come across a decimal such as 2.4, we get rid of the decimal point, by just using division by 10.

Thus $2.4 = \frac{24}{10}$, which is simplified as $\frac{12}{5}$. But, when we have a decimal such as $2.\overline{4}$, the problem is that we have infinite number of 4s and hence will need infinite number of 0s in the denominator. For example,

$$2.4 \quad = 2 + \frac{4}{10}$$

$$2.44 \quad = 2 + \frac{4}{10} + \frac{4}{100}$$

$$2.444 = 2 + \frac{4}{10} + \frac{4}{100} + \frac{4}{1000}$$

$$\vdots$$

How tough it is to have infinite 4's and work with them. We need to get rid of the infinite sequence in some way. The good thing about the infinite sequence is that even if we pull away one , two or more 4 out of it, the sequence still remains infinite.

Let $x = 2.\overline{4}$ ...(1)

Then, $10x = 24.\overline{4}$ ...(2) [When you multiply by 10, the decimal moves one place to the right but you still have infinite 4s left over).

Subtract the first equation from the second to get,

$9x = 24.\overline{4} - 2.\overline{4} = 22$ (Infinite 48 subtract out the infinite 4s and the left out is $24 - 2 = 22$)

$x = \frac{22}{9}$, the required value.

We use the same exact logic to convert any number with a non terminating repeating part into a fraction.

Example 2.5

Convert the following decimal numbers in the form of $\frac{p}{q}$ $(p, q \in Z$ and $q \neq 0)$.

(i) $0.\overline{3}$ (ii) $2.12\overline{4}$ (iii) $0.\overline{45}$ (iv) $0.\overline{568}$

Solution

(i) Let $x = 0.\overline{3} = 0.3333\ldots$ (1)

(Here period of decimal is 1, multiply equation (1) by 10)

$10\,x = 3.3333...$ (2)

$(2) - (1):\quad 9\,x = 3$ or $x = \frac{1}{3}$

(ii) Let $x = 2.\overline{124} = 2.124124124\ldots$ (1)

(Here period of decimal is 3, multiply equation (1) by 1000.)

$1000\,x = 2124.124124124\ldots$ (2)

$(2)-(1):\quad 999\,x = 2122$ $\quad x = \frac{2122}{999}$

(iii) Let $x = 0.4\overline{5} = 0.45555\ldots$ (1)

(Here the repeating decimal digit is 5, which is the second digit after the decimal point, multiply equation (1) by 10)

$10\,x = 4.5555\ldots$ (2)

(Now period of decimal is 1, multiply equation (2) by 10)

$100\,x = 45.5555\ldots$ (3)

$(3) - (2):\quad 90\,x = 41$ or $x = \frac{41}{90}$

(iv) Let $x = 0.5\overline{68} = 0.5686868\ldots$ (1)

(Here the repeating decimal digit is 68, which is the second digit after the decimal point, so multiply equation (1) by 10)

$10\,x = 5.686868\ldots$ (2)

(Now period of decimal is 2, multiply equation (2) by 100)

$1000\,x = 568.686868\ldots$ (3)

$(3) - (2):\quad 990\,x = 563$ or $x = \frac{563}{990}$.

Note

To determine whether the decimal form of a rational number will terminate or non - terminate, we can make use of the following rule

If a rational number $\frac{p}{q}$, $q \neq 0$ can be expressed in the form $\frac{p}{2^m \times 5^n}$, where $p \in \mathbb{Z}$ and $m, n \in \mathbb{W}$, then rational number will have a terminating decimal expansion. Otherwise, the rational number will have a non- terminating and recurring decimal expansion

Example 2.6

Without actual division, classify the decimal expansion of the following numbers as terminating or non – terminating and recurring.

(i) $\frac{13}{64}$ (ii) $\frac{-71}{125}$ (iii) $\frac{43}{375}$ (iv) $\frac{31}{400}$

Solution

(a) $\frac{13}{64} = \frac{13}{2^6}$, So $\frac{13}{64}$ has a terminating decimal expansion.

(b) $\frac{-71}{125} = \frac{-71}{5^3}$, So $\frac{-71}{125}$ has a terminating decimal expansion.

(c) $\frac{43}{375} = \frac{43}{3^1 \times 5^3}$, So $\frac{43}{375}$ has a non – terminating recurring decimal expansion.

(d) $\frac{31}{400} = \frac{31}{2^4 \times 5^2}$, So $\frac{31}{400}$ has a terminating decimal expansion.

Example 2.7

Verify that $1 = 0.\overline{9}$

Solution

Let $x = 0.\overline{9} = 0.99999\ldots$ (1)

(Multiply equation (1) by 10)

$10\,x = 9.99999\ldots$ (2)

Subtract (1) from (2)

$9x = 9$ or $x = 1$

Thus, $0.\overline{9} = 1$

$1 = 0.9999\ldots$

$7 = 6.9999\ldots$

$3.7 = 3.6999\ldots$

The pattern suggests that any terminating decimal can be represented as a non-terminating and recurring decimal expansion with an endless block of 9's.

Exercise 2.2

1. Express the following rational numbers into decimal and state the kind of decimal expansion

   (i) $\frac{2}{7}$ (ii) $\frac{35}{100}$ (iii) $-5\frac{3}{11}$ (iv) $\frac{22}{3}$ (v) $\frac{-9}{32}$ (vi) $\frac{327}{200}$

2. Express $\frac{1}{13}$ in decimal form. Find the length of the period of decimals.

3. Express the rational number $\frac{1}{33}$ in recurring decimal form by using the recurring decimal expansion of $\frac{1}{11}$. Hence write $\frac{71}{33}$ in recurring decimal form.

4. Express the following decimal expression into rational numbers.

   (i) $0.\overline{24}$ (ii) $2.\overline{327}$ (iii) 0.86 (iv) -5.132

   (v) $3.1\overline{7}$ (vi) $17.2\overline{15}$ (vii) $0.\overline{0001}$ (viii) $-21.213\overline{7}$

5. Without actual division, find which of the following rational numbers have terminating decimal expansion.

(i) $\frac{7}{128}$ (ii) $\frac{21}{15}$ (iii) $\frac{19}{125}$ (iv) $4\frac{9}{35}$

(v) $\frac{387}{800}$ (vi) $\frac{219}{2200}$

2.3.6 Decimal Representation to Identify Irrational Numbers

It can be shown that irrational numbers, when expressed as decimal numbers, do not terminate, nor do they repeat. For example, the decimal representation of the number π starts with 3.14159265358979, but no finite number of digits can represent π exactly, nor does it repeat.

Consider the following decimal expansions:

(i) 0.10110011100011111… (ii) 3.012012120121212…

(iii) 12.230223300222333000… (in) $\sqrt{2}$ = 1.4142135624…

Are the above numbers terminating (or) recurring and non- terminating? No… They are neither terminating, nor non–terminating and recurring. Hence they are not rational numbers. They cannot be written in the form of $\frac{p}{q}$,where p, q ,$\in \mathbb{Z}$ and $q \neq 0$. They are irrational numbers.

A number having non- terminating and non- recurring decimal expansion is an irrational number.

Example2.8

Find the decimal expansion of $\sqrt{3}$.

Solution

	1.7320508…
1	3.00,00,00,00,00,…
	1
27	200
	189
343	1100
	1029
3462	7100
	6924
346405	1760000
	1732025
34641008	279750000
	277128064
	2621936

We often write $\sqrt{2}$ = 1.414…., $\sqrt{3}$ = 1.732…, π = 3.141, etc. These are only approximate values and not exact values. In the case of the irrational number π, we take frequently $\frac{22}{7}$ (which gives the value 3.142857…) to be its correct value but in reality these are only approximations. This is because, the decimal expansion of an irrational number is non-terminating and non-recurring. None of them gives an exact value!

Thus, by division method, $\sqrt{3} = 1.7320508\ldots$

It is found that the square root of every positive non perfect square number is an irrational number. $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \ldots$ are all irrational numbers.

Activity 1

Constructing the 'square root spiral'

When calculators were not yet invented, it was a tough job to approximate irrational numbers. The irrational numbers was the offshoot of the discovery that the side length of the diagonal of a square with side length 1 is irrational. But how did mathematicians of the ancient time approximate a segment with length, say, $\sqrt{7}$?

Here is a procedure to construct what is known as the 'square root spiral'

Step:1 Start with point A and draw a line segment $\overline{AB}$ of 1 unit length.

Step:2 Draw a line segment $\overline{BC}$, perpendicular to $\overline{AB}$ of 1 unit length (where $AB=BC=1$)

Step:3 Join AC. ($AC = \sqrt{2}$)

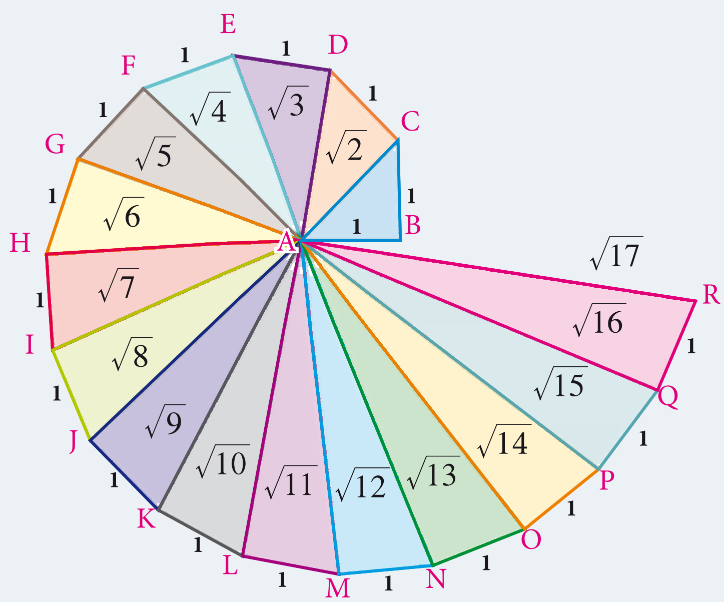


Fig.2.8

Step:4 Draw a line segment $\overline{CD}$ of 1 unit length perpendicular to $\overline{AC}$.

Step:5 Join AD. ($AD = \sqrt{3}$).

Step:6 Continue in this manner for more number of steps , you will create a beautiful spiral made of line segments $AC, AD, AE, \ldots$ etc.

Observe that, $\overline{AC}, \overline{AD}, \overline{AE}, \ldots$ denote the lengths $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \ldots$ respectively

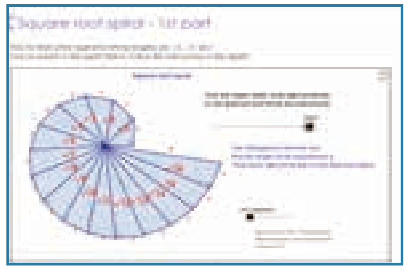
Example 2.9

Classify the numbers as rational or irrational:

(i) $\sqrt{10}$ (ii) $\sqrt{49}$ (iii) 0.025 (iv) $0.7\overline{6}$ (v) 2.505500555... (vi) $\frac{\sqrt{2}}{2}$

ICT Corner

Expected Result is shown in this picture



Step – 1

Open the Browser and copy and paste the Link given below (or) by typing the URL given (or) Scan the QR Code.

Step - 2

GeoGebra workbook named "Real Numbers" will open. There are several worksheets in the workbook. Open the worksheet named "Square root spiral – 1st part"

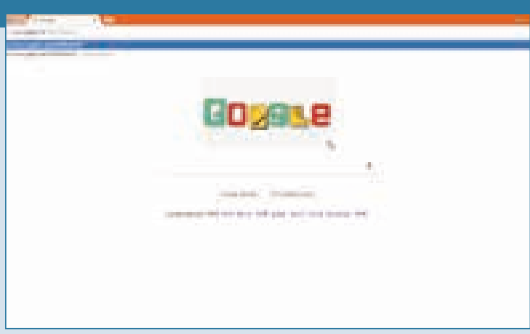
Step-3

Drag the slider named "Steps ". The construction of Square root of numbers 2,3,4,5,…. will appear step by step.

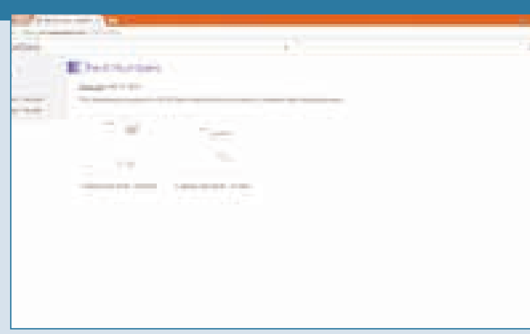
Step-4

By dragging the slider named "Unit segment" you can enlarge the diagram for more clarity. Now you can draw the same in a paper and measure the values obtained

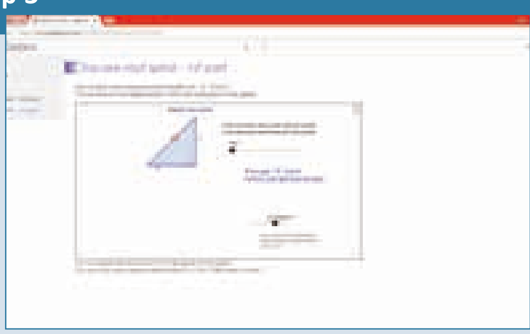
Step 1



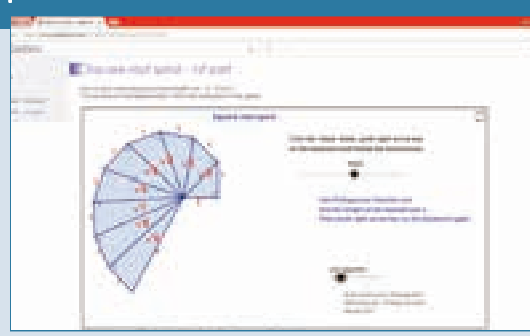
Step 2



Step 3



Step 4



Browse in the link

Union of Sets: https://www.ggbm.at/m6GQc6mQ

Solution

(i) $\sqrt{10}$ is an irrational number (since 10 is not a perfect square number).

(ii) $\sqrt{49} = 7 = \frac{7}{1}$, a rational number(since 49 is a perfect square number).

(iii) 0.025 is a rational number (since it is a terminating decimal).

(iv) $0.7\overline{6} = 0.7666....$ is a rational number (since it is a non – terminating and recurring decimal expansion).

(v) 2.505500555.... is an irrational number (since it is a non – terminating and non–recurring decimal).

(vi) $\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \frac{1}{\sqrt{2}}$ is an irrational number (since 2 is not a perfect square number).

Note

The above example(vi) it is not to be misunderstood as $\frac{p}{q}$ form, because both p and q must be integers and not an irrational number.

Example 2.10

Locate an irrational number between two rational numbers $\frac{23}{10}$ and $\frac{12}{5}$.

Solution

$\frac{23}{10}$ is 2.3 and $\frac{12}{5}$ is 2.4

You need an irrational number greater than 2.3 but less than 2.4

One such irrational number is

2.301001000100001000001000000100000001........

How do you write it?

See the given box for details.

Is this newly found number irrational?

Yes, observe that it is non-terminating and non-recurring.

After 2.3
place 01
then 001
then 0001
then 00001
⋮
and so on

Note

This procedure is possible whenever any two rational numbers are given.
Write the smaller rational number as a terminating decimal; after its last decimal place put 01 then 001 then 0001 then 00001 ... ... so on and on. That will be an irrational number between the two given rational numbers.)

Can you find an irrational number between 63 and 64 now?

Could it be 63.0100100010000100000100000001… … …   ?

Between the rational numbers 7.568903 and 7.568904, is it possible that there is this irrational number 7.568903 01 001 0001 00001 000001 0000001… … …?

Example 2.11

Find any 4 irrational numbers between $\frac{1}{4}$ and $\frac{1}{3}$.

Solution

$\frac{1}{4} = 0.25$ and $\frac{1}{3} = 0.3333\ldots.. = 0.\overline{3}$

In between 0.25 and $0.\overline{3}$, there are infinitely many irrational numbers .

Four irrational numbers between 0.25 and $0.\overline{3}$ are

0.2601001000100001…..

0.2701001000100001…..

0.2801001000100001…..

0.3101001000100001…..

Example 2.12

Find any 3 irrational numbers between 0.12 and 0.13 .

Solution

Three irrational numbers between 0.12 and 0.13 are 0.12010010001…, 0.12040040004…, 0.12070070007…

Example 2.13

Give any two rational numbers lying between 0.5151151115…. and 0.5353353335…

Solution

Two rational numbers between the given two irrational numbers are 0.5152 and 0.5352

Note

We state (without proof) an important result worth remembering.

If 'a' is a rational number and $\sqrt{b}$ is an irrational number then each one of the following is an irrational number:

(i) $a+\sqrt{b}$; (ii) $a-\sqrt{b}$; (iii) $a\sqrt{b}$; (iv) $\frac{a}{\sqrt{b}}$; (v) $\frac{\sqrt{b}}{a}$.

For example, when you consider the rational number 4 and the irrational number $\sqrt{5}$, then $4+\sqrt{5}$, $4-\sqrt{5}$, $4\sqrt{5}$, $\frac{4}{\sqrt{5}}$, $\frac{\sqrt{5}}{4}$ ------- are all irrational numbers.

Progress check

1. i) We used to write π as $\frac{22}{7}$. Can we say π is a rational number ?
2. ii) We used to write $\sqrt{2} = 1.414 = \frac{1414}{1000}$. Can we say $\sqrt{2}$ is a rational number?
3. Identify the type of numbers needed to solve the simple equations given here. Question 1 and 2 are solved for you, as examples. (Perhaps the number systems developed gradually depending on the needs of 'answers' during such problem solving!)

S.No.	Equation	Solution	Type of number
1	$x - 7 = 17$	$x = 24$	Natural number
2	$x + 5 = 5$	$x = 0$	Whole number
3	$x + 1 = 9$		
4	$x + 9 = 1$		
5	$7x = 19$		
6	$5x = -3$		
7	$x^2 - 2 = 0$		

4. Create a word problem whose solution is an irrational number.

Exercise 2.3

1. Represent the following irrational numbers on the number line.

   (i) $\sqrt{3}$ (ii) $\sqrt{4.7}$ (iii) $\sqrt{6.5}$

2. Find any two irrational numbers between

   (i) 0.3010011000111…. and 0.3020020002….

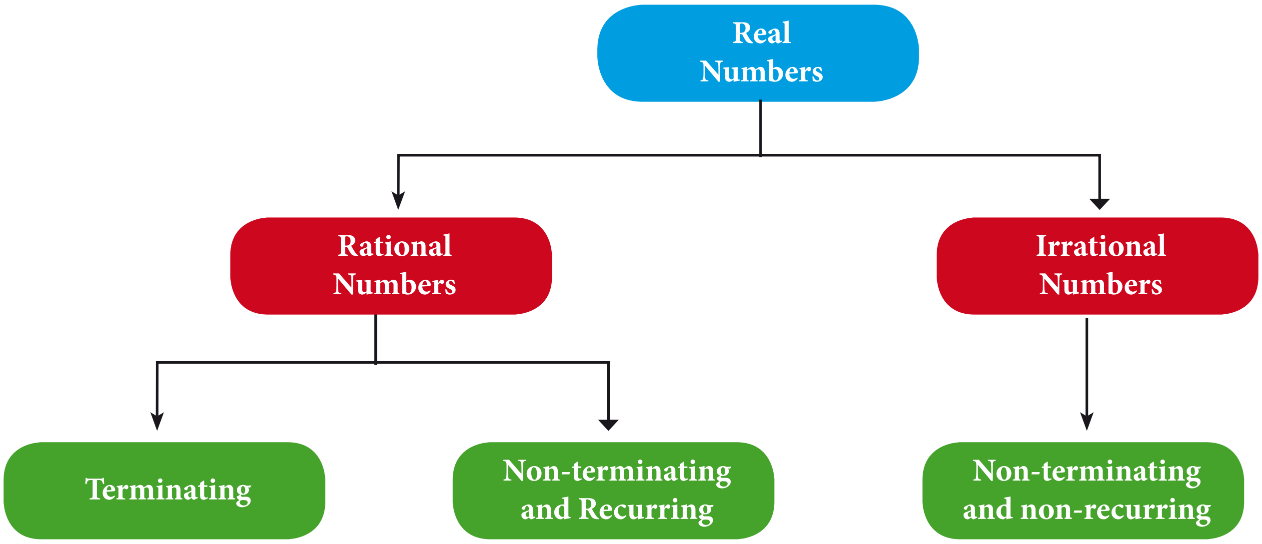
   (ii) $\frac{6}{7}$ and $\frac{12}{13}$ (iii) $\sqrt{2}$ and $\sqrt{3}$ (iv) 0.12 and 0.13

3. Find any two rational numbers between 2.2360679….. and 2.236505500….

2.4 Real Numbers

The real numbers consist of all the rational numbers and all the irrational numbers.

Real numbers can be thought of as points on an infinitely long number line called the real line, where the points corresponding to integers are equally spaced.



Any real number can be determined by a possibly infinite decimal representation, (as we have already seen decimal representation of the rational numbers and the irrational numbers).

A Puzzle

$$\begin{aligned}1 &= \sqrt{1}\\ &= \sqrt{(-1)(-1)}\\ &= \sqrt{(-1)}\,\sqrt{(-1)}\\ &= -1\end{aligned}$$

How? Look at the third line . Is $\sqrt{(-1)}$ defined?

Example 2.14

Represent $\sqrt{9.3}$ on a number line.

Solution

- Draw a line and mark a point A on it.
- Mark a point B such that $AB = 9.3$ cm.
- Mark a point C on this line such that $BC = 1$ unit.
- Find the midpoint of AC by drawing perpendicular bisector of AC and let it be O
- With O as center and $OC = OA$ as radius, draw a semicircle.
- Draw a line BD, which is perpendicular to AB at B.
- Now $BD = \sqrt{9.3}$, which can be marked in the number line as the value of $BE = BD = \sqrt{9.3}$

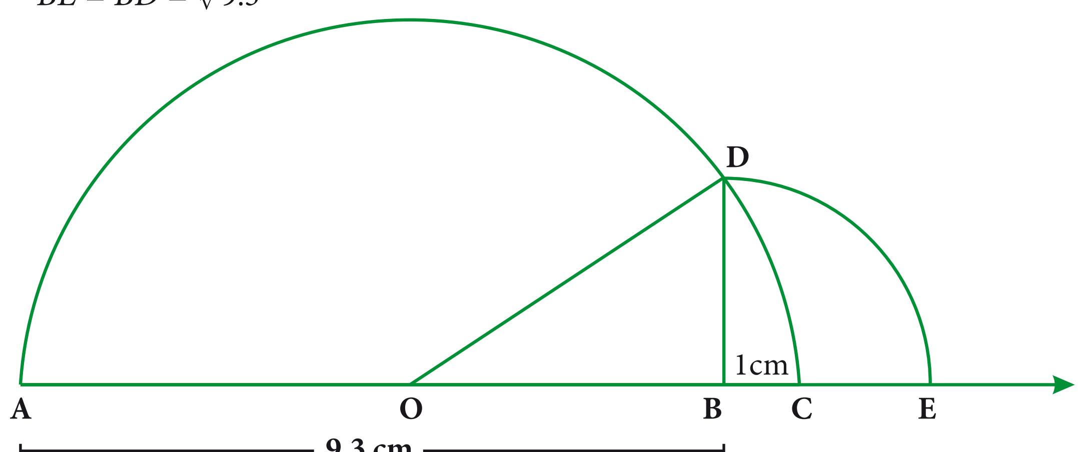


Fig.2.9

Activity 2

Consider the ten real numbers,

$\sqrt{2}, \frac{7}{9}, -1.32, \frac{6}{7}, -\sqrt{3}, 2.151155\ldots, \frac{23}{6}, \frac{48}{5}, -3.010010001\ldots$ and $12.353553555.$

(i) Arrange the ten real numbers in the given boxes in ascending order.

(ii) Arrange the same numbers in the boxes given below in descending order.

2.4.1 The Square Root of a Real Number

You have already come across the concept of the square root of a whole number, decimal fractions etc. While you easily compute values like $\sqrt{169}$ (which give whole number answers), you also encounter values like $\sqrt{5}, \sqrt{18}, \ldots$ etc that yield irrational solutions.

The number 25 has two square roots 5 and –5. However, when we write $\sqrt{25}$, we always mean the positive square root 5 (and not the negative square root –5). The symbol $\sqrt{\ }$ denotes the positive square root only.

If a is a rational number, any irrational number of the form $\sqrt{a}$ will sometimes be referred to as a surd. A real number such as $2\sqrt{3}$ will be loosely referred to as surd, since it can be expressed as $\sqrt{12}$ (How?). We may also say an expression such as $\sqrt{3}+\sqrt{2}$ is a surd, although technically we should say that it is a sum of two surds.

Is the number π a surd? No. It cannot be expressed as the root of a rational number, or a finite combination of such numbers.

Recall the basic rules for square roots:

If a,b are positive numbers, then

$$(\sqrt{a})^2 = a$$

$$\sqrt{a^2} = a$$

Note

These rules are true only when a, b are positive numbers. In general $\sqrt{a^2} = (a^2)^{\frac{1}{2}} = a$.

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

$$\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}}$$

Example2.15

Find whether x and y are rational or irrational in the following.

(i) $a = 2 + \sqrt{3}\,,\ b = 2 - \sqrt{3}\,;\ x = a + b,\ \ y = a - b$

(ii) $a = \sqrt{2} + 7,\ b = \sqrt{2} - 7;\ x = a + b,\ \ y = a - b$

(iii) $a = \sqrt{75}\,,\ b = \sqrt{3}\,;\ \ x = ab,\ \ y = \frac{a}{b}$

(iv) $a = \sqrt{18}\,,\ b = \sqrt{3}\,;\ x = ab,\ \ y = \frac{a}{b}$

Solution

(i) Given that $a = 2 + \sqrt{3}, b = 2 - \sqrt{3}$

$x = a + b = (2 + \sqrt{3}) + (2 - \sqrt{3}) = 4$, a rational number.

$y = a - b = (2 + \sqrt{3}) - (2 - \sqrt{3}) = 2\sqrt{3}$, an irrational number.

(ii) Given that $a = \sqrt{2} + 7$, $b = \sqrt{2} - 7$

$x = a + b = (\sqrt{2} + 7) + (\sqrt{2} - 7) = 2\sqrt{2}$, an irrational number.

$y = a - b = (\sqrt{2} + 7) - (\sqrt{2} - 7) = 14$, a rational number.

(iii) Given that $a = \sqrt{75}$, $b = \sqrt{3}$

$x = ab = \sqrt{75} \times \sqrt{3} = \sqrt{75 \times 3} = \sqrt{5 \times 5 \times 3 \times 3} = 5 \times 3 = 15$, a rational number.

$y = \frac{a}{b} = \frac{\sqrt{75}}{\sqrt{3}} = \sqrt{\frac{75}{3}} = \sqrt{25} = 5$, rational number.

(iv) Given that $a = \sqrt{18}$, $b = \sqrt{3}$

$x = ab = \sqrt{18} \times \sqrt{3} = \sqrt{18 \times 3} = \sqrt{6 \times 3 \times 3} = 3\sqrt{6}$, an irrational number.

$y = \frac{a}{b} = \frac{\sqrt{18}}{\sqrt{3}} = \sqrt{\frac{18}{3}} = \sqrt{6}$, an irrational number.

Note

(i) From the above examples, it is clear that the sum , difference, product, quotient of any two irrational numbers could be rational or irrational.

(ii) Usually surds that have perfect squares as factors are given in a simplified form as far as possible.

2.4.2 The Real Number Line

Visualisation through Successive Magnification.

We can visualise the representation of numbers on the number line, as if we glimpse through a magnifying glass.

Example 2.16

Represent 4.863 on the number line.

Solution

4.863 lies between 4 and 5(see Fig. 2.10)

(i) Divide the distance between 4 and 5 into 10 equal intervals.

(ii) Mark the point 4.8 which is second from the left of 5 and eighth from the right of 4

(iii) 4.86 lies between 4.8 and 4.9. Divide the distance into 10 equal intervals.

(iv) Mark the point 4.86 which is fourth from the left of 4.9 and sixth from the right of 4.8

(v) 4.863 lies between 4.86 and 4.87. Divide the distance into 10 equal intervals.

(vi) Mark point 4.863 which is seventh from the left of 4.87 and third from the right of 4.86.

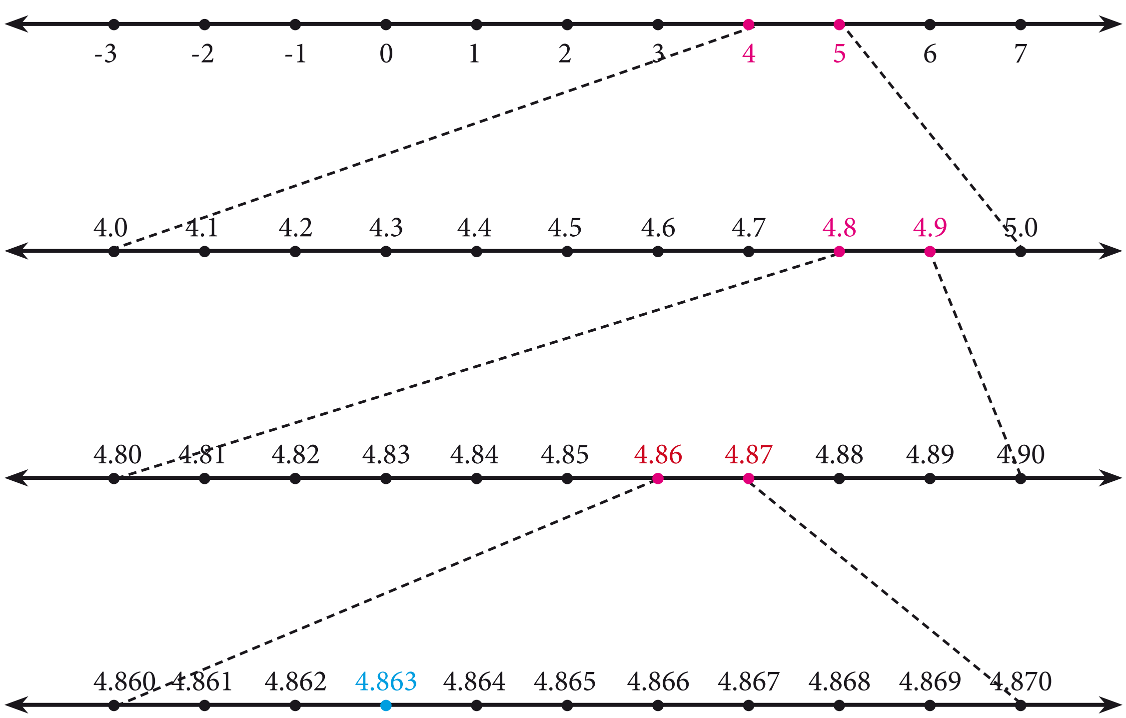


Fig. 2.10

Example 2.17

Represent $3.\overline{45}$ on the number line upto 4 decimal places.

Solution

E53615

$3.\overline{45} = 3.45454545\ldots..$

$= 3.4545$ (correct to 4 decimal places).

The number lies between 3 and 4

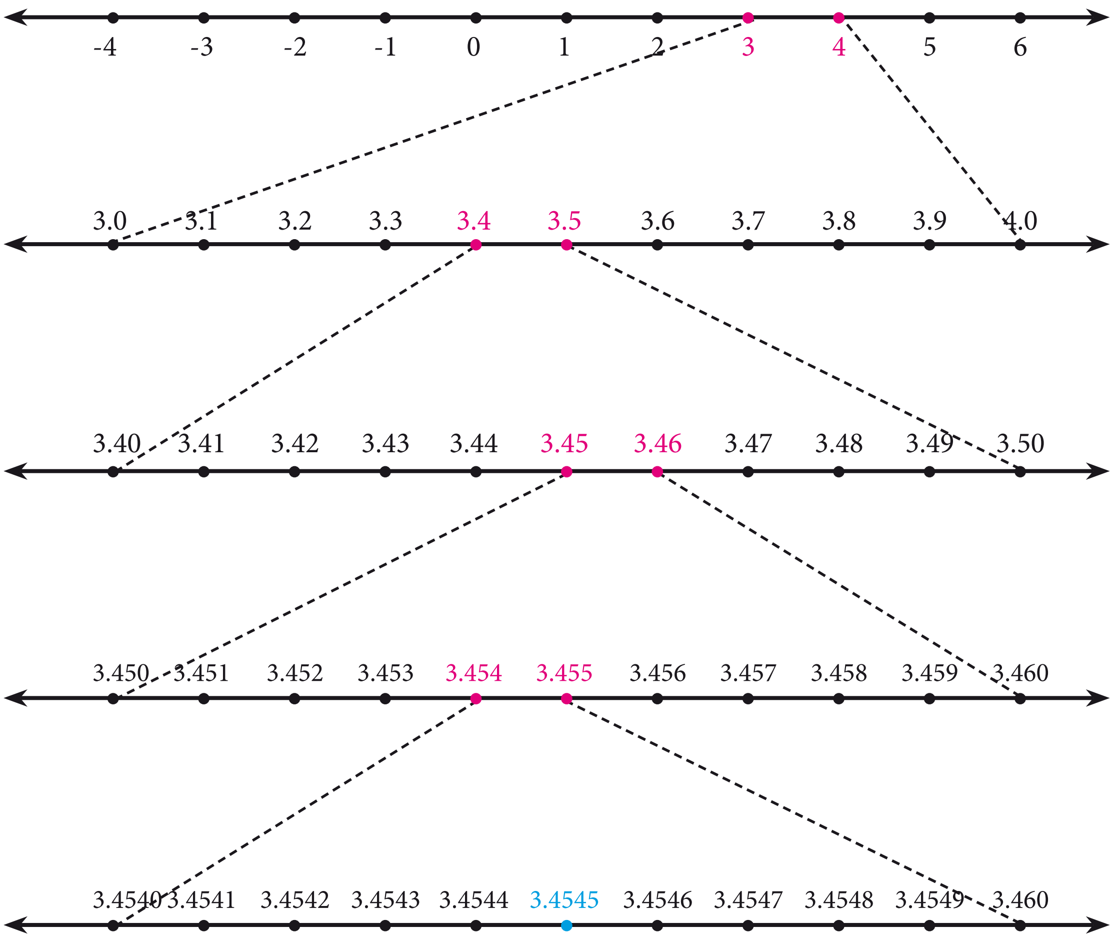


Fig. 2.11

Exercise 2.4

1. Represent the following numbers on the number line.

   (i) 5.348 (ii) $6.\overline{4}$ upto 3 decimal places. (iii) $4.\overline{73}$ upto 4 decimal places.

Exercise 2.5

Multiple Choice Questions

1. If n is a natural number then $\sqrt{n}$ is
   (a) always a natural number. (b) always an irrational number.
   (c) always a rational number (d) may be rational or irrational

2. Which of the following is not true?.
   (a) Every rational number is a real number. (b) Every integer is a rational number.
   (c) Every real number is an irrational number. (d) Every natural number is a whole number.

3. Which one of the following, regarding sum of two irrational numbers, is true?
   (a) always an irrational number. (b) may be a rational or irrational number.
   (c) always a rational number. (d) always an integer.

4. Which one of the following has a terminating decimal expansion?.
   (a) $\frac{5}{64}$ (b) $\frac{8}{9}$ (c) $\frac{14}{15}$ (d) $\frac{1}{12}$

5. Which one of the following is an irrational number
   (a) $\sqrt{25}$ (b) $\sqrt{\frac{9}{4}}$ (c) $\frac{7}{11}$ (d) π

6. An irrational number between 2 and 2.5 is
   (a) $\sqrt{11}$ (b) $\sqrt{5}$ (c) $\sqrt{2.5}$ (d) $\sqrt{8}$

7. The smallest rational number by which $\frac{1}{3}$ should be multiplied so that its decimal expansion terminates after one place of decimal is
   (a) $\frac{1}{10}$ (b) $\frac{3}{10}$ (c) 3 (d) 30

8. The number $0.\overline{3}$ in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$ is
   (a) $\frac{33}{100}$ (b) $\frac{3}{10}$ (c) $\frac{1}{3}$ (d) $\frac{3}{100}$

9. The value of $0.\overline{23} + 0.\overline{22}$ is
   (a) $0.\overline{43}$ (b) 0.45 (c) $0.4\overline{5}$ (d) $0.\overline{45}$

10. If $\frac{1}{7} = 0.\overline{142857}$ then the value of $\frac{5}{7}$ is
    (a) $0.\overline{142857}$ (b) $0.\overline{714285}$ (c) $0.\overline{571428}$ (d) 0.714285

11. Find the odd one out of the following.
    (a) $\sqrt{32} \times \sqrt{2}$ (b) $\frac{\sqrt{27}}{\sqrt{3}}$ (c) $\sqrt{72} \times \sqrt{8}$ (d) $\frac{\sqrt{54}}{\sqrt{18}}$

12. $0.\overline{34} + 0.3\overline{4} =$

(a) $0.6\overline{87}$ (b) $0.\overline{68}$ (c) $0.6\overline{8}$ (d) $0.68\overline{7}$

Points to remember

1. When the decimal expansion of $\frac{p}{q}$, $q \neq 0$ terminates that is, comes to an end, the decimal is called a terminating decimal.
2. In the decimal expansion of $\frac{p}{q}$, $q \neq 0$ when the remainder is not zero, we have a repeating (recurring) block of digits in the quotient. In this case, the decimal expansion is called non-terminating and recurring.
3. If a rational number $\frac{p}{q}$, $q \neq 0$ can be expressed in the form $\frac{p}{2^m \times 5^n}$, where $p \in \mathbb{Z}$ and $m, n \in \mathbb{W}$, then the rational number will have a terminating decimals. Otherwise, the rational number will have a non-terminating repeating (recurring) decimal.
4. A rational number can be expressed either a terminating or a non- terminating repeating decimal.
5. An irrational number is a non-terminating and non-recuring decimal, i.e. it cannot be written in form $\frac{p}{q}$, where p and q are both integers and $q \neq 0$.
6. The union of all rational numbers and all irrational numbers is called the set of real numbers.
7. Every real number is either a rational number or an irratonal number.
8. If a real number is not rational number, then it must be an irrational number.
9. The sum or difference of a rational number and an irrational number is always an irrational number.
10. The product or quotient of non-zero rational number and an irrational number is also an irrational number.
11. The sum, difference, product or quotient of two irrational numbers need not be irrational. The result may be rational or irrational.
12. The product or quotient of two rational number and an irrational number is also an irrational number.
13. The sum, difference, product or quotient of two irrational numbers need not be irrational. The result may be rational or irrational.

Answers

Exercise 2.1

1. D 2. $-\frac{6}{11}, -\frac{5}{11}, -\frac{4}{11}, \ldots \frac{1}{11}$

3. (i) $\frac{9}{40}, \frac{19}{80}, \frac{39}{160}, \frac{79}{320}, \frac{159}{640}$;

The given answer is one of the answers. There can be many more answers

(ii) 0.101, 0.102, ... 0.109

The given answer is one of the answers. There can be many more answers

(iii) $-\frac{3}{2}, -\frac{5}{4}, -\frac{9}{8}, -\frac{17}{16}, -\frac{33}{32}$

The given answer is one of the answers. There can be many more answers

Exercise 2.2

1. (i) 0.2857142..., Non terminating and recurring (ii) 0.35, Terminating

(iii) $-5.\overline{27}$, Non terminating and recurring (iv) $7.\overline{3}$, Non terminating and recurring

(v) −0.28125, Terminating (vi) 1.635, Terminating 2. 6 3. $2.\overline{15}$

4. (i) $\frac{24}{99}$ (ii) $\frac{2325}{999}$ (iii) $\frac{43}{50}$ (iv) $-\frac{1283}{250}$ (v) $\frac{143}{45}$

(vi) $\frac{5681}{330}$ (vii) $\frac{1}{9999}$ (viii) $-\frac{190924}{9000}$

5. (i) Terminating (ii) Terminating (iii) Terminating

(iv) Non terminating (v) Terminating (vi) Non terminating

Exercise 2.3

2. (i) 0.301202200222..., 0.301303300333... (ii) 0.8616611666111 ..., 0.8717711777111 ...

(iii) 1.515511555..., 1.616611666... (iv) 0.12201100111..., 0.12301100111...

3. 2.2362, 2.2363

Exercise 2.5

1. (d) 2. (c) 3. (b) 4. (a) 5. (d) 6. (b) 7. (b) 8. (c) 9. (d) 10. (b)

11. (d) 12. (a)

3 ALGEBRA

Algebra is the metaphysics of Arithmetic

- John Ray

Can all polynomial equations be solved with real numbers? Obviously not, since we can see that even $x^2+1=0$ has no solution! However, if we go beyond real numbers to include what are called complex numbers, then in fact every polynomial equation can be solved.

This was proved by the German mathematician Carl Friedrich Gauss in 1799. This fact is so important that his theorem is called the *Fundamental theorem of Algebra*.

Johann Carl Friedrich Gauss (1977 – 1855)

Learning Outcomes

- To understand a polynomial in one variable.
- To understand the classification of polynomials based on degree and number of terms.
- To evaluate a polynomial for the given value.
- To understand the zeros of polynomial.
- To perform the basic operations on polynomials.
- To understand the remainder theorem.

3.1 Introduction

Why study polynomials?

This chapter is going to be all about polynomial expressions in algebra. These are your friends, you have already met, without being properly introduced! We will properly introduce them to you, and they are going to be your friends in whatever mathematical journey you undertake from here on.

$(a+1)^2 = a^2 + 2a + 1$

Now that's a polynomial. That does not look very special, does it? We have seen a lots of algebraic expressions already, so why to bother about these? There are many reasons why polynomials are interesting and important in mathematics.

For now, we will just take one example showing their use. Remember, we studied lots of arithmetic and then came to algebra, thinking of variables as unknown numbers. Actually we can now get back to numbers and try to write them in the language of algebra.

Consider a number like 5418. It is actually 5 thousand 4 hundred and eighteen.

Write it as:

$5 \times 1000 + 4 \times 100 + 1 \times 10 + 8$

which again can be written as:

$5 \times 10^3 + 4 \times 10^2 + 1 \times 10^1 + 8$

Now it should be clear what this is about. This is of the form $5x^3 + 4x^2 + x + 8$, which is a polynomial. How does writing in this form help? We always write numbers in decimal system, and hence always $x = 10$. Then what is the fun? Remember divisibility rules? Recall that a number is divisible by 3 only if the sum of its digits is divisible by 3. Now notice that if x divided by 3 gives 1 as remainder, then it is the same for x^2, x^3, etc. They all give remainder 1 when divided by 3. So you get each digit multiplied by 1, added together, which is the sum of digits. If that is divisible by 3, so is the whole number. You can check that the rule for divisibility by 9, or even divisibility by 2 or 5, can be proved similarly with great ease. Our purpose is not to prove divisibility rules but to show that representing numbers as polynomials shows us many new number patterns. In fact, many many objects of study, not just numbers, can be represented as polynomials and then we can learn many things about them.

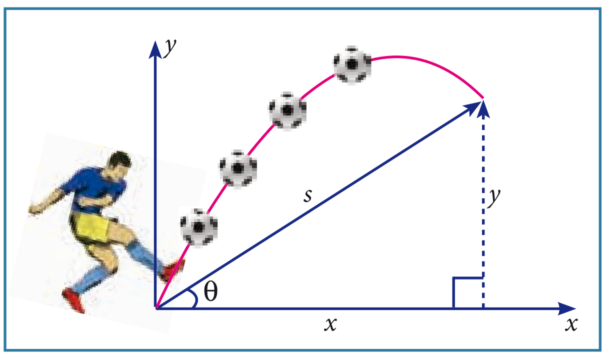


Fig. 3.1

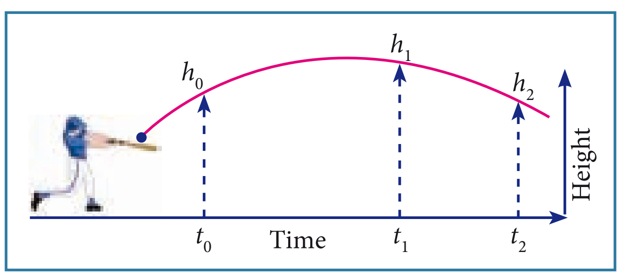


Fig. 3.2

In algebra we think of x^2, $5x^2-3$, $2x+7$ etc as *functions* of x. We draw pictures to see how the function varies as x varies, and this is very helpful to *understand* the function. And now, it turns out that a good number of functions that we encounter in science, engineering, business studies, economics, and of course in mathematics, all can be approximated by polynomials, if not actually be represented as polynomials. In fact, approximating functions using polynomials is a fundamental theme in all of higher mathematics and a large number of people make a living simply by working on this idea.

Polynomials are extensively used in biology, computer science, communication systems ... the list goes on. The given pictures (Fig. 3.1, 3.2 & 3.3) may be repsresented as a quardartic polynomial. We will not only learn what polynomials are but also how we can use them like in numbers, we add them, multiply them, divide one by another, etc.

Fig. 3.3

Observe the given figures.

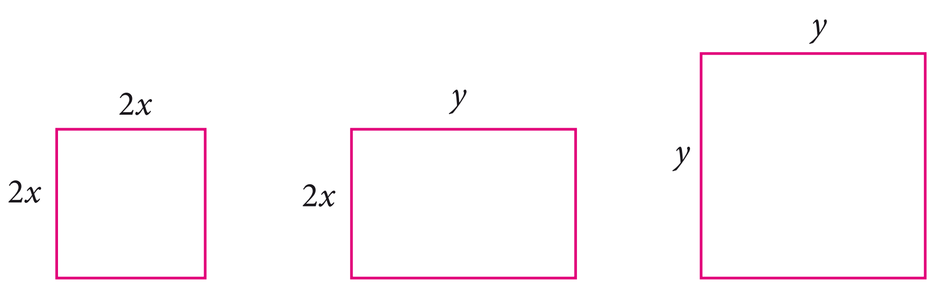


Fig. 3.4 Fig. 3.5 Fig. 3.6

The total area of the above figures is $4x^2 + 2xy + y^2$, we call this expression as an algebraic expression. Here for different values of x and y we get different values of areas. Since the sides x and y can have different values, they are called variables. Thus, a variable is a symbol which can have various numerical values.

Variables are usually denoted by letters such as x, y, z, etc. In the above algebraic expression the numbers 4, 2 are called constants. Hence the constant is a symbol, which has a fixed numeric value.

Algebraic Expression

An **algebraic expression** is a combination of constants and variables combined together with the help of the four fundamental signs.

Examples of algebraic expression are

$$x^3 - 4x^2 + 8x - 1, \; 4xy^2 + 3x^2y - \frac{5}{4}xy + 9, \; 5x^2 - 7x + 6$$

Constants

Any real number is a constant. We can form numerical expressions using constants and the four arithmetical operations.

Examples of constant are 1, 5, -32, $\frac{3}{7}$, $-\sqrt{2}$, 8.432, 1000000 and so on.

Variables

The use of variables and constants together in expressions give us ways of representing a range of numbers, one for each value of the variable. For instance, we know the expression $2\pi r$, it stands for the circumference of a circle of radius r. As we vary r, say, 1cm, 4cm, 9cm etc, we get larger and larger circles of circumference 2π, 8π, 18π etc.

The single expression $2\pi r$ its a short and compact description for the circumference of all these circles. We can use arithmetical operations to combine algebraic expressions and get a rich language of functions and numbers. Letters used for representing unknown real numbers called variables are x, y, a, b and so on.

Coefficients

Any part of a term that is multiplied by the remaining part of the term is called the coefficient of the remaining term.

For example,

$x^2 + 5x - 24$ is an algebraic expression containing three terms. The variable of this expression is x, coefficient of x^2 is 1 and the coefficient of x is 5 and the constant is -24 (not 24).

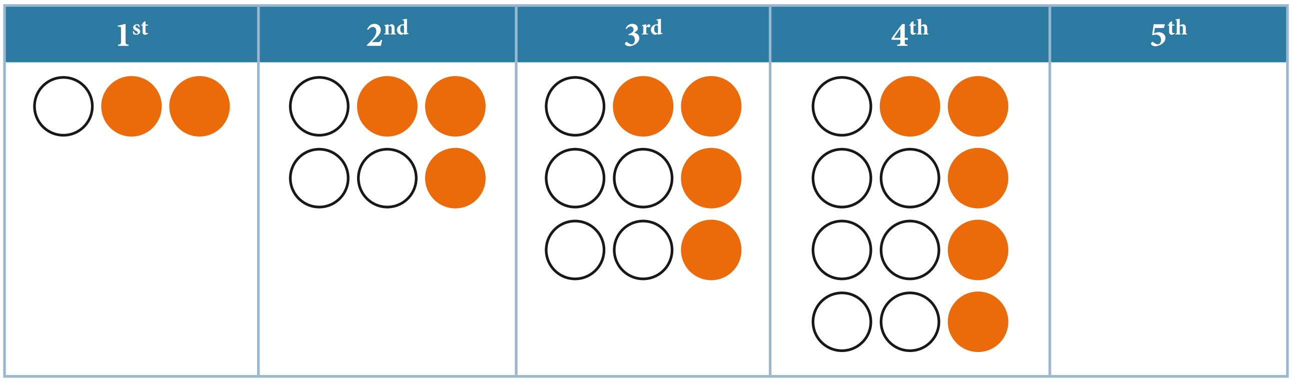
Activity-1

Write the Variable, Coefficient and Constant in the given algebraic expression

Expression	$x + 7$	$3y - 2$	$5x^2$	$2xy + 11$	$-\frac{1}{2}p + 7$	$-8 + 3a$
Variable	x			x, y		
Coefficient	1				$-\frac{1}{2}$	
Constant	7					-8

3.2 Polynomials

The following dot diagrams show the sequence of patterns.



(a) Draw the fifth pattern in the sequence in the space above.

(b) How many dots of each colour will be there in the 10^{th} pattern?
White: ____________ Colour : ____________

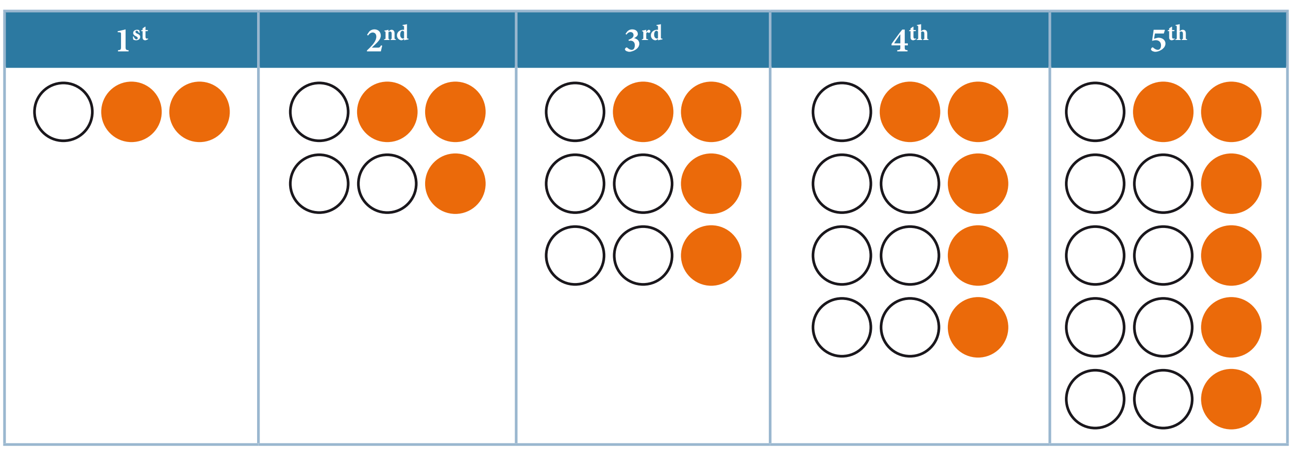
(c) The number of colour dots in the n^{th} diagram is given by the expression ____________

(d) Write algebraic expressions for the number of white dots and for the total number of dots.

Number of white dots	+	Number of colour dots	=	Total number of dots
__________		__________		__________

Solution:

(a) Draw the fifth pattern in the sequence in the space above.



(b) How many dots of each colour will be there in the 10^{th} pattern?
White : <u>19</u> Colour : <u>11</u>

(c) The number of colour dots in the n[th] diagram is given by the expression $n + 1$

(d) Write algebraic expressions for the number of white dots and for the total number of dots.

Number of white dots		Number of colour dots		Total number of dots
$2n - 1$	+	$n + 1$	=	$3n$

Activity - 2

Study the following pattern and write the algebraic expression

(i)

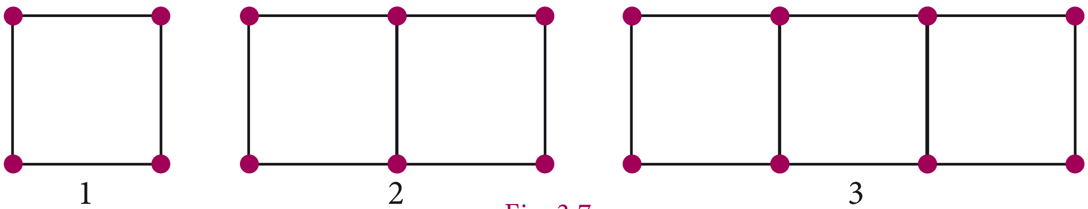


Fig. 3.7

Shape number	1	2	3	4	5
Number of Matchsticks	4	7	10	13	16

(ii)

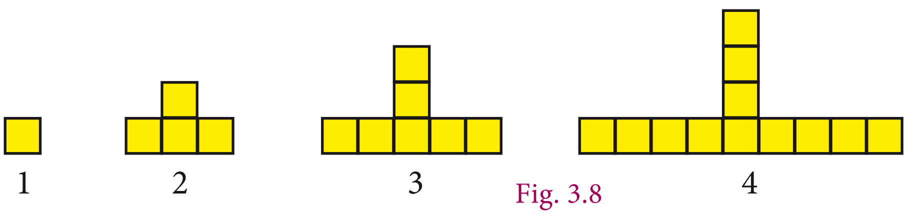


Fig. 3.8

Shape number	1	2	3	4	5
Number of square boxes	1	4	7	10	13

3.2.1 Polynomial in One Variable

An algebraic expression of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ is called Polynomial in one variable x of degree 'n' where $a_0, a_1, a_2, \ldots a_n$ are constants ($a_n \neq 0$) and n is a whole number.

In general polynomials are denoted by $f(x), g(x), p(t), q(z)$ and $r(x)$ and so on.

Note

The coefficient of variables in the algebraic expression may have any real numbers, where as the powers of variables in polynomial must have only non-negative integral powers that is, only whole numbers. Recall that $a^0 = 1$ for all a.

For example,

S.No	Given expression	Polynomial / not a polynomial	Reason
1	$4y^3 + 2y^2 + 3y + 6$	Polynomial	Non- negative integral power
2	$4x^{-4} + 5x^4$	Not a polynomial	One of the powers is negative (-4)
3	$m^2 + \frac{4}{5}m + 8$	Polynomial	Non- negative integral power
4	$\sqrt{5}\,y^2$	Polynomial	Non- negative integral power
5	$2r^2 + 3r - 1 + \frac{1}{r}$	Not a polynomial	One of the power is negative $\left(\frac{1}{r} = r^{-1}\right)$
6	$8 + \sqrt{q}$	Not a polynomial	power of q is fraction $\left(\sqrt{q} = q^{\frac{1}{2}}\right)$
7	$\sqrt{8}\,p^2 + 5p - 7$	Polynomial	Non- negative integral power
8	$5n^{\frac{4}{5}} + 6n - 1$	Not a polynomial	One of the power of n is a fraction $\frac{4}{5}$

3.2.2 Standard Form of a Polynomial

We can write a polynomial $p(x)$ in the decreasing or increasing order of the powers of x. This way of writing the polynomial is called the standard form of a polynomial.

For example

(i) $8x^4 + 4x^3 - 7x^2 - 9x + 6$ (ii) $5 - 3y + 6y^2 + 4y^3 - y^4$

Activity-3

Write the following polynomials in standard form.

Sl.No.	Polynomial	Standard Form
1	$5m^4 - 3m + 7m^2 + 8$	
2	$\frac{2}{3}y + 8y^3 - 12 + \sqrt{5}\,y^2$	
3	$12p^2 - 8p^5 - 10p^4 - 7$	

3.2.3 Degree of the Polynomial

In a polynomial of one variable, the highest power of the variable is called the **degree of** the polynomial.

In case of a polynomial of more than one variable, the sum of the powers of the variables in each term is considered and the **highest sum** so obtained is called the degree of the polynomial.

This is intended as the most **significant** power of the polynomial. Obviously when we write x^2+5x the value of x^2 becomes much larger than $5x$ for large values of x. So we could think of $x^2 + 5x$ being almost the same as x^2 for large values of x. So the higher the power, the more it dominates. That is why we use the highest power as important information about the polynomial and give it a name.

Example 3.1

Find the degree of each term for the following polynomial and also find the degree of the polynomial

$$6ab^8 + 5a^2b^3c^2 - 7ab + 4b^2c + 2$$

Solution

Given polynomial is $6ab^8 + 5a^2b^3c^2 - 7ab + 4b^2c + 2$

Degree of each of the terms is given below.

$6ab^8$ has degree $(1+8) = 9$

$5a^2b^3c^2$ has degree $(2+3+2) = 7$

$7ab$ has degree $(1+1) = 2$

$4b^2c$ has degree $(2+1) = 3$

The constant term 2 is always regarded as having degree Zero.

The **degree** of the polynomial $6ab^8 + 5a^2b^3c^2 - 7ab + 4b^2c + 2$.

= the largest exponent in the polynomial

$= 9$

3.2.4 A very Special Polynomial

We have said that coefficients can be any real numbers. What if the coefficient is zero? Well that term becomes zero, so we won't write it. What if all the coefficients are zero? We acknowledge that it exists and give it a name.

E5KX4F

It is the polynomial having all its coefficients to be Zero.

$$g(t) = 0t^4 + 0t^2 - 0t \;, \quad h(p) = 0p^2 - 0p + 0$$

From the above example we see that we cannot talk of the degree of the Zero polynomial, since the above two have different degrees but both are Zero polynomial. So we say that the degree of the Zero polynomial is not defined

The degree of the Zero polynomial is not defined

3.2.5 Types of Polynomials

(i) Polynomial on the basis of number of terms	
MONOMIAL	A polynomial having one term is called a monomial Examples : 5, 6m, 12ab
BINOMIAL	A polynomial having two terms is called a Binomial Examples : $5x + 3,\ 4a - 2,\ 10p + 1$
TRINOMIAL	A polynomial having three terms is called a Trinomial Example : $4x^2 + 8x - 12,\ \ 3a^2 + 4a + 10$
(ii) Polynomial based on degree	
CONSTANT	A polynomial of degree zero is called constant polynomial Examples : $5, -7, \frac{2}{3}, \sqrt{5}$
LINEAR	A polynomial of degree one is called linear polynomial Examples : $410x - 7$
QUADRATIC	A polynomial of degree two is called quadratic polynomial Example : $2\sqrt{5}\,x^2 + 8x - 4$
CUBIC	A polynomial of degree three is called cubic polynomial Example : $12y^3,\qquad 6m^3 - 7m + 4$

Example 3.2

Classify the following polynomials based on number of terms.

S.No.	Polynomial	No of Terms	Type of polynomial based of terms
(i)	$5t^3 + 6t + 8t^2$	3 Terms	Trinomial
(ii)	$y-7$	2 Terms	Binomial
(iii)	$\frac{2}{3}r^4$	1 Term	Monomial
(iv)	$6y^5 + 3y - 7$	3 Terms	Trinomial
(v)	$8m^2 + 7m^2$	Like Terms. So, it is $15m^2$ which is 1 term only	Monomial

Example 3.3

Classify the following polynomials based on their degree.

S.No.	Polynomial	Degree	Type
(i)	$\sqrt[3]{4}\,z + 7$	Degree one	Linear polynomial
(ii)	$z^3 - z^2 + 3$	Degree three	Cubic polynomial
(iii)	$\sqrt{7}$	Degree zero	Constant polynomial
(iv)	$y^2 - \sqrt{8}$	Degree two	Quadratic polynomial

Activity-4

Classify the following polynomials .

S.No.	Polynomial	Based on number of terms	Based on their degree
(i)	$\frac{x^2}{\sqrt{2}} + x + 1$		
(ii)	$\sqrt{5}\,x^2 - \frac{6}{5}x$		
(iii)	1		
(iv)	$m^4 - \frac{6}{7}m^2 - 6m + 2.7$		
(v)	$\frac{t^3}{\sqrt{6}} + \frac{t^4}{4} - t$		

3.3 Arithmetic of Polynomials

We now have a rich language of polynomials, and we have seen that they can be classified in many ways as well. Now, what can we do with polynomials? Consider a polynomial on x.

We can evaluate the polynomial at a particular value of x. We can ask how the function given by the polynomial changes as x varies. Write the polynomial equation $p(x) = 0$ and solve for x. All this is interesting, and we will be doing plenty of all this as we go along. But there is something else we can do with polynomials, and that is *to treat them like numbers*! We already have a clue to this at the beginning of the chapter when we saw that every positive integer could be represented as a polynomial.

Following arithmetic, we can try to add polynomials, subtract one from another, multiply polynomials, divide one by another. As it turns out, the analogy between numbers and polynomials runs deep, with many interesting properties relating them. For now, it is fun to simply try and define these operations on polynomials and work with them.

3.3.1 Addition of Polynomials

The addition of two polynomials is also a polynomial.

Note

Only like terms can be added. $3x^2 + 5x^2$ gives $8x^2$ but unlike terms such as $3x^2$ and $5x^3$ when added gives $3x^2 + 5x^3$, a new polynomial.

Example 3.4

Let $p(x) = 4x^2 - 3x + 2x^3 + 5$ and $q(x) = x^2 + 2x + 4$ find $p(x) + q(x)$

Solution

Given Polynomial	Standard form
$p(x) = 4x^2 - 3x + 2x^3 + 5$	$2x^3 + 4x^2 - 3x + 5$
$q(x) = x^2 + 2x + 4$	$x^2 + 2x + 4$
$p(x) + q(x) = 2x^3 + 5x^2 - x + 9$	

We see that $p(x) + q(x)$ is also a polynomial. Hence the sum of any two polynomials is also a polynomial.

Activity-5

Add the following polynomials and find the degree of the resultant polynomial.

S.No.	Polynomial	Addition	Degree of the resultant polynomial
(i)	$p(x) = 5x^3 - 3x + x^2 + 4$ $q(x) = 7x - 4x^2 + 2$		
(ii)	$p(m) = 6m - 7$ $q(m) = 7m^2 - 12 + 4m$		
(iii)	$p(x) = 4x^2 - 6x^3 - 4x + 6$ $q(x) = 8x^3 + 2x^2 - 2$		
(iv)	$r(y) = 7y^4 + 5y^2 + 4y$ $s(y) = 2y - 3y^2$		
(v)	$p(m) = 12m^3 - 10m^2 - 7$ $q(m) = 7m^3 + 5m^2 - 3$		

3.3.2 Subtraction of Polynomials

The subtraction of two polynomials is also a polynomial.

Note

Only like terms can be subtracted. $8x^2 - 5x^2$ gives $3x^2$ but when $5x^3$ is subtracted from $3x^2$ we get, $3x^2 - 5x^3$, a new polynomial.

Example 3.5

Let $p(x) = 4x^2 - 3x + 2x^3 + 5$ and $q(x) = x^2 + 2x + 4$ find $p(x) - q(x)$

Solution

Given Polynomial	Standard form
$p(x) = 4x^2 - 3x + 2x^3 + 5$	$2x^3 + 4x^2 - 3x + 5$
$q(x) = x^2 + 2x + 4$	$x^2 + 2x + 4$
$p(x) - q(x) = 2x^3 + 3x^2 - 5x + 1$	

We see that $p(x) - q(x)$ is also a polynomial. Hence the subtraction of any two polynomials is also a polynomial.

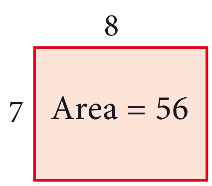
Activity-6

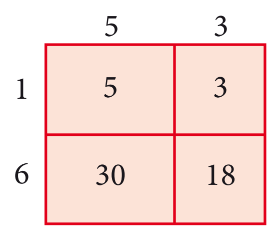
Subtract the following polynomials and find the degree of the resultant polynomial.

S.No.	Polynomial	Difference	Degree of the resultant polynomial
(i)	$p(x) = 5x^3 - 3x + x^2 + 4$ $q(x) = 7x - 4x^2 + 2$		
(ii)	$p(m) = 6m - 7$ $q(m) = 7m^2 - 12 + 4m$		
(iii)	$p(x) = 4x^2 - 6x^3 - 4x + 6$ $q(x) = 8x^3 + 2x^2 - 2$		
(iv)	$r(y) = 7y^4 + 5y^2 + 4y$ $s(y) = 2y - 3y^2$		
(v)	$p(m) = 12m^3 - 10m^2 - 7$ $q(m) = 7m^3 + 5m^2 - 3$		

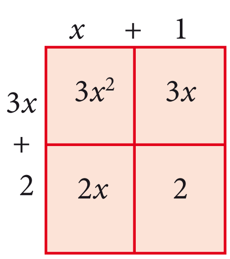
3.3.3 Multiplication of Two Polynomials

Divide a rectangle with 8 units of length and 7 units of breadth into 4 rectangles as shown below, and observe that the area is same, this motivates us to study the multiplication of polynomials.





For example, considering length as $(x+1)$ and breadth as $(3x+2)$ the area of the rectangle can be found by the following way.



Hence the area of the rectangle is

$= 3x^2 + 3x + 2x + 2$

$= 3x^2 + 5x + 2$

If x is a variable and m, n are positive integers then $x^m \times x^n = x^{m+n}$.

When two polynomials are multiplied the product will also be a polynomial.

Example 3.6

Find the product $(4x - 5)$ and $(2x^2 + 3x - 6)$.

Solution

To multiply $(4x - 5)$ and $(2x^2 + 3x - 6)$ distribute each term of the first polynomial to every term of the second polynomial. In this case, we need to distribute the terms $4x$ and -5. Then gather the like terms and combine them:

$$(4x - 5)(2x^2 + 3x - 6) = 4x(2x^2 + 3x - 6) - 5(2x^2 + 3x - 6)$$

$$= 8x^3 + 12x^2 - 24x - 10x^2 - 15x + 30$$

$$= 8x^3 + 2x^2 - 39x + 30$$

Or, you may also use the method of detached coefficients:

$$\begin{array}{rrrr}
 & 2 & +3 & -6 \\
 & & +4 & -5 \\
\hline
 & -10 & -15 & +30 \\
8 & +12 & -24 & \\
\hline
8 & +2 & -39 & +30 \\
\hline
\end{array}$$

$\therefore \quad (4x - 5)(2x^2 + 3x - 6) = 8x^3 + 2x^2 - 39x + 30$

Activity-7

Multiply the following polynomials and check whether their product is also a polynomial. Also find the degree of the resultant polynomial.

S.No.	Polynomial	Product	Degree of the resultant polynomial
(i)	$p(x) = 3x^3 + 2x - x^2 + 8$ $q(x) = 7x + 2$		
(ii)	$r(x) = 5y^3 - 3y^2 + 4$ $s(m) = 9y^2 - 2y + 6$		
(iii)	$p(m) = 8m - 9$ $q(m) = 9m^2 - 1 + 2m$		

Exercise 3.1

1. Which of the following expressions are polynomials. If not give reason:

   (i) $\frac{1}{x^2}+3x-4$ (ii) $x^2(x-1)$

   (iii) $\frac{1}{x}(x+5)$ (iv) $\frac{1}{x^{-2}}+\frac{1}{x^{-1}}+7$

   (v) $\sqrt{5}\,x^2+\sqrt{3}\,x+\sqrt{2}$ (vi) $m^2-\sqrt[3]{m}+7m-10$

2. Write the coefficient of x^2 and x in each of the following polynomials.

   (i) $4+\frac{2}{5}x^2-3x$ (ii) $6-2x^2+3x^3-\sqrt{7}\,x$

   (iii) πx^2-x+2 (iv) $\sqrt{3}\,x^2+\sqrt{2}\,x+0.5$

   (v) $x^2-\frac{7}{2}x+8$

3. Find the degree of the following polynomials.

   (i) $1-\sqrt{2}\,y^2+y^7$ (ii) $\frac{x^3-x^4+6x^6}{x^2}$

   (iii) $x^3(x^2+x)$ (iv) $3x^4+9x^2+27x^6$

   (v) $2\sqrt{5}\,p^4-\frac{8p^3}{\sqrt{3}}+\frac{2p^2}{7}$

4. Rewrite the following polynomial in standard form.

   (i) $x-9+\sqrt{7}\,x^3+6x^2$ (ii) $\sqrt{2}\,x^2-\frac{7}{2}x^4+x-5x^3$

   (iii) $7x^3-\frac{6}{5}x^2+4x-1$ (iv) $y^2+\sqrt{5}\,y^3-11-\frac{7}{3}y+9y^4$

5. Add the following polynomials and find the degree of the resultant polynomial.

   (i) $p(x)=6x^2-7x+2$ $q(x)=6x^3-7x+15$

   (ii) $h(x)=7x^3-6x+1$ $f(x)=7x^2+17x-9$

   (iii) $f(x)=16x^4-5x^2+9$ $g(x)=-6x^3+7x-15$

6. Subtract the second polynomial from the first polynomial and find the degree of the resultant polynomial.

   (i) $p(x)=7x^2+6x-1$ $q(x)=6x-9$

   (ii) $f(y)=6y^2-7y+2$ $g(y)=7y+y^3$

   (iii) $h(z)=z^5-6z^4+z$ $f(z)=6z^2+10z-7$

7. What should be added to $2x^3+6x^2-5x+8$ to get $3x^3-2x^2+6x+15$?

8. What must be subtracted from $2x^4+4x^2-3x+7$ to get $3x^3-x^2+2x+1$?

9. Multiply the following polynomials and find the degree of the resultant polynomial:

(i) $p(x) = x^2 - 9$ $\qquad$ $q(x) = 6x^2 + 7x - 2$

(ii) $f(x) = 7x + 2$ $\qquad$ $g(x) = 15x - 9$

(iii) $h(x) = 6x^2 - 7x + 1$ $\qquad$ $f(x) = 5x - 7$

10. The cost of a chocolate is Rs. $(x + y)$ and Amir bought $(x + y)$ chocolates. Find the total amount paid by him in terms of x and y. If x =10, y =5 find the amount paid by him.

11. The length of a rectangle is $(3x+2)$ units and it's breadth is $(3x-2)$ units. Find its area in terms of x. What will be the area if $x = 20$ units.

12. $p(x)$is a polynomial of degree 1 and $q(x)$ is a polynomial of degree 2. What kind of the polynomial $p(x) \times q(x)$ is ?

3.4 Value and Zeros of a Polynomial

Consider the two graphs given below. The first is linear, the second is quadratic. The first intersects the X axis at one point ($x = -3$) and the second at two points (x = -1 and x = 2). They both intersect the Y axis only at one point. In general, every polynomial has a graph and the graph is shown as a picture (since we all like pictures more than formulas, don't we?). But also, the graph contains a lot of useful information – like whether it is a straight line, what is the shape of the curve, how many places it cuts the x-axis, etc.

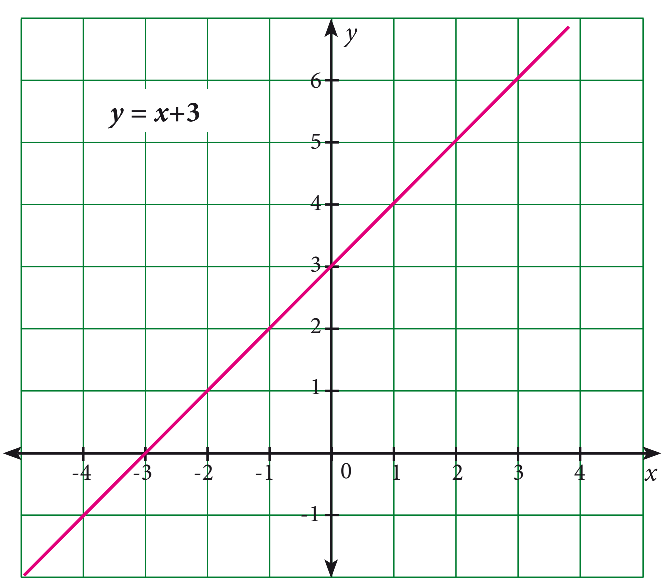


Fig. 3.9

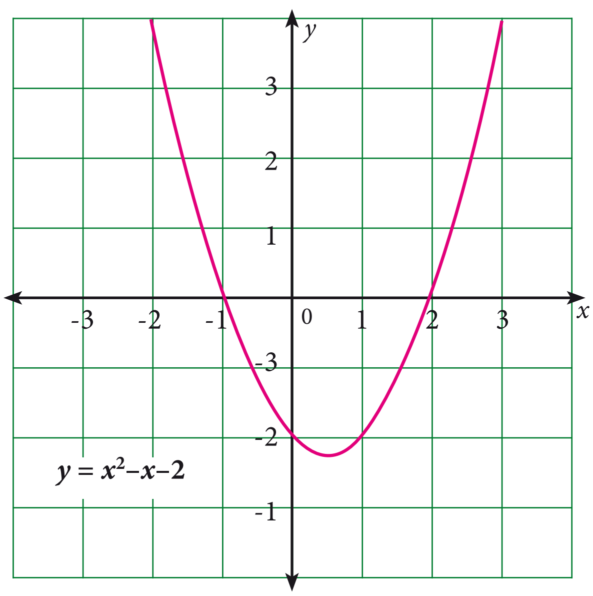


Fig. 3.10

How do you plot the graph of the polynomial ? You already know the answer. Put in different values of x and you get different values for the polynomial. If we put x=5 and calculate, we get the value of the polynomial at x=5. In general, the value of a polynomial $p(x)$ at x=a, denoted $p(a)$, is obtained by replacing x by a, where a is any real number.

Notice that the value of $p(x)$ can be zero for many possible values of x (as in the second graph below). So it is interesting to ask, for how many values of x, does p(x) become zero, and for which values ? We call these values of x, the Zeros of the polynomial $p(x)$.

Once we see that the values of the polynomial are what we plot in the graph of the polynomial, it is also easy to notice that the polynomial becomes zero exactly when the plot intersects the X-axis.

Can a polynomial $p(x)$ have no zeros at all ? Now you have to think like a mathematician. If a polynomial has degree 1, it is going to be of the form $ax+b$, and you can solve $ax+b = 0$ to see that it will have some value of x that solves the equation. So it has one zero. What is the polynomial that has degree zero ? That means it has only a constant, some number r. But if r is non-zero, then it can never have value zero ! So yes, the constant polynomial can have no zeroes at all.

Can we argue that every polynomial $p(x)$ with positive degree has a zero ? That is the same as saying that every polynomial $p(x) = 0$ has a solution ! But this is not true, since $x^2 +1 = 0$ has no solution in the set of real numbers. The polynomial x^2+1 has only positive values, never intersects the X-axis.

So we see that a polynomial may or may not have zeros and may have one or more zeroes. So it is very interesting, given any polynomial, to figure out whether it has zeros or not, and how many it can have.

Can a polynomial have infinitely many zeros ?

That is, it is a special polynomial that keeps intersecting the X-axis again and again and again and .... at newer and newer values of x. Now, there is a theorem that assures us that a polynomial of degree has atmost many zeros. (We will have to learn a good deal of Mathematics before we can prove that theorem, but when you learn you will see that it is beautiful and worth the wait !)

We will now state all this in formal definitions.

(The number of zeros depends on the line or curves intersecting x axis.)

For Fig. 3.9, Number of zeros is equal to 1

For Fig. 3.10, Number of zeros is equal to 2

Note

Number of zeros of a polynomial $\leq$ the degree of the polynomial

3.4.1 Value of a Polynomial

Value of a polynomial $p(x)$ at $x = a$ is $p\ (a)$ obtained on replacing x by a $(a \in R)$

For example,

Consider $f(x) = x^2 + 3x - 1$.

The value of $f(x)$ at $x = 2$ is

$f(2) = 2^2+3(2)-1 = 4+6-1 = 9$.

3.4.2 Zeros of Polynomial

(i) Consider the polynomial $p(x) = 4x^3 - 6x^2 + 3x - 14$

The value of $p(x)$ at $x = 1$ is $p(1) = 4(1)^3 - 6(1)^2 + 3(1) - 14$

$= 4 - 6 + 3 - 14$

$= -13$

Then, we say that the value of $p(x)$ at $x = 1$ is -11.

If we replace x by 0, we get $p(0) = 4(0)^3 - 6(0)^2 + 3(0) - 14$

$= 0 - 0 + 0 - 14$

$= -14$

we say that the value of $p(x)$ at $x = 0$ is -14.

The value of $p(x)$ at $x = 2$ is $p(2) = 4(2)^3 - 6(2)^2 + 3(2) - 14$

$= 32 - 24 + 6 - 14$

$= 0$

Since the value of $p(x)$ at $x = 0$ is zero, we can say that 2 is one of the zeros of $p(x)$ where $p(x) = 4x^3 - 6x^2 + 3x - 14$.

3.4.3 Roots of a Polynomial Equation

In general, if $p(a) = 0$ we say that $\boldsymbol{a}$ is zero of polynomial $p(x)$ or $\boldsymbol{a}$ is the root of polynomial equation $p(x) = 0$

Example 3.7

If $f(x) = x^2 - 4x + 3$, find the values of $f(1), f(-1), f(2), f(3)$. Also find the zeros of the polynomial $f(x)$.

Solution

$f(x) = x^2 - 4x + 3$

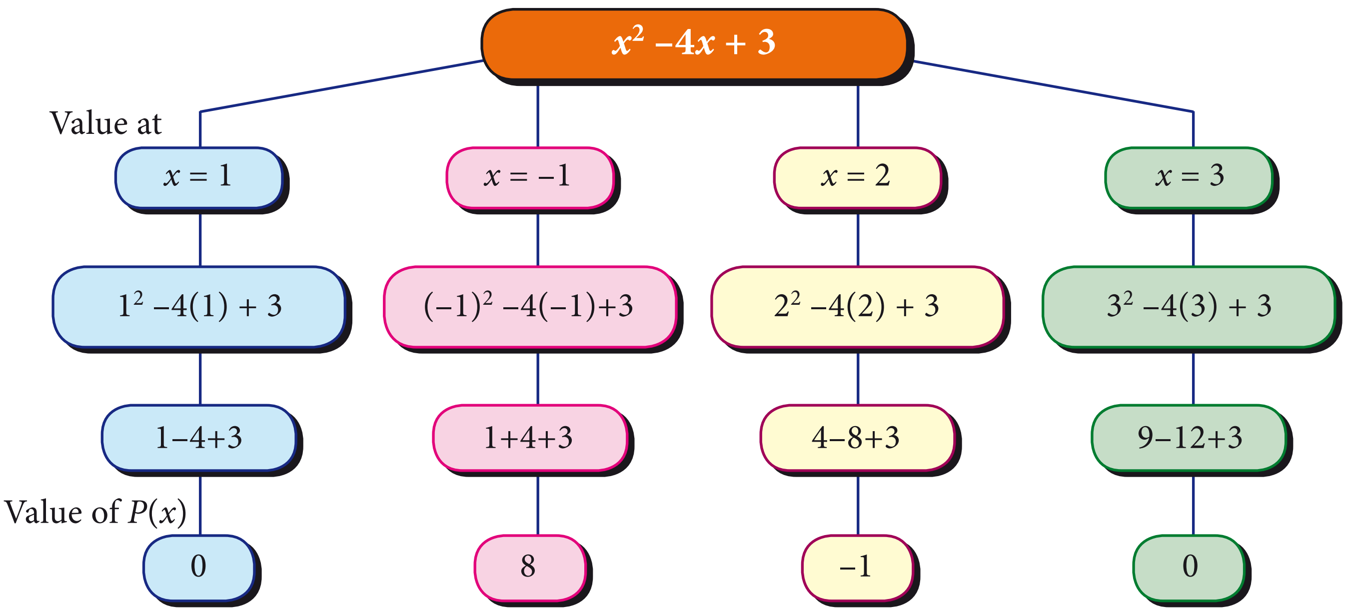


Fig. 3.11

Since the value of the polynomial $f(x)$ at $x = 1$ and $x = 3$ is zero, as the zeros of polynomial $f(x)$ are 1 and 3.

Activity-8

Find $p(0)$, $p(1)$ and $p(-1)$ for each of the following polynomials.

S.No.	Polynomial	$P(0)$	$P(1)$	$P(-1)$
(i)	$p(x) = x^2 + 2x - 1$			
(ii)	$p(x) = x^3 - 1$			
(iii)	$p(x) = x + 1$			

Example 3.8

Find the Zeros of the following polynomials.

(i) $f(x) = 2x + 1$ (ii) $f(x) = 3x - 5$

Solution

(i) Given that $f(x) = 2x + 1 = 2\left(x + \frac{1}{2}\right)$

$$= 2\left(x - \left(-\frac{1}{2}\right)\right)$$

$$f\left(-\frac{1}{2}\right) = 2\left[-\frac{1}{2} - \left(-\frac{1}{2}\right)\right] = 2(0) = 0$$

Since $f\left(-\frac{1}{2}\right) = 0$, $x = -\frac{1}{2}$ is the zero of $f(x)$

(ii) Given that $f(x) = 3x - 5 = 3\left(x - \frac{5}{3}\right)$

$$f\left(\frac{5}{3}\right) = 3\left(\frac{5}{3} - \frac{5}{3}\right) = 3(0) = 0$$

Hence, $x = \frac{5}{3}$ is the zero of $f(x)$

Example 3.9

Find the roots of the following polynomial equations.

(i) $5x - 3 = 0$ (ii) $-7 - 4x = 0$

Solution

(i) $5x - 3 = 0$

(or) $5x = 3$

Then, $x = \frac{3}{5}$

(ii) $-7 - 4x = 0$

(or) $4x = -7$

Then, $x = \frac{7}{-4} = -\frac{7}{4}$

Note

(i) A zero of a polynomial can be any real number not necessarily zero.

(ii) A non zero constant polynomial has no zero.

(iii) By convention, every real number is zero of the zero polynomial

Example 3.10

Check whether -3 and 3 are zeros of the polynomial $x^2 - 9$

Solution

Let $f(x) = x^2 - 9$

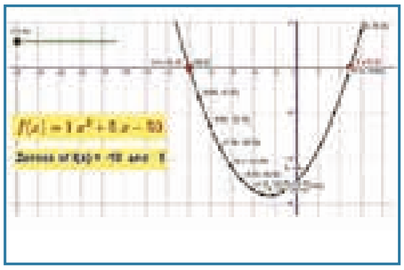
Then, $f(-3) = (-3)^2 - 9 = 9 - 9 = 0$

$f(+3) = 3^2 - 9 = 9 - 9 = 0$

$\therefore$ -3 and 3 are zeros of the polynomial $x^2 - 9$

ICT Corner

Expected Result is shown in this picture



Step – 1

Open the Browser and copy and paste the Link given below (or) by typing the URL given (or) Scan the QR Code.

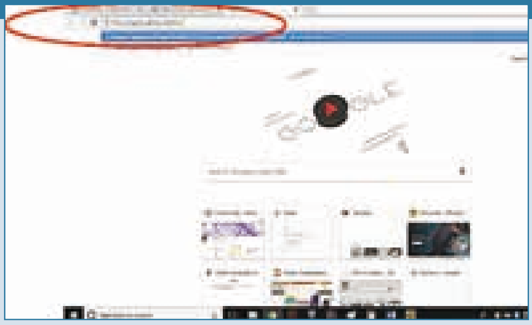
Step - 2

GeoGebra Work Book called "Polynomials and Quadratic Equations" will appear. There are several work sheets in this work Book. Open the worksheet named "Zeroes: Quadratic Polynomial".

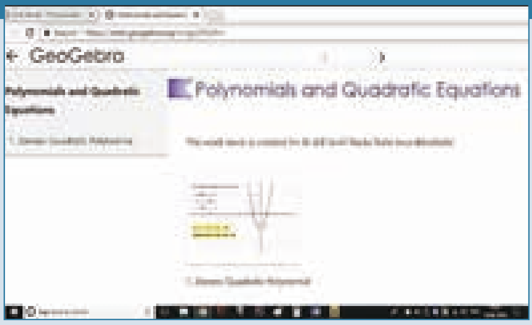
Step-3

Drag the sliders a, b and c to change the quadratic co-efficient. Follow the changes in points A and B where the curve cuts the x-axis. These points are called Zeroes of a Polynomial.

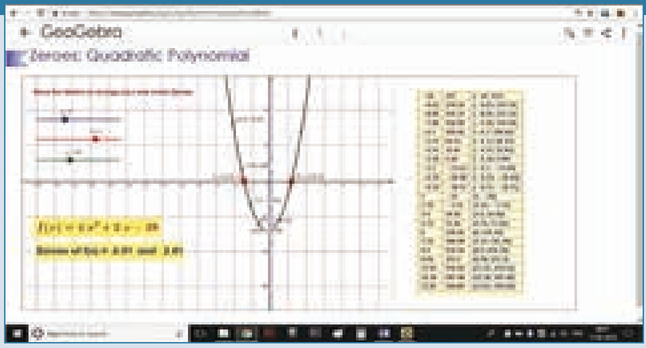
Step 1



Step 2



Step 3



Similarly you can check other worksheets in the Workbook related to your lesson

Browse in the link

Zeroes of Polynomials : https://ggbm.at/tgu3PpWm

Exercise 3.2

1. Find the value of the polynomial $f(y) = 6y - 3y^2 + 3$ at

   (i) $y = 1$ (ii) $y = -1$ (iii) $y = 0$

2. If $p(x) = x^2 - 2\sqrt{2}x + 1$, find $p(2\sqrt{2})$.

3. Find the zeros of the polynomial in each of the following :

   (i) $p(x) = x - 3$ (ii) $p(x) = 2x + 5$ (iii) $q(y) = 2y - 3$

   (iv) $f(z) = 8z$ (v) $p(x) = ax$ when $a \neq 0$ (vi) $h(x) = ax + b,\ a \neq 0,\ a, b \in R$

4. Find the roots of the polynomial equations .

   (i) $5x - 6 = 0$ (ii) $x + 3 = 0$

   (iii) $10x + 9 = 0$ (iv) $9x - 4 = 0$

5. Verify whether the following are zeros of the polynomial indicated against them, or not.

   (i) $p(x) = 2x - 1,\ x = \frac{1}{2}$ (ii) $p(x) = x^3 - 1,\ x = 1$

   (iii) $p(x) = ax + b,\ x = \frac{-b}{a}$ (iv) $p(x) = (x + 3)(x - 4),\ x = 4, x = -3$

6. Find the number of zeros of the following polynomials represented by their graphs.

(i)

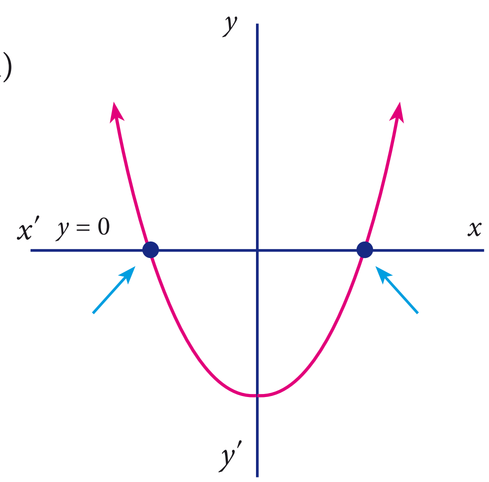


Fig. 3.12

(ii)

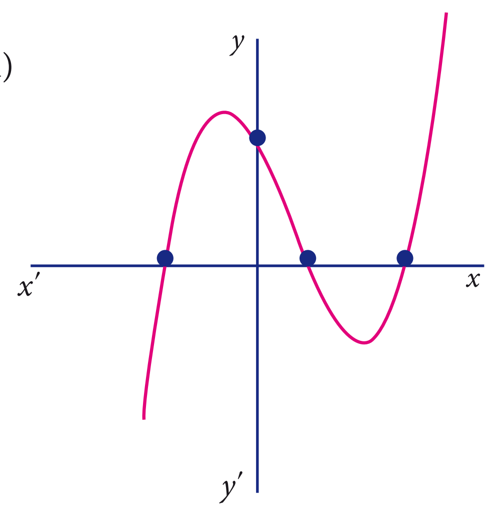


Fig. 3.13

(iii)

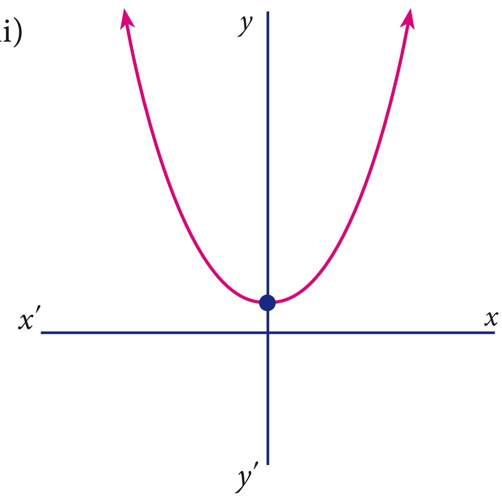


Fig. 3.14

(iv)

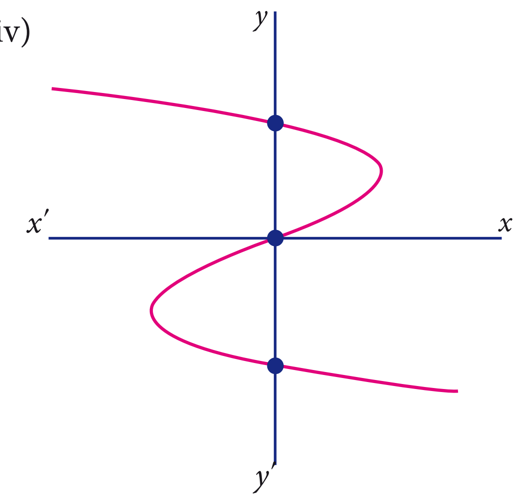


Fig. 3.15

(v)

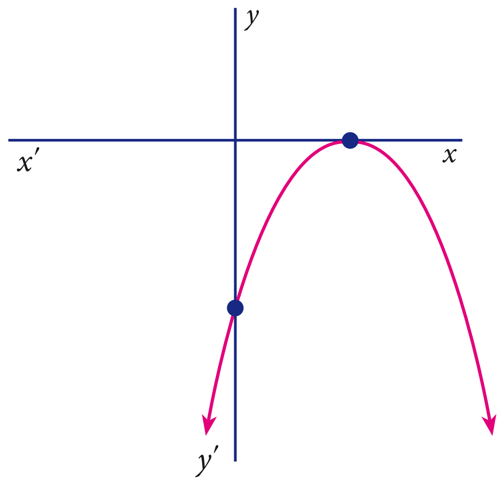


Fig. 3.16

3.5 Division of Polynomials

Let us consider the numbers 13 and 5. When 13 is divided by 5 what is the quotient and remainder.?

Yes, of course, the quotient is 2 and the remainder is3.We write 13 = (5×2)+3

Let us try.

Divide	Expressed as	Remainder	Divisor
11 by 4	(4×2)+3	3	4
24 by 5	(5×4)+4	4	5
18 by 3	(3×6)+0	0	3
22 by 11	(11×2)+0	0	11

From the above examples, we observe that the remainder is less than the divisor.

Can we say ? that when the remainder is 0, then the dividend is the multiple of the divisor?

Dividend = (Divisor × Quotient) + Remainder.

Is it possible to divide one polynomial by another ?

Of course, yes, and the way to do it is just the way similar to what you do with numbers!

Let us start with the division of a polynomial by a monomial.

Example 3.11

Divide $x^3 - 4x^2 + 6x$ by x, where , $x \neq 0$

Solution

We have

$$\frac{x^3 - 4x^2 + 6x}{x} = \frac{x^3}{x} - \frac{4x^2}{x} + \frac{6x}{x}, x \neq 0$$

$$= x^2 - 4x + 6$$

3.5.1 Division Algorithm for Polynomials

Let $p(x)$ and $g(x)$ be two polynomials such that degree of $p(x) \geq$ degree of $g(x)$ and $g(x) \neq 0$. Then there exists unique polynomials $q(x)$ and $r(x)$ such that

$$p(x) = g(x) \times q(x) + r(x) \qquad \ldots (1)$$

where $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$.

The polynomial $p(x)$ is the Dividend, $g(x)$ is the Divisor, $q(x)$ is the Quotient and $r(x)$ is the Remainder. Now (1) can be written as

Dividend = (Divisor × Quotient) + Remainder.

> If $r(x)$ is zero, then we say $p(x)$ is a multiple of $g(x)$. In other words, $g(x)$ divides $p(x)$.

If it looks complicated, don't worry ! it is important to know how to divide polynomials, and that comes easily with practice. The examples below will help you.

Example 3.12

Find the quotient and the remainder when $(5x^2 - 7x + 2) \div (x - 1)$

Solution

$(5x^2 - 7x + 2) \div (x - 1)$

$$
\begin{array}{r|l}
 & 5x-2 \\
\hline
x-1 & 5x^2 - 7x + 2 \\
 & 5x^2 - 5x \\
 & (-)\ \ (+) \\
\hline
 & \quad -2x+2 \\
 & \quad -2x+2 \\
 & \quad (+)\ \ (-) \\
\hline
 & \qquad 0 \\
\hline
\end{array}
$$

(i) $\dfrac{5x^2}{x} = 5x$

(ii) $5x(x-1) = 5x^2 - 5x$

(iii) $-\dfrac{2x}{x} = -2$

(iv) $-2(x-1) = (-2x+2)$

$\therefore$ Quotient $= 5x-2$ Remainder $= 0$

Example 3.13

Find quotient and the remainder when $f(x)$ is divided by $g(x)$

(i) $f(x) = (8x^3 - 6x^2 + 15x - 7)$, $g(x) = 2x+1$. (ii) $f(x) = x^3 + 1$, $g(x) = x+1$

Solution

(i) $f(x) = (8x^3 - 6x^2 + 15x - 7)$, $g(x) = 2x+1$

$$
\begin{array}{r|l}
 & 4x^2 - 5x + 10 \\
\hline
2x+1 & 8x^3 - 6x^2 + 15x - 7 \\
 & 8x^3 + 4x^2 \\
 & (-)\ \ (-) \\
\hline
 & \quad -10x^2 + 15x \\
 & \quad -10x^2 - 5x \\
 & \quad (+)\ \ \ (+) \\
\hline
 & \qquad\quad 20x - 7 \\
 & \qquad\quad 20x + 10 \\
 & \qquad\quad (-)\ \ \ (-) \\
\hline
 & \qquad\qquad -17 \\
\hline
\end{array}
$$

$\therefore$ Quotient $= 4x^2 - 5x + 10$ and

Remainder $= -17$

(ii) $f(x) = x^3 + 1$, $g(x) = x+1$

$$
\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
x+1 & x^3 + 0x^2 + 0x + 1 \\
 & x^3 + x^2 \\
 & (-)\ \ (-) \\
\hline
 & \quad -x^2 + 0x \\
 & \quad -x^2 - x \\
 & \quad (+)\ \ \ (+) \\
\hline
 & \qquad\quad x + 1 \\
 & \qquad\quad x + 1 \\
 & \qquad\quad (-)\ (-) \\
\hline
 & \qquad\qquad 0 \\
\hline
\end{array}
$$

Quotient $= x^2 - x + 1$ and

Remainder $= 0$

Example 3.14

If $x^4 - 3x^3 + 5x^2 - 7$ is divided by $x^2 + x + 1$ then find the quotient and the remainder.

Solution

$$
\begin{array}{r|l}
 & x^2 - 4x + 8 \\
\hline
x^2 + x + 1 & x^4 - 3x^3 + 5x^2 + 0x - 7 \\
 & x^4 + x^3 + x^2 \\
 & (-)\ \ (-)\ \ (-) \\
\hline
 & \quad -4x^3 + 4x^2 + 0x \\
 & \quad -4x^3 - 4x^2 - 4x \\
 & \quad\ (+)\ \ \ (+)\ \ \ (+) \\
\hline
 & \qquad\qquad 8x^2 + 4x - 7 \\
 & \qquad\qquad 8x^2 + 8x + 8 \\
 & \qquad\qquad (-)\ \ (-)\ \ (-) \\
\hline
 & \qquad\qquad\qquad -4x - 15 \\
\hline
\end{array}
$$

$\therefore$ Quotient $= x^2 - 4x + 8$ and

Remainder $= -4x - 15$

Exercise 3.3

1. Find the quotient and remainder of the following.

   (i) $(4x^3 + 6x^2 - 23x + 18) \div (x+3)$

   (ii) $(x^3 + 3x^2 - 31x + 12) \div (x-4)$

   (iii) $(8y^3 - 16y^2 + 16y - 15) \div (2y-1)$

   (iv) $(8x^3 - 1) \div (2x-1)$

   (v) $(-18z + 14z^2 + 24z^3 + 18) \div (3z+4)$

2. The area of a rectangle is $x^2 + 7x + 12$. If its breadth is $(x+3)$, then find its length.

3. The base of a parallelogram is $(5x+4)$. Find its height, if the area is $25x^2-16$.

4. The sum of $(x+5)$ observations is (x^3+125). Find the mean of the observations.

3.6 Remainder Theorem

In the previous section , we have learnt the division of a polynomial by another non – zero polynomial.

In this section , we shall study a simple and an elegant method of finding the remainder.

In the case of divisibility of a polynomial by a linear polynomial we use a well known theorem called **Remainder Theorem.**

If a polynomial $p(x)$ of degree greater than or equal to one is divided by a linear polynomial $(x-a)$ then the remainder is $p(a)$, where a is any real number.

Significance of Remainder theorem : It enables us to find the remainder without actually following the cumbersome process of long division.

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It leads to another well known theorem called 'Factor theorem'.

Note

(i) If $p(x)$ is divided by $(x+a)$, then the remainder is $p(-a)$

(ii) If $p(x)$ is divided by $(ax-b)$, then the remainder is $p(\frac{b}{a})$

(iii) If $p(x)$ is divided by $(ax+b)$, then the remainder is $p(-\frac{b}{a})$

Example: 3.15

S.No.	Question	Solution	Hint
1	Find the remainder when $f(x) = x^3 + 3x^2 + 3x + 1$ is divided by $x+1$.	$f(x) = x^3 + 3x^2 + 3x + 1$ $f(-1) = (-1)^3+3(-1)^2+3(-1)+1$ $= -1+3-3+1 = 0$ Hence, the remainder is 0	$g(x) = x+1$ $g(x) = 0$ $x+1 = 0$ $x = -1$
2	Check whether $f(x) = x^3 - x + 1$ is a multiple of $g(x) = 2 - 3x$	$\therefore f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 - \frac{2}{3} + 1$ $= \frac{8}{27} - \frac{2}{3} + 1$ $= \frac{8-18+27}{27} = \frac{17}{27} \neq 0$ $\Rightarrow f(x)$ is not multiple of $g(x)$	$g(x) = 2-3x$ $= 0$ gives $x = \frac{2}{3}$
3	Find the remainder when $f(x) = x^3 - ax^2 + 6x - a$ is divided by $(x - a)$	We have $f(x) = x^3 - ax^2 + 6x - a$ $f(a) = a^3 - a(a)^2 + 6a - a$ $= a^3 - a^3 + 5a$ $= 5a$ Hence the required remainder is $5a$	Let $g(x) = x-a$ $g(x) = 0$ $x - a = 0$ $x = a$

4	For what value of k is the polynominal $2x^4 + 3x^3 + 2kx^2 + 3x + 6$ exactly divisible by $(x + 2)$?	Let $f(x) = 2x^4 + 3x^3 + 2kx^2 + 3x + 6$ If $f(x)$ is exactly divisible by $(x+2)$, then the remainder must be zero i.e ., $f(-2) = 0$ i.e ., $2(-2)^4 + 3(-2)^3 + 2k(-2)^2 + 3(-2)+6=0$ $2(16) + 3(-8) + 2k(4) -6 + 6=0$ $32-24 + 8k = 0$ $8k = -8$, $k = -1$ Hence $f(x)$ is exactly divisible by $(x-2)$ when $k = -1$	Let $g(x)=x+2$ $g(x)=0$ $x+2 = 0$ $x = -2$

Example 3.16

If the polynomials $f(x) = ax^3 + 4x^2 + 3x -4$ and $g(x) = x^3- 4x + a$ leave the same remainder when divided by $x-3$, find the value of a. Also find the remainder.

Solution :

Let $f(x) = ax^3 + 4x^2 + 3x -4$ and $g(x) = x^3- 4x + a$, *When* $f(x)$ is divided by $(x-3)$, the remainder is $f(3)$.

Now $f(3) = a(3)^3 + 4(3)^2 + 3(3) - 4$

$= 27a + 36 + 9 - 4$

$f(3) = 27a + 41$ (1)

When $g(x)$ is divided by $(x-3)$, the remainder is $g(3)$.

Now $g(3) = 3^3 - 4(3) + a$

$= 27 - 12 + a$

$= 15 + a$ (2)

Since the remainders are same, (1) = (2)

Given that, $f(3) = g(3)$

That is $27a + 41 = 15 + a$

$$27a - a = 15 - 41$$

$$26a = -26$$

$$a = \frac{-26}{26} = -1$$

Substituting $a = -1$, in $f(3)$, we get

$$f(3) = 27(-1) + 41$$

$$= -27 + 41$$

$$f(3) = 14$$

$\therefore$ The remainder is 14.

Example 3.17

Without actual division , prove that $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$ is exactly divisible by $x^2 - 3x + 2$

Solution :

Let $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$

$$g(x) = x^2 - 3x + 2$$

$$= x^2 - 2x - x + 2$$

$$= x(x-2) - 1(x-2)$$

$$= (x-2)(x-1)$$

we show that $f(x)$ is exactly divisible by $(x-1)$ and $(x-2)$ using remainder theorem

$$f(1) = 2(1)^4 - 6(1)^3 + 3(1)^2 + 3(1) - 2$$

$$= 2 - 6 + 3 + 3 - 2$$

$$= 0$$

$$f(2) = 2(2)^4 - 6(2)^3 + 3(2)^2 + 3(2) - 2$$

$$= 32 - 48 + 12 + 6 - 2$$

$$f(2) = 0$$

$\therefore$ $f(x)$ is exactly divisible by $(x-1)(x-2)$

i.e., $f(x)$ is exactly divisible by $x^2 - 3x + 2$

Exercise 3.4

1. Check whether $p(x)$ is a multiple of $g(x)$ or not .

   (i) $p(x) = x^3 - 5x^2 + 4x - 3$; $g(x) = x - 2$

   (ii) $p(x) = 2x^3 - 11x^2 - 4x + 3$; $g(x) = 2x + 3$

2. By remainder theorem, find the remainder when, $p(x)$ is divided by $g(x)$ where,

   (i) $p(x) = x^3 - 2x^2 - 4x - 1$; $g(x) = x + 1$

   (ii) $p(x) = 4x^3 - 12x^2 + 14x - 3$; $g(x) = 2x - 1$

   (iii) $p(x) = x^3 - 3x^2 + 4x + 50$; $g(x) = x - 3$

   (iv) $p(x) = 27x^3 - 54x^2 + 3x - 4$; $g(x) = 1 - \frac{3}{2}x$

3. Find the remainder when $3x^3 - 4x^2 + 7x - 5$ is divided by $(x+3)$

4. What is the remainder when $x^{2018} + 2018$ is divided by $x-1$

5. For what value of k is the polynomial

   $p(x) = 2x^3 - kx^2 + 3x + 10$ exactly divisible by $(x-2)$

6. The polynomials $ax^3 - 3x^2 + 4$ and $2x^3 - 5x + a$ when divided by $(x-2)$ leave the remainders p and q respectively, if $p - 2q = 4$; find the value of a.

7. If two polynomials $2x^3 + ax^2 + 4x - 12$ and $x^3 + x^2 - 2x + a$ leave the same remainder when divided by $(x - 3)$, find the value of a and also find the remainder.

Exercise 3.5

Multiple Choice Questions

1. If $x^3 + 6x^2 + kx + 6$ is exactly divisible by $(x + 2)$, then k= ?

   (a) −6 (b) −7 (c) −8 d) 11

2. The root of the polynomial equation $2x + 3 = 0$ is

   (a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $-\frac{3}{2}$ (d) $-\frac{2}{3}$

3. The type of the polynomial $4-3x^3$ is

   (a) constant polynomial (b) linear polynomial

   (c) quadratic polynomial (d) cubic polynomial.

4. $x^3 - x^2$ is a …………..

(a) monomial (b) binomial (c) trinomial (d) constant polynomial

5. If $x^{51} + 51$ is divided by $x + 1$, then the remainder is

(a) 0 (b) 1 (c) 49 (d) 50

6. The zero of the polynomial $2x+5$ is

(a) $\frac{5}{2}$ (b) $-\frac{5}{2}$ (c) $\frac{2}{5}$ (d) $-\frac{2}{5}$

7. The sum of the polynomials $p(x) = x^3 - x^2 - 2$, $q(x) = x^2 - 3x + 1$

(a) $x^3 - 3x - 1$ (b) $x^3 + 2x^2 - 1$ (c) $x^3 - 2x^2 - 3x$ (d) $x^3 - 2x^2 + 3x - 1$

8. The product of the polynomials $p(x) = 4x - 3$ $q(x) = 4x + 3$

(a) $1 - x - 8$ (b) $16x^2 - 9$

(c) $18x^3 + 12x^2 - 12x - 8$ (d) $18x^3 - 12x^2 + 12x + 8$

9. The remainder when $p(x) = x^3 - ax^2 + 6x - a$ is divided by $(x - a)$ is

(a) $-5a$ (b) $\frac{1}{5}$ (c) 5 (d) $5a$

10. The Auto fare is found as minimum ₹25 for 3 kilo meter and thereafter ₹ 12 for per kilo meter.Which of the following equations represents the relationship between the total cost 'c' in rupees and the number of kilometers n?

(a) $c = 25 + n$ (b) $c = 25 + 12n$ (c) $c = 25 + (n-3)12$ (d) $c = (n-3)12$

11. Degree of the polynomial $(y^3-2)(y^3 + 1)$ is

(a) 9 (b) 2 (c) 3 (d) 6

12. Let the polynomials be

(A) $-13q^5 + 4q^2 + 12q$ (B) $(x^2 + 4)(x^2 + 9)$

(C) $4q^8 - q^6 + q^2$ (D) $-\frac{5}{7}y^{12} + y^3 + y^5$

Then ascending order of their degree is

(a) A,B,D,C (b) A,B,C,D (c) B,C,D,A (d) B,A,C,D

Points to remember

- An algebraic expression of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ is called Polynomial in one variable x of degree 'n' where $a_0, a_1, a_2, \ldots a_n$ are constants $(a_n \neq 0)$ and n is a whole number.

- Let $p(x)$ be a polynominal. If $p(a) = 0$ then we say that 'a' is a zero of the polynomial $p(x)$

- If $x = a$ statisfies the polynominal $p(x) = 0$ then $x = a$ is called a root of the polynominal equation $p(x) = 0$.

- Remainder Theorem: If a polynomial $p(x)$ of degree greater than or equal to one is divided by a linear polynomial $(x-a)$, then the remainder is $p(a)$, where a is any real number.

Answers

Exercise 3.1

1. (i) not a polynomial (ii) polynomial (iii) not a polynomial (iv) polynomial

(v) polynomial (vi) not a polynomial

2.

	Cooefficient of x^2	Cooefficient of x
(i)	$\frac{2}{5}$	-3
(ii)	-2	$-\sqrt{7}$
(iii)	π	-1
(iv)	$\sqrt{3}$	$\sqrt{2}$
(v)	1	$-\frac{7}{2}$

3. (i) 7 (ii) 4 (iii) 5 (iv) 6 (v) 4

4.

	Descending order	Ascending order
(i)	$\sqrt{7}x^3 + 6x^2 + x - 9$	$-9 + x + 6x^2 + \sqrt{7}x^3$
(ii)	$-\frac{7}{2}x^4 - 5x^3 + \sqrt{2}x^2 + x$	$x + \sqrt{2}x^2 - 5x^3 - \frac{7}{2}x^4$

(iii) $7x^3 - \frac{6}{5}x^2 + 4x - 1$ $\quad -1 + 4x - \frac{6}{5}x^2 + 7x^3$

(iv) $9y^4 + \sqrt{5}\,y^3 + y^2 - \frac{7}{3}y - 11$ $\quad -11 - \frac{7}{3}y + y^2 + \sqrt{5}\,y^3 + 9y^4$

5. (i) $6x^3+6x^2-14x+17$, 3 (ii) $7x^3+7x^2+11x-8$, 3 (iii) $16x^4-6x^3-5x^2+7x-6$, 4

6. (i) $7x^2+8$, 2 (ii) $-y^3+6y^2-14y+2$, 3 (iii) $z^5-6z^4-6z^2-9z+7$, 5

7. $x^3-8x^2+11x+7$ 8. $2x^4-3x^3+5x^2-5x+6$

9. (i) $6x^4+7x^3-56x^2-63x+18$, 4 (ii) $105x^2-33x-18$, 2 (iii) $30x^3-77x^2+54x-7$, 3

10. x^2+y^2+2xy, ₹.225 11. $9x^2-4$, 3596 sq. units

12. cubic polynomial or polynomial of degree 3

Exercise 3.2

1. (i) 6 (ii) –6 (iii) 3 2. 1 3. (i) 3 (ii) $-\frac{5}{2}$ (iii) $\frac{3}{2}$ (iv) 0 (v) 0 (vi) $-\frac{b}{a}$

4. (i) $\frac{6}{5}$ (ii) –3 (iii) $-\frac{9}{10}$ (iv) $\frac{4}{9}$

5. (i) yes zero (ii) yes zero (iii) yes zero (iv) yes zero

6. (i) 2 (ii) 3 (iii) 0 (iv) 1 (v) 1

Exercise 3.3

1. (i) Quotient : $4x^2-6x-5$, Remainder : 33 (ii) Quotient : x^2+7x-3, Remainder : 0

(iii) Quotient : $4y^2-6y+5$, Remainder : –10 (iv) Quotient : $4x^2+2x+1$, Remainder : 0

(v) Quotient : $8z^2-6z+2$, Remainder : 10 2. Length : $x+4$ 3. Height : $5x-4$

4. Mean : $x^2-5x+25$

Exercise 3.4

1. (i) P(x) is not a multiple of g(x) (ii) P(x) is not a multiple of g(x)

2. (i) Remainder : 0 (ii) Remainder : $\frac{3}{2}$ (iii) Remainder : 62 (iv) Remainder : –18

3. Remainder : –143 4. Remainder : 2019 5. $K = 8$ 6. $a = 4$ 7. $a = -3$ Remainder : 27

Exercise 3.5

1. (d) 2. (c) 3. (c) 4. (b) 5. (d) 6. (b) 7. (a) 8. (b) 9. (d) 10. (c)

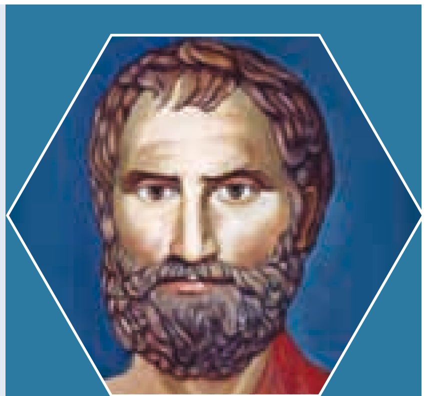
11. (d) 12. (d)

4 GEOMETRY

E64P7Q

There is geometry in humming of the strings,
there is music in the spacing of the spheres.
-Pythagoras

Thales (Pronounced THAYLEES) was born in the Greek city of Miletus. He was known for theoretical and practical understanding of geometry, especially triangles. He used geometry to solve many problems such as calculating the height of pyramids and the distance of ships from the sea shore. He was one of the so-called Seven Sagas or Seven Wise Men of Greece and many regarded him as the first philosopher in the western tradition.



Thales
(Period 620 – 546 BC)

Learning Outcomes

- To understand theorems on linear pairs and vertically opposite angles.
- To understand the angle sum property of triangle.
- To classify quadrilaterals.
- To understand the properties of quadrilaterals and use them in problem solving.
- To construct the Circumcentre of a triangle.
- To construct the Orthocentre of a triangle.

4.1. Introduction

In geometry, we study **shapes**. But what is there to *study* in shapes, you may ask. Think first, what are all the things we do with shapes? We draw shapes, we compare shapes, we *measure* shapes. *What* do we measure in shapes?

Take some shapes like this:

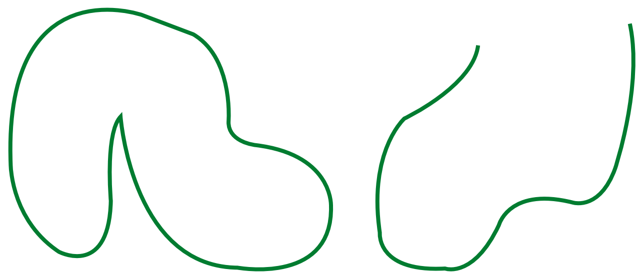


Fig. 4.1

In both of them, there is a *curve* forming the shape: one is a closed curve, enclosing a region, and the other is an open curve. We can use a rope (or a thick string) and measure the **length** of the open curve and the length of the boundary of the region in the case of the closed curve.

Curves are tricky, aren't they? It is so much easier to measure length of straight lines using the scale, isn't it? Consider the two shapes below.

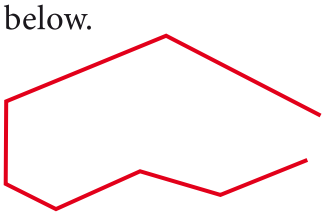


Fig. 4.2

We are going to focus our attention for now only on shapes made using straight lines, and only closed figures. As you will see, there is plenty of interesting things to do already? Fig.4.2 shows an open figure.

We not only want to draw such shapes, we want to compare them, measure them and do much more. For doing so, we want to **describe** them. How would you describe these closed shapes? (See Fig 4.3) They are all made of straight lines, all closed.

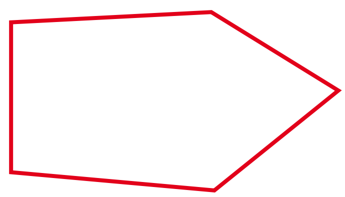


Fig. 4.3

Activity 1

Your friend draws a figure such as this on a piece of paper.

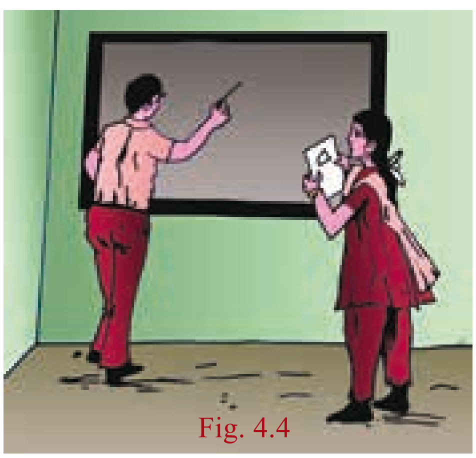


Fig. 4.4

You don't know what it is, and stand at the board.

She has to describe the figure to you so that you draw exactly the **same** figure on the board.

You both know rectangles, so it is easy to describe and draw a rectangle. The rest are not easy.

Can we describe any such shape we can *ever think of*, in a way that another person hearing it can reproduce it **exactly?**

Yes, yes, yes! Now, isn't that exciting?

The answer is so simple that it is breathtaking.

We describe different shapes by their properties.

It is like talking to your friend about someone you saw: "Tall, thin man, big moustache, talks fast, has a deep voice ...."

We will do a similar thing with shapes.

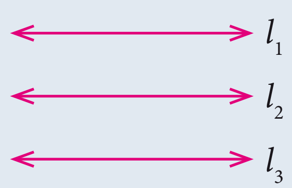
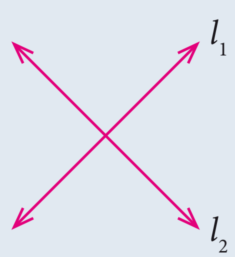
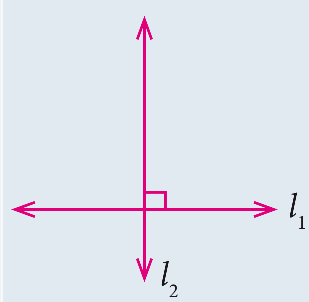
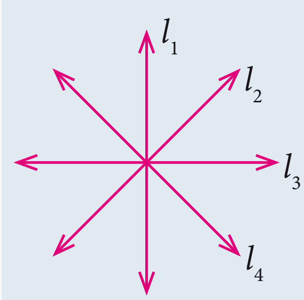
In science we learn that air has certain properties – for instance, it occupies space. Water has the property that it takes the shape of the vessel that contains it, flows from a height to down below. Similarly, triangles, rectangles, circles, and all the shapes we draw, however different they look, have properties that describe them **uniquely.** By the time you learn Geometry well, you can give shape for everything on earth – indeed (***Aadhaar card***), not only on earth but anywhere! But for now, have some patience. Just as we started with small numbers and arithmetical operations on them in primary level classes and now you know it is the same for **all** numbers, similarly we will learn properties of very **simple** shapes now, and slowly but surely we will learn more and when we are done, you will have techniques to describe any shape in 2 dimensions (like what we draw on paper), or in 3 dimensions (like the solids we use in life), or indeed, in any dimensions.

Why should we bother to learn this? One very practical reason is that all science and engineering demands it. We cannot design buildings, or even tables and chairs, or lay out the circuits inside our mobile phones, without a mathematical understanding of shapes. Another important reason is that geometry gives you a new way of looking at the world, at everything in it. You begin to see that all objects, however complicated they look, are made of very simple shapes. You learn to think not only like a mathematician, but also learn to appreciate symmetry and order, as artists do. Yes, geometry helps you become an artist too.

4.2 Geometry Basics –Recall

Draw two lines on a plane. They can be either parallel or intersecting.

Parallel lines	Two or more lines lying in the same plane that never meet.
Intersecting lines	Two lines which meet at a common point.
Perpendicular lines	Two lines which intersect each other at right angle.
Concurrent lines	Three or more lines passing through the same point.

Parallel Lines	Intersecting Lines	Perpendicular Lines	Concurrent Lines
			
$l_1 \parallel l_2 \parallel l_3$		$l_1 \perp l_2$	

4.2.1 Types of Angles

Plumbers measure the angle between connecting pipes to make a good fitting. Wood workers adjust their saw blades to cut wood at the correct angle. Air Traffic Controllers (ATC) use angles to direct planes. Carom and billiards players must know their angles to plan their shots. An angle is formed by two rays that share a common end point provided that the two rays are non-collinear.

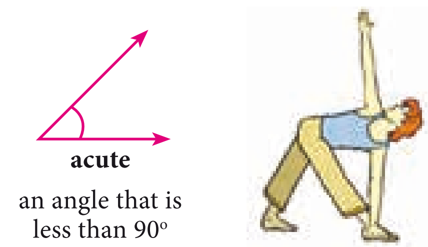


Fig. 4.5

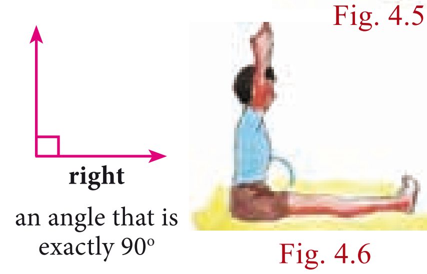
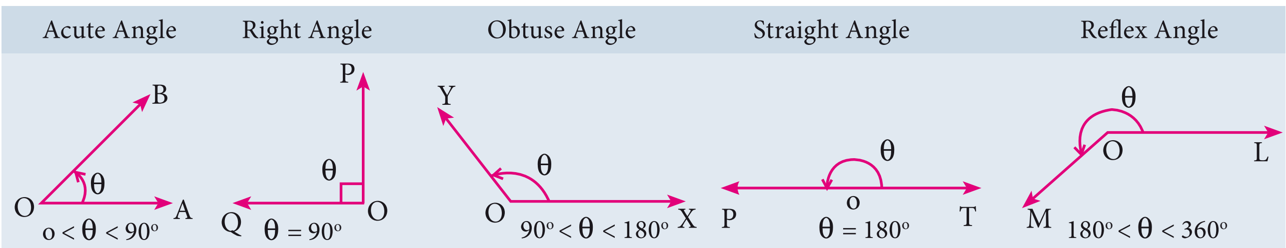


Fig. 4.6



Complementary Angles

Two angles are Complementary if their sum is 90°. For example, if $\angle ABC$=64° and $\angle DEF$=26°, then angles $\angle ABC$ and $\angle DEF$ are complementary to each other because $\angle ABC + \angle DEF = 90°$

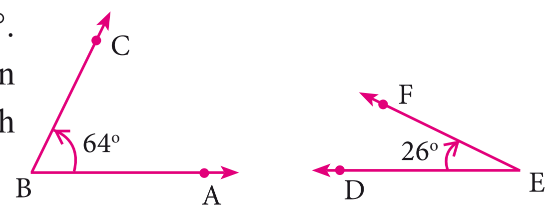


Fig. 4.7

Supplementary Angles

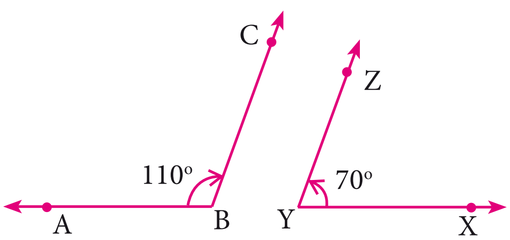


Fig. 4.8

Two angles are Supplementary if their sum is 180°. For example if $\angle ABC$=110° and $\angle XYZ$=70°

Here $\angle ABC + \angle XYZ = 180°$

$\therefore \angle ABC$ and $\angle XYZ$ are supplementary to each other

Adjacent Angles

Two angles are called adjacent angles if

(i) They have a common vertex.

(ii) They have a common arm.

(iii) The common arm lies between the two non-common arms.

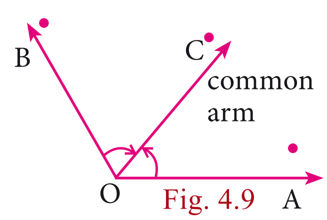


Fig. 4.9

Linear Pair of Angles

If a ray stands on a straight line then the sum of two adjacent angle is 180°. We then say that the angles so formed is a linear pair.

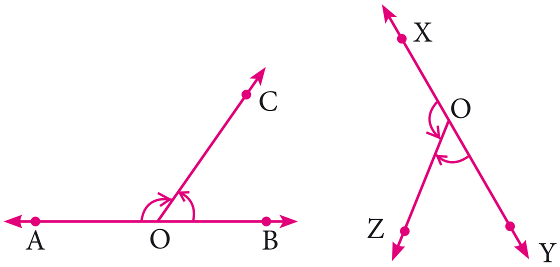


Fig. 4.10

$\angle AOC + \angle BOC = 180°$

$\therefore \angle AOC$ and $\angle BOC$ form a linear pair

$\angle XOZ + \angle YOZ = 180°$

$\angle XOZ$ and $\angle YOZ$ form a linear pair

Vertically Opposite Angles

If two lines intersect each other, then vertically opposite angles are equal.

In this figure $\angle POQ = \angle SOR$

$\angle POS = \angle QOR$

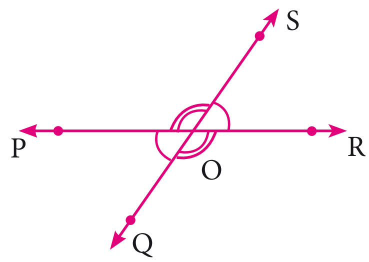


Fig. 4.11

4.2.2 Transversal

A line which intersects two or more lines at a distinct points is called a transversal of those lines.

Case (i) When a transversal intersect two lines, we get eight angles.

In the figure the line l is the transversal for the lines m and n

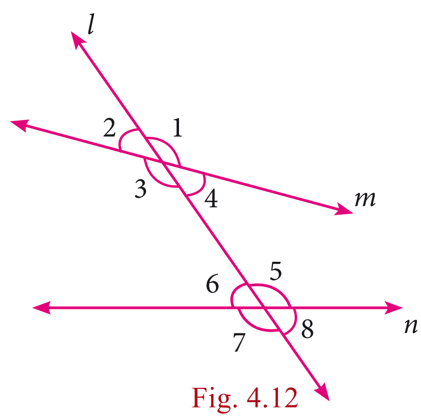


Fig. 4.12

(i) Corresponding Angles: $\angle 1$ and $\angle 5$, $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$, $\angle 4$ and $\angle 8$

(ii) Alternate Interior Angles: $\angle 4$ and $\angle 6$, $\angle 3$ and $\angle 5$

(iii) Alternate Exterior Angles: $\angle 1$ and $\angle 7$, $\angle 2$ and $\angle 8$

(iv) $\angle 4$ and $\angle 5$, $\angle 3$ and $\angle 6$ are interior angles on the same side of the transversal.

(v) $\angle 1$ and $\angle 8$, $\angle 2$ and $\angle 7$ are exterior angles on the same side of the transversal.

Case (ii) If a transversal intersects two parallel lines. The transversal forms different pairs of angles.

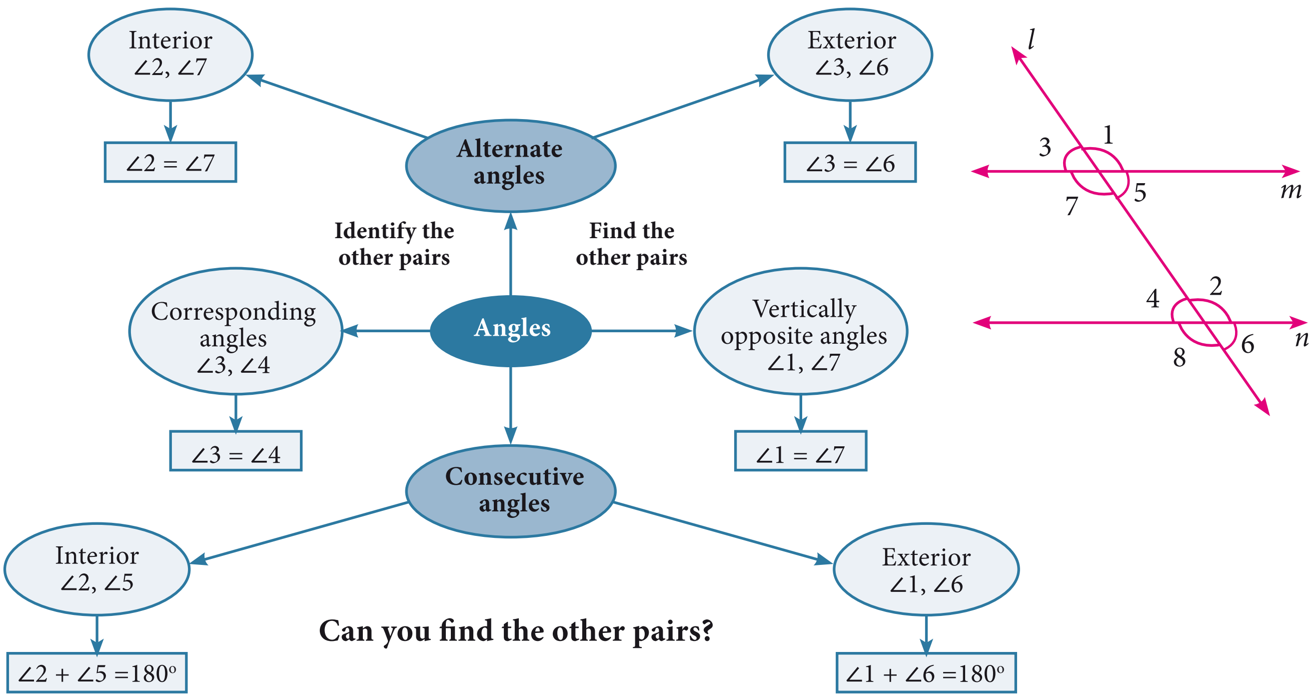


Fig. 4.13

4.2.3 Triangles

Activity 2

Take three different colour sheets; place one over the other and draw a triangle on the top sheet. Cut the sheets to get triangles of different colour which are identical. Mark the vertices and the angles as shown. Place the interior angles ∠1, ∠2 and ∠3 on a straight line, adjacent to each other, without leaving any gap. What can you say about the total measure of the three angles ∠1, ∠2 and ∠3?

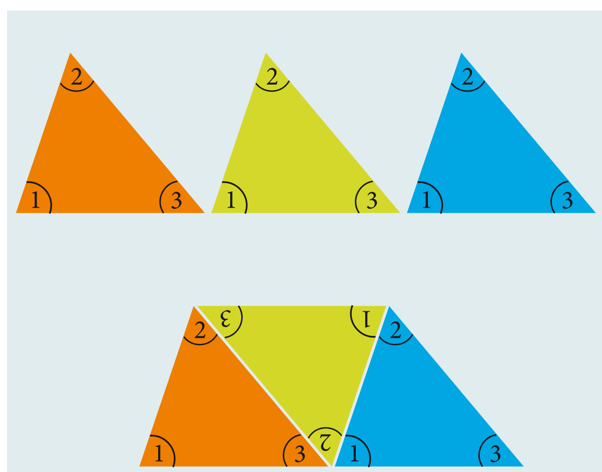


Fig. 4.14

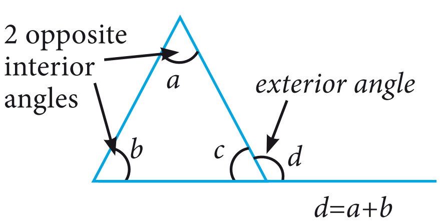


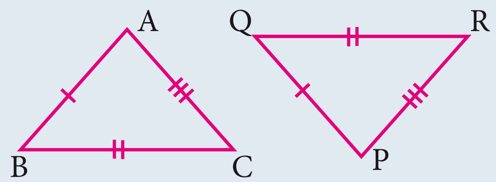
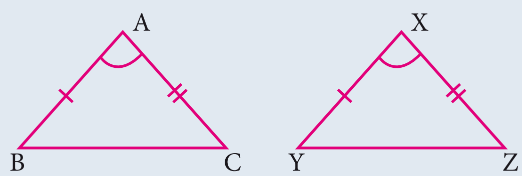
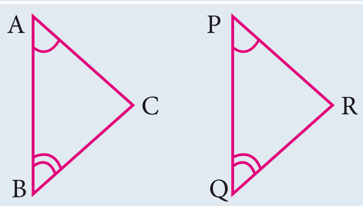
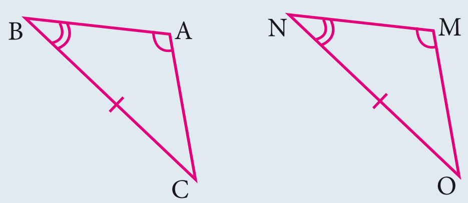
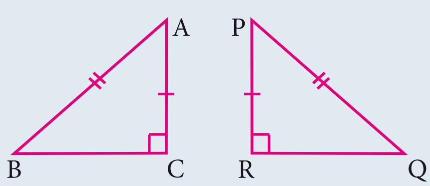
Fig. 4.15

Can you use the same figure to explain the "**Exterior angle property**" of a triangle?

If a side of a triangle is stretched, the exterior angle so formed is equal to the sum of the two remote interior angles.

4.2.4 Congruent Triangles

Two triangles are congruent if the sides and angles of one triangle are equal to the corresponding sides and angles of another triangle.

Rule	Diagrams	Reason
SSS		$AB = PQ$ $BC = QR$ $AC = PR$ $\Delta ABC \cong \Delta PQR$
SAS		$AB = XY$ $\angle BAC = \angle YXZ$ $AC = XZ$ $\Delta ABC \cong \Delta XYZ$
ASA		$\angle A = \angle P$ $AB = PQ$ $\angle B = \angle Q$ $\Delta ABC \cong \Delta PQR$
AAS		$\angle A = \angle M$ $\angle B = \angle N$ $BC = NO$ $\Delta ABC \cong \Delta MNO$
RHS		$\angle ACB = \angle PRQ = 90°(R)$ $AB =$ PQ hypotenuse (H) $AC = PR$ (S) $\Delta ABC \cong \Delta PQR$

Exercise 4.1

1. Find the complement of the following angles $(1° = 60'$ minutes, $1' = 60''$ seconds)

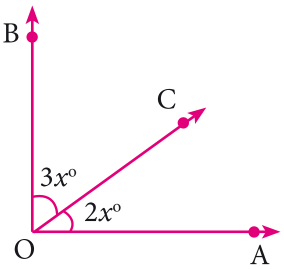
   (i) 70° (ii) 27° (iii) 45° (iv) 62°32′

2. Find the supplement of the following angles.

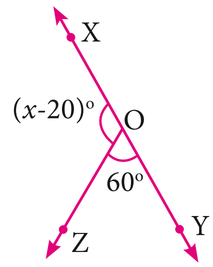
   (i) 140° (ii) 34° (iii) Right angle (iv) 121°48′

3. Find the value of x

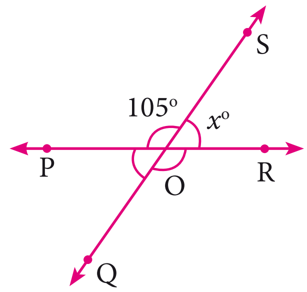
(i)



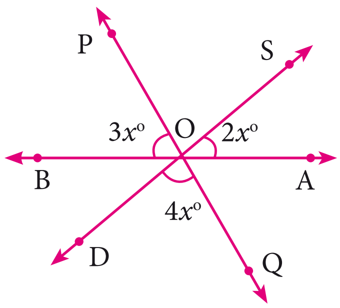
(ii)



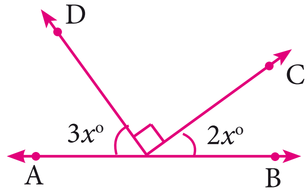
(iii)



(iv)

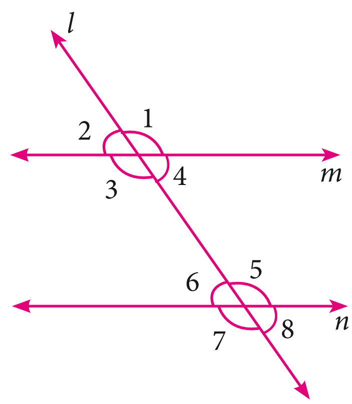


(v)



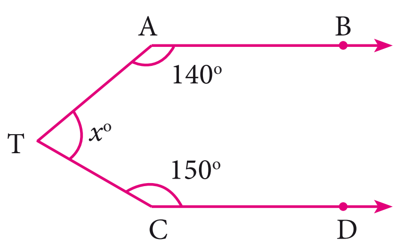
4. Let $m \| n$ and l is a transversal

Such that $\angle 1 : \angle 2 = 11 : 7$. Determine all the eight angles.

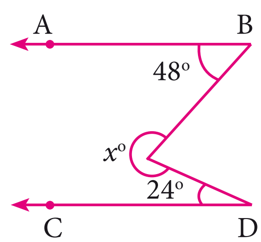


5. In the figure, AB is parallel to CD, find x

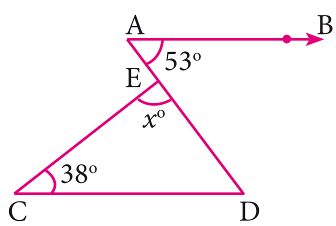
(i)



(ii)



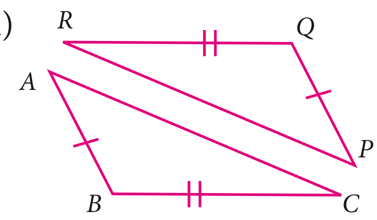
(iii)

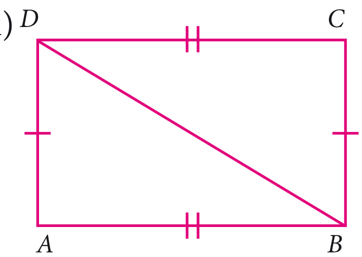


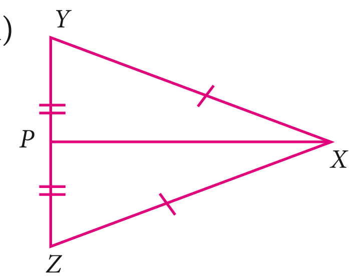
E6DK9D

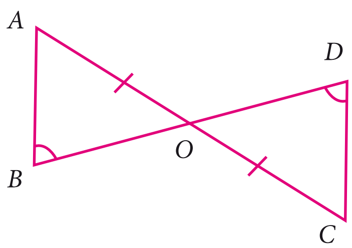
6. The angles of a triangle are in the ratio 1: 2 : 3, find the measure of each angle of the triangle.

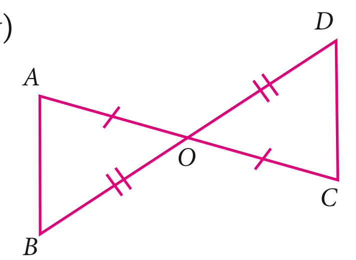
7. Consider the given pairs of triangles and say whether each pair is that of congruent triangles. If the triangles are congruent, say 'how'; if they are not congruent say 'why' and also say if a small modification would make them congruent:

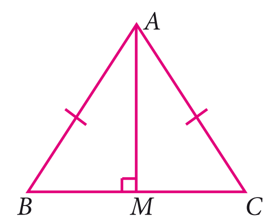
(i)


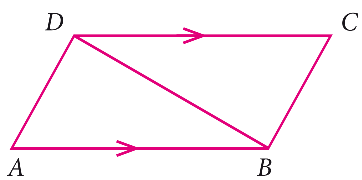
(ii)


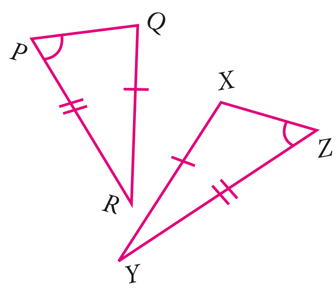
(iii)


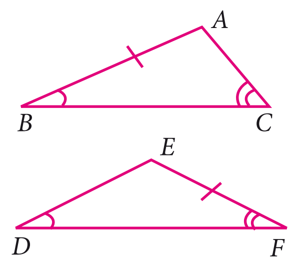
(iv)


(v)


(vi)


(vii)


(viii)


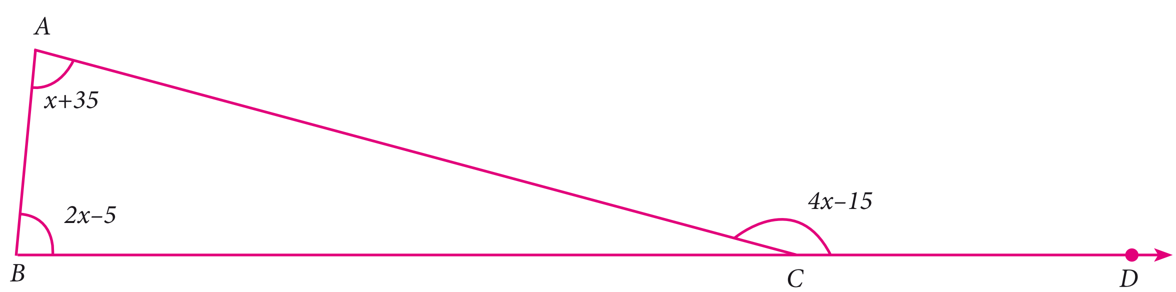
(ix)


8. ΔABC and ΔDEF are two triangles in which

$AB=DF$, $\angle ACB=70°$, $\angle ABC=60°$; $\angle DEF=70°$ and $\angle EDF=60°$.

Prove that the triangles are congruent.

9. Find all the three angles of the ΔABC



4.3 Quadrilaterals

Activity 3

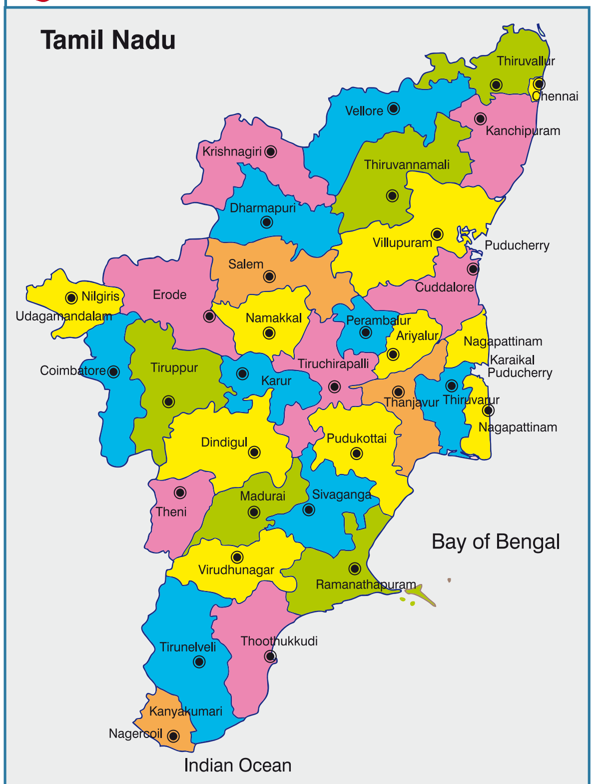


Fig. 4.16

Four Tamil Nadu State Transport buses take the following routes. The first is a one-way journey, and the rest are round trips. Find the places on the map, put points on them and connect them by lines to draw the routes. The places connecting four different routes are given as follows.

(i) Nagercoil, Tirunelveli, Virudhunagar, Madurai

(ii) Sivagangai, Puthukottai, Thanjavur, Dindigul

(iii) Erode, Coimbatore, Dharmapuri, Karur

(iv) Chennai, Cuddalore, Krishnagiri, Vellore

You will get the following shapes.

(i) 

Fig. 4.17

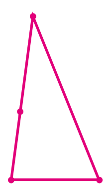
(ii) 

Fig. 4.18

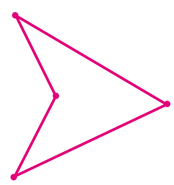
(iii) 

Fig. 4.19

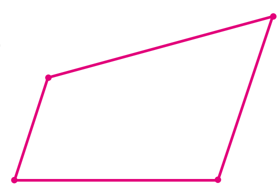
(iv) 

Fig. 4.20

Label the vertices with city names, draw the shapes exactly as they are shown on the map without rotations.

We observe that the first is a single line, the four points are collinear. The other three are closed shapes made of straight lines, of the kind we have seen before. We need names to call such closed shapes, we will call them **polygons** from now on.

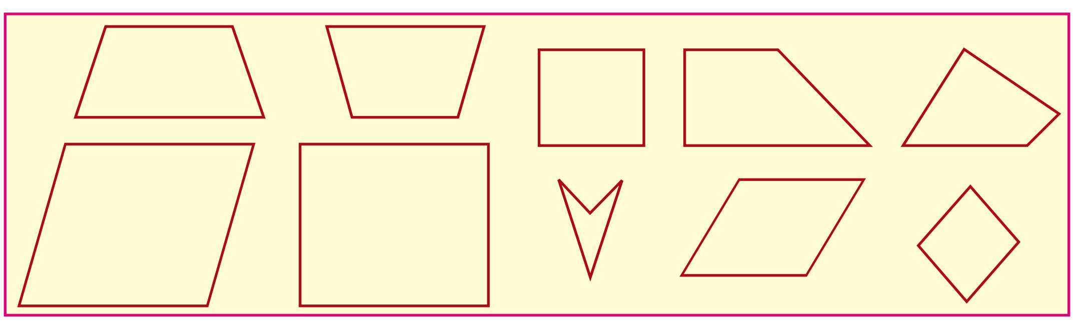


Fig. 4.21

How do polygons look? They have sides, with points at either end. We call these points **vertices** of the polygon. The sides are line segments joining the vertices. The word *poly* stands for many, and a polygon is a many-sided figure.

Note

Concave polygon: *Polygon having any one of the interior angle greater than 180º*

Convex Polygon: *Polygon having each interior angle less than 180º*

(Diagonals should be inside the polygon)

How many sides can a polygon have? One? But that is just a line segment. Two ? But how can you get a closed shape with two sides? Three? Yes, and this is what we know as a triangle. Four sides?

Squares and rectangles are examples of polygons with 4 sides but they are not the only ones. Here are some examples of 4-sided polygons. We call them ***quadrilaterals.***

Thinking Corner

You know *bi*-cycles and *tri*-cycles, don't you ? When we attach these to the front of any word, they stand for 2 (bi) or 3 (tri) of them. Similarly *quadri* stands for 4 of them. We should really speak of quadri-cycles also, but we don't. *Lateral* stands for sideways, thus quadrilateral means a 4-sided figure. You know *trilaterals*; they are also called triangles !

After 4 ? We have: 5 – *penta*, 6 – *hexa*, 7 – *hepta*, 8 – *octa*, 9 – *nano*, 10 – *deca*.

Conventions are made by history. Trigons are called triangles, quadrigons are called quadrilaterals, but then we have pentagons, hexagons, heptagons, octagons, nanogons and decagons. Beyond these, we have 11-gons, 12-gons etc. Perhaps you can draw a 23-gon !

Activity 4

This is a copy of the tangram puzzle. The tangram puzzle consists of 7 geometric pieces which are normally boxed in the shape of a square. The pieces, called 'tans', are used to create different patterns including **animals, people, numbers, geometric shapes** and many more.

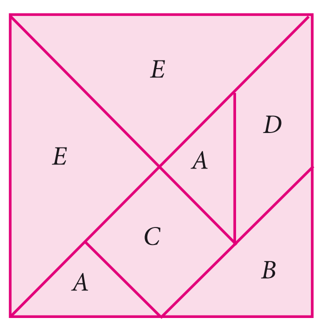


Fig. 4.22

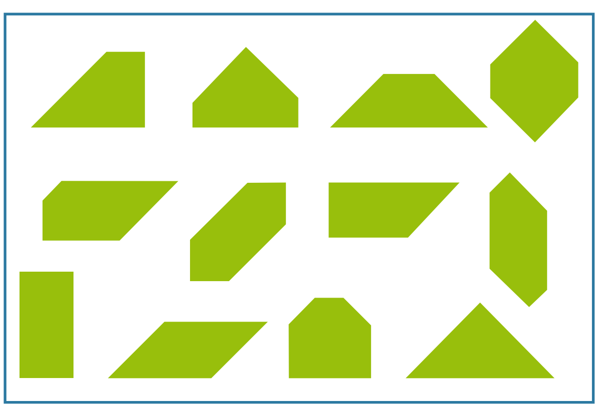


Fig. 4.23

You can make several polygons using the pieces in different ways.

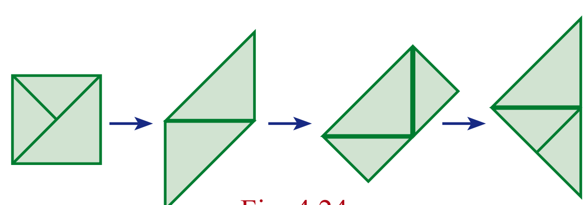


Fig. 4.24

Here are some examples.(See Fig.4.23)

Use three triangular pieces to make a square, a triangle, a rectangle or a parallelogram.

Use 5 of the pieces to make a trapezium.

Use 7 of the pieces to make a hexagon.

Here (See Fig.4.24) is how one can transform a square (made of 3 triangles) into a large triangle. Try to explore similar transformations.

Activity 5

Angle sum for a polygon

Draw any quadrilateral $ABCD$.

Mark a point P in its interior.

Join the segments PA, PB, PC and PD.

You have 4 triangles now.

How much is the sum of all the angles of the 4 triangles?

How much is the sum of the angles at their vertex, now P?

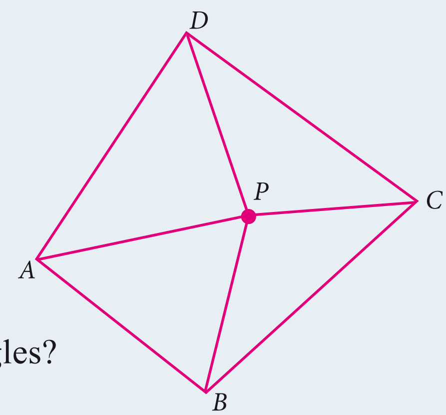


Fig. 4.25

Can you now find the 'angle sum' of the quadrilateral $ABCD$?

Can you extend this idea to any polygon?

Thinking Corner

1. If there is a polygon of n sides ($n \geq 3$), then the sum of all interior angles is $(n-2) \times 180°$

2. For the regular polygon

- Each interior angle is $\dfrac{(n-2)}{n} \times 180°$
- Each exterior angle is $\dfrac{360°}{n}$
- The sum of all the exterior angles formed by producing the sides of a convex polygon in the order is 360° .
- If a polygon has 'n' sides , then the number of diagonals of the polygon is $\dfrac{n(n-3)}{2}$

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Activity 6

Draw the following special quadrilaterals on a graph sheet, measure the sides and angles and complete the table to explore the properties of the quadrilaterals with respect to sides and angles.

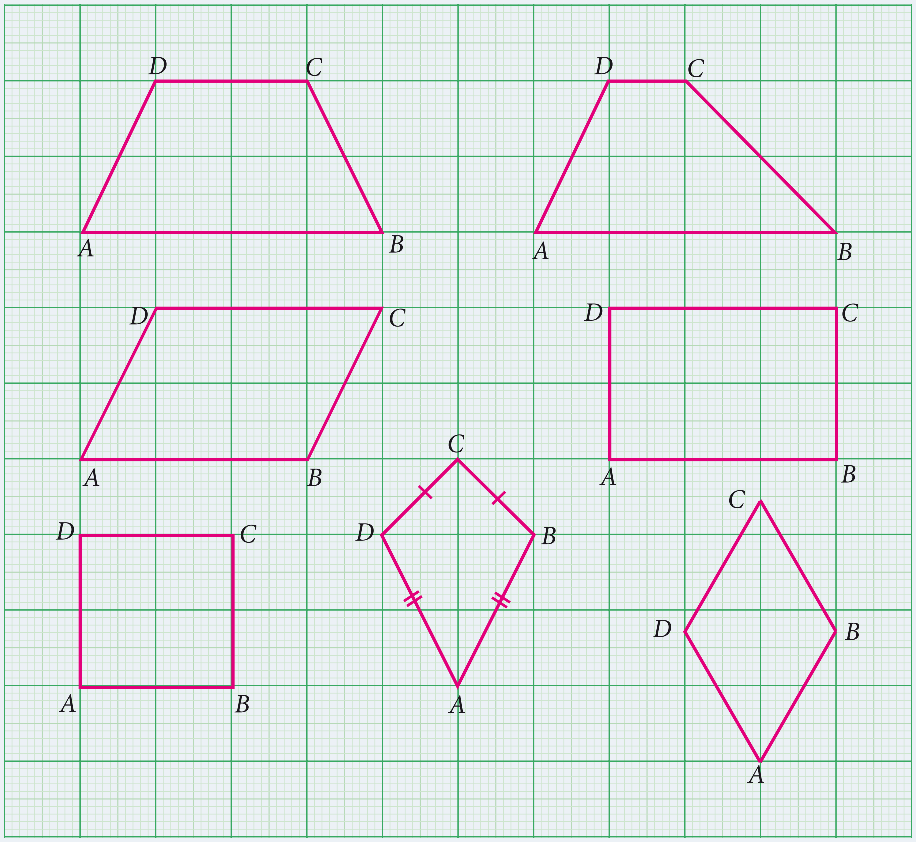


Fig. 4.26

S. No.	Name of the quadrilateral	Length of sides				Measure of angles			
		AB	BC	CD	DA	$\angle A$	$\angle B$	$\angle C$	$\angle D$
1	Trapezium								
2	Isosceles Trapezium								
3	Parallelogram								
4	Rectangle								
5	Rhombus								
6	Square								
7	Kite								

Do you see some patterns in all the data you have recorded? We see many interesting properties, but how do we know whether these are true in general, or happen to hold only for these figures? It is not even clear **what** properties we should look for. The best way to answer this is to go back to what we already know and look at it from this viewpoint. We know *rectangles*, so we can ask what properties rectangles have. Here they are:

- Opposite sides are equal. (1)
- *All* angles are equal, each is 90 degrees. (2)
- Adjacent sides may or may not be equal. (3)

Among these, the last statement really says nothing! (Mathematicians call such statements **trivial**, and they prefer not to write them down.) Note that adjacent sides have a vertex in common, and opposite sides have no vertex in common. A square has all these properties but the third is replaced by; Adjacent sides are equal. (4) Now we can combine (1) and (4) and say that in a square, all sides are equal.

Thus we see that every square is a rectangle but a rectangle need not be a square. This suggests a way of classifying quadrilaterals, of grouping them according to whether some sides are equal or not, some angles are equal or not.

4.3.1 Special Names for Some Quadrilaterals

1. A **parallelogram** is a quadrilateral in which opposite sides are parallel and equal.
2. A **rhombus** is a quadrilateral in which opposite sides are parallel and all sides are equal.
3. A **trapezium** is a quadrilateral in which *one pair of* opposite sides are parallel.

Draw a few parallellograms, a few rhombuses (correctly called rhombii, like cactus and cactii) and a few trapeziums (correctly written trapezia).

Fig. 4.27

The great advantage of listing properties is that we can see the relationships among them immediately.

- Every parallelogram is a trapezium, but not necessarily the other way.
- Every rhombus is a parallelogram, but not necessarily the other way.
- Every rectangle is a parallelogram, but not necessarily the other way.
- Every square is a rhombus and hence every square is a parallelogram as well.

For "not necessarily the other way" mathematicians usually say "the converse is not true". A smart question then is: just *when* is the other way also true? For instance, when is a parallelogram also a rectangle? Any parallelogram in which all angles are also equal is a rectangle. (Do you see why?) Now we can observe many more interesting properties. For instance, we see that a rhombus is a parallelogram in which all **sides** are also equal.

4.3.2 More Special Names

When all sides of a quadrilateral are equal, we call it **equilateral**. When all angles of a quadrilateral are equal, we call it **equiangular**. In triangles, we talked of equilateral triangles as those with all sides equal. Now we can call them equiangular triangles as well!

We thus have:

A rhombus is an equilateral parallelogram.

A rectangle is an equiangular parallelogram.

A square is an equilateral and equiangular parallelogram.

Here are two more special quadrilaterals, called **kite** and **isosceles trapezium**.

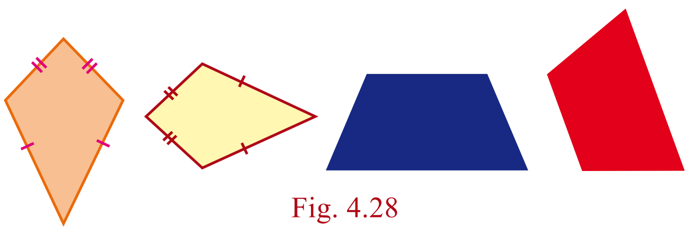


Fig. 4.28

4.3.3 Types of Quadrilaterals

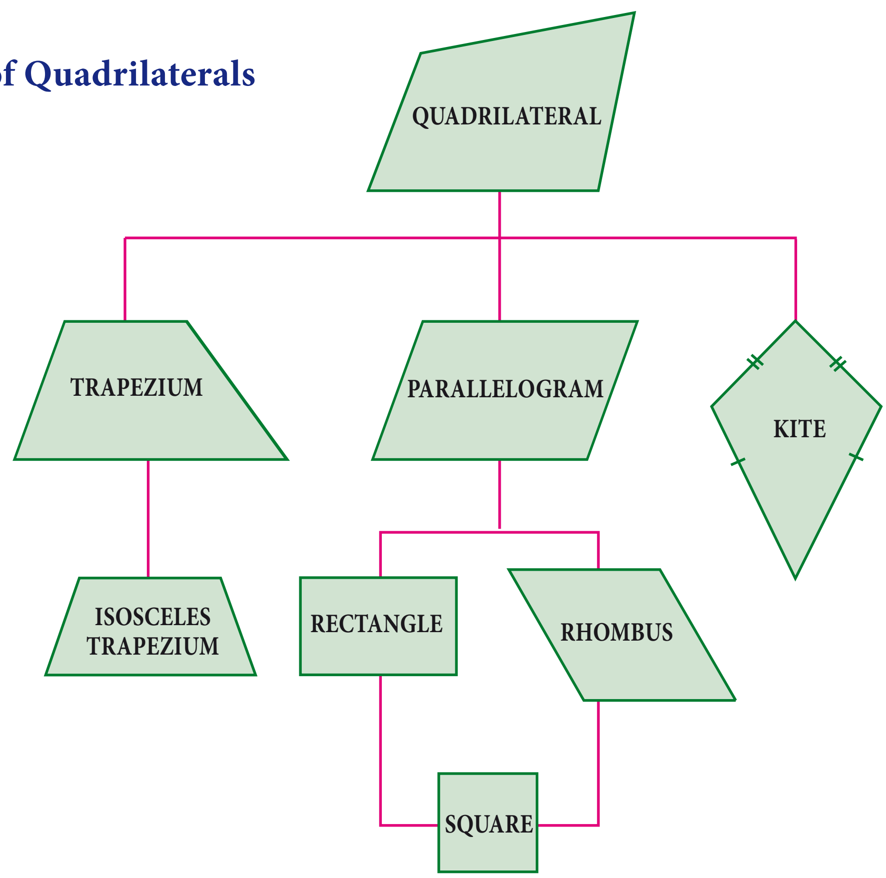


Fig. 4.29

Progress Check

Answer the following question.

(i) Are the opposite angles of a rhombus equal?

(ii) A quadrilateral is a ______________ if a pair of opposite sides are equal and parallel.

(iii) Are the opposite sides of a kite equal?

(iv) Which is an equiangular but not an equilateral parallelogram?

(v) Which is an equilateral but not an equiangular parallelogram?

(vi) Which is an equilateral and equiangular parallelogram?

(vii) _________ is a rectangle, a rhombus and a parallelogram.

Activity 7

Step – 1

Cut out four different quadrilaterals from coloured glazed papers.

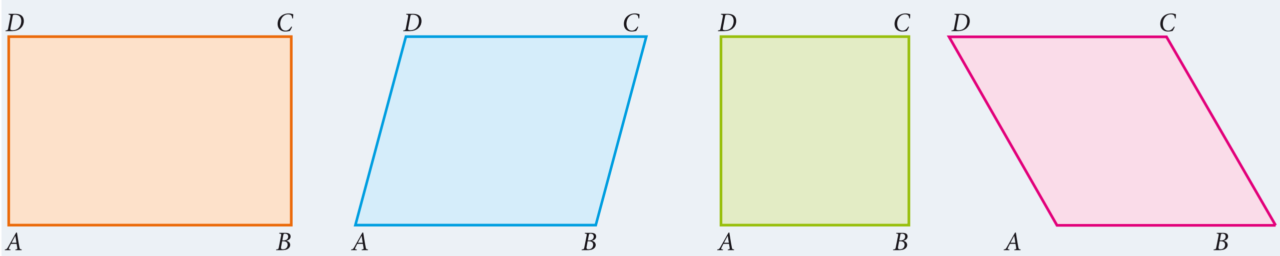


Fig. 4.30

Step – 2

Fold the quadrilaterals along their respective diagonals. Press to make creases. Here, dotted line represent the creases.

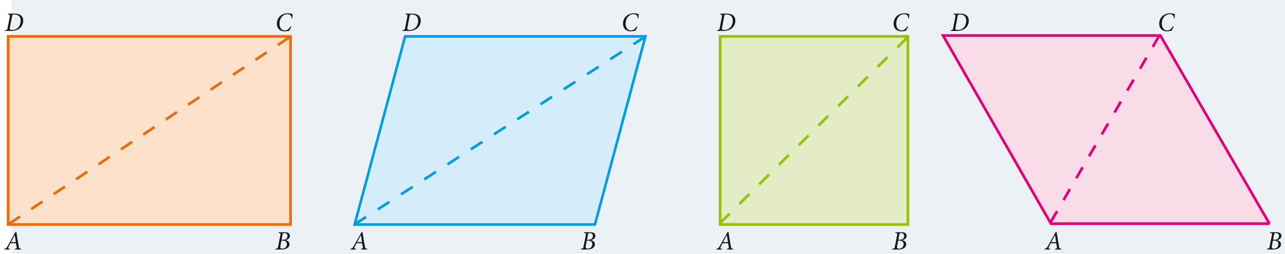


Fig. 4.31

Step – 3

Fold the quadrilaterals along both of their diagonals. Press to make creases.

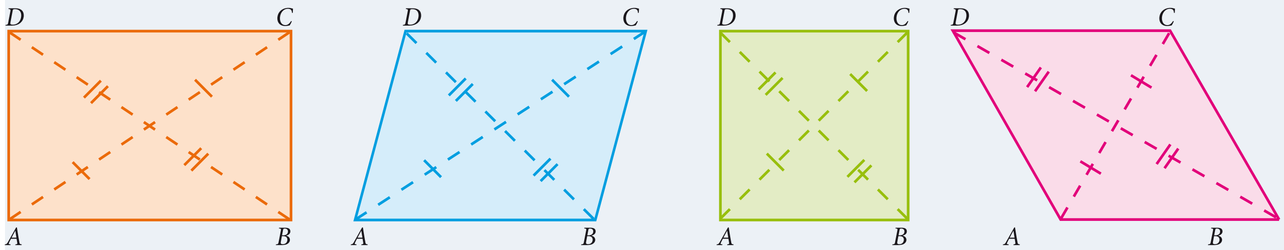


Fig. 4.32

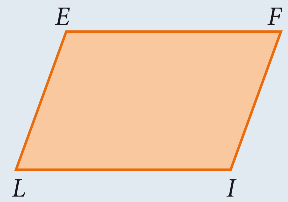
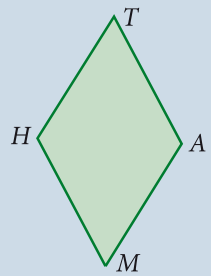
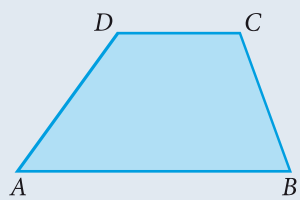
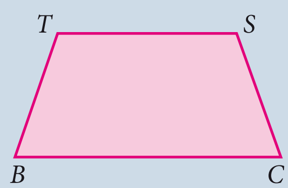
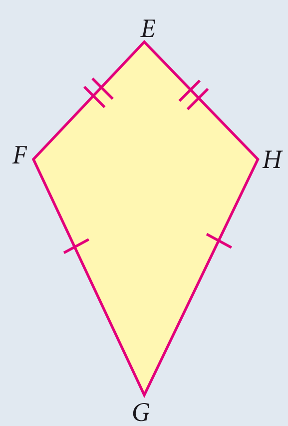
We observe that two imposed triangles are congruent to each other. Measure the lengths of portions of diagonals and angles between the diagonals.

Also do the same for the quadrilaterals such as Trapezium, Isosceles Trapezium and Kite.

From the above activity, measure the lengths of diagonals and angles between the diagonals and record them in the table below:

S. No.	Name of the quadrilateral	Length along diagonals						Measure of angles			
		AC	BD	OA	OB	OC	OD	$\angle AOB$	$\angle BOC$	$\angle COD$	$\angle DOA$
1	Trapezium										
2	Isosceles Trapezium										
3	Parallelogram										
4	Rectangle										
5	Rhombus										
6	Square										
7	Kite										

4.3.4 Properties of Quadrilaterals

Name	Diagram	Sides	Angles	Diagonals
Parallelogram		Opposite sides are parallel and equal	Opposite angles are equal and sum of any two adjacent angles is 180°	Diagonals bisect each other.
Rhombus		All sides are equal and opposite sides are parallel	Opposite angles are equal and sum of any two adjacent angles is 180°	Diagonals bisect each other at right angle.
Trapezium		One pair of opposite sides are parallel	The angles at the ends of each non-parallel sides are supplementary	Diagonals need not be equal
Isosceles Trapezium		One pair of opposite sides are parallel and non-parallel sides are equal in length.	The angles at the ends of each parallel sides are equal.	Diagonals are of equal length.
Kite		Two pairs of adjacent sides are equal	One pair of opposite angles are equal	1. Diagonals intersect at right angle. 2. Shorter diagonal bisected by longer diagonal 3. Longer diagonal divides the kite into two congruent triangles

Note

(i) A rectangle is an equiangular parallelogram.

(ii) A rhombus is an equilateral parallelogram.

(iii) A square is an equilateral and equiangular parallelogram.

(iv) A square is a rectangle , a rhombus and a parallelogram.

Activity 8

Procedure

(i) Make a parallelogram on a chart/graph paper and cut it.

(ii) Draw diagonal of the parallelogram.

(iii) Cut along the diagonal and obtain two triangles.

(iv) Superimpose one triangle onto the other.

What do you conclude ?

Activity 9

Procedure

Cut off a quadrilateral $ABCD$ in a paper with prescribed dimensions. Mark the mid-points P, Q, R and S of the sides AB, BC, CD and DA respectively. By folding the sides appropriately cut off the quadrilateral $PQRS$. Fold the quadrilateral (i) Does PQ lie on SR? similarly fold the other way and verify QR lies on PS.

Observation

(i) What do you conclude?

(ii) Name the resulting figure?

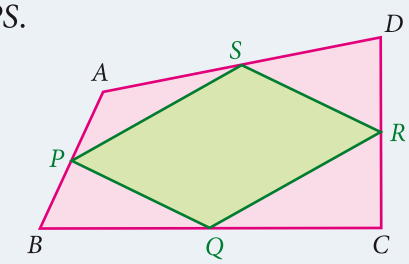


Fig. 4.33

Draw the following quadrilaterals and join the mid points of all its sides. Find the resultant shape and complete the table.

Name of the quadrilateral	Shape obtained by joining the midpoints
Parallelogram	
Rectangle	
Square	
Kite	
Trapezium	
Rhombus	

Progress Check

State the reasons for the following.

(i) A square is a special kind of a rectangle.

(ii) A rhombus is a special kind of a parallelogram.

(iii) A rhombus and a kite have one common property.

(iv) A square and a rhombus have one common property.

What type of quadrilateral is formed when the following pairs of triangles are joined together?

(i) Equilateral triangle.

(ii) Right angled triangle.

(iii) Isosceles triangle.

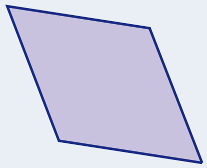
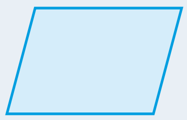
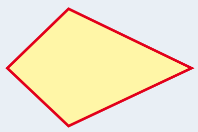
Activity 10

Complete the given table by placing Yes or No

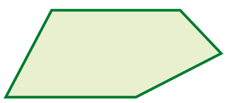
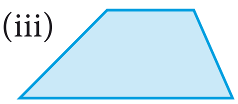
Quadrilaterals	Opposite sides		All sides Equal	Diagonals			Angles		
	Parallel	Equal		Equal	Perpendicular to each other	Bisect each other	All angles are equal	Opposite angles equal	Adjacent angles are supplementary
Parallelogram									
Rectangle									
Square									
Rhombus									
Trapezium									
Isosceles trapezium									
Kite									

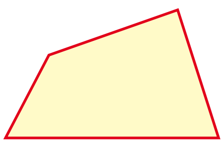
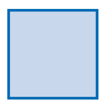
Exercise 4.2

1. Match the name of the shapes with its figure on the right.

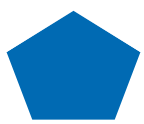
Rhombus	
Kite	
Parallelogram	
Trapezium	
Rectangle	

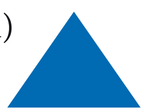
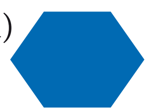
2. Identify which ones are parallelograms and which are not.

(i)  (ii)  (iii) 

(iv)  (v)  (vi) 

3. Which ones are not quadrilaterals?

(i)  (ii)  (iii)  (iv) 

(v)  (vi)  (vii)  (viii) 

4. Identify which ones are trapeziums and which are not.

(i)  (ii)  (iii) 

(iv)  (v)  (vi) 

4.3.5 Properties of Parallelogram

We can now embark on an interesting journey. We can tour among lots of quadrilaterals, noting down interesting properties. What properties do we look for, and how do we know they are true?

For instance, opposite sides of a parallelogram are parallel, but are they also **equal**? We could draw any number of parallelograms and verify whether this is true or not. In fact, we see that opposite sides are equal in **all of** them. Can we then conclude that opposite sides are equal in *all* parallelograms? No, because we might later find a parallelogram, one which we had not thought of until then, in which opposite sides are unequal. So, we need an argument, a **proof**.

Consider the parallelogram $ABCD$ in the given Fig. 4.34. We believe that $AB = CD$ and $AD = BC$, but how can we be sure? We know triangles and their properties. So we can try and see if we can use that knowledge. But we don't have any triangles in the parallelogram $ABCD$.

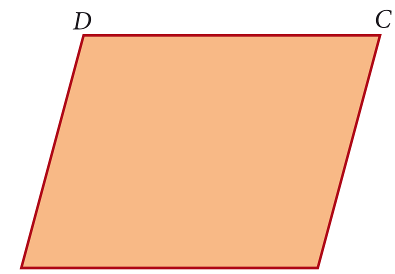


Fig. 4.34

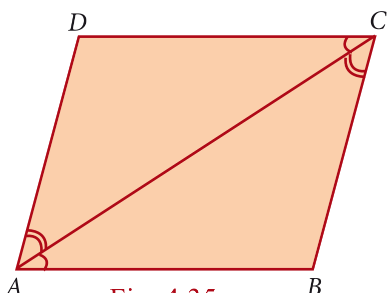


Fig. 4.35

This is easily taken care of by joining AC. (We could equally well have joined BD, but let it be AC for now.) We now have 2 triangles ADC and ABC with a common side AC. If we could somehow prove that these two triangles are congruent, we would get $AB = CD$ and $AD = BC$, which is what we want!

Is there any hope of proving that ΔADC and ΔABC are congruent? There are many criteria for congruence, it is not clear which one is relevant here.

So far we have not used the fact that $ABCD$ is a parallelogram at all. So we need to use the facts that $AB \parallel DC$ and $AD \parallel BC$ to show that ΔADC and ΔABC are congruent. From sides being parallel we have to get to some angles being equal. Do we know any such properties? Yes we do, and that is all about **transversals!**

Now we can see it clearly. $AD \parallel BC$ and AC is a transversal, hence $\angle DAC = \angle BCA$. Similarly, $AB \parallel DC$, AC is a transversal, hence $\angle BAC = \angle DCA$. With AC as common side, the ASA criterion tells us that ΔADC and ΔABC are congruent, just what we needed. From this we can conclude that $AB = CD$ and $AD = BC$.

Thus opposite sides are indeed equal in a parallelogram.

The argument we now constructed is written down as a ***formal proof*** in the following manner.

Theorem 1

In a parallelogram, opposite sides are equal

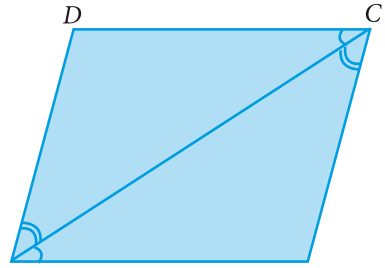


Fig. 4.36

Given $ABCD$ is a parallelogram

To Prove $AB=CD$ and $DA=BC$

Construction Join AC

Proof

Since $ABCD$ is a parallelogram

$AD \parallel BC$ and AC is the transversal

$\angle DAC = \angle BCA$ →(1) (alternate angles are equal)

$AB \parallel DC$ and AC is the transversal

$\angle BAC = \angle DCA$ →(2) (alternate angles are equal)

In ΔADC and ΔCBA

$\angle DAC = \angle BCA$ from (1)

AC is common

$\angle DCA = \angle BAC$ from (2)

$\Delta ADC \cong \Delta CBA$ (By ASA)

Hence $AD = CB$ and $DC = BA$ (Corresponding sides are equal)

Along the way in the proof above, we have proved another property that is worth recording as a theorem.

Theorem 2

A diagonal of a parallelogram divides it into two congruent triangles.

Notice that the proof above established that $\angle DAC = \angle BCA$ and $\angle BAC = \angle DCA$. Hence we also have, in the figure above,

$$\angle BCA + \angle BAC = \angle DCA + \angle DAC$$

But we know that:

$$\angle B + \angle BCA + \angle BAC = 180$$

and $\angle D + \angle DCA + \angle DAC = 180$

Therefore we must have that $\angle B = \angle D$.

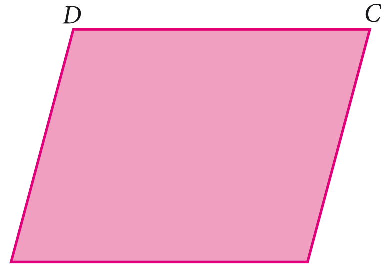


Fig. 4.37

With a little bit of work, proceeding similarly, we could have shown that $\angle A = \angle C$ as well. Thus we have managed to prove the following theorem:

Theorem 3

The opposite angles of a parallelogram are equal.

Now that we see congruence of triangles as a good "strategy", we can look for more triangles. Consider both diagonals AC and DB. We already know that ΔADC and ΔCBA are congruent. By a similar argument we can show that ΔDAB and ΔBCD are congruent as well. Are there more congruent triangles to be found in this figure ?

Yes. The two diagonals intersect at point O. We now see 4 new ΔAOB, ΔBOC, ΔCOD and ΔDOA. Can you see any congruent pairs among them?

Since AB and CD are parallel and equal, one good guess is that ΔAOB and ΔCOD are congruent. We could again try the ASA crierion, in which case we want $\angle OAB = \angle OCD$ and $\angle ABO = \angle CDO$. But the first of these follows from the fact that $\angle CAB = \angle ACD$ (which we already established) and observing that $\angle CAB$ and $\angle OAB$ are the same (and so also $\angle OCD$ and $\angle ACD$). We now use the fact that BD is a transversal to get that $\angle ABD = \angle CDB$, but then $\angle ABD$ is the same as $\angle ABO$, $\angle CDB$ is the same as $\angle CDO$, and we are done.

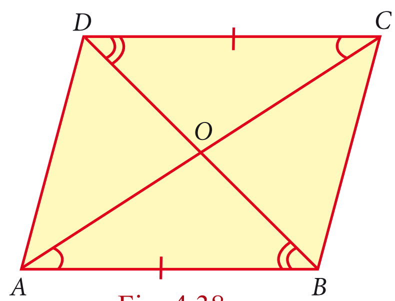


Fig. 4.38

Again, we need to write down the formal proof, and we have another theorem.

Theorem 4

The diagonals of a parallelogram bisect each other.

It is time now to reinforce our concepts on parallelograms. Consider each of the given statements, in the adjacent box, one by one. For each statement, we can conclude that it is a quadrilateral. If the quadrilateral happens to be a parallelogram, what type of parallelogram is it?

- Each pair of its opposite sides are parallel.
- Each pair of opposite sides is equal.
- All of its angles are right angles.
- Its diagonals bisect each other.
- The diagonals are equal.
- The diagonals are perpendicular and equal.
- The diagonals are perpendicular bisectors of each other.
- Each pair of its consecutive angles is supplementary.

Now we begin with lots of interesting properties of parallelograms. Can we try and prove some property relating to two or more parallelograms ? A simple case to try is when two parallelograms share the same base, as in Fig.4.39

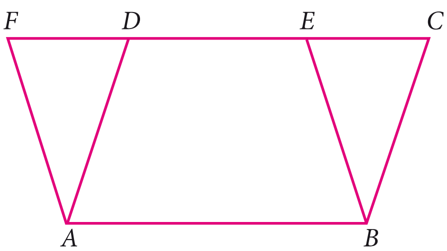


Fig. 4.39

We see parallelograms $ABCD$ and $ABEF$ are on the common base AB. At once we can see a pair of triangles for being congruent ΔADF and ΔBCE. We already have that $AD = BC$ and $AF = BE$. But then since $AD \parallel BC$ and $AF \parallel BE$, the angle formed by AD and AF must be the same as the angle formed by BC tand BE. Therefore $\angle DAF = \angle CBE$. Thus ΔADF and ΔBCE are congruent.

That is an interesting observation; can we infer anything more from this ? Yes, we know that congruent triangles have the *same area*. This makes us think about the areas of the parallelograms $ABCD$ and $ABEF$.

$$\begin{aligned} \text{Area of } ABCD &= \text{area of quadrilateral } ABED + \text{area of } \Delta BCE \\ &= \text{area of quadrilateral } ABED + \text{area of } \Delta ADF \\ &= \text{area of } ABEF \end{aligned}$$

Thus we have proved another interesting theorem:

Theorem 5:

Parallelograms on the same base and between the same parallels are equal in area.

In this process, we have also proved other interesting statements. These are called *Corollaries*, which do not need separate detailed proofs.

Corollary 1: Triangles on the same base and between the same parallels are equal in area.

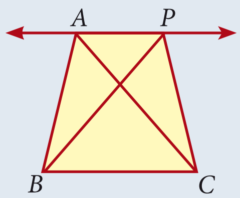
Corollary 2: A rectangle and a parallelogram on the same base and between the same parallels are equal in area.

These statements that we called Theorems and Corollaries, hold for all parallelograms, however large or small, with whatever be the lengths of sides and angles at vertices.

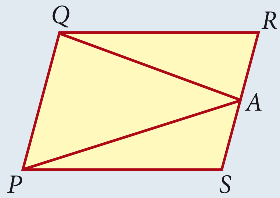
Progress Check

Which of the following figures lie on the same base and between the same parallels? In such a case, write the common base and the two parallel line segments:

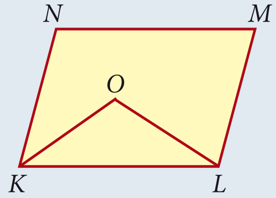
(i)



(ii)



(iii)



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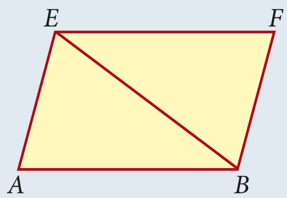
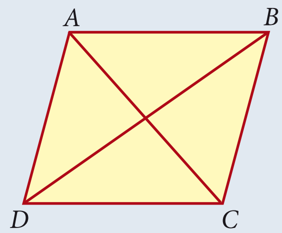
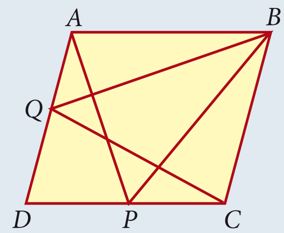


(v)



(vi)



We can now apply this knowledge to find out properties of specific quadrilateral that is parallelogram.

Example 4.1

In a parallelogram $ABCD$, the bisectors of the consecutive angles $\angle A$ and $\angle B$ insersect at P. Show that $\angle APB = 90^\circ$

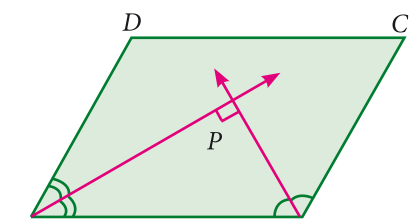


Fig. 4.40

Solution

$ABCD$ is a parallelogram AP and BP are bisectors of consecutive angles $\angle A$ and $\angle B$.

Since the consecutive angles of a parallelogram are supplementary

$$\angle A + \angle B = 180^\circ$$

$$\frac{1}{2}\angle A + \frac{1}{2}\angle B = \frac{180^\circ}{2}$$

$$\Longrightarrow \quad \angle PAB + \angle PBA = 90^\circ$$

In ΔAPB,

$$\angle PAB + \angle APB + \angle PBA = 180^\circ \text{ (angle sum property of triangle)}$$

$$\angle APB = 180^\circ - [\angle PAB + \angle PBA]$$

$$= 180^\circ - 90^\circ = 90^\circ$$

Hence Proved.

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Example 4.2

In the Fig.4.41 $ABCD$ is a parallelogram, P and Q are the mid-points of sides AB and DC respectively. Show that $APCQ$ is a parallelogram.

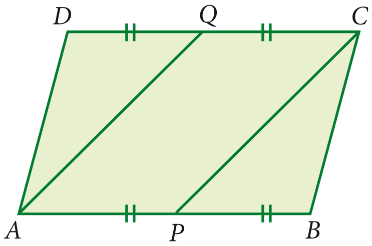


Fig. 4.41

Solution

Since P and Q are the mid points of

AB and DC respectively

Therefore $\quad AP = \frac{1}{2} AB$ and

$QC = \frac{1}{2} DC \qquad (1)$

But $\quad AB = DC \qquad$ (Opposite sides of a parallelogram are equal)

$\Longrightarrow \quad \frac{1}{2} AB = \frac{1}{2} DC$

$\Longrightarrow \quad AP = QC \qquad (2)$

Also , $\quad AB \parallel DC$

$\Longrightarrow \quad AP \parallel QC \qquad$ (3) [$\because$ $ABCD$ is a parallelogram]

Thus , in quadrilateral $APCQ$ we have $AP = QC$ and $AP \parallel QC$ [from (2) and (3)]

Hence , quadrilateral $APCQ$ is a parallelogram.

Example 4.3

$ABCD$ is a parallelogram Fig.4.42 such that $\angle BAD = 120^\circ$ and AC bisects $\angle BAD$ show that $ABCD$ is a rhombus.

Solution

Given $\quad \angle BAD = 120^\circ$ and AC bisects $\angle BAD$

$\angle BAC = \frac{1}{2} \times 120^\circ = 60^\circ$

$\angle 1 = \angle 2 = 60^\circ$

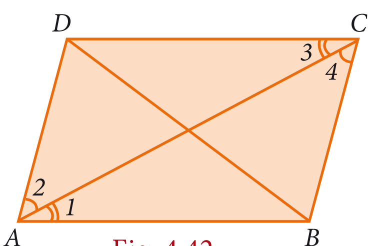


Fig. 4.42

$AD \parallel BC$ and AC is the traversal

$\angle 2 = \angle 4 = 60^\circ$

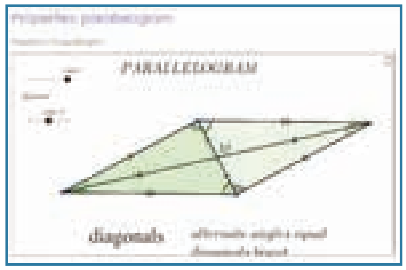
$\Delta\, ABC$ is isosceles triangle $\qquad [\because \angle 1 = \angle 4 = 60^\circ]$

$\Longrightarrow \quad AB = BC$

Parallelogram $ABCD$ is a rhombus.

ICT Corner

Expected Result is shown in this picture



Step – 1

Open the Browser and copy and paste the Link given below (or) by typing the URL given (or) Scan the QR Code.

Step - 2

GeoGebra worksheet "Properties: Parallelogram" will appear. There are two sliders named "Rotate" and "Page"

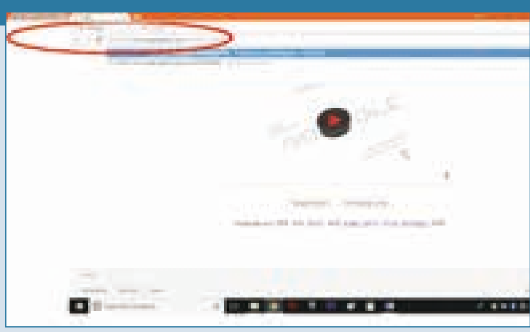
Step-3

Drag the slider named "Rotate "and see that the triangle is doubled as parallelogram.

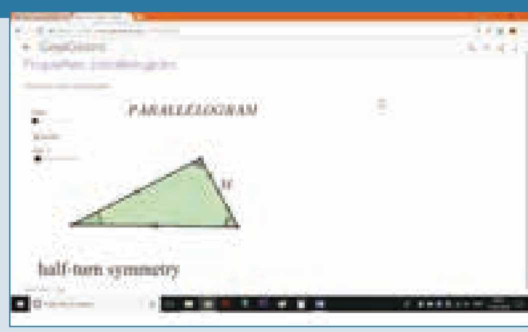
Step-4

Drag the slider named "Page" and you will get three pages in which the Properties are explained.

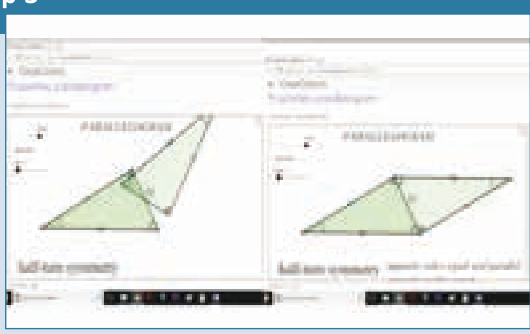
Step 1



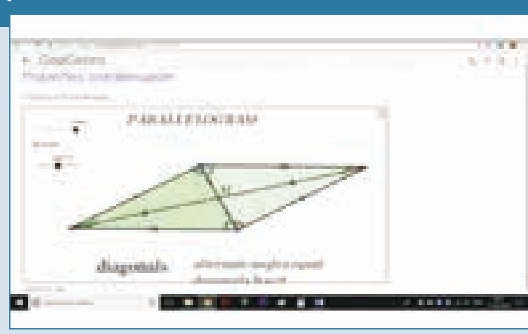
Step 2



Step 3



Step 4



Similarly you can check other worksheets in the Workbook related to your lesson

Browse in the link

Properties: Parallelogram: https://www.geogebra.org/m/m9Q2QpWD

Example 4.4

In a parallelogram $ABCD$, P and Q are the points on line DB such that $PD = BQ$ show that $APCQ$ is a parallelogram

Solution

$ABCD$ is a parallelogram.

$$OA = OC \text{ and}$$

$$OB = OD \;(\because \text{Diagonals bisect each other})$$

now $$OB + BQ = OD + DP$$

$$OQ = OP \text{ and } OA = OC$$

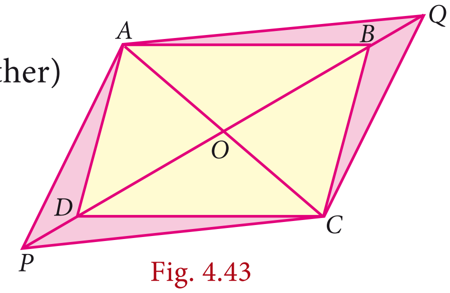


Fig. 4.43

$APCQ$ is a parallelogram.

Exercise 4.3

1. The angles of a quadrilateral are in the ratio 2 : 4 : 5 : 7. Find all the angles.

2. In a quadrilateral $ABCD$, $\angle A = 72°$ and $\angle C$ is the supplementary of $\angle A$. The other two angles are $2x–10$ and $x + 4$. Find the value of x and the measure of all the angles.

3. The side of a rhombus is 13 cm and the length of one of the diagonal is 24 cm. Find the length of the other diagonal?

4. $ABCD$ is a rectangle whose diagonals AC and BD intersect at O. If $\angle OAB = 46°$, find $\angle OBC$

5. The lengths of the diagonals of a Rhombus are 12 cm and 16 cm . Find the side of the rhombus.

6. Show that the bisectors of angles of a parallelogram form a rectangle .

7. If a triangle and a parallelogram lie on the same base and between the same parallels, then prove that the area of the triangle is equal to half of the area of parallelogram.

8. The legs of a stool make angle 35 ° with the floor as shown in the figure. Find the angles x and y.

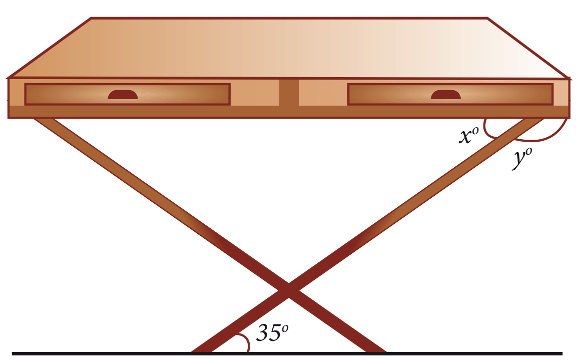


Fig. 4.44

9. Iron rods a, b, c, d, e, and f are making a design in a bridge as shown in the figure. If $a \parallel b$, $c \parallel d$, $e \parallel f$, find the marked angles between

(i) b and c

(ii) d and e

(iii) d and f

(iv) c and f

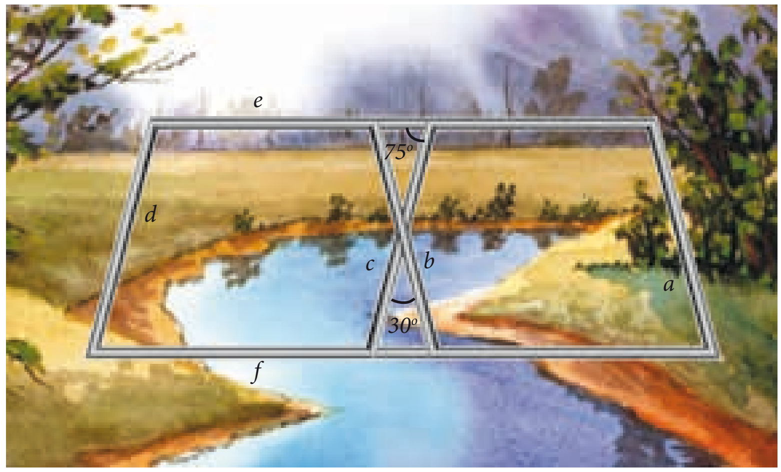


Fig. 4.45

10. In the given Fig. 4.46, $\angle A = 64°$, $\angle ABC = 58°$. If BO and CO are the bisectors of $\angle ABC$ and $\angle ACB$ respectively of ΔABC, find $x°$ and $y°$

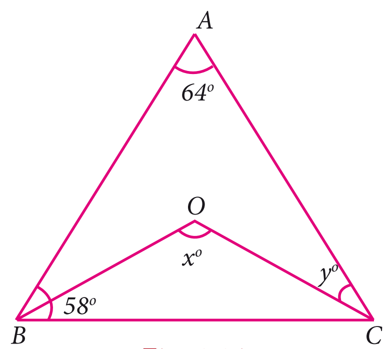


Fig. 4.46

11. Which type of quadrilateral satisfies the following properties?

(i) Both pairs of opposite angles are equal in size.

(ii) Both pairs of opposite sides are equal in length.

(iii) Each diagonal is an angle bisector.

(iv) The diagonals bisect each other.

(v) Each pair of consecutive angles is supplementary.

(vi) The diagonals are equal.

(vii) Can be divided into two congruent triangles.

12. In the given Fig. 4.47, if $AB = 2$, $BC = 6$, $AE = 6$, $BF = 8$, $CE = 7$, and $CF = 7$, compute the ratio of the area of quadrilateral $ABDE$ to the area of ΔCDF.

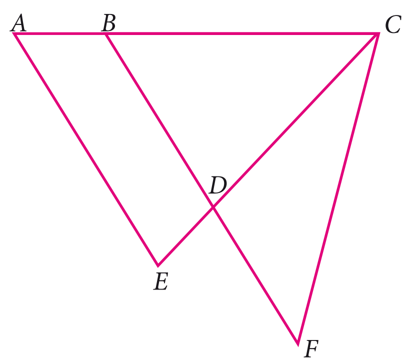


Fig. 4.47

13. In the Fig. 4.48, $ABCD$ is a rectangle and $EFGH$ is a parallelogram. Using the measurements given in the figure, what is the length d of the segment that is perpendicular to $\overline{HE}$ and $\overline{FG}$?

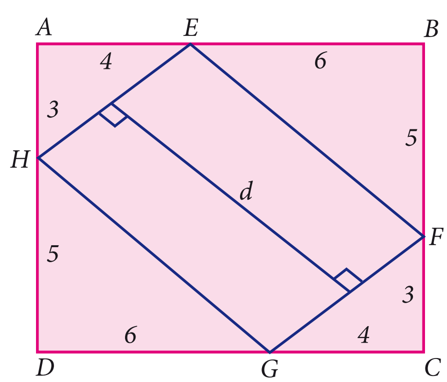


Fig. 4.48

14. $ABCD$ is a parallelogram such that AB is parallel to DC and DA parallel to CB. The length of side AB is 20 cm. E is a point between A and B such that the length of AE is 3 cm. F is a point between points D and C. Find the length of DF such that the segment EF divides the parallelogram in two regions with equal areas.

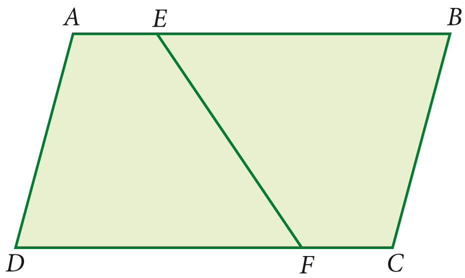


Fig. 4.49

15. In parallelogram $ABCD$ of the accompanying diagram, line DP is drawn bisecting BC at N and meeting AB (extended) at P. From vertex C, line CQ is drawn bisecting side AD at M and meeting AB (extended) at Q. Lines DP and CQ meet at O. Show that the area of triangle QPO is $\frac{9}{8}$ of the area of the parallelogram $ABCD$.

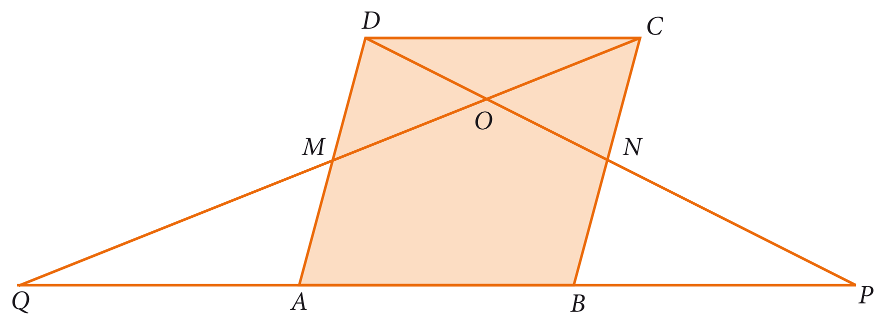


Fig. 4.50

4.4 Constructions

Practical Geometry is the method of applying the rules of Geometry dealt with the properties of Points, Lines and other figures to construct geometrical figures. "Construction" in Geometry means to draw shapes, angles or lines accurately. The geometric constructions have been discussed in detail in Euclid's book 'Elements'. Hence these constructions are also known as Euclidean constructions. These constructions use only compass and straightedge (i.e. ruler). The compass establishes equidistance and the straightedge establishes collinearity. All geometric constructions are based on those two concepts.

It is possible to construct rational and irrational numbers using straightedge and a compass as seen in Chapter II. In 1913 the Indian mathematical Genius, Ramanujan gave a geometrical construction for $355/113 = \pi$. Today with all our accumulated skill in exact measurements. It is a noteworthy feature that lines driven through a mountain meet and make a tunnel.In the earlier classes, we have learnt the construction of angles and triangles with the given measurements.

In this chapter we learn to construct Circumcentre and Orthocentre of a triangle by using concurrent lines.

4.4.1 Construction of the Circumcentre of a Triangle

Circumcentre

The Circumcentre is the point of concurrency of the Perpendicular bisectors of the sides of a triangle.

It is usually denoted by S.

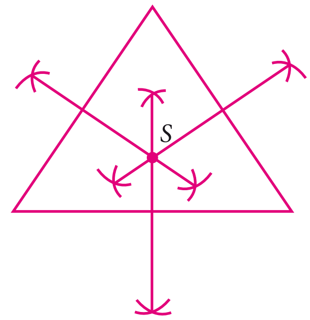


Fig. 4.51

Circumcircle

The circle passing through all the three vertices of the triangle with circumcentre (S) as centre is called circumcircle.

Circumradius

The line segment from any vertex of a triangle to the Circumcentre of a given triangle is called circumradius of the circumcircle.

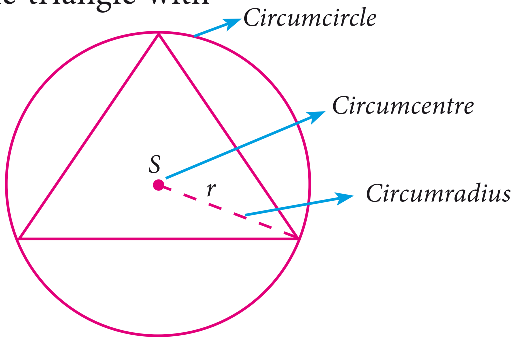


Fig. 4.52

Activity 11

Objective To find the mid-point of a line segment using paper folding

Procedure Make a line segment on a paper by folding it and name it PQ. Fold the line segment PQ in such a way that P falls on Q and mark the point of intersection of the line segment and the crease formed by folding the paper as M. M is the midpoint of PQ.

Activity 12

Objective To construct a perpendicular bisector of a line segment using paper folding.

Procedure Make a line segment on a paper by folding it and name it as PQ. Fold PQ to such a way that P falls on Q and thereby creating a crease RS. This line RS is the perpendicular bisector of PQ.

Activity 13

Objective To construct a perpendicular to a line segment from an external point using paper folding.

Procedure Draw a line segment AB and mark an external point P. Move B along BA till the fold passes through P and crease it along that line. The crease thus formed is the perpendicular to AB through the external point P.

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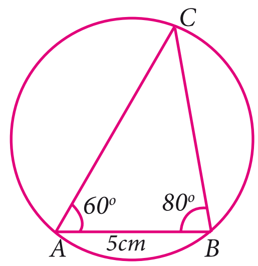
Activity 14

Objective To locate the circumcentre of a triangle using paper folding.

Procedure Using Activity 12, find the perpendicular bisectors for any two sides of the given triangle. The meeting point of these is the circumcentre of the given triangle.

Example 4.5

Construct the circumcentre of the ΔABC with $AB = 5$ cm, $\angle A = 60^0$ and $\angle B = 80^0$. Also draw the circumcircle and find the circumradius of the ΔABC.



Rough Diagram

Solution

Step 1 Draw the ΔABC with the given measurements

Step 2

Construct the perpendicular bisector of any two sides (AC and BC) and let them meet at S which is the circumcentre.

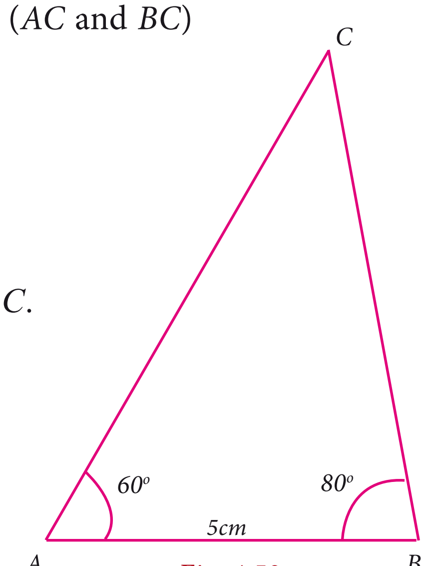


Fig. 4.53

Step 3

S as centre and $SA = SB = SC$ as radius,

draw the Circumcircle to passes through A,B and C.

Circumradius = 3.9 cm.

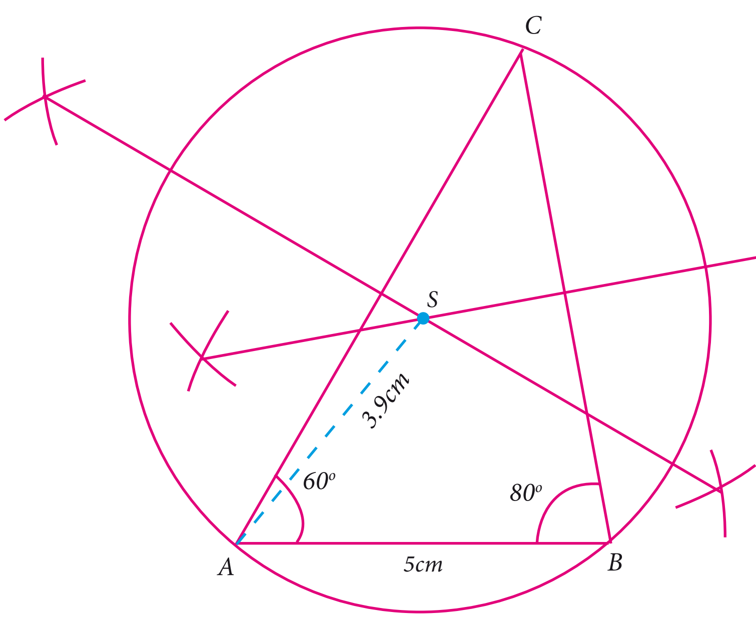


Fig. 4.54

Exercise 4.4

1 Draw the circumcircle for

(i) An equilateral triangle of side 7 cm.

(ii) An isosceles right triangle having 6 cm as the length of the equal sides.

2 Draw a triangle ABC, where $AB = 8$ cm, $BC = 6$ cm and $\angle B = 70^0$ and locate its circumcentre and draw the circumcircle.

3 Construct the right triangle PQR whose perpendicular sides are 4.5 cm and 6 cm. Also locate its circumcentre and draw the circumcircle.

4. Construct ΔABC with $AB = 5$ cm $\angle B = 100^0$ and $BC = 6$ cm. Also locate its circumcentre draw circumcircle.

5. Construct an isosceles triangle PQR where $PQ = PR$ and $\angle Q = 50^0$, $QR = 7$cm. Also draw its circumcircle.

4.4.2 Construction of Orthocentre of a Triangle

Orthocentre

The orthocentre is the point of concurrency of the altitudes of a triangle. Usually it is denoted by H.

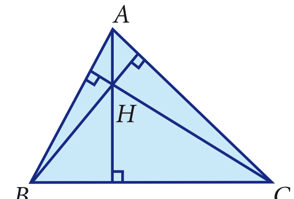


Fig. 4.55

Activity 15

Objective To locate the Orthocentre of a triangle using paper folding.

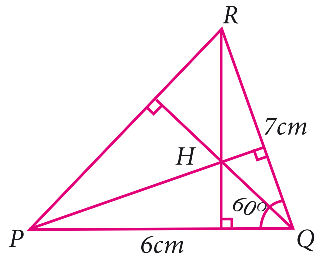
Procedure Using Activity 12 with any two vertices of the triangle as external points, construct the perpendiculars to opposite sides. The point of intersection of the perpendiculars is the Orthocentre of the given triangle.

Example 4.6

Construct ΔPQR whose sides are $PQ = 6$ cm $\angle Q = 60^0$ and $QR = 7$ cm and locate its Orthocentre.

Solution

Step 1 Draw the ΔPQR with the given measurements.



Rough Diagram

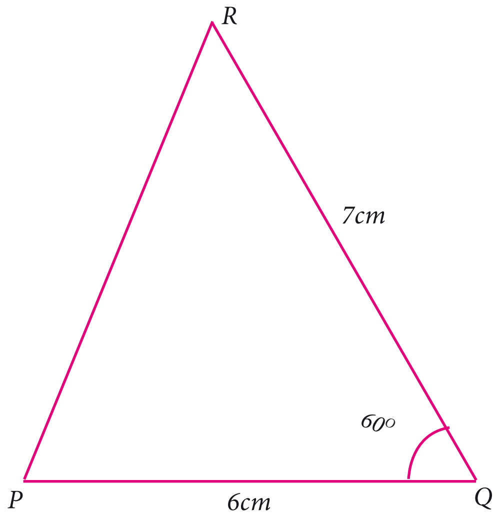


Fig. 4.56

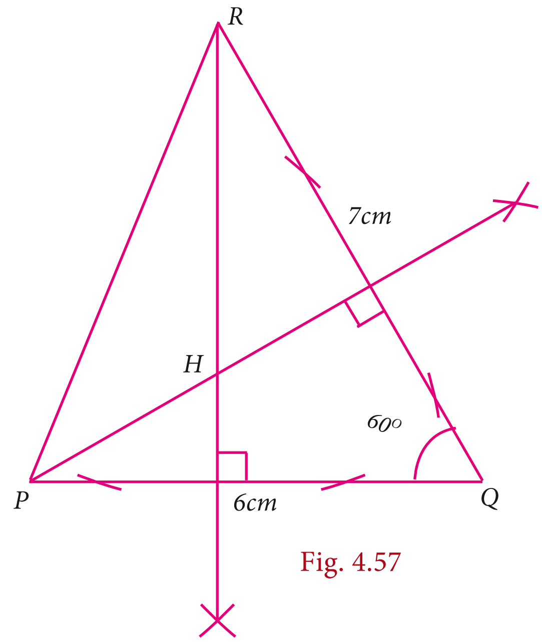


Fig. 4.57

Step 2:

Construct altitudes from any two vertices R and P, to their opposite sides PQ and QR respectively.

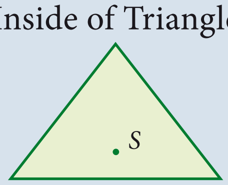
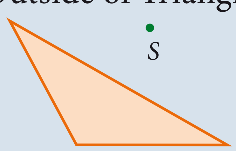
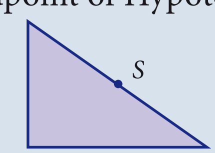
The point of intersection of the altitude H is the Orthocentre of the given ΔPQR.

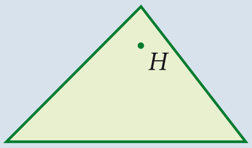
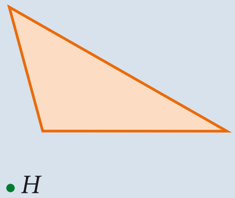
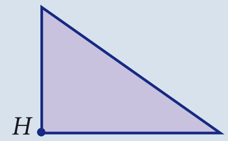
Exercise 4.5

1. Draw ΔPQR with sides $PQ = 7$ cm, $QR = 8$ cm and $PR = 5$ cm and construct its Orthocentre.
2. Draw an equilateral triangle of sides 6.5 cm and locate its Orthocentre.
3. Draw ΔABC, where $AB = 6$ cm, $\angle B = 110^0$ and $BC = 5$ cm and construct its Orthocentre.
4. Draw and locate the Orthocentre of a right triangle PQR where $PQ = 4.5$ cm, $QR = 6$ cm and $PR = 7.5$ cm.
5. Construct an isosceles triangle ABC with $AB = BC$ of sides 7 cm and $\angle B = 70^0$ and locate its Orthocentre.

Note

Where do the Circumcentre and Orthocentre lie in the given triangles.

	Acute Triangle	Obtuse Triangle	Right Triangle
Circumcentre	Inside of Triangle 	Outside of Triangle 	Midpoint of Hypotenuse 

	Inside of Triangle	Outside of Triangle	Vertex at Right Angle
Orthocentre			

Exercise 4.6

Multiple Choice Questions

1. It is not possible to construct a triangle when its sides are

   (a) 8.2 cm, 3.5 cm, 6.5 cm (b) 6.3 cm, 3.1 cm, 3.2 cm

   (c) 7 cm, 8 cm, 10 cm (d) 4 cm, 6 cm, 6 cm

2. The exterior angle of a triangle is equal to the sum of two

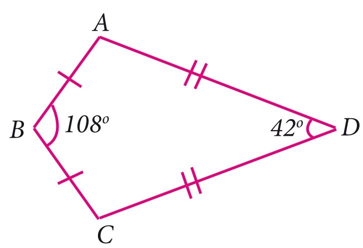
   (a) Exterior angles (b) Interior opposite angles

   (c) Alternate angles (d) Interior angles

3. In the quadrilateral $ABCD$, $AB = BC$ and $AD = DC$ Measure of $\angle BCD$ is

   (a) 150° (b) 30°

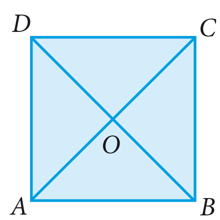
   (c) 105° (d) 72°



4. $ABCD$ is a square, diagonals AC and BD meet at O. The number of pairs of congruent triangles are

   (a) 6 (b) 8

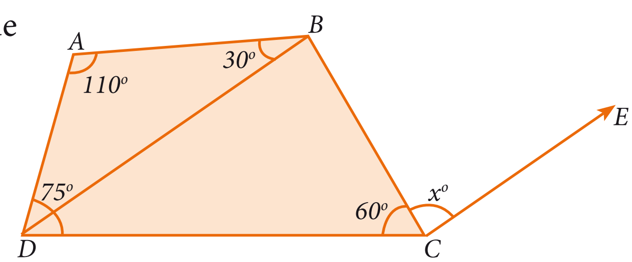
   (c) 4 (d) 12



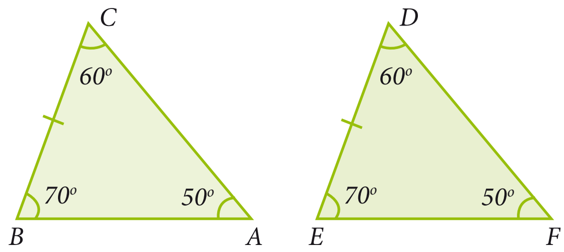
5. In the given figure $CE \parallel DB$ then the value of x^o is

   (a) 45° (b) 30°

   (c) 75° (d) 85°



6. The correct statement out of the following is



(a) $\Delta ABC \cong \Delta DEF$ (b) $\Delta ABC \cong \Delta DEF$

(c) $\Delta ABC \cong \Delta FDE$ (d) $\Delta ABC \cong \Delta FED$

7. If the diagonal of a rhombus are equal, then the rhombus is a

(a) Parallelogram but not a rectangle

(b) Rectangle but not a square

(c) Square

(d) Parallelogram but not a square

8. If bisectors of $\angle A$ and $\angle B$ of a quadrilateral $ABCD$ meet at O, then $\angle AOB$ is

(a) $\angle C + \angle D$ (b) $\frac{1}{2}(\angle C + \angle D)$

(c) $\frac{1}{2}\angle C + \frac{1}{3}\angle D$ (d) $\frac{1}{3}\angle C + \frac{1}{2}\angle D$

9. The interior angle made by the side in a parallelogram is 90° then the parallelogram is a

(a) rhombus (b) rectangle

(c) trapezium (d) kite

10. Which of the following statement is correct?

(a) Opposite angles of a parallelogram are not equal.

(b) Adjacent angles of a parallelogram are complementary.

(c) Diagonals of a parallelogram are always equal.

(d) Both pairs of opposite sides of a parallelogram are always equal.

11. The angles of the triangle are $3x-40$, $x+20$ and $2x-10$ then the value of x is

(a) 40 (b) 35 (c) 50 (d) 45

Points to remember

- In a parallelogram the opposite sides are equal.
- In a parallelogram the opposite angles are equal.
- The diagonals of a parallelogram bisect each other.

- The diagonals of a parallelogram divides it into two congruent triangles
- A quadrilateral is a parallelogram if its opposite sides are equal .
- Parallelogram on the same base and between same parallel are equal in area.
- Triangles on the same base and between same parallel are equal in area.
- Parallelogram is a rhombus if its diagonals are perpendicular.
- A diagonal of a parallelogram divides it into two triangles of equal area.
- The circumcentre is the point of concurrency of the Perpendicular bisectors of the sides of a triangle.
- The orthocentre is the point of concurrency of the altitudes of a triangle.

Answers

Exercise 4.1

1. (i) $20°$ (ii) $63°$ (iii) $45°$ (iv) $27°28'$

2. (i) $40°$ (ii) $146°$ (iii) $90°$ (iv) $58°12'$

3. (i) $18°$ (ii) $140°$ (iii) $75°$ (iv) $20°$ (v) $18°$

4. $\angle 1 = 110°$, $\angle 2 = 70°$, $\angle 3 = 110°$, $\angle 4 = 70°$, $\angle 5 = 110°$, $\angle 6 = 70°$, $\angle 7 = 110°$ $\angle 8 = 70°$

5. (i) $70°$ (ii) $288°$ (iii) $89°$ 6. $30°, 60°, 90°$ 9. $80°, 85°, 15°$

Exercise 4.2

2. (i), (v) and (vi) are parallelograms 3. (iii), (vi) and (viii) are not quadrilaterals

4. (i), (v), and (vi) are trapeziums, (ii), (iii) and (iv) are not trapeziums

Exercise 4.3

1. (i) $40°, 80°, 100°, 140°$ 2. $62°, 114°, 66°$ 3. 10cm

4. $44°$ 5. 10cm

8. $35°, 145°$ 9. (i) $30°$ (ii) $105°$ (iii) $75°$ (iv) $105°$ 10. $122°, 29°$

11. The quadrilateral which satisfies all the properties is a square 12. Ratios are equal

13. $d = \sqrt{61}$ 14. DF = 17 cm

Exercise 4.6

1. (b) 2. (b) 3. (c) 4. (a) 5. (d) 6. (d) 7. (c) 8. (b) 9. (b) 10. (d) 11. (b)

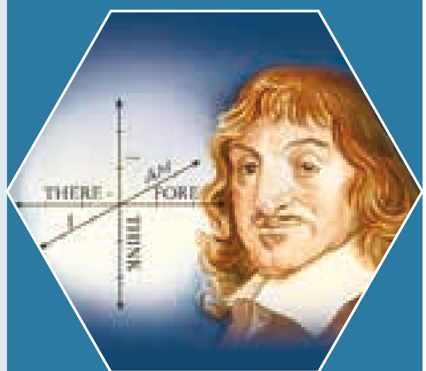
5 COORDINATE GEOMETRY

E6HCCN

Geometry is that part of universal mechanics which accurately proposes and demonstrates the art of measuring.

- " Sir Isacc Newton"

The French Mathematician Rene Descartes (pronounced "Day- CART") developed a new branch of Mathematics known as Analytical Geometry or Coordinate Geometry which combined all arithmetic, algebra and geometry of the past ages in a single technique of visualising as points on a graph and equations as geometrical shapes. The fixing of a point position in the plane by assigning two numbers, coordinates, giving its distance from two lines perpendicular to each other, was entirely Descartes' invention.



Rene Descartes
1596-1650

Learning Outcomes

- To understand the Cartesian coordinate system.
- To identify the abscissa, ordinate and coordinates of any given point.
- To find the distance between any two points in the Cartesian plane using formula.

5.1 Mapping the Plane

How do you write your address ? Here is one.

Sarakkalvilai Primary School
135, Sarakkalvilai,
Sarakkalvilai Housing Board Road,
Keezha Sarakkalvilai,
Nagercoil 629002, Kanyakumari Dist.
Tamil Nadu, India.

Fig. 5.1

Somehow, this information is enough for anyone in the world from anywhere to locate the school one studied. Just consider there are crores and crores of buildings on the Earth. But yet, we can use an address system to locate a particular person's studied place, however interior it is.

How is this possible? Let us work out the procedure of locating a particular address. We know the World is divided into countries. One among them is India. Subsequently India is divided into States. Among these States we can locate our State Tamilnadu.

Further going deeper, we find our State is divided into Districts. Districts into taluks, taluks into villages proceeding further in this way, one could easily locate "Sarakkalvilai" among the villages in that Taluk. Further among the roads in that village, 'Housing Board Road' is the specific road which we are interested to explore. Finally we end up the search by the locating Government Primary School building bearing the door number 135 to enable us precisely among the buildings in that road.

In New York city of USA, there is an area called Manhattan. The map shows Avenues run in the North – South direction and the Streets run in the East – West direction. So, if you know that the place you are looking for is on 57th street between 9th and 10th Avenues, you can find it immediately on the map. Similarly it is easy to find a place on 2nd Avenue between 34th and 35th streets. In fact, New Yorkers make it even simpler. From the door number on a street, you can actually calculate which avenues it lies between, and from the door number on an avenue, you can calculate which streets it stands.

Fig. 5.2

All maps do just this for us. They help us in finding our way and locate a place easily by using information of any landmark which is nearer to our search to make us understand whether we are near or far, how far are we, or what is in between etc. We use latitudes (east – west, like streets in Manhattan) and longitudes (north – south, like avenues in Manhattan) to pin point places on Earth. It is interesting to see how using numbers in maps helps us so much.

This idea, of using numbers to map places, comes from geometry. Mathematicians wanted to build maps of planes, solids and shapes of all kinds. Why would they want such maps? When we work with a geometric figure, we want to observe wheather a point lies inside the region or outside or on the boundary. Given two points on the boundary and a point outside, we would like to examine which of the two points on the boundary is closer to the one outside, and how much closer and so on. With solids like cubes, you can imagine how interesting and complicated such questions can be.

Mathematicians asked such questions and answered them only to develop their own understanding of circles, polygons and spheres. But the mathematical tools and techniques were used to find immense applications in the day to day life. Mapping the world using latitudes and longitudes would not have been developed at all in 18th century, if the co-ordinate system had not been developed mathematically in the 17th century.

You already know the map of the real number system is the number line. It extends infinitely on both directions. In between any two points, on a number line, there lies infinite number of points. We are now going to build a map of the plane so that we can discuss about the points on the plane, of the distance between the points etc. We can then draw on the plane all the geometrical shapes we have discussed so far, precisely.

Arithmetic introduced us to the world of numbers and operations on them. Algebra taught us how to work with unknown values and find them using equations. Geometry taught us to describe shapes by their properties. Co-ordinate geometry will teach us how to use numbers and algebraic equations for studying geometry and beautiful integration of many techniques in one place. In a way, that is also great fun as an activity. Can't wait ? Let us plunge in.

5.2 Devising a Coordinate System

You ask your friend to draw a rectangle on a blank sheet of paper, 5 cm by 3 cm. He says, "Sure, but where on this sheet ?" How would you answer him ?

Now look at the picture. How will you describe it to another person?

Fig. 5.3

Let us analyse the given picture (Fig. 5.3). Just like that a particular house is to be pointed out, it is going to be a difficult task. Instead, if any place or an object is fixed for identification then it is easy to identify any other place or object relative to it. For example, you fix the flag and talk about the house to the left of it, the hotel below it, the antenna on the house to right of it etc.

As shown Fig. 5.4 you draw two perpendicular lines in such a way that the flag is pointed out, near the intersection point. Now if you tell your friend the total length and width of the picture frame, keeping the flagpole as landmark, you can also say 2 cm to the right, 3 cm above etc. Since you know directions, you can also say 2 cm east, 3 cm north.

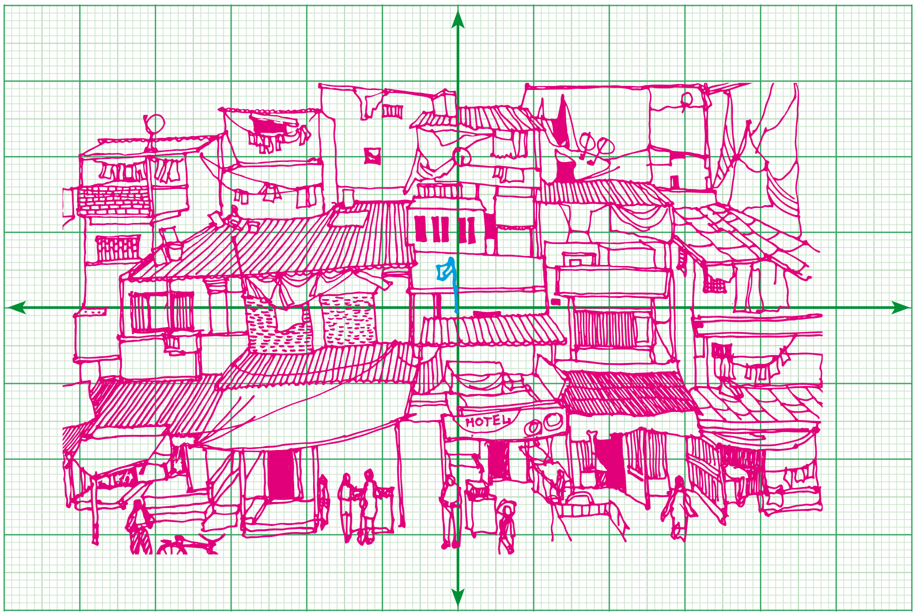


Fig. 5.4

This is what we are going to do. A number line is usually represented as horizontal line on which the positive numbers always lie on the right side of zero, negative on the left side of zero. Now consider another copy of the number line, but drawn vertically: the positive integers are represented above zero and the negative integers are below zero (fig 5.5).

Where do the two number lines meet ? Obviously at zero for both lines. That will be our "flag", the fixed location. We can talk of other numbers relative to it, on both the lines. But now you see that we talk not only of numbers on the two number lines but lots more !

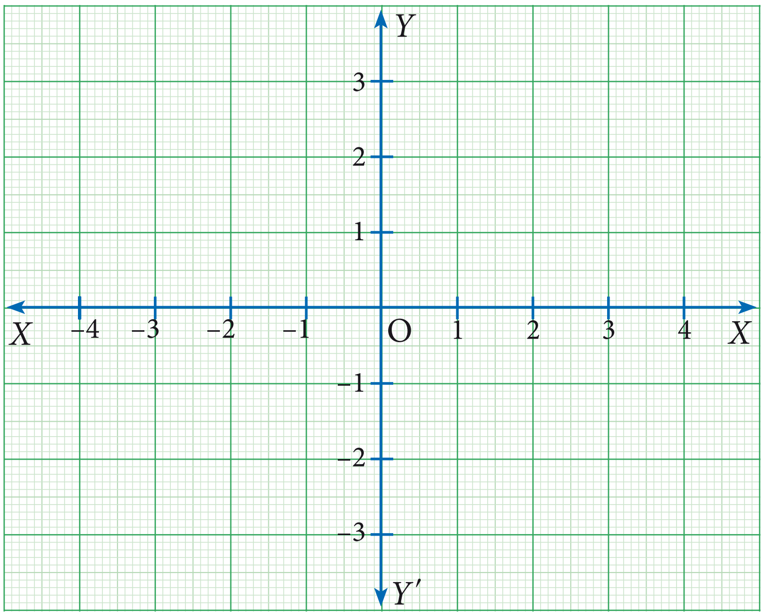


Fig. 5.5

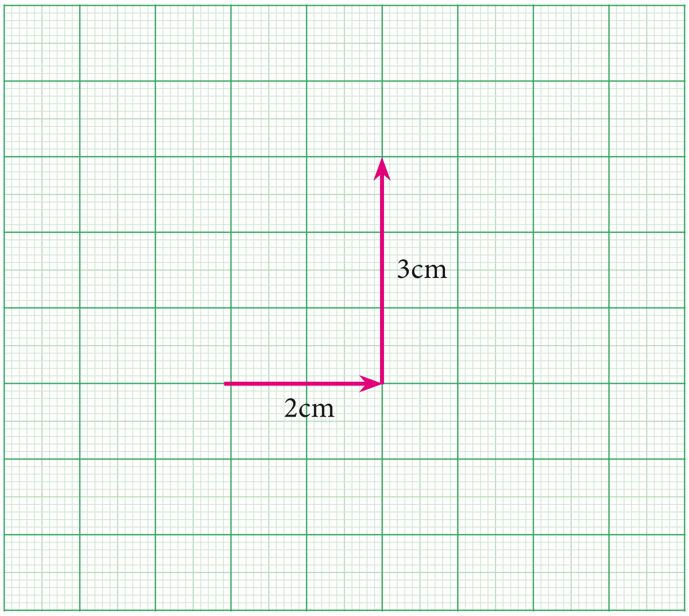


Fig. 5.6

Suppose we go 2 to the right and then 3 to the top. We would call this place ($\rightarrow$ 2, $\uparrow$ 3).

All this vertical, horizontal, up, down etc is all very cumbersome. We simply say (2,3) and understand this as 2 to the right and then 3 up. Notice that we would reach the same place if we first went 3 up and then 2 right, so for us, the instruction (2,3) is not the same as the instruction (3,2). What about (–2,3) ? It would mean 2 left and 3 up. From where ? Always from (0,0). What about the instruction (2,–3)? It would mean 2 right and 3 down. We need names for horizontal and vertical number lines too. We call the horizontal number line the x-axis and the vertical number line the y-axis. To the right we mark it as X, to the left as X', to the top as Y, to the bottom as Y'.

The x-co-ordinate is called the *abscissa* and the y-co-ordinate is called the *ordinate*. We call the meeting point of the axes (0,0) the origin.

Now we can describe any point on a sheet of paper by a pair (x,y). However, what do (1, 2) etc mean on our paper ? We need to choose some convenient unit and represent these numbers. For instance we can choose 1 unit to be 1 cm. Thus (2,3) is the instruction to move 2 cm to the right of (0,0) and then to move 3 cm up. Please remember that the choice of units is arbitrary: if we fix 1 unit to be 2 cm, our figures will be larger, but the relative distances will remain the same.

Note

Whether we place (0,0) at the centre of the sheet, or somewhere else does not matter, (0,0) is always the origin for us, and all "instructions" are relative to that point. We usually denote the origin by the letter 'O'.

In fact, we now have a language to describe all the infinitely many points on the plane, not just our sheet of paper !

Since the x-axis and the y-axis divide the plane into four regions, we call them quadrants. (Remember, quadrilateral has 4 sides, quadrants are 4 regions.) They are usually numbered as I, II, III and IV, with I for upper east side, II for upper west side, III for lower west side and IV for lower east side, thus making an anti-clockwise tour of them all.

Region	Quadrant	Nature of x,y	Signs of the coordinates
XOY	I	$x>0, y>0$	(+,+)
$X'OY$	II	$x<0, y>0$	(–,+)
$X'OY'$	III	$x<0, y<0$	(–,–)
XOY'	IV	$x>0, y<0$	(+,–)

Note

Why this way (anti-clockwise) and not clockwise, or not starting from any of the other quadrants ? It does not matter at all, but it is good to follow some convention, and this is what we have been doing for a few centuries now.

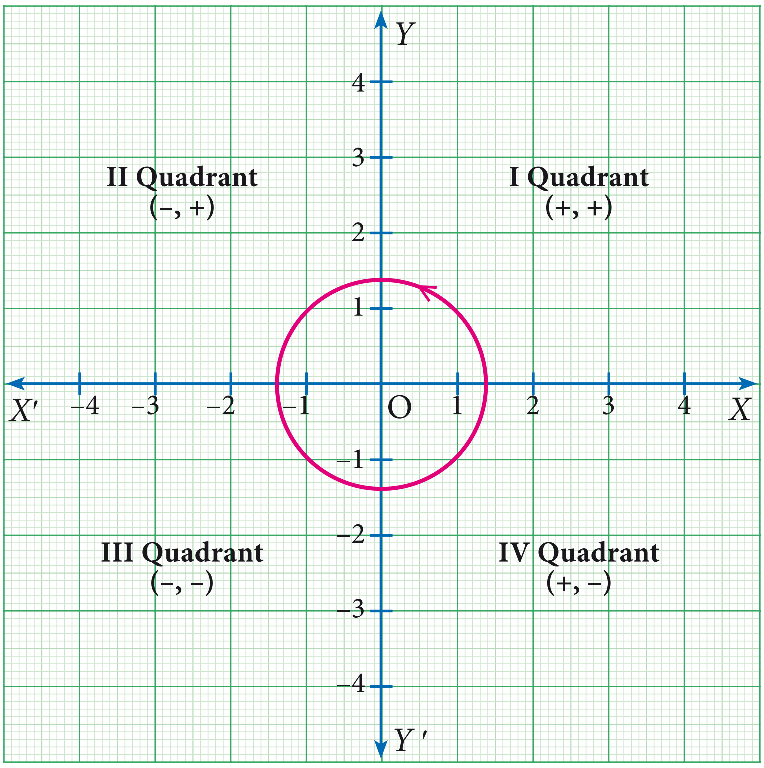


Fig. 5.7

Note

1. *For any point P on the x axis,the value of y coordinate (ordinate) is zero ie. P (x,0).*
2. *For any point Q on the y axis, the value of x coordinate (abscissa) is zero. ie. Q (0, y)*
3. *(x, y) ≠ (y, x) unless x= y*
4. *A plane with the rectangular coordinate system is called the Cartesian plane.*

5.2.1 Plotting Points in Cartesian Coordinate Plane

To plot the points (4, 5) in the Cartesian coordinate plane.

We follow the x – axis until we reach 4 and draw a vertical line at $x = 4$.

Similarly, we follow the y – axis until we reach 5 and draw a horizontal line at $y = 5$.

The intersection of these two lines is the position of (4, 5) in the Cartesian plane.

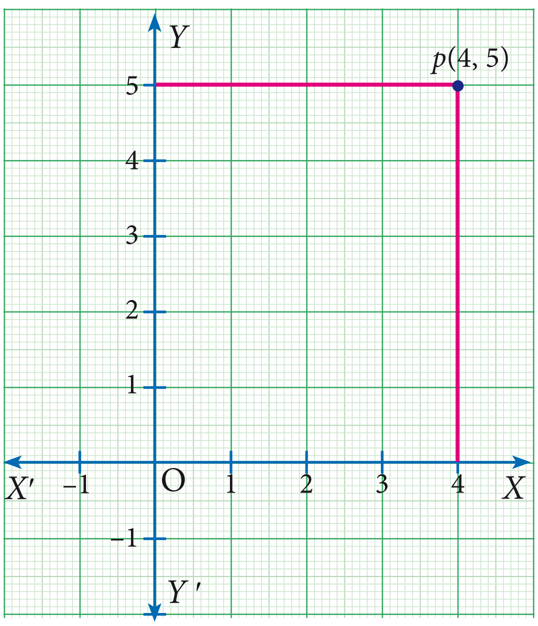


Fig. 5.8

This point is at a distance of 4 units from the y-axis and 5 units from the x-axis. Thus the position of (4, 5) is located in the Cartesian plane.

Activity 1

Observe the given graph and complete the following table:

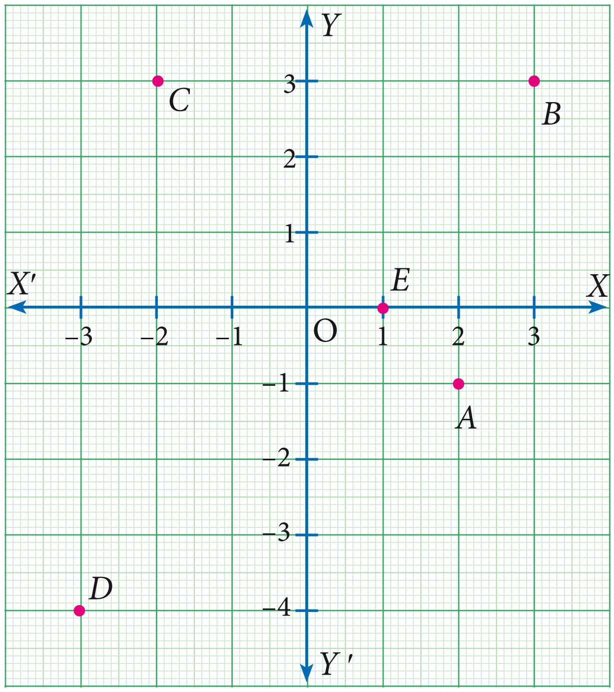


Fig. 5.9

Points	Quadrant	Ordered Pair
A	IV	
B		(3,3)
C		(-2,3)
D	III	

Activity 2

1. Join the points taken in order and name the figure obtained. Name at least any one point which is lying : (i) On x- axis (ii) On y- axis (iii) In each of the Quadrants.

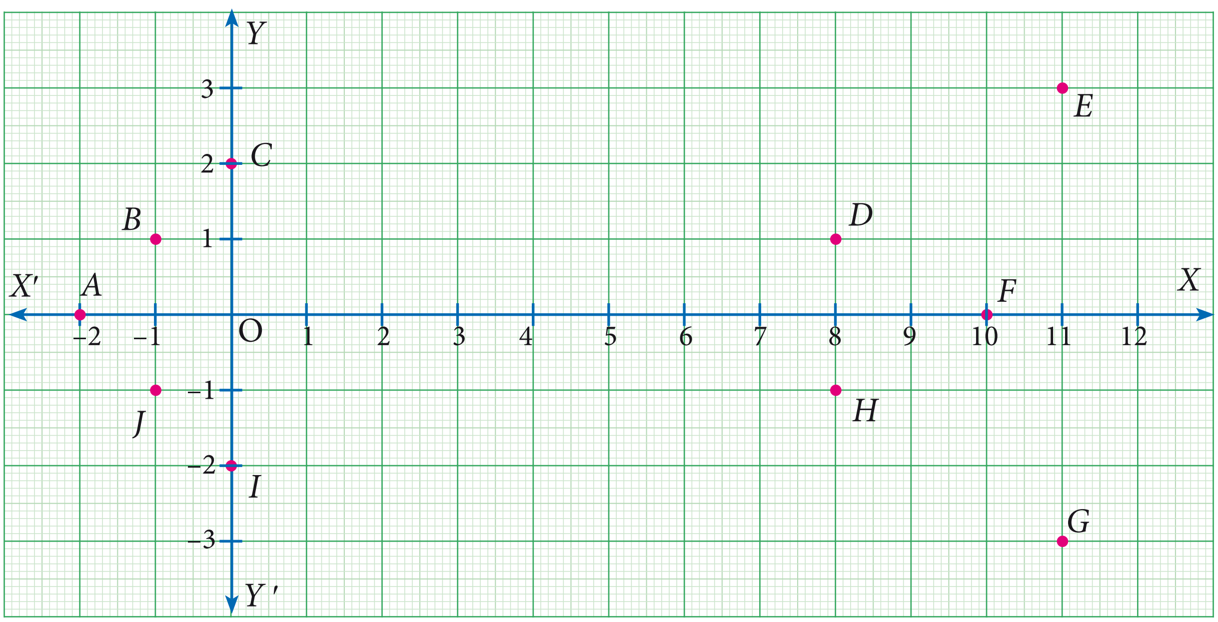


Fig. 5.10

Example 5.1

In which quadrant does the following points lie?

(a) (3,–8) (b) (–1,–3) (c) (2, 5) (d) (–7, 3)

Solution

(a) The x- coordinate is positive and y – coordinate is negative. So, Point(3,–8) lies in the IV quadrant.

(b) The x-coordinate is negative and y – coordinate is negative. So, Point(–1,–3) lies in the III quadrant.

(c) The x-coordinate is positive and y – coordinate is positive. So Point(2,5) lies in the I quadrant.

(d) The x-coordinate is negative and y – coordinate is positive. So, Point(–7,3) lies in the II quadrant

Example 5.2

Plot the points

$A(2,4)$, $B(-3,5)$, $C(-4,-5)$, $D(4,-2)$

in the Cartesian plane.

Solution

(i) To plot (2, 4), draw a vertical line at $x = 2$ and draw a horizontal line at $y = 4$. The intersection of these two lines is the position of (2, 4) in the Cartesian plane. Thus, the Point A (2, 4) is located in the I quadrant of Cartesian plane.

(ii) To plot (–3, 5), draw a vertical line at $x = -3$ and draw a horizontal line at $y = 5$. The intersection of these two lines

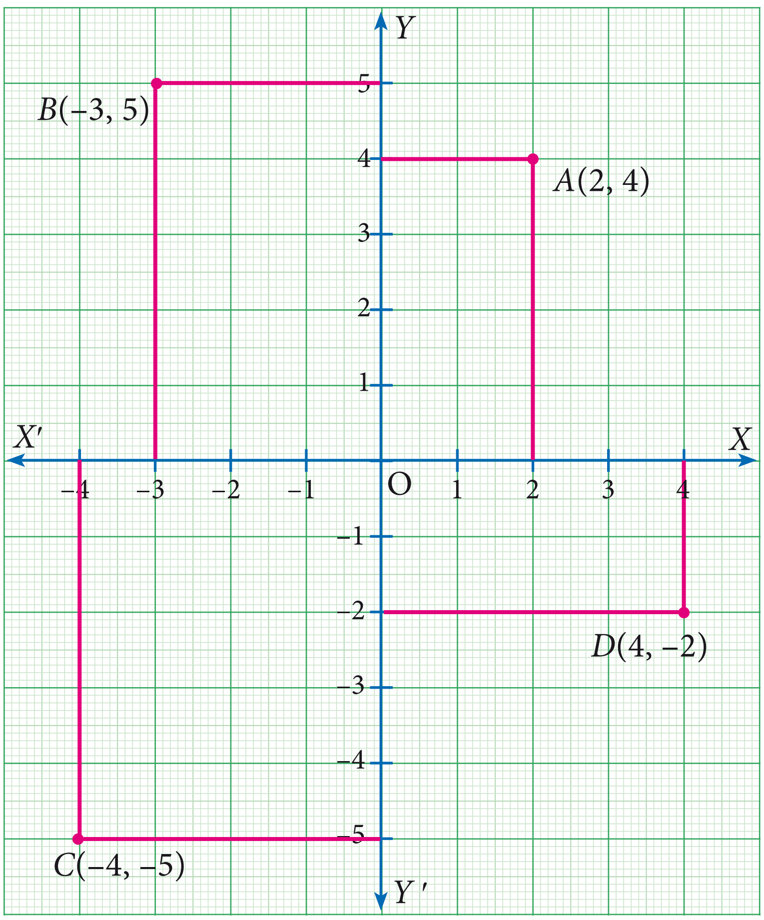


Fig. 5.11

is the position of (–3, 5) in the Cartesian plane. Thus, the Point B (–3, 5) is located in the II quadrant of Cartesian plane.

(iii) To plot (–4, –5), draw a vertical line at x =–4 and draw a horizontal line at y = –5. The intersection of these two lines is the position of (-4, 5) in the Cartesian plane. Thus, the Point C (–4, –5) is located in the III quadrant of Cartesian plane.

(iv) To plot (4, –2), draw a vertical line at x = 4 and draw a horizontal line at y = –2. The Intersection of these two lines is the position of (4,–2) in the Cartesian plane. Thus, the Point D (4,–2) is located in the IV quadrant of Cartesian plane.

Example 5.3

Locate the points

(i) (2,–5) and (–5, 2) (ii) (–3, 4) and (4,–3) in the rectangular coordinate system.

Solution

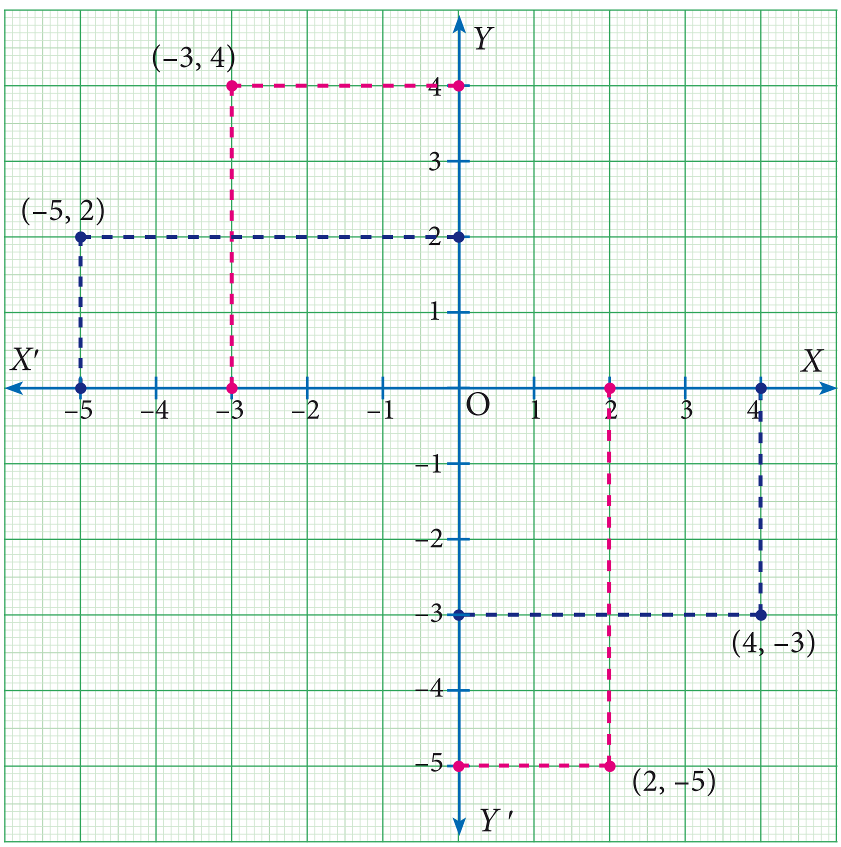


Fig. 5.12

Note

Observe that if we interchange the x and y coordinates of a point, then it will represent a different point in the Cartesian plane. Think! When it will be the same?

Example 5.4

Plot the following points (2, 0), (–5, 0), (3, 0), (–1, 0) in the Cartesian plane. Where do they lie?

Solution

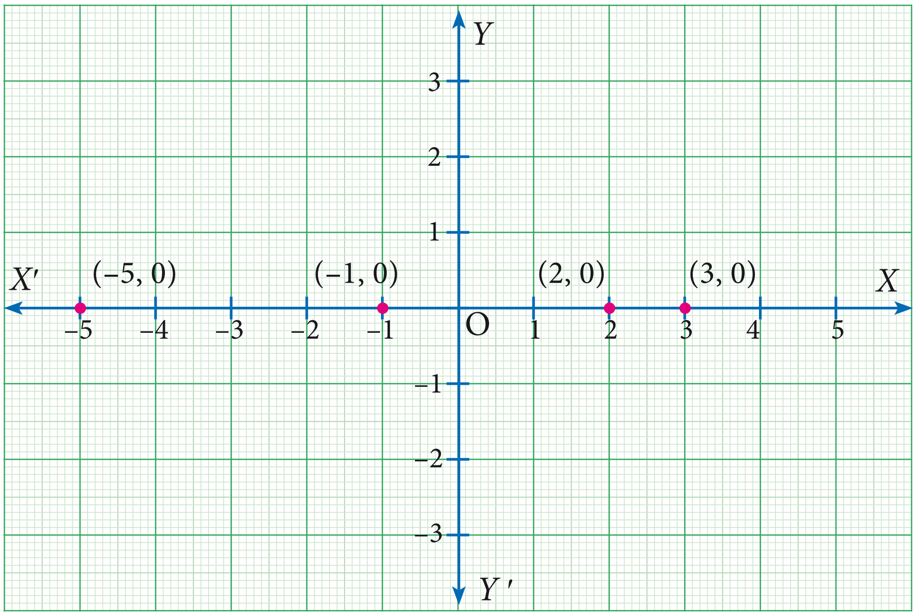


Fig. 5.13

All points lie on x-axis.

Example 5.5

Plot the following points (0,–3),(0,4),(0,–1),(0,5) in the Cartesian plane. Where do they lie?

Solution

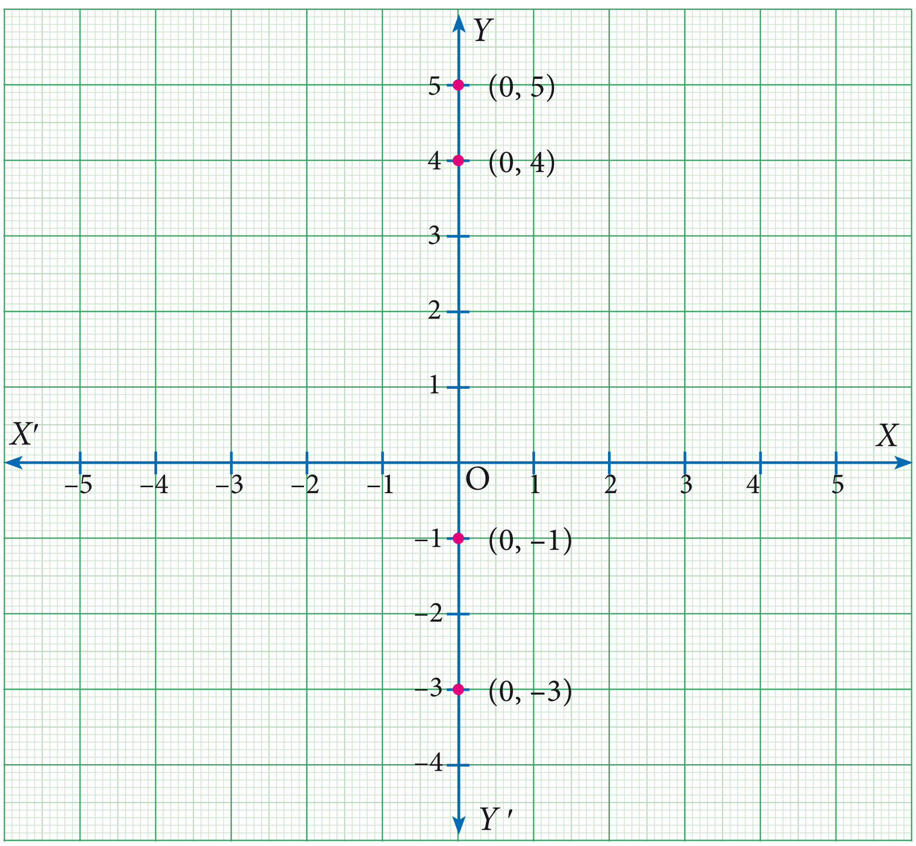


Fig. 5.14

All points lie on y-axis.

Example 5.6

Plot the points (–4,3),(–3,3),(–1,3),(0,3),(3,3) in the Cartesian Plane. What can you say about the position of these points?

Solution

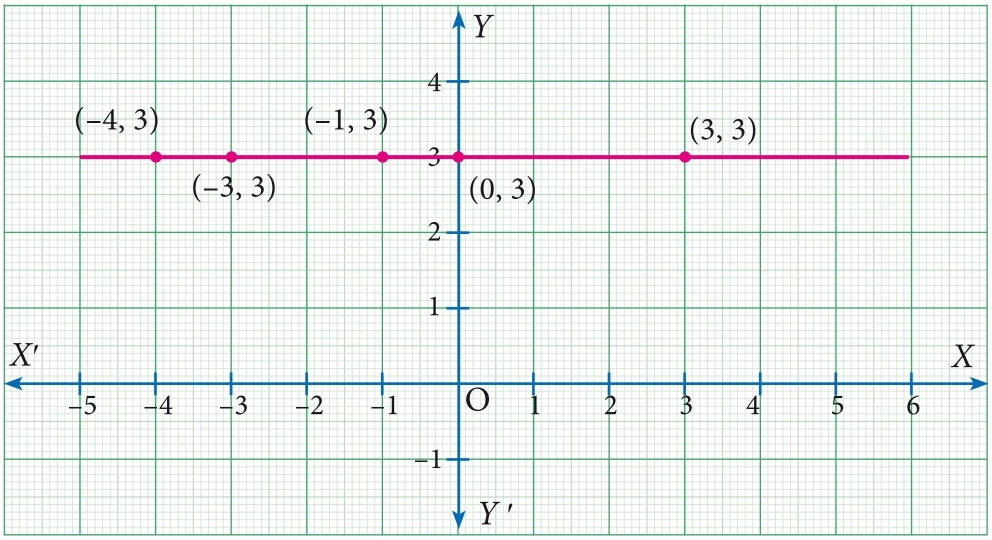


Fig. 5.15

When you join these points, you see that they lie on a line which is parallel to x-axis.

> **Note**
>
> *For points on a line parallel to x-axis, the y- coordinates are equal.*

Example 5.7

Plot the following points $A(2,2)$, $B(-2,2)$, $C(-2,-1)$, $D(2,-1)$ in the Cartesian plane . Discuss the type of the diagram by joining all the points taken in order.

Solution

Point	*A*	*B*	*C*	*D*
Quadrant	I	II	III	IV

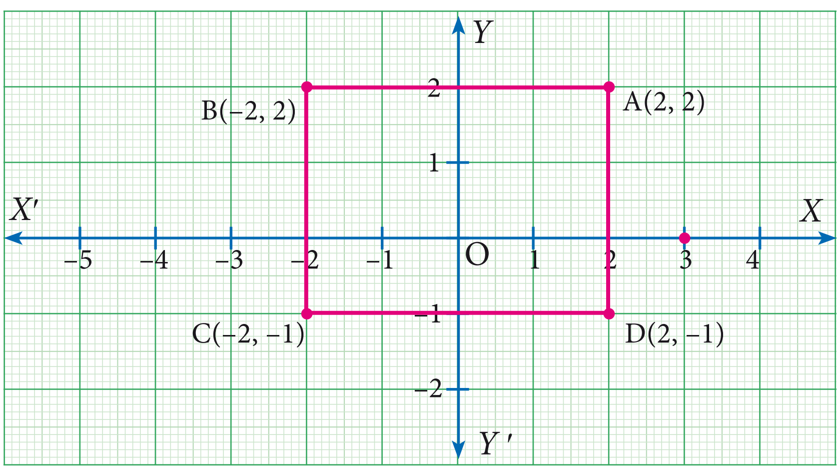


Fig. 5.16

ABCD is a rectangle.

Can you find the length, breath and area of the rectangle?

Exercise 5.1

1. Plot the following points in the coordinate system and identify the quadrants $P(-7,6)$, $Q(7,-2)$, $R(-6,-7)$, $S(3,5)$ and $T(3,9)$

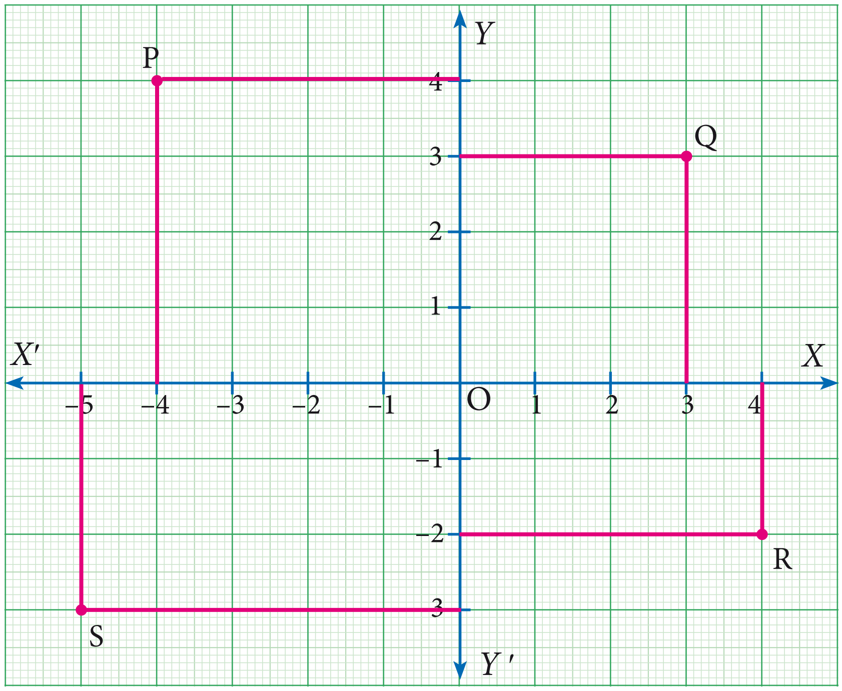


Fig. 5.17

2. Write down the abscissa and ordinate of the following.

   (i) P (ii) Q

   (iii) R (iv) S

3. Plot the following points in the coordinate plane and join them. What is your conclusion about the resulting figure?

   (i) (–5,3) (–1,3) (0,3) (5,3) (ii) (0,–4) (0,–2) (0,4) (0,5)

4. Plot the following points in the coordinate plane. Join them in order. What type of geometrical shape is formed?

   (i) (0,0) (–4,0) (–4,–4) (0,–4) (ii) (–3,3) (2,3) (–6,–1) (5,–1)

Activity 3

Plot the following points on a graph sheet by taking the scale as 1cm = 1 unit.

Find how far the points are from each other?

A (1, 0) and D (4, 0). Find AD and also DA.

Is $AD = DA$?

You plot another set of points and verify your Result.

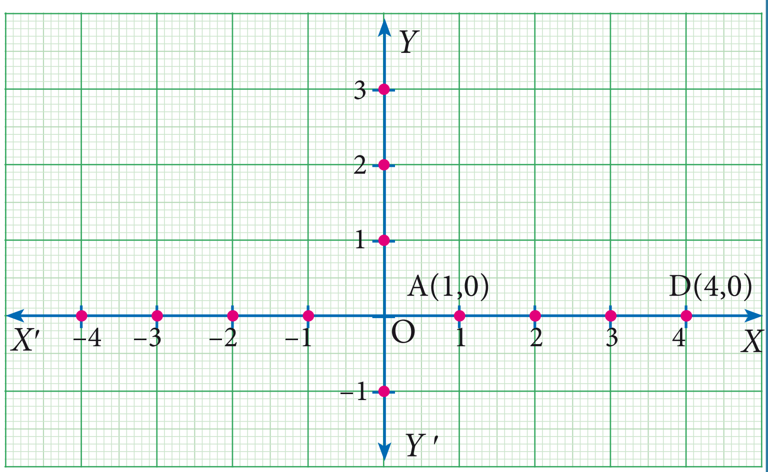


Fig. 5.18

5.3 Distance between any Two Points

Akila and Shanmugam are friends living on the same street in Sathyamangalam. Shanmugam's house is at the intersection of one street with another street on which there is a library. They both study in the same school, and that is not far from Shanmugam's house. Try to draw a picture of their houses, library and school by yourself before looking at the map below.

Consider the school as the origin. (We can do this ! That is the whole point about the co-ordinate language we are using.)

Now fix the scale as 1 unit = 50 metres. Here are some questions for you to answer by studying the given figure (Fig 5.19).

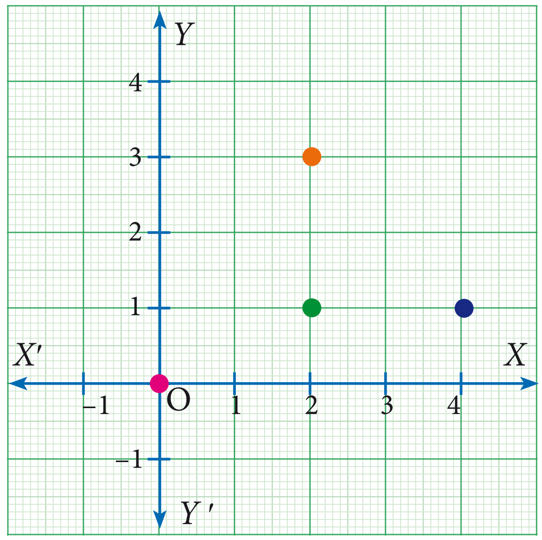


Fig. 5.19

1. How far is Akila's house from Shanmugam's house ?
2. How far is the library from Shanmugam's house?
3. How far is the school from Shanmugam's and Akila's house ?
4. How far is the library from Akila's house ?
5. How far is Shanmugam's house from Akila's house ?

Question 5 is not needed after answering question 1. Obviously, the distance from point *A* to *B* is the same as the distance from point *B* to *A*, and we usually call it the distance between points *A* and *B*. But as mathematicians we are supposed to note down properties as and when we see them, so it is better to note this too: distance (A,B) = distance (B,A). This is true for all points *A* and *B* on the plane, so of course question 5 is same as question 1.

Note

The equation distance(A,B)=distance(B,A) is not always obvious. Suppose that the road from A to B is a one-way street on which you cannot go the other way? Then the distance from B to A might be longer ! But we will avoid all these complications and assume that we can go both ways.

What about the other questions ? They are not the same. Since we know that the two houses are on the same street which is running north – south, the y-distance tells us the answer to question 1. Similarly, we know that the library and Shanmugam's house are on the same street running east – west, we can take the x-distance to answer question 2.

Questions 3 and 4 depend on what kind of routes are available. If we assume that the only streets available are parallel to the x and y axes at the points marked 1, 2, 3 etc then we answer these questions by adding the x and y distances. But consider the large field east of Akila's house.

If she can walk across the field, of course she would prefer it. Now there are many ways of going from one place to another, so when we talk of the distance between them, it is not precise. We need some way to fix what we mean. When there are many routes between A and B, we will use distance(A,B) to denote the distance on the shortest route between A and B.

Once we think of distance(A,B) as the "straight line distance" between A and B, there is an elegant way of understanding it for any points A and B on the plane. This is the important reason for using the co-ordinate system at all ! Before that, 2 more questions from our example.

1. With the school as origin, define the coordinates of the two houses, the school and the library.
2. Use the coordinates to give the distance between any one of these and another.

The "straight line distance" is usually called "as the crow flies". This is to mean that we don't worry about any obstacles and routes on the ground, but how we would get from A to B if we could fly. No bird ever flies on straight lines, though.

We can give a systematic answer to this: given any two points $A = (x,y)$ and $B = (x',y')$ on the plane, find distance(A,B). It is easy to derive a formula in terms of the four numbers x, y, x' and y'. This is what we set out to do now

5.3.1 Distance between Two Points on the Coordinate Axes

Points on x – axis: If two points lie on the x- axis, then the distance between them is equal to the difference between the x- coordinates.

Consider two points $A\ (x_1,0)$ and $B\ (x_2,0)$ on the x-axis .

The distance of B from A is

$$AB = OB - OA = x_2 - x_1 \text{ if } x_2 > x_1 \text{ or}$$
$$= x_1 - x_2 \text{ if } x_1 > x_2$$
$$AB = |x_2 - x_1|$$

(Read as modulus or absolute value of $x_2 - x_1$)

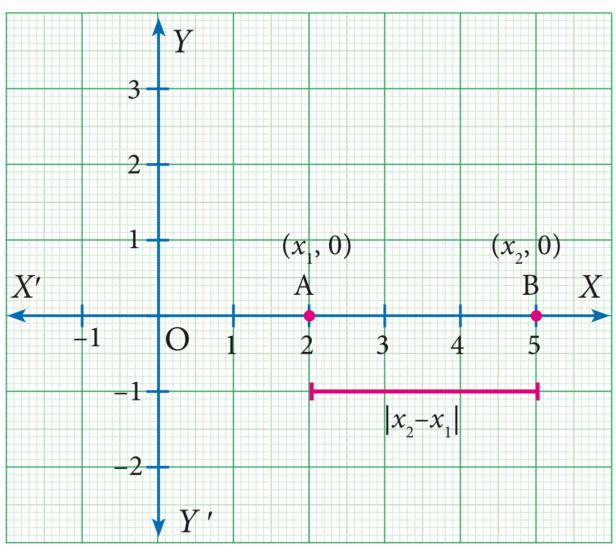


Fig. 5.20

Activity 4

(i) Where do the following points lie?

$P(-2, 0)$, $Q(2,0)$, and $R(3,0)$.

(ii) Find the distance between the following points using coordinates given in question (i)

(a) P and R (b) Q and R.

Points on x – axis: If two points lie on y-axis then the distance between them is equal to the difference between the y-coordinates.

Consider two points $P(0,y_1)$ and $Q(0,y_2)$

The distance Q from P is

$$PQ = OQ - OP.$$

$$= y_2 - y_1 \text{ if } y_2 > y_1 \text{ or}$$

$$= y_1 - y_2 \text{ if } y_1 > y_2$$

$$PQ = |y_2 - y_1|$$

(Read as modulus or absolute value of $y_2 - y_1$)

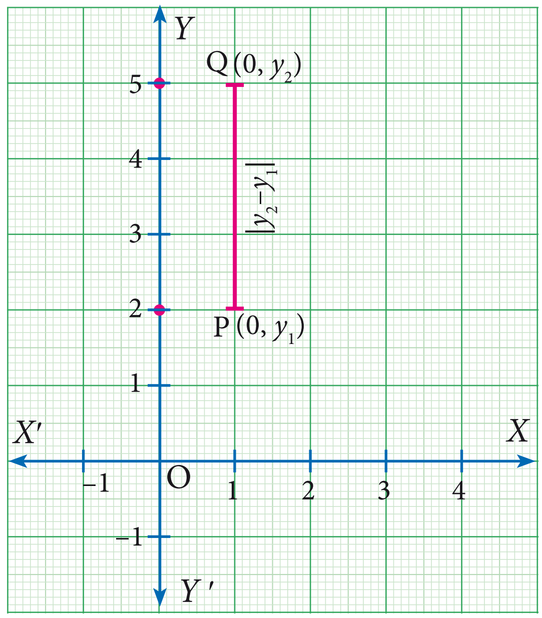


Fig. 5.21

Activity 5

(i). Where do the following points lie?

$P(0,-4)$, $Q(0,-1)$, $R(0,3)$, $S(0,6)$.

(ii). Find the distance between the following points using coordinates given in question (i).

(a) P and R (b) Q and S

(iii). Draw the line diagram for the given above points.

5.3.2 Distance Between Two Points Lying on a Line Parallel to Coordinate Axes

Consider the points $A(x_1, y_1)$ and $B(x_2, y_1)$. Since the y - coordinates are equal the points lie on a line parallel to x- axis. From A and B draw AP and BQ perpendicular to x- axis respectively. Observe the given figure (Fig. 5.22), it is obvious that the distance AB is same as the distance PQ

Distance AB = Distance between $PQ = |x_2 - x_1|$

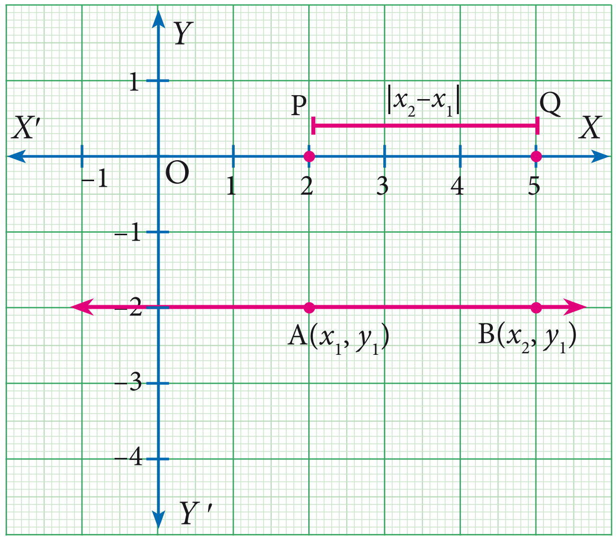


Fig. 5.22

[The difference between x coordinates]

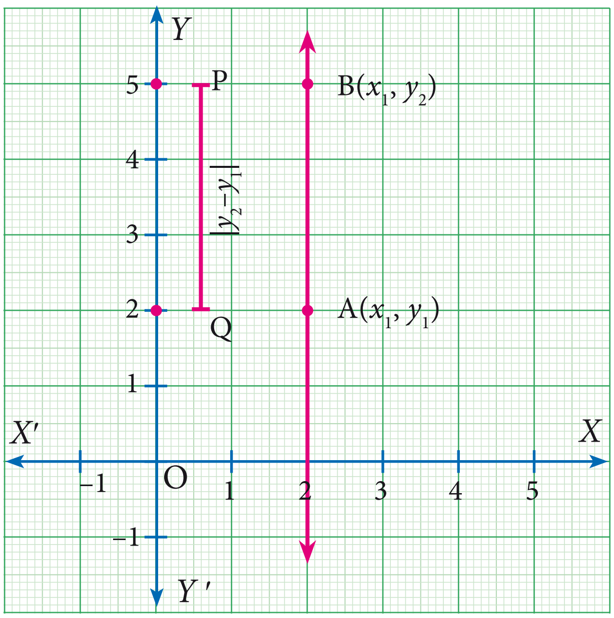


Fig. 5.23

Similarly consider the line joining the two points

$A(x_1, y_1)$ and $B(x_1, y_2)$, parallel to y – axis.

Then the distance between these two points is

$$|y_2 - y_1|$$

[The difference between y coordinates]

Activity 6

Find the distance between the two given points

(i) $A(8,3)$ and $B(-4, 3)$

(ii) $A(3,4)$ and $B(3,8)$

5.3.3 Distance Between the Two Points on a Plane.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the Cartesian plane (or xy – plane), at a distance 'd' apart such that $d = PQ$.

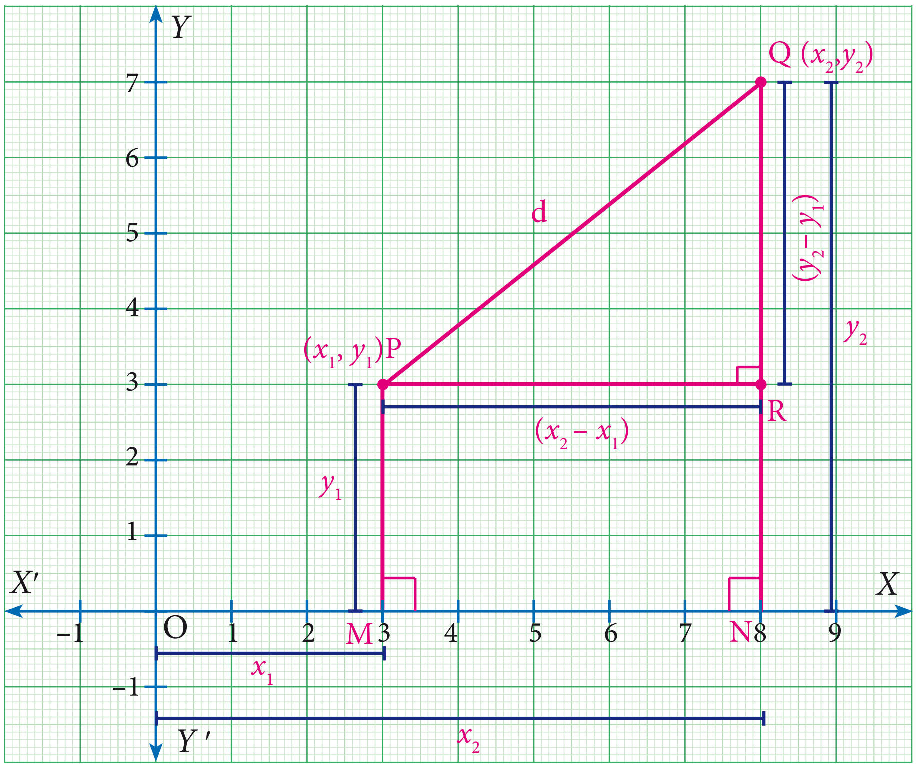


Fig. 5.24

Step 1

By the definition of coordinates,

$OM = x_1 \qquad MP = y_1$

$ON = x_2 \qquad NQ = y_2$

Now $PR = MN$

(Opposite sides of the rectangle $MNRP$)

$= ON - OM$ (Measuring the distance from O)

$= x_2 - x_1$ ……..(1)

And $RQ = NQ - NR$

$= NQ - MP$ (Opposite sides of the rectangle $MNRP$)

$= y_2 - y_1$ ………..(2)

Thinking Corner

A man goes 3 km. towards north and then 4 km. towards each. How far is he away from the initial position?

Step 2

Triangle PQR is right angled at R. ($PR \perp NQ$)

$PQ^2 = PR^2 + RQ^2$ (By Pythagoras theorem)

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (Taking positive square root)

E768EB

Note

Distance between two points

1. Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the distance between these points is given by the formula $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$.
2. The distance between PQ = The distance between QP i.e. $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$
3. The distance of a point $P(x_1, y_1)$ from the origin $O(0,0)$ is $OP = \sqrt{x_1^2 + y_1^2}$

5.3.4 Properties of Distances

We have already seen that distance (A,B) = distance (B,A) for any points A, B on the plane. What other properties have you noticed ? In case you have missed them, here are some:

distance $(A,B) = 0$ exactly when A and B denote the identical point: $A = B$.

distance $(A,B) > 0$ for any two distinct points A and B.

Now consider three points A, B and C. If we are given their co-ordinates and we find that their x-co-ordinates are the same then we know that they are collinear, and lie on a line parallel to the y-axis. Similarly, if their y-co-ordinates are the same then we know that they are collinear, and lie on a line parallel to the x-axis. But these are not the only conditions. Points (0,0), (1,1) and (2,2) are collinear as well. Can you think of what relationship should exist between these coordinates for the points to be collinear ?

The distance formula comes to our help here. We know that when A, B and C are the vertices of a triangle, we get,

distance(A,B) + distance(B,C) >distance(A,C) (after renaming the vertices suitably).

When do three points on the plane not form a triangle ? When they are collinear, of course. In fact, we can show that when,

distance(A,B) + distance(B,C) = distance(A,C), the points A, B and C must be collinear.

Similarly, when A, B and C are the vertices of a right angled triangle, $\angle ABC = 90^\circ$ we know that:

distance$(AB)^2$+ distance$(BC)^2$ = distance$(AC)^2$

with appropriate naming of vertices. We can also show that the converse holds: whenever the equality here holds for A, B and C, they must be the vertices of a right angled triangle.

The following examples illustrate how these properties of distances are useful for answering questions about specific geometric shapes.

Example 5.8

Find the distance between the points $(-4, 3), (2,-3)$.

Solution

The distance between the points $(-4, 3), (2,-3)$ is

$$
\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(2+4)^2 + (-3-3)^2} \\
&= \sqrt{(6^2 + (-6)^2)} = \sqrt{(36+36)} \\
&= \sqrt{(36 \times 2)} \\
&= 6\sqrt{2}
\end{aligned}
$$

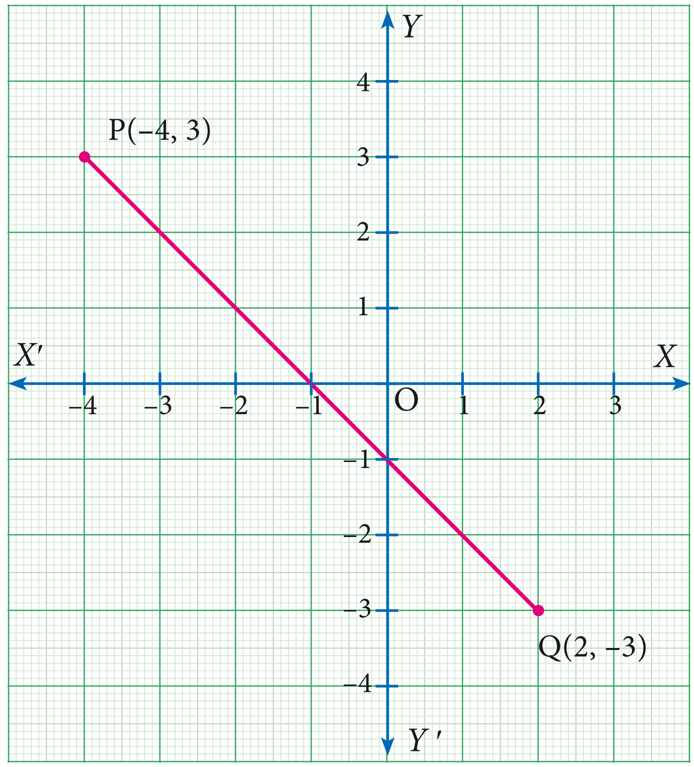


Fig. 5.25

Example 5.9

Show that the following points $A(3,1)$, $B(6,4)$ and $C(8,6)$ lies on a straight line.

Solution

Using the distance formula, we have

$$AB = \sqrt{(6-3)^2 + (4-1)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

$$BC = \sqrt{(8-6)^2 + (6-4)^2} = \sqrt{(4+4)} = \sqrt{8} = 2\sqrt{2}$$

$$AC = \sqrt{(8-3)^2 + (6-1)^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$$

$$AB + BC = 3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2} = AC$$

Therefore the points lie on a straight line.

Collinear points

To show the collinearity of three points, we prove that the sum of the distance between two pairs of points is equal to the third pair of points.

In otherwords, points A, B, C are collinear if $AB + BC = AC$

Example 5.10

Show that the points $A(7,10)$, $B(-2,5)$, $C(3,-4)$ are the vertices of a right angled triangle.

Solution

Here $A = (7, 10)$, $B = (-2, 5)$, $C = (3,-4)$

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(-2-7)^2 + (5-10)^2}$$

$$= \sqrt{(-9)^2 + (-5)^2}$$

$$= \sqrt{(81+25)}$$

$$= \sqrt{106}$$

$$AB^2 = 106 \qquad \text{... (1)}$$

$$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(3-(-2))^2 + (-4-5)^2} = \sqrt{(5)^2 + (-9)^2}$$

$$= \sqrt{25+81} = \sqrt{106}$$

$$BC^2 = 106 \qquad \text{... (2)}$$

$$AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(3-7)^2 + (-4-10)^2} = \sqrt{(-4)^2 + (-14)^2}$$

$$= \sqrt{16+196} = \sqrt{212}$$

$$AC^2 = 212 \qquad \text{... (3)}$$

From (1), (2) & (3) we get,

$$AB^2 + BC^2 = 106 + 106 = 212 = AC^2$$

Since $AB^2 + BC^2 = AC^2$

$\therefore$ ΔABC is a right angled triangle, right angled at B.

Right angled triangle

We know that the sum of the squares of two sides is equal to the square of the third side,which is the hypotenuse of a right angled triangle.

Example 5.11

Show that the points

$A(-4,-3)$, $B(3,1)$, $C(3,6)$, $D(-4,2)$ taken in that order form the vertices of a parallelogram.

Solution

Let $A(-4,-3)$, $B(3,1)$, $C(3,6)$, $D(-4,2)$ be the four vertices of any quadrilateral $ABCD$. Using the distance formula,

Parallelogram

We know that opposite sides are equal

$$\text{Let} \quad d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$AB = \sqrt{(3+4)^2+(1+3)^2} = \sqrt{49+16} = \sqrt{65}$$

$$BC = \sqrt{(3-3)^2+(6-1)^2}$$

$$= \sqrt{0+25} = \sqrt{25} = 5$$

$$CD = \sqrt{(-4-3)^2+(2-6)^2}$$

$$= \sqrt{(-7)^2+(-4)^2} = \sqrt{49+16} = \sqrt{65}$$

$$AD = \sqrt{(-4+4)^2+(2+3)^2} = \sqrt{(0)^2+(5)^2} = \sqrt{25} = 5$$

$AB = CD = \sqrt{65}$ and $BC = AD = 5$

Here, the opposite sides are equal. Hence $ABCD$ is a parallelogram.

Example 5.12

Prove that the points $A(3,5)$, $B(6,2)$, $C(3,-1)$, and $D(0,2)$ taken in order are the vertices of a square.

Solution

Let $A(3,5)$, $B(6,2)$, $C(3,-1)$, and $D(0,2)$ be the vertices of any quadrilateral $ABCD$.

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Square

We know that four sides are equal and the diagonals are also equal

By using the distance formula we get,

$$AB = \sqrt{(6-3)^2+(2-5)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

$$BC = \sqrt{(3-6)^2+(-1-2)^2} = \sqrt{(-3)^2+(-3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

$$CD = \sqrt{(0-3)^2+(2+1)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

$$AD = \sqrt{(0-3)^2+(2-5)^2} = \sqrt{(-3)^2+(-3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

From the above results, we see that $AB=BC=CD=DA= 3\sqrt{2}$

(i.e.) All the four sides are equal.

Further, $A(3,5)$, $C(3,-1)$

Diagonal $AC = \sqrt{(3-3)^2+(-1-5)^2} = \sqrt{(0)^2+(-6)^2} = \sqrt{36} = 6$

Diagonal $BD = \sqrt{(0-6)^2+(2-2)^2} = \sqrt{(-6)^2+(0)^2} = \sqrt{36} = 6$

From the above we see that $AB = CD = 6$

Hence $ABCD$ is a square.

Example 5.13

If the distance between the points (5,–2), (1, a), is 5 units, find the values of a.

Solution

The two given points are (5,-2), (1, a) and $d = 5$.

By distance formula

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$\sqrt{(1-5)^2+(a+2)^2} = 5$$

$$\sqrt{16+(a+2)^2} = 5$$

$$16+(a+2)^2 = 25 \text{ (By squaring on both the sides)}$$

$$(a+2)^2 = 25-16$$

$$(a+2)^2 = 9$$

$$(a+2) = \pm 3 \text{ (By taking the square root on both side)}$$

$$a = -2 \pm 3$$

$$a = -2+3 \text{ (or) } a = -2-3$$

$$a = 1 \text{ or } -5.$$

Example 5.14

Calculate the distance between the points A (7, 3) and B which lies on the x-axis whose abscissa is 11.

Solution

Since B is on the x-axis, the y-coordinate of B is 0.

So, the coordinates of the point B is (11, 0)

By the distance formula the distance between the points A (7, 3), B (11, 0) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AB = \sqrt{(11-7)^2 + (0-3)^2} = \sqrt{(4)^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25}$$

$$= 5$$

Example 5.15

Find the value of 'a' such that $PQ = QR$ where P, Q, and R are the points whose coordinates are (6, –1), (1, 3) and (a, 8) respectively.

Solution

Given P (6, –1), Q (1, 3) and R (a, 8)

$$PQ = \sqrt{(1-6)^2 + (3+1)^2} = \sqrt{(-5)^2 + (4)^2} = \sqrt{41}$$

$$QR = \sqrt{(a-1)^2 + (8-3)^2} = \sqrt{(a-1)^2 + (5)^2}$$

$$\text{Given } PQ = QR$$

$$\text{Therefore } \sqrt{41} = \sqrt{(a-1)^2 + (5)^2}$$

$$41 = (a-1)^2 + 25 \quad \text{[Squaring both sides]}$$

$$(a-1)^2 + 25 = 41$$

$$(a-1)^2 = 41 - 25$$

$$(a-1)^2 = 16$$

$$(a-1) = \pm 4 \quad \text{[taking square root on both sides]}$$

$$a = 1 \pm 4$$

$$a = 1 + 4 \text{ or } a = 1 - 4$$

$$a = 5, -3$$

Example 5.16

Let $A(2, 2)$, $B(8, -4)$ be two given points in a plane. If a point P lies on the X- axis (in positive side), and divides AB in the ratio 1: 2, then find the coordinates of P.

Solution

Given points are $A(2, 2)$ and $B(8, -4)$ and let $P = (x, 0)$ [P lies on x axis]

By the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AP = \sqrt{(x-2)^2 + (0-2)^2} = \sqrt{x^2 - 4x + 4 + 4} = \sqrt{x^2 - 4x + 8}$$

$$BP = \sqrt{(x-8)^2 + (0+4)^2} = \sqrt{x^2 - 16x + 64 + 16} = \sqrt{x^2 - 16x + 80}$$

Given $\quad AP : PB = 1 : 2$

i.e. $\quad \dfrac{AP}{BP} = \dfrac{1}{2} \quad (\because BP = PB)$

$$\frac{\sqrt{x^2 - 4x + 8}}{\sqrt{x^2 - 16x + 80}} = \frac{1}{2}$$

squaring on both sides,

$$\frac{x^2 - 4x + 8}{x^2 - 16x + 80} = \frac{1}{4}$$

$$4x^2 - 16x + 32 = x^2 - 16x + 80$$

$$3x^2 - 48 = 0$$

$$3x^2 = 48$$

$$x^2 = 16$$

$$x = \pm 4$$

As the point P lies on x-axis (positive side), its x- coordinate cannot be -4.

Hence the coordinates of P is$(4, 0)$

E7F4FY

Example 5.17

Show that (4, 3) is the centre of the circle passing through the points (9, 3), (7,–1), (–1,3). Find the radius.

Solution

Let $P(4, 3)$, $A(9, 3)$, $B(7, -1)$ and $C(-1, 3)$

If P is the centre of the circle which passes through the points A, B, and C, then P is equidistant from A, B and C (i.e.) $PA = PB = PC$

By distance formula,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AP = PA = \sqrt{(4-9)^2+(3-3)^2} = \sqrt{(-5)^2+0} = \sqrt{25} = 5$$

$$BP = PB = \sqrt{(4-7)^2+(3+1)^2} = \sqrt{(3)^2+(4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$CP = PC = \sqrt{(4+1)^2+(3-3)^2} = \sqrt{(5)^2+0} = \sqrt{25} = 5$$

$$PA = PB = PC$$

Therefore P is the centre of the circle, passing through A, B and C

Radius = PA = 5.

Exercise 5.2

1. Find the distance between the following pairs of points.

   (i) (1, 2) and (4, 3) (ii) (3,4) and (– 7, 2)

   (ii) (a, b) and (c, b) (iv) (3,– 9) and (–2, 3)

2. Determine whether the given set of points in each case are collinear or not.

   (i) (7,–2),(5,1),(3,4) (ii) (–2, –8), (2,–3) (6,2)

   (iii) $(a,-2)$, $(a,3)$, $(a,0)$

3. Show that the following points taken in order form an isosceles triangle.

   (i) A (5,4), B(2,0), C (–2,3) (ii) A (6,–4), B (–2, –4), C (2,10)

4. Show that the following points taken in order form an equilateral triangle in each case.

   (i) $A(2, 2)$, $B(-2, -2)$, $C(-2\sqrt{3}, 2\sqrt{3})$ (ii) $A(\sqrt{3}, 2)$, B (0,1), C(0,3)

5. Show that the following points taken in order form the vertices of a parallelogram.

   (i) $A(-3, 1)$, $B(-6, -7)$, C (3, –9) and $D(6, -1)$

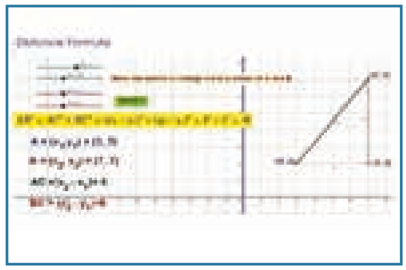
   (ii) A (–7, –3), B(5,10), C(15,8) and D(3, –5)

6. Verify that the following points taken in order form the vertices of a rhombus.

   (i) A(3,–2), B (7,6),C (–1,2) and D (–5, –6)

   (ii) A (1,1), B(2,1),C (2,2) and D(1,2)

ICT Corner



Expected Result is shown in this picture

Step – 1

Open the Browser and copy and paste the Link given below (or) by typing the URL given (or) Scan the QR Code.

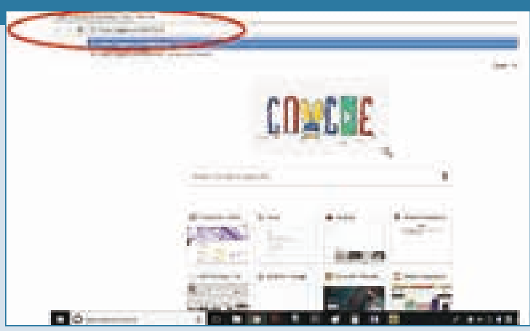
Step - 2

GeoGebra work book called “IX Analytical Geometry” will open. There are several worksheets given. Select the one you want. For example, open” Distance Formula”

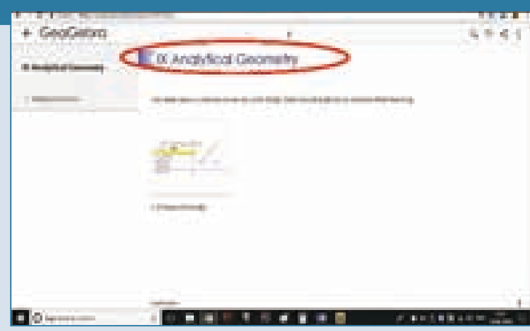
Step-3

Move the sliders x_1, x_2, y_1, y_2 to change the co-ordinates of A and B. Now you calculate the distance AB using theDistance formula in a piece of paper and check your answer

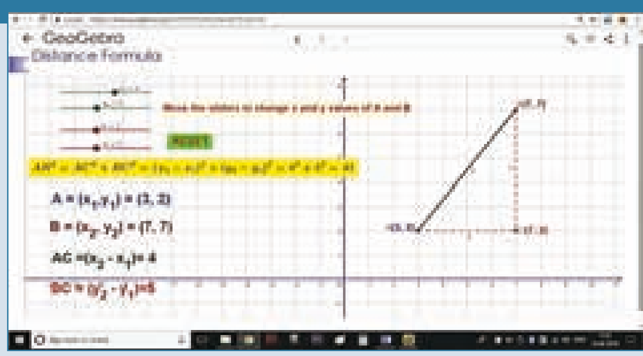
Step 1



Step 2



Step 3



Similarly you can check other worksheets in the Workbook related to your lesson

Browse in the link

Co-Ordinate Geometry: https://ggbm.at/V9HfY4v8

7. If A, B, C are points, $(-1, 1)$, $(1, 3)$ and $(3, a)$ respectively and if $AB = BC$, then find 'a'.

8. The abscissa of a point A is equal to its ordinate, and its distance from the point $B(1, 3)$ is 10 units, What are the coordinates of A?

9. The point (x, y) is equidistant from the points (3,4)and(–5,6). Find a relation between x and y.

10. Let $A(2, 3)$ and $B(2,-4)$ be two points. If P lies on the x-axis, such that $AP = \frac{3}{7}AB$, find the coordinates of P.

11. Find the perimeter of the triangle whose vertices are(3,2), (7,2) and (7, 5).

12. Show that the point (11,2) is the centre of the circle passing through the points (1,2), (3,–4) and (5,–6)

13. The radius of a circle with centre at origin is 30 units. Write the coordinates of the points where the circle intersects the axes. Find the distance between any two such points.

14. Points A $(-1, y)$ and B $(5, 7)$ lie on a circle with centre C $(2, -3y)$. Find the radius of the circle.

Exercise 5.3

Multiple Choice Questions

1. Point (–3,5) lie in the ________ quadrant.

   (a) I (b) II (c) III (d) IV

2. Signs of the abscissa and ordinate of a point in the fourth quadrant are respectively _____

   (a) (+,+) (b) (–, –) (c) (–, +) (d) (+, –)

3. Point (0, –7) lies ________________________________

   (a) on the x-axis (b) in the II quadrant

   (c) on the y-axis (d) in the IV quadrant.

4. Point (–10, 0) lies ________________________________

   (a) on the negative direction of x-axis (b) on the negative direction of y-axis

   (c)in the III quadrant (d) in the IV quadrant

5. If the y-coordinate of a point is zero, then the point always lies ______

(a)in the I quadrant (b) in the II quadrant (c)on x-axis (d) on y-axis

6. The point M lies in the IV quadrant. The coordinates of M is _______

(a) (a,b) (b) $(-a, b)$ (c) $(a, -b)$ (d) $(-a, -b)$

7. The points $(-5, 2)$ and $(2, -5)$ lie in the ________

(a) same quadrant (b) II and III quadrant respectively

(c) II and IV quadrant respectively (d) IV and II quadrant respectively

8. On plotting the points $O(0,0)$, $A(3, -4)$, $B(3, 4)$ and $C(0, 4)$ and joining OA, AB, BC and CO, which of the following figure is obtained?

(a) Square (b) Rectangle (c) Trapezium (d) Rhombus

9. If $P(-1,1)$, $Q(3,-4)$, $R(1, -1)$, $S(-2, -3)$ and $T(-4, 4)$ are plotted on a graph paper, then the points in the fourth quadrant are __________

(a) P and T (b) Q and R (c) only S (d) P and Q

10. The point whose ordinate is 4 and which lies on the y-axis is _______________

(a)$(4, 0)$ (b) $(0, 4)$ (c) $(1, 4)$ (d) $(4, 2)$

11. The distance between the two points $(2, 3)$ and $(1, 4)$ is ______

(a) 2 (b) $\sqrt{56}$ (c) $\sqrt{10}$ (d) $\sqrt{2}$

12. If the points A (2,0), B (-6,0), C $(3, a-3)$ lie on the x-axis then the value of a is _____

(a) 0 (b) 2 (c) 3 (d) -6

13. If $(x+2, 4) = (5, y-2)$, then the coordinates (x,y) are _____

(a) (7, 12) (b) (6, 3) (c) (3, 6) (d) (2, 1)

14. If Q_1, Q_2, Q_3, Q_4 are the quadrants in a Cartesian plane then $Q_2 \cap Q_3$ is ___________

(a) $Q_1 \cup Q_2$ (b) $Q_2 \cup Q_3$ (c) Null set (d) Negative x-axis.

15. The distance between the point $(5, -1)$ and the origin is _________

(a) $\sqrt{24}$ (b) $\sqrt{37}$ (c) $\sqrt{26}$ (d) $\sqrt{17}$

Activity 7

Plot the points $A(-1, 0)$, $B(3, 0)$, $C(3, 4)$, and $D(-1, 4)$ on a graph sheet. Join them to form a rectangle. Draw the mirror image of the diagram in clockwise direction:

(i) about x-axis. (ii) about y-axis.

What is your observation on the coordinates of the mirror image?

Activity 8

Plot the points $A(1, 0)$, $B(-7, 2)$, $C(-3, 7)$ on a graph sheet and join them to form a triangle.

Plot the point $G(-3, 3)$.

Join AG and extend it to intersect BC at D.

Join BG and extend it to intersect AC at E.

What do you infer when you measure the distance between BD and DC and the distance between CE and EA?

Using distance formula find the lengths of CG and GF, where F in on AB.

Write your inference about $AG: GD$, $BG: GE$ and $CG: GF$.

Note: ***G*** is the **centroid** of the triangle and AD, BE and CF are the three medians of the triangle.

Points to remember

- If x_1, x_2 are the x-coordinates of two points on the x-axis then the distance between them is $x_2 - x_1$, if $x_2 > x_1$.
- If y_1, y_2 are the y-coordinates of two points on the y-axis then the distance between them is $|y_1 - y_2|$.
- Distance between the two points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- Distance between (x_1, y_1) and the origin $(0, 0)$ is $\sqrt{x_1^2 + y_1^2}$

If three or four points are given, in order then to prove that a given figure is:

- Triangle- We prove that the sum of the lengths of any two sides is greater than the length of the third side.
- Isosceles triangle- We prove that the length of any two sides are equal.
- Equilateral triangle- We prove that the length of all the three sides are equal.
- Square - We prove that four sides are equal.
- Rectangle - We prove that opposite sides are equal and the diagonals are also equal.
- Parallelogram (not a rectangle) - We prove that opposite sides are equal but the diagonals are not equal.
- Rhombus (not a square)- We prove that all sides are equal but the diagonals are not equal.

Answers

Exercise 5.1

1. $P(-7,6)$ = II Quadrant; $Q(7,-2)$ = IV Quadrant; $R(-6,-7)$ = III Quadrant; $S(3,5)$ = I Quadrant; and $T(3,9)$ = I Quadrant

2. (i) $P=(-4,4)$ (ii) $Q=(3,3)$ (iii) $R=(4,-2)$ (iv) $S=(-5,-3)$

3. (i) Straight line parallel to x -axis (ii) Straight line which lie on y -axis.

4. (i) Square (ii) Trapezium

Exercise 5.2

1. (i) $\sqrt{10}$ units (ii) $2\sqrt{26}$ units (iii) $c-a$ (iv) 13 units

2. (i) Collinear (ii) Collinear (iii) Collinear 7. 5 or 1

8. Coordinates of A (9, 9) or (–5,–5) 9. $y=4x+9$ 10. Coordinates of $P(2,0)$

11. 12 units 13. $30\sqrt{2}$ 14. $y=7$, Radius = $\sqrt{793}$, $y=-1$, Radius = 5

Exercise 5.3

1. (b)	2. (d)	3. (c)	4. (a)	5. (c)
6. (c)	7. (c)	8. (c)	9. (b)	10. (b)
11. (d)	12. (c)	13. (c)	14. (c)	15. (c)

MATHEMATICAL TERMS	
Abscissa	X-அச்சின் தொலைவு (கிடைஅச்சு தொலைவு)
Acute triangle	குறுங்கோண முக்கோணம்
Adjacent angles	அடுத்துள்ள கோணங்கள்
Algebraic expression	இயற்கணிதக் கோவை
Alternate angles	ஒன்றுவிட்ட கோணங்கள்
Altitudes of a triangle	முக்கோணத்தின் குத்துக்கோடுகள்
Angle sum property of triangle	முக்கோணத்தின் கோணங்களின் கூடுதல் பண்பு
Binomial expression	ஈருறுப்புக் கோவை
Cardinal number of a set	கணத்தின் ஆதி எண்
Cartesian plane	கார்டீசியன் தளம்
Cartesian coordinate system	கார்டீசியன் அச்சுத் தொகுப்பு முறை
Centriod	நடுக்கோட்டு மையம்
Circum radius	முக்கோணத்தின் சுற்றுவட்ட ஆரம்
Circumcentre of a triangle	முக்கோணத்தின் சுற்றுவட்டமையம்
Circumcircle	சுற்றுவட்டம்
Ordinate	Y–அச்சின் தொலைவு (செங்குத்து அச்சுத்தொலைவு)
Co-efficient	கெழு
Collections	தொகுப்பு
Complement of a set	நிரப்புக் கணம்
Concurrent lines	ஒரு புள்ளி வழிக் கோடுகள்
Congruent triangles	சர்வசம முக்கோணங்கள்
Constant	மாறிலி
Coordinate axes	ஆய அச்சுகள்
Corresponding angles	ஒத்த கோணங்கள்
Cubic polynomial	முப்படி பல்லுறுப்புக் கோவை
Decimal expansion	தசம விரிவாக்கம்
Decimal representation	தசம குறியீடு
Degree of polynomial	பல்லுறுப்புக்கோவையின் படி
Denseness property	அடர்த்திப் பண்பு
Descriptive form	விவரித்தல் முறை
Diagonal	மூலைவிட்டம்
Difference of two sets	கணங்களின் வித்தியாசம்
Disjoint sets	வெட்டா கணங்கள்
Division Algoritham of polynomial	பல்லுறுப்புக் கோவையின் வகுத்தல் படிமுறை
Empty set/Null set	வெற்றுக் கணம்
Equal sets	சமகணங்கள்
Equiangular triangle	சமகோண முக்கோணம்
Equilateral triangle	சமபக்க முக்கோணம்
Equivalent sets	சமான கணங்கள்
Finite set	முடிவுறு கணம்
Incentre	உள்வட்ட மையம்
Infinite set	முடிவிலி கணம்
Interior angles	உட்கோணங்கள்
Intersection of two sets	கணங்களின் வெட்டு
Irrational numbers	விகிதமுறா எண்கள்
Isosceles Trapezium	இருசமபக்க சரிவகம்
Isosceles triangles	இருசமபக்க முக்கோணம்

Linear pair of angles	நேர்கோட்டுக் கோணங்கள்
Linear polynomial expression	ஒருபடி பல்லுறுப்புக் கோவை
Median	நடுக்கோடு
Monomial expression	ஓருறுப்புக் கோவை
Negative integers	குறை முழுக்கள்
Non-terminating decimals	முடிவுறா தசம எண்கள்
Number of terms	உறுப்புகளின் எண்ணிக்கை
Obtuse traingle	விரிகோண முக்கோணம்
Operation of polynomial	பல்லுறுப்புக்கோவையின் செயல்பாடு
Ortho centre of a triangle	முக்கோணத்தின் செங்கோட்டு மையம்
Period of decimals	தசம எண்களின் காலமுறைமை
Plotting points	புள்ளிகளைக் குறித்தல்
Polygon	பல கோணம்
Polynomial equation	பல்லுறுப்புக்கோவைச் சமன்பாடு
Positive integers	மிகை முழுக்கள்
Power set	அடுக்குக்கணம்
Proper sub set	தகு உட்கணம்
Quadrant	காற்பகுதி
Quadratic polynomial	இருபடி பல்லுறுப்புக் கோவை
Quadrilateral	நாற்கரம்
Rational numbers	விகிதமுறு எண்கள்
Rationalization	விகிதப்படுத்துதல்
Real number	மெய்யெண்கள்
Real number line	மெய் எண் கோடு
Recurring decimals	சுழல் தன்மையுள்ள தசம எண்கள்
Remainder theroem	மீதித் தேற்றம்
Right triangle	செங்கோண முக்கோணம்
Roots of a polynomial	பல்லுறுப்புக் கோவையின் மூலங்கள்
Roster Form/Tabular form	பட்டியல் முறை
Set builder form/Rule form	கணக்கட்டமைப்பு முறை
Set operations	கணச்செயல்கள்
Sets	கணங்கள்
Singular set/Singleton set	ஓருறுப்புக் கணம்
Square root	வர்க்க மூலம்
Sub set	உட்கணம்
Surd / Irrational roots	விகிதமுறா மூலம்
Symmetric difference of sets	கணங்களின் சமச்சீர் வித்தியாசம்
Terminating Decimals	முடிவுறு தசம எண்கள்
Transversal	குறுக்குவெட்டி
Trapezium	சரிவகம்
Trinomial expression	மூவுறுப்புக் கோவை
Union of sets	கணங்களின் சேர்ப்பு
Universal set	அனைத்துக் கணம்
Venn Diagram	வென்படம்
Vertically opposite angles	குத்தெதிர் கோணங்கள்
Well defined	நன்கு வரையறுக்கப்பட்ட
Zeros of polynomial	பல்லுறுப்புக்கோவையின் பூச்சியங்கள்

Secondary Mathematics - Class 9
Text Book Development Team

Reviewers

Prof. R. Ramanujam
Professor IMSC, Chennai

Dr. Aaloka Kanhere
Assistant Professor,
Homi Bhabha Centre for Science Education, Mumbai.

Athmaraman R
Mathematics Educational Consultant,
Association of Mathematics Teachers of India, Chennai -05

Domain Experts

Dr. K.Kumarasamy.
Associate Professor, R.K.M. Vivekanandha college, Chennai.

Thulasiram
Associate Professor, A.M. Jain college, Chennai.

Academic Coordinator

Tamilselvi B
Deputy Director, SCERT, Chennai

Coordinator

Rajkamal E
Junior Lecturer, SCERT, Chennai – 06

ICT Coordinators

Sankar K
B.T. Assistant., GHSS., Vellore

Art and Design Team

Chief Co-ordinator and Creative Head

Srinivasan Natarajan

Illustration

Shanmugam B
Art Teacher, PUMS, Thirumalai Kuppam,
Madhanur Block, Vellore Dt.

Art Teachers,
Government of Tamil Nadu.
Students,
Government College of Fine Arts,
Chennai & Kumbakonam.

Layout

Joy Graphics, Chennai

In-House

QC - Gopu Rasuvel
- Balaji

Typist

Kanimozhi T
Computer Teacher, GBHSS, SSA, Avadi.

Content Writers

K.S.A Mohamed Yousuf Jainulabudeen,
B.T. Asst, Manbaul Uloom HSS, Fort, Coimbatore.

A.Senthil Kumar
B.T. Asst, GHSS, Thiruthuraiyur, Cuddalore Dt.

N.P. Sivakumar,
PGT, Sir M Venkata Subba Rao MHSS, T. Nagar, Chennai.

S. Sudhakar,
B.T.Asst, JG Garodia National HSS, Tambaram(E).

Rani Desikan,
PGT, E.O Jaigopal Garodia Hindu Vidyalaya,
West Mambalam, Chennai

R. Ragothaman
B.T. Asst, GHSS, Theethipalayam, Coimbatore.

H. Shanawas,
B.T. Asst, Model School, Karimangalam, Dharmapuri Dt.

N. Kalavalli,
Headmistress, VSA GHS, Uthukkadu, Kancheepuram Dt.

K.P. Ganesh,
B.T. Asst, Avvaiyar GGHSS, Dharmapuri

M. Thiruvengadathan,
B.T. Asst, Sethupathi HSS, Madurai.

S. M. Mani,
B.T.Asst, GHSS, Thirukandalam, Tiruvallur Dt.

N. Venkatraman ,
PGT, Sri Gopal Naidu HSS, Peelamedu, Coimbatore.

C. Kirubavathy,
B.T.Asst, ORGN GBHSS, Redhills, Tiruvallur Dt.

K. Krishna Kumar ,
B.T.Asst, GHS, Egattur, Tiruvallur Dt.

Content Readers

Mr. P. Jeyaraman,
Assistant Professor in Mathematics,
L.N. Govt Arts College, Ponneri – 601 204

Dr. N. Geetha,
Assistant Professor in Mathematics,
L.N. Govt Arts College, Ponneri – 601 204

S. Karthikeyan,
Assistant Professor in Mathematics,
L.N. Govt Arts College, Ponneri – 601 204

Dr. K. Kavitha
Assistant Professor in Mathematics,
Bharathi Women's College, Chennai.

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