GOVERNMENT OF TAMILNADU

HIGHER SECONDARY SECOND YEAR

BUSINESS MATHEMATICS AND STATISTICS

VOLUME - I

A publication under Free Textbook Programme of Government of Tamil Nadu

Department of School Education

Untouchability is Inhuman and a Crime

Government of Tamil Nadu

First Edition - 2019

(Published under new syllabus)

NOT FOR SALE

Content Creation

State Council of Educational Research and Training

Printing & Publishing

Tamil Nadu Textbook and Educational Services Corporation

www.textbooksonline.tn.nic.in

HOW TO USE THE BOOK?

Career Options	List of Further Studies & Professions.
Learning Objectives:	Learning objectives are brief statements that describe what students will be expected to learn by the end of school year, course, unit, lesson or class period.
	Additional information about the concept.
DO YOU KNOW?	Amazing facts, Rhetorical questions to lead students to Mathematical inquiry
Exercise	Assess students' critical thinking and their understanding
ICT	To enhance digital skills among students
Web links	List of digital resources
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Glossary	Tamil translation of Mathematical terms
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CAREER OPTIONS IN BUSINESS MATHEMATICS and STATISTICS

Higher Secondary students who have taken commerce with Business Mathematics and Statiscs can take up careers in BCA, B.Com., and B.Sc. Statistics. Students who have taken up commerce stream, have a good future in banking and financial institutions.

A lot of students choose to do B.Com with a specialization in computers. Higher Secondary Commerce students planning for further studies can take up careers in professional fields such as Company Secretary , Chartered Accountant (CA), ICAI and so on. Others can take up bachelor's degree in commerce (B.Com), followed by M.Com, Ph.D and M.Phil. There are wide range of career opportunities for B.Com graduates.

After graduation in commerce, one can choose MBA, MA Economics, MA Operational and Research Statistics at Postgraduate level. Apart from these, there are several diploma, certificate and vocational courses which provide entry level jobs in the field of commerce.

Career chart for Higher Secondary students who have taken commerce with Business Mathematics and Statistics.

Courses	Institutions	Scope for further studies
B.Com., B.B.A., B.B.M., B.C.A., B.Com (Computer), B.A.	• Government Arts & Science Colleges, Aided Colleges, Self financing Colleges. • Shri Ram College of Commerce (SRCC), Delhi • Symbiosis Society's College of Arts & Commerce, Pune. • St. Joseph's College, Bangalore	**C.A., I.C.W.A, C.S.**
B.Sc Statistics	• Presidency College, Chepauk, Chennai. • Madras Christian College, Tambaram • Loyola College, Chennai. • D.R.B.C.C Hindu College, Pattabiram, Chennai.	**M.Sc., Statistics**
B.B.A., LLB, B.A., LLB, B.Com., LL.B. (Five years integrated Course)	• Government Law College. • School of excellence, Affiliated to Dr.Ambethkar Law University	**M.L.**
M.A. Economics (Integrated Five Year course) – Admission based on All India Entrance Examination	• Madras School of Economics, Kotturpuram, Chennai.	**Ph.D.,**
B.S.W.	• School of Social studies, Egmore, Chennai	M.S.W

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SYLLABUS

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	Rank of a Matrix - Concept of Rank of a matrix- Elementary Transformations and Equivalent matrices - Echelon form and Finding the rank of the matrix up to the order of 3 x 4 -Testing the consistency of non homogeneous linear equations (two and three variables) by rank method. **Cramer's Rule**- Non - Homogeneous linear equations up to3 Variables. **Transition Probability Matrices** -Forecasting the succeeding state when theinitial market share is given.	
2.	**INTEGRAL CALCULUS – I**	**(25 periods)**
	Indefinite Integrals : Concept of Indefinite Integral - Some standard results - Integration by decomposition - Integration by parts - Substitution method - Some special type of Integrals. **Definite Integrals :** The fundamental theorems of Integral calculus - Properties of Definite Integrals -Gamma Integral (Statement only)-Definite Integral as the limit of a sum (Statement only)	
3.	**INTEGRAL CALCULUS – II**	**(21 periods)**
	The Area of the region bounded by the curves - Geometrical interpretation of the definite integral as area under a curve. **Application of Integration in Economics and Commerce**-Cost functions from marginal cost functions-. Revenue functions from marginal revenue functions-The demand function from elasticity of demand -Consumer's Surplus-Producer's Surplus.	
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5.	**Numerical Methods**	**(15 periods)**
	Finite differences - Forward difference Operator, Backward difference Operator and Shifting Operator - Finding missing terms. **Interpolation :** Methods of interpolation - Graphical method - Algebraic Method - Gregory Newton's forward and backward formulae - Lagrange's formula	

1 Applications of Matrices and Determinants

Brahmagupta

(c.598 AD(CE) - c.668 AD(CE))

Introduction

In our daily life, We use Matrices for taking seismic surveys. They are used for plotting graphs, statistics and also to do Scientific studies in almost different fields. Matrices are used in representing the real world data like the traits of people's population, habits etc..

Determinants have wonderful algebraic properties and occupied their proud place in linear algebra, because of their role in higher level algebraic thinking.

Brahmagupta (born c.598 AD(CE) died c.668 AD(CE)) was an Indian Mathematician and astronomer. He was the first to give rules to compute with zero. His contribution in Matrix is called as Brahmagupta Matrix.

$$B(x\,,y) = \begin{pmatrix} x & y \\ \pm ty & \pm x \end{pmatrix}.$$

Learning Objectives

On Completion of this chapter , the students are able to understand

- the concept of rank of a matrix.
- elementary transformations and equivalent matrices.
- echelon form of a matrix.
- the rank of the matrix.
- testing the consistency of a non- homogeneous linear equations.
- applications of linear equations
- the concept of Cramer's rule to solve non- homogenous linear equations.
- forecasting the succeeding state when the initial market share is given.

8VNMAI

1.1 Rank of a Matrix

Matrices are one of the most commonly used tools in many fields such as Economics, Commerce and Industry.

We have already studied the basic properties of matrices. In this chapter we will study about the elementary transformations to develop new methods for various applications of matrices.

1.1.1 Concept

With each matrix, we can associate a non-negative integer called its rank.

Definition 1.1

The rank of a matrix A is the order of the largest non-zero minor of A and is denoted by $\rho(A)$

In other words, A positive integer 'r' is said to be the rank of a non- zero matrix A, if

(i) there is atleast one minor of A of order 'r' which is not zero and

(ii) every minor of A of order greater than 'r' is zero.

Note

(i) $\rho(A) \geq 0$

(ii) If A is a matrix of order $m \times n$, then $\rho(A) \leq$ minimum of $\{m, n\}$

(iii) The rank of a zero matrix is '0'

(iv) The rank of a non- singular matrix of order $n \times n$ is 'n'

Example 1.1

Find the rank of the matrix $\begin{pmatrix} 1 & 5 \\ 3 & 9 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 1 & 5 \\ 3 & 9 \end{pmatrix}$

Order of A is 2×2 $\quad \therefore \rho(A) \leq 2$

Consider the second order minor

$$\begin{vmatrix} 1 & 5 \\ 3 & 9 \end{vmatrix} = -6 \neq 0$$

There is a minor of order 2, which is not zero. $\therefore \rho(A) = 2$

DO YOU KNOW?

In cryptography, we are using matrix concepts.

Example 1.2

Find the rank of the matrix $\begin{pmatrix} -5 & -7 \\ 5 & 7 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} -5 & -7 \\ 5 & 7 \end{pmatrix}$

Order of A is 2×2 $\quad \therefore \rho(A) \leq 2$

Consider the second order minor $\begin{vmatrix} -5 & -7 \\ 5 & 7 \end{vmatrix} = 0$

Since the second order minor vanishes, $\rho(A) \neq 2$

Consider a first order minor $|-5| \neq 0$

There is a minor of order 1, which is not zero

$\therefore \rho\ (A) = 1$

Example 1.3

Find the rank of the matrix $\begin{pmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{pmatrix}$

Order of A is 3×3.

$\therefore \rho(A) \leq 3$

Consider the third order minor $\begin{vmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{vmatrix} = 6 \neq 0$

There is a minor of order 3, which is not zero

$\therefore \rho(A) = 3.$

Example 1.4

Find the rank of the matrix $\begin{pmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{pmatrix}$

Order of A is 3×3.

$\therefore \rho(A) \leq 3.$

Consider the third order minor $\begin{vmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{vmatrix} = 0$

Since the third order minor vanishes, therefore $\rho(A) \neq 3$

Consider a second order minor $\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 7 \neq 0$

There is a minor of order 2, which is not zero.

$\therefore \rho(A) = 2.$

Example 1.5

Find the rank of the matrix $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

Order of A is 3×4

$\therefore \rho(A) \leq 3.$

> **DO YOU KNOW?**
>
> A is a square matrix of order 3. If A is of rank 2, then *adj* A is of rank 1.

Consider the third order minors

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ 3 & 6 & 3 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & -1 & 3 \\ 2 & 1 & -2 \\ 3 & 3 & -7 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & -2 \\ 3 & 6 & -7 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ 6 & 3 & -7 \end{vmatrix} = 0$$

Since all third order minors vanishes, $\rho(A) \neq 3$.

Now, let us consider the second order minors, Consider one of the second order minors $\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 6 \neq 0$

There is a minor of order 2 which is not zero.

$\therefore \rho(A) = 2.$

1.1.2 Elementary Transformations and Equivalent matrices

Elementary transformations of a matrix

(i) Interchange of any two rows (or columns): $R_i \leftrightarrow R_j$ $(or\ C_i \leftrightarrow C_j)$.

(ii) Multiplication of each element of a row (or column) by any non-zero scalar k: $R_i \to kR_i$ $(or\ C_i \to kC_i)$

(iii) Addition to the elements of any row (or column) the same scalar multiples of corresponding elements of any other row (or column):
$R_i \to R_i + kR_j.(or\, C_i \to C_i + kC_j)$

Equivalent Matrices

Two matrices A and B are said to be equivalent if one is obtained from the another by applying a finite number of elementary transformations and we write it as $A \sim B$ or $B \sim A$.

1.1.3 Echelon form and finding the rank of the matrix (upto the order of 3×4)

A matrix A of order $m \times n$ is said to be in echelon form if

(i) Every row of A which has all its entries 0 occurs below every row which has a non-zero entry.

(ii) The number of zeros before the first non-zero element in a row is less then the number of such zeros in the next row.

Example 1.6

Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$

Solution :

The order of A is 3×3.

$\therefore \rho(A) \leq 3.$

Let us transform the matrix A to an echelon form by using elementary transformations.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - R_2$
The above matrix is in echelon form	

The number of non zero rows is 2

$\therefore$ Rank of A *is* 2.

$\rho(A) = 2.$

Note

A row having atleast one non-zero element is called as non-zero row.

Example 1.7

Find the rank of the matrix $A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$

Solution:

The order of A is 3×4.

$\therefore \rho(A) \leq 3.$

Let us transform the matrix A to an echelon form

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	
$A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	$R_1 \leftrightarrow R_2$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{pmatrix}$	$R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$	$R_3 \to R_3 + 5R_2$

The number of non zero rows is 3. $\therefore \rho(A) = 3.$

Example 1.8

Find the rank of the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{pmatrix}$

Solution:

The order of A is 3×4.

$\therefore \rho(A) \leq 3.$

Let us transform the matrix A to an echelon form

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -2 \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$R_3 \to R_3 - R_2$

The number of non zero rows is 3.

$\therefore \rho(A) = 3.$

Consistency of Equations

System of linear equations in two variables

We have already studied , how to solve two simultaneous linear equations by matrix inversion method.

Recall

Linear equations can be written in matrix form AX=B, then the solution is $X = A^{-1}B$, provided $|A| \neq 0$

Consider a system of linear equations with two variables,

$$ax + by = h$$
$$cx + dy = k \qquad (1)$$

Where a, b, c, d, h and k are real constants and neither a and b nor c and d are both zero.

For any tow given lines L_1 and L_2, one and only one of the following may occur.

L_1 and L_2 intersect at exactly one point

L_1 and L_2 are coincident

L_1 and L_2 are parallel and distinct.

(see Fig 1.1) In the first case, the system has a unique solution corresponding to the single point of intersection of the two lines.

In the second case, the system has infinitely many solutions corresponding to the points lying on the same line

Finally in the third case, the system has no solution because the two lines do not intersect.

Let us illustrate each of these possibilities by considering some specific examples.

(a) A system of equations with exactly one solution

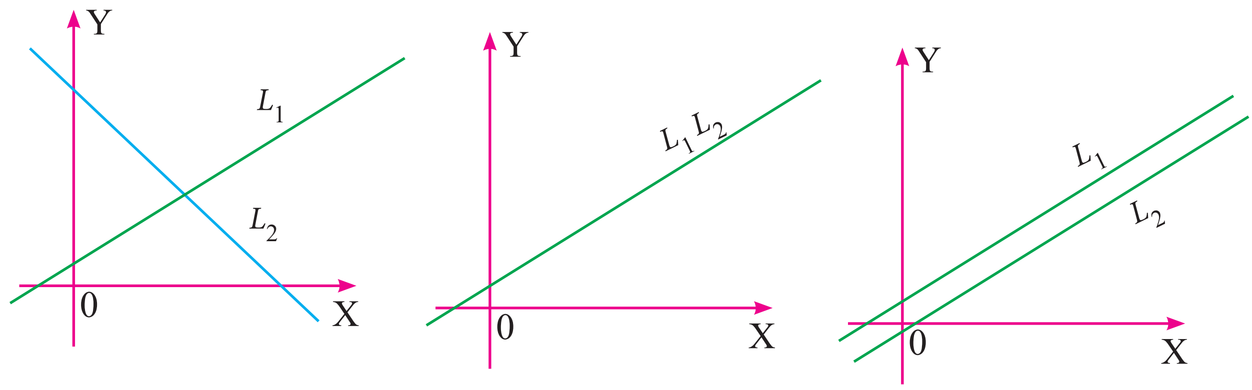
Consider the system $2x - y = 1,\ 3x + 2y = 12$ which represents two lines intersecting at (2, 3) i.e (2, 3) lies on both lines. So the equations are consistent and have unique solution.

(b) A system of equations with infinitely many solutions

Consider the system $2x - y = 1,\ 6x - 3y = 3$ which represents two coincident lines. We find that any point on the line is a solution. The equations are consistent and have infinite sets of solutions such as $(0,\ -1),\ (1,\ 1)$ and so on. Such a system is said to be dependent.

(c) A system of equations that has no solution

Consider the system $2x - y = 1,\ 6x - 3y = 12$ which represents two parallel straight lines. The equations are inconsistent and have no solution.



(a) Unique solution (b) Infinitely many solutions (c) No solution

Fig. 1.1

System of non Homogeneous Equations in three variables

A linear system composed of three linear equations with three variables x, y and z has the general form

$$\begin{aligned} a_1x+b_1y+c_1z&=d_1 \\ a_2x+b_2y+c_2z&=d_2 \\ a_3x+b_3y+c_3z&=d_3 \end{aligned} \qquad (2)$$

A linear equation $ax+by+cz=d$ (a, b and c not all equal to zero) in three variables represents a plane in three dimensional space. Thus, each equation in system (2) represents a plane in three dimensional space, and the solution(s) of the system is precisely the point(s) of intersection of the three planes defined by the three linear equations that make up the system. This system has one and only one solution, infinitely many solutions, or no solution, depending on whether and how the planes intersect one another. Figure 1.2 illustrates each of these possibilities.

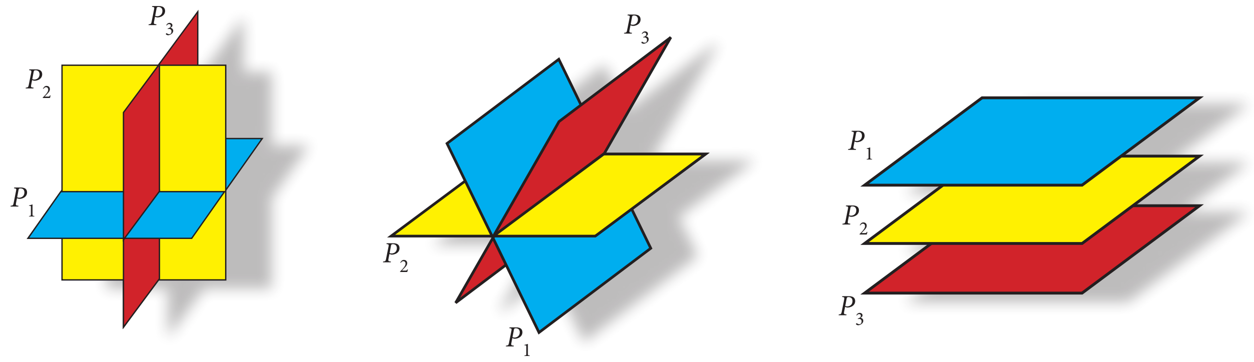


Fig.1.2

(a) A Unique solution (b) Infinitely many solutions (c) No solution

In Figure 1.2(a), the three planes intersect at a point corresponding to the situation in which system (2) has a unique solution.

Figure 1.2 (b) depicts a situation in which there are infinitely many solutions to the system. Here the three planes intersect along a line, and the solutions are represented by the infinitely many points lying on this line.

In Figure 1.2 (c), the three planes are parallel and distinct, so there is no point in common to all three planes; system (2) has no solution in this case.

Note

Every system of linear equations has no solution, or has exactly one solution or has infinitely many solutions.

An arbitrary system of 'm' linear equations in 'n' unknowns can be written as

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1\\
&a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2\\
&\vdots \qquad \vdots \qquad\qquad\quad \vdots \qquad \vdots\\
&a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m
\end{aligned}$$

where $x_1, x_2, \dots, x_n$ are the unknowns and the subscripted a's and b's denote the constants.

Augmented matrices

A system of 'm' linear equations in 'n' unknowns can be abbreviated by writing only the rectangular array of numbers.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_n \end{bmatrix}$$

This is called the augmented matrix for the system and $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ is the coefficient matrix.

Consider the following system of equations

$$\begin{aligned}
x + y + 2z &= 9\\
2x + 4y - 3z &= 1\\
3x + 6y - 5z &= 0
\end{aligned}$$

$$\underset{A}{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} \underset{=}{=} \underset{B}{\begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}}$$

$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$ is the coefficient matrix

and $[A, B] = \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$ is the augmented matrix.

1.1.4 Testing the consistency of non homogeneous linear equations (two and three variables) by rank method.

Consider the equations $A X = B$ in 'n' unknowns.

(i) If $\rho\left(\left[A,\, B\right]\right) = \rho\left(A\right)$, then the equations are consistent.

(ii) If $\rho\left(\left[A,\, B\right]\right) = \rho\left(A\right) = n$, then the equations are consistent and have unique solution.

(iii) If $\rho\left(\left[A,\, B\right]\right) = \rho\left(A\right) < n$, then the equations are consistent and have infinitely many solutions.

(iv) If $\rho\left(\left[A,\, B\right]\right) \neq \rho\left(A\right)$ then the equations are inconsistent and has no solution.

Example 1.9

Show that the equations $x + y = 5,\ 2x + y = 8$ are consistent and solve them.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

$$A \quad X \quad = \quad B$$

Matrix A	Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 5 \\ 2 & 1 & 8 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$	$\sim \begin{pmatrix} 1 & 1 & 5 \\ 0 & -1 & -2 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$
$\rho(A) = 2$	$\rho([A,B]) = 2$	

Number of non-zero rows is 2.

$\rho(A) = \rho([A,B]) = 2 =$ Number of unknowns.

The given system is consistent and has unique solution.

Now, the given system is transformed into

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$$x + y = 5$$

$$y = 2$$

$$\therefore (1) \Rightarrow x + 2 = 5$$

$$x = 3$$

Solution is $x = 3,\ y = 2$

Example 1.10

Show that the equations $2x+y=5,\ 4x+2y=10$ are consistent and solve them.

Solution:

The matrix equation corresponding to the system is

$$\underset{A}{\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}} \underset{X}{\begin{pmatrix} x \\ y \end{pmatrix}} = \underset{B}{\begin{pmatrix} 5 \\ 10 \end{pmatrix}}$$

Matrix A	Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 5 \\ 4 & 2 & 10 \end{pmatrix}$	
$\sim\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$	$\sim\begin{pmatrix} 2 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$
$\rho(A)=1$	$\rho([A,B])=1$	

$$\rho(A)=\rho([A,B]) = 1 < \text{number of unknowns}$$

$\therefore$ The given system is consistent and has infinitely many solutions.

Now, the given system is transformed into the matrix equation.

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$\Rightarrow 2x+y=5$$

Let us take $y=k, k \in R$

$$\Rightarrow 2x+k=5$$

$$x=\frac{1}{2}(5-k)$$

$$x=\frac{1}{2}(5-k),\ y=k \text{ for all } k \in R$$

Thus by giving different values for k, we get different solution. Hence the system has infinite number of solutions.

Example 1.11

Show that the equations $3x-2y=6,\ 6x-4y=10$ are inconsistent.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 3 & -2 \\ 6 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix}$$

$$\text{AX} = \text{B}$$

Matrix A	**Augmented matrix $[A,B]$**	**Elementary Transformation**
$\begin{pmatrix} 3 & -2 \\ 6 & -4 \end{pmatrix}$	$\begin{pmatrix} 3 & -2 & 6 \\ 6 & -4 & 10 \end{pmatrix}$	
$\sim\begin{pmatrix} 3 & -2 \\ 0 & 0 \end{pmatrix}$	$\sim\begin{pmatrix} 3 & -2 & 6 \\ 0 & 0 & -2 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$
$\rho(A) = 1$	$\rho([A,B]) = 2$	

$$\therefore \rho([A,\ B]) = 2, \qquad \rho\ (A) = 1$$

$$\rho\ (A) \neq \rho([A, B])$$

$\therefore$ The given system is inconsistent and has no solution.

Example 1.12

Show that the equations $2x + y + z = 5,\ x + y + z = 4,\ x - y + 2z = 1$ are consistent and hence solve them.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix}$$

$$A \qquad X \quad = \quad B$$

Augmented matrix $[A,B]$	**Elementary Transformation**
$\begin{pmatrix} 2 & 1 & 1 & 5 \\ 1 & 1 & 1 & 4 \\ 1 & -1 & 2 & 1 \end{pmatrix}$	
$\sim\begin{pmatrix} 1 & 1 & 1 & 4 \\ 2 & 1 & 1 & 5 \\ 1 & -1 & 2 & 1 \end{pmatrix}$	$R_1 \leftrightarrow R_2$

$\sim\begin{pmatrix}1 & 1 & 1 & 4\\0 & -1 & -1 & -3\\0 & -2 & 1 & -3\end{pmatrix}$	$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - R_1$
$\sim\begin{pmatrix}1 & 1 & 1 & 4\\0 & -1 & -1 & -3\\0 & 0 & 3 & 3\end{pmatrix}$	$R_3 \to R_3 - 2R_2$
$\rho(A)=3,\ \ \rho([A,B])=3$	

Obviously the last equivalent matrix is in the echelon form. It has three non-zero rows.

$\rho(A) = \rho([A,B]) = 3 = \text{Number of unknowns}$.

The given system is consistent and has unique solution.

To find the solution, let us rewrite the above echelon form into the matrix form.

$$\begin{pmatrix}1 & 1 & 1\\0 & -1 & -1\\0 & 0 & 3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4\\-3\\3\end{pmatrix}$$

$$x+y+z=4 \qquad (1)$$
$$y+z=3 \qquad (2)$$
$$3z=3 \qquad (3)$$

$(3) \Rightarrow\ z=1$

$(2) \Rightarrow\ y=3-z=2$

$(1) \Rightarrow x=4-y-z$

$x=1$

$\therefore\ x=1,\quad y=2,\quad z=1$

Example 1.13

Show that the equations $x+y+z=6$, $x+2y+3z=14$, $x+4y+7z=30$ are consistent and solve them.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 4 & 7\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}6\\14\\30\end{pmatrix}$$

$$A \qquad X \quad = \quad B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 4 & 16 \end{pmatrix}$	$R_2 \to R_2 - R_1$ $R_3 \to R_3 - R_2$
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - 2R_2$
$\rho(A)=2,\ \rho([A,B])=2$	

Obviously the last equivalent matrix is in the echelon form. It has two non-zero rows.

$$\therefore \qquad \rho([A,B])=2,\ \rho(A)=2$$

$$\rho(A)=\rho([A,\ B])=2<\text{Number of unknowns.}$$

The given system is consistent and has infinitely many solutions.

The given system is equivalent to the matrix equation,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix}$$

$$x+y+z=6 \qquad (1)$$

$$y+2z=8 \qquad (2)$$

$$(2) \Rightarrow\ y=8-2z,$$

$$(1) \Rightarrow\ x=6-y-z=6-(8-2z)-z=z-2$$

Let us take $z=k, k\in R$, we get $x=k-2, y=8-2k$, Thus by giving different values for k we get different solutions. Hence the given system has infinitely many solutions.

Example 1.14

Show that the equations $x-4y+7z=14,\ 3x+8y-2z=13,\ 7x-8y+26z=5$ are inconsistent.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 5 \end{pmatrix}$$

$$A \qquad X = B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 1 & -4 & 7 & 14 \\ 3 & 8 & -2 & 13 \\ 7 & -8 & 26 & 5 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 20 & -23 & -93 \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 7R_1$
$\sim \begin{pmatrix} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 0 & 0 & 64 \end{pmatrix}$	$R_3 \to R_3 - R_2$
$\rho(A) = 2, \ \rho([A,B]) = 3$	

The last equivalent matrix is in the echelon form. $[A, B]$ has 3 non-zero rows and $[A]$ has 2 non-zero rows.

$$\therefore \quad \rho([A, B]) = 3, \qquad \rho(A) = 2$$

$$\rho(A) \neq \rho([A, B])$$

The system is inconsistent and has no solution.

Example 1.15

Find k, if the equations $x + 2y - 3z = -2,\ 3x - y - 2z = 1,\ 2x + 3y - 5z = k$ are consistent.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & -2 \\ 2 & 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ k \end{pmatrix}$$

$$A \qquad X = B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 1 & 2 & -3 & -2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & k \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & -3 & -2 \\ 0 & -7 & 7 & 7 \\ 0 & -1 & 1 & 4+k \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 2 & -3 & -2 \\ 0 & -7 & 7 & 7 \\ 0 & 0 & 0 & 21+7k \end{pmatrix}$	$R_3 \to 7R_3 - R_2$
$\rho(A)=2,\ \rho([A,\ B])=2$ or 3	

For the equations to be consistent, $\rho([A,\ B]) = \rho(A) = 2$

$$\therefore \quad 21+7k=0$$
$$7k=-21.$$
$$k=-3$$

Example 1.16

Find k, if the equations $x+y+z=7$, $x+2y+3z=18$, $y+kz=6$ are inconsistent.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & k \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 18 \\ 6 \end{pmatrix}$$
$$A \qquad X \quad = \quad B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 7 \\ 1 & 2 & 3 & 18 \\ 0 & 1 & k & 6 \end{pmatrix}$	

$\sim\begin{pmatrix}1&1&1&7\\0&1&2&11\\0&1&k&6\end{pmatrix}$	$R_2 \to R_2 - R_1$
$\sim\begin{pmatrix}1&1&1&7\\0&1&2&11\\0&0&k-2&-5\end{pmatrix}$	$R_3 \to R_3 - R_2$

For the equations to be inconsistent

$$\rho([A, B]) \neq \rho(A)$$

It is possible if $k-2=0$.

$$\therefore \boxed{k=2}$$

Example 1.17

Investigate for what values of 'a' and 'b' the following system of equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+az=b$ have

(i) no solution (ii) a unique solution (iii) an infinite number of solutions.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix}1&1&1\\1&2&3\\1&2&a\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}6\\10\\b\end{pmatrix}$$

$$A \quad X = B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix}1&1&1&6\\1&2&3&10\\1&2&a&b\end{pmatrix}$	
$\sim\begin{pmatrix}1&1&1&6\\0&1&2&4\\0&1&a-1&b-6\end{pmatrix}$	$R_2 \to R_2 - R_1$
$\sim\begin{pmatrix}1&1&1&6\\0&1&2&4\\0&0&a-3&b-10\end{pmatrix}$	$R_3 \to R_3 - R_1$

Case (i) For no solution:

The system possesses no solution only when $\rho(A) \neq \rho([A,B])$ which is possible only when $a-3=0$ and $b-10 \neq 0$

Hence for $a=3,\ b \neq 10$, the system possesses no solution.

Case (ii) For a unique solution:

The system possesses a unique solution only when $\rho(A)=\rho([A,B])=$ number of unknowns.

i.e when $\rho(A)=\rho([A,B])=3$

Which is possible only when $a-3 \neq 0$ and b may be any real number as we can observe .

Hence for $a \neq 3$ and $b \in R$, the system possesses a unique solution.

Case (iii) For an infinite number of solutions:

The system possesses an infinite number of solutions only when

$\rho(A)=\rho([A,B]) <$ number of unknowns

i,e when $\rho(A)=\rho([A,B])=2<3$ (number of unknowns) which is possible only when $a-3=0,\ \ b-10=0$

Hence for $a=3,\ b=10$, the system possesses infinite number of solutions.

Example 1.18

The total number of units produced (P) is a linear function of amount of over times in labour (in hours) (l), amount of additional machine time (m) and fixed finishing time (a)

i.e, $P = a + bl + cm$

From the data given below, find the values of constants a, b and c

Day	Production (in Units P)	Labour (in Hrs l)	Additional Machine Time (in Hrs m)
Monday	6,950	40	10
Tuesday	6,725	35	9
Wednesday	7,100	40	12

Estimate the production when overtime in labour is 50 hrs and additional machine time is 15 hrs.

Solution:

We have, $P = a + bl + cm$

Putting above values we have

$6{,}950 = a + 40b + 10c$

$6{,}725 = a + 35b + 9c$

$7{,}100 = a + 40b + 12c$

The Matrix equation corresponding to the given system is

$$\underset{A}{\begin{pmatrix} 1 & 40 & 10 \\ 1 & 35 & 9 \\ 1 & 40 & 12 \end{pmatrix}} \underset{X}{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} \underset{=}{=} \underset{B}{\begin{pmatrix} 6950 \\ 6725 \\ 7100 \end{pmatrix}}$$

Augmented matrix $[A,B]$	**Elementary Transformation**
$\begin{pmatrix} 1 & 40 & 10 & 6950 \\ 1 & 35 & 9 & 6725 \\ 1 & 40 & 12 & 7100 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 40 & 10 & 6950 \\ 0 & -5 & -1 & -225 \\ 0 & 0 & 2 & 150 \end{pmatrix}$	$R_2 \to R_2 - R_1$ $R_3 \to R_3 - R_1$
$\rho(A)=3, \rho([A,B])=3$	

$\therefore$ The given system is equivalent to the matrix equation

$$\begin{pmatrix} 1 & 40 & 10 \\ 0 & -5 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 6950 \\ -225 \\ 150 \end{pmatrix}$$

$$a + 40b + 10c = 6950 \qquad (1)$$

$$-5b - c = -225 \qquad (2)$$

$$2c = 150 \qquad (3)$$

$$\boxed{c = 75}$$

Now, $(2) \Rightarrow -5b - 75 = -225$

$$\boxed{b = 30}$$

and $(1) \Rightarrow a + 1200 + 750 = 6950$

$$a = 5000$$

$a = 5000, \ b = 30, \ c = 75$

$\therefore$ The production equation is $P = 5000 + 30l + 75m$

$\therefore$ $P_{\text{at } l = 50, \ m=15} = 5000 + 30(50) + 75(15)$

$= 7625$ units.

$\therefore$ The production = 7,625 units.

Exercise 1.1

1. Find the rank of each of the following matrices

i) $\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ ii) $\begin{pmatrix} 1 & -1 \\ 3 & -6 \end{pmatrix}$ iii) $\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$

iv) $\begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix}$ v) $\begin{pmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{pmatrix}$ vi) $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

vii) $\begin{pmatrix} 3 & 1 & -5 & -1 \\ 1 & -2 & 1 & -5 \\ 1 & 5 & -7 & 2 \end{pmatrix}$ viii) $\begin{pmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{pmatrix}$

2. If $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$, then find the rank of AB and the rank of BA.

3. Solve the following system of equations by rank method
$$x + y + z = 9, \ 2x + 5y + 7z = 52, \ 2x - y - z = 0$$

4. Show that the equations $5x + 3y + 7z = 4, \ 3x + 26y + 2z = 9, \ 7x + 2y + 10z = 5$ are consistent and solve them by rank method.

5. Show that the following system of equations have unique solution:
$$x + y + z = 3, x + 2y + 3z = 4, \ x + 4y + 9z = 6 \text{ by rank method.}$$

6. For what values of the parameter λ, will the following equations fail to have unique solution: $3x - y + \lambda z = 1$, $2x + y + z = 2$, $x + 2y - \lambda z = -1$ by rank method.

7. The price of three commodities X, Y and Z are x, y and z respectively Mr.Anand purchases 6 units of Z and sells 2 units of X and 3 units of Y. Mr.Amar purchases a unit of Y and sells 3 units of X and 2units of Z. Mr.Amit purchases a unit of X and sells 3 units of Y and a unit of Z. In the process they earn ₹5,000/-, ₹2,000/- and ₹5,500/- respectively Find the prices per unit of three commodities by rank method.

8. An amount of ₹5,000/- is to be deposited in three different bonds bearing 6%, 7% and 8% per year respectively. Total annual income is ₹358/-. If the income from first two investments is ₹70/- more than the income from the third, then find the amount of investment in each bond by rank method.

1.2 Cramer's Rule

Gabriel Cramer, a Swiss mathematician born in the city Geneva in 31 July 1704. He edited the works of the two elder Bernoulli's, and wrote on the physical cause of the spheriodal shape of the planets and the motion of their apsides (1730), and on Newton's treatment of cubic curves (1746).

In 1750 he published Cramer's Rule, giving a general formula for the solution of certain linear system of n equations in n unknowns having a unique solution in terms of determinants. Advantages of Cramer's rule is that we can find the value of x, y or z without knowing any of the other values of x, y or z. Cramer's rule is applicable only when $\Delta \neq 0$ (Δ is the determinant value of the coefficient matrix) for unique solution.

Theorem (without proof) Cramer's Rule:

Consider, $$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If $AX = B$ is a system of n linear equations in 'n' unknowns such that det $(A) \neq 0$, then the system has a unique solution.

This solution is, $$x_1 = \frac{\det(A_1)}{\det A}, \quad x_2 = \frac{\det(A_2)}{\det A}, \ldots\ldots, \quad x_n = \frac{\det(A_n)}{\det A}$$

where A_j is the matrix obtained by replacing the entries in the j^{th} column of A by the entries in the matrix

$$B = \begin{bmatrix} b_1 \\ b_2 \\ : \\ b_n \end{bmatrix}$$

1.2.1 Non Homogeneous linear equations upto three variables.

(a) Consider the system of two linear equations with two unknowns.

$$a_1x + b_1y = d_1$$

$$a_2x + b_2y = d_2$$

Let $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ $\quad \Delta_x = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}$ $\quad \Delta_y = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$

The solution of unknown by Cramer's rule is

$$x = \frac{\Delta x}{\Delta}, \qquad y = \frac{\Delta y}{\Delta} \qquad \text{provided } \Delta \neq 0$$

(b) Consider the system of three linear equations with three unknowns

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ $\quad \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Solution of unknown by Cramer Rule is $x = \dfrac{\Delta x}{\Delta},\ y = \dfrac{\Delta y}{\Delta}, z = \dfrac{\Delta z}{\Delta}$

Example 1.19

Solve the equations $2x + 3y = 7$, $3x + 5y = 9$ by Cramer's rule.

Solution:

The equations are

$$2x + 3y = 7$$

$$3x + 5y = 9$$

$$\text{Here} \quad \Delta = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 1$$

$$\neq 0$$

$\therefore$ we can apply Cramer's Rule

$$\text{Now} \quad \Delta_x = \begin{vmatrix} 7 & 3 \\ 9 & 5 \end{vmatrix} = 8 \qquad \Delta_y = \begin{vmatrix} 2 & 7 \\ 3 & 9 \end{vmatrix} = -3$$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta x}{\Delta} = \frac{8}{1} = 8 \qquad y = \frac{\Delta y}{\Delta} = \frac{-3}{1} = -3$$

$\therefore$ Solution is $x = 8, y = -3$

Example 1.20

The following table represents the number of shares of two companies A and B during the month of January and February and it also gives the amount in rupees invested by Ravi during these two months for the purchase of shares of two companies. Find the the price per share of A and B purchased during both the months

Months	Number of Shares of the company		Amount invested by Ravi (in ₹)
	A	B	
January	10	5	125
February	9	12	150

Solution:

Let the price of one share of A be x

Let the price of one share of B be y

$\therefore$ By given data, we get the following equations

$$10x + 5y = 125$$

$$9x + 12y = 150$$

$$\Delta = \begin{vmatrix} 10 & 5 \\ 9 & 12 \end{vmatrix} = 75 \neq 0 \qquad \Delta_x = \begin{vmatrix} 125 & 5 \\ 150 & 12 \end{vmatrix} = 750 \qquad \Delta_y = \begin{vmatrix} 10 & 125 \\ 9 & 150 \end{vmatrix} = 375$$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta x}{\Delta} = \frac{750}{75} = 10 \qquad y = \frac{\Delta y}{\Delta} = \frac{375}{75} = 5$$

The price of the share A is ₹10 and the price of the share B is ₹5.

Example 1.21

The total cost of 11 pencils and 3 erasers is ₹ 64 and the total cost of 8 pencils and 3 erasers is ₹49. Find the cost of each pencil and each eraser by Cramer's rule.

Solution:

Let 'x' be the cost of a pencil

Let 'y' be the cost of an eraser

$\therefore$ By given data, we get the following equations

$$11x+3y=64$$
$$8x+3y=49$$

$$\Delta=\begin{vmatrix}11 & 3\\ 8 & 3\end{vmatrix}=9\neq 0.\ \text{It has unique solution.}$$

$$\Delta_x=\begin{vmatrix}64 & 3\\ 49 & 3\end{vmatrix}=45 \qquad \Delta_y=\begin{vmatrix}11 & 64\\ 8 & 49\end{vmatrix}=27$$

$\therefore$ By Cramer's rule

$$x=\frac{\Delta x}{\Delta}=\frac{45}{9}=5$$
$$y=\frac{\Delta y}{\Delta}=\frac{27}{9}=3$$

$\therefore$ The cost of a pencil is ₹ 5 and the cost of an eraser is ₹ 3.

Example 1.22

Solve by Cramer's rule $x+y+z=4,\ 2x-y+3z=1,\ 3x+2y-z=1$

Solution:

Here $\Delta=\begin{vmatrix}1 & 1 & 1\\ 2 & -1 & 3\\ 3 & 2 & -1\end{vmatrix}=13\neq 0$

$\therefore$ We can apply Cramer's Rule and the system is consistent and it has unique solution.

> **DO YOU KNOW?**
>
> If $|A|=0$, then the system of equations has either no solution or infinitely many solutions.

$$\Delta_x=\begin{vmatrix}4 & 1 & 1\\ 1 & -1 & 3\\ 1 & 2 & -1\end{vmatrix}=-13 \qquad \Delta_y=\begin{vmatrix}1 & 4 & 1\\ 2 & 1 & 3\\ 3 & 1 & -1\end{vmatrix}=39 \qquad \Delta_z=\begin{vmatrix}1 & 1 & 4\\ 2 & -1 & 1\\ 3 & 2 & 1\end{vmatrix}=26$$

$\therefore$ By Cramer's rule

$$x=\frac{\Delta x}{\Delta}=\frac{-13}{13}=-1 \qquad y=\frac{\Delta y}{\Delta}=\frac{39}{13}=3 \qquad z=\frac{\Delta z}{\Delta}=\frac{26}{13}=2$$

$\therefore$ The solution is $(x,y,z)=(-1,3,2)$

Example 1.23

The price of 3 Business Mathematics books, 2 Accountancy books and one Commerce book is ₹840. The price of 2 Business Mathematics books, one Accountancy book and one Commerce book is ₹570. The price of one Business Mathematics book, one Accountancy book and 2 Commerce books is ₹630. Find the cost of each book by using Cramer's rule.

Solution:

Let 'x' be the cost of a Business Mathematics book

Let 'y' be the cost of a Accountancy book.

Let 'z' be the cost of a Commerce book.

$$\therefore \quad 3x+2y+z=840$$
$$2x+y+z=570$$
$$x+y+2z=630$$

Here $\Delta = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2 \neq 0$ $\qquad \Delta_x = \begin{vmatrix} 840 & 2 & 1 \\ 570 & 1 & 1 \\ 630 & 1 & 2 \end{vmatrix} = -240$

$\Delta_y = \begin{vmatrix} 3 & 840 & 1 \\ 2 & 570 & 1 \\ 1 & 630 & 2 \end{vmatrix} = -300$ $\qquad \Delta_z = \begin{vmatrix} 3 & 2 & 840 \\ 2 & 1 & 570 \\ 1 & 1 & 630 \end{vmatrix} = -360$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta x}{\Delta} = \frac{-240}{-2} = 120 \qquad y = \frac{\Delta y}{\Delta} = \frac{-300}{-2} = 150 \qquad z = \frac{\Delta z}{\Delta} = \frac{-360}{-2} = 180$$

$\therefore$ The cost of a Business Mathematics book is ₹120,

the cost of a Accountancy book is ₹150 and

the cost of a Commerce book is ₹180.

Example 1.24

An automobile company uses three types of Steel S_1, S_2 and S_3 for providing three different types of Cars C_1, C_2 and C_3. Steel requirement R (in tonnes) for each type of car and total available steel of all the three types are summarized in the following table.

Types of Steel	Types of Car			Total Steel available
	C_1	C_2	C_3	
S_1	3	2	4	28
S_2	1	1	2	13
S_3	2	2	1	14

Determine the number of Cars of each type which can be produced by Cramer's rule.

Solution:

Let 'x' be the number of cars of type C_1

Let 'y' be the number of cars of type C_2

Let 'z' be the number of cars of type C_3

$$3x+2y+4z=28$$
$$x+y+2z=13$$
$$2x+2y+z=14$$

Here $\Delta=\begin{vmatrix} 3 & 2 & 4 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}=-3\neq 0$ $\qquad$ $\Delta_x=\begin{vmatrix} 28 & 2 & 4 \\ 13 & 1 & 2 \\ 14 & 2 & 1 \end{vmatrix}=-6$

$\Delta_y=\begin{vmatrix} 3 & 28 & 4 \\ 1 & 13 & 2 \\ 2 & 14 & 1 \end{vmatrix}=-9$ $\qquad$ $\Delta_z=\begin{vmatrix} 3 & 2 & 28 \\ 1 & 1 & 13 \\ 2 & 2 & 14 \end{vmatrix}=-12$

$\therefore$ By Cramer's rule

$$x=\frac{\Delta x}{\Delta}=\frac{-6}{-3}=2 \qquad y=\frac{\Delta y}{\Delta}=\frac{-9}{-3}=3 \qquad z=\frac{\Delta z}{\Delta}=\frac{-12}{-3}=4$$

$\therefore$ The number of cars of each type which can be produced are 2, 3 and 4.

Exercise 1.2

1. Solve the following equations by using Cramer's rule

   (i) $2x+3y=7;\quad 3x+5y=9$

   (ii) $5x+3y=17;\quad 3x+7y=31$

   (iii) $2x+y-z=3,\quad x+y+z=1,\quad x-2y-3z=4$

   (iv) $x+y+z=6,\quad 2x+3y-z=5,\quad 6x-2y-3z=-7$

   (v) $x+4y+3z=2,\quad 2x-6y+6z=-3,\quad 5x-2y+3z=-5$

2. A commodity was produced by using 3 units of labour and 2 units of capital, the total cost is ₹62. If the commodity had been produced by using 4 units of labour and one unit of capital, the cost is ₹56. What is the cost per unit of labour and capital? (Use determinant method).

3. A total of ₹8,600 was invested in two accounts. One account earned $4\frac{3}{4}\%$ annual interest and the other earned $6\frac{1}{2}\%$ annual interest. If the total interest for one year was ₹431.25, how much was invested in each account? (Use determinant method).

4. At marina two types of games viz., Horse riding and Quad Bikes riding are available on hourly rent. Keren and Benita spent ₹780 and ₹560 during the month of May.

Name	Number of hours		Total amount spent (in ₹)
	Horse Riding	Quad Bike Riding	
Keren	3	4	780
Benita	2	3	560

Find the hourly charges for the two games (rides). (Use determinant method).

5. In a market survey three commodities A, B and C were considered. In finding out the index number some fixed weights were assigned to the three varieties in each of the commodities. The table below provides the information regarding the consumption of three commodities according to the three varieties and also the total weight received by the commodity

Commodity Variety	Variety			Total weight
	I	II	III	
A	1	2	3	11
B	2	4	5	21
C	3	5	6	27

Find the weights assigned to the three varieties by using Cramer's Rule.

6. A total of ₹8,500 was invested in three interest earning accounts. The interest rates were 2%, 3% and 6% if the total simple interest for one year was ₹380 and the amount invested at 6% was equal to the sum of the amounts in the other two accounts, then how much was invested in each account? (use Cramer's rule).

1.3 Transition Probability Matrices

1.3.1 Forecasting the succeeding state when the initial market share is given

One stage Transition Probability

The Occurrence of an event at a specified point in time, put the system in state S_n; if after the passage of one unit of time, another event occurs, that is the system moved from the state S_n to S_{n+1}. This movement is related to a probability distribution, there is a probability associated with each (move) transition from event S_n to S_{n+1}. This probability distribution is called one stage transition probability.

Transition Matrix

The transition Probabilities P_{jk} satisfy $P_{jk} > 0$, $\sum_{k} P_{jk} = 1$ for all j

These probabilities may be written in the matrix form

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots \\ P_{21} & P_{22} & P_{23} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

This is called the transition probability matrix

Example 1.25

Consider the matrix of transition probabilities of a product available in the market in two brands A and B.

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix} \end{array}$$

Determine the market share of each brand in equilibrium position.

Solution:

Transition probability matrix

$$T = \begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix} \end{array}$$

At equilibrium, $\begin{pmatrix} A & B \end{pmatrix} T = \begin{pmatrix} A & B \end{pmatrix}$ where $A + B = 1$

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix} = \begin{pmatrix} A & B \end{pmatrix}$$

$$\begin{aligned} 0.9A + 0.3B &= A \\ 0.9A + 0.3(1 - A) &= A \\ 0.9A - 0.3A + 0.3 &= A \\ 0.6A + 0.3 &= A \\ 0.4A &= 0.3 \\ A &= \frac{0.3}{0.4} = \frac{3}{4} \end{aligned}$$

$$B = 1-\frac{3}{4}=\frac{1}{4}$$

Hence the market share of brand A is 75% and the market share of brand B is 25%

Example 1.26

Parithi is either sad (S) or happy (H) each day. If he is happy in one day, he is sad on the next day by four times out of five. If he is sad on one day, he is happy on the next day by two times out of three. Over a long run, what are the chances that Parithi is happy on any given day?

Solution:

The transition porbability matrix is $T=\begin{pmatrix}\frac{4}{5} & \frac{1}{5}\\ \frac{2}{3} & \frac{1}{3}\end{pmatrix}$

At equilibrium, $\begin{pmatrix}S & H\end{pmatrix}\begin{pmatrix}\frac{4}{5} & \frac{1}{5}\\ \frac{2}{3} & \frac{1}{3}\end{pmatrix} = \begin{pmatrix}S & H\end{pmatrix}$ where $S+H=1$

$$\frac{4}{5}S+\frac{2}{3}H = S$$

$$\frac{4}{5}S+\frac{2}{3}(1-S) = S$$

On solving this, we get

$$S = \frac{10}{13} \text{ and } H = \frac{3}{13}$$

In the long run, on a randomly selected day, his chances of being happy is $\frac{10}{13}$.

Example 1.27

Akash bats according to the following traits. If he makes a hit (S), there is a 25% chance that he will make a hit his next time at bat. If he fails to hit (F), there is a 35% chance that he will make a hit his next time at bat. Find the transition probability matrix for the data and determine Akash's long- range batting average.

Solution:

The Transition probability matrix is $T = \begin{pmatrix}0.25 & 0.75\\ 0.35 & 0.65\end{pmatrix}$

At equilibrium, $\begin{pmatrix}S & F\end{pmatrix}\begin{pmatrix}0.25 & 0.75\\ 0.35 & 0.65\end{pmatrix} = \begin{pmatrix}S & F\end{pmatrix}$ where $S+F=1$

$$0.25\,S + 0.35\,F = S$$

$$0.25\,S + 0.35\,(1-S) = S$$

On solving this, we get $S = \dfrac{0.35}{1.10}$

$$\Rightarrow \quad S = 0.318 \quad \text{and} \quad F = 0.682$$

$\therefore$ Akash's batting average is 31.8%

Example 1.28

80% of students who do maths work during one study period, will do the maths work at the next study period. 30% of students who do english work during one study period, will do the english work at the next study period.

Initially there were 60 students do maths work and 40 students do english work. Calculate,

(i) The transition probability matrix

(ii) The number of students who do maths work, english work for the next subsequent 2 study periods.

Solution

(i) Transition probability matrix

$$T = \begin{array}{c} \\ M \\ E \end{array}\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} \end{array}$$

After one study period,

$$\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 60 & 40 \end{pmatrix} \end{array} \begin{array}{c} \\ M \\ E \end{array}\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 76 & 24 \end{pmatrix} \end{array}$$

So in the very next study period, there will be 76 students do maths work and 24 students do the English work.

After two study periods,

$$\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 76 & 24 \end{pmatrix} \end{array} \begin{array}{c} \\ M \\ E \end{array}\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} \end{array} = \begin{pmatrix} 60.8+16.8 & 15.2+7.2 \end{pmatrix}$$

$$= \begin{pmatrix} 77.6 & 22.4 \end{pmatrix}$$

After two study periods there will be 78 (approx) students do maths work and 22 (approx) students do English work.

Aliter

$$\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 60 & 40 \end{pmatrix} \end{array} \begin{array}{c} \\ M \\ E \end{array}\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{pmatrix}^2 \end{array}$$

$$\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 60 & 40 \end{pmatrix} \end{array} \begin{array}{c} \\ M \\ E \end{array}\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 0.78 & 0.22 \\ 0.77 & 0.23 \end{pmatrix} \end{array}$$

$= (46.8 + 30.8 \quad 13.2 + 9.2)$

$= (77.6 \quad 22.4)$

Exercise 1.3

1. The subscription department of a magazine sends out a letter to a large mailing list inviting subscriptions for the magazine. Some of the people receiving this letter already subscribe to the magazine while others do not. From this mailing list, 45% of those who already subscribe will subscribe again while 30% of those who do not now subscribe will subscribe. On the last letter, it was found that 40% of those receiving it ordered a subscription. What percent of those receiving the current letter can be expected to order a subscription?

2. A new transit system has just gone into operation in Chennai. Of those who use the transit system this year, 30% will switch over to using metro train next year and 70% will continue to use the transit system. Of those who use metro train this year, 70% will continue to use metro train next year and 30% will switch over to the transit system. Suppose the population of Chennai city remains constant and that 60% of the commuters use the transit system and 40% of the commuters use metro train this year.
   (i) What percent of commuters will be using the transit system after one year?
   (ii) What percent of commuters will be using the transit system in the long run?

3. Two types of soaps A and B are in the market. Their present market shares are 15% for A and 85% for B. Of those who bought A the previous year, 65% continue to buy it again while 35% switch over to B. Of those who bought B the previous year, 55% buy it again and 45% switch over to A. Find their market shares after one year and when is the equilibrium reached?

4. Two products A and B currently share the market with shares 50% and 50% each respectively. Each week some brand switching takes place. Of those who bought A the previous week, 60% buy it again whereas 40% switch over to B. Of those who bought B the previous week, 80% buy it again where as 20% switch over to A. Find their shares after one week and after two weeks. If the price war continues, when is the equilibrium reached?

Exercise 1.4

BUTUGH

Choose the correct answer

1. If $A=(1\ \ 2\ \ 3)$, then the rank of AA^T is

   (a) 0 (b) 2 (c) 3 (d) 1

2. The rank of $m \times n$ matrix whose elements are unity is

(a) 0 (b) 1 (c) m (d) n

3. If $T = \begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{pmatrix} \end{array}$ is a transition probability matrix, then at equilibrium A is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{5}$ (c) $\frac{1}{6}$ (d) $\frac{1}{8}$

4. If $A = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$, then $\rho(A)$ is

(a) 0 (b) 1 (c) 2 (d) n

5. The rank of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ is

(a) 0 (b) 1 (c) 2 (d)3

6. The rank of the unit matrix of order n is

(a) $n-1$ (b) n (c) $n+1$ (d) n^2

7. If $\rho(A) = r$ then which of the following is correct?

(a) all the minors of order r which does not vanish

(b) A has at least one minor of order r which does not vanish

(c) A has at least one $(r+1)$ order minor which vanishes

(d) all $(r+1)$ and higher order minors should not vanish

8. If A = $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ then the rank of AA^T is

(a) 0 (b) 1 (c) 2 (d) 3

9. If the rank of the matrix $\begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{pmatrix}$ is 2. Then λ is

(a) 1 (b) 2 (c) 3 (d) only real number

10. The rank of the diagonal matrix $\begin{pmatrix} 1 & & & & & \\ & 2 & & & & \\ & & -3 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$

(a) 0 (b) 2 (c) 3 (d) 5

11. If $T = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & x \end{pmatrix} \end{array}$ is a transition probability matrix, then the value of x is

(a) 0.2 (b) 0.3 (c) 0.4 (d) 0.7

12. Which of the following is not an elementary transformation?

(a) $R_i \leftrightarrow R_j$ (b) $R_i \rightarrow 2R_i + 2C_j$

(c) $R_i \rightarrow 2R_i - 4R_j$ (d) $C_i \rightarrow C_i + 5C_j$

13. If $\rho(A) = \rho(A, B)$ then the system is

(a) Consistent and has infinitely many solutions

(b) Consistent and has a unique solution

(c) Consistent (d) inconsistent

14. If $\rho(A) = \rho(A, B) =$ the number of unknowns, then the system is

(a) Consistent and has infinitely many solutions

(b) Consistent and has a unique solution

(c) inconsistent (d) consistent

15. If $\rho(A) \neq \rho(A, B)$, then the system is

(a) Consistent and has infinitely many solutions

(b) Consistent and has a unique solution

(c) inconsistent (d) consistent

16. In a transition probability matrix, all the entries are greater than or equal to

(a) 2 (b) 1 (c) 0 (d) 3

17. If the number of variables in a non- homogeneous system $AX = B$ is n, then the system possesses a unique solution only when

(a) $\rho(A) = \rho(A,B) > n$ (b) $\rho(A) = \rho(A,B) = n$

(c) $\rho(A) = \rho(A,B) < n$ (d) none of these

18. The system of equations $4x+6y=5, 6x+9y=7$ has

(a) a unique solution (b) no solution

(c) infinitely many solutions (d) none of these

19. For the system of equations $x+2y+3z=1, \quad 2x+y+3z=2 \; 5x+5y+9z=4$

(a) there is only one solution

(b) there exists infinitely many solutions

(c) there is no solution (d) None of these

20. If $|A| \neq 0$, then A is

(a) non- singular matrix (b) singular matrix

(c) zero matrix (d) none of these

21. The system of linear equations $x+y+z=2, \; 2x+y-z=3, \; 3x+2y+k=4$ has unique solution, if k is not equal to

(a) 4 (b) 0 (c) −4 (d) 1

22. Cramer's rule is applicable only to get an unique solution when

(a) $\Delta_z \neq 0$ (b) $\Delta_x \neq 0$ (c) $\Delta \neq 0$ (d) $\Delta_y \neq 0$

23. If $\frac{a_1}{x}+\frac{b_1}{y}=c_1, \; \frac{a_2}{x}+\frac{b_2}{y}=c_2, \; \Delta_1=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}; \; \Delta_2=\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \; \Delta_3=\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$ then (x,y) is

(a) $\left(\frac{\Delta_2}{\Delta_1},\frac{\Delta_3}{\Delta_1}\right)$ (b) $\left(\frac{\Delta_3}{\Delta_1},\frac{\Delta_2}{\Delta_1}\right)$ (c) $\left(\frac{\Delta_1}{\Delta_2},\frac{\Delta_1}{\Delta_3}\right)$ (d) $\left(\frac{-\Delta_1}{\Delta_2},\frac{-\Delta_1}{\Delta_3}\right)$

24. $|A_{n\times n}| = 3 \quad |adjA| = 243$ then the value n is

(a) 4 (b) 5 (c) 6 (d) 7

25. Rank of a null matrix is

(a) 0 (b) −1 (c) ∞ (d) 1

Miscellaneous problems

1. Find the rank of the matrix $A=\begin{pmatrix}1 & -3 & 4 & 7\\ 9 & 1 & 2 & 0\end{pmatrix}$.

2. Find the rank of the matrix $A=\begin{pmatrix}-2 & 1 & 3 & 4\\ 0 & 1 & 1 & 2\\ 1 & 3 & 4 & 7\end{pmatrix}$.

3. Find the rank of the matrix $A=\begin{pmatrix}4 & 5 & 2 & 2\\ 3 & 2 & 1 & 6\\ 4 & 4 & 8 & 0\end{pmatrix}$.

4. Examine the consistency of the system of equations:
$x+y+z=7$, $x+2y+3z=18$, $y+2z=6$.

5. Find k if the equations $2x+3y-z=5$, $3x-y+4z=2$, $x+7y-6z=k$ are consistent.

6. Find k if the equations $x+y+z=1$, $3x-y-z=4$, $x+5y+5z=k$ are inconsistent.

7. Solve the equations $x+2y+z=7$, $2x-y+2z=4$, $x+y-2z=-1$ by using Cramer's rule

8. The cost of 2kg. of wheat and 1kg. of sugar is ₹100. The cost of 1kg. of wheat and 1kg. of rice is ₹80. The cost of 3kg. of wheat, 2kg. of sugar and 1kg of rice is ₹220. Find the cost of each per kg., using Cramer's rule.

9. A salesman has the following record of sales during three months for three items A,B and C, which have different rates of commission.

Months	Sales of units			Total commission drawn (in ₹)
	A	B	C	
January	90	100	20	800
February	130	50	40	900
March	60	100	30	850

Find out the rate of commission on the items A,B and C by using Cramer's rule.

10. The subscription department of a magazine sends out a letter to a large mailing list inviting subscriptions for the magazine. Some of the people receiving this letter already subscribe to the magazine while others do not. From this mailing list, 60% of those who already subscribe will subscribe again while 25% of those who do not now subscribe will subscribe. On the last letter it was found that 40% of those receiving it ordered a subscription. What percent of those receiving the current letter can be expected to order a subscription?

Summary

In this chapter we have acquired the knowledge of

- **Rank of a matrix**

  The rank of a matrix A is the order of the largest non-zero minor of A
- The rank of a matrix A is the order of the largest non-zero minor of A
- $\rho(A) \geq 0$
- If A is a matrix of order $m \times n$, then $\rho(A) \leq$ minimum of $\{m, n\}$
- The rank of a zero matrix is '0'
- The rank of a non- singular matrix of order $n \times n$ is 'n'
- **Equivalent Matrices**

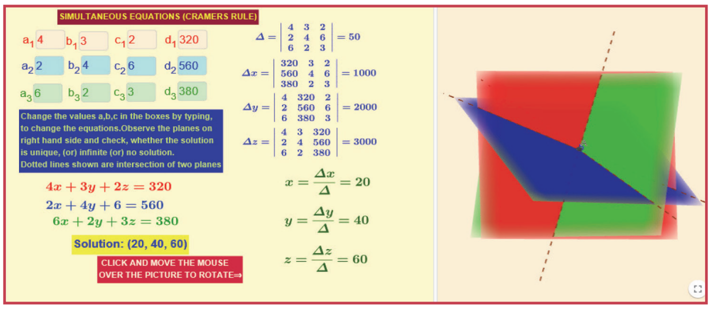
  Two Matrices A and B are said to be equivalent if one can be obtained from another by a finite number of elementary transformations and we write it as $A \sim B$.
- **Echelon form**

  A matrix of order $m \times n$ is said to be in echelon form if the row having all its entries zero will lie below the row having non-zero entry.
- A system of equations is said to be consistent if it has at least one set of solution. Otherwise, it is said to be inconsistent
  - If $\rho([A, B]) = \rho(A)$, then the equations are consistent.
  - If $\rho([A, B]) = \rho(A) = n$, then the equations are consistent and have unique solution.
  - If $\rho([A, B]) = \rho(A) < n$, then the equations are consistent and have infinitely many solutions.
  - If $\rho([A, B]) \neq \rho(A)$ then the equations are inconsistent and has no solution.
- $|adjA| = |A|^{n-1}$
- If $|A| = 0$ then A is a singular matrix. Otherwise, A is a non singular matrix.
- In $AX = B$ if $|A| \neq 0$ then the system is consistent and it has unique solution.
- Cramer's rule is applicable only when $\Delta \neq 0$.

GLOSSARY	
Matrix equation	அணி சமன்பாடு
Transition	நிலை மாற்றம்
Commodities	பொருட்கள்
Consistent	ஒருங்கமைவு
Inconsistent	ஒருங்கமைவு அற்ற
Rank	தரம்
Production	உற்பத்தி
Augmented	மிகுதிப்படுத்திய,விரிவுபடுத்தப்பட்ட
Echelon form	ஏறுபடி வடிவம்
Unknown	தெரியாத
Transit system	போக்குவரத்து அமைப்பு
Intersecting	வெட்டிக்கொள்ளும்
Singular matrix	ஒருமை அணி
Elementary Transformations	அடிப்படை உருமாற்றங்கள்
Linear equations	நேரிய சமன்பாடுகள்
Equilibrium	சமநிலை
Commuters	பயணிகள்
Subsequent	தொடர்ச்சியான
Determinant	அணிக்கோவை
Singular matrix	பூஜ்ஜியக்கோவை அணி
Non- Singular matrix	பூஜ்ஜியமற்ற கோவை அணி
Square matrix	சதுர அணி
Matrix	அணி
Order	வரிசை
Consistent	ஒருங்கமைவுள்ள
Variables	மாறிகள்
Non homogeneous	அசமப்படித்தான
Homogenerous	சமப்படித்தான
Unique solution	ஒரே ஒரு தீர்வு
Trivial solution	வெளிப்படைத்தீர்வு
Non trivial solution	வெளிப்படையற்ற தீர்வு

ICT Corner

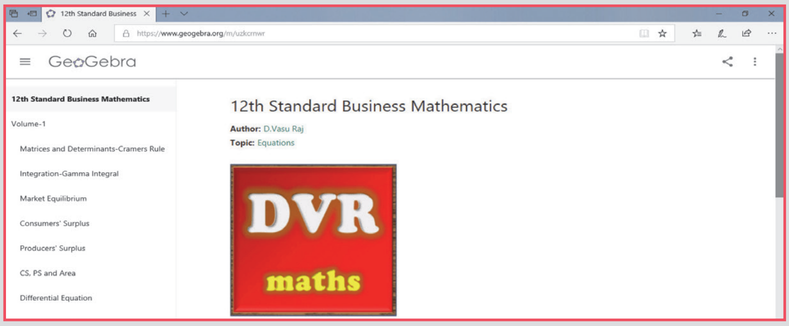
Expected Result is shown in this picture



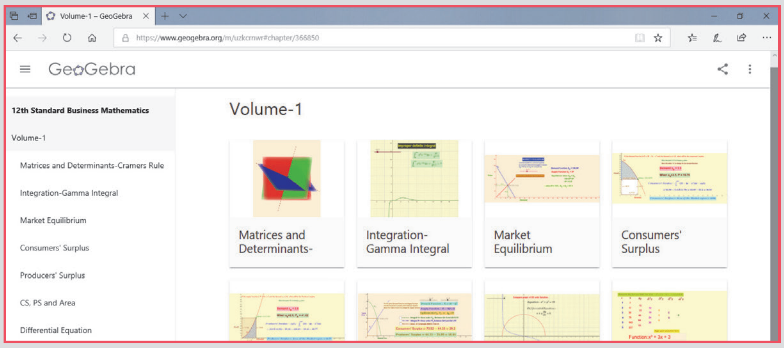
Step – 1 : Open the Browser, type the URL Link given below (or) Scan the QR Code. GeoGebra work Book named "12th Standard Business Mathematics " will open. In the work book there are two Volumes. Select "Volume-1".

Step - 2 : Select the worksheet named"Matrices and Determinants-Cramer's rule" . There is a question based on Cramer's rule.Change the coefficients of x, y and z by typing the no.s in the respective boxes. Move the 3D picture and see the nature of solution.

Step 1



Step 2



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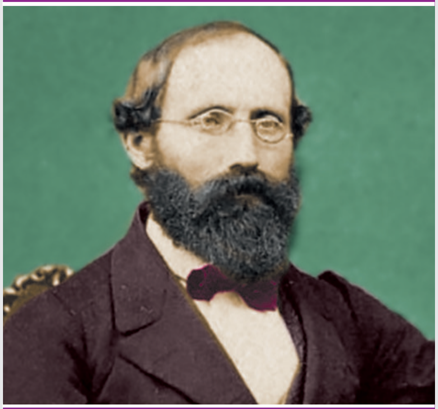
2 Integral Calculus – I



Bernhard Riemann
(17^{th} September 1826 - 20^{th} July 1866)

Introduction

Georg Friedrich Bernhard Riemann (Nineteenth century) was an inspiring German mathematician. He was very much recognised for his contribution in calculus.

Calculus divides naturally into two parts, namely (i) differential calculus and (ii) integral calculus. Differential calculus deals with the derivatives of a function whereas, integral calculus deals with the anti derivative of the derived function that is, finding a function when its rate of change / marginal function is known. So integration is the technique to find the original function from the derived function, the function obtained is called the indefinite integral. Definite integral is the evaluation of the indefinite integral between the specified limits, and is equal to the area bounded by the graph of the function (curve) between the specified limits and the axis. The area under the curve is approximately equal to the area obtained by summing the area of the number of inscribed rectangles and the approximation becomes exact in the limit that the number of rectangles approaches infinity. Therefore both differential and integral calculus are based on the theory of limits.

The word 'integrate' literally means that 'to find the sum'. So, we believe that the name "Integral Calculus" has its origin from this process of summation. Calculus is the mathematical tool used to test theories about the origins of the universe, the development of tornadoes and hurricanes. It is also used to find the surplus of consumer and producer, identifying the probability density function of a continuous random variable, obtain an original function from its marginal function and etc., in business applications.

In this chapter, we will study about the concept of integral and some types of method of indefinite and definite integrals.

Learning Objectives

After studying this chapter, the students will be able to understand

- the indefinite integral.
- how to find the indefinite integral of a function involving sum, difference and constant multiples.
- how to use and where to apply the substitution technique in indefinite integrals.
- the techniques involved in integration by parts and some special type of integrals.
- the fundamental theorems of integral calculus.
- the properties of definite integral and its applications.
- the application of a particular case of gamma integral.
- the properties of gamma function.
- the evaluation of the definite integral as the limit of a sum.

2.1 Indefinite Integrals

2.1.1 Concept of Indefinite Integral

In differential calculus, we have learned how to calculate the differential coefficient $f'(x)$ of a given function $f(x)$ with respect to x. In this chapter, we have to find out the primitive function $f(x)$ (i.e. original function) whenever its derived function $f'(x)$ (i.e. derivative of a function) is given, such process is called integration or anti differentiation.

$\therefore$ Integration is the reverse process of differentiation

We know that $\frac{d}{dx}(\sin x) = \cos x$. Here $\cos x$ is known as **Derived function**, and $\sin x$ is known as **Primitive function** [also called as Anti derivative function (or) Integral function].

Definition 2.1

A function $F(x)$ is said to be a primitive function of the derived function $f(x)$, if $\frac{d}{dx}[F(x)] = f(x)$

Now, consider the following examples which are already known to us.

$\frac{d}{dx}\left(x^3\right) = 3x^2$, $\quad \frac{d}{dx}\left(x^3+5\right) = 3x^2$, $\quad \frac{d}{dx}\left(x^3-\frac{3}{2}\right) = 3x^2$,

$\frac{d}{dx}\left(x^3+e\right) = 3x^2$, $\quad \frac{d}{dx}\left(x^3-\pi\right) = 3x^2$, $\quad \ldots$

From the above examples, we observe that $3x^2$ is the derived function of the primitive functions x^3, x^3+5, $x^3-\frac{3}{2}$, x^3+e, $x^3-\pi$, ... and which indicates that the primitive functions are need not be unique, even though the derived function is unique. So we come to a conclusion that x^3+c is the primitive function of the derived function $3x^2$.

$\therefore$ For every derived function, there are infinitely many primitives by choosing c arbitrarily from the set of real numbers $\mathbb{R}$. So we called these integrals as indefinite integrals.

> In general, $\frac{d}{dx}\left[F(x)\right]=f(x) \;\Rightarrow\; \int f(x)\,dx = F(x)+c,$
>
> where c is called the constant of integration.

Remarks

- If two different primitive functions $F(x)$ and $G(x)$ have the same derived function $f(x)$, then they differ only by a constant term.
- $\int f(x)\,dx$ is called as indefinite integral.
- The symbol looks like an elongated S [$\int$], which stands for "summation" is the sign of integration.
- $\int f(x)\,dx$ is read as integral of $f(x)$ with respect to x.
- $f(x)$ in $\int f(x)\,dx$ [i.e. the function to be integrated] is called as integrand.
- x in $\int f(x)\,dx$ is the variable of integration.
- The term integration means the process of finding the integral.
- The term constant of integration means any real number c, considered as a constant function.

Definition 2.2

The process of determining an integral of a given function is defined as integration of a function.

2.1.2 Two important properties of Integral Calculus

(i) if k is any constant, then $\int k\,f(x)\,dx = k\int f(x)\,dx$

(ii) if $f(x)$ and $g(x)$ are any two functions, then

$$\int\left[f(x)\pm g(x)\right]dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Methods of Integration

The following are the four principal methods of integration.

(i) Integration by decomposition.

(ii) Integration by parts.

(iii) Integration by substitution.

(iv) Integration by successive reduction.

Remember !

Integrate a function $f(x)$ means, finding a function $F(x)$ such that

$$\frac{d}{dx}\left[F(x)+c\right]=f(x)$$

Note

Here, we discuss only the first three methods of integration, because the method of integration by successive reduction is beyond the scope of the syllabus.

2.1.3 Integration by decomposition

Apply this method, whenever the integrand is impossible to integrate directly, but it can be integrated after decomposing into a sum or difference of integrands whose individual integrals are already known.

Note

The integrals which are directly obtained from their corresponding derivatives are known as the standard results of integration.

Some standard results of integration

Type: I

(i) $\int x^n\,dx=\dfrac{x^{n+1}}{n+1}+c,\quad n\neq -1$

(ii) $\int (ax+b)^n\,dx=\dfrac{(ax+b)^{n+1}}{a(n+1)}+c,\quad n\neq -1$

DO YOU KNOW?

$y=\int f(x)\;dx\;=F(x)+c$ denotes family of curves having parallel tangents at $x=k$

Example 2.1

Evaluate $\displaystyle\int \frac{ax^2+bx+c}{\sqrt{x}}dx$

Solution:

$$\int \frac{ax^2+bx+c}{\sqrt{x}}dx\;=\;\int\left(ax^{\frac{3}{2}}+bx^{\frac{1}{2}}+cx^{-\frac{1}{2}}\right)dx$$

DO YOU KNOW?

$\int dx=x+c$

$$= a\int x^{\frac{3}{2}}dx + b\int x^{\frac{1}{2}}dx + c\int x^{-\frac{1}{2}}dx$$

$$= \frac{2ax^{\frac{5}{2}}}{5} + \frac{2bx^{\frac{3}{2}}}{3} + 2c\,x^{\frac{1}{2}} + k$$

Example 2.2

Evaluate $\int \sqrt{2x+1}\,dx$

Solution:

$$\int \sqrt{2x+1}\,dx = \int (2x+1)^{\frac{1}{2}}dx$$

$$= \frac{(2x+1)^{\frac{3}{2}}}{3} + c$$

DO YOU KNOW?

$$\int a(ax+b)^n\,dx = \int (ax+b)^n\,d(ax+b)$$

$$= \frac{(ax+b)^{n+1}}{n+1} + c$$

Example 2.3

Evaluate $\int \frac{dx}{(2x+3)^2}$

Solution:

$$\int \frac{dx}{(2x+3)^2} = \int (2x+3)^{-2}dx$$

$$= -\frac{1}{2(2x+3)} + c$$

DO YOU KNOW?

(i) $\int \frac{1}{x^n}dx = -\frac{1}{(n-1)x^{n-1}} + c\ ,\ n \neq 1$

(ii) $\int \frac{a}{(ax+b)^n}dx = \int \frac{d(ax+b)}{(ax+b)^n}$

$$= -\frac{1}{(n-1)(ax+b)^{n-1}} + c\ ,\ n \neq 1$$

Example 2.4

Evaluate $\int \left(x + \frac{1}{x}\right)^2 dx$

Solution:

$$\int \left(x+\frac{1}{x}\right)^2 dx = \int \left(x^2 + 2 + \frac{1}{x^2}\right)dx$$

$$= \int x^2 dx + 2\int dx + \int \frac{1}{x^2}dx$$

$$= \frac{x^3}{3} + 2x - \frac{1}{x} + c$$

Example 2.5

Evaluate $\int (x^3+7)(x\ -4)\,dx$

Solution:

$$\int (x^3+7)(x-4)\,dx = \int (x^4-4x^3+7x-28)\,dx$$

$$= \frac{x^5}{5} - x^4 + \frac{7x^2}{2} - 28x + c$$

Example 2.6

Evaluate $\int \frac{2x^2-14x+24}{x-3}\,dx$

Solution:

$$\int \frac{2x^2-14x+24}{x-3}\,dx = \int \frac{(x-3)(2x-8)}{x-3}\,dx$$

$$= \int (2x-8)\,dx$$

$$= x^2 - 8x + c$$

By factorisation,

$2x^2-14x+24 = (x-3)(2x-8)$

Example 2.7

Evaluate $\int \frac{x+2}{\sqrt{2x+3}}\,dx$

Solution:

$$\int \frac{x+2}{\sqrt{2x+3}}\,dx = \int \frac{1}{2}\left\{(2x+3)^{\frac{1}{2}} + (2x+3)^{-\frac{1}{2}}\right\}dx$$

$$= \frac{1}{2}\left\{\frac{(2x+3)^{\frac{3}{2}}}{3} + (2x+3)^{\frac{1}{2}}\right\} + c$$

Split into simple integrands

$$\frac{x+2}{\sqrt{2x+3}} = \frac{\frac{1}{2}(2x+4)}{(2x+3)^{\frac{1}{2}}}$$

$$= \frac{1}{2}\left\{\frac{(2x+3)+1}{(2x+3)^{\frac{1}{2}}}\right\}$$

$$= \frac{1}{2}\left\{(2x+3)^{\frac{1}{2}} + (2x+3)^{-\frac{1}{2}}\right\}$$

Example 2.8

Evaluate $\int \frac{1}{\sqrt{x+2}-\sqrt{x-2}}\,dx$

Solution:

$$\int \frac{1}{\sqrt{x+2}-\sqrt{x-2}}\,dx$$

$$= \int \frac{\sqrt{x+2}+\sqrt{x-2}}{4}\,dx$$

$$= \frac{1}{6}\left\{(x+2)^{\frac{3}{2}} + (x-2)^{\frac{3}{2}}\right\} + c$$

By rationalisation,

$$\frac{1}{\sqrt{x+2}-\sqrt{x-2}} = \frac{1}{\sqrt{x+2}-\sqrt{x-2}} \times \frac{\sqrt{x+2}+\sqrt{x-2}}{\sqrt{x+2}+\sqrt{x-2}}$$

$$= \frac{\sqrt{x+2}+\sqrt{x-2}}{4}$$

Exercise 2.1

Integrate the following with respect to x.

1. $\sqrt{3x+5}$
2. $\left(9x^2-\frac{4}{x^2}\right)^2$
3. $(3+x)(2-5x)$
4. $\sqrt{x}\left(x^3-2x+3\right)$
5. $\frac{8x+13}{\sqrt{4x+7}}$
6. $\frac{1}{\sqrt{x+1}+\sqrt{x-1}}$
7. If $f'(x)=x+b, f(1)=5$ and $f(2)=13$, then find $f(x)$
8. If $f'(x)=8x^3-2x$ and $f(2)=8$, then find $f(x)$

Type: II

(i) $\int \frac{1}{x}dx = \log|x|+c$

(ii) $\int \frac{1}{ax+b}dx = \frac{1}{a}\log|ax+b|+c$

Example 2.9

Evaluate $\int \frac{3x^2+2x+1}{x}dx$

Solution:

$$\int \frac{3x^2+2x+1}{x}dx = \int\left(3x+2+\frac{1}{x}\right)dx$$

$$= \frac{3x^2}{2}+2x+\log|x|+c$$

Example 2.10

Evaluate $\int \frac{2}{3x+5}dx$

Solution:

$$\int \frac{2}{3x+5}dx = 2\int \frac{1}{3x+5}dx$$

$$= \frac{2}{3}\log|3x+5|+c$$

DO YOU KNOW?

$$\int \frac{a}{ax+b}dx = \int \frac{d(ax+b)}{ax+b}$$

$$= \log|ax+b|+c$$

Example 2.11

Evaluate $\int \frac{x^2+2x+3}{x+1}dx$

Solution:

$$\int \frac{x^2+2x+3}{x+1}\,dx = \int\left\{(x+1)+\frac{2}{x+1}\right\}dx$$

$$= \frac{x^2}{2} + x + 2\log|x+1| + c$$

Split into simple integrands

$$\frac{x^2+2x+3}{x+1} = \frac{(x^2+2x+1)+2}{x+1}$$

$$= (x+1)+\frac{2}{x+1}$$

Example 2.12

Evaluate $\int \frac{x^3+5x^2-9}{x+2}\,dx$

Solution:

$$\int \frac{x^3+5x^2-9}{x+2}\,dx = \int\left[x^2+3x-6+\frac{3}{x+2}\right]dx$$

$$= \frac{x^3}{3} + \frac{3x^2}{2} - 6x + 3\log|x+2| + c$$

By simple division,

$$\frac{x^3+5x^2-9}{x+2} = x^2+3x-6+\frac{3}{x+2}$$

Example 2.13

Evaluate $\int \frac{7x-1}{x^2-5x+6}\,dx$

Solution:

$$\int \frac{7x-1}{x^2-5x+6}\,dx = \int\left[\frac{20}{x-3}-\frac{13}{x-2}\right]dx$$

$$= 20\int\frac{dx}{x-3} - 13\int\frac{dx}{x-2}$$

$$= 20\log|x-3| - 13\log|x-2| + c$$

By partial fractions,

$$\frac{7x-1}{x^2-5x+6} = \frac{A}{x-3}+\frac{B}{x-2}$$

$$\Rightarrow \frac{7x-1}{x^2-5x+6} = \frac{20}{x-3}-\frac{13}{x-2}$$

Example 2.14

Evaluate $\int \frac{3x+2}{(x-2)^2(x-3)}\,dx$

Solution:

By partial fractions,

$$\frac{3x+2}{(x-2)^2(x-3)} = \frac{A}{(x-2)}+\frac{B}{(x-2)^2}+\frac{C}{(x-3)}$$

$$\Rightarrow \frac{3x+2}{(x-2)^2(x-3)} = -\frac{11}{(x-2)}-\frac{8}{(x-2)^2}+\frac{11}{(x-3)}$$

$$\int \frac{3x+2}{(x-2)^2(x-3)}\,dx$$

$$= \int\left[-\frac{11}{(x-2)}-\frac{8}{(x-2)^2}+\frac{11}{(x-3)}\right]dx$$

$$= -11\int\frac{dx}{(x-2)} - 8\int\frac{dx}{(x-2)^2} + 11\int\frac{dx}{(x-3)}$$

$$= -11\log|x-2| + \frac{8}{x-2} + 11\log|x-3| + c = 11\log\left|\frac{x-3}{x-2}\right| + \frac{8}{x-2} + c$$

Example 2.15

Evaluate $\displaystyle\int \frac{3x^2+6x+1}{(x+3)(x^2+1)}dx$

Solution:

$$\int \frac{3x^2+6x+1}{(x+3)(x^2+1)}dx = \int\left[\frac{1}{(x+3)}+\frac{2x}{(x^2+1)}\right]dx$$
$$= \int\frac{dx}{(x+3)}+\int\frac{2x}{(x^2+1)}dx$$
$$= \log|x+3|+\log|x^2+1|+c$$
$$= \log|(x+3)(x^2+1)|+c$$
$$= \log|x^3+3x^2+x+3|+c$$

By partial fractions,
$$\frac{3x^2+6x+1}{(x+3)(x^2+1)} = \frac{A}{(x+3)}+\frac{Bx+C}{(x^2+1)}$$
$$\Rightarrow \frac{3x^2+6x+1}{(x+3)(x^2+1)} = \frac{1}{(x+3)}+\frac{2x}{(x^2+1)}$$

Exercise 2.2

Integrate the following with respect to x.

1. $\left(\sqrt{2x}-\dfrac{1}{\sqrt{2x}}\right)^2$
2. $\dfrac{x^4-x^2+2}{x-1}$
3. $\dfrac{x^3}{x+2}$
4. $\dfrac{x^3+3x^2-7x+11}{x+5}$
5. $\dfrac{3x+2}{(x-2)(x-3)}$
6. $\dfrac{4x^2+2x+6}{(x+1)^2(x-3)}$
7. $\dfrac{3x^2-2x+5}{(x-1)(x^2+5)}$
8. If $f'(x)=\dfrac{1}{x}$ and $f(1)=\dfrac{\pi}{4}$, then find $f(x)$

Type: III

(i) $\int e^x\,dx = e^x+c$

(ii) $\int e^{ax+b}dx = \dfrac{1}{a}e^{ax+b}+c$

(iii) $\int a^x\,dx = \dfrac{1}{\log a}a^x+c$, $a>0$ and $a\neq 1$

(iv) $\int a^{mx+n}dx = \dfrac{1}{m\log a}a^{mx+n}+c$, $a>0$ and $a\neq 1$

Example 2.16

Evaluate $\int 3^{2x+3}\,dx$

Solution:

$$\int 3^{2x+3}\,dx = \int 3^{2x}\cdot 3^3\,dx$$

Do You Know?

$$\int m a^{mx+n}\,dx = \int a^{mx+n}\,d(mx+n)$$
$$= \frac{1}{\log a}a^{mx+n}+c,$$
$a>0$ and $a\neq 1$

$$= 3^3\int 3^{2x}\,dx$$

$$= 27\frac{3^{2x}}{2\log 3} + c$$

Example 2.17

Evaluate $\int \frac{e^x + 7}{e^x}\,dx$

Solution:

$$\int \frac{e^x+7}{e^x}\,dx = \int\left(1+7e^{-x}\right)dx$$
$$= x - 7e^{-x} + c$$

Example 2.18

Evaluate $\int \frac{5+5e^{2x}}{e^x+e^{-x}}\,dx$

Solution:

$$\int \frac{5+5e^{2x}}{e^x+e^{-x}}\,dx = 5\int \frac{e^x\left(e^{-x}+e^x\right)}{e^x+e^{-x}}\,dx$$
$$= 5\int e^x\,dx$$
$$= 5e^x + c$$

Example 2.19

Evaluate $\int\left(e^x + \frac{1}{e^x}\right)^2 dx$

Solution:

$$\int\left(e^x+\frac{1}{e^x}\right)^2 dx = \int\left(e^{2x}+\frac{1}{e^{2x}}+2\right)dx$$
$$= \int\left(e^{2x}+e^{-2x}+2\right)dx$$
$$= \frac{e^{2x}}{2} - \frac{e^{-2x}}{2} + 2x + c$$

DO YOU KNOW?

$$\int (2ax+b)e^{ax^2+bx+c}\,dx$$
$$= \int e^{ax^2+bx+c}\,d\left(ax^2+bx+c\right)$$
$$= e^{ax^2+bx+c} + k$$

Exercise 2.3

Integrate the following with respect to x.

1. $e^{x\log a} + e^{a\log a} - e^{n\log x}$
2. $\frac{a^x - e^{x\log b}}{e^{x\log a}\ b^x}$
3. $\left(e^x+1\right)^2 e^x$

4. $\dfrac{e^{3x}-e^{-3x}}{e^x}$

5. $\dfrac{e^{3x}+e^{5x}}{e^x+e^{-x}}$

6. $\left[1-\frac{1}{x^2}\right]e^{\left(x+\frac{1}{x}\right)}$

7. $\dfrac{1}{x(\log x)^2}$

8. If $f'(x)=e^x$ and $f(0)=2$, then find $f(x)$

Type: IV

(i) $\int \sin x\,dx = -\cos x + c$

(ii) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+c$

(iii) $\int \cos x\,dx = \sin x + c$

(iv) $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c$

(v) $\int \sec^2 x\,dx = \tan x + c$

(vi) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)+c$

(vi) $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

(viii) $\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b)+c$

Example 2.20

Evaluate $\int(2\sin x - 5\cos x)\,dx$

Solution:

$$\int(2\sin x - 5\cos x)\,dx = 2\int \sin x\,dx - 5\int \cos x\,dx$$
$$= -2\cos x - 5\sin x + c$$

Example 2.21

Evaluate $\int \sin^2 x\,dx$

Solution:

$$\int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx$$
$$= \frac{1}{2}\left[\int dx - \int \cos 2x\,dx\right]$$
$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]+c$$

Change into simple integrands

$$\cos 2x = 1-2\sin^2 x$$
$$\therefore \sin^2 x = \frac{1}{2}(1-\cos 2x)$$

Example 2.22

Evaluate $\int \dfrac{\cos 2x}{\sin^2 x\cos^2 x}\,dx$

Solution:

$$\int \frac{\cos 2x}{\sin^2 x\cos^2 x}\,dx = \int(\operatorname{cosec}^2 x - \sec^2 x)\,dx$$
$$= -\cot x - \tan x + c$$

Change into simple integrands

$$\frac{\cos 2x}{\sin^2 x\cos^2 x} = \frac{\cos^2 x - \sin^2 x}{\sin^2 x\cos^2 x}$$
$$= \frac{1}{\sin^2 x} - \frac{1}{\cos^2 x}$$
$$= \operatorname{cosec}^2 x - \sec^2 x$$

Example 2.23

Evaluate $\int \sqrt{1+\sin 2x}\, dx$

Solution:

$$\int \sqrt{1+\sin 2x}\, dx = \int \sqrt{(\sin x+\cos x)^2}\, dx$$
$$= \int (\sin x+\cos x) dx$$
$$= -\cos x + \sin x + c$$

> Change into simple integrands
>
> $1+\sin 2x = \sin^2 x+\cos^2 x+2\sin x\cos x$
>
> $= (\sin x+\cos x)^2$

Example 2.24

Evaluate $\int \cos^3 x\, dx$

Solution:

$$\int \cos^3 x\, dx = \frac{1}{4}\int \cos 3x\, dx + \frac{3}{4}\int \cos x\, dx$$
$$= \frac{\sin 3x}{12} + \frac{3\sin x}{4} + c$$

> Change into simple integrands
>
> $\cos 3x = 4\cos^3 x - 3\cos x$
>
> $\cos^3 x = \frac{1}{4}[\cos 3x + 3\cos x]$
>
> $= \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$

Exercise 2.4

Integrate the following with respect to x.

1. $2\cos x - 3\sin x + 4\sec^2 x - 5\operatorname{cosec}^2 x$
2. $\sin^3 x$
3. $\dfrac{\cos 2x + 2\sin^2 x}{\cos^2 x}$
4. $\dfrac{1}{\sin^2 x\cos^2 x}$ $\left[\text{Hint}: \sin^2 x+\cos^2 x = 1\right]$
5. $\sqrt{1-\sin 2x}$

2.1.4 Integration by parts

Type: V

(1) We know that, if u and v are two differentiable functions of x,

then $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

Integrate on both sides with respect to x.

$$\int \frac{d}{dx}(uv)dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$
$$\Rightarrow \quad \int d(uv) = \int u\,dv + \int v\,du$$
$$uv = \int udv + \int vdu$$
$$\therefore \int udv = uv - \int vdu$$

This method is very useful when the integrand is a product of two different types of functions or a function which is not directly integrable. The success of this method depends on the proper choice of u. So we can choose the function u by using the following guidelines.

(i) If the integrand contains only a function which is directly not integrable, then take this as u.

(ii) If the integrand contains both directly integrable and non integrable functions, then take non integrable function as u.

(iii) If the integrand contains both the functions are integrable and one of them is of the form x^n, n is a positive integer, then take this x^n as u.

(iv) for all other cases, the choice of u is ours.

(Or) we can also choose u as the function which comes first in the word "I L A T E" Where,

I stands for the inverse trigonometric function

L stands for the logarithmic function

A stands for the algebraic function

T stands for the trigonometric function

E stands for the exponential function and

take the remaining part of the function and dx as dv.

(2) If u and v are functions of x, then $\int u dv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots$

where $u', u'', u''', \ldots$ are the successive derivatives of u and $v_1, v_2, v_3, \ldots$ are the repeated integrals of v.

Note

- The above mentioned formula is well known as Bernoulli's formula.
- Bernoulli's formula is applied when $u = x^n$ where n is a positive integer.

Example 2.25

Evaluate $\int xe^x \, dx$

Solution:

$$\int xe^x \, dx = \int u dv$$

$$= uv - \int v\,du$$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x + c$$

$$= e^x (x-1) + c$$

Take $u = x$	and $dv = e^x dx$
Differentiate	Integrate
$du = dx$	$v = e^x$

Example 2.26

Evaluate $\int x^3 e^x dx$

Solution:

$$\int x^3 e^x dx = \int u\,dv$$

$$= uv - u'v_1 + u''v_2 - u'''v_3 + \ldots$$

$$= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + c$$

$$= e^x \left(x^3 - 3x^2 + 6x - 6\right) + c$$

Successive derivatives	Repeated integrals
	and $dv = e^x dx$
Take $u = x^3$	$v = e^x$
$u' = 3x^2$	$v_1 = e^x$
$u'' = 6x$	$v_2 = e^x$
$u''' = 6$	$v_3 = e^x$

Example 2.27

Evaluate $\int x^3 \log x \; dx$

Solution:

$$\int x^3 \log x \; dx = \int u\,dv$$

$$= uv - \int v\,du$$

$$= \frac{x^4}{4} \log x - \frac{1}{4} \int x^3 dx$$

$$= \frac{x^4}{4} \log x - \frac{1}{4} \left(\frac{x^4}{4}\right) + c$$

$$= \frac{x^4}{4} \left[\log x - \frac{1}{4}\right] + c$$

Take $u = \log x$	and $dv = x^3 dx$
Differentiate	Integrate
$du = \frac{1}{x} dx$	$v = \frac{x^4}{4}$

Example 2.28

Evaluate $\int (\log x)^2 \; dx$

Solution:

Take $u = (\log x)^2$	and $dv = dx$
Differentiate	Integrate
$du = (2\log x)\left(\frac{1}{x}dx\right)$	$v = x$

$$\int(\log x)^2\,dx = \int udv$$
$$= uv - \int vdu$$
$$= x(\log x)^2 - 2\int \log x\,dx \ldots (*)$$
$$= x(\log x)^2 - 2\int udv$$
$$= x(\log x)^2 - 2\left[uv - \int vdu\right]$$
$$= x(\log x)^2 - 2\left[x\log x - \int dx\right]$$
$$= x(\log x)^2 - 2x\log x + 2x + c$$
$$= x\left[(\log x)^2 - \log x^2 + 2\right] + c$$

For $\int \log x\,dx$ in (*)	
Take $u = (\log x)$	and $dv = dx$
Differentiate	Integrate
$du = \frac{1}{x}dx$	$v = x$

Example 2.29

Evaluate $\int(x^2 - 2x + 5)e^{-x}dx$

Solution:

Successive derivatives	Repeated integrals
	and $dv = e^{-x}dx$
Take $u = x^2 - 2x + 5$	$v = -e^{-x}$
$u' = 2x - 2$	$v_1 = e^{-x}$
$u'' = 2$	$v_2 = -e^{-x}$

$$\int(x^2 - 2x + 5)e^{-x}dx = \int udv$$
$$= uv - u'v_1 + u''v_2 - u'''v_3 + \ldots$$
$$= (x^2 - 2x + 5)(-e^{-x}) - (2x - 2)e^{-x} + 2(-e^{-x}) + c$$
$$= e^{-x}(-x^2 - 5) + c$$

Exercise 2.5

Integrate the following with respect to x.

1. xe^{-x}
2. x^3e^{3x}
3. $\log x$
4. $x\log x$
5. $x^n\log x$
6. $x^5e^{x^2}$

2.1.5 Integration by substitution (or) change of variable method

Integrals of certain functions cannot be obtained directly, because they are not in any one of the standard forms as discussed above, but may be reduced to a standard form by suitable substitution. The method of evaluating an integral by reducing it to a standard form by suitable substitution is called integration by substitution.

Type: VI

1. $\int [f(x)]^n f'(x)\,dx = \dfrac{[f(x)]^{n+1}}{n+1} + c, n \neq -1$

2. $\int \dfrac{f'(x)}{f(x)}\,dx = \log|f(x)| + c$

3. $\int \dfrac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$

4. $\int e^x [f(x) + f'(x)]\,dx = e^x f(x) + c$

5. $\int e^{ax} [a f(x) + f'(x)]\,dx = e^{ax} f(x) + c$

Example 2.30

Evaluate $\int \dfrac{x}{x^2+1}\,dx$

Solution:

Take $f(x) = x^2 + 1$

$\therefore \quad f'(x) = 2x$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$$
$$= \frac{1}{2}\int \frac{f'(x)}{f(x)}\,dx$$
$$= \frac{1}{2}\log[f(x)] + c$$
$$= \frac{1}{2}\log|x^2+1| + c$$

DO YOU KNOW?

$$\int \frac{nax^{n-1}}{ax^n+b}\,dx = \int \frac{d(ax^n+b)}{ax^n+b}$$
$$= \log|ax^n+b| + c$$

Example 2.31

Evaluate $\int \dfrac{x}{\sqrt{x^2+1}}\,dx$

Solution:

Take $f(x) = x^2 + 1$

$\therefore \quad f'(x) = 2x$

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2+1}}\,dx$$
$$= \frac{1}{2}\int \frac{f'(x)}{\sqrt{f(x)}}\,dx$$
$$= \frac{1}{2}\left[2\sqrt{f(x)}\right] + c$$
$$= \sqrt{x^2+1} + c$$

DO YOU KNOW?

$$\int \frac{nax^{n-1}}{\sqrt{ax^n+b}}\,dx = \int \frac{d(ax^n+b)}{\sqrt{ax^n+b}}$$
$$= 2\sqrt{ax^n+b} + c$$

Example 2.32

Evaluate $\int x\sqrt{x^2+1}\; dx$

Solution:

$$\int x\sqrt{x^2+1}\; dx = \frac{1}{2}\int \left(x^2+1\right)^{\frac{1}{2}} (2x)\, dx$$

$$= \frac{1}{2}\int \left[f(x)\right]^{\frac{1}{2}} f'(x)\, dx$$

$$= \frac{1}{2}\frac{\left[f(x)\right]^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{1}{3}\left(x^2+1\right)^{\frac{3}{2}} + c$$

Take $f(x) = x^2+1$

$\therefore \quad f'(x) = 2x$

Example 2.33

Evaluate $\int \frac{x^3}{\left(x^2+1\right)^3}\, dx$

Solution:

$$\int \frac{x^3}{\left(x^2+1\right)^3}\, dx = \int \frac{x^2}{\left(x^2+1\right)^3}\, x\, dx$$

$$= \frac{1}{2}\int \frac{z-1}{z^3}\, dz$$

$$= \frac{1}{2}\int \left[\frac{1}{z^2} - \frac{1}{z^3}\right] dz$$

$$= \frac{1}{2}\left[-\frac{1}{z} + \frac{1}{2z^2}\right] + c$$

$$= \frac{1}{4\left(x^2+1\right)^2} - \frac{1}{2\left(x^2+1\right)} + c$$

Take $z = x^2+1$

$\therefore x^2 = z-1$ *and*

$dz = 2x\, dx$

$\Rightarrow \frac{dz}{2} = x\, dx$

Example 2.34

Evaluate $\int \frac{dx}{x\left(x^3+1\right)}$

Solution:

$$\int \frac{dx}{x\left(x^3+1\right)} = \int \frac{x^2}{x^3\left(x^3+1\right)}dx$$

$$= \frac{1}{3}\int \frac{dz}{z\left(z+1\right)}dz$$

$$= \frac{1}{3}\int \left[\frac{1}{z} - \frac{1}{z+1}\right]dz$$

$$= \frac{1}{3}\left[\log\left|z\right| - \log\left|z+1\right|\right] + c$$

$$= \frac{1}{3}\log\left|\frac{z}{z+1}\right| + c$$

$$= \frac{1}{3}\log\left|\frac{x^3}{x^3+1}\right| + c$$

Take $z = x^3$	and by partial fractions,
$\therefore\ dz = 3x^2\,dx$	$\frac{1}{z(z+1)} = \frac{A}{z} + \frac{B}{z+1}$
$\Rightarrow \frac{dz}{3} = x^2\,dx$	$\Rightarrow \frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1}$

Example 2.35

Evaluate $\int x^3 e^{x^2}\,dx$

Solution:

> Take $z = x^2$
>
> $\therefore\ dz = 2x\,dx$
>
> $\Rightarrow \frac{dz}{2} = x\,dx$

$$\int x^3 e^{x^2}\,dx = \int x^2 e^{x^2}\,(x\,dx)$$

$$= \frac{1}{2}\int z\,e^z\,dz$$

$$= \frac{1}{2}\left[e^z\,(z-1)\right] + c \qquad \text{[By applying Integration by parts]}$$

$$= \frac{1}{2}\left[e^{x^2}\,(x^2-1)\right] + c$$

Example 2.36

Evaluate $\int e^x\left(x^2+2x\right)dx$

Solution:

> Take $f(x) = x^2$
>
> $\therefore\ f'(x) = 2x$

$$\int e^x\left(x^2+2x\right)dx = \int e^x\left[f(x) + f'(x)\right]dx$$

$$= e^x f(x) + c$$

$$= e^x x^2 + c$$

Example 2.37

Evaluate $\int \frac{xe^x}{(1+x)^2}dx$

Solution:

$$\int \frac{xe^x}{(1+x)^2}dx$$

$$= \int e^x\left[\frac{1}{1+x} + \frac{-1}{(1+x)^2}\right]dx$$

$$= \int e^x\left[f(x) + f'(x)\right]dx$$

$$= e^x f(x) + c$$

$$= \frac{e^x}{1+x} + c$$

By partial fractions,	and Take
$\frac{x}{(1+x)^2} = \frac{A}{1+x} + \frac{B}{(1+x)^2}$ $\Rightarrow \frac{x}{(1+x)^2} = \frac{1}{1+x} + \frac{-1}{(1+x)^2}$	$f(x) = \frac{1}{1+x}$ $\therefore \quad f'(x) = \frac{-1}{(1+x)^2}$

Example 2.38

Evaluate $\int e^{2x}\left[\frac{2x-1}{4x^2}\right]dx$

Solution:

$$\int e^{2x}\left[\frac{2x-1}{4x^2}\right]dx$$

$$= \frac{1}{4}\int e^{2x}\left[2\left(\frac{1}{x}\right) + \frac{-1}{x^2}\right]dx$$

$$= \frac{1}{4}\int e^{ax}\left[a\,f(x) + f'(x)\right]dx$$

$$= \frac{1}{4}\left[e^{ax}\ f(x)\right] + c$$

$$= \frac{1}{4x}\,e^{2x} + c$$

	and Take $a = 2$,
Here, $\frac{2x-1}{4x^2} = \frac{1}{2x} + \frac{-1}{4x^2}$ $= \frac{1}{4}\left[2\left(\frac{1}{x}\right) + \frac{-1}{x^2}\right]$	$f(x) = \frac{1}{x}$ $\therefore f'(x) = -\frac{1}{x^2}$

Example 2.39

Evaluate $\int\left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right]dx$

Solution:

$$\int\left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right]dx \quad = \int\left[\frac{1}{z} - \frac{1}{z^2}\right]e^z dz$$

$$= \int e^z \left[f(z) + f'(z) \right] dz$$

$$= e^z f(z) + c$$

$$= e^z \left[\frac{1}{z} \right] + c$$

$$= \frac{x}{\log x} + c$$

Take $z = \log x$

$\therefore dz = \frac{1}{x} dx$

$\Rightarrow dx = e^z dz \quad \left[\because x = e^z \right]$

and $f(z) = \frac{1}{z}$

$\therefore f'(z) = -\frac{1}{z^2}$

Exercise 2.6

Integrate the following with respect to x.

1. $\dfrac{2x+5}{x^2+5x-7}$
2. $\dfrac{e^{3\log x}}{x^4+1}$
3. $\dfrac{e^{2x}}{e^{2x}-2}$
4. $\dfrac{(\log x)^3}{x}$
5. $\dfrac{6x+7}{\sqrt{3x^2+7x-1}}$
6. $(4x+2)\sqrt{x^2+x+1}$
7. $x^8\left(1+x^9\right)^5$
8. $\dfrac{x^{e-1}+e^{x-1}}{x^e+e^x}$
9. $\dfrac{1}{x\log x}$
10. $\dfrac{x}{2x^4-3x^2-2}$
11. $e^x(1+x)\log\left(xe^x\right)$
12. $\dfrac{1}{x\left(x^2+1\right)}$
13. $e^x\left[\dfrac{1}{x^2}-\dfrac{2}{x^3}\right]$
14. $e^x\left[\dfrac{x-1}{(x+1)^3}\right]$
15. $e^{3x}\left[\dfrac{3x-1}{9x^2}\right]$

2.1.6 Some special types of Integrals

Type: VII

To evaluate the integrals of the form $\int \frac{dx}{ax^2+bx+c}$, $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ and $\int \sqrt{ax^2+bx+c}\, dx$, first we have to express ax^2+bx+c as the sum or difference of two square terms [completing the squares], that is $(x+\alpha)^2+\beta^2$ (or) $(x+\alpha)^2-\beta^2$ (or) $\beta^2-(x+\alpha)^2$ and apply the suitable formula from the formulae given below.

1. $\displaystyle \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$
2. $\displaystyle \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$
3. $\displaystyle \int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + c$
4. $\displaystyle \int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + c$
5. $\displaystyle \int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$
6. $\displaystyle \int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$

Example 2.40

Evaluate $\int \frac{dx}{16-x^2}$

Solution:

$$\int \frac{dx}{16-x^2} = \int \frac{dx}{4^2-x^2}$$

$$= \frac{1}{2(4)} \log\left|\frac{4+x}{4-x}\right| + c$$

$$= \frac{1}{8} \log\left|\frac{4+x}{4-x}\right| + c$$

Example 2.41

Evaluate $\int \frac{dx}{1-25x^2}$

Solution:

$$\int \frac{dx}{1-25x^2} = \frac{1}{25}\int \frac{dx}{\left(\frac{1}{5}\right)^2 - x^2}$$

$$= \frac{1}{25}\left[\frac{1}{2\left(\frac{1}{5}\right)} \log\left|\frac{\frac{1}{5}+x}{\frac{1}{5}-x}\right|\right] + c$$

$$= \frac{1}{10} \log\left|\frac{1+5x}{1-5x}\right| + c$$

DO YOU KNOW?

$$\int \frac{m}{a^2-(mx)^2}\, dx = \int \frac{d(mx)}{a^2-(mx)^2}$$

$$= \frac{1}{2a} \log\left|\frac{a+mx}{a-mx}\right| + c$$

Example 2.42

Evaluate $\int \frac{dx}{2+x-x^2}$

Solution:

$$\int \frac{dx}{2+x-x^2} = \int \frac{dx}{\left(\frac{3}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\left(\frac{3}{2}\right)} \log\left|\frac{\frac{3}{2}+\left(x-\frac{1}{2}\right)}{\frac{3}{2}-\left(x-\frac{1}{2}\right)}\right| + c$$

By completing the squares

$$2+x-x^2 = 2-\left[x^2-x\right]$$

$$= 2-\left[\left(x-\frac{1}{2}\right)^2 - \frac{1}{4}\right]$$

$$= \left(\frac{3}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2$$

$$= \frac{1}{3}\log\left|\frac{2+2x}{4-2x}\right| + c$$

$$= \frac{1}{3}\log\left|\frac{1+x}{2-x}\right| + c$$

Example 2.43

Evaluate $\displaystyle\int \frac{dx}{4x^2-1}$

Solution:

$$\int \frac{dx}{4x^2-1} = \int \frac{dx}{4\left(x^2-\frac{1}{4}\right)}$$

$$= \frac{1}{4}\int \frac{dx}{x^2-\left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{4}\left[\frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{x-\frac{1}{2}}{x+\frac{1}{2}}\right|\right] + c$$

$$= \frac{1}{4}\log\left|\frac{2x-1}{2x+1}\right| + c$$

DO YOU KNOW?

$$\int \frac{m}{(mx)^2-a^2}\,dx = \int \frac{d(mx)}{(mx)^2-a^2}$$

$$= \frac{1}{2a}\log\left|\frac{mx-a}{mx+a}\right| + c$$

Example 2.44

Evaluate $\displaystyle\int \frac{x^2}{x^2-25}\,dx$

Solution:

$$\int \frac{x^2}{x^2-25}\,dx = \int \frac{\left(x^2-25\right)+25}{x^2-25}\,dx$$

$$= \int \left\{1+\frac{25}{x^2-25}\right\}dx$$

$$= \int dx + 25\int \frac{dx}{x^2-25}$$

$$= x + 25\left[\frac{1}{2(5)}\log\left|\frac{x-5}{x+5}\right|\right] + c$$

$$= x + \frac{5}{2}\log\left|\frac{x-5}{x+5}\right| + c$$

Example 2.45

Evaluate $\int \frac{dx}{x^2-3x+2}$

Solution:

$$\int \frac{dx}{x^2-3x+2} = \int \frac{dx}{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\left(\frac{1}{2}\right)} \log\left|\frac{\left(x-\frac{3}{2}\right)-\frac{1}{2}}{\left(x-\frac{3}{2}\right)+\frac{1}{2}}\right| + c$$

$$= \log\left|\frac{2x-4}{2x-2}\right| + c$$

$$= \log\left|\frac{x-2}{x-1}\right| + c$$

By completing the squares
$x^2-3x+2 = \left(x-\frac{3}{2}\right)^2-\frac{9}{4}+2$
$= \left(x-\frac{3}{2}\right)^2-\frac{1}{4}$
$= \left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$

Example 2.46

Evaluate $\int \frac{dx}{\sqrt{4x^2-9}}$

Solution:

$$\int \frac{dx}{\sqrt{4x^2-9}} = \int \frac{dx}{\sqrt{4\left[x^2-\frac{9}{4}\right]}}$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{x^2-\left(\frac{3}{2}\right)^2}}$$

$$= \frac{1}{2}\log\left|x+\sqrt{x^2-\left(\frac{3}{2}\right)^2}\right| + c$$

$$= \frac{1}{2}\log\left|x+\sqrt{x^2-\frac{9}{4}}\right| + c$$

$$= \frac{1}{2}\log\left|2x+\sqrt{4x^2-9}\right| + c$$

DO YOU KNOW?

$$\int \frac{m}{\sqrt{(mx)^2-a^2}}dx = \int \frac{d(mx)}{\sqrt{(mx)^2-a^2}}$$

$$= \log\left|mx+\sqrt{(mx)^2-a^2}\right| + c$$

Note

$m \log n \pm k = c$

Example 2.47

Evaluate $\int \frac{dx}{\sqrt{x^2-3x+2}}$

Solution:

$$\int \frac{dx}{\sqrt{x^2-3x+2}} = \int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+c$$

$$= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+c$$

By completing the squares
$x^2-3x+2=\left(x-\frac{3}{2}\right)^2-\frac{9}{4}+2$ $=\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$

Example 2.48

Evaluate $\int \frac{dx}{\sqrt{x^2+25}}$

Solution:

$$\int \frac{dx}{\sqrt{x^2+25}} = \int \frac{dx}{\sqrt{x^2+5^2}}$$

$$= \log\left|x+\sqrt{x^2+5^2}\right|+c \quad = \log\left|x+\sqrt{x^2+25}\right|+c$$

Example 2.49

Evaluate $\int \frac{dx}{\sqrt{x^2+4x+8}}$

Solution:

$$\int \frac{dx}{\sqrt{x^2+4x+8}} = \int \frac{dx}{\sqrt{(x+2)^2+2^2}}$$

$$= \log\left|(x+2)+\sqrt{(x+2)^2+2^2}\right|+c$$

$$= \log\left|(x+2)+\sqrt{x^2+4x+8}\right|+c$$

By completing the squares
$x^2+4x+8=(x+2)^2-4+8$ $=(x+2)^2+2^2$

DO YOU KNOW?

$$\int \frac{m}{\sqrt{(mx)^2+a^2}}dx = \int \frac{d(mx)}{\sqrt{(mx)^2+a^2}}$$

$$= \log\left|mx+\sqrt{(mx)^2+a^2}\right|+c$$

Example 2.50

Evaluate $\int \frac{x^3dx}{\sqrt{x^8+1}}$

Solution:

$$\int \frac{x^3dx}{\sqrt{x^8+1}} = \frac{1}{4}\int \frac{4x^3dx}{\sqrt{(x^4)^2+1^2}}$$

$$= \frac{1}{4}\log\left|x^4+\sqrt{(x^4)^2+1^2}\right|+c$$

$$= \frac{1}{4}\log\left|x^4+\sqrt{x^8+1}\right|+c$$

Example 2.51

Evaluate $\int \sqrt{x^2-16}\,dx$

Solution:

$$\int \sqrt{x^2-16}\,dx = \int \sqrt{x^2-4^2}\,dx$$

$$= \frac{x}{2}\sqrt{x^2-4^2} - \frac{4^2}{2}\log\left|x+\sqrt{x^2-4^2}\right| + c$$

$$= \frac{x}{2}\sqrt{x^2-16} - 8\log\left|x+\sqrt{x^2-16}\right| + c$$

Example 2.52

Evaluate $\int \sqrt{x^2+5}\,dx$

Solution:

$$\int \sqrt{x^2+5}\,dx = \int \sqrt{x^2+\left(\sqrt{5}\right)^2}\,dx$$

$$= \frac{x}{2}\sqrt{x^2+\left(\sqrt{5}\right)^2} + \frac{\left(\sqrt{5}\right)^2}{2}\log\left|x+\sqrt{x^2+\left(\sqrt{5}\right)^2}\right| + c$$

$$= \frac{x}{2}\sqrt{x^2+5} + \frac{5}{2}\log\left|x+\sqrt{x^2+5}\right| + c$$

Example 2.53

Evaluate $\int \sqrt{4x^2+9}\,dx$

Solution:

$$\int \sqrt{4x^2+9}\,dx$$

$$= \frac{1}{2}\int \sqrt{(2x)^2+3^2}\,d(2x)$$

$$= \frac{1}{2}\left[\frac{2x}{2}\sqrt{(2x)^2+3^2} + \frac{3^2}{2}\log\left|2x+\sqrt{(2x)^2+3^2}\right|\right] + c$$

$$= \frac{x}{2}\sqrt{4x^2+9} + \frac{9}{4}\log\left|2x+\sqrt{4x^2+9}\right| + c$$

DO YOU KNOW?

$$\int m\sqrt{(mx)^2+a^2}\,dx = \int \sqrt{(mx)^2+a^2}\,d(mx)$$

$$= \frac{mx}{2}\sqrt{(mx)^2+a^2} + \frac{a^2}{2}\log\left|mx+\sqrt{(mx)^2+a^2}\right| + c$$

Example 2.54

Evaluate $\int \sqrt{x^2-4x+3}\,dx$

Solution:

$$\int \sqrt{x^2-4x+3}\,dx$$

$$= \int \sqrt{(x-2)^2-1^2}\,dx$$

$$= \frac{(x-2)}{2}\sqrt{(x-2)^2-1^2} - \frac{1}{2}\log\left|(x-2)+\sqrt{(x-2)^2-1^2}\right| + c$$

$$= \frac{(x-2)}{2}\sqrt{x^2-4x+3} - \frac{1}{2}\log\left|(x-2)+\sqrt{x^2-4x+3}\right| + c$$

By completing the squares
$x^2-4x+3 = (x-2)^2-4+3$ $= (x-2)^2-1^2$ $\therefore \sqrt{x^2-4x+3} = \sqrt{(x-2)^2-1^2}$

Example 2.55

Evaluate $\int \frac{1}{x-\sqrt{x^2-1}}\,dx$

Solution:

$$\int \frac{1}{x-\sqrt{x^2-1}}\,dx$$

$$= \int\left[x+\sqrt{x^2-1}\right]dx$$

$$= \int x\,dx + \int\sqrt{x^2-1}\,dx$$

$$= \frac{x^2}{2} + \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\log\left|x+\sqrt{x^2-1}\right| + c$$

By rationalisation,
$\frac{1}{x-\sqrt{x^2-1}} = \frac{1}{x-\sqrt{x^2-1}} \times \frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$ $= x+\sqrt{x^2-1}$

Exercise 2.7

Integrate the following with respect to x

1. $\frac{1}{9-16x^2}$
2. $\frac{1}{9-8x-x^2}$
3. $\frac{1}{2x^2-9}$
4. $\frac{1}{x^2-x-2}$
5. $\frac{1}{x^2+3x+2}$
6. $\frac{1}{2x^2+6x-8}$
7. $\frac{e^x}{e^{2x}-9}$
8. $\frac{1}{\sqrt{9x^2-7}}$
9. $\frac{1}{\sqrt{x^2+6x+13}}$
10. $\frac{1}{\sqrt{x^2-3x+2}}$
11. $\frac{x^3}{\sqrt{x^8-1}}$
12. $\sqrt{1+x+x^2}$
13. $\sqrt{x^2-2}$
14. $\sqrt{4x^2-5}$
15. $\sqrt{2x^2+4x+1}$
16. $\frac{1}{x+\sqrt{x^2-1}}$

2.2 Definite integrals

So for we learnt about indefinite integrals on elementary algebraic, exponential, trigonometric and logarithmic functions. Now we are going to study about the definite integrals.

Geometrically, definite integral $\int_a^b f(x)dx$ represents the limit of a sum. It is also represented as the area bounded the curve $y = f(x)$, the axis of x, and the ordinates $x = a$ and $x = b$.

2.2.1 The fundamental theorems of Integral Calculus

Theorem 2.1 First fundamental theorem of Integral Calculus:

If $f(x)$ is a continuous function and $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$.

Theorem 2.2 Second fundamental theorem of Integral Calculus:

Let $f(x)$ be a continuous function on [a,b], if $F(x)$ is anti derivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Here a and b are known as the lower limit and upper limit of the definite integral.

Note

$\int_a^b f(x)dx$ is a definite constant, whereas $\int_a^x f(t)dt$ which is a function of the variable x.

Example 2.56

Evaluate $\int_0^1 \left(x^3 + 7x^2 - 5x\right)dx$

Solution:

We have already learnt about the evaluation of the integral $\int \left(x^3 + 7x^2 - 5x\right)dx$ in the previous section.

$$\therefore \int \left(x^3 + 7x^2 - 5x\right)dx = \frac{x^4}{4} + \frac{7x^3}{3} - \frac{5x^2}{2} + c$$

$$\text{Now, } \int_0^1 \left(x^3 + 7x^2 - 5x\right)dx = \left[\frac{x^4}{4} + \frac{7x^3}{3} - \frac{5x^2}{2}\right]_0^1$$

DO YOU KNOW?

$$\int f(x)dx = F(x) + c$$

$$\Rightarrow \int_a^b f(x)dx = F(b) - F(a)$$

$$= \left[\frac{1}{4}+\frac{7}{3}-\frac{5}{2}\right]-\left[\frac{0}{4}+\frac{0}{3}-\frac{0}{2}\right]$$

$$= \left[\frac{1}{4}+\frac{7}{3}-\frac{5}{2}\right]=\frac{1}{12}$$

Example 2.57

Find the integration for $\frac{dy}{dx}=\frac{2x}{5x^2+1}$ with limiting values as 0 and 1

Solution:

Here, $\frac{dy}{dx} = \frac{2x}{5x^2+1}$

$\therefore$ $$y = \int_0^1 \frac{2x}{5x^2+1}dx$$

$$= \frac{1}{5}\int_0^1 \frac{10x}{5x^2+1}dx$$

$$= \frac{1}{5}\left[\log\left(5x^2+1\right)\right]_0^1$$

$$= \frac{1}{5}\left[\log 6-\log 1\right]$$

$$= \frac{1}{5}\log 6$$

Example 2.58

Evaluate $\int_0^1\left(e^x-4a^x+2+\sqrt[3]{x}\right)dx$

Solution:

$$\int_0^1\left(e^x-4a^x+2+\sqrt[3]{x}\right)dx = \left[e^x-4\frac{a^x}{\log a}+2x+3\frac{x^{\frac{4}{3}}}{4}\right]_0^1$$

$$= e - \frac{4a}{\log a} + 2 + \frac{3}{4} - 1 + \frac{4}{\log a}$$

$$= e + \frac{4\left(1-a\right)}{\log a} + \frac{7}{4}$$

Example 2.59

Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sin x\,dx$

Solution:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x\,dx = \left[-\cos x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= -\left(\cos\frac{\pi}{3} - \cos\frac{\pi}{6}\right)$$

$$= \frac{1}{2}\left(\sqrt{3} - 1\right)$$

Example 2.60

Evaluate $\int_{0}^{\frac{\pi}{2}} \cos^2 x\,dx$

Solution:

$$\int_{0}^{\frac{\pi}{2}} \cos^2 x\,dx = \int_{0}^{\frac{\pi}{2}} \frac{1}{2}[1+\cos 2x]\,dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + 0\right] = \frac{\pi}{4}$$

> Change into simple integrands
>
> $\cos 2x = 2\cos^2 x - 1$
>
> $\therefore \cos^2 x = \frac{1}{2}[1+\cos 2x]$

Example 2.61

Evaluate $\int_{0}^{1} \left[e^{a\log x} + e^{x\log a}\right]dx$

Solution:

$$\int_{0}^{1} \left[e^{a\log x} + e^{x\log a}\right]dx = \int_{0}^{1}\left(x^a + a^x\right)dx$$

$$= \left[\frac{x^{a+1}}{a+1} + \frac{a^x}{\log a}\right]_0^1$$

$$= \left(\frac{1}{a+1} + \frac{a}{\log a}\right) - \left(0 + \frac{1}{\log a}\right)$$

$$= \frac{1}{a+1} + \frac{a}{\log a} - \frac{1}{\log a}$$

$$= \frac{1}{a+1} + \frac{(a-1)}{\log a}$$

> **Note**
>
> $e^{a\log x} = x^a$

Example 2.62

Evaluate $\int_{2}^{3} \frac{x^4+1}{x^2} dx$

Solution:

$$\int_{2}^{3} \frac{x^4+1}{x^2} dx = \int_{2}^{3} \left(x^2 + x^{-2}\right) dx$$

$$= \left[\frac{x^3}{3} - \frac{1}{x}\right]_2^3$$

$$= \left(9 - \frac{1}{3}\right) - \left(\frac{8}{3} - \frac{1}{2}\right) = \frac{13}{2}$$

Example 2.63

Evaluate $\int_{-1}^{1} \left(x^3+3x^2\right)^3 \left(x^2+2x\right) dx$

Solution:

$$\int_{-1}^{1} \left(x^3+3x^2\right)^3 \left(x^2+2x\right) dx = \left[\frac{1}{3} \frac{\left(x^3+3x^2\right)^4}{4}\right]_{-1}^{1} \qquad \left[\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c\right]$$

$$= \frac{1}{3}(64-4)$$

$$= 20$$

Example 2.64

Evaluate $\int_{a}^{b} \frac{\sqrt{\log x}}{x} dx \quad a, b > 0$

Solution:

$$\int_{a}^{b} \frac{\sqrt{\log x}}{x} dx = \int_{a}^{b} (\log x)^{\frac{1}{2}} \frac{dx}{x} \qquad \left[\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c\right]$$

$$= \left[2 \frac{(\log x)^{\frac{3}{2}}}{3}\right]_a^b$$

$$= \frac{2}{3}\left[(\log b)^{\frac{3}{2}} - (\log a)^{\frac{3}{2}}\right]$$

DO YOU KNOW?

$\log b^{\frac{3}{2}} = \frac{3}{2} \log b$ but,

$(\log b)^{\frac{3}{2}} \neq \frac{3}{2} \log b$ Further,

$(\log b)^{\frac{3}{2}} - (\log a)^{\frac{3}{2}} \neq \left(\log \frac{b}{a}\right)^{\frac{3}{2}}$

Example 2.65

Evaluate $\int_{-1}^{1} x\sqrt{x+1}\,dx$

Solution:

$$\int_{-1}^{1} x\sqrt{x+1}\,dx = \int_{0}^{2}(t-1)\sqrt{t}\,dt$$

$$= \int_{0}^{2}\left(t^{\frac{3}{2}} - t^{\frac{1}{2}}\right)dt$$

$$= \left[\frac{2t^{\frac{5}{2}}}{5} - \frac{2t^{\frac{3}{2}}}{3}\right]_{0}^{2}$$

$$= \frac{8\sqrt{2}}{5} - \frac{4\sqrt{2}}{3}$$

$$= \frac{4\sqrt{2}}{15}$$

Take $t = x+1$

$dt = dx$

and

x	-1	1
t	0	2

Example 2.66

Evaluate $\int_{0}^{\infty} e^{-\frac{x}{2}}\,dx$

Solution:

$$\int_{0}^{\infty} e^{-\frac{x}{2}}\,dx = -2\left[e^{-\frac{x}{2}}\right]_{0}^{\infty}$$

$$= -2[0-1] = 2$$

Example 2.67

Evaluate $\int_{0}^{\infty} x^2 e^{-x^3}\,dx$

Solution:

$$\int_{0}^{\infty} x^2 e^{-x^3}\,dx = \int_{0}^{\infty} e^{-t}\frac{dt}{3}$$

$$= \frac{1}{3}\left[-e^{-t}\right]_{0}^{\infty}$$

$$= \frac{-1}{3}[0-1]$$

$$= \frac{1}{3}$$

Take $x^3 = t$

$3x^2\,dx = dt$

$\Rightarrow x^2\,dx = \frac{dt}{3}$ and

x	0	∞
t	0	∞

Example 2.68

Evaluate $\int_1^2 \frac{1}{(x+1)(x+2)}dx$

Solution:

$$\int_1^2 \frac{1}{(x+1)(x+2)}dx = \int_1^2 \left[\frac{1}{x+1}-\frac{1}{x+2}\right] dx$$

$$= \left[\log|x+1| - \log|x+2|\right]_1^2$$

$$= \log\frac{3}{4} - \log\frac{2}{3}$$

$$= \log\frac{9}{8}$$

By partial fractions,
$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$ $\Rightarrow \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$

Example 2.69

Evaluate $\int_1^e \log x \, dx$

Solution:

$$\int_1^e \log x \, dx = \int_1^e u \, dv$$

$$= [uv]_1^e - \int_1^e v du$$

$$= [x\log x]_1^e - \int_1^e x\frac{1}{x} dx$$

$$= (e\log e - 1\log 1) - [x]_1^e$$

$$= (e-0) - (e-1)$$

$$= 1$$

Take $u = \log x$	and $dv = dx$
Differentiate	Integrate
$du = \frac{1}{x}dx$	$v = x$

Example 2.70

Evaluate $\int_0^{\frac{\pi}{2}} x \sin x \, dx$

Solution:

$$\int_0^{\frac{\pi}{2}} x \sin x \, dx = \int_0^{\frac{\pi}{2}} u dv$$

Take $u = x$	and $dv = \sin x \, dx$
Differentiate	Integrate
$du = dx$	$v = -\cos x$

$$= [uv]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} v\,du$$

$$= [-x \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\, dx$$

$$= 0 + [\sin x]_0^{\frac{\pi}{2}} = 1$$

Example 2.71

If $\int_1^a 3x^2\, dx = -1$, then find the value of a ($a \in R$).

Solution:

$$\text{Given that } \int_1^a 3x^2\, dx = -1$$

$$[x^3]_1^a = -1$$

$$a^3 - 1 = -1$$

$$a^3 = 0 \Rightarrow a = 0$$

Example 2.72

If $\int_a^b dx = 1$ and $\int_a^b x\,dx = 1$, then find a and b

Solution:

Given that $\int_a^b dx = 1$

$$[x]_a^b = 1$$

$$b - a = 1 \quad \ldots (1)$$

Now, $\int_a^b x\,dx = 1$

$$\left[\frac{x^2}{2}\right]_a^b = 1$$

$$b^2 - a^2 = 2$$

$$(b+a)(b-a) = 2$$

$$b + a = 2 \quad \ldots (2) \left[\because b - a = 1\right]$$

$$(1) + (2) \Rightarrow \quad 2b = 3$$

$$\therefore b = \frac{3}{2}$$

Now, $$\frac{3}{2} - a = 1 \quad \left[\because \text{from}(1)\right]$$

$$\therefore a = \frac{1}{2}$$

Example 2.73

Evaluate $\int_1^4 f(x)\, dx$, where $f(x) = \begin{cases} 7x+3, & if\ 1 \le x \le 3 \\ 8x & ,if\ 3 \le x \le 4 \end{cases}$

Solution:

$$\begin{aligned} \int_1^4 f(x)\, dx &= \int_1^3 f(x)\, dx + \int_3^4 f(x)\, dx \\ &= \int_1^3 (7x+3)\, dx + \int_3^4 8x\, dx \\ &= \left[\frac{7x^2}{2} + 3x\right]_1^3 + \left[\frac{8x^2}{2}\right]_3^4 \\ &= \frac{63}{2} + 9 - \frac{13}{2} + 64 - 36 \\ &= 62 \end{aligned}$$

Example 2.74

If $f(x) = \begin{cases} x^2, & -2 \le x < 1 \\ x, & 1 \le x < 2 \\ x-4, & 2 \le x \le 4 \end{cases}$, then find the following

(i) $\int_{-2}^{1} f(x)dx$ (ii) $\int_1^2 f(x)dx$ (iii) $\int_2^3 f(x)dx$ (iv) $\int_{-2}^{1.5} f(x)dx$ (v) $\int_1^3 f(x)dx$

Solution:

(i) $$\int_{-2}^{1} f(x)dx = \int_{-2}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_{-2}^{1} = \frac{1}{3} - \left(\frac{-8}{3}\right) = 3$$

(ii) $$\int_1^2 f(x)dx = \int_1^2 x dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

(iii) $\int_2^3 f(x)dx = \int_2^3 (x-4)dx = \left[\frac{x^2}{2} - 4x\right]_2^3 = \left(\frac{9}{2} - 12\right) - \left(\frac{4}{2} - 8\right) = -\frac{15}{2} + 6 = \frac{-3}{2}$

(iv) $\int_{-2}^{1.5} f(x)dx = \int_{-2}^{1} f(x)dx + \int_{1}^{1.5} f(x)dx = 3 + \int_1^{1.5} xdx$ using (i)

$$= 3 + \left[\frac{x^2}{2}\right]_1^{1.5} = 3 + \frac{2.25}{2} - \frac{1}{2} = 3 + \frac{1.25}{2} = 3.625$$

(v) $\int_1^3 f(x)dx = \int_1^2 f(x)dx + \int_2^3 f(x)dx$

$$= \frac{3}{2} + \left(\frac{-3}{2}\right) = 0 \text{ using (ii) and (iii)}$$

Exercise 2.8

I Using second fundamental theorem, evaluate the following:

1. $\int_0^1 e^{2x}dx$ 2. $\int_0^{\frac{1}{4}} \sqrt{1-4x}\, dx$ 3. $\int_1^2 \frac{xdx}{x^2+1}$ 4. $\int_0^3 \frac{e^x dx}{1+e^x}$

5. $\int_0^1 xe^{x^2} dx$ 6. $\int_1^e \frac{dx}{x(1+\log x)^3}$ 7. $\int_{-1}^1 \frac{2x+3}{x^2+3x+7}dx$ 8. $\int_0^{\frac{\pi}{2}} \sqrt{1+\cos x}\, dx$

9. $\int_1^2 \frac{x-1}{x^2} dx$

II Evaluate the following:

1. $\int_1^4 f(x)\, dx$ where $f(x) = \begin{cases} 4x+3, & 1 \le x \le 2 \\ 3x+5, & 2 < x \le 4 \end{cases}$

2. $\int_0^2 f(x)\, dx$ where $f(x) = \begin{cases} 3-2x-x^2, & x \le 1 \\ x^2+2x-3, & 1 < x \le 2 \end{cases}$

3. $\int_{-1}^1 f(x)\, dx$ where $f(x) = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$

4. $f(x) = \begin{cases} cx, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$ Find 'c' if $\int_0^1 f(x)dx = 2$

2.2.2 Properties of definite integrals

(i) $\int_a^b f(x)dx = \int_a^b f(t)dt$

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

DO YOU KNOW?

$$\int_a^b dx = b - a$$

(iii) If $f(x)$ and $g(x)$ are continuous functions in $[a,b]$, then

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

(iv) If $f(x)$ is a continuous function in $[a,b]$ and a < c < b, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

(v) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof:

$$\begin{aligned} \text{R.H.S.} &= \int_0^a f(a-x)dx \\ &= \int_a^0 f(t)(-dt) \\ &= \int_0^a f(t)dt \\ &= \int_0^a f(x)dx \qquad \text{[using the Property (i)]} \\ &= \text{L.H.S.} \end{aligned}$$

Take $a-x=t \Rightarrow dx=-dt$		
x	0	a
t	a	0

(vi) (a) If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

(b) If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

Proof :

(a) If $f(x)$ is an even function, then $f(x) = f(-x)$ ...(1)

$$\begin{aligned} \int_{-a}^{a} f(x)dx &= \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx \\ &= \int_{-a}^{0} f(-x)dx + \int_0^a f(x)dx \qquad \text{using (1)} \end{aligned}$$

Take $-x=t \Rightarrow dx=-dt$		
x	$-a$	0
t	a	0

$$\therefore \int_{-a}^{a} f(x)\,dx = \int_{a}^{0} f(t)(-dt) + \int_{0}^{a} f(x)\,dx$$

Apply the above mentioned substitution only in first part of integral of the R.H.S

$$= \int_{0}^{a} f(t)\,dt + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \text{[using the Property (i)]}$$

$$= 2\int_{0}^{a} f(x)\,dx$$

(b) If $f(x)$ is an odd function, then $f(-x) = -f(x)$ ...(2)

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{-a}^{0} -f(-x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \text{using (2)}$$

Take $-x = t \Rightarrow dx = -dt$		
x	$-a$	0
t	a	0

$$\therefore \int_{-a}^{a} f(x)\,dx = -\int_{a}^{0} f(t)(-dt) + \int_{0}^{a} f(x)\,dx$$

Apply the above mentioned substitution only in first part of integral of the R.H.S

$$= -\int_{0}^{a} f(t)\,dt + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \text{[using the Property (i)]}$$

$$= 0$$

(vii) $\displaystyle\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx$

Proof:

Let $a+b-x = t$ and

$-dx = dt$

$dx = -dt$

$t = a+b-x$		
x	a	b
t	b	a

$$\therefore \int_{a}^{b} f(a+b-x)\,dx = -\int_{b}^{a} f(t)\,dt = \int_{a}^{b} f(t)\,dt$$

$$= \int_{a}^{b} f(x)\,dx \qquad \text{[using the Property (i)]}$$

$$\therefore \int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx$$

Evaluate the following using properties of definite integrals:

Example 2.75

Evaluate $\displaystyle\int_{-1}^{1} \frac{x^5\,dx}{a^2 - x^2}$

Solution:

Let $\quad f(x) = \dfrac{x^5}{a^2 - x^2}$

$$f(-x) = \frac{(-x)^5}{a^2 - (-x)^2} = \frac{-x^5}{a^2 - x^2} = -f(x)$$

Here $\quad f(x) = -f(x)$

$\therefore f(x)$ is an odd function

$$\Rightarrow \int_{-1}^{1} \frac{x^5\,dx}{a^2 - x^2} = 0$$

Example 2.76

Evaluate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$

Solution:

Let $\quad f(x) = \cos x$

$$f(-x) = \cos(-x) = \cos x$$

$$\Rightarrow f(x) = -f(x)$$

$\therefore f(x)$ is an even function

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx = 2\int_{0}^{\frac{\pi}{2}} \cos x\,dx$$

$$= 2\left[\sin x\right]_0^{\frac{\pi}{2}}$$

$$= 2\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= 2$$

Example 2.77

Evaluate $\int_{-1}^{1}\left(x^2+x\right)dx$

Solution:

$$\int_{-1}^{1}\left(x^2+x\right)dx = \int_{-1}^{1} x^2\,dx + \int_{-1}^{1} x\,dx$$

$$= 2\int_{0}^{1} x^2\,dx + 0 \quad [\because x^2 \text{ is an even function and } x \text{ is an odd function}]$$

$$= 2\left[\frac{x^3}{3}\right]_0^1 = 2\left[\frac{1}{3}-0\right]$$

$$= \frac{2}{3}$$

Example 2.78

Evaluate $\int_{0}^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx$

Solution:

Let $\quad I = \int_{0}^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx \qquad \ldots (1)$

$$I = \int_{0}^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}\,dx \qquad \ldots (2)$$

$(1)+(2) \Rightarrow$

$$2I = \int_{0}^{\frac{\pi}{2}}\left[\frac{\sin x}{\sin x+\cos x}+\frac{\cos x}{\cos x+\sin x}\right]dx$$

$$= \int_{0}^{\frac{\pi}{2}} dx \quad = [x]_0^{\frac{\pi}{2}} \quad = \frac{\pi}{2}$$

$\therefore \quad I = \frac{\pi}{4}$

Example 2.79

Evaluate $\int_{2}^{5} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}}\,dx$

Solution:

$$\text{Let} \qquad I = \int_{2}^{5} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}}\,dx \quad \ldots(1)$$

$$I = \int_{2}^{5} \frac{\sqrt{2+5-x}}{\sqrt{2+5-x}+\sqrt{7-(2+5-x)}}\,dx \qquad \left[\because \int_{a}^{b} f(x)dx = \int_{a}^{b} f(a+b-x)dx\right]$$

$$I = \int_{2}^{5} \frac{\sqrt{7-x}}{\sqrt{7-x}+\sqrt{x}}\,dx \quad \ldots(2)$$

$(1)+(2) \Rightarrow$

$$2I = \int_{2}^{5} \left[\frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}} + \frac{\sqrt{7-x}}{\sqrt{7-x}+\sqrt{x}}\right] dx$$

$$= \int_{2}^{5} \left[\frac{\sqrt{x}+\sqrt{7-x}}{\sqrt{x}+\sqrt{7-x}}\right] dx$$

$$= \int_{2}^{5} dx \quad = [x]_{2}^{5} \quad = 3$$

$$\therefore \qquad I = \frac{3}{2}$$

Exercise 2.9

Evaluate the following using properties of definite integrals:

1. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^3 \cos^3 x\,dx$
2. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \theta\, d\theta$
3. $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) dx$
4. $\int_{0}^{\frac{\pi}{2}} \frac{\sin^7 x}{\sin^7 x+\cos^7 x}\,dx$
5. $\int_{0}^{1} \log\left(\frac{1}{x}-1\right) dx$
6. $\int_{0}^{1} \frac{x}{(1-x)^{\frac{3}{4}}}\,dx$

2.1.3 Gamma Integral

Gamma integral is an important result which is very useful in the evaluation of a particular type of an improper definite integrals.

First, let us know about the concepts of indefinite integrals, proper definite integrals and improper definite integrals

Indefinite integral:

An integral function which is expressed without limits, and so containing an arbitrary constant is called an indefinite integral

Example: $\int e^{-t} dt$

Proper definite integral:

Proper definite integral is an integral function, which has both the limits *a* and *b* are finite and the integrand *f*(*x*) is continuous in [*a*, *b*].

Example: $\int_0^1 e^{-t} dt$

Improper definite integral:

An improper definite integral is an integral function, in which the limits either *a* or *b* or both are infinite, or the integrand *f*(*x*) becomes infinite at some points of the interval [*a*, *b*].

Example: $\int_0^\infty e^{-t} dt$

Definition 2.3

For $n > 0$, $\int_0^\infty x^{n-1}e^{-x}dx$ is known known as Gamma function and is denoted by $\Gamma(n)$ [read as Gamma of n].

Note

If n is a positive integer, then $\int_0^\infty x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}$ is the particular case of Gamma Integral.

Properties:

1. $\Gamma(n) = (n-1)\Gamma(n-1),\ n>1$
2. $\Gamma(n+1) = n\Gamma(n),\ n>0$

DO YOU KNOW?

$\Gamma(1)=1$

3. $\Gamma(n+1) = n!$, n is a positive integer.

4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Example 2.80

Evaluate (i) $\Gamma(6)$ (ii) $\Gamma\left(\frac{7}{2}\right)$ (iii) $\int_0^{\infty} e^{-2x} x^5 dx$ (iv) $\int_0^{\infty} e^{-x^2} dx$

Solution:

(i)
$$\Gamma(6) = 5! = 120$$

(ii)
$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi} = \frac{15}{8}\sqrt{\pi}$$

(iii) we know that

$$\int_0^{\infty} x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}$$

$$\therefore \int_0^{\infty} e^{-2x} x^5\, dx = \frac{5!}{2^{5+1}} = \frac{5!}{2^6}$$

(iv)
$$\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} dt$$

Put $t = x^2 \Rightarrow dt = 2x dx$

$$\therefore \quad \Gamma(n) = \int_0^{\infty} e^{-x^2} \left(x^2\right)^{n-1} 2x\, dx = \int_0^{\infty} e^{-x^2} x^{2n-2} 2x\, dx = 2\int_0^{\infty} e^{-x^2} x^{2n-1}\, dx$$

Put $n = \frac{1}{2}$, we have

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}dx$$

$$\Rightarrow \sqrt{\pi} = 2\int_0^\infty e^{-x^2}dx$$

$$\therefore \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

Exercise 2.10

1. Evaluate the following:

(i) $\Gamma(4)$ (ii) $\Gamma\left(\frac{9}{2}\right)$ (iii) $\int_0^\infty e^{-mx}x^6dx$ (iv) $\int_0^\infty e^{-4x}x^4dx$ (v) $\int_0^\infty e^{-\frac{x}{2}}x^5dx$

2. If $f(x) = \begin{cases} x^2 e^{-2x}, x \geq 0 \\ 0, \text{ otherwise} \end{cases}$, then evaluate $\int_0^\infty f(x)dx$

2.2.4 Definite integral as the limit of a sum

Let $f(x)$ be a continuous real valued function in $[a, b]$, which is divided into n equal parts of width h, then

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ h\to 0}} h\left[f(a)+f(a+h)+f(a+2h)+.....+f\left(a+(n-1)h\right)\right] \text{, where } h=\frac{b-a}{n}$$

(or)

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{r=1}^{n} h\, f(a+rh)\text{, where } h=\frac{b-a}{n}$$

The following results are very useful in evaluating definite integral as the limit of a sum

(i) $1+2+3+......n = \frac{n(n+1)}{2} = \sum_{r=1}^{n} r$

(ii) $1^2+2^2+3^2+......n^2 = \frac{n(n+1)(2n+1)}{6} = \sum_{r=1}^{r=n} r^2$

(iii) $1^3+2^3+3^3+......n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \sum_{r=1}^{r=n} r^3$

Example 2.81

Evaluate the integral as the limit of a sum: $\int_0^1 x\,dx$

Solution:

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{r=1}^{n} h\, f(a+rh)$$

Here $a=0,\ b=1,\ h=\dfrac{b-a}{n}=\dfrac{1-0}{n}=\dfrac{1}{n}$ and $f(x)=x$

Now $f(a+rh)=f\left(0+\dfrac{r}{n}\right)=f\left(\dfrac{r}{n}\right)=\dfrac{r}{n}$

On substituting in (1) we have

$$\int_0^1 x\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n}.\frac{r}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \sum_{r=1}^{n} r$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n^2\left(1+\frac{1}{n}\right)}{2}$$

$$= \frac{1+0}{2} = \frac{1}{2}$$

$$\therefore \qquad \int_0^1 x\,dx = \frac{1}{2}$$

Example 2.82

Evaluate the integral as the limit of a sum: $\displaystyle\int_1^2 (2x+1)\,dx$

Solution:

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{r=1}^{n} h\, f(a+rh)$$

Here $a=1,\ b=2,\ h=\dfrac{b-a}{n}=\dfrac{2-1}{n}=\dfrac{1}{n}$ and $f(x)=2x+1$

$$f(a+rh) = f\left(1+\frac{r}{n}\right) \quad]$$

$$= 2\left(1+\frac{r}{n}\right)+1$$

$$= 2+\frac{2r}{n}+1$$

$$f(a+rh) = 3+\frac{2r}{n}$$

$$\int_1^2 f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n}\left(3+\frac{2r}{n}\right)$$

$$= \underset{n\to\infty}{lt} \sum_{r=1}^{n}\left(\frac{3}{n}+\frac{2r}{n^2}\right)$$

$$= \lim_{n\to\infty}\left[\frac{3}{n}\sum_{r=1}^{n} 1 + \frac{2}{n^2}\sum_{r=1}^{n} r\right]$$

$$= \lim_{n\leftarrow\infty}\left[\frac{3}{n} . n+\frac{2}{n^2}\frac{n(n+1)}{2}\right]$$

$$= 3+\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$$\int_1^2 f(x)\,dx = 3+1 = 4$$

Example 2.83

Evaluate the integral as the limit of a sum: $\int_1^2 x^2\,dx$

Solution:

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{r=1}^{n} h\, f(a+rh)$$

Here $a=1,\;\; b=2,\;\; h=\frac{b-a}{n}=\frac{2-1}{n}=\frac{1}{n}$ and $f(x)=x^2$

Now $f(a+rh)= f\left(1+\frac{r}{n}\right) = \left(1+\frac{r}{n}\right)^2 = 1+\frac{2r}{n}+\frac{r^2}{n^2}$

$$\therefore \qquad \int_1^2 x^2\,dx = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\left(1+\frac{2r}{n}+\frac{r^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{1}{n}+\frac{2r}{n^2}+\frac{r^2}{n^3}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{r=1}^{n}1+\frac{2}{n^2}\sum_{r=1}^{n}r+\frac{1}{n^3}\sum_{r=1}^{n}r^2\right)$$

$$= \lim_{n\leftarrow\infty}\left(\frac{1}{n}(n)+\frac{2}{n^2}\frac{n(n+1)}{2}+\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty}\left[1+\left(1+\frac{1}{n}\right)+\frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}\right]$$

$$= \left[1+1+\frac{(1)(2)}{6}\right]$$

$$\therefore \int_1^2 x^2\, dx = \frac{7}{3}$$

Exercise 2.11

Evaluate the following integrals as the limit of the sum:

1. $\int_0^1 (x+4)\, dx$ 2. $\int_1^3 x\, dx$ 3. $\int_1^3 (2x+3)\, dx$ 4. $\int_0^1 x^2\, dx$

G4AQPS

Exercise 2.12

Choose the correct answer:

1. $\int \frac{1}{x^3} dx$ is

(a) $\frac{-3}{x^2}+c$ (b) $\frac{-1}{2x^2}+c$ (c) $\frac{-1}{3x^2}+c$ (d) $\frac{-2}{x^2}+c$

2. $\int 2^x dx$ is

(a) $2^x \log 2+c$ (b) 2^x+c (c) $\frac{2^x}{\log 2}+c$ (d) $\frac{\log 2}{2^x}+c$

3. $\int \frac{\sin 2x}{2\sin x} dx$ is

(a) $\sin x+c$ (b) $\frac{1}{2}\sin x+c$ (c) $\cos x+c$ (d) $\frac{1}{2}\cos x+c$

4. $\int \frac{\sin 5x-\sin x}{\cos 3x} dx$ is

(a) $-\cos 2x+c$ (b) $-\cos 2x+c$ (c) $-\frac{1}{4}\cos 2x+c$ (d) $-4\cos 2x+c$

5. $\int \frac{\log x}{x} dx\ , x>0$ is

(a) $\frac{1}{2}(\log x)^2+c$ (b) $-\frac{1}{2}(\log x)^2$ (c) $\frac{2}{x^2}+c$ (d) $\frac{2}{x^2}+c$

6. $\int \frac{e^x}{\sqrt{1+e^x}} dx$ is

(a) $\frac{e^x}{\sqrt{1+e^x}}+c$ (b) $2\sqrt{1+e^x}+c$ (c) $\sqrt{1+e^x}+c$ (d) $e^x\sqrt{1+e^x}+c$

7. $\int \sqrt{e^x}\, dx$ is

(a) $\sqrt{e^x}+c$ (b) $2\sqrt{e^x}+c$ (c) $\frac{1}{2}\sqrt{e^x}+c$ (d) $\frac{1}{2\sqrt{e^x}}+c$

8. $\int e^{2x}[2x^2+2x]dx$

(a) $e^{2x}x^2+c$ (b) $xe^{2x}+c$ (c) $2x^2e^2+c$ (d) $\frac{x^2e^x}{2}+c$

9. $\int \frac{e^x}{e^x+1}dx$ is

(a) $\log\left|\frac{e^x}{e^x+1}\right|+c$ (b) $\log\left|\frac{e^x+1}{e^x}\right|+c$ (c) $\log\left|e^x\right|+c$ (d) $\log\left|e^x+1\right|+c$

10. $\int\left[\frac{9}{x-3}-\frac{1}{x+1}\right]dx$ is

(a) $\log|x-3|-\log|x+1|+c$ (b) $\log|x-3|+\log|x+1|+c$

(c) $9\log|x-3|-\log|x+1|+c$ (d) $9\log|x-3|+\log|x+1|+c$

11. $\int \frac{2x^3}{4+x^4}dx$ is

(a) $\log\left|4+x^4\right|+c$ (b) $\frac{1}{2}\log\left|4+x^4\right|+c$ (c) $\frac{1}{4}\log\left|4+x^4\right|+c$ (d) $\log\left|\frac{2x^3}{4+x^4}\right|+c$

12. $\int \frac{dx}{\sqrt{x^2-36}}$ is

(a) $\sqrt{x^2-36}+c$ (b) $\log\left|x+\sqrt{x^2-36}\right|+c$

(c) $\log\left|x-\sqrt{x^2-36}\right|+c$ (d) $\log\left|x^2+\sqrt{x^2-36}\right|+c$

13. $\int \frac{2x+3}{\sqrt{x^2+3x+2}}dx$ is

(a) $\sqrt{x^2+3x+2}+c$ (b) $2\sqrt{x^2+3x+2}+c$

(c) $\log\left(x^2+3x+2\right)+c$ (d) $\frac{2}{3}\left(x^2+3x+2\right)^{\frac{3}{2}}+c$

14. $\int_0^1 (2x+1)\, dx$ is

(a) 1 (b) 2 (c) 3 (d) 4

15. $\int_2^4 \frac{dx}{x}$ is

(a) log 4 (b) 0 (c) log 2 (d) log 8

16. $\int_{0}^{\infty} e^{-2x}\, dx$ is

(a) 0 (b) 1 (c) 2 (d) $\frac{1}{2}$

17. $\int_{-1}^{1} x^3 e^{x^4}\, dx$ is

(a) 1 (b) $2\int_{0}^{1} x^3 e^{x^4}\, dx$ (c) 0 (d) e^{x^4}

18. If $f(x)$ is a continuous function and $a < c < b$, then $\int_{a}^{c} f(x)\, dx + \int_{c}^{b} f(x)\, dx$ is

(a) $\int_{a}^{b} f(x)\, dx - \int_{a}^{c} f(x)\, dx$ (b) $\int_{a}^{c} f(x)\, dx - \int_{a}^{b} f(x)\, dx$

(c) $\int_{a}^{b} f(x)\, dx$ (d) 0

19. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx$ is

(a) 0 (b) 2 (c) 1 (d) 4

20. $\int_{0}^{1} \sqrt{x^4 (1-x)^2}\, dx$ is

(a) $\frac{1}{12}$ (b) $\frac{-7}{12}$ (c) $\frac{7}{12}$ (d) $\frac{-1}{12}$

21. If $\int_{0}^{1} f(x)\, dx = 1$, $\int_{0}^{1} x\, f(x)\, dx = a$ and $\int_{0}^{1} x^2\, f(x)\, dx = a^2$, then $\int_{0}^{1} (a-x)^2\, f(x)\, dx$ is

(a) $4a^2$ (b) 0 (c) $2a^2$ (d) 1

22. The value of $\int_{2}^{3} f(5-x)\, dx - \int_{2}^{3} f(x)\, dx$ is

(a) 1 (b) 0 (c) -1 (d) 5

23. $\int_{0}^{4} \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$ is

(a) $\frac{20}{3}$ *(b)* $\frac{21}{3}$ *(c)* $\frac{28}{3}$ *(d)* $\frac{1}{3}$

24. $\int_{0}^{\frac{\pi}{3}} \tan x\, dx$ is

(a) log 2 (b) 0 (c) $\log\sqrt{2}$ (d) 2 log 2

25. Using the factorial representation of the gamma function, which of the following is the solution for the gamma function $\Gamma(n)$ when $n=8$

(a) 5040 (b) 5400 (c) 4500 (d) 5540

26. $\Gamma(n)$ is

(a) $(n-1)!$ (b) $n!$ (c) $n\Gamma(n)$ (d) $(n-1)\Gamma(n)$

27. $\Gamma(1)$ is

(a) 0 (b) 1 (c) n (d) $n!$

28. If $n>0$, then $\Gamma(n)$ is

(a) $\int_0^1 e^{-x}x^{n-1}dx$ (b) $\int_0^1 e^{-x}x^{n}dx$ (c) $\int_0^\infty e^{x}x^{-n}dx$ (d) $\int_0^\infty e^{-x}x^{n-1}dx$

29. $\Gamma\left(\frac{3}{2}\right)$

(a) $\sqrt{\pi}$ (b) $\frac{\sqrt{\pi}}{2}$ (c) $2\sqrt{\pi}$ (d) $\frac{3}{2}$

30. $\int_0^\infty x^4 e^{-x}dx$ is

(a) 12 (b) 4 (c) 4! (d) 64

Miscellaneous problems

Evaluate the following integrals:

1. $\int \frac{1}{\sqrt{x+2}-\sqrt{x+3}}dx$

2. $\int \frac{dx}{2-3x-2x^2}$

3. $\int \frac{dx}{e^x+6+5e^{-x}}$

4. $\int \sqrt{2x^2-3}\,dx$

5. $\int \sqrt{9x^2+12x+3}\,dx$

6. $\int (x+1)^2 \log x\,dx$

7. $\int \log\left(x-\sqrt{x^2-1}\right)dx$

8. $\int_0^1 \sqrt{x(x-1)}\,dx$

9. $\int_{-1}^1 x^2 e^{-2x}dx$

10. $\int_0^3 \frac{x\,dx}{\sqrt{x+1}+\sqrt{5x+1}}$

Summary

In this chapter, we have acquired the knowledge of

- **The relation between the Primitive function and the derived function:**

  A function $F(x)$ is said to be a primitive function of the derived function $f(x)$, if $\frac{d}{dx}[F(x)] = f(x)$

- **Integration of a function:**

  The process of determining an integral of a given function is defined as integration of a function

- **Properties of indefinite integrals:**

  $\int a\, f(x)\, dx = a\int f(x)\, dx$

  $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

- **Standard results of indefinite integrals:**

1. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$
2. $\int \frac{1}{x}\, dx = \log|x| + c$
3. $\int e^x\, dx = e^x + c$
4. $\int a^x\, dx = \frac{1}{\log a} a^x + c, a > 0$ and $a \neq 1$
5. $\int \sin x\, dx = -\cos x + c$
6. $\int \cos x\, dx = \sin x + c$
7. $\int \sec^2 x\, dx = \tan x + c$
8. $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$
9. $\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c, n \neq -1$
10. $\int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c$
11. $\int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)} + c$
12. $\int u\, dv = uv - \int v\, du$
13. $\int u\, dv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots$
14. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$
15. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$
16. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + c$
17. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + c$
18. $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$
19. $\int e^{ax} [a\, f(x) + f'(x)]\, dx = e^{ax} f(x) + c$
20. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$
21. $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + c$

- **Definite integral:**

  Let $f(x)$ be a continuous function on [a,b] and if $F(x)$ is anti derivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

- **Properties of definite integrals:**

  (i) $\int_a^b f(x)dx = \int_a^b f(t)dt$ (ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

  (iii) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$ (iv) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

  (v) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ (vi) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

  (vii) a) If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

  b) If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

- **Particular case of Gamma Integral:**

  If n is a positive integer, then $\int_0^\infty x^n e^{-ax}\,dx = \dfrac{n!}{a^{n+1}}$

- **Properties of gamma function:**

  (i) $\Gamma(n) = (n-1)\Gamma(n-1),\ n>1$ (ii) $\Gamma(n+1) = n\Gamma(n),\ n>0$

  (iii) $\Gamma(n+1) = n!$, n is a positive integer (iv) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

- **Definite integral as the limit of a sum:**

  Let $f(x)$ be a continuous real valued function in $[a, b]$, which is divided into n equal parts each of width h, then

  $$\int_a^b f(x)dx = \lim_{\substack{h\to 0\\ n\to\infty}} \sum_{r=1}^{n} h\, f(a+rh) \quad \text{where } h = \frac{b-a}{n}$$

- **Results:**

  (i) $1+2+3+......n = \dfrac{n(n+1)}{2} = \sum_{r=1}^{n} r$

  (ii) $1^2+2^2+3^2+......n^2 = \dfrac{n(n+1)(2n+1)}{6} = \sum_{r=1}^{r=n} r^2$

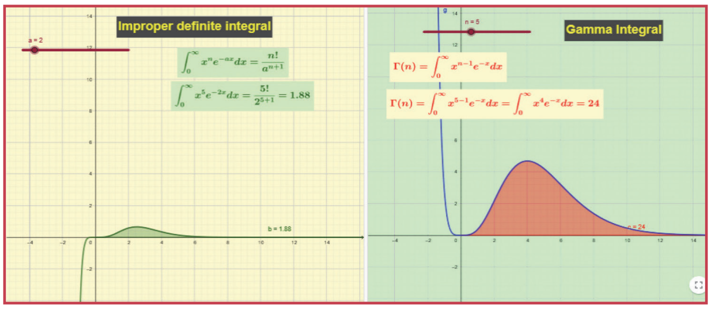
  (iii) $1^3+2^3+3^3+......n^3 = \left[\dfrac{n(n+1)}{2}\right]^2 = \sum_{r=1}^{r=n} r^3$

GLOSSARY	
Abscissa	மட்டாயம் / கிடை தூரம்
Abstract value	நுண்ம மதிப்பு / புலனாகாத மதிப்பு
Algebraic function	அறமச் சார்பு / இயற் கணிதச் சார்பு
Anti derivative	எதிர்மறை வகையீடு
Approaches	அணுகுதல் / நெருங்குதல்
Approximate	தோராயம்
Arbitrary	ஏதேனுமொரு / தன்னிச்சையான
Between the specified limits	குறிப்பிடப்பட்ட எல்லைகளுக்குள்
Bounded region	வரம்புள்ள பகுதி
Certain	நிச்சயமான / உறுதியான
Change of variable	மாறியின் மாற்றம்
Closed interval	மூடிய இடைவெளி
Completing the square	வர்க்க நிறைவாக்கல்
Concept	கருத்துரு
Constant of integration	தொகையிடல் மாறிலி
Constant term	மாறா உறுப்பு
Continuous function	தொடர்ச்சியான சார்பு
Decomposition	பிரித்தல் / கூறாக்கல்
Definite integral	வரையறுத்த தொகை / வரையறுத்த தொகையீடு
Derived function	வருவித்தச் சார்பு
Differentiable function	வகையிடத்தக்கச் சார்பு
Differential coefficient	வகைக்கெழு / வகையீட்டு கெழு
Differentiation	வகையிடல்
Directly integrate	நேரிடையாக தொகையிட
Exponential function	அடுக்கைச் சார்பு / அடுக்குகுறிச் சார்பு
Family of curves	வளைவரைகளின் தொகுதி
Indefinite integral	அறுதியிடப்படாத தொகையீடு / வரையறாதத் தொகையீடு
Infinity	முடிவிலி / கந்தழி / எண்ணிலி
Inscribe	உள்வரை
Instantaneous	உடனடியயாக
Integrable function	தொகையிடத்தக்கச் சார்பு
Integral	தொகை / தொகையீடு
Integrand	தொகைக்காண் சார்பு / தொகைக் காண்பான்
Integrate	தொகையிடு / தொகையிட
Integration	தொகையிடல்
Integration by parts	பகுதிப் படுத்தித் தொகையிடல் / பகுதித் தொகையிடல்
Integration by substitution	பிரதியிட்டுத் தொகையிடல்

Integrator	தொகைப்பான்
Inverse trigonometric function	நேர் மாற்று கோணவியல் சார்பு / நேர்மாறு திரிகோணமிதி சார்பு
Logarithmic function	மடக்கை சார்பு
Lower limit	கீழ் எல்லை
Marginal function	இறுதிநிலைச் சார்பு
Natural logarithm	இயல் மடக்கை
Open interval	திறந்த இடைவெளி
Ordinate	குத்தாயம் / நிலைத் தூரம்
Parallel tangents	இணைத் தொடுகோடுகள்
Partial fraction	பகுதிப் பின்னம்
Positive integer	மிகை முழு எண்
Power rule	அடுக்கு விதி
Primitive function	மூலமுதலான சார்பு / தொடக்க நிலைச் சார்பு
Quantity	கணியம் / அளவு
Rationalisation method	காரணக்காரீய முறை
Reduction	குறைப்பு / சுருக்கல்
Repeated integral	தொடர் முறைத் தொகையீடு / மீண்டும் மீண்டும் தொகையிடப்பட்ட
Reverse process	எதிர் மறை செயல்
Standard form	திட்ட வடிவம்
Substitution	பிரதியிடல் / ஈடு செய்தல்
Successive	அடுத்தடுத்த / தொடர்ச்சியான
Successive derivatives	தொடர் வகையிடல்கள் / அடுத்தடுத்த வகையிடல்கள்
Suitable	ஏற்புடைய / பொருத்தமான
Summation	கூடுதல் / கூட்டல் தொகை
Technique	உத்தி / நுட்பம்
Trigonometric function	முக்கோண கணிப்பு சார்பு / திரிகோணமிதிச்சார்பு
Unique function	ஒருமைத் தன்மை கொண்ட சார்பு
Upper limit	மேல் எல்லை
Variable of integration	தொகையிடல் மாறி

ICT Corner

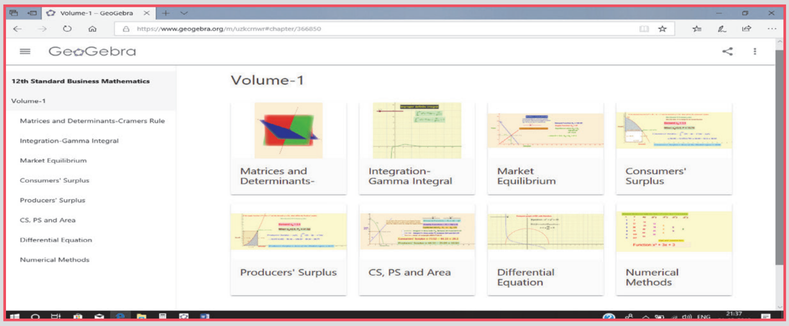
Expected Result is shown in this picture



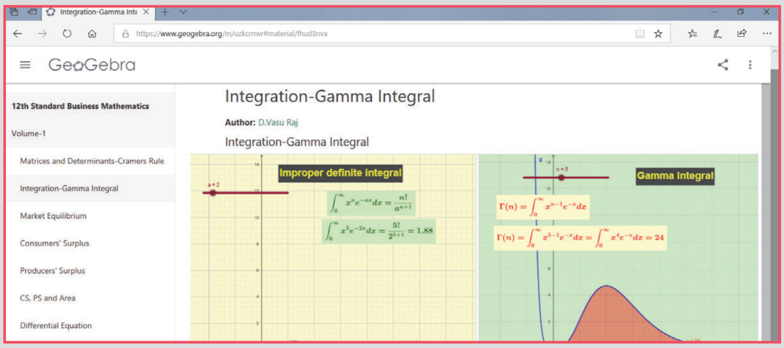
Step – 1 : Open the Browser, type the URL Link given below (or) Scan the QR Code. GeoGebra work Book named "12th Standard Business Mathematics " will open. In the work book there are two Volumes. Select "Volume-1".

Step - 2 : Select the worksheet named" Integration-Gamma Integral". There is a problem based on Gamma Integral and Improper definite Integral. Move the sliders to change "a" and "n" value. Observe the graph.

Step 1



Step 2

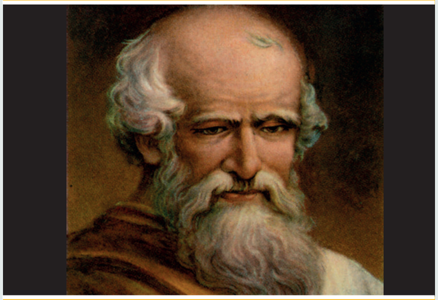


Browse in the link

12th standard Business Mathematics :https://ggbm.at/uzkcrnwr or Scan the QR Code.

B284_12_BUS_MAT_EM

3 Integral Calculus – II



Archimedes

(287 BC(BCE)-217 BC(BCE))

Introduction

History of integration begins over 2500 years ago. The first Greek Mathematician Antiphon (around 430 BC(BCE)) introduced the "method of exhaustion" to find areas of simple polygons and more complicated curves. (method of exhaustion means dividing the given area into infinite number of triangles). Though Antiphon invented the method of exhaustion to find area bounded by complicated curves, the mathematician Eudoxus did the logical development in this method of exhaustion. Later Euclid used this method to calculate the area of circle.

Using the same method of exhaustion Archimedes (287 BC(BCE)-217 BC(BCE)) find the area bounded by parabola. Thus using integration area bounded by curves is developed. We will see the method of finding area by using integration in this chapter.

Learning Objectives

After studying this chapter , the students will be able to understand

- the geometrical interpretation of definite integral.
- the applications of integration in finding the area bounded by a curve.
- the concept of consumer's & producer's surplus.
- the applications of integration in Economics and Commerce.

3.1 The area of the region bounded by the curves

HT59TF

Using integration we can evaluate the area bounded by the curves with coordinate axes. We can also calculate the area between two given curves.

3.1.1 Geometrical Interpretation of Definite Integral as Area under a curve:

Suppose we want to find out the area of the region which is bounded above by a curve $y = f(x)$, below by the $x-axis$ and the lines $x = a$ and $x = b$.

Now from Fig 3.1 let the interval $[a\ b]$ is divided into n subintervals $[x_{i-1},\ x_i]$ of equal length Δx_i *i.e* $x_i - x_{i-1} = \Delta_{xi}$ for any $x_i' \in [x_{i-1},\ x_i]$ let $f(x_i')$ be the height of n rectangles having $x_i - x_{i-1} = \Delta x_i$ as its base. Then area $A_i = \Delta x_i f(x_i')$. Now the total area $A = \sum_{i=1}^{n} f(x_i')\Delta x_i$

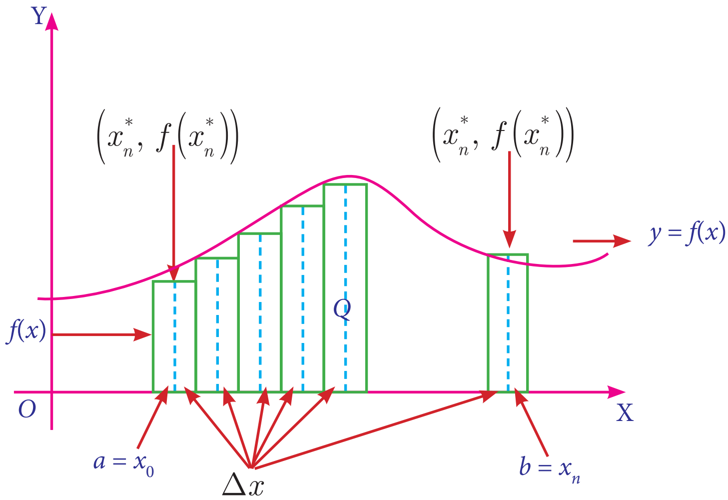


Fig. 3.1

Now from the definition of definite integral, if $f(x)$ is a function defined on $[a,b]$ with $a < b$ then the definite integral is

$$\int_a^b f(x)dx = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i')\Delta x_i$$

The area under the curve is exhausted by increasing the number of rectangular strips to ∞

Thus the geometrical interpretation of definite integral is the area under the curve between the given limits.

The area of the region bounded by the curve $y=f(x)$, with x- axis and the ordinates at $x = a$ and $x = b$ given by

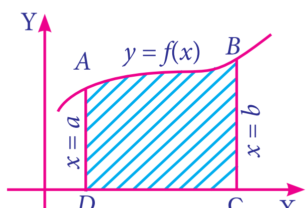


Fig.3.2

$$\text{Area} \quad A = \int_a^b y dx$$

$$= \int_a^b f(x)dx$$

Note

(i) The area of the region bounded by the curve $y = f(x)$ between the limits $x = a$, $x = b$ and lies below x-axis, is

$$A = \int_a^b -y\, dx = -\int_a^b f(x)dx$$

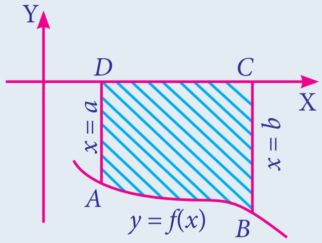


Fig.3.3

(ii) The area of the region bounded by the curve $x = f(y)$ between the limits $y = c$ and $y = d$ *with* $y - axis$ and the area lies lies to the right of y- axis, is

$$A = \int_c^d x dy \quad = \int_c^d f(y)dy$$

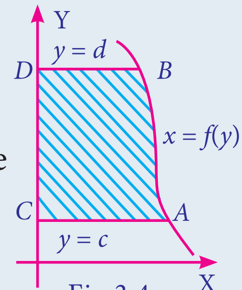


Fig.3.4

(iii) The area bounded by the curve $x = f(y)$ between the limits $y = c$ and $y = d$ *with* $y - axis$ and the area lies to the left of y- axis, is

$$A = \int_c^d -x dy = -\int_c^d f(y)dy$$

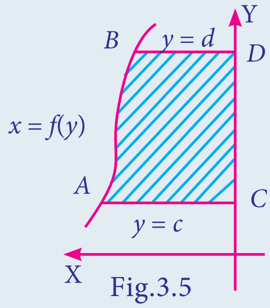


Fig.3.5

Area between two curves

Let $f(x)$ and $g(x)$ be two continuous functions defined on x in the interval $[a, b]$. Also $f(x) > g(x), a \le b$

Then the area between these two curves from $x = a$ *to* $x = b$, is

$$A = \int_a^b [f(x) - g(x)]dx.$$

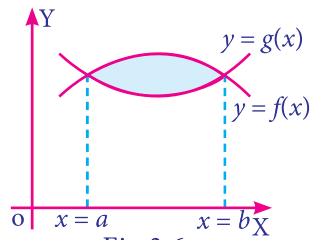


Fig.3.6

Example 3.1

Find the area bounded by $y = 4x + 3$ with x- axis between the lines $x = 1$ and $x = 4$

Solution:

$$\text{Area} = \int_1^4 y\,dx$$

$$= \int_1^4 (4x+3)\,dx$$

$$= \left[2x^2+3x\right]_1^4 = 32+12-2-3$$

$$= 39 \text{ sq.units}$$

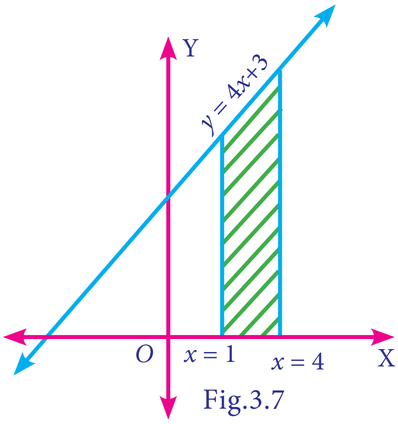


Fig.3.7

Example 3.2

Find the area of the region bounded by the line $x-2y-12=0$, the y-axis and the lines $y = 2$, $y = 5$.

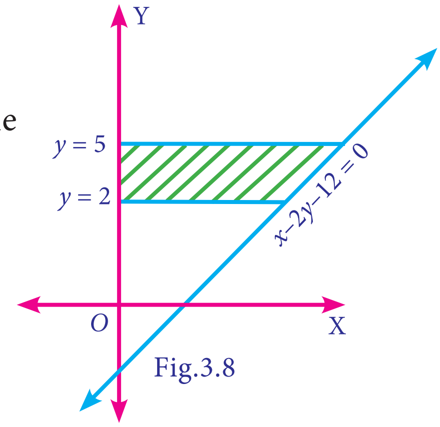


Fig.3.8

Solution:

$$x-2y-12=0$$
$$x=2y+12$$

Required Area

$$= \int_2^5 x\,dy$$

$$= \int_2^5 (2y+12)\,dy = \left[y^2+12y\right]_2^5$$

$$= (25+60)-(4+24) = 57 \text{ sq.units}$$

DO YOU KNOW?

The area bounded by the line $y = mx+c$ with x-axis between lines $x = 0$ and $x = a$ is
$= \left|\frac{1}{2}[a][(\text{value of } y \text{ at } x = 0) + [(\text{value of } y \text{ at } x = a)]\right|$

Example 3.3

Find the area of the region bounded by the parabola $y = 4-x^2$, $x-axis$ and the lines $x=0, x=2$

Solutions:

$$y = 4-x^2$$

$$\text{Required area} = \int_0^2 y\,dx = \int_0^2 (4-x^2)\,dx$$

$$= \left[4x-\frac{x^3}{3}\right]_0^2 = 8-\frac{8}{3}$$

$$= \frac{16}{3}\,sq.units$$

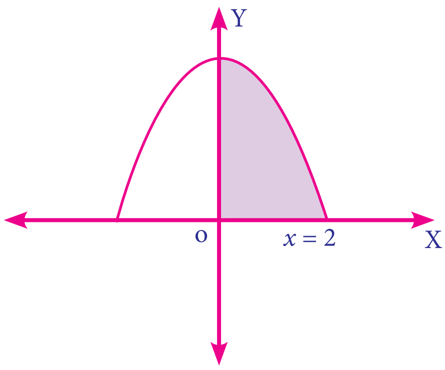


Fig. Fig.3.9

Example 3.4

Find the area bounded by $y = x$ between the lines $x = -1 \, and \, x = 2$ with x -axis.

Solution:

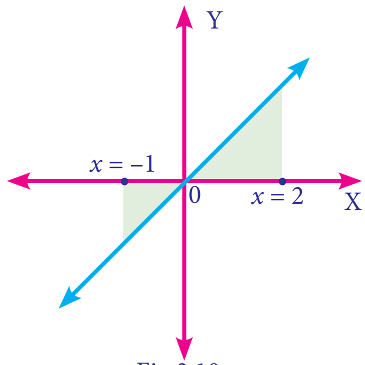


Fig.3.10

$$\text{Required area} = \int_{-1}^{0} -x dx + \int_{0}^{2} x\, dx$$

$$= -\left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{2} = -\left[0 - \frac{1}{2}\right] + \left[\frac{4}{2} - 0\right]$$

$$= \frac{5}{2} \text{ sq.units}$$

Example 3.5

Find the area of the parabola $y^2 = 8x$ bounded by its latus rectum.

Solution

$$y^2 = 8x \qquad (1)$$

Comparing this with the standard form $y^2 = 4ax$,

$4a = 8$

$a = 2$

Equation of latus rectum is $x = 2$

Since equation (1) is symmetrical about x- axis

Required Area = 2[Area in the first quadrant between the limits $x = 0$ and $x = 2$]

$$= 2\int_{0}^{2} y\, dx$$

$$= 2\int_{0}^{2} \sqrt{8x} dx = 2\left(2\sqrt{2}\right)\int_{0}^{2} x^{1/2}\, dx$$

$$= 4\sqrt{2}\left[\frac{2x^{\frac{3}{2}}}{3}\right]_{0}^{2} = 4\sqrt{2} \times 2 \times \frac{2^{\frac{3}{2}}}{3}$$

$$= \frac{32}{3} \text{ sq. units.}$$

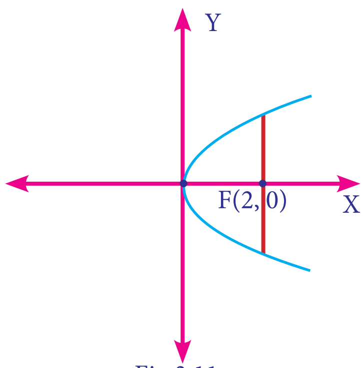


Fig.3.11

Example 3.6

Sketch the graph $y=|x+3|$ and evaluate $\int_{-6}^{0}|x+3|\,dx$.

Solution:

$$y=|x+3| = \begin{cases} x+3 & \text{if} \quad x\geq -3 \\ -(x+3) & \text{if} \quad x<-3 \end{cases}$$

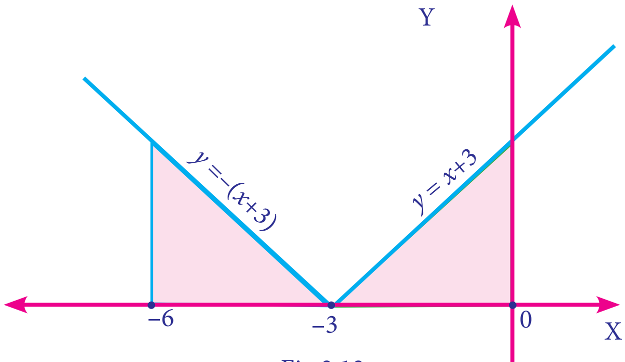


Fig.3.12

$$\text{Required area} = \int_{b}^{a} y\,dx = \int_{-6}^{0} y\,dx$$

$$= \int_{-6}^{-3} y\,dx + \int_{-3}^{0} y\,dx$$

$$= \int_{-6}^{-3} -(x+3)\,dx + \int_{-3}^{0} (x+3)\,dx$$

$$= -\left[\frac{(x+3)^2}{2}\right]_{-6}^{-3} + \left[\frac{(x+3)^2}{2}\right]_{-3}^{0} = -\left[0-\frac{9}{2}\right]+\left[\frac{9}{2}-0\right] = 9 \text{ sq. units}$$

Example 3.7

Using integration find the area of the circle whose center is at the origin and the radius is a units.

Solution

Equation of the required circle is $x^2+y^2=a^2$ (1)

put $y = 0,\ x^2 = a^2$

$\Rightarrow \quad x = \pm a$

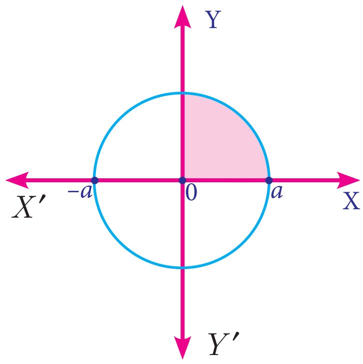


Fig.3.13

Since equation (1) is symmetrical about both the axes

The required area = 4 [Area in the first quadrant between the limit 0 and a.]

$$= 4\int_0^a y\,dx$$

$$= 4\int_0^a \sqrt{a^2-x^2}\,dx = 4\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[0+\frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right] = 4\left(\frac{a^2}{2}\sin^{-1}(1)\right) = 4.\frac{a^2}{2}\frac{\pi}{2}$$

$$= \pi a^2 \text{ sq. units}$$

Example 3.8

Using integration find the area of the region bounded between the line $x = 4$ and the parabola $y^2 = 16x$.

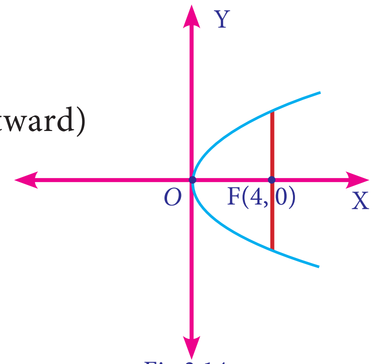


Fig.3.14

Solution:

The equation $y^2 = 16x$ represents a parabola (Open rightward)

$$\text{Required Area } = 2\int_a^b y\,dx$$

$$= 2\int_0^4 \sqrt{16x}\,dx$$

$$= 8\int_0^4 x^{\frac{1}{2}}\,dx = 8\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 = \frac{16}{3}\left((4)^{\frac{3}{2}}\right) = \frac{128}{3.} \text{ sq. units}$$

Exercise 3.1

1. Using Integration, find the area of the region bounded the line $2y + x = 8$, the x axis and the lines $x = 2$, $x = 4$.

2. Find the area bounded by the lines $y - 2x - 4 = 0, y = 1, \ y = 3$ and the y-axis

3. Calculate the area bounded by the parabola $y^2 = 4ax$ and its latusrectum.

4. Find the area bounded by the line $y = x$, the x-axis and the ordinates $x = 1$, $x = 2$.

5. Using integration, find the area of the region bounded by the line $y - 1 = x$, *the* x *axis* and the ordinates $x = -2$, $x = 3$.

6. Find the area of the region lying in the first quadrant bounded by the region $y = 4x^2, x = 0, y = 0$ and $y = 4$.

7. Find the area bounded by the curve $y = x^2$ and the line $y = 4$

3.2 Application of Integration in Economics and Commerce.

Integration helps us to find out the total cost function and total revenue function from the marginal cost. It is possible to find out consumer's surplus and producer's surplus from the demand and supply function. Cost and revenue functions are calculated through indefinite integral.

We learnt already that the marginal function is obtained by differentiating the total cost function. Now we shall obtain the total cost function when marginal cost function is given, by integration.

3.2.1 Cost functions from marginal cost functions

If C is the cost of producing an output x, then marginal cost function $MC = \frac{dc}{dx}$. Using integration, as the reverse process of differentiation, we obtain,

$$\text{Cost function} \quad C = \int (MC)\, dx + k$$

Where k is the constant of integration which is to be evaluated,

Average cost function $AC = \frac{C}{x},\ x \neq 0$

Example 3.9

The marginal cost function of manufacturing x shoes is $6 + 10x - 6x^2$. The cost producing a pair of shoes is ₹12. Find the total and average cost function.

Solution:

Given,

$$
\begin{aligned}
\text{Marginal cost } MC &= 6 + 10x - 6x^2 \\
C &= \int MC\, dx + k \\
&= \int \left(6 + 10x - 6x^2\right) dx + k \\
&= 6x + 5x^2 - 2x^3 + k \qquad (1)
\end{aligned}
$$

when $x = 2$, $\quad C = 12$ (given)

$$12 = 12 + 20 - 16 + k$$

$$\boxed{k = -4}$$

$$C = 6x + 5x^2 - 2x^3 - 4$$

$$
\begin{aligned}
\text{Average cost} &= \frac{C}{x} = \frac{6x + 5x^2 - 2x^3 - 4}{x} \\
&= 6 + 5x - 2x^2 - \frac{4}{x}
\end{aligned}
$$

Example 3.10

A company has determined that the marginal cost function for a product of a particular commodity is given by $MC = 125 + 10x - \frac{x^2}{9}$ where C rupees is the cost of producing x units of the commodity. If the fixed cost is ₹250 what is the cost of producing 15 units.

Solution:

$$MC = 125 + 10x - \frac{x^2}{9}$$

$$C = \int MC\,dx + k$$

$$= \int \left(125 + 10x - \frac{x^2}{9}\right) dx + k$$

$$= 125x + 5x^2 - \frac{x^3}{27} + k$$

Fixed cost $k = 250$

$$C = 125x + 5x^2 - \frac{x^3}{27} + 250$$

When $x = 15$

$$C = 125\,(15) + 5\,(15)^2 - \frac{(15)^3}{27} + 250$$

$$= 1875 + 1125 - 125 + 250$$

C = ₹3,125

Example 3.11

The marginal cost function $MC = 2 + 5e^x$ (i) Find C if $C\,(0) = 100$ (ii) Find AC.

Solution:

Given $MC = 2 + 5e^x$

$$C = \int MC\,dx + k$$

$$= \int \left(2 + 5e^x\right) dx + k$$

$$= 2x + 5e^x + k$$

$x = 0 \Rightarrow C = 100$,

$$100 = 2(0) + 5\left(e^0\right) + k$$

$$\boxed{k = 95}$$

$$C = 2x + 5e^{x} + 95.$$

$$\text{Average cost} = \frac{C}{x} = \frac{2x + 5e^{x} + 95}{x}$$

$$AC = 2 + \frac{5e^{x}}{x} + \frac{95}{x}.$$

Rate of growth or sale

If the rate of growth or sale of a function is a known function of t say $f(t)$ where t is a time measure, then total growth (or) sale of a product over a time period t is given by,

$$\text{Total sale} = \int_0^r f(t)\,dt,\ 0 \le t \le r$$

Example 3.12

The rate of new product is given by $f(x) = 100 - 90\,e^{-x}$ where x is the number of days the product is on the market. Find the total sale during the first four days. (e^{-4}=0.018)

Solution:

$$\text{Total sale} = \int_0^4 \left(100 - 90e^{-x}\right) dx$$

$$= \left(100x + 90e^{-x}\right)_0^4$$

$$= 400 + 90e^{-4} - (0 + 90)$$

$$= 400 + 90(0.018) - 90$$

$$= 311.62 \text{ units}$$

Example 3.13

A company produces 50,000 units per week with 200 workers. The rate of change of productions with respect to the change in the number of additional labour x is represented as $300 - 5x^{2/3}$. If 64 additional labours are employed, find out the additional number of units, the company can produce.

Solution:

Let p be the additional product produced for additional of x labour,

$$\frac{dp}{dx} = 300 - 5x^{2/3}$$

$$p = \int_0^{64}\left(300 - 5x^{\frac{2}{3}}\right)dx$$

$$= \left[300x - 3x^{\frac{5}{3}}\right]_0^{64}$$

$$= 300\times 64 - 3(64)^{5/3}$$

$$= 16128$$

∴ The number of additional units produced 16128

Total number of units produced by 264 workers

= 50,000 + 16,128 = 66128 units

Example 3.14

The rate of change of sales of a company after an advertisement campaign is represented as, $f(t) = 3000e^{-0.3t}$ where t represents the number of months after the advertisement. Find out the total cumulative sales after 4 months and the sales during the fifth month. Also find out the total sales due to the advertisement campaign $\left[e^{-1.2} = 0.3012,\ e^{-1.5} = 0.2231\right]$.

Solution:

Assume that $F(t)$ is the total sales after t months, sales rate is $\frac{d}{dt}F(t) = f(t)$

∴ $$F(t) = \int_0^t f(t)dt$$

Total cumulative sales after 4 months.

$$F(4) = \int_0^4 f(t)dt$$

$$= \int_0^4 3000e^{-0.3t}\,dt$$

$$= 3000\left[\frac{e^{-0.3t}}{-0.3}\right]_0^4$$

$$= -10{,}000\ \left[e^{-1.2} - e^0\right]$$

$$= -10{,}000\left[0.3012 - 1\right]$$

$$= 6988 \text{ units}$$

(ii) Sales during the 5th month $= \int_4^5 3000 e^{-0.3t}\, dt$

$$= 3000 \left[\frac{e^{-0.3t}}{-0.3} \right]_4^5$$

$$= -10{,}000 \left[e^{-1.5} - e^{-1.2} \right]$$

$$= -10{,}000 [0.2231 - 0.3012]$$

$$= 781 \text{ units}$$

Total sales due to the advertisement campaign.

$$= \int_0^{\infty} 3000 e^{-0.3t}\, dt = \frac{3000}{-0.3} \left[e^{-0.3t} \right]_0^{\infty}$$

$$= -10000[0 - 1]$$

$$= 10{,}000 \text{ untis.}$$

Example 3.15

The price of a machine is 6,40,000 if the rate of cost saving is represented by the function $f(t) = 20{,}000\, t$. Find out the number of years required to recoup the cost of the function.

Solution:

Saving Cost $\quad S(t) = \int_0^t 20000 t\, dt$

$$= 10000\, t^2$$

To recoup the total price,

$$10000\, t^2 = 640000$$

$$t^2 = 64$$

$$t = 8$$

When $t = 8$ years, one can recoup the price.

3.2.2 Revenue functions from Marginal revenue functions

If R is the total revenue function when the output is x, then marginal revenue $MR = \frac{dR}{dx}$ Integrating with respect to 'x' we get

Revenue Function, $R = \int (MR)\, dx + k.$

Where 'k' is the constant of integration which can be evaluated under given conditions, when $x = 0$, the total revenue $R = 0$,

$$\text{Demand Function, } P = \frac{R}{x}, \quad x \neq 0.$$

Example 3.16

For the marginal revenue function $MR = 35 + 7x - 3x^2$, find the revenue function and demand function.

Solution:

$$\text{Given} \quad MR = 35 + 7x - 3x^2$$

$$R = \int (MR)\, dx + k$$

$$= \int (35 + 7x - 3x^2)\, dx + k$$

$$R = 35x + \frac{7}{2}x^2 - x^3 + k$$

Since R = 0 when $x = 0$, $\quad k = 0$

$$R = 35x + \frac{7}{2}x^2 - x^3$$

$$\text{Demand function } P = \frac{R}{x}$$

$$P = 35 + \frac{7}{2}x - x^2.$$

Example 3.17

A firm has the marginal revenue function given by $MR = \frac{a}{(x+b)^2} - c$ where x is the output and a, b, c are constants. Show that the demand function is given by $x = \frac{a}{b\,(p+c)} - b$.

Solution:

$$\text{Given} \quad \text{MR} = a\,(x+b)^{-2} - c$$

$$R = \int a(x+b)^{-2}\, dx - c\int dx$$

$$R = \frac{a\,(x+b)^{-1}}{-1} - cx + k$$

$$R = -\frac{a}{x+b} - cx + k$$

When $\quad x = 0, \ R = 0$

$$\therefore \qquad 0 = -\frac{a}{b} - c(0) + k$$

$$k = \frac{a}{b}$$

$$R = -\frac{a}{x+b} - cx + \frac{a}{b}$$

$$= \frac{-ab + a\left(x+b\right)}{b\left(x+b\right)} - cx$$

$$R = \frac{ax}{b\left(x+b\right)} - cx$$

Demand function $\qquad P = \dfrac{R}{x}$

$$P = \frac{a}{b\left(x+b\right)} - c$$

$$P+c = \frac{a}{b\left(x+b\right)}$$

$$b(x+b) = \frac{a}{P+c}$$

$$x = \frac{a}{b\left(P+c\right)} - b.$$

To find the Maximum Profit if Marginal Revenue and Marginal cost function are given:

If 'P' denotes the profit function, then $\dfrac{dP}{dx} = \dfrac{d}{dx}\left(R-C\right) = \dfrac{dR}{dx} - \dfrac{dC}{dx} = MR - MC$

Integrating both sides with respect to x gives , $P = \int\left(MR - MC\right)dx + k$

Where k is the constant of integration. However if we are given additional information, such as fixed cost or loss at zero level of output, we can determine the constant k. Once P is known, it can be maximum by using the concept of maxima and minima.

Example 3.18

The marginal cost $C'\left(x\right)$ and marginal revenue $R'\left(x\right)$ are given by $C'\left(x\right) = 50 + \dfrac{x}{50}$ *and* $R'\left(x\right) = 60$. The fixed cost is ₹200. Determine the maximum profit.

Solution:

Given $\qquad C(x) = \int C'\left(x\right)dx + k_1$

$$= \int\left(50+\frac{x}{50}\right)dx + k_1$$

$$C(x) = 50x+\frac{x^2}{100}+k_1$$

When quantity produced is zero, then the fixed cost is 200.

i.e. When $x = 0, c = 200$

$$\Rightarrow \quad k_1 = 200$$

Cost function is $C(x) = 50x+\frac{x^2}{100}+200$ (1)

The Revenue $R'(x) = 60$

$$R(x) = \int R'(x)\,dx+k_2$$

$$= \int 60\,dx+k_2$$

$$= 60x + k_2$$

When no product is sold, revenue = 0

i.e. When $x=0, R=0.$

Revenue $R(x) = 60x$ (2)

Profit P = Total Revenue – Total cost

$$= 60x-50x-\frac{x^2}{100}-200$$

$$= 10x-\frac{x^2}{100}-200$$

$$\frac{dp}{dx} = 10-\frac{x}{50}$$

To get profit maximum, $\frac{dp}{dx} = 0 \quad \Rightarrow \quad \boxed{x=500.}$

$$\frac{d^2P}{dx^2} = \frac{-1}{50}<0$$

$\therefore$ Profit is maximum when $x=500$ and

Maximum Profit is $P = 10(500)-\frac{(500)^2}{100}-200$

$$= 5000 - 2500 - 200$$

$$= 2300$$

$$\text{Profit} = ₹\ 2,300.$$

Example 3.19

The marginal cost and marginal revenue with respect to commodity of a firm are given by $C'(x) = 8 + 6x$ and $R'(x) = 24$. Find the total Profit given that the total cost at zero output is zero.

Solution:

Given $\quad MC = 8 + 6x$

$$C(x) = \int (8 + 6x)\,dx + k_1$$

$$= 8x + 3x^2 + k_1 \qquad (1)$$

But given when $x = 0, C = 0 \quad \Rightarrow \quad k_1 = 0$

$$\therefore \quad C(x) = 8x + 3x^2 \qquad (2)$$

Given that $\quad MR = 24$

$$R(x) = \int MR\,dx + k_2$$

$$= \int 24\,dx + k_2$$

$$= 24x + k_2$$

Revenue = 0, when $x = 0 \quad \Rightarrow \quad k_2 = 0$

$$R(x) = 24x \qquad (3)$$

Total Profit functions $\quad P(x) = R(x) - C(x)$

$$P(x) = 24x - 8x - 3x^2$$

$$= 16x - 3x^2$$

Example 3.20

The marginal revenue function (in thousand of rupees) of a commodity is $10 + e^{-0.05x}$ Where x is the number of units sold. Find the total revenue from the sale of 100 units $(e^{-5} = 0.0067)$

Solution:

Given, Marginal revenue $R'(x) = 10 + e^{-0.05x}$

Total revenue from sale of 100 units is

$$R = \int_0^{100} \left(10 + e^{-0.05x}\right) dx$$

$$= \left[10x + \frac{e^{-0.05x}}{-0.05}\right]_0^{100}$$

$$= \left(1000 - \frac{e^{-5}}{0.05}\right) - \left(0 - \frac{100}{5}\right)$$

$$= 1000 + 20 - (20 \times 0.0067)$$

$$= 1019.87$$

$$\text{Total revenue} = 1019.87 \times 1000$$

$$= ₹10,19,870$$

Example 3.21

The price of a machine is ₹5,00,000 with an estimated life of 12 years. The estimated salvage value is ₹30,000. The machine can be rented at ₹72,000 per year. The present value of the rental payment is calculated at 9% interest rate. Find out whether it is advisable to rent the machine. $(e^{-1.08} = 0.3396)$.

Solution:

The present value of payment for t year $= \int_0^t 72000 e^{-0.09t}\, dt$

$$\text{Present value of 12 years} = \int_0^{12} 72000 e^{-0.09t}\, dt$$

$$= 72000 \left[\frac{e^{-0.09t}}{-0.09}\right]_0^{12}$$

$$= \frac{72000}{-0.09}\left[e^{-0.09(12)} - e^0\right]$$

$$= -8,00,000\left[e^{-1.08} - e^0\right]$$

$$= -8,00,000[0.3396 - 1]$$

$$= 5,28,320$$

$$\text{Cost of the machine} = 5,00,000 - 30,000$$

$$= 4,\ 70,\ 000$$

Hence it not advisable to rent the machine

It is better to buy the machine.

Inventory :

Given the inventory on hand $I(x)$ and the unit holding cost (C_1), the total inventory carrying cost is $C_1 \int_0^T I(x)dx$, where T is the time period under consideration.

Example 3.22

A company receives a shipment of 200 cars every 30 days. From experience it is known that the inventory on hand is related to the number of days. Since the last shipment, $I(x) = 200 - 0.2x$. Find the daily holding cost for maintaining inventory for 30 days if the daily holding cost is ₹3.5

Solution:

Here $\quad I(x) = 200 - 0.2x$

$C_1 = ₹\ 3.5$

$T = 30$

Total inventory carrying cost $= C_1 \int_0^r I(x)dx = 3.5 \int_0^{30} (200 - 0.2x)dx$

$$= 3.5\left(200x - \frac{0.2x^2}{2}\right)_0^{30} = 20{,}685$$

Amount of an Annuity

The amount of an annuity is the sum of all payments made plus all interest accumulated. Let an annuity consist of equal payments of Rs. p and let the interest rate of r percent annually be compounded continuously.

Amount of annuity after N payments $A = \int_0^N pe^{rt}\, dt$

Example 3.23

Mr. Arul invests ₹10,000 in ABC Bank each year, which pays an interest of 10% per annum compounded continuously for 5 years. How much amount will there be after 5 years. $\left(e^{0.5} = 1.6487\right)$

Solution:

$$p = 10000, \quad r = 0.1, \qquad N = 5$$

$$\text{Annuity} = \int_0^5 10000 e^{0.1t}\, dt$$

$$= \frac{10000}{0.1}\left(e^{0.1t}\right)_0^5$$

$$= 100000\left[e^{0.1\times 5} - e^{0}\right]$$

$$= 100000\left(e^{0.5} - 1\right)$$

$$= 100000\left[0.6487\right]$$

$$= ₹\ 64{,}870$$

Consumption of a Natural Resource

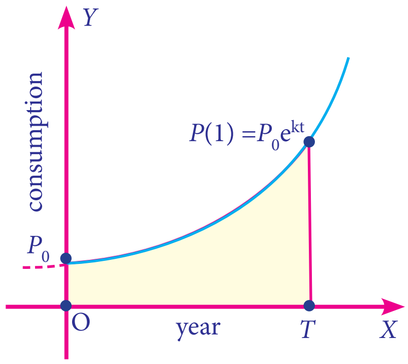


Fig. 3.15

Suppose that $p(t)$ is the annual consumption of a natural resource in year t. If the consumption of the resource is growing exponentially at growth rate k, then the total consumption of the resource after T years is given by

$$\int_0^T p_0 e^{kt}\, dt = \frac{p_0}{k}\left(e^{kT} - 1\right)$$

Where p_0 is the initial annual consumption at time $t = 0$.

Example 3.24

In year 2000 world gold production was 2547 metric tons and it was growing exponentially at the rate of 0.6% per year. If the growth continues at this rate, how many tons of gold will be produced from 2000 to 2013? [$e^{0.078} = 1.0811$)

Solution:

Annual consumption at time $t = 0$ (In the year 2000) $= p_0 = 2547$ metric ton.

Total production of Gold from 2000 to 2013 $= \int_0^{13} 2547 e^{0.006t}\, dt$

$$= \frac{2547}{0.006}\left[e^{0.006t}\right]_0^{13}$$

$$= 424500\left(e^{0.078} - 1\right)$$

= 34,426.95 metric tons approximately.

3.2.3 The demand functions from elasticity of demand

Elasticity of the function $y = f(x)$ at a point x is defined as the limiting case of ratio of the relative change in y to the relative change in x.

$$\therefore \qquad \eta = \frac{E_y}{E_x} = \lim_{\Delta_x \to 0} \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \frac{\frac{dy}{y}}{\frac{dx}{x}}$$

$$\Rightarrow \qquad \eta = \frac{x}{y} \cdot \frac{dy}{dx}$$

Elasticity of demand $\eta_d = \frac{-p}{x} \frac{dx}{dp}$

$$\frac{-dp}{p} = \frac{dx}{x} \cdot \frac{1}{\eta_d}$$

Integrating both sides w.r. to x

$$-\int \frac{dp}{p} = \frac{1}{\eta_d} \int \frac{dx}{x}$$

Equation yields the demand function 'p' as a function of x.

The revenue function can be found out by using integration.

Example 3.25

When the Elasticity function is $\frac{x}{x-2}$. Find the function when $x = 6$ and $y = 16$.

Solution:

$$\frac{E_y}{E_x} = \frac{x}{x-2}$$

$$\frac{x}{y} \frac{dy}{dx} = \frac{x}{x-2}$$

$$\frac{dy}{y} = \frac{x}{x-2} \cdot \frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x-2}$$

$$\log y = \log(x-2) + \log k$$

$$y = k(x-2)$$

when $x = 6$, $y = 16$ $\Rightarrow$ $16 = k(6-2)$

$$k = 4$$

$$y = 4\,(x-2)$$

Example 3.26

The elasticity of demand with respect to price p for a commodity is $\eta_d = \dfrac{p+2p^2}{100-p-p^2}$. Find demand function where price is ₹5 and the demand is 70.

Solution

$$\eta_d = \frac{p+2p^2}{100-p-p^2}$$

$$\frac{-p}{x}\frac{dx}{dp} = \frac{p(2p+1)}{100-p-p^2}$$

$$\frac{-dx}{x} = \frac{-(2p+1)}{p^2+p-100}dp$$

$$\int \frac{dx}{x} = \int \frac{2p+1}{p^2+p-100}dp$$

$$\log x = \log(p^2+p=100)+\log k$$

$$\therefore \quad x = k(p^2+p-100)$$

When $x=70, \quad p=5,$

$$70 = k(25+5-100)$$

$$\Rightarrow k = -1$$

Hence $\quad x = 100-p-p^2$

$$R = px$$

Revenue $\quad = p(100-p-p^2)$

Exercise 3.2

1. The cost of over haul of an engine is ₹10,000 The operating cost per hour is at the rate of $2x-240$ where the engine has run x km. Find out the total cost if the engine run for 300 hours after overhaul.

2. Elasticity of a function $\dfrac{Ey}{Ex}$ is given by $\dfrac{Ey}{Ex} = \dfrac{-7x}{(1-2x)(2+3x)}$. Find the function when $x=2, y=\dfrac{3}{8}$.

3. The elasticity of demand with respect to price for a commodity is given by $\frac{(4-x)}{x}$, where p is the price when demand is x. Find the demand function when price is 4 and the demand is 2. Also find the revenue function.

4. A company receives a shipment of 500 scooters every 30 days. From experience it is known that the inventory on hand is related to the number of days x. Since the shipment, $I(x)=500-0.03x^2$, the daily holding cost per scooter is ₹ 0.3. Determine the total cost for maintaining inventory for 30 days.

5. An account fetches interest at the rate of 5% per annum compounded continuously An individual deposits ₹1,000 each year in his account. How much will be in the account after 5 years. $(e^{0.25}=1.284)$.

6. The marginal cost function of a product is given by $\frac{dC}{dx}=100-10x+0.1x^2$ where x is the output. Obtain the total and the average cost function of the firm under the assumption, that its fixed cost is ₹ 500.

7. The marginal cost function is $MC=300\,x^{\frac{2}{5}}$ and fixed cost is zero. Find out the total cost and average cost functions.

8. If the marginal cost function of x units of output is $\frac{a}{\sqrt{ax+b}}$ and if the cost of output is zero. Find the total cost as a function of x.

9. Determine the cost of producing 200 air conditioners if the marginal cost (is per unit) is $C'(x)=\frac{x^2}{200}+4$.

10. The marginal revenue (in thousands of Rupees) functions for a particular commodity is $5+3\,e^{-0.03\,x}$ where x denotes the number of units sold. Determine the total revenue from the sale of 100 units. (Given $e^{-3}=0.05$ approximately)

11. If the marginal revenue function for a commodity is $MR=9-4x^2$. Find the demand function.

12. Given the marginal revenue function $\frac{4}{(2x+3)^2}-1$, show that the average revenue function is $P=\frac{4}{6x+9}-1$.

13. A firm's marginal revenue function is $MR=20e^{-x/10}\left(1-\frac{x}{10}\right)$. Find the corresponding demand function.

14. The marginal cost of production of a firm is given by $C'(x) = 5 + 0.13x$, the marginal revenue is given by $R'(x) = 18$ and the fixed cost is ₹ 120. Find the profit function.

15. If the marginal revenue function is $R'(x) = 1500 - 4x - 3x^2$. Find the revenue function and average revenue function.

16. Find the revenue function and the demand function if the marginal revenue for x units is $MR = 10 + 3x - x^2$.

17. The marginal cost function of a commodity is given by $MC = \dfrac{14000}{\sqrt{7x+4}}$ and the fixed cost is ₹18,000.Find the total cost and average cost.

18. If the marginal cost (MC) of a production of the company is directly proportional to the number of units (x) produced, then find the total cost function, when the fixed cost is ₹ 5,000 and the cost of producing 50 units is ₹ 5,625.

19. If $MR = 20 - 5x + 3x^2$, find total revenue function.

20. If $MR = 14 - 6x + 9x^2$, find the demand function.

3.2.4 Consumer's surplus:

This theory was developed by the great economist Marshal. The demand function reveals the relationship between the quantities that the people would buy at a given price. It can be expressed as $p = f(x)$

Let us assume that the demand of the product $x = x_0$ when the price is p_0. But there can be some consumer who is ready to pay q_0 which is more than p_0 for the same quantity x_0. Any consumer who is ready to pay the price more than p_0 gains from the fact that the price is only p_0. This gain is called the consumer's surplus.

It is represented in the following diagram

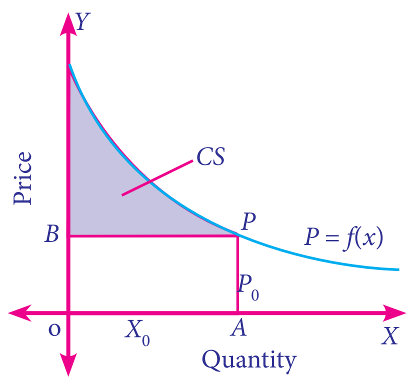


Fig. 3.16

Mathematically the Consumer's Surplus (CS) can be defined as

CS = (Area under the demand curve from $x = 0$ to $x = x_0$)

– (Area of the rectangle OAPB)

$$CS = \left[\int_0^{x_0} f(x)\,dx\right] - x_0 p_0$$

Example 3.27

The demand function of a commodity *is* $y = 36 - x^2$. Find the consumer's surplus for $y_0 = 11$

Solution:

Given $y = 36 - x^2$ and $y_0 = 11$

$$11 = 36 - x^2$$

$$x^2 = 25$$

$$x = 5$$

$$\text{CS} = \int_0^x (\text{ demand function})dx - (\text{Price} \times \text{quantity demanded})$$

$$= \int_0^5 \left(36 - x^2\right) dx - 5 \times 11$$

$$= \left[36x - \frac{x^3}{3}\right]_0^5 - 55$$

$$= \left[36(5) - \frac{5^3}{3}\right] - 55$$

$$= 180 - \frac{125}{3} - 55 = \frac{250}{3}$$

Hence the consumer's surplus is $= \dfrac{250}{3}$ units.

DO YOU KNOW?

The point of intersection of demand and supply curves is called equilibrium point.

At equilibrium point $q_d = q_s$

3.2.5 Producer surplus

A supply function $g(x)$ represents the quantity that can be supplied at a price p. Let p_0 be the market price for the corresponding supply x_o. But there can be some producers who are willing to supply the commodity below the market price gain from the fact that the price is p_0. This gain is called the producer's surplus. It is represented in the following diagram.

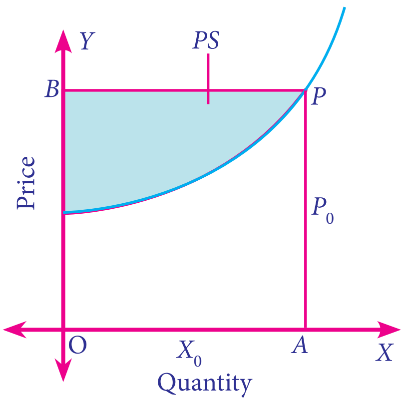


Fig. 3.17

Mathematically, producer's surplus (PS) can be defined as,

PS = (Area of the rectangle OAPB) − (Area below the supply function from $x = 0$ to $x = x_0$)

$$\text{PS} = x_0\, p_0 - \int_0^{x_0} g(x)dx$$

Example 3.28

Find the producer's surplus defined by the supply curve $g(x) = 4x+8$ when $x_o = 5$.

Solution:

$$g(x) = 4x + 8 \text{ and } x_0 = 5$$

$$p_0 = 4(5) + 8 = 28$$

$$\text{PS} = x_0 p_0 - \int_0^{x_0} g(x)dx$$

$$= (5 \times 28) - \int_0^5 (4x+8)dx$$

$$= 140 - \left[4\left(\frac{x^2}{2}\right) + 8x\right]_0^5$$

$$= 140 - (50 + 40)$$

$$= 50 \text{ units}$$

Hence the producer's surplus= 50 units.

Example 3.29

The demand and supply function of a commodity are $p_d = 18 - 2x - x^2$ and $p_s = 2x - 3$. Find the consumer's surplus and producer's surplus at equilibrium price.

Solution:

Given $P_d = 18 - 2x - x^2$; $P_s = 2x - 3$

We know that at equilibrium prices $p_d = p_s$

$$18 - 2x - x^2 = 2x - 3$$

$$x^2 + 4x - 21 = 0$$

$$(x-3)(x+7) = 0$$

$$x = -7 \text{ or } 3$$

The value of x cannot be negative, $x = 3$

When $x_0 = 3$

$$\therefore \quad p_0 = 18 - 2(3) - (3)^2 = 3$$

$$CS = \int_0^x f(x)dx - x_0 p_0$$

$$= \int_0^3 (18 - 2x - x^2)dx \quad - 3 \times 3$$

$$= \left[18x - x^{2} - \frac{x^3}{3}\right]_0^3 - 9$$

$$= 18(3) - (3)^2 - \left(\frac{3^3}{3}\right) - 9$$

$$CS = 27 \text{ units}$$

$$PS = x_0 P_0 - \int_0^x g(x)\,dx$$

$$= (3 \times 3) - \int_0^3 (2x - 3)\,dx$$

$$= 9 - \left(x^2 - 3x\right)_0^3$$

$$= 9 \text{ units}$$

Hence at equilibrium price,

(i) the consumer's surplus is 27 units

(ii) the producer's surplus is 9 units.

Exercise 3.3

1. Calculate consumer's surplus if the demand function $p = 50 - 2x$ and $x = 20$

2. Calculate consumer's surplus if the demand function $p = 122 - 5x - 2x^2$ and $x = 6$

3. The demand function $p = 85 - 5x$ and supply function $p = 3x - 35$. Calculate the equilibrium price and quantity demanded .Also calculate consumer's surplus.

4. The demand function for a commodity is $p = e^{-x}$. Find the consumer's surplus when $p = 0.5$.

5. Calculate the producer's surplus at $x = 5$ for the supply function $p = 7 + x$.

6. If the supply function for a product is $p = 3x + 5x^2$ .Find the producer's surplus when $x = 4$.

7. The demand function for a commodity is $p = \frac{36}{x+4}$. Find the consumer's surplus when the prevailing market price is ₹6.

8. The demand and supply functions under perfect competition are $p_d = 1600 - x^2$ and $p_s = 2x^2 + 400$ respectively. Find the producer's surplus.

9. Under perfect competition for a commodity the demand and supply laws are $p_d = \frac{8}{x+1} - 2$ and $p_s = \frac{x+3}{2}$ respectively. Find the consumer's and producer's surplus.

10. The demand equation for a products is $x = \sqrt{100-p}$ and the supply equation is $x = \frac{p}{2} - 10$. Determine the consumer's surplus and producer's surplus, under market equilibrium.

11. Find the consumer's surplus and producer's surplus for the demand function $p_d = 25 - 3x$ and supply function $p_s = 5 + 2x$.

J3FXVR

Exercise 3.4

Choose the best answer form the given alternatives

1. Area bounded by the curve $y = x(4-x)$ between the limits 0 and 4 with $x-$ axis is

   (a) $\frac{30}{3}$ sq.units (b) $\frac{31}{2}$ sq.units (c) $\frac{32}{3}$ sq.units (d) $\frac{15}{2}$ sq.units

2. Area bounded by the curve $y = e^{-2x}$ between the limits $0 \le x \le \infty$ is

   (a) 1 sq.units (b) $\frac{1}{2}$ sq.unit (c) 5 sq.units (d) 2 sq.units

3. Area bounded by the curve $y = \frac{1}{x}$ between the limits 1 and 2 is

   (a) log2 sq.units (b) log5 sq.units (c) log3 sq.units (d) log 4 sq.units

4. If the marginal revenue function of a firm is $MR = e^{\frac{-x}{10}}$, then revenue is

   (a) $-10e^{\frac{-x}{10}}$ (b) $1 - e^{\frac{-x}{10}}$ (c) $10\left(1 - e^{\frac{-x}{10}}\right)$ (d) $e^{\frac{-x}{10}} + 10$

5. If MR and MC denotes the marginal revenue and marginal cost functions, then the profit functions is

   (a) $P = \int (MR - MC)\, dx + k$ (b) $P = \int (MR + MC)\, dx + k$

   (c) $P = \int (MR)(MC) dx + k$ (d) $P = \int (R - C) dx + k$

6. The demand and supply functions are given by $D(x) = 16 - x^2$ and $S(x) = 2x^2 + 4$ are under perfect competition, then the equilibrium price x is

   (a) 2 (b) 3 (c) 4 (d) 5

7. The marginal revenue and marginal cost functions of a company are $MR = 30 - 6x$ and $MC = -24 + 3x$ where x is the product, then the profit function is

(a) $9x^2 + 54x$ (b) $9x^2 - 54x$ (c) $54x - \frac{9x^2}{2}$ (d) $54x - \frac{9x^2}{2} + k$

8. The given demand and supply function are given by $D(x) = 20 - 5x$ *and* $S(x) = 4x + 8$ if they are under perfect competition then the equilibrium demand is

(a) 40 (b) $\frac{41}{2}$ (c) $\frac{40}{3}$ (d) $\frac{41}{5}$

9. If the marginal revenue $MR = 35 + 7x - 3x^2$, then the average revenue AR is

(a) $35x + \frac{7x^2}{2} - x^3$ (b) $35 + \frac{7x}{2} - x^2$

(c) $35 + \frac{7x}{2} + x^2$ (d) $35 + 7x + x^2$

10. The profit of a function $p(x)$ is maximum when

(a) $MC - MR = 0$ (b) MC=0 (c) MR=0 (d) MC+MR=0

11. For the demand function $p(x)$, the elasticity of demand with respect to price is unity then

(a) revenue is constant (b) cost function is constant

(c) profit is constant (d) none of these

12. The demand function for the marginal function $MR = 100 - 9x^2$ is

(a) $100 - 3x^2$ (b) $100x - 3x^2$ (c) $100x - 9x^2$ (d) $100 + 9x^2$

13. When $x_0 = 5$ *and* $p_0 = 3$ the consumer's surplus for the demand function $p_d = 28 - x^2$ is

(a) 250 units (b) $\frac{250}{3}$ *units* (c) $\frac{251}{2}$ *units* (d) $\frac{251}{3}$ *units*

14. When $x_0 = 2$ *and* $P_0 = 12$ the producer's surplus for the supply function $P_s = 2x^2 + 4$ is

(a) $\frac{31}{5}$ *units* (b) $\frac{31}{2}$ *units* (c) $\frac{32}{3}$ *units* (d) $\frac{30}{7}$ *units*

15. Area bounded by $y = x$ between the lines $y = 1, y = 2$ with $y = axis$ is

(a) $\frac{1}{2}$ sq.units (b) $\frac{5}{2}$ sq.units (c) $\frac{3}{2}$ sq.units (d) 1 sq.unit

16. The producer's surplus when the supply function for a commodity is $P = 3 + x \ and \ x_0 = 3$ is

(a) $\frac{5}{2}$ (b) $\frac{9}{2}$ (c) $\frac{3}{2}$ (d) $\frac{7}{2}$

17. The marginal cost function is $MC = 100\sqrt{x}$. find AC given that TC =0 when the out put is zero is

(a) $\frac{200}{3}x^{\frac{1}{2}}$ (b) $\frac{200}{3}x^{\frac{3}{2}}$ (c) $\frac{200}{3x^{\frac{3}{2}}}$ (d) $\frac{200}{3x^{\frac{1}{2}}}$

18. The demand and supply function of a commodity are $P(x) = (x-5)^2$ and $S(x) = x^2 + x + 3$ then the equilibrium quantity x_0 is

(a) 5 (b) 2 (c) 3 (d) 19

19. The demand and supply function of a commodity are $D(x) = 25 - 2x$ and $S(x) = \frac{10+x}{4}$ then the equilibrium price P_0 is

(a) 5 (b) 2 (c) 3 (d) 10

20. If MR and MC denote the marginal revenue and marginal cost and $MR - MC = 36x - 3x^2 - 81$, then the maximum profit at x is equal to

(a) 3 (b) 6 (c) 9 (d)5

21. If the marginal revenue of a firm is constant, then the demand function is

(a) MR (b) MC (c) $C(x)$ (d) AC

22. For a demand function p, if $\int \frac{dp}{p} = k \int \frac{dx}{x}$ then k is equal to

(a) η_d (b) $-\eta_d$ (c) $\frac{-1}{\eta_d}$ (d) $\frac{1}{\eta_d}$

23. Area bounded by $y = e^x$ between the limits 0 to 1 is

(a) $(e-1)$ sq.units (b) $(e+1)$ sq.units (c) $\left(1-\frac{1}{e}\right)$ sq.units (d) $\left(1+\frac{1}{e}\right)$ sq.units

24. The area bounded by the parabola $y^2 = 4x$ bounded by its latus rectum is

(a) $\frac{16}{3}$ *sq.units* (b) $\frac{8}{3}$ *sq.units* (c) $\frac{72}{3}$ *sq.units* (d) $\frac{1}{3}$ *sq.units*

25. Area bounded by $y = |x|$ between the limits 0 and 2 is

(a) 1sq.units (b) 3 sq.units (c) 2 sq.units (d) 4 sq.units

Miscellaneous problems

1. A manufacture's marginal revenue function is given by MR$=275-x-0.3x^2$. Find the increase in the manufactures total revenue if the production is increased from 10 to 20 units.

2. A company has determined that marginal cost function for x product of a particular commodity is given by $MC=125+10x-\frac{x^2}{9}$. Where C is the cost of producing x units of the commodity. If the fixed cost is ₹ 250 what is cost of producing 15 units

3. The marginal revenue function for a firm is given by $MR=\frac{2}{x+3}-\frac{2x}{(x+3)^2}+5$. Show that the demand function is $P=\frac{2}{x+3}+5$.

4. For the marginal revenue function $MR=6-3x^2-x^3$, Find the revenue function and demand function.

5. The marginal cost of production of a firm is given by $C'(x)=20+\frac{x}{20}$ the marginal revenue is given by $R'(x)=30$ and the fixed cost is ₹ 100. Find the profit function.

6. The demand equation for a product is $p_d=20-5x$ and the supply equation is $p_s=4x+8$. Determine the consumer's surplus and producer's surplus under market equilibrium.

7. A company requires $f(x)$ number of hours to produce 500 units. It is represented by $f(x)=1800x^{-0.4}$. Find out the number of hours required to produce additional 400 units. [$(900)^{0.6}$=59.22, $(500)^{0.6}$=41.63]

8. The price elasticity of demand for a commodity is $\frac{p}{x^3}$. Find the demand function if the quantity of demand is 3, when the price is ₹2.

9. Find the area of the region bounded by the curve between the parabola $y=8x^2-4x+6$ the y-axis and the ordinate at x = 2.

10. Find the area of the region bounded by the curve $y^2=27x^3$ and the lines x = 0, y = 1 and y = 2.

Summary

- The area of the region bounded by the curve $y=f(x)$ between limits $x=a$ and $x=b$ with x –axis if area lies above x-axis is $=\int_a^b y\,dx$.
- The area of the region bounded by the curve $y=f(x)$ between the limits $x=a$ and $x=b$ with $x-axis$ if area lies below $x-axis$ is = $\int_a^b -y dx$

- The area of the region bounded by the curve $x = g(y)$ between te limits $y = c$ and $y = d$ with y - axis if the area lies to the right of y- axis is $= \int_c^d x \, dy$
- The area of the region bounded by the curve $x = g(y)$ between the limits $y=d$ and $y=e$ with y-axis if the area lies to the left of y- axis is $= \int_d^e -x \, dy$
- The area between the two given curves $y=f(x)$ and $y=g(x)$ from $x = a$ to $x = b$, is $\int_a^b (f(x) - g(x)) \, dx$.
- If the rate of growth or sale of a function is a known function of t say $f(t)$ where t is a time measure, then total growth (or) sale of a product over a time period t is given by,

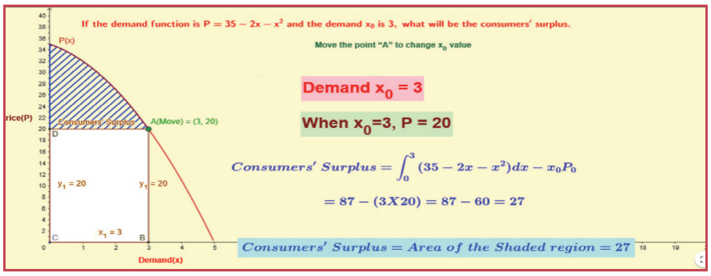
  Total sale = $\int_0^r f(t) dt, \; 0 \le t \le r$
- Elasticity of demand is $\eta_d = \frac{-p}{x} \frac{dx}{dp}$
- Total inventory carrying cost $= c_1 \int_0^T I(x) dx$
- Amount of annuity after N Payment is $A = \int_o^N pe^{rt} \, dt$
- Cost function is $C = \int (MC) \, dx + k$.
- Average cost function is $AC = \frac{C}{x}, x \neq 0$
- Revenue function is $R = \int (MR) dx + k$.
- Demand function is $P = \frac{R}{x}$
- Profit function is =MR–MC $= R'(x) - C'(x)$
- Consumer's surplus = $\int_0^{x_0} f(x) \, dx - x_0 p_0$
- Producer's surplus = $x_0 p_0 - \int_0^{x_0} p(x) \, dx$

GLOSSARY	
Cost function	செலவுச்சார்பு
Marginal cost function	இறுதிநிலை செலவுச்சார்பு
Integration	தொகையீடல்
Average cost function	சராசரி செலவுச்சார்பு
Fixed cost	மாறாச்செலவு
Out put	வெளியீடு
Reveunue function	வருவாய்ச்சார்பு

Marginal revenue function	இறுதிநிலை வருவாய்ச்சார்பு
Demand function	தேவைச்சார்பு
Supply function	அளிப்புச்சார்பு
Consumer's surplus	நுகர்வோர் உபரி
Producer's surplus	உற்பத்தியாளர் உபரி
Equilibrium	சமநிலை
Profit	இலாபம்
Maximum Profit	அதிகபட்ச இலாபம்
Manufacturer	உற்பத்தியாளர்
Production	உற்பத்தி
Inventory	சரக்குஇருப்பு
Annuity	பங்கீட்டு தவணைத்தொகை

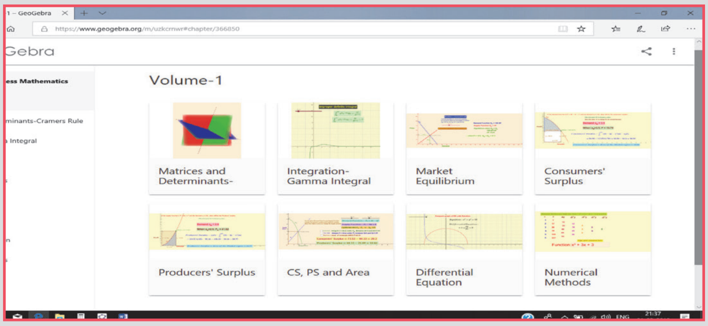
ICT Corner

Expected Result is shown in this picture



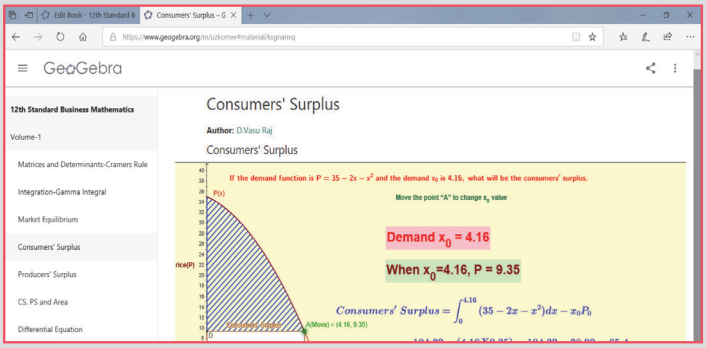
Step 1

Open the Browser, type the URL Link given below (or) Scan the QR Code. GeoGebra work Book named "12th Standard Business Mathematics " will open. In the work book there are two Volumes. Select "Volume-1".



Step 2

Select the worksheet named" Consumers' Surplus". There is a problem based on Consumers' Surplus using Integration. Move the point "A" on the curve. Observe the graph, formula applied and the result.



Browse in the link

12th standard Business Mathematics :https://ggbm.at/uzkcrnwr or Scan the QR Code.

B284_12_BUS_MAT_EM

4 Differential Equations

Gottfried Wilhelm Leibniz

(1st July 1646 - 14th November 1716)

Introduction

Differential equations, began with Gottfried Wilhelm Leibniz. He was a Germen Philosopher, mathematicians, and logician.

In lower class, we have studied algebraic equations like $5x-3(x-6)=4x$, $x^2-7x+12=0$, $|18x-5|=3$. The goal here was to solve the equation, which meant to find the value (or values) of the variable that makes the equation true.

For example, $x=9$ is the solution to the first equation because only where 9 is substituted for x both sides of the equation are identical.

In general each type of algebraic equation had its own particular method of solution; quadratic equations were solved by one method equations involving absolute values by another, and so on. In each case, an equation was presented, and a certain method was employed to arrive at a solution, a method appropriate or the particular equation at hand.

These same general ideas carry over to differential equations, which are equations involving derivatives. There are different types of differential equations, and each type requires its own particular solution method.

Many problems related to economics, commerce and engineering, when formulated in mathematical forms, lead to differential equations. Many of these problems are complex in nature and very difficult to understand. But when they are described by differential equations, it is easy to analyse them.

M2L62Q

Learning Objectives

After studying this chapter , the students will be able to understand

- order of the differential equations
- degree of the differential equations
- general and Particular solution of Differential equations
- formation of Differential equations
- differential equations with variable separable
- homogeneous differential equations
- linear differential equations
- second order linear differential equations with constant coefficients.

A differential equation is an equation with a function and one or more of its derivatives. (i.e) an equation with the function $y = f(x)$ and its derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2} \ldots$ is called differential equation.

For example,

(i) $\frac{dy}{dx} = x + 5$ (ii) $\frac{dy}{dx} = \frac{\sqrt{1-x^2}}{\sqrt{2-y^2}}$ (iii) $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$ (iv) $\frac{d^2x}{dt^2} + m^2x = 0$

Differential equations are of two types. One is ordinary differential equations and other one partial differential equations. Here we study only ordinary differential equations.

4.1 Formation of ordinary differential equations

4.1.1 Definition of ordinary differential equation

An ordinary differential equation is an equation that involves some ordinary derivatives $\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\right)$ of a function $y = f(x)$. Here we have one independent variable.

4.1.2 Order and degree of a differential equation

The highest order derivative present in the differential equation is the order of the differential equation.

Degree is the highest power of the highest order derivative in the differential equation, after the equation has been cleared from fractions and the radicals as for as the derivatives are concerned.

For example, consider the differential equation

$$\frac{d^3y}{dx^3}+\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^5+y=7$$

Here the highest order derivatives is $\frac{d^3y}{dx^3}$ (i.e 3rd order derivative). So the order of the differential equation is 3.

Now the power of highest order derivative $\frac{d^3y}{dx^3}$ is 1.

$\therefore$ The degree of the differential equation is 1.

Example 4.1

Find the order and degree of the following differential equations.

(i) $\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2+4y=0$

(ii) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+3y=0$

(iii) $\frac{d^3y}{dx^3}-3\left(\frac{dy}{dx}\right)^6+2y=x^2$

(iv) $\left[1+\frac{d^2y}{dx^2}\right]^{\frac{3}{2}}=a\frac{d^2y}{dx^2}$

(v) $y'+(y'')^2=(x+y'')^2$

(vi) $\frac{d^3y}{dx^3}-\left(\frac{dy}{dx}\right)^{\frac{1}{2}}=0$

(vii) $y=2\left(\frac{dy}{dx}\right)^2+4x\frac{dx}{dy}$

Solution

(i) $\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2+4y=0$

Highest order derivative is $\frac{d^2y}{dx^2}$

$\therefore$ order = 2

Power of the highest order derivative $\frac{d^2y}{dx^2}$ is 1.

$\therefore$ Degree =1

(ii) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+3y=0$

Highest order derivative is $\frac{d^2y}{dx^2}$

$\therefore$ order =2

DO YOU KNOW?

Order and degree (if defined) of a differential equation are always positive integers.

Power of the highest order derivative $\frac{d^2y}{dx^2}$ is 1.

$\therefore$ degree $= 1$

(iii) $\frac{d^3y}{dx^3} - 3\left(\frac{dy}{dx}\right)^6 + 2y = x^2$

$\therefore$ order $=3$, Degree $=1$

(iv) $\left[1+\frac{d^2y}{dx^2}\right]^{\frac{3}{2}} = a\frac{d^2y}{dx^2}$

Here we eliminate the radical sign.

Squaring both sides, we get

$$\left[1+\frac{d^2y}{dx^2}\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$$

$\therefore$ Order$=2$, degree $=3$

(v) $y' + (y'')^2 = (x + y'')^2$

$$y' + (y'')^2 = x^2 + 2xy'' + (y'')^2$$

$$y' = x^2 + 2xy'' \Rightarrow \frac{dy}{dx} = x^2 + 2x\frac{d^2y}{dx^2}$$

$\therefore$ Order$=2$, degree$=1$

(vi) $\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^{\frac{1}{2}} = 0$

Here we eliminate the radical sign.

For this write the equation as

$$\frac{d^3y}{dx^3} = \left(\frac{dy}{dx}\right)^{\frac{1}{2}}$$

Squaring both sides, we get

$$\left(\frac{d^3y}{dx^3}\right)^2 = \frac{dy}{dx}$$

$\therefore$ Order$=3$, degree$=2$

(vii) $y=2\left(\frac{dy}{dx}\right)^2+4x\frac{dx}{dy}$

$$y=2\left(\frac{dy}{dx}\right)^2+4x\frac{1}{\left(\frac{dy}{dx}\right)}$$

$$y\frac{dy}{dx}=2\left(\frac{dy}{dx}\right)^3+4x$$

$\therefore$ order=1, degree=3

Family of Curves

Sometimes a family of curves can be represented by a single equation with one or more arbitrary constants. By assigning different values for constants, we get a family of curves. The arbitrary constants are called the parameters of the family.

For example,

(i) $y^2=4ax$ represents the equation of a family of parabolas having the origin as vertex where 'a' is the parameter.

(ii) $x^2+y^2=a^2$ represents the equation of family of circles having the origin as centre, where 'a' is the parameter.

(iii) $y= mx+c$ represents the equation of a family of straight lines in a plane, where m and c are parameters.

4.1.3 Formation of ordinary differential equation:

Consider the equation $f\left(x,y,c_1\right)=0$ -------(1) where c_1 is the arbitrary constant. We form the differential equation from this equation. For this, differentiate equation (1) with respect to the independent variable occur in the equation.

Eliminate the arbitrary constant c from (1) and its derivative. Then we get the required differential equation.

Suppose we have $f\left(x,y,c_1,c_2\right)=0$. Here we have two arbitrary constants c_1 and c_2. So, find the first two successive derivatives. Eliminate c_1 and c_2 from the given function and the successive derivatives. We get the required differential equation.

Note

The order of the differential equation to be formed is equal to the number of arbitrary constants present in the equation of the family of curves.

Example 4.2

Find the differential equation of the family of straight lines $y = mx + c$ when (i) m is the arbitrary constant (ii) c is the arbitrary constant (iii) m and c both are arbitrary constants.

Solution:

(i) m is an arbitrary constant

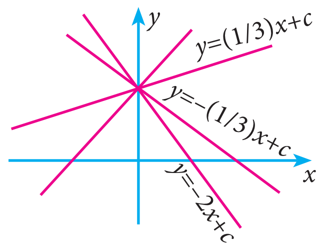


Fig. 4.1

$$y = mx + c \qquad \ldots(1)$$

Differentiating w.r.t. to x, we get $\dfrac{dy}{dx} = m \quad \ldots(2)$

Now we eliminate m from (1) and (2)

For this substitute (2) in (1)

$$y = x\frac{dy}{dx} + c$$

$x\dfrac{dy}{dx} - y + c = 0$ which is the required differential equation of first order .

(ii) c is an arbitrary constant

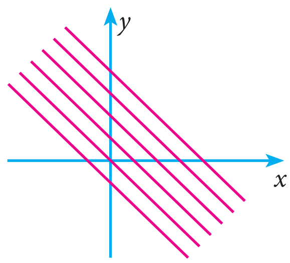


Fig. 4.2

Differentiating (1), we get $\dfrac{dy}{dx} = m$

Here c is eliminated from the given equation

$\therefore \dfrac{dy}{dx} = m$ is the required differential equation.

(iii) both m and c are arbitrary constants

Since m and c are two arbitrary constants differentiating (1) twice we get

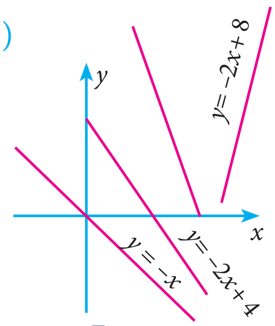


Fig. 4.3

$$\frac{dy}{dx} = m$$

$$\frac{d^2y}{dx^2} = 0$$

Here m and c are eliminated from the given equation.

$\dfrac{d^2y}{dx^2} = 0$ which is the required differential equation.

Example 4.3

Find the differential equation of the family of curves $y=\frac{a}{x}+b$ where a and b are arbitrary constants.

Solution:

Given $y=\frac{a}{x}+b$

Differentiating w.r.t x, we get

$$\frac{dy}{dx}=\frac{-a}{x^2}$$

$$x^2\frac{dy}{dx}=-a$$

Again differentiating w.r.t x we get

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=0$$

$\Rightarrow x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$ which is the required differential equation

Example 4.4

Find the differential equation corresponding to $y=ae^{4x}+be^{-x}$ where a, b are arbitrary constants,

Solution:

Given $y=ae^{4x}+be^{-x}$. (1)

Here a and b are arbitrary constants

From (1), $\frac{dy}{dx} = 4ae^{4x}-be^{-x}$ (2)

and $\frac{d^2y}{dx^2} = 16ae^{4x}+be^{-x}$ (3)

(1)+(2) $\Rightarrow$ $y+\frac{dy}{dx} = 5ae^{4x}$ (4)

(2)+(3) $\Rightarrow$ $\frac{dy}{dx}+\frac{d^2y}{dx^2} = 20ae^{4x}$

$$= 4\left(5ae^{4x}\right)$$

$$= 4\left(y+\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}+\frac{d^2y}{dx^2} = 4y+4\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}-3\frac{dy}{dx}-4y = 0,$$ which is the required differential equation.

Example 4.5

Find the differential equation of the family of curves $y=e^x\left(a\cos x+b\sin x\right)$ where a and b are arbitrary constants.

Solution :

$$y=e^x\left(a\cos x+b\sin x\right) \qquad (1)$$

Differentiating (1) w.r.t x, we get

$$\frac{dy}{dx} = e^x\left(a\cos x+b\sin x\right)+e^x\left(-a\sin x+b\cos x\right)$$

$$= y+e^x(-a\sin x+b\cos x) \quad \text{(from (1))}$$

$$\Rightarrow \quad \frac{dy}{dx}-y = e^x(-a\sin x+b\cos x) \qquad (2)$$

Again differentiating, we get

$$\frac{d^2y}{dx^2}-\frac{dy}{dx} = e^x(-a\sin x+b\cos x)+e^x(-a\cos x-b\sin x)$$

$$\frac{d^2y}{dx^2}-\frac{dy}{dx} = e^x(-a\sin x+b\cos x)-e^x(a\cos x+b\sin x)$$

$$\frac{d^2y}{dx^2}-\frac{dy}{dx} = \left(\frac{dy}{dx}-y\right)-y \qquad \text{(from (1) and (2))}$$

$$\Rightarrow \frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y = 0,$$ which is the requited differential equation.

Exercise 4.1

1. Find the order and degree of the following differential equations.

(i) $\frac{dy}{dx}+2y=x^3$

(ii) $\frac{d^3y}{dx^3}+3\left(\frac{dy}{dx}\right)^3+2\frac{dy}{dx}=0$

(iii) $\frac{d^2y}{dx^2}=\sqrt{y-\frac{dy}{dx}}$

(iv) $\frac{d^3y}{dx^3}=0$

(v) $\frac{d^2y}{dx^2}+y+\left(\frac{dy}{dx}-\frac{d^3y}{dx^3}\right)^{3/2}=0$

(vi) $\left(2-y''\right)^2=y''^2+2y'$

(vii) $\left(\frac{dy}{dx}\right)^3+y=x-\frac{dx}{dy}$

2. Find the differential equation of the following

   (i) $y = cx + c - c^3$ (ii) $y = c(x-c)^2$ (iii) $xy = c^2$ (iv) $x^2 + y^2 = a^2$

3. Form the differential equation by eliminating α and β from $(x-\alpha)^2 + (y-\beta)^2 = r^2$

4. Find the differential equation of the family of all straight lines passing through the origin.

5. Form the differential equation that represents all parabolas each of which has a latus rectum $4a$ and whose axes are parallel to the x axis.

6. Find the differential equation of all circles passing through the origin and having their centers on the y axis.

7. Find the differential equation of the family of parabola with foci at the origin and axis along the x-axis.

Solution of a Differential Equation:

The relation between the dependent and independent variables not involving derivatives is called the solution of the differential equation.

Solution of the differential equation must contain the same number of arbitrary constants as the order of the equation. Such a solution is called General (complete) solution of the differential equation.

4.2 First order and first degree differential equations

A differential equation of first order and first degree can be written as $f\left(x, y, \frac{dy}{dx}\right) = 0$. Here we will discuss the solution of few types of equations.

4.2.1 General solution and particular solution

For any differential equations it is possible to find the general solution and particular solution.

4.2.2 Differential Equation in which variables are separable

If in an equation it is possible to collect all the terms of x and dx on one side and all the terms of y and dy on the other side, then the variables are said to be separable. Thus the general form of such an equation is

$$f(x)dx = g(y)dy \text{ (or) } f(x)dx + g(y)dy = 0$$

By direct integration we get the solution.

Example 4.6

Solve: $(x^2+x+1)dx+(y^2-y+3)dy=0$

Solution:

Given $\quad (x^2+x+1)dx+(y^2-y+3)dy = 0$

It is of the form $\quad f(x)dx+g(y)dy = 0$

Integrating , we get

$$\int(x^2+x+1)dx+\int(y^2-y+3)\,dy = c$$

$$\left(\frac{x^3}{3}+\frac{x^2}{2}+x\right)+\left(\frac{y^3}{3}-\frac{y^2}{2}+3y\right) = c$$

Example 4.7

Solve $\dfrac{dy}{dx}=e^{x-y}+x^2e^{-y}$

Solution :

Given $\dfrac{dy}{dx}=e^{x-y}+x^2e^{-y} = e^{-y}e^{x} + e^{-y}x^2$

$$=e^{-y}\left(e^x+x^2\right)$$

Separating the variables, we get $e^y dy=\left(e^x+x^2\right)dx$

Integrating, we get $\quad \int e^y dy=\int\left(e^x+x^2\right)dx$

$$e^y=e^x+\frac{x^3}{3}+c$$

Example 4.8

Solve $3e^x\tan y\,dx+(1+e^x)\sec^2 y\,dy=0$ given $y(0)=\dfrac{\pi}{4}$

Solution:

Given $\quad 3e^x\tan y\,dx+(1+e^x)\sec^2 y\,dy = 0$

$$3e^x\tan y\;dx = -(1+e^x)\sec^2 y\;dy$$

$$\frac{3e^x}{1+e^x}dx = -\frac{\sec^2 y}{\tan y}dy$$

Integrating, we get $3\int \frac{e^x}{1+e^x}dx = -\int \frac{\sec^2 y}{\tan y}dy + c$

$$3\log(1+e^x) = -\log\tan y + \log c \quad \left[\because \int \frac{f'(x)}{f(x)}dx = \log f(x)\right]$$

$$\log(1+e^x)^3 + \log\tan y = \log c$$

$$\log\left[\left(1+e^x\right)^3 \tan y\right] = \log c$$

$$(1+e^x)^3 \tan y = c \qquad (1)$$

Given $y(0)=\frac{\pi}{4}(i.e) \quad y=\frac{\pi}{4} \;\; at \;\; x=0$

$$(1) \Rightarrow \quad (1+e^0)^3 \tan\frac{\pi}{4} = c$$

$$2^3(1) = c$$

$$\Rightarrow \quad c = 8$$

Hence the required solution is $(1+e^x)^3 \tan y = 8$

Example 4.9

Solve $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

Solution:

Separating the variables, we get

$$\frac{\sec^2 x}{\tan x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

Integrating, we get

$$\int \frac{\sec^2 x}{\tan x}dx + \int \frac{\sec^2 y}{\tan y}dy = c$$

$$\log\tan x + \log\tan y = \log c$$

$$\log(\tan x \tan y) = \log c$$

$$\tan x \tan y = c$$

Example 4.10

Solve $ydx - xdy - 3x^2y^2e^{x^3}dx = 0$

Solution:

Given equation can be written as $\dfrac{ydx - xdy}{y^2} - 3x^2 e^{x^3}\, dx = 0$

Integrating, $\displaystyle\int \frac{ydx - xdy}{y^2} - \int 3x^2 e^{x^3}\, dx = c$

$$\int d\left(\frac{x}{y}\right) - \int e^t dt = c \qquad (\text{where } t = x^3 \text{ and } dt = 3x^2 dx\ )$$

$$\frac{x}{y} - e^t = c$$

$$\frac{x}{y} - e^{x^3} = c$$

Example 4.11

Solve : $x - y\dfrac{dx}{dy} = a\left(x^2 + \dfrac{dx}{dy}\right)$

Solution:

Given $x - y\dfrac{dx}{dy} = a\left(x^2 + \dfrac{dx}{dy}\right)$

$$x - y\frac{dx}{dy} = ax^2 + a\frac{dx}{dy}$$

$$x - ax^2 = a\frac{dx}{dy} + y\frac{dx}{dy}$$

$$x(1 - ax) = (a + y)\frac{dx}{dy}$$

By separating the variables, we get

$$\frac{dx}{x(1-ax)} = \frac{dy}{a+y}$$

$$\left(\frac{a}{1-ax} + \frac{1}{x}\right)dx = \frac{dy}{a+y}$$

Integrating, $\displaystyle\int\left(\frac{a}{1-ax} + \frac{1}{x}\right)dx = \frac{dy}{a+y}$

$$-\log(1-ax) + \log x = \log(a+y) + \log c$$

$$\log\left(\frac{x}{1-ax}\right) = \log\left(c\left(a+y\right)\right)$$

$$\left(\frac{x}{1-ax}\right) = c\,(a+y)$$

$x = (1-ax)(a+y)c$ which is the required solution

Example 4.12

The marginal cost function of manufacturing x gloves is $6+10x-6x^2$. The total cost of producing a pair of gloves is ₹100. Find the total and average cost function.

Solution:

Given $\quad MC = 6+10x-6x^2$

i.e., $\quad \dfrac{dc}{dx} = 6+10x-6x^2$

$$dc = (6+10x-6x^2)dx$$

$$\int dc = \int (6+10x-6x^2)\,dx + k$$

$$c = 6x+10\frac{x^2}{2}-6\frac{x^3}{3}+k$$

$$c = 6x+5x^2-2x^3+k \qquad (1)$$

Given $\quad c = 100$ when $x = 2$

$\therefore \quad (1) \Rightarrow \quad 100 = 12+5(4)-2(8)+k$

$\Rightarrow \quad k = 84$

$\therefore \quad (1) \Rightarrow \quad c\,(x) = 6x+5x^2-2x^3+84$

Average Cost $\quad AC = \dfrac{c}{x} = 6+5x-2x^2+\dfrac{84}{x}$

Example 4.13

The normal lines to a given curve at each point (x, y) on the curve pass through the point (1,0). The curve passes through the point (1,2). Formulate the differential equation representing the problem and hence find the equation of the curve.

Solution:

Slope of the normal at any point $P(x, y) = -\dfrac{dx}{dy}$

Let Q be (1,0)

Slope of the normal PQ *is* $\dfrac{y_2-y_1}{x_2-x_1}$

i.e., $\quad \dfrac{y-0}{x-1} = \dfrac{y}{x-1}$

$$\therefore\ -\frac{dx}{dy}=\frac{y}{x-1} \Rightarrow \frac{dx}{dy}=\frac{y}{1-x}\text{, which is the differential equation}$$

i.e., $(1-x)dx = ydy$

$$\int(1-x)dx = \int ydy + c$$

$$x-\frac{x^2}{2} = \frac{y^2}{2}+c \qquad (1)$$

Since it passes through (1,2)

$$1-\frac{1}{2} = \frac{4}{2}+c$$

$$c=\frac{1}{2}-2 = -\frac{3}{2}$$

Put $c = \frac{-3}{2}$ in (1)

$$x-\frac{x^2}{2} = \frac{y^2}{2}-\frac{3}{2}$$

$$2x-x^2 = y^2-3$$

$\Rightarrow \quad y^2 = 2x-x^2+3$, which is the equation of the curve

Example 4.14

The sum of ₹2,000 is compounded continuously, the nominal rate of interest being 5% per annum. In how many years will the amount be double the original principal? $(\log_e 2 = 0.6931)$

Solution:

Let P be the principal at time 't'

$$\frac{dP}{dt}=\frac{5}{100}P=0.05P$$

$$\Rightarrow \quad \int\frac{dP}{P} = \int 0.05\, dt + c$$

$$\log_e P = 0.05t + c$$

$$P = e^{0.05t}e^{c}$$

$$P = c_1 e^{0.05t} \qquad (1)$$

Given $P = 2000$ when $t = 0$

$$\Rightarrow \qquad c_1 = 2000$$

$$\therefore (1) \Rightarrow \qquad P = 2000e^{0.05t}$$

To find t, when $P = 4000$

$$(2) \Rightarrow \qquad 4000 = 2{,}000e^{0.05t}$$

$$2 = e^{0.05t}$$

$$0.05t = \log 2$$

$$t = \frac{0.6931}{0.05} = 14 \text{ years (approximately)}$$

Exercise 4.2

1. Solve: (i) $\frac{dy}{dx} = ae^y$ (ii) $\frac{1+x^2}{1+y} = xy\frac{dy}{dx}$

2. Solve : $y(1-x) - x\frac{dy}{dx} = o$

3. Solve: (i) $ydx - xdy = 0$ (ii) $\frac{dy}{dx} + e^x + ye^x = 0$

4. Solve: $\cos x(1+\cos y)dx - \sin y(1+\sin x)dy = 0$

5. Solve: $(1-x)dy - (1+y)dx = 0$

6. Solve: (i) $\frac{dy}{dx} = y\sin 2x$ (ii) $\log\left(\frac{dy}{dx}\right) = ax + by$

7. Find the curve whose gradient at any point $P(x, y)$ on it is $\frac{x-a}{y-b}$ and which passes through the origin.

4.2.3 Homogeneous Differential Equations

A differential equation of the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ is called homogeneous differential equation if $f(x, y)$ and $g(x, y)$ are homogeneous functions of the same degree in x and y.
(or)

Homogeneous differential can be written as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

Method of solving first order Homogeneous differential equation

Check $f(x,y)$ and $g(x,y)$ are homogeneous functions of same degree.

i.e. $\dfrac{dy}{dx} = F\left(\dfrac{y}{x}\right)$

Put $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

The given differential equation becomes $v + x\dfrac{dv}{dx} = F(v)$

Separating the variables, we get

$$x\frac{dv}{dx} = F(v) - v \quad \Rightarrow \quad \frac{dv}{F(v) - v} = \frac{dx}{x}$$

By integrating we get the solution in terms of v and x.

Replacing v by $\dfrac{y}{x}$ we get the solution.

Note

Sometimes it becomes easier by taking the Homogeneous differential equation as $\dfrac{dx}{dy} = F\left(\dfrac{x}{y}\right)$ (1)

In this method we have to substitute $x = vy$ and $\dfrac{dx}{dy} = v + x\dfrac{dv}{dy}$ then (1) reduces to variable separable type. By integrating, we get the solution in terms of v and y. The solution is deduced by replacing $v = \dfrac{x}{y}$.

Example 4.15

Solve the differential equation $y^2dx + (xy + x^2)dy = 0$

Solution

$$y^2dx + (xy + x^2)\,dy = 0$$

$$(xy + x^2)\,dy = -y^2dx$$

$$\frac{dy}{dx} = \frac{-y^2}{xy + x^2} \qquad (1)$$

It is a homogeneous differential equation, same degree in x and y

Put $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ (1) becomes

$$v + x\frac{dv}{dx} = \frac{-v^2x^2}{x\ vx + x^2}$$

$$= \frac{-v^2}{v+1}$$

$$x\frac{dv}{dx} = \frac{-v^2}{v+1} - v$$

$$= \frac{-v^2 - v^2 - v}{v+1}$$

$$x\frac{dv}{dx} = \frac{-\left(v+2v^2\right)}{1+v}$$

Now, separating the variables

$$\frac{1+v}{v(1+2v)}\,dv = \frac{-dx}{x}$$

$$\frac{(1+2v)-v}{v(1+2v)}\,dv = \frac{-dx}{x} \qquad (\because\ 1+v = 1+2v-v)$$

$$\frac{1}{v} - \frac{1}{1+2v}\,dv = \frac{-dx}{x}$$

On Integration we have

$$\int\left(\frac{1}{v} - \frac{1}{1+2v}\right)dv = -\int\frac{dx}{x}$$

$$\log v - \frac{1}{2}\log(1+2v) = -\log x + \log c$$

$$\log\left(\frac{v}{\sqrt{1+2v}}\right) = \log\left(\frac{c}{x}\right)$$

$$\frac{v}{\sqrt{1+2v}} = \frac{c}{x}$$

Replace $v = \frac{y}{x}$ we get

$$\frac{\frac{y}{x}}{\sqrt{1+\frac{2y}{x}}} = \frac{c}{x}$$

$$\frac{y\sqrt{x}}{\sqrt{x+2y}} = c$$

$$\frac{y^2x}{x+2y} = k$$

where $k = c^2$

Note

$\int \frac{1+v}{v^2+2v}\,dv$ can be done by the method of partial fraction also.

Example 4.16

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x-y}{x+y}$.

Solution:

$$\frac{dy}{dx} = \frac{x-y}{x+y} \qquad (1)$$

This is a homogeneous differential equation.

Now put $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\therefore \quad (1) \Rightarrow \quad v + x\frac{dv}{dx} = \frac{x - vx}{x + vx}$$

$$= \frac{1-v}{1+v}$$

$$x\frac{dv}{dx} = \frac{1-v}{1+v} - v$$

$$= \frac{1-2v-v^2}{1+v}$$

$$\frac{1+v}{v^2+2v-1}dv = \frac{-dx}{x}$$

Multiply 2 on both sides

$$\frac{2+2v}{v^2+2v-1}dv = -2\frac{dx}{x}$$

On Integration

$$\int \frac{2+2v}{v^2+2v-1}dv = -2\int \frac{dx}{x}$$

$$\log(v^2+2v-1) = -2\log x + \log c$$

$$v^2+2v-1 = \frac{c}{x^2}$$

$$x^2\,(v^2+2v-1) = c$$

Now, Replace $\quad v = \dfrac{y}{x}$

$$x^2\left[\frac{y^2}{x^2} + \frac{2y}{x} - 1\right] = c$$

$y^2 + 2xy - x^2 = c$ is the solution.

Example 4.17

Find the particular solution of the differential equation $x^2\ dy + y\left(x+y\right)dx = 0$ given that $x=1,\ y=1$

Solution

$$x^2dy + y\left(x+y\right)dx = 0$$

$$x^2dy = -y\left(x+y\right)dx$$

$$\frac{dy}{dx} = \frac{-(xy+y^2)}{x^2} \qquad (1)$$

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (1)

$$v + x\frac{dv}{dx} = \frac{-(x\,vx + v^2x^2)}{x^2}$$

$$= -\left(v+v^2\right)$$

$$x\frac{dv}{dx} = -v^2 - v - v$$

$$= -(v^2+2v)$$

On separating the variables

$$\frac{dv}{v^2+2v} = \frac{-dx}{x}$$

$$\frac{dv}{v\left(v+2\right)} = \frac{-dx}{x}$$

$$\frac{1}{2}\left[\frac{\left(v+2\right)-v}{v\left(v+2\right)}\right]dv = \frac{-dx}{x}$$

$$\frac{1}{2}\int\left(\frac{1}{v} - \frac{1}{v+2}\right)dv = -\int\frac{dx}{x}$$

$$\frac{1}{2}\left[\log v - \log\left(v+2\right)\right] = -\log x + \log c$$

$$\frac{1}{2}\log\frac{v}{v+2} = \log\frac{c}{x}$$

We have

$$\frac{v}{v+2} = \frac{c^2}{x^2}$$

Replace $v = \dfrac{y}{x}$, we get

$$\frac{y}{x\left(\frac{y}{x}+2\right)} = \frac{k}{x^2} \qquad \text{where } c^2 = k$$

$$\frac{y\,x^2}{y+2x} = \text{k} \qquad (2)$$

When $x=1\,, y=1$

$$\therefore (2) \Rightarrow \qquad k = \frac{1}{1+2}$$

$$k = \frac{1}{3}$$

$\therefore$ The solution is $3x^2\, y = 2x + y$

Example 4.18

If the marginal cost of producing x shoes is given by $(3xy + y^2)\, dx + (x^2 + xy)\, dy = 0$. and the total cost of producing a pair of shoes is given by ₹12. Then find the total cost function.

Solution:

Given marginal cost function is $(x^2 + xy)\, dy + (3xy + y^2)\, dx = 0$

$$\frac{dy}{dx} = \frac{-(3xy + y^2)}{x^2 + xy} \qquad (1)$$

Put $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in (1)

$$v + x\frac{dv}{dx} = \frac{-(3x\;vx + v^2x^2)}{x^2 + x\;vx}$$

$$= \frac{-(3v + v^2)}{1+v}$$

Now,
$$x\frac{dv}{dx} = \frac{-3v - v^2}{1+v} - v$$

$$= \frac{-3v - v^2 - v - v^2}{1+v}$$

$$x\frac{dv}{dx} = \frac{-4v - 2v^2}{1+v}$$

$$\frac{1+v}{4v+2v^2}\,dv = \frac{-dx}{x}$$

On Integration

$$\int \frac{1+v}{4v+2v^2}\,dv = -\int \frac{dx}{x}$$

Now, multiply 4 on both sides

$$\int \frac{4+4v}{4v+2v^2}\,dv = -4\int \frac{dx}{x}$$

$$\log\,(4v+2v^2) = -4\,\log\,x+\log\,c$$

$$4v+2v^2 = \frac{c}{x^4}$$

$$x^4\,(4v+2v^2) = c$$

Replace $v = \dfrac{y}{x}$

$$x^4\left(4\,\frac{y}{x}+2\,\frac{y^2}{x^2}\right) = c$$

$$x^4\left[\frac{4\,xy+2y^2}{x^2}\right] = c$$

$$c = 2x^2\,(2xy+y^2) \qquad (2)$$

Cost of producing a pair of shoes = ₹12

$(i.e)$ $y=12$ when $x=2$

$$c = 8\,[48+144] = 1536$$

$\therefore$ The cost function is $x^2\,(2xy+y^2) = 768$

Example 4.19

The marginal revenue 'y' of output 'q' is given by the equation $\dfrac{dy}{dq} = \dfrac{q^2+3y^2}{2qy}$. Find the total Revenue function when output is 1 unit and Revenue is ₹5.

Solution:

Given that $MR = \dfrac{dy}{dq} = \dfrac{q^2+3y^2}{2qy} \qquad (1)$

Put $y = vq$ and $\frac{dy}{dq} = v + q\frac{dv}{dq}$ in (1)

Now (1) becomes

$$v + q\frac{dv}{dq} = \frac{q^2 + 3v^2q^2}{2q\,vq}$$

$$= \frac{1+3v^2}{2v}$$

$$q\frac{dv}{dq} = \frac{1+3v^2}{2v} - v$$

$$= \frac{1+3v^2-2v^2}{2v}$$

$$= \frac{1+v^2}{2v}$$

$$\frac{2v}{1+v^2}\,dv = \frac{dq}{q}$$

On Integration

$$\int\frac{2v}{1+v^2}\,dv = \int\frac{dq}{q}$$

$$\log(1+v^2) = \log q + \log c$$

$$1+v^2 = cq$$

Replace $v = \frac{y}{q}$

$$1+\frac{y^2}{q^2} = cq$$

$$q^2 + y^2 = c\,q^3 \qquad (2)$$

Given output is 1 unit and revenue is ₹5

$\therefore$ (2) $\Rightarrow$ $1+25 = c \Rightarrow c = 26$

$\therefore$ The total revenue function is $q^2 + y^2 = 26q^3$

Exercise 4.3

Solve the following homogeneous differential equations.

1. $x\dfrac{dy}{dx} = x + y$

2. $(x - y)\dfrac{dy}{dx} = x + 3y$

3. $x\dfrac{dy}{dx} - y = \sqrt{x^2 + y^2}$

4. $\dfrac{dy}{dx} = \dfrac{3x - 2y}{2x - 3y}$

5. $(y^2 - 2xy)\,dx = (x^2 - 2xy)\,dy$

6. The slope of the tangent to a curve at any point (x, y) on it is given by $(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy = 0$ and the curve passes through $(1, 2)$. Find the equation of the curve.

7. An electric manufacturing company makes small household switches. The company estimates the marginal revenue function for these switches to be $(x^2 + y^2)dy = xydx$ where x represents the number of units (in thousands). What is the total revenue function?

4.2.4 Linear differential equations of first order:

A differential equation is said to be linear when the dependent variable and its derivatives occur only in the first degree and no product of these occur.

The most general form of a linear equation of the first order is $\dfrac{dy}{dx} + Py = Q$ (1)

P and Q are functions of x alone.

Equation (1) is linear in y. The solution is given by $ye^{\int pdx} = \int Qe^{\int pdx}dx + c$. Here $e^{\int pdx}$ is known as an integrating factor and is denoted by I.F.

Note

For the differential equation $\dfrac{dx}{dy} + Px = Q$ (linear in x) where P and Q are functions of y alone, the solution is $xe^{\int pdy} = \int Qe^{\int pdy}dy + c$

Example 4.20

Solve $\dfrac{dy}{dx} + \dfrac{y}{x} = x^3$

Solution:

$$\text{Given } \frac{dy}{dx} + \frac{1}{x}y = x^3$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $P = \frac{1}{x}, \quad Q = x^3$

$$\int P dx = \int \frac{1}{x} dx = \log x$$

$$I.F = e^{\int p dx} = e^{\log x} = x$$

The required solution is $y(I.F) = \int Q(I.F) dx + c$

$$yx = \int x^3 . x \, dx + c$$

$$= \int x^4 dx + c$$

$$= \frac{x^5}{5} + c$$

$$\therefore \quad yx = \frac{x^5}{5} + c$$

DO YOU KNOW?

A linear differential equation is always of first degree but every differential equation of the first degree need not be linear

$$\frac{d^2 y}{dx^2} - \frac{3dy}{dx} + 2y^2 = 0$$

is not linear.

Example 4.21

Solve $\cos^2 x \frac{dy}{dx} + y = \tan x$

Solution:

The given equation can be written as $\frac{dy}{dx} + \frac{1}{\cos^2 x} y = \frac{\tan x}{\cos^2 x}$

$$\frac{dy}{dx} + y \sec^2 x = \tan x \sec^2 x$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $\quad P = \sec^2 x, Q = \tan x \sec^2 x$

$$\int P dx = \int \sec^2 x \, dx = \tan x$$

$$I.F = e^{\int p dx} = e^{\tan x}$$

The required solution is $y(I.F) = \int Q(I.F) dx + c$

$$y e^{\tan x} = \int \tan x \sec^2 x \, e^{\tan x} \, dx + c$$

Put $\quad \tan x = t$

Then $\quad \sec^2 x dx = dt$

$$\therefore ye^{\tan x} = \int te^t \; dt + c$$
$$= \int td(e^t) + c$$
$$= te^t - e^t + c$$
$$= \tan x \, e^{\tan x} - e^{\tan x} + c$$
$$ye^{\tan x} = e^{\tan x}\left(\tan x - 1\right) + c$$

Example 4.22

Solve $\left(x^2+1\right)\dfrac{dy}{dx} + 2xy = 4x^2$

Solution:

The given equation can be reduced to

$$\frac{dy}{dx} + \frac{2x}{x^2+1}y = \frac{4x^2}{x^2+1}$$

It is of the form $\dfrac{dy}{dx} + Py = Q$

Here $P = \dfrac{2x}{x^2+1}, Q = \dfrac{4x^2}{x^2+1}$

$$\int Pdx = \int \frac{2x}{x^2+1}dx = \log(x^2+1)$$
$$I.F = e^{\int pdx} = e^{\log\left(x^2+1\right)} = x^2+1$$

The required solution is $y(IF) = \int Q\left(I.F\right)dx + c$

$$y(x^2+1) = \int \frac{4x^2}{x^2+1}\left(x^2+1\right)dx + c$$
$$y(x^2+1) = \frac{4x^3}{3} + c$$

Example 4.23

Solve $\dfrac{dy}{dx} - 3y\cot x = \sin 2x$ given that $y = 2$ when $x = \dfrac{\pi}{2}$

Solution:

Given $\dfrac{dy}{dx} - \left(3\cot x\right).y = \sin 2x$

It is of the form $\dfrac{dy}{dx}+Py = Q$

Here $P=-3\cot x,\ Q=\sin 2x$

$$\int Pdx = \int -3\cot x dx = -3\log\sin x = -\log\sin^3 x = \log\frac{1}{\sin^3 x}$$

$$\text{I.F.} = e^{\log\frac{1}{\sin^3 x}} = \frac{1}{\sin^3 x}$$

The required solution is $y\,(I.F) = \int Q(I.F)dx + c$

$$y\frac{1}{\sin^3 x} = \int \sin 2x \frac{1}{\sin^3 x}\,dx + c$$

$$y\frac{1}{\sin^3 x} = \int 2\sin x\cos x \times \frac{1}{\sin^3 x}dx + c$$

$$= 2\int \frac{1}{\sin x}\times\frac{\cos x}{\sin x}\,dx + c$$

$$= 2\int \operatorname{cosec} x\cot x dx + c$$

$$y\frac{1}{\sin^3 x} = -2\operatorname{cosec} x + c \qquad (1)$$

Now $\qquad y = 2$ when $x = \dfrac{\pi}{2}$

(1) $\Rightarrow\qquad 2\left(\dfrac{1}{1}\right) = -2\times 1 + c \Rightarrow c = 4$

$\therefore$ (1) $\Rightarrow\qquad y\dfrac{1}{\sin^3 x} = -2\operatorname{cosec} x + 4$

Example 4.24

A firm has found that the cost C of producing x tons of certain product by the equation $x\dfrac{dC}{dx} = \dfrac{3}{x} - C$ and $C = 2$ when $x = 1$. Find the relationship between C and x.

Solution:

$$x\frac{dC}{dx} = \frac{3}{x} - C$$

$$\frac{dC}{dx} = \frac{3}{x^2} - \frac{C}{x}$$

$$\frac{dC}{dx} + \frac{C}{x} = \frac{3}{x^2}$$

i.e., $$\frac{dC}{dx} + \frac{1}{x}C = \frac{3}{x^2}$$

It is of the form $\quad \dfrac{dC}{dx} + PC = Q$

Here, $P = \dfrac{1}{x}$, $Q = \dfrac{3}{x^2}$

$$\int P dx = \int \frac{1}{x} dx = \log x$$

$$I.F = e^{\int pdx} = e^{\log x} = x$$

The Solution is

$$C(I.F) = \int Q(I.F)dx + k \text{ where } k \text{ is constant}$$

$$Cx = \int \frac{3}{x^2} x dx + k$$

$$= 3\int \frac{1}{x} dx + k$$

$$Cx = 3\log x + k \qquad (1)$$

Given $C = 2$ When $x = 1$

$(1) \quad \Rightarrow \quad 2\times 1 = k \Rightarrow k = 2$

$\therefore$ The relationship between C and x is

$$Cx = 3\log x + 2$$

Exercise 4.4

Solve the following:

1. $\dfrac{dy}{dx} - \dfrac{y}{x} = x$

2. $\dfrac{dy}{dx} + y \cos x = \sin x \cos x$

3. $x\dfrac{dy}{dx} + 2y = x^4$

4. $\dfrac{dy}{dx} + \dfrac{3x^2}{1+x^3} y = \dfrac{1+x^2}{1+x^3}$

5. $\dfrac{dy}{dx} + \dfrac{y}{x} = xe^x$

6. $\dfrac{dy}{dx} + y\tan x = \cos^3 x$

7. If $\dfrac{dy}{dx} + 2y\tan x = \sin x$ and if $y = 0$ when $x = \dfrac{\pi}{3}$ express y in terms of x

8. $\dfrac{dy}{dx} + \dfrac{y}{x} = xe^x$

9. A bank pays interest by continuous compounding , that is by treating the interest rate as the instantaneous rate of change of principal. A man invests ₹1,00,000 in the bank deposit which accures interest, 8% per year compounded continuously. How much will he get after 10 years.

4.3 Second Order first degree differential equations with constant coefficients:

4.3.1 A general second order linear differential equation with constant coefficients is of the form

$$\boxed{a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)}$$

$$aD^2y+bDy+cy=f(x),\quad \text{where}\quad \frac{d}{dx}=D,\frac{d^2}{dx^2}=D^2$$

$$\phi(D)y=f(x) \qquad (1)$$

where $\phi(D)=aD^2+bD+c$ (a,b and c are constants)

To solve the equation (1), we first solve the equation $\phi(D)y=0$. The solution so obtained is called complementary function ($C.F$).

Next we operate on $f(x)$ with $\frac{1}{\phi(D)}$, the solution so obtained is called particular integral ($P.I$)

$$PI=\frac{1}{\phi(D)}f(x)$$

General solution is $y=C.F+P.I$

Type 1 : $f(x)=0$

$$(i.e)\phi(D)y=0$$

To solve this, put $\phi(D)=0$

Replace D by m. This equation is called auxiliary equation $.\phi(m)=0$ *is a* quadratic equation. So we have two roots, say m_1 and m_2.

Now we have the following three cases

	Nature of roots	Complementary function
1	Real and different $(m_1\neq m_2)$	$Ae^{m_1x}+Be^{m_2x}$
2	Real and equal $m_1=m_2=m(say)$	$(Ax+B)e^{mx}$
3	Complex roots $(\alpha\pm i\beta)$	$e^{\alpha x}(A\cos\beta x+B\sin\beta x)$

Here A and B are arbitrary constants

Example 4.25

Solve $(D^2-3D-4)y=0$

Solution:

Given $\quad (D^2-3D-4)y = 0$

The auxiliary equations is

$$m^2-3m-4 = 0$$

$$\Rightarrow \quad (m-4)(m+1) = 0$$

$$m = -1, 4$$

Roots are real and different

$\therefore$ The complementary function is $Ae^{-x}+Be^{4x}$

The general solution is $y = Ae^{-x}+Be^{4x}$

Example 4.26

Solve $9y''-12y'+4y=0$

Solution:

Given $\quad \left(9D^2-12D+4\right)y = 0$

The auxiliary equation is

$$(3m-2)^2 = 0$$

$$(3m-2)(3m-2) = 0 \quad \Rightarrow m=\frac{2}{3},\frac{2}{3}$$

Roots are real and equal.

The C.F. is $(Ax+B)e^{\frac{2}{3}x}$

The general solution is $y = (Ax+B)e^{\frac{2}{3}x}$

Example 4.27

Solve $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+5y=0$

Solution:

Given $$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+5y = 0$$

$$\left(D^2-4D+5\right)y = 0$$

The auxiliary equation is

$$m^2-4m+5 = 0$$

$$\Rightarrow (m-2)^2-4+5 = 0$$

$$(m-2)^2 = -1$$

$$m-2 = \pm\sqrt{-1}$$

$m = 2\pm i$, it is if the form $\alpha\pm i\beta$

$\therefore$ $$\text{C.F} = e^{2x}\left[A\cos x+B\sin x\right]$$

The general solution is $y = e^{2x}\left[A\cos x+B\sin x\right]$

Example 4.28

Solve $\dfrac{d^2x}{dt^2}-\dfrac{3dx}{dt}+2x=0$ given that when $t=0,\ x=0$ *and* $\dfrac{dx}{dt}=1$

Solution:

$$\frac{d^2x}{dt^2}-3\frac{dx}{dt}+2x = 0$$

Given $(D^2-3D+2)x = 0$ where $D=\dfrac{d}{dt}$

A.E is $m^2-3m+2 = 0$

$$(m-1)(m-2) = 0$$

$$m = 1,2$$

$$C.F = Ae^{t}+Be^{2t}$$

The general solution is $x = Ae^{t}+Be^{2t}$ (1)

Now when $t=0,\ x=0$ (given)

(1) $\Rightarrow$ $0 = A+B$ (2)

Differentiating (1) w.r.t 't'

$$\frac{dx}{dt} = Ae^{t} + 2Be^{2t}$$

When t=0, $$\frac{dx}{dt} = 1$$

$$A + 2B = 1 \qquad (3)$$

Thus we have $A + B = 0$ and $A + 2B = 1$

Solving, we get $A = -1$, $B = 1$

$\therefore$ (1) $\Rightarrow$ $$x = -e^{t} + e^{2t}$$

(i.e.) $$x = e^{2t} - e^{t}$$

Type II : $f(x) = e^{ax}\ (i.e)\phi(D)y = e^{ax}$

$$\text{P.I} = \frac{1}{\phi(D)}e^{ax}$$

Replace D by a, provided $\phi(D) \neq 0$ when D = a

If $\phi(D) = 0$ when $D = a$, then

$$\text{P.I} = x\frac{1}{\phi'(D)}e^{ax}$$

Replace D by a, provided $\phi'(D) \neq 0$ when $D = a$

If $\phi'(D) = 0$ when $D = a$, then

$$\text{P.I} = x^{2}\frac{1}{\phi''(D)}e^{ax} \text{ and so on}$$

Example 4.29

Solve : $(D^{2} - 4D - 1)y = e^{-3x}$

Solution:

$$(D^{2} - 4D - 1)y = e^{-3x}$$

The auxiliary equation is

$$m^{2} - 4m - 1 = 0$$

$$(m-2)^{2} - 4 - 1 = 0$$

$$
\begin{aligned}
(m-2)^2 &= 5 \\
m-2 &= \pm\sqrt{5} \\
m &= 2\pm\sqrt{5} \\
\text{C.F} &= Ae^{(2+\sqrt{5})x} + Be^{(2-\sqrt{5})x} \\
\text{PI} &= \frac{1}{\phi(D)} f(x) \\
&= \frac{1}{D^2-4D-1} e^{-3x} \\
&= \frac{1}{(-3)^2-4(-3)-1} e^{-3x} \text{ (Replace } D \text{ by} -3) \\
&= \frac{1}{9+12-1} e^{-3x} \\
&= \frac{e^{-3x}}{20}
\end{aligned}
$$

Hence the general solution is $y = \text{C.F+P.I}$

$$\Rightarrow \quad y = Ae^{(2+\sqrt{5})x} + Be^{(2-\sqrt{5})x} + \frac{e^{-3x}}{20}$$

Example 4.30

Solve: $(D^2-2D+1)y = e^{2x}+e^x$

Solution:

$$(D^2-2D+1)y = e^{2x}+e^x$$

The auxiliary equation is

$$
\begin{aligned}
m^2-2m+1 &= 0 \\
\Rightarrow \quad (m-1)(m-1) &= 0 \\
m &= 1, 1 \\
\text{C.F} &= (Ax+B)e^x \\
\text{PI} &= \frac{1}{\phi(D)} f(x) = \frac{1}{D^2-2D+1}\left(e^{2x}+e^x\right)
\end{aligned}
$$

Now, $$P.I_1 = \frac{1}{D^2-2D+1} e^{2x}$$

$$= \frac{1}{4-4+1} e^{2x} \text{ (replace D by 2)}$$
$$= e^{2x}$$

and $$P.I_2 = \frac{1}{D^2 - 2D + 1} e^x$$
$$= \frac{1}{(D-1)^2} e^x$$

Replace D by 1. $(D-1)^2 = 0$ when $D = 1$

$$\therefore \quad P.I_2 = x.\frac{1}{2(D-1)} e^x$$

Replace D by 1. $(D-1) = 0$ when $D = 1$

$$\therefore \quad P.I_2 = x^2 \frac{1}{2} e^x$$

The general solution is

$$y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$$
$$y = (Ax + B)e^x + e^{2x} + \frac{x^2}{2} e^x$$

Example 4.31

Solve: $(3D^2 + D - 14)y = 4 - 13e^{\frac{-7}{3}x}$

Solution:

$$(3D^2 + D - 14)y = 4 - 13e^{\frac{-7}{3}x}$$

The auxiliary equation is

$$3m^2 + m - 14 = 0$$
$$(3m+7)(m-2) = 0$$
$$m = \frac{-7}{3}, 2$$
$$\text{C.F} = Ae^{\frac{-7}{3}x} + Be^{2x}$$

$$\Rightarrow \quad \text{PI} = \frac{1}{\phi(D)} f(x) = \frac{1}{3D^2 + D - 14}\left(4 - 13e^{\frac{-7}{3}x}\right)$$

$$= \frac{1}{3D^2+D-14}(4) + \frac{1}{3D^2+D-14}\left(-13e^{\frac{-7}{3}x}\right)$$

$$= PI_1 + PI_2$$

$$\text{P.I}_1 = \frac{1}{3D^2+D-14} 4e^{0x}$$

$$= \frac{1}{0+0-14} 4e^{0x} \text{ (Replace } D \text{ by 0)}$$

$$\text{P.I}_1 = \frac{-4}{14}$$

$$P.I_2 = \frac{1}{3D^2+D-14} \times (-13)e^{\frac{-7}{3}x}$$

Replace D by $\frac{-7}{3}$ Here $3D^2+D-14=0$ when $D=-\frac{7}{3}$

$$\therefore \quad P.I_2 = x.\frac{1}{6D+1}\left(-13e^{\frac{-7}{3}x}\right)$$

Replace D by $\frac{-7}{3}$

$$\therefore \quad P.I_2 = x\frac{1}{6\left(\frac{-7}{3}\right)+1}\left(-13e^{\frac{-7}{3}x}\right)$$

$$= x\frac{1}{-13}\left(-13e^{\frac{-7}{3}x}\right)$$

$$= xe^{\frac{-7}{3}x}$$

The general solution is $y = C.F. + P.I_1 + P.I_2$

$$y = Ae^{\frac{-7}{3}x} + Be^{2x} - \frac{2}{7} + xe^{\frac{-7}{3}x}$$

Example 4.32

Suppose that the quantity demanded $Q_d = 29 - 2p - 5\frac{dp}{dt} + \frac{d^2p}{dt^2}$ and quantity supplied $Q_s = 5 + 4p$ where p is the price. Find the equilibrium price for market clearance.

Solution:

For market clearance, the required condition is $Q_d = Q_s$

$$\Rightarrow \quad 29-2p-5\frac{dp}{dt}+\frac{d^2p}{dt} = 5+4p$$

$$\Rightarrow \quad 24-6p-5\frac{dp}{dt}+\frac{d^2p}{dt^2} = 0$$

$$\Rightarrow \quad \frac{d^2p}{dt^2}-5\frac{dp}{dt}-6p = -24$$

$$(D^2-5D-6)\,p = -24$$

The auxiliary equation is

$$m^2-5m-6 = 0$$

$$(m-6)(m+1) = 0$$

$$\Rightarrow \quad m = 6, -1$$

$$\text{C.F} = \text{A}e^{6t}+Be^{-t}$$

$$\text{P.I} = \frac{1}{\phi(D)}f(x)$$

$$= \frac{1}{D^2-5D-6}(-24)e^{0t}$$

$$= \frac{-24}{-6} \quad \text{(Replace D by 0)}$$

$$= 4$$

The general solution is $\quad p = C.F+P.I$

$$= Ae^{6t}+Be^{-t}+4$$

Exercise 4.5

Solve the following differential equations

(1) $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+8y=0$

(2) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$

(3) $\left(D^2+2D+3\right)y=0$

(4) $\frac{d^2y}{dx^2}-2k\frac{dy}{dx}+k^2y=0$

(5) $\left(D^2-2D-15\right)y=0$ given that $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}=2$ when $x=0$

(6) $\left(4D^2+4D-3\right)y=e^{2x}$

(7) $\frac{d^2y}{dx^2}+16y=0$

(8) $\left(D^2-3D+2\right)y=e^{3x}$ which shall vanish for $x = 0$ and for $x = \log 2$

(9) $\left(D^2+D-6\right)y=e^{3x}+e^{-3x}$

(10) $\left(D^2-10D+25\right)y=4e^{5x}+5$

(11) $\left(4D^2+16D+15\right)y=4e^{\frac{-3}{2}x}$

(12) $\left(3D^2+D-14\right)y=13e^{2x}$

(13) Suppose that the quantity demanded $Q_d = 13-6p+2\frac{dp}{dt}+\frac{d^2p}{dt^2}$ and quantity supplied $Q_s=-3+2p$ where p is the price. Find the equilibrium price for market clearance.

Exercise 4.6

Choose the Correct answer

1. The degree of the differential equation $\frac{d^4y}{dx^4}-\left(\frac{d^2y}{dx^2}\right)^4+\frac{dy}{dx}=3$

(a) 1 (b) 2 (c) 3 (d) 4

2. The order and degree of the differential equation $\sqrt{\frac{d^2y}{dx^2}}=\sqrt{\frac{dy}{dx}+5}$ are respectively

(a) 2 and 3 (b) 3 and 2 (c) 2 and 1 (d) 2 and 2

3. The order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}}-\sqrt{\left(\frac{dy}{dx}\right)}-4=0$ are respectively.

(a) 2 and 6 (b) 3 and 6 (c) 1 and 4 (d) 2 and 4

4. The differential equation $\left(\frac{dx}{dy}\right)^3+2y^{\frac{1}{2}}=x$ is

(a) of order 2 and degree 1
(b) of order 1 and degree 3
(c) of order 1 and degree 6
(d) of order 1 and degree 2

5. The differential equation formed by eliminating a and b from $y=ae^x+be^{-x}$ is

(a) $\frac{d^2y}{dx^2}-y=0$ (b) $\frac{d^2y}{dx^2}-\frac{dy}{dx}=0$ (c) $\frac{d^2y}{dx^2}=0$ (d) $\frac{d^2y}{dx^2}-x=0$

6. If $y=cx+c-c^3$ then its differential equation is

(a) $y=x\frac{dy}{dx}+\frac{dy}{dx}-\left(\frac{dy}{dx}\right)^3$
(b) $y+\left(\frac{dy}{dx}\right)^3=x\frac{dy}{dx}-\frac{dy}{dx}$
(c) $\frac{dy}{dx}+y=\left(\frac{dy}{dx}\right)^3-x\frac{dy}{dx}$
(d) $\frac{d^3y}{dx^3}=0$

7. The integrating factor of the differential equation $\frac{dx}{dy}+Px=Q$ is

(a) $e^{\int Pdx}$ (b) $e^{\int Pdx}$ (c) $\int Pdy$ (d) $e^{\int Pdy}$

8. The complementary function of $\left(D^2+4\right)y=e^{2x}$ is

(a) $\left(Ax+B\right)e^{2x}$ (b) $\left(Ax+B\right)e^{-2x}$ (c) $A\cos 2x+B\sin 2x$ (d) $Ae^{-2x}+Be^{2x}$

9. The differential equation of $y=mx+c$ is (m and c are arbitrary constants)

(a) $\frac{d^2y}{dx^2}=o$ (b) $y=x\frac{dy}{dx}+c$ (c) $xdy+ydx=0$ (d) $ydx-xdy=0$

10. The particular integral of the differential equation is $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+16y=2e^{4x}$

(a) $\frac{x^2e^{4x}}{2!}$ (b) $\frac{e^{4x}}{2!}$ (c) x^2e^{4x} (d) xe^{4x}

11. Solution of $\frac{dx}{dy}+Px=0$

`(a) $x=ce^{py}$ (b) $x=ce^{-py}$ (c) $x=py+c$ (d) $x=cy$

12. If $\sec^2 x$ is an integrating factor of the differential equation $\frac{dy}{dx}+Py=Q$ then $P=$

(a) $2\tan x$ (b) $\sec x$ (c) $\cos^2 x$ (d) $\tan^2 x$

13. The integrating factor of $x\frac{dy}{dx}-y=x^2$ is

(a) $\frac{-1}{x}$ (b) $\frac{1}{x}$ (c) $\log x$ (d) x

14. The solution of the differential equation $\frac{dy}{dx}+Py=Q$ where P and Q are the function of x is

(a) $y=\int Qe^{\int Pdx}dx+c$ (b) $y=\int Qe^{-\int Pdx}dx+c$

(c) $ye^{\int Pdx}=\int Qe^{\int Pdx}dx+c$ (d) $ye^{\int Pdx}=\int Qe^{-\int Pdx}dx+C$

15. The differential equation formed by eliminating A and B from $y=e^{-2x}\left(A\cos x+B\sin x\right)$ is

(a) $y_2-4y_1+5=0$ (b) $y_2+4y-5=0$

(c) $y_2-4y_1-5=0$ (d) $y_2+4y_1+5=0$

16. The particular integral of the differential equation $f(D)y = e^{ax}$ where $f(D) = (D-a)^2$

(a) $\frac{x^2}{2}e^{ax}$ (b) xe^{ax} (c) $\frac{x}{2}e^{ax}$ (d) x^2e^{ax}

17. The differential equation of $x^2 + y^2 = a^2$

(a) $xdy+ydx=0$ (b) $ydx-xdy=0$ (c) $xdx-ydx=0$ (d) $xdx+ydy=0$

18. The complementary function of $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$ is

(a) $A + Be^x$ (b) $(A+B)e^x$ (c) $(Ax+B)e^x$ (d) $Ae^x + B$

19. The P.I of $(3D^2 + D - 14)y = 13e^{2x}$ is

(a) $\frac{x}{2}e^{2x}$ (b) xe^{2x} (c) $\frac{x^2}{2}e^{2x}$ (d) $13xe^{2x}$

20. The general solution of the differential equation $\frac{dy}{dx} = \cos x$ is

(a) $y = \sin x + 1$ (b) $y = \sin x - 2$

(c) $y = \cos x + c$, c is an arbitrary constant

(d) $y = \sin x + c$, c is an arbitrary constant

21. A homogeneous differential equation of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ can be solved by making substitution,

(a) $y = v\,x$ (b) $v = y\,x$ (c) $x = v\,y$ (d) $x = v$

22. A homogeneous differential equation of the form $\frac{dx}{dy} = f\left(\frac{x}{y}\right)$ can be solved by making substitution,

(a) $x = v\,y$ (b) $y = v\,x$ (c) $y = v$ (d) $x = v$

23. The variable separable form of $\frac{dy}{dx} = \frac{y(x-y)}{x(x+y)}$ by taking $y = vx$ *and* $\frac{dy}{dx} = v + x\frac{dv}{dx}$ is

(a) $\frac{2v^2}{1+v}dv = \frac{dx}{x}$ (b) $\frac{2v^2}{1+v}dv = -\frac{dx}{x}$

(c) $\frac{2v^2}{1-v}dv = \frac{dx}{x}$ (d) $\frac{1+v}{2v^2}dv = -\frac{dx}{x}$

24. Which of the following is the homogeneous differential equation?

(a) $(3x-5)\,dx = (4y-1)\,dy$ (b) $xy\,dx-(x^3+y^3)\,dy = 0$

(c) $y^2\,dx+(x^2-xy-y^2)\,dy = 0$ (d) $(x^2+y)\,dx = (y^2+x)\,dy$

25. The solution of the differential equation $\frac{dy}{dx}=\frac{y}{x}+\frac{f\left(\frac{y}{x}\right)}{f'\left(\frac{y}{x}\right)}$ is

(a) $f\left(\frac{y}{x}\right)=kx$ (b) $x\,f\left(\frac{y}{x}\right)=k$ (c) $f\left(\frac{y}{x}\right)=ky$ (d) $y\,f\left(\frac{y}{x}\right)=k$

Miscellaneous Problems

1. Suppose that $Q_d=30-5P+2\frac{dP}{dt}+\frac{d^2P}{dt^2}$ and $Q_s=6+3P$. Find the equilibrium price for market clearance.

2. Form the differential equation having for its general solution $y=ax^2+bx$

3. Solve $yx^2dx+e^{-x}dy=0$

4. Solve $(x^2+y^2)dx+2xy\,dy=0$

5. Solve $x\frac{dy}{dx}+2y=x^4$

6. A manufacturing company has found that the cost C of operating and maintaining the equipment is related to the length 'm' of intervals between overhauls by the equation $m^2\frac{dC}{dm}+2mC=2$ and c = 4 and when m = 2. Find the relationship between C and m.

7. Solve $(D^2-3D+2)y=e^{4x}$ given y = 0 when x = 0 and x = 1.

8. Solve $\frac{dy}{dx}+y\cos x+x=2\cos x$

9. Solve $x^2ydx-(x^3+y^3)dy=0$

10. Solve $\frac{dy}{dx}=xy+x+y+1$

Summary

- A differential equation is an equation with a function and one or more of its derivatives.

  (i.e) an equation with the function $y=f(x)$ and its derivatives $\frac{dy}{dx},\frac{d^2y}{dx^2}\ldots$ is called differential equation.

- Order of the highest order derivative present in the differential equation is the order of the differential equation.

- Degree is the highest power of the highest order derivative in the differential equation, after the equation has been cleared from fractions and the radicals as for as the derivatives are concerned.

- A function which satisfies the given differential equation is called its solution. The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution and the solution free from arbitrary constants is called particular solution.

- To form a differential equation from a given function we differentiate the function successively as many times as the number of arbitrary constants in the given function and then eliminate the arbitrary constants.

- In an equation it is possible to collect all the terns of x and dx on one side and all the terms of y and dy on the other side, then the variables are said to be separable. Thus the general form of such an equation is $f(x)dx = g(y)dy$ (or) $f(x)dx + g(y)dy = 0$ By direct integration, we get the solution.

- A differential equation which can be expressed in the form $\frac{dy}{dx} = f\left(x, y\right)$ or $\frac{dx}{dy} = g\left(x, y\right)$ where $f\left(x, y\right)$ and $g\left(x, y\right)$ are homogeneous function of degree zero is called a homogeneous differential equation.

- A differential equation of the form $\frac{dy}{dx} + Py = Q$ where P and Q are constants or functions of x only is called a first order linear differential equation.

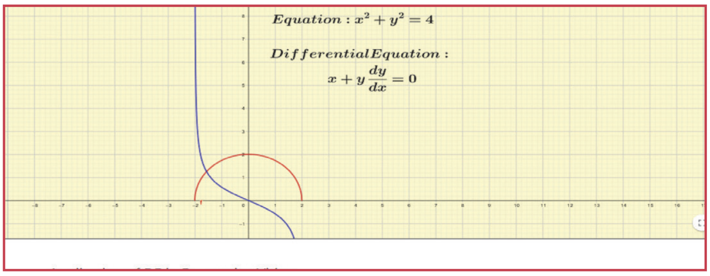
- A general second order linear differential equation with constant coefficients is of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f\left(x\right)$

GLOSSARY	
Differential equation	வகைக்கெழுச் சமன்பாடுகள்
Ordinary differential equation	சாதாரண வகைக்கெழுச் சமன்பாடுகள்
Partial differential equation	பகுதி வகைக்கெழுச் சமன்பாடுகள்
Order	வரிசை
Degree	படி
Variable	மாறி
Constant	மாறிலி
Fixed constant	நிலையான மாறிலி

Arbitrary constant	மாறத்தக்க மாறிலி
General solution	பொதுத் தீர்வு
Particular integra	சிறப்புத் தீர்வு
Abscissa	கிடை அச்சுத் தொலைவு
Ordinate	குத்தாயம்
Variable separable	மாறிகள் பிரிக்கக் கூடியன
Homogeneous equations	சமபடித்தான சமன்பாடுகள்
Linear differential equations	நேரிய வகைக்கெழுச் சமன்பாடு
Second order linear differential equations	இரண்டாம் வரிசை நேரிய வகைக்கெழுச் சமன்பாடுகள்
Auxiliary equation	துணைச் சமன்பாடு
Complementary function	நிரப்புச் சார்பு

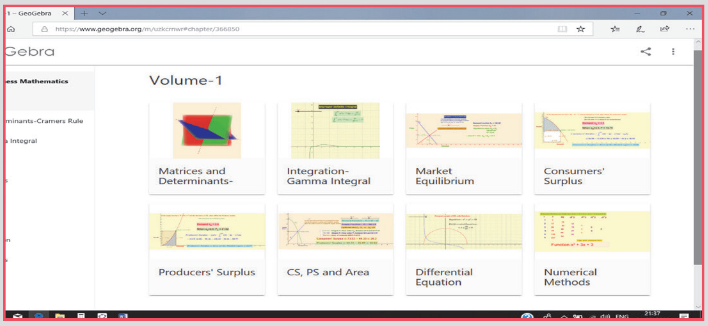
ICT Corner

Expected Result is shown in this picture



Step 1

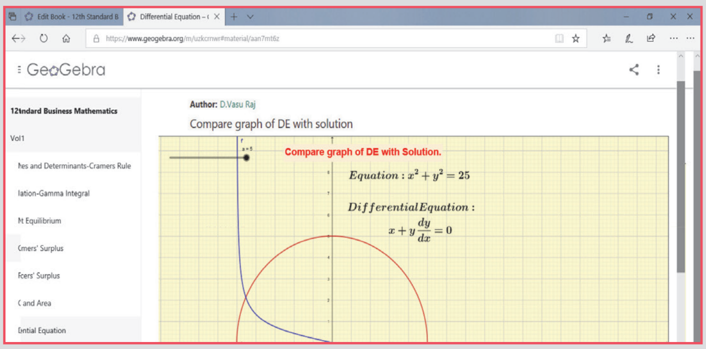
Open the Browser, type the URL Link given below (or) Scan the QR Code. GeoGebra work Book named "12th Standard Business Mathematics " will open. In the work book there are two Volumes. Select "Volume-1".



Step 2

Select the worksheet named" Differential Equation"

There is a video explanation and a graph in a single worksheet. Observe and learn the concepts.

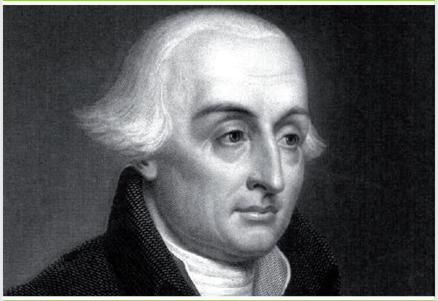


Browse in the link

12th standard Business Mathematics :https://ggbm.at/uzkcrnwr or Scan the QR Code.

B284_12_BUS_MAT_EM

5 Numerical Methods



Joseph-Louis Lagrange
(25.01.1736 - 10.04.1813)

Introduction

Numerical Analysis is a branch of Mathematics which leads to approximate solution by repeated applications of four basic operations of Algebra. The knowledge of finite differences is essential for the study of Numerical Analysis.

Joseph-Louis Lagrange was an Italian mathematician and astronomer. he made significant contributions to the fields of analysis, number theory, and both classical and celestial mechanics.

Learning Objectives

After studying this chapter, students will be able to understand

- the finite differences
- how to find the polynomial using finite differences
- how to find the relations between the operators
- how to find the missing terms
- how to interpolate the values of a given series using Newton's interpolation formulae
- how to apply the Lagrange's interpolation formula

SZWKEN

5.1 FINITE DIFFERENCES

Consider the arguments $x_0, x_1, x_2, \cdots, x_n$ and the entries $y_0, y_1, y_2, \cdots, y_n$. $y = f(x)$ be a function of x. Let us assume that the values of x are in increasing order and equally spaced with a space length h. Then the values of x may be taken to be $x_0, x_0+h, x_0+2h, \cdots, x_0+nh$ and the function assumes the values

$f(x_0), f(x_0+h), f(x_0+2h), \cdots, f(x_0+nh)$. Here we study some of the finite differences of the function $y = f\left(x\right)$

5.1.1 Forward Difference Operator, Backward Difference Operator and Shifting Operator

Forward Difference Operator(Δ):

Let $y = f(x)$ be a given function of x. Let $y_0, y_1, y_2, \cdots, y_n$ be the values of y at $x = x_0, x_1, x_2, \ldots, x_n$ respectively. Then $y_1 - y_0,\ y_2 - y_1,\ y_3 - y_2, \ldots, y_n - y_{n-1}$ are called the first (forward) differences of the function y. They are denoted by $\Delta y_0, \Delta y_1, \Delta y_2, \ldots, \Delta y_{n-1}$ respectively.

(i.e) $\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2, \ldots, \Delta y_{n-1} = y_n - y_{n-1}$

In general, $\Delta y_n = y_{n+1} - y_n,\ n = 0, 1, 2, 3, \ldots$

The symbol Δ is called the forward difference operator and pronounced as delta.

The forward difference operator Δ can also be defined as $\Delta f\left(x\right) = f\left(x+h\right) - f\left(x\right)$, h is the equal interval of spacing.

Proof of these properties are not included in our syllabus:

Properties of the operator Δ :

Property 1: If c is a constant then $\Delta c = 0$

Proof: Let $f(x) = c$

$\therefore f\left(x+h\right) = c$ (where 'h' is the interval of difference)

$\Delta\ f\left(x\right) = f\left(x+h\right) - f\left(x\right)$

$\Delta c = c - c = 0$

Property 2: Δ is distributive i.e. $\Delta\left(f(x) + g(x)\right) = \Delta f(x) + \Delta g(x)$

Proof:
$$\begin{aligned}\Delta\left[f(x) + g(x)\right] &= \left[f(x+h) + g(x+h)\right] - \left[f(x) + g(x)\right]\\ &= f(x+h) + g(x+h) - f(x) - g(x)\\ &= f(x+h) - f(x) + g(x+h) - g(x)\\ &= \Delta f(x) + \Delta g(x)\end{aligned}$$

Similarly we can show that $\Delta\left[f(x)-g(x)\right]=\Delta f(x)-\Delta g(x)$

In general, $\Delta\left[f_1(x)+f_2(x)......+f_n(x)\right]=\Delta f_1(x)+\Delta f_2(x)+....+\Delta f_n(x)$

Property 3: If c is a constant then $\Delta c\, f(x)=c\,\Delta f(x)$

Proof:

$$\begin{aligned}\Delta[c\,f(x)] &= c\,f(x+h)-c\,f(x)\\ &= c[f(x+h)-f(x)]\\ &= c\,\Delta f(x)\end{aligned}$$

Results without proof

1. If m and n are positive integers then $\Delta^m.\Delta^n\ f(x)=\Delta^{m+n}f(x)$
2. $\Delta\left[f(x)\ g(x)\right]=f(x)\Delta g(x)+g(x)\Delta f(x)$
3. $\Delta\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{g(x)\Delta f(x)-f(x)\Delta g(x)}{g(x).\ g(x+h).}$

The differences of the first differences denoted by $\Delta^2 y_0,\Delta^2 y_1,...,\Delta^2 y_n$ are called second differences, where

$$\begin{aligned}\Delta^2 y_n &= \Delta(\Delta y_n)=\Delta(y_{n+1}-y_n)\\ &= \Delta y_{n+1}-\Delta y_n \qquad (1)\\ \Delta^2 y_n &= \Delta y_{n+1}-\Delta y_n,\ n=0,1,2,\ldots\\ \Delta^2 y_0 &= \Delta y_1-\Delta y_0\\ \Delta^2 y_1 &= \Delta y_2-\Delta y_1,...\end{aligned}$$

Similarly the differences of second differences are called third differences.

$$\Delta^3 y_n=\Delta^2 y_{n+1}-\Delta^2 y_n,\ n=0,1,2,\ldots$$

In particular,

$$\begin{aligned}\Delta^3 y_0 &= \Delta^2 y_1-\Delta^2 y_0\\ \Delta^3 y_0 &= \Delta^2 y_2-\Delta^2 y_1\end{aligned}$$

Note

$\Delta^k f(x)=\Delta^{k-1}f(x+h)-\Delta^{k-1}f(x)$

In general k^{th} differences of y_n is

$$\Delta^k y_n=\Delta^{k-1}y_{n+1}-\Delta^{k-1}y_n,\ n=0,1,2,\ldots$$

It is convenient to represent the above differences in a table as shown below.

Forward Difference Table for y :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

The forward difference table for $f(x)$ is given below.

x	$f(x)$				
		$\Delta f(x)$			
$x+h$	$f(x+h)$		$\Delta^2 f(x)$		
		$\Delta f(x+h)$		$\Delta^3 f(x)$	
$x+2h$	$f(x+2h)$		$\Delta^2 f(x+h)$		$\Delta^4 f(x)$
		$\Delta f(x+2h)$		$\Delta^3 f(x+h)$	
$x+3h$	$f(x+3h)$		$\Delta^2 f(x+2h)$		
		$\Delta f(x+3h)$			
$x+4h$	$f(x+4h)$				

Backward Difference operator (∇) :

Let $y = f(x)$ be a given function of x. Let $y_0, y_1, \ldots, y_n$ be the values of y at $x = x_0, x_1, x_2, \ldots, x_n$ respectively. Then

$$y_1 - y_0 = \nabla y_1$$

$$y_2 - y_1 = \nabla y_2$$

$$y_n - y_{n-1} = \nabla y_n$$

are called the first(backward) differences.

The operator ∇ is called backward difference operator and pronounced as nepla.

Second(backward) differences: $\nabla^2 y_n = \nabla y_n - \nabla y_{n+1}, \quad n = 1,2,3,\ldots$

Third (backward) differences: $\nabla^3 y_n = \nabla^2 y_n - \nabla^2 y_{n-1} \; n = 1,2,3,\ldots$

In general, k^{th} (backward) differences: $\nabla^k y_n = \nabla^{k-1} y_n - \nabla^{k-1} y_{n-1} \; n = 1,2,3,\ldots$

Backward difference table:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

Backward differences can also be defined as follows.

$$\nabla f(x) = f(x) - f(x-h)$$

First differences: $\nabla f(x+h) = f(x+h) - f(x)$

$\nabla f(x+2h) = f(x+2h) - f(x+h), \ldots,$ h is the interval of spacing.

Second differences:

$$\nabla^2 f(x+h) = \nabla(\nabla f(x+h) = \nabla(f(x+h) - f(x))$$

$$= \nabla f(x+h) - \nabla f(x)$$

$$\nabla^2 f(x+2h) = \nabla f(x+2h) - \nabla f(x+h)$$

Third differences:

$$\nabla^3 f(x+h) = \nabla^2 f(x+h) - \nabla^2 f(x)$$

$$\nabla^3 f(x+2h) = \nabla^2 f(x+2h) - \nabla^2 f(x+h)$$

Here we note that, $\nabla f(x+h) = f(x+h) - f(x) = \Delta f(x)$

$$\nabla f(x+2h) = f(x+2h) - f(x+h) = \Delta f(x+h)$$

$$\nabla^2 f(x+2h) = \nabla f(x+2h) - \nabla f(x+h) = \Delta f(x+h) - \Delta f(x)$$

$$= \Delta^2 f(x)$$

In general, $\nabla^n f(x+nh) = \Delta^n f(x)$

Shifting operator (E):

Let $y = f(x)$ be a given function of x and $x_0, x_0+h, x_0+2h, x_0+3h, \ldots, x_0+nh$ be the consecutive values of x. Then the operator E is defined as

$$\mathrm{E}\left[f(x_0)\right] = f(x_0+h)$$

E is called the shifting operator . It is also called the displacement operator.

$$E\left[f\left(x_0+h\right)\right] = f(x_0+2h), E[f(x_0+2h)] = f(x_0+3h), \ldots,$$

$$\mathrm{E}[f(x_0+(n-1)h)] = f(x_0+nh)$$

$\mathrm{E}\left[f(x)\right] = f(x+h)$, h is the (equal) interval of spacing

$E^2 f(x)$ means that the operator E is applied twice on $f(x)$

(i.e) $\mathrm{E}^2 f(x) = \mathrm{E}\left[\mathrm{E} f(x)\right] = \mathrm{E}[f(x+h) = f(x+2h)$

In general ,

$$\mathrm{E}^n f(x) = f(x+nh) \text{ and } \mathrm{E}^{-n} f(x) = f(x-nh)$$

Properties of the operator E:

1. $\mathrm{E}\left[f_1(x) + f_2(x) + \ldots.. + f_n(x)\right] = \mathrm{E}f_1(x) + \mathrm{E}f_2(x) \ldots. + \ldots. + \mathrm{E}\left[f_n(x)\right]$

2. $\mathrm{E}\left[c\, f(x)\right] = cE\left[f(x)\right]$ c is a constant

3. $E^m\left[E^n f(x)\right]=E^n\left(E^m f(x)\right)=E^{m+n} f\left(x\right)$

4. If 'n' is a positive integer, then $E^n\left[E^{-n}\left(f(x)\right)\right]=f(x)$

Note

Let $y = f(x)$ be given function of x. Let $y_0, y_1, y_2, \cdots, y_n$ be the values of y at $x = x_0, x_1, x_2, ..., x_n$. Then E can also be defined as

$$Ey_0 = y_1, \quad Ey_1 = y_2, ..., Ey_{n-1} = y_n$$

$$E\left[Ey_0\right] = E\left(y_1\right) = y_2 \text{ and in general } E^n y_0 = y_n$$

Relations between the operators Δ, ∇ and E:

1. $\Delta \equiv \boldsymbol{E} - 1$

Proof: From the definition of Δ we know that

$$\Delta f(x) = f(x+h) - f(x)$$

and $E[f(x)] = f(x+h)$

where h is the interval of difference.

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta f(x) = Ef(x) - f(x)$$

$$\Rightarrow \Delta f(x) = (E-1)f(x)$$

$$\Delta \equiv E - 1$$

$$\therefore \quad E \equiv 1 + \Delta$$

2. $E\Delta \equiv \Delta E$

Proof:

$$\begin{aligned} E\left(\Delta f(x)\right) &= E\left[f(x+h) - f(x)\right] \\ &= E f(x+h) - E f(x) \\ &= f(x+2h) - f(x+h) \\ &= \Delta f(x+h) \\ &= \Delta E f(x) \end{aligned}$$

$$\therefore E\Delta \equiv \Delta E$$

3. $$\nabla \equiv \frac{E-1}{E}$$

Proof:

$$\nabla f(x) = f(x) - f(x-h)$$

$$= f(x) - E^{-1} f(x)$$

$$= (1 - E^{-1}) f(x)$$

$$\Rightarrow \quad \nabla \equiv 1 - E^{-1}$$

i.e., $$\nabla \equiv 1 - \frac{1}{E}$$

Hence, $$\nabla \equiv \frac{E-1}{E}$$

DO YOU KNOW?

(i) $(1+\Delta)(1-\nabla) = 1$

(ii) $\Delta\nabla \equiv \Delta - \nabla$

(iii) $\nabla \equiv E^{-1}\Delta$

Example 5.1

Construct a forward difference table for the following data

x	0	10	20	30
y	0	0.174	0.347	0.518

Solution:

The Forward difference table is given below

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	0			
		0.174		
10	0.174		-0.001	
		0.173		-0.001
20	0.347		-0.002	
		0.171		
30	0.518			

Example 5.2

Construct a forward difference table for $y = f(x) = x^3 + 2x + 1$ for $x = 1,2,3,4,5$

Solution

$y = f(x) = x^3 + 2x + 1$ for $x = 1,2,3,4,5$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	4				
		9			
2	13		12		
		21		6	
3	34		18		0
		39		6	
4	73		24		
		63			
5	136				

Example 5.3

By constructing a difference table and using the second order differences as constant, find the sixth term of the series 8,12,19,29,42…

Solution:

Let k be the sixth term of the series in the difference table

First we find the forward differences.

x	y	Δ	Δ^2
1	8		
		4	
2	12		3
		7	
3	19		3
		10	
4	29		3
		13	
5	42		$k-55$
		$k-42$	
6	k		

Given that the second differences are constant

$\therefore \quad k - 55 = 3$

$k = 58$

$\therefore$ the sixth term of the series is 58

Example 5.4

Find (i) Δe^{ax} (ii) $\Delta^2 e^x$ (iii) $\Delta \log x$

Solution:

(i) $\Delta e^{ax} = e^{a(x+h)} - e^x$

$= e^{ax}.e^h - e^{ax}$ $\qquad [\because a^{m+n} = a^m.a^n]$

$= e^{ax}\left[e^h - 1\right]$

(ii) $\Delta^2 e^x = \Delta.\left[\Delta e^x\right]$

$= \Delta\left[e^{x+h} - e^x\right] = \Delta\left[e^x e^h - e^x\right]$

$= \Delta e^x\left[e^h - 1\right]$

$= \left(e^h - 1\right)\Delta e^x$

$= \left(e^h - 1\right).\left(e^h - 1\right).e^x$

$= \left(e^h - 1\right)^2.e^x$

(iii) $\Delta \log x = \log(x+h) - \log x$

$= \log\frac{x+h}{x}$

$= \log\left(\frac{x}{x} + \frac{h}{x}\right)$

$= \log\left(1 + \frac{h}{x}\right)$

Example 5.5

Evaluate $\Delta\left[\frac{5x+12}{x^2+5x+6}\right]$ by taking '1' as the interval of differencing.

Solution:

$$\Delta\left[\frac{5x+12}{x^2+5x+6}\right]$$

By Partial fraction method

$$\frac{5x+12}{x^2+5x+6} = \frac{A}{x+3} + \frac{B}{x+2}$$

$$A = \frac{5x+12}{x+2}[x=-3] = \frac{-15+12}{-1} = \frac{-3}{-1} = 3$$

$$B = \frac{5x+12}{x+3}\left[x=-2\right]$$

$$= \frac{2}{1} = 2$$

$$\frac{5x+12}{x^2+5x+6} = \left[\frac{3}{x+3}+\frac{2}{x+2}\right]$$

$$\Delta\left[\frac{5x+12}{x^2+5x+6}\right] = \Delta\left[\frac{3}{x+3}+\frac{2}{x+2}\right]$$

$$= \left[\frac{3}{x+1+3}-\frac{3}{x+3}\right]+\left\{\frac{2}{x+1+2}-\frac{2}{x+2}\right\}$$

$$= 3\left[\frac{1}{x+4}-\frac{1}{x+3}\right]+2\left[\frac{1}{x+3}-\frac{1}{x+2}\right]$$

$$= \left[\frac{-3}{(x+4)(x+3)}-\frac{2}{(x+3)(x+2)}\right]$$

$$= \frac{-5x-14}{(x+2)(x+3)(x+4)}$$

Example 5.6

Evaluate $\Delta^2\left(\frac{1}{x}\right)$ by taking '1' as the interval of differencing.

Solution:

$$\Delta^2\left(\frac{1}{x}\right) = \Delta\left(\Delta\left(\frac{1}{x}\right)\right)$$

Now $$\Delta\left[\frac{1}{x}\right] = \frac{1}{x+1}-\frac{1}{x}$$

$$\Delta^2\left(\frac{1}{x}\right) = \Delta\left(\frac{1}{1+x}-\frac{1}{x}\right)$$

$$= \Delta\left(\frac{1}{1+x}\right)-\Delta\left(\frac{1}{x}\right)$$

Similarly $$\Delta^2\left(\frac{1}{x}\right) = \frac{2}{x(x+1)(x+2)}$$

DO YOU KNOW?

$$\Delta^n\left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x(x+1)(x+2)\ldots\ldots(x+n)}$$

Example 5.7

Prove that $f(4) = f(3)+\Delta f(2)+\Delta^2 f(1)+\Delta^3 f(1)$ taking '1' as the interval of differencing.

Solution:

We know that $f(4)-f(3)=\Delta f(3)$

$$
\begin{aligned}
f(4)-f(3) &= \Delta f(3)\\
&= \Delta\left[f(2)+\Delta f(2)\right] \qquad \because \left[f(3)-f(2)=\Delta f(2)\right]\\
&= \Delta f(2)+\Delta^2 f(2)\\
&= \Delta f(2)+\Delta^2\left[f(1)+\Delta f(1)\right]\\
\therefore f(4) &= f(3)+\Delta f(2)+\Delta^2 f(1)+\Delta^3 f(1).
\end{aligned}
$$

Example 5.8

Given $U_0=1, U_1,=11, U_2=21, U_3=28$ and $U_4=29$ find $\Delta^4 U_0$

Solution :

$$
\begin{aligned}
\Delta^4 U_0 &= (E-1)^4 U_0\\
&= \left(E^4-4E^3+6E^2-4E+1\right)U_0\\
&= E^4U_0-4E^3U_0+6E^2U_0-4EU_0+U_0\\
&= U_4-4U_3+6U_2-4U_1+U_0\\
&= 29-4(28)+6(21)-4(11)+1.\\
&= 156-156=0
\end{aligned}
$$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Example 5.9

Given $y_3=2, y_4=-6, y_5=8, y_6=9$ and $y_7=17$ Calculate $\Delta^4 y_3$

Solution :

Given $y_3=2, y_4=-6, y_5=8, y_6=9$ and $y_7=17$

$$
\begin{aligned}
\Delta^4 y_3 &= (E-1)^4 y_3\\
&= (\mathrm{E}^4-4E^3+6E^2-4E+1)y_3\\
&= E^4y_3-4E^3y_3+6E^2y_3-4Ey_3+y_3\\
&= y_7-4y_6+6y_5-4y_4+y_3\\
&= 17-4(9)+6(8)-4(-6)+2\\
&= 17-36+48+24+2=55
\end{aligned}
$$

5.1.2 Finding the missing terms

Using the difference operators and shifting operator we can able to find the missing terms.

Example 5.10

From the following table find the missing value

x	2	3	4	5	6
$f(x)$	45.0	49.2	54.1	-	67.4

Solution:

Since only four values of $f(x)$ are given, the polynomial which fits the data is of degree three. Hence fourth differences are zeros.

(ie) $\Delta^4 y_0 = 0, \therefore (E-1)^4 y_0 = 0$

$$(E^4 - 4E^3 + 6E^2 - 4E + 1)\, y_0 = 0$$

$$E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4Ey_0 + y_0 = 0$$

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$

$$67.4 - 4y_3 + 6(54.1) - 4(4.2) + 45 = 0$$

$$240.2 = 4y_3 \qquad \therefore y_3 = 60.05$$

Example 5.11

Estimate the production for 1964 and 1966 from the following data

Year	1961	1962	1963	1964	1965	1966	1967
Production	200	220	260	-	350	-	430

Solution:

Since five values are given, the polynomial which fits the data is of degree four.

Hence $\Delta^5 y_k = 0$ (ie) $(E-1)^5 y_k = 0$

i.e., $(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)\, y_k = 0$

$$E^5 y_k - 5\,E^4 y_k + 10E^3 y_k - 10E^2 y_k + 5Ey_k - y_k = 0 \qquad (1)$$

Put $k = 0$ in (1)

$$E^5y_0 - 5E^4y_0 + 10E^3y_0 - 10E^2y_0 + 5Ey_0 - y_0 = 0$$

$$y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$$

$$y_5 - 5(350) + 10y_3 - 10(260) + 5(220) - 200 = 0$$

$$y_5 + 10y_3 = 3450 \qquad (2)$$

Put $k = 1$ in (1)

$$E^5y_1 - 5E^4y_1 + 10E^3y_1 - 10E^3y_1 - y_1 = 0$$

$$y_6 - 5y_5 + 10y_4 - 10y_3 - y_1 = 0$$

$$430 - 5y_5 + 10(350) - 10y_3 + 5(260) - 220 = 0$$

$$5y_5 + 10y_3 = 5010 \qquad (3)$$

$$(3) - (2) \Rightarrow \quad 4y_5 = 1560$$

$$y_5 = 390$$

From $\quad 390 + 10y_3 = 3450$

$$10y_3 = 3450 - 390$$

$$y_3 \cong 306$$

					1					
				1		2		1		
			1		3		3		1	
		1		4		6		4		1
1	5		10				10		5	1

Exercise 5.1

1. Evaluate $\Delta(\log ax)$.

2. If $y = x^3 - x^2 + x - 1$ calculate the values of y for $x = 0,1,2,3,4,5$ and form the forward differences table.

3. If $h = 1$ then prove that $(E^{-1}\Delta)x^3 = 3x^2 - 3x + 1$.

4. If $f(x) = x^2 + 3x$ than show that $\Delta f(x) = 2x + 4$

5. Evaluate $\Delta\left[\frac{1}{(x+1)(x+2)}\right]$ by taking '1' as the interval of differencing

6. Find the missing entry in the following table

x	0	1	2	3	4
y_x	1	3	9	-	81

7. Following are the population of a district

Year (x)	1881	1891	1901	1911	1921	1931
Population (y) Thousands	363	391	421	-	467	501

Find the population of the year 1911

8. Find the missing entries from the following.

x	0	1	2	3	4	5
$y = f(x)$	0	-	8	15	-	35

5.2 Interpolation

Consider the profit of a manufacturing company in various years as given below

Year (x)	1986	1987	1988	1990	1991	1992
Profit (Rs. in lakhs)	25	29	24	30	32	31

The profit for the year 1989 is not available .To estimate the profit for 1989 we use the technique called interpolation. Let x denote the year and y denote the profit. The independent variable x is called the argument and the dependent variable y is called the entry. If y is to be estimated for the value of x between two extreme points in a set of values, it is called interpolation.

DO YOU KNOW?

If y is to be estimated for the values of x which lies outside the given set of the values of it, is called extrapolation.

5.2.1 Methods of interpolation

There are two methods for interpolation. One is Graphical method and the other one is algebraic method.

5.2.2 Graphical method

We are given the 'n' values of x and the corresponding values of y for given function $y = f(x)$. we plot these n observed points $(x_i, y_i), i = 1,2,3....$ and draw a free hand curve passing through these plotted points. From the graph so obtained, we can find out the value of y for any intermediate value of x. There is one drawback in the graphic method which states that the value of y obtained is the estimated value of y. The estimated value of y differs from the actual value of y.

Example 5.12

Using graphic method, find the value of y when x = 38 from the following data:

x	10	20	30	40	50	60
y	63	55	44	34	29	22

Solution:

From the graph in Fig. 5.1 we find that for x = 38, the value of y is equal to 35

Steps in Graphic method:

Take a suitable scale for the values of x and y, and plot the various points on the graph paper for given values of x and y.

Draw a suitable curve passing through the plotted points.

Find the point corresponding to the value x = 38 on the curve and then read the corresponding value of y on the y- axis, which will be the required interpolated valu

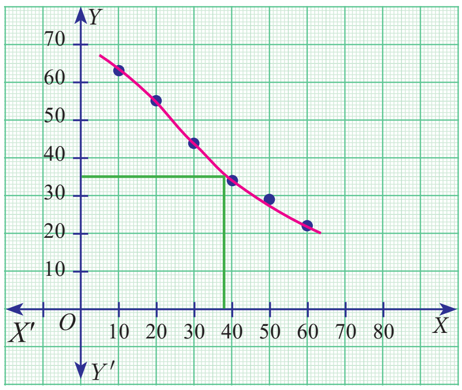


Fig. 5.1

5.2.3 Algebraic method

Newton's Gregory forward interpolation formula (or) Newton's forward interpolation formula (for equal intervals).

Let $y = f(x)$ denote a polynomial of degree n which takes $(n+1)$ values. Let them be $y_0, y_1, y_2, \dots y_n$ corresponding to the values $x_0, x_1, \dots x_n$ respectively.

The values of x ($x_0, x_1, x_2, \dots x_n$) are at equidistant.

DO YOU KNOW?

The first two terms will give the linear interpolation and the first three terms will give a parabolic interpolation and so on.

(i.e.) $x_1 = x_0 + h,\ x_2 = x_0 + 2h,\ x_3 = x_0 + 3h, \dots, x_n = x_0 + nh$

Then the value of $f(x)$ at $x = x_0 + nh$ is given by

$$f(x_0 + nh) = f(x_0) + \frac{n}{1!}\Delta f(x_0) + \frac{n(n-1)}{2!}\Delta^2 f(x_0) + \frac{n(n-1)(n-2)}{3!}\Delta^3 f(x_0) + \dots$$

(or) $y_{(x=x_0+nh)} = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \dots$ where $n = \frac{x - x_0}{h}$

Newton's Gregory backward interpolation Formula.

Note

Newton's forward interpolation formula is used when the value of y is required near the beginning of the table.

Newton's forward interpolation formula cannot be used when the value of y is required near the end of the table. For this we use another formula, called Newton's Gregory backward interpolation formula.

Then the value of $f(x)$ at $x = x_n + nh$ is given by

$$f(x_n + nh) = f(x_n) + \frac{n}{1!}\nabla f(x_n) + \frac{n(n+1)}{2!}\nabla^2 f(x_n) + \frac{n(n+1)(n+2)}{3!}\nabla^3 f(x_n) + \dots$$

(or) $y_{(x=x_n+nh)} = y_n + \frac{n}{1!}\nabla y_n + \frac{n(n+1)}{2!}\nabla^2 y_n + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_n + \dots$ when $n = \frac{x - x_n}{h}$

Note

Newton's backward interpolation formula is used when the value of y is required near the end of the table.

Example 5.13

Using Newton's formula for interpolation estimate the population for the year 1905 from the table:

Year	1891	1901	1911	1921	1931
Population	98,752	1,32,285	1,68,076	1,95,670	2,46,050

Solution

To find the population for the year 1905 (i.e) the value of y at $x = 1905$

Since the value of y is required near the beginning of the table, we use the Newton's forward interpolation formula.

$$y_{(x=x_0+nh)} = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots$$

To find y at $x = 1905$ $\therefore x_0 + nh = 1905$, $x_0 = 1891, h = 10$

$1891 + n(10) = 1905 \quad \Rightarrow n = 1.4$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	98,752				
		33,533			
1901	1,32,285		2,258		
		35,791		−10,435	
1911	1,68,076		-8,177		41,376
		27,614			
1921	1,95,690			30,941	
			22,764		
		50,360			
1931	2,46,050				

$$\boldsymbol{y}_{(x=1905)} = 98,752 + (1.4)(33533) + \frac{(1.4)(0.4)}{2}(2258) + \frac{(1.4)(0.4)\,(-0.6)}{6}(-10435) + \frac{(1.4)(0.6)(-0.6)(-1.6)}{24}(41376)$$

$$= 98,752 + 46946.2 + 639.8 + 584.36 + 1390.23$$

$$= 1,48,312.59$$

$$\cong 1,48,313$$

Example 5.14

The values of $y = f(x)$ for $x = 0, 1, 2, \ldots, 6$ are given by

x	0	1	2	3	4	5	6
y	2	4	10	16	20	24	38

Estimate the value of y (3.2) using forward interpolation formula by choosing the four values that will give the best approximation.

Solution:

Since we apply the forward interpolation formula,last four values of $f(x)$ are taken into consideration (Take the values from $x = 3$).

The forward interpolation formula is

$$y_{(x=x_0+nh)} = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots$$

$$x_0 + nh = 3.2, x_0 = 3, h = 1$$

$$\therefore \qquad n = \frac{1}{5}$$

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
3	16			
		4		
4	20		0	
		4		10
5	24		10	
		14		
6	38			

$$y_{(x=3.2)} = 16 + \frac{1}{5}(4) + \frac{\frac{1}{5}\left(\frac{-4}{5}\right)}{2}(0) + \frac{\frac{1}{5}\left(\frac{-4}{5}\right)\left(\frac{-9}{5}\right)}{6} \times 10$$

$$= 16 + 0.8 + 0 + 0.48$$

$$= 17.28$$

Example 5.15

From the following table find the number of students who obtained marks less than 45.

Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	31	42	51	35	31

Solution:

Let x be the marks and y be the number of students

By converting the given series into cumulative frequency distribution, the difference table is as follows.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Less than 40	31				
		42			
50	73		9		
		51		–25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

$$y_{(x=x_0+nh)} = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots$$

To find y at $x = 45$ $\therefore x_0 + nh = 45$, $x_0 = 40, h = 10$ $\Rightarrow n = \dfrac{1}{2}$

$$y_{(x=45)} = 31 + \frac{1}{2} \times 42 + \frac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2}(9) + \frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{6} \times (-25)$$
$$+ \frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{24} \times (37)$$

$$= 31 + 21 - \frac{9}{8} - \frac{25}{16} - \frac{37 \times 15}{128}$$

$$= 47.867 \cong 48$$

Example 5.16

Using appropriate interpolation formula find the number of students whose weight is between 60 and 70 from the data given below

Weight in lbs	0-40	40-60	60-80	80-100	100-120
No. of Students	250	120	100	70	50

Solution:

Let x be the weight and y be the number of students.

Difference table of cumulative frequencies are given below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
60	370		−20		
		100		−10	
80	470		−30		20
		70		10	
100	540		−20		
		50			
120	590				

Let us calculate the number of students whose weight is below 70. For this we use forward difference formula.

$$y_{(x=x_0+nh)} = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots$$

To find y at $x = 70 \therefore\ x_0 + nh = 70$, $x_0 = 40, h = 20$

$$40+n(20) = 70 \Rightarrow n = 1.5$$

$$\therefore \quad y_{(x=70)} = 250 + 1.5(120) + \frac{(1.5)(0.5)}{2!}(-20) + \frac{(1.5)(0.5)(-0.5)}{3!}(-10) + \frac{(1.5)(0.5)(-0.5)(-1.5)}{4!}(20)$$

$$= 250 + 180 - 7.5 - 0.625 + 0.46875$$

$$= 423.59$$

$$\cong 424.$$

Number of students whose weight is between

$$60 \text{ and } 70 = y(70) - y(60) = 424 - 370 = 54$$

Example 5.17

The population of a certain town is as follows

Year : x	1941	1951	1961	1971	1981	1991
Population in lakhs : y	20	24	29	36	46	51

Using appropriate interpolation formula, estimate the population during the period 1946.

Solution:

x	1941	1951	1961	1971	1981	1991
y	20	24	29	36	46	51

Here we find the population for year1946. (i.e) the value of y at x=1946. Since the value of y is required near the beginning of the table, we use the Newton's forward interpolation formula.

$$y_{(x=x_0+nh)} = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots$$

To find y at $x = 1946$ $\therefore\ x_0 + nh = 1946,\ x_0 = 1941,\ h = 10$

$1941 + n(10) = 1946 \Rightarrow n = 0.5$

x	y	Δy	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\Delta^5 y$
1941	20					
		4				
1951	24		1			
		5		1		
1961	29		2		0	
		7		1		-9
1971	36		3		-9	
		10		-8		
1981	46		-5			
		5				
1991	51					

$$y_{(x=1946)} = 20 + \frac{0.5}{1!}(4) + \frac{0.5(0.5-1)}{2!}(1) + \frac{0.5(0.5-1)(0.5-2)}{3!}(1)$$

$$+\frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}(0) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)(0.5-4)}{5!}(-9)$$

$$= 20 + 2 - 0.125 + 0.0625 - 0.24609$$

$$= 21.69 \text{ lakhs}$$

Example 5.18

The following data are taken from the steam table.

Tempreture C^0	140	150	160	170	180
Pressure kg f/cm^2	3.685	4.854	6.302	8.076	10.225

Find the pressure at temperature $t = 175^0$

Solution:

Since the pressure required is at the end of the table, we apply Backward interpolation formula. Let temperature be x and the pressure be y.

$$y_{(x=x_n+nh)} = y_n + \frac{n}{1!}\nabla y_n + \frac{n(n+1)}{2!}\nabla^2 y_n + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_n + \ldots$$

To find y at $x = 175$

$$\therefore x_n + nh = 175,\ x_n = 180, h = 10 \ \Rightarrow \ n = -0.5$$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
140	3.685				
		1.169			
150	4.854		0.279		
		1.448		0.047	
160	6.032		0.326		0.002
		1.774		0.049	
170	8.076		0.375		
		2.149			
180	10.225				

$$\therefore \qquad y_{(x=175)} = 10.225 + (-0.5)(2.149) + \frac{(-0.5)(0-5)}{2!}(0.375)$$

$$+ \frac{(-0.5)(0-5)(1.5)}{3!}(0.049) + \frac{(-0-5)(0.5)(1.5)(2.5)}{4!}(0.002)$$

$$= 10.225 - 1.0745 - 0.046875 - 0.0030625 - 0.000078125$$

$$= 9.10048438$$

$$= 9.1$$

Example 5.19

Calculate the value of y when $x = 7.5$ from the table given below

x	1	2	3	4	5	6	7	8
y	1	8	27	64	125	216	343	512

Solution:

Since the required value is at the end of the table, apply backward interpolation formula.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	
6	216		36		0
		127		6	
7	343		42		
		169			
8	512				

$$y_{(x=x_n+nh)} = y_n + \frac{n}{1!}\nabla y_n + \frac{n(n+1)}{2!}\nabla^2 y_n + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_n + \ldots$$

To find y at $x = 7.5 \therefore x_n + nh = 7.5,\ x_n = 8, h = 1 \Rightarrow n = -0.5$

$$y_{(x=7.5)} = 512 + \frac{-0.5}{1!}169 + \frac{-0.5(-0.5+1)}{2!}42 + \frac{-0.5(-0.5+1)(-0.5+2)}{3!}6 + \ldots$$

$$= 421.87$$

Example 5.20

From the following table of half- yearly premium for policies maturing at different ages. Estimate the premium for policies maturing at the age of 63.

Age	45	50	55	60	65
Premium	114.84	96.16	83.32	74.48	68.48

Solution:

Let age $= x$ and premium $= y$

To find y at $x = 63$. So apply Newton's backward interpolation formula

$$y_{(x=x_n+nh)} = y_n + \frac{n}{1!}\nabla y_n + \frac{n(n+1)}{2!}\nabla^2 y_n + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_n + \ldots$$

To find y at $x = 63$ $\therefore x_n + nh = 63$, $x_n = 68.48, h = 5 \Rightarrow n = -\frac{2}{5}$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
45	114.84				
		-18.68			
50	96.16		5.84		
		-12.84		-1.84	
55	83.32		4		0.68
		-8.84		-1.16	
60	74.48		2.84		
		-6			
65	68.48				

$$y_{(x=63)} = 68.48 + \frac{\frac{-2}{5}}{1!}(-6) + \frac{\frac{-2}{5}\left(\frac{-2}{5}+1\right)}{2!}2.84 + \frac{\frac{-2}{5}\left(\frac{-2}{5}+1\right)\left(\frac{-2}{5}+2\right)}{3!}(-1.16)$$

$$+ \frac{\frac{-2}{5}\left(\frac{-2}{5}+1\right)\left(\frac{-2}{5}+2\right)\left(\frac{-2}{5}+3\right)}{3!}(0.68)$$

$$= 68.48 + 2.4 - 0.3408 + 0.07424 - 0 - 0.028288$$

$$y(63) = 70.437$$

Example 5.21

Find a polynomial of degree two which takes the values

x	0	1	2	3	4	5	6	7
y	1	2	4	7	11	16	22	29

Solution:

We will use Newton's backward interpolation formula to find the polynomial.

$$y_{(x=x_n+nh)} = y_n + \frac{n}{1!}\nabla y_n + \frac{n(n+1)}{2!}\nabla^2 y_n + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_n + \ldots$$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
0	1			
		1		
1	2		1	
		2		0
2	4		1	
		3		0
3	7		1	
		4		0
4	11		1	
		5		0
5	16		1	
		6		0
6	22		1	
		7		
7	29			

To find y in terms of x $\therefore x_n + nh = x$, $x_n = 7, h = 1 \Rightarrow n = x - 7$

$$\begin{aligned} y_{(x)} &= 29 + (x-7)(7) + \frac{(x-7)(x-6)}{2}(1) \\ &= 29 + 7x - 49 + \frac{1}{2}\left(x^2 - 13x + 42\right) \\ &= \frac{1}{2}\left[58 + 14x - 98 + x^2 - 13x + 42\right] \\ &= \frac{1}{2}\left[x^2 + x + 2\right] \end{aligned}$$

5.2.4 Lagrange's interpolation formula

The Newton's forward and backward interpolation formulae can be used only when the values of x are at equidistant. If the values of x are at equidistant or not at equidistant, we use Lagrange's interpolation formula.

Let $y = f(x)$ be a function such that $f(x)$ takes the values $y_0, y_1, y_2, \ldots\ldots, y_n$ corresponding to $x = x_0,\ x_1, x_2 \ldots, x_n$ That is $y_i = f(x_i), i = 0,1,2,\ldots,n$. Now, there are $(n+1)$ paired values $(x_i, y_i), i = 0,1,2,\ldots,n$ and hence $f(x)$ can be represented by a polynomial function of degree n in x.

Then the Lagrange's formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} y_1$$

$$+ \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} y_n$$

Example 5.22

Using Lagrange's interpolation formula find y(10) from the following table:

x	5	6	9	11
y	12	13	14	16

Solution:

Here the intervals are unequal. By Lagrange's interpolation formula we have

$$x_0 = 5, x_1 = 6, x_2 = 9, x_3 = 11$$

$$y_0 = 12, y_1 = 13, y_2 = 14, y_3 = 16$$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$= \frac{(x-6)(x-9)(x-11)}{(5-6)(5-6)(5-11)}(12) + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-9)}(13)$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16)$$

Put $x = 10$

$$y(10) = f(10) = \frac{4(1)(-1)}{(-1)(-4)(-6)}(12) + \frac{(5)(1)(-1)}{(1)(-3)(-5)}(13) + \frac{5(4)(-1)}{4(3)(-2)}(14) + \frac{(5)(4)(1)}{6(5)(2)}(16)$$

$$= \frac{1}{6}(12) - \frac{13}{3} + \frac{5(14)}{3 \times 2} + \frac{4 \times 16}{12}$$

$$= 14.6663$$

Exercise 5.2

1. Using graphic method, find the value of y when $x = 48$ from the following data:

x	40	50	60	70
y	6.2	7.2	9.1	12

2. The following data relates to indirect labour expenses and the level of output

Months	Jan	Feb	Mar	Apr	May	June
Units of output	200	300	400	640	540	580
Indirect labour expenses (Rs)	2500	2800	3100	3820	3220	3640

Estimate the expenses at a level of output of 350 units, by using graphic method.

3. Using Newton's forward interpolation formula find the cubic polynomial.

x	0	1	2	3
$f(x)$	1	2	1	10

4. The population of a city in a censes taken once in 10 years is given below. Estimate the population in the year 1955.

Year	1951	1961	1971	1981
Population in lakhs	35	42	58	84

5. In an examination the number of candidates who secured marks between certain interval were as follows:

Marks	0-19	20-39	40-59	60-79	80-99
No. of candidates	41	62	65	50	17

Estimate the number of candidates whose marks are lessthan 70.

6. Find the value of $f(x)$ when $x = 32$ from the following table

x	30	35	40	45	50
$f(x)$	15.9	14.9	14.1	13.3	12.5

7. The following data gives the melting point of a alloy of lead and zinc where 't' is the temperature in degree c and P is the percentage of lead in the alloy

P	40	50	60	70	80	90
T	180	204	226	250	276	304

Find the melting point of the alloy containing 84 percent lead.

8. Find $f(2.8)$ from the following table.

x	0	1	2	3
$f(x)$	1	2	11	34

9. Using interpolation estimate the output of a factory in 1986 from the following data

Year	1974	1978	1982	1990
Output in 1000 tones	25	60	80	170

10. Use Lagrange's formula and estimate from the following data the number of workers getting income not exceeding Rs. 26 per month.

Income not exceeding (₹)	15	25	30	35
No. of workers	36	40	45	48

11. Using interpolation estimate the business done in 1985 from the following data

Year	1982	1983	1984	1986
Business done (in lakhs)	150	235	365	525

12. Using interpolation, find the value of $f(x)$ when $x = 15$

x	3	7	11	19
$f(x)$	42	43	47	60

VZ2SKM

Exercise 5.3

Choose the correct Answer

1. $\Delta^2 y_0 =$

(a) $y_2 - 2y_1 + y_0$ (b) $y_2 + 2y_1 - y_0$ (c) $y_2 + 2y_1 + y_0$ (d) $y_2 + y_1 + 2y_0$

2. $\Delta f(x) =$

(a) $f(x+h)$ (b) $f(x) - f(x+h)$ (c) $f(x+h) - f(x)$ (d) $f(x) - f(x-h)$

3. $E \equiv$

(a) $1 + \Delta$ (b) $1 - \Delta$ (c) $1 + \nabla$ (d) $1 - \nabla$

4. If $h=1$, then $\Delta(x^2) =$

(a) $2x$ (b) $2x - 1$ (c) $2x + 1$ (d) 1

5. If c is a constant then $\Delta c =$

(a) c (b) Δ (c) Δ^2 (d) 0

6. If m and n are positive integers then $\Delta^m \Delta^n f(x) =$

(a) $\Delta^{m+n} f(x)$ (b) $\Delta^m f(x)$ (c) $\Delta^n f(x)$ (d) $\Delta^{m-n} f(x)$

7. If 'n' is a positive integer $\Delta^n \left[\Delta^{-n} f(x)\right]$

(a) $f(2x)$ (b) $f(x+h)$ (c) $f(x)$ (d) $\Delta f(x)$

8. $E f(x) =$

(a) $f(x-h)$ (b) $f(x)$ (c) $f(x+h)$ (d) $f(x+2h)$

9. $\nabla \equiv$

(a) 1+E　(b) 1-E　(c) $1-E^{-1}$　(d) $1+E^{-1}$

10. $\nabla f(a) =$

(a) $f(a)+f(a-h)$　(b) $f(a)-f(a+h)$

(c) $f(a)-f(a-h)$　(d) $f(a)$

11. For the given points (x_0, y_0) and (x_1, y_1) the Lagrange's formula is

(a) $y(x) = \dfrac{x-x_1}{x_0-x_1}y_0 + \dfrac{x-x_0}{x_1-x_0}y_1$　(b) $y(x) = \dfrac{x_1-x}{x_0-x_1}y_0 + \dfrac{x-x_0}{x_1-x_0}y_1$

(c) $y(x) = \dfrac{x-x_1}{x_0-x_1}y_1 + \dfrac{x-x_0}{x_1-x_0}y_0$　(d) $y(x) = \dfrac{x_1-x}{x_0-x_1}y_1 + \dfrac{x-x_0}{x_1-x_0}y_0$

12. Lagrange's interpolation formula can be used for

(a) equal intervals only　(b) unequal intervals only

(c) both equal and unequal intervals　(d) none of these.

13. If $f(x) = x^2+2x+2$ and the interval of differencing is unity then $\Delta f(x)$

(a) $2x-3$　(b) $2x+3$　(c) $x+3$　(d) $x-3$

14. For the given data find the value of $\Delta^3 y_0$ is

x	5	6	9	11
y	12	13	15	18

(a) 1　(b) 0　(c) 2　(d) –1

Miscellaneous Problems

1. If $f(x) = e^{ax}$ then show that $f(0)$, $\Delta f(0)$, $\Delta^2 f(0)$ are in G.P

2. Prove that $i)(1+\Delta)(1-\nabla)=1$　$ii)\Delta\nabla = \Delta-\nabla$　(iii) $E\nabla = \Delta = \nabla E$

3. A second degree polynomial passes though the point (1,-1) (2,-1) (3,1) (4,5). Find the polynomial.

4. Find the missing figures in the following table

x	0	5	10	15	20	25
y	7	11	-	18	-	32

5. Find $f(0.5)$ if $f(-1)=202$, $f(0)=175$, $f(1)=82$ and $f(2)=55$

6. From the following data find y at $x = 43$ and $x = 84$

x	40	50	60	70	80	90
y	184	204	226	250	276	304

7. The area A of circle of diameter 'd' is given for the following values

D	80	85	90	95	100
A	5026	5674	6362	7088	7854

Find the approximate values for the areas of circles of diameter 82 and 91 respectively.

8. If $u_0 = 560,\ u_1 = 556,\ u_2 = 520,\ u_4 = 385$, show that $u_3 = 465$

9. From the following table obtain a polynomial of degree y in x

x	1	2	3	4	5
y	1	-1	1	-1	1

10. Using Lagrange's interpolation formula find a polynomial which passes through the points $(0, -12), (1, 0), (3, 6)$ and $(4,12)$.

Summary

In this chapter we have acquired the knowledge of

- $\Delta f(x) = f(x+h) - f(x)$
- $\nabla f(x) = f(x) - f(x-h)$
- $\nabla f(x+h) = \Delta f(x)$
- $Ef(x) = f(x+h)$
- $E^n f(x) = f(x+nh)$
- Newton's forward interpolation formula:

$$y_{(x=x_0+nh)} = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots$$

- Newton's backward interpolation formula:

$$y_{(x=x_n+nh)} = y_n + \frac{n}{1!}\nabla y_n + \frac{n(n+1)}{2!}\nabla^2 y_n + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_n + \ldots$$

- Lagrange's interpolation formula:

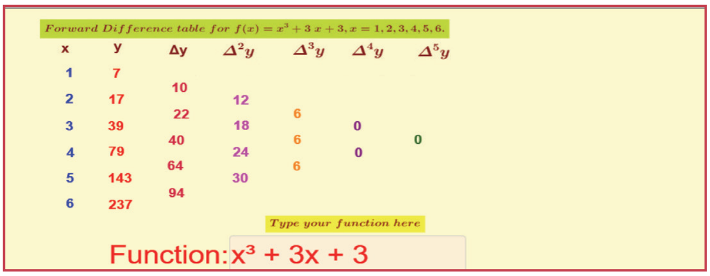
$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}y_1 + \ldots$$

$$+ \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}y_n$$

GLOSSARY	
Numerical	எண்ணியியல்
Interpolation	இடைச்செருகல்
Extrapolation	புறச்செருகல்
Graphic method	வரைபட முறை
Algebraic methods	இயற்கணித முறைகள்
Finite differences	திட்டமான வேறுபாடுகள்
Gregory- Newton's formulae	கிரிகோரி–நியுட்டனின் சூத்திரங்கள்
Lagrange's formula	இலக்ராஞ்சியின் சூத்திரம்
Forward difference operator	முன்நோக்கு வேறுபாட்டுச் செயலி
Backward difference operator	பின்நோக்கு வேறுபாட்டுச் செயலி
Shifting operator	இடப்பெயர்வுச் செயலி
Policy	காப்பீடு

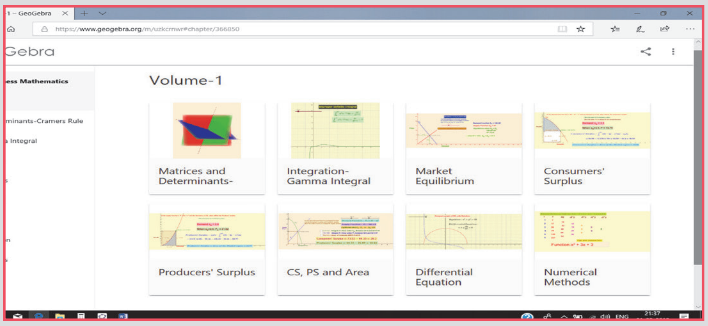
ICT Corner

Expected Result is shown in this picture



Step 1

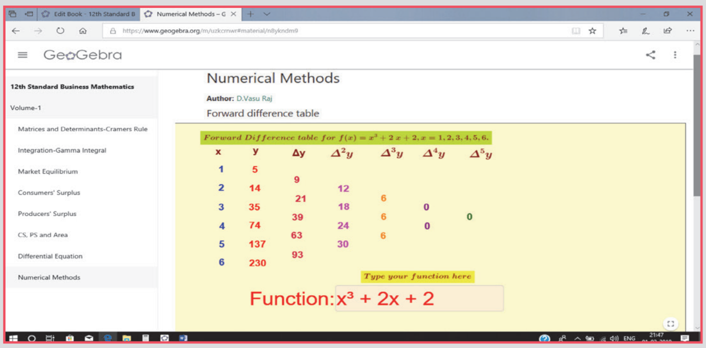
Open the Browser, type the URL Link given below (or) Scan the QR Code. GeoGebra work Book named "12th Standard Business Mathematics " will open. In the work book there are two Volumes. Select "Volume-1".



Step 2

Select the worksheet named" Numerical Methods"

In this Forward Difference table is given. You can type new function in the box given. Calculate and Check the table.



Browse in the link

12th standard Business Mathematics :https://ggbm.at/uzkcrnwr or Scan the QR Code.

B284_12_BUS_MAT_EM

Answers

1. Applications of Matrices and Determinants

Exercise 1.1

1.(i) $\rho(A)=2$ (ii) $\rho(A)=2$ (iii) $\rho(A)=1$ (iv) $\rho(A)=3$

(v) $\rho(A)=3$ (vi) $\rho(A)=3$ (vii) $\rho(A)=2$ (viii) $\rho(A)=3$ (ix) $\rho(A)=2$

2. $\rho(AB)=3,\ \rho(BA)=2$
3. $x=1, y=3, z=5$
4. $x=\frac{1}{11}(7-16k),\quad y=\frac{1}{11}(3+k),\quad z=k$
5. $x=2,\ y=1,\ z=0,$,
6. $\lambda=\frac{-7}{2}$
7. $x=1000,\ y=2000,\quad z=500$
8. $x=1000,\ y=2200,\ z=1800$

Exercise 1.2

1.(i) $x=8,\ y=-3$ (ii) $x=1,\ y=4$ (iii) $(x,\ y,\ z)=(2,\ -1,\ 0)$

(iv) $(x,\ y,\ z)=(1,\ 2,\ 3)$ (v) $(x,\ y,\ z)=\left(-1,\ \frac{1}{2},\frac{1}{3}\right)$

2. Cost per unit of labour is ₹10 Cost per unit of capital is ₹ 16
3. Amount invested at $4\frac{3}{4}\%$ is ₹7,300 Amount invested at $6\frac{1}{2}\%$ is ₹1,300
4. hourly charges for horse riding is ₹100 and AVT riding is ₹120
5. $(x, y, z)=(2,3,1)$
6. Amount invested at 2% is ₹250
   Amount invested at 3% is ₹4,000
   Amount invested at 6% is ₹4,250

Exercise 1.3

1. 36%
2. (i) 54% , 46% (ii) 50%
3. A=56.25%, B=43.75%
4. A=33%,B= 67%

Exercise 1.4

1	2	3	4	5	6	7	8	9	10	11	12	13
(d)	(b)	(a)	(c)	(d)	(b)	(b)	(d)	(a)	(c)	(c)	(b)	(c)
14	15	16	17	18	19	20	21	22	23	24	25	
(b)	(c)	(c)	(b)	(b)	(a)	(b)	(b)	(c)	(d)	(c)	(a)	

Miscellaneous problems

1. $\rho(A)=2$ 2. $\rho(A)=3$ 3. $\rho(A)=3$
4. The given system is inconsistent and has no solution. 5. $k = 8$.
6. The equations are inconsistent when k assumes any real value other than 0.
7. $x=1,\ y=2$ and $z=2$
8. Cost of wheat is ₹30/kg, Cost of sugar is ₹40/kg and Cost of rice is ₹50/kg.
9. The rates of commission for A,B and C are ₹2, ₹4 and ₹11 respectively
10. 39%

2. Integral Calculus – I

Exercise: 2.1

1. $\frac{2}{9}(3x+5)^{\frac{3}{2}}+c$ 2. $\frac{81x^5}{5}-\frac{16}{3x^3}-72x+c$ 3. $6x-\frac{13x^2}{2}-\frac{5x^3}{3}+c$
4. $\frac{2x^{\frac{9}{2}}}{9}-\frac{4x^{\frac{5}{2}}}{5}+2x^{\frac{3}{2}}+c$ 5. $\frac{(4x+7)^{\frac{3}{2}}}{3}-\frac{(4x+7)^{\frac{1}{2}}}{2}+c$
6. $\frac{1}{3}\left[(x+1)^{\frac{3}{2}}-(x-1)^{\frac{3}{2}}\right]+c$ 7. $b=\frac{13}{2},c=-2,f(x)=\frac{x^2}{2}+\frac{13}{2}x-2$
8. $c=-20, f(x)=2x^4-x^2-20$

Exercise: 2.2

1. $x^2+\frac{1}{2}\log|x|-2x+c$ 2. $\frac{x^4}{4}+\frac{x^3}{3}+2\log|x-1|+c$
3. $\frac{x^3}{3}-x^2+4x-8\log|x+2|+c$ 4. $\frac{x^3}{3}-x^2+3x-4\log|x+5|+c$
5. $11\log|x-3|-8\log|x-2|+c$ 6. $\log|x+1|+3\log|x-3|+\frac{2}{(x+1)}+c$
7. $\log|x^3-x^2+5x-5|+c$ 8. $c=\frac{\pi}{4}, f(x)=\log|x|+\frac{\pi}{4}$

Exercise: 2.3

1. $\dfrac{a^x}{\log a}+a^a x-\dfrac{x^{n+1}}{n+1}+c$
2. $\dfrac{1}{a^x\log a}-\dfrac{1}{b^x\log b}+c$
3. $e^x+e^{2x}+\dfrac{e^{3x}}{3}+c$
4. $\dfrac{e^{2x}}{2}+\dfrac{e^{-4x}}{4}+c$
5. $\dfrac{e^{4x}}{4}+c$
6. $e^{\left(x+\frac{1}{x}\right)}+c$
7. $-\dfrac{1}{\log x}+c$
8. $c=1, f(x)=e^x+1$

Exercise: 2.4

1. $2\sin x+3\cos x+4\tan x+5\cot x+c$
2. $-\dfrac{3}{4}\cos x+\dfrac{1}{12}\cos 3x+c$
3. $\tan x+c$
4. $\tan x-\cot x+c$
5. $-[\sin x+\cos x]+c$

Exercise: 2.5

1. $-e^{-x}(x+1)+c$
2. $e^{3x}\left[\dfrac{x^3}{3}-\dfrac{x^2}{3}+\dfrac{2x}{9}-\dfrac{2}{27}\right]+c$
3. $x(\log x-1)+c$
4. $\dfrac{x^2}{2}\left[\log x-\dfrac{1}{2}\right]+c$
5. $\dfrac{x^{n+1}}{n+1}\left(\log x-\dfrac{1}{n+1}\right)+c$
6. $e^{x^2}\left(x^4-2x^2+2\right)+c$

Exercise: 2.6

1. $\log\left|x^2+5x-7\right|+c$
2. $\dfrac{1}{4}\log\left|x^4+1\right|+c$
3. $\dfrac{1}{2}\log\left|e^{2x}-2\right|+c$
4. $\dfrac{(\log x)^4}{4}+c$
5. $2\sqrt{3x^2+7x-1}+c$
6. $\dfrac{4}{3}\left(x^2+x+1\right)^{\frac{3}{2}}+c$
7. $\dfrac{1}{54}\left(1+x^9\right)^6+c$
8. $\dfrac{1}{e}\log\left|x^e+e^x\right|+c$
9. $\log\left|\log x\right|+c$
10. $\dfrac{1}{10}\log\left|\dfrac{x^2-2}{2x^2+1}\right|+c$
11. $xe^x\left[\log\left(xe^x\right)-1\right]+c$
12. $\log|x|-\dfrac{1}{2}\log\left|x^2+1\right|+c$
13. $\dfrac{e^x}{x^2}+c$
14. $\dfrac{e^x}{(x+1)^2}+c$
15. $\dfrac{e^{3x}}{9x}+c$

Exercise 2.7

1. $\dfrac{1}{24}\log\left|\dfrac{3+4x}{3-4x}\right|+c$
2. $\dfrac{1}{10}\log\left|\dfrac{9+x}{1-x}\right|+c$
3. $\dfrac{1}{6\sqrt{2}}\log\left|\dfrac{\sqrt{2}x-3}{\sqrt{2}x+3}\right|+c$
4. $\dfrac{1}{3}\log\left|\dfrac{x-2}{x+1}\right|+c$
5. $\log\left|\dfrac{x+1}{x+2}\right|+c$
6. $\dfrac{1}{10}\log\left|\dfrac{x-1}{x+4}\right|+c$
7. $\dfrac{1}{6}\log\left|\dfrac{e^x-3}{e^x+3}\right|+c$
8. $\dfrac{1}{3}\log\left|3x+\sqrt{9x^2-7}\right|+c$
9. $\log\left|(x+3)+\sqrt{x^2+6x+13}\right|+c$
10. $\log\left|\left(x-\dfrac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+c$
11. $\dfrac{1}{4}\log\left|x^4+\sqrt{x^8-1}\right|+c$

12. $\dfrac{\left(x+\frac{1}{2}\right)}{2}\sqrt{1+x+x^2}+\frac{3}{8}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{1+x+x^2}\right|+c$

13. $\frac{x}{2}\sqrt{x^2-2}-\log\left|x+\sqrt{x^2-2}\right|+c$ 14. $\frac{1}{4}\left[2x\sqrt{4x^2-5}-5\log\left|2x+\sqrt{4x^2-5}\right|\right]+c$

15. $\left(\frac{x+1}{2}\right)\sqrt{2x^2+4x+1}-\frac{\sqrt{2}}{4}\log\left|\sqrt{2}(x+1)+\sqrt{2x^2+4x+1}\right|+c$

16. $\frac{x^2}{2}-\frac{x}{2}\sqrt{x^2-1}+\frac{1}{2}\log\left|x+\sqrt{x^2-1}\right|+c$

Exercise 2.8

I:1. $\frac{1}{2}\left[e^2-1\right]$ 2. $\frac{1}{6}$ 3. $\frac{1}{2}\log\left[\frac{5}{2}\right]$ 4. $\log\left[\frac{1+e^3}{2}\right]$

5. $\frac{1}{2}[e-1]$ 6. $\frac{3}{8}$ 7. $\log\left[\frac{11}{5}\right]$ 8. 2 9. $\frac{1}{2}[2\log 2-1]$

II:1. 37 2. 0 3. 1 4. $c=4$

Exercise 2.9

1. 0 2. $\frac{\pi}{2}$ 3. 0 4. $\frac{\pi}{4}$ 5. 0 6. $\frac{16}{5}$

Exercise 2.10

1.(i) 6 (ii) $\frac{105\sqrt{\pi}}{16}$ (iii) $\frac{6!}{m^7}$ (iv) $\frac{3}{128}$ (v) $\left(2^6\right)5!$ 2. $\frac{1}{4}$

Exercise 2.11

1. $\frac{9}{2}$ 2. $\frac{3}{2}$ 3. 14 4. $\frac{1}{3}$

Exercise 2.12

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(b)	(c)	(a)	(a)	(a)	(b)	(b)	(a)	(d)	(c)	(b)	(b)	(b)	(b)	(c)
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
(d)	(c)	(c)	(b)	(a)	(b)	(b)	(c)	(a)	(a)	(a)	(b)	(d)	(b)	(c)

Miscellaneous problems

1. $\frac{2}{15}\left[(x+2)^{\frac{3}{2}}+(x+3)^{\frac{3}{2}}\right]+c$ 2. $\frac{1}{5}\log\left|\frac{2+x}{1-2x}\right|+c$ 3. $\frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right|+c$

4. $\frac{x}{2}\sqrt{2x^2-3}-\frac{3\sqrt{2}}{4}\log\left|\sqrt{2}x+\sqrt{2x^2-3}\right|+c$

5. $\frac{(3x+2)}{6}\sqrt{9x^2+12x+3}-\frac{1}{6}\log\left|(3x+2)+\sqrt{9x^2+12x+3}\right|+c$

6. $\frac{1}{3}\left[(x+1)^3\log x-\frac{x^3}{3}-\frac{3x^2}{2}-3x-\log|x|\right]+c$

7. $x\log\left(x-\sqrt{x^2-1}\right)+\sqrt{x^2-1}+c$ 8. 0   9. $\frac{1}{4}\left[\frac{e^4-5}{e^2}\right]$ 10. $\frac{14}{15}$

3. Integral Calculus – II

Exercise 3.1

1. 5 sq.units   2. 2 sq.units   3. $\frac{8a^2}{3}$ *sq.units*   4. $\frac{3}{2}$ *sq.units*

5. $\frac{17}{2}$ *sq.units*   6. $\frac{8}{3}$ *sq.units*   7. $\frac{8\sqrt{2}}{3}$ *sq.units*   8. $\frac{1}{3}$ *sq.units*

Exercise 3.2

1. ₹28,000   2. $y=\left(\frac{2x-1}{3x+2}\right)$ 3. $P=8-2x,\ R=8x-2x^2$

4. ₹4,419   5. ₹5,680

Exercise 3.3

1. Total cost $=100x-5x^2+\frac{0.1x^3}{3}+500$, Average Cost$==100x-5x+\frac{x^2}{30}+\frac{500}{x}$

2. Total Cost $=\frac{1500}{7}x^{\frac{7}{5}}$, Average Cost$=\frac{1500}{7}x^{\frac{2}{5}}$

3. Cost function $C=2\sqrt{ax+b}-2\sqrt{b}$ 4. ₹14,133.33

5. Total Revenue=₹5.95   6. Demand function $P=9-\frac{4x^2}{3}$

8. Demand function $P=20e^{-\frac{x}{10}}$ 9. Profit function $=13x-0.065x^2-120$

10. Revenue function $R=1500x-2x^2-x^3$, Average revenue function $P=1500-2x-x^2$

11. Revenue function $R=10x+\frac{3x^2}{2}-\frac{x^3}{3}$, Demand function $P=10+\frac{3x}{2}-\frac{x^2}{3}$

12. Total Cost $C=4000\sqrt{7x+4}+10000$, Average Cost $A.C=\frac{4000}{x}\sqrt{7x+4}+\frac{10000}{x}$

13. $C=\frac{x^2}{4}+5000$ 14. Revenue function $R=20x-\frac{5x^2}{2}+x^3$

15. Demand function $P=14-3x+3x^2$

Exercise 3.4

1. cs= 400 units
2. C.S=378 units
3. C.S=562.50 units
4. C.S=$\frac{1}{2}\left[1\text{-}\log_e 2\right]$ units
5. P.S=$\frac{25}{2}$units.
6. P.S=237.3 units
7. $C.S = 36\log\frac{3}{2} - 12\,units$
8. $=\frac{32000}{3}$units
9. $C.S = (8\log 2 - 4)$units, P.S=$\frac{1}{4}$units
10. C.S=$\frac{1024}{3}$units, P.S=64 units
11. C.S=24 units, P.S=16 units

Exercise 3.5

1	2	3	4	5	6	7	8	9	10	11	12	13
(c)	(b)	(a)	(c)	(a)	(a)	(d)	(c)	(b)	(a)	(a)	(a)	(b)
14	15	16	17	18	19	20	21	22	23	24	25	
(c)	(c)	(b)	(a)	(b)	(a)	(c)	(a)	(c)	(a)	(b)	(c)	

Miscellaneous problems

1. ₹1,900
2. C= ₹3,125
4. $R = 6x - x^3 - \frac{x^4}{4}$, $p = 6 - x^2 - \frac{x^3}{4}$
5. Profit function is $10x - \frac{x^2}{40} - 100$.
6. $C.S = \frac{40}{9}$ units, $P.S = \frac{32}{9}$units.
7. 52,770 units
8. $P = 11 - \frac{x^3}{3}$
9. $\frac{76}{3}$ sq.units
10. $\frac{1}{5}\left[(2)^{\frac{5}{3}} - 1\right]$ sq.units

4. Differential Equations

Exercise 4.1

1.(i) (1 , 1) (ii) (3 , 1) (iii) (2 , 2) (iv) (3 , 1)

(v) (3 , 3) (vi) (2 , 1) (vii) (1 , 4).

2.(i) $y = x\frac{dy}{dx} + \frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$ (ii) $\left(\frac{dy}{dx}\right)^3 - 4xy\frac{dy}{dx} + 8y^2 = 0$ (iii) $y + x\frac{dy}{dx} = 0$

(iv) $x + y\frac{dy}{dx} = 0$

3. $r^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$

4. $y = x\frac{dy}{dx}$

5. $2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$

6. $y^2 = x^2 + 2xy\frac{dy}{dx}$

7. $y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2$

Exercise 4.2

1.(i) $e^{-y} + ax + C = 0$ (ii) $\log x + \frac{x^2}{2} = \frac{y^2}{2} + \frac{y^3}{3} + C$

2. $\log x - x = \log y + C$

3.(i) $x = Cy$ (ii) $\log(1+y) = -e^x + C$

4. $(1 + \sin x) = C(1 + \cos y)$

5. $(x-1)(y+1)=C$ 6.(i) $\log y=\frac{-\cos 2x}{2}+C$ (ii) $\frac{e^{ax}}{a}=\frac{-e^{by}}{b}+C$

7. $(y-b)^2=(x-a)^2+b^2-a^2$

Exercise 4.3

1. $x=Ce^{\frac{y}{x}}$ 2. $x+y=Ke^{\frac{-2x}{x+y}}$ 3. $y+\sqrt{x^2+y^2}=x^2C$

4. $3y^2-4yx+3x^2=x^3C$ 5. $\left(xy-y^2\right)x=C$ 6. $y\sqrt{y^2-x^2}=2\sqrt{3}\,x^5$

7. $y=ce^{\frac{x^2}{2y^2}}$

Exercise 4.4

1. $\frac{y}{x}=x+C$ 2. $ye^{\sin x}=e^{\sin x}(\sin x-1)+C$ 3. $x^2y=\frac{x^6}{6}+C$

4. $y\left(1+x^3\right)=x+\frac{x^3}{3}+C$ 5. $xy=e^x\left(x^2-2x+2\right)+C$ 6. $y\sec x=\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)+C$

7. $y\sec^2 x=\sec x-2$ 8. $x^2e^x-2xe^x+2e^x+c$ 9. ₹ 2,22,550

Exercise 4.5

1. $y=Ae^{2x}+Be^{4x}$ 2. $y=(Ax+B)e^{2x}$ 3. $y=e^{-x}(A\cos\sqrt{2}\,x+B\sin\sqrt{2}\,x)$

4. $y=(Ax+B)e^{kx}$ 5. $y=\frac{e^{-3x}}{12}+\frac{e^{5x}}{20}$ 6. $y=Ae^{\frac{1}{2}x}+Be^{\frac{3}{2}x}+\frac{e^{2x}}{4}$

7. $y=A\cos 4x+B\sin 4x$ 8. $y=3e^x-\frac{7}{2}e^{2x}+\frac{e^{3x}}{2}$

9. $y=Ae^{-3x}+Be^{2x}+\frac{e^{3x}}{6}-\frac{x}{5}e^{-3x}$ 10. $y=(Ax+B)e^{5x}+2xe^{5x}+\frac{1}{5}$

11. $y=Ae^{\frac{-3}{2}x}+Be^{\frac{-5}{2}x}+4xe^{-\frac{3}{2}x}$ 12. $y=Ae^{2x}+Be^{\frac{-7}{3}x}+xe^{2x}$

13. $P=Ae^{-4t}+Be^{2t}+2$

Exercise 4.6

1	2	3	4	5	6	7	8	9	10	11	12	13
(a)	(d)	(a)	(b)	(a)	(a)	(d)	(c)	(a)	(c)	(b)	(a)	(b)
14	15	16	17	18	19	20	21	22	23	24	25	
(c)	(d)	(a)	(d)	(a)	(b)	(d)	(a)	(a)	(d)	(c)	(a)	

Miscellaneous Problems

1. $p=Ae^{-4t}+Be^{2t}+3$ 2. $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$ 3. $e^x\left(x^2-2x+2\right)+\log y=c$

4. $x\left(1+\frac{3y^2}{x^2}\right)^{\frac{1}{3}}=c$ 5. $yx^2=\frac{x^6}{6}+c$ 6. $cm^2=2(m+6)$

7. $6y=\left(e^2+e\right)e^x-\left(e^2+e+1\right)e^{2x}+e^{4x}$

8. $ye^{\sin x}=2e^{\sin x}+c$

9. $\log y=\dfrac{x^3}{3y^2}+c$

10. $\log\left|1+y\right|=x+\dfrac{x^2}{2}+c$

5. Numerical Methods

Exercise 5.1

1. $\log\left(1+\dfrac{h}{ax}\right)$

2

x	y	Δy	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\Delta^5 y$
0	-1					
		1				
1	0		3			
		4		8		
					-3	
2	5		11			-4
		15		5		
3	20		16		1	
		31		6		
4	51		22			
5	104					

5. $\dfrac{-2}{(x+1)(x+2)(x+3)}$

6. 31

7. 445 lakhs

8. 3 and 24

Exercise 5.2

1. 6.8
2. ₹2,900
3. $f\left(x\right)=2x^3-7x^2+6x+1$
4. 36.784 (lakhs)
5. 197
6. 15.45
7. 286.96
8. 22.0948
9. 100 (Thousand tones)
10. 42persons
11. 444.7Lakhs
12. 53

Exercise 5.3

1	2	3	4	5	6	7	8	9	10	11	12	13	14
(a)	(c)	(a)	(c)	(d)	(a)	(c)	(c)	(c)	(c)	(a)	(c)	(b)	(b)

Miscellaneous Problems

3. $f\left(x\right)=x^2-3x+1$

4. 14.25, 23.5

5. 128.5

6. 189.79, 286.96

7. 5281, 6504

9. $y=\dfrac{2}{3}x^4-8x^3+\dfrac{100}{3}x^2-56x+31$

10. $y=x^3-8x^2+19x-12$

Books for Reference

1. Introduction to Matrices, S.P.Gupta , S.Chand & Company
2. Matrices, Shanthi Narayanan,S.Chand & Company
3. Matrices and Determinants,P.N.Arora, S.Chand & Company
4. Stochastic Processes, J. Medhi, New age international publishers
5. A Text Book on differential Calculus – S. K. Goyal -Jai Prakash Nath Publications.
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Business Mathematics and Statistics - Higher Secondary Second Year
Text Book Development Team

Domain Expert

- **Mr.N. RAMESH,**
  Associate Professor (Retd),
  Department of Mathematics,
  Government Arts College(Men), Nandanam,
  Chennai-600 035

Reviewers

- **Dr. M.R. Sreenivasan**
  Professor & HOD,
  Department of Statistics
  University of Madras, Chennai-600 005.
- **Dr. D. Arivudainambi**
  Professor in Mathematics
  Department of Mathematics,
  Anna University, Chennai-600 025

Content Experts

- **Dr. VENU PRAKASH**
  Associate Professor & HOD,
  Department of Statistics
  Presidency College, Chennai-600 005.
- **Dr. R. THIRUMALAISAMY**
  Associate Professor & HOD,
  Department of Mathematics,
  Government Arts College(Men), Nandanam,
  Chennai-600 035
- **Dr. S. J. VENKATESAN**
  Associate Professor,
  Department of Mathematics,
  Government Arts College(Men), Nandanam,
  Chennai-600 035
- **Mrs. K. GOKILA**
  Assistant Professor & HOD
  Dr.Ambedkar Government Arts College
  Vyasarpadi, Chennai
- **Dr. N. SUNDARAM**
  Assistant Professor
  Department of Statistics
  Presidency College, Chennai-600 005.
- **Dr. L.PARIDAYAL**
  Assistant Professor
  Department of Statistics
  MCC, Tambaram,Chennai
- **Mrs. M. THILAGAM**
  Assistant Professor
  Department of Statistics
  Presidency College, Chennai-600 005.

Text Book Group In-Charge

- **Mr.A.RAVI KUMAR**
  Principal, DIET
  Mayanoor, Karur Dist.

Academic Co-ordinator

- **Mr. S.BABU**
  Assistant Professor
  State Council of Educational Research and
  Training, Chennai-600 006

Authors

- **Mr.R.VILLAVANKOTHAI**
  Head Master, GHSS, Korukkai, Nagapattinam District.
- **Mr. T.P. SWAMINATHAN**
  Post Graduate Teacher
  MMA Govt. Hr. Sec. School, Pallavaram, Chennai-43
- **Mr. H. VENKATESH,**
  Post Graduate Teacher,
  Sir Ramaswami Mudaliar Hr. Sec. School,
  Ambattur, Chennai 600 053.
- **Mrs. C.BAGAVATHY**
  Post Graduate Teacher
  Ahobila Math Oriental HSS., West Mambalam,
  Chennai - 600 033.
- **Mr. V.KALAI SELVAN**
  Post Graduate Teacher
  GHSS,Tiruninravur, Thiruvallur Dist
- **Mr. S.F. SULAIMAN**
  Post Graduate Teacher
  MMA Govt. Hr. Sec. School, Pallavaram, Chennai-43
- **Mr. T. RAJA SEKHAR**
  Post Graduate Teacher
  Govt. Boys Hr. Sec. School, Chrompet, Chennai-44
- **Mrs. A.SUGANYA**
  Post Graduate Teacher
  GHSS, Kovilambakkam, Kancheepuram District
- **Mr. V.GANESAN**
  Post Graduate Teacher
  Got. Boys Hr. Sec. School, Nanganallur, Chennai-114
- **Mr. M.K.THRILOGACHANDRAN**
  Post Graduate Teacher, GHSS, Meyyur, Thiruvallur Dist.

ICT Co-ordinator

- **Mr. D. VASURAJ**
  PGT - Maths Teacher & HOD
  KRM Public School, Chennai- 600 011

QR Management Team

- **J.F. Paul Edwin Roy, B.T Asst.**
  Pums -Rakkipatty, Salem.
- **S. Albert Valavan Babu, B.T Asst.**
  G.h.s, Perumal Kovil, Paramakudi, Ramanathapuram.
- **A. Devi Jesintha, B.T Asst.**
  G.H.S, N.M.kovil, Vellore.

Layout Designer

- **JOY GRAPHICS,**
  Chindatripet, Chennai-600 002.

In House QC

- **T. Rajesh**
  C. Jerald Wilson
  P. Prasanth
  P. Arun Kamaraj

Typing

- **K. Nagavelu.**

This book has been printed on 80 GSM
Elegant Maplitho paper
Printed by offset at :

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