BUSINESS MATHEMATICS

HIGHER SECONDARY - SECOND YEAR

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Preface

'The most distinct and beautiful statement of any truth must atlast take the Mathematical form' -Thoreau.

Among the Nobel Laureates in Economics more than 60% were Economists who have done pioneering work in Mathematical Economics.These Economists not only learnt Higher Mathematics with perfection but also applied it successfully in their higher pursuits of both Macroeconomics and Econometrics.

A Mathematical formula (involving stochastic differential equations) was discovered in 1970 by Stanford University Professor of Finance Dr.Scholes and Economist Dr.Merton.This achievement led to their winning Nobel Prize for Economics in 1997.This formula takes four input variables-duration of the option,prices,interest rates and market volatility-and produces a price that should be charged for the option. Not only did the formula work ,it transformed American Stock Market.

Economics was considered as a deductive science using verbal logic grounded on a few basic axioms.But today the transformation of Economics is complete. Extensive use of graphs,equations and Statistics replaced the verbal deductive method.Mathematics is used in Economics by beginning with a few variables,gradually introducing other variables and then deriving the inter relations and the internal logic of an economic model.Thus Economic knowledge can be discovered and extended by means of mathematical formulations.

Modern Risk Management including Insurance,Stock Trading and Investment depend on Mathematics and it is a fact that one can use Mathematics advantageously to predict the future with more precision!Not with 100% accuracy, of course.But well enough so that one can make a wise decision as to where to invest money.The idea of using Mathematics to predict the future goes back to two 17th Century French Mathematicians Pascal and Fermat.They worked out probabilities of the various outcomes in a game where two dice are thrown a fixed number of times.

In view of the increasing complexity of modern economic problems,the need to learn and explore the possibilities of the new methods is becoming ever more pressing. If methods based on Mathematics and Statistics are used suitably according to the needs of Social Sciences they can prove to be compact, consistent and powerful tools especially in the fields of Economics, Commerce and Industry. Further these methods not only guarantee a deeper insight into the subject but also lead us towards exact and analytical solutions to problems treated.

This text book has been designed in conformity with the revised syllabus of Business Mathematics(XII) (to come into force from 2005 - 2006)-http:/www.tn.gov. in/schoolsyllabus/. Each topic is developed systematically rigorously treated from

first principles and many worked out examples are provided at every stage to enable the students grasp the concepts and terminology and equip themselves to encounter problems. Questions compiled in the Exercises will provide students sufficient practice and self confidence.

Students are advised to read and simultaneously adopt pen and paper for carrying out actual mathematical calculations step by step. As the Statistics component of this Text Book involves problems based on numerical calculations,Business Mathematics students are advised to use calculators.Those students who succeed in solving the problems on their own efforts will surely find a phenomenal increase in their knowledge, understanding capacity and problem solving ability. They will find it effortless to reproduce the solutions in the Public Examination.

We thank the Almighty God for blessing our endeavour and we do hope that the academic community will find this textbook triggering their interests on the subject!

"The direct application of Mathematical reasoning to the discovery of economic truth has recently rendered great services in the hands of master Mathematicians"

– Alfred Marshall.

| **Malini** | **Amali Raja** | **Raman** | **Padmanabhan** | **Ramachandran** |
| **Prakash** | **Murthy** | **Ramesh** | **Srinivasan** | **Antony Raj** |

CONTENTS

Calculator should be used for solving problems in chapters 7- 10

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APPLICATIONS OF MATRICES AND DETERMINANTS

1

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The concept of matrices and determinants has extensive applications in many fields such as Economics, Commerce and Industry. In this chapter we shall develop some new techniques based on matrices and determinants and discuss their applications.

1.1 INVERSE OF A MATRIX

1.1.1 Minors and Cofactors of the elements of a determinant.

The minor of an element a_{ij} of a determinant A is denoted by M_{ij} and is the determinant obtained from A by deleting the row and the column where a_{ij} occurs.

The cofactor of an element a_{ij} with minor M_{ij} is denoted by C_{ij} and is defined as

$$C_{ij} = \begin{cases} M_{ij}, & \text{if } i+j \text{ is even} \\ -M_{ij}, & \text{if } i+j \text{ is odd} \end{cases}$$

Thus, cofactors are signed minors.

In the case of $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ we have

$$M_{11} = a_{22}, \qquad M_{12} = a_{21}, \qquad M_{21} = a_{12}, \qquad M_{22} = a_{11}$$

Also $\quad C_{11} = a_{22}, \qquad C_{12} = -a_{21}, \qquad C_{21} = -a_{12}, \qquad C_{22} = a_{11}$

In the case of $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, we have

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \qquad C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix};$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \qquad C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix};$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \qquad C_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix};$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \qquad C_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \text{ and so on.}$$

1.1.2 Adjoint of a square matrix

The transpose of the matrix got by replacing all the elements of a square matrix A by their corresponding cofactors in | A | is called the **Adjoint** of A or **Adjugate** of A and is denoted by Adj A.

Thus, $\text{Adj } A = A^t_c$

Note

(i) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A_c = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

$\text{Adj } A = A^t_c = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Thus the Adjoint of a 2 x 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

can be written instantly as $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

(ii) Adj I = I, where I is the unit matrix.

(iii) $A \,(\text{Adj } A) = (\text{Adj } A)\, A = | A | \, I$

(iv) $\text{Adj } (AB) = (\text{Adj } B)\,(\text{Adj } A)$

(v) If A is a square matrix of order 2, then $|\text{Adj } A| = |A|$

If A is a square matrix of order 3 , then $|\text{Adj } A| = |A|^2$

Example 1

Write the Adjoint of the matrix A = $\begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix}$

Solution :

$Adj\ A = \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix}$

Example 2

Find the Adjoint of the matrix A = $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$

Solution :

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}, \quad \text{Adj } A = A_c^t$$

Now, $C_{11} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1, \quad C_{12} = - \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8, \quad C_{13} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5,$

$C_{21} = - \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{22} = \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -6, \quad C_{23} = - \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 3,$

$C_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1, \quad C_{32} = - \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2, \quad C_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1,$

$$\therefore A_c = \begin{pmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{pmatrix}$$

Hence

$$\text{Adj } A = \begin{pmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{pmatrix}^t = \begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix}$$

1.1.3 Inverse of a non singular matrix

The inverse of a non singular matrix A is the matrix B such that AB = BA = I. B is then called the inverse of A and denoted by A^{-1}.

Note

(i)   A non square matrix has no inverse.

(ii)   The inverse of a square matrix A exists only when $|A| \neq 0$ that is, if A is a singular matrix then A^{-1} does not exist.

(iii)   If B is the inverse of A then A is the inverse of B. That is $B = A^{-1} \Rightarrow A = B^{-1}$.

(iv) $A A^{-1} = I = A^{-1} A$

(v)   The inverse of a matrix, if it exists, is unique. That is, no matrix can have more than one inverse.

(vi)   The order of the matrix A^{-1} will be the same as that of A.

(vii) $I^{-1} = I$

(viii) $(AB)^{-1} = B^{-1} A^{-1}$, provided the inverses exist.

(ix) $A^2 = I$ implies $A^{-1} = A$

(x)   If AB = C then

　　　(a) A = CB^{-1} (b) B = A^{-1}C, provided the inverses exist.

(xi)   We have seen that

　　　A(Adj A) = (Adj A) A = |A| I

　　　$\therefore A\dfrac{1}{|A|}(\text{Adj A}) = \dfrac{1}{|A|}(\text{Adj A})A = I$　　　(|A| ≠ 0)

This suggests that

　　　$A^{-1} = \dfrac{1}{|A|}(\text{Adj A}).$ That is, $A^{-1} = \dfrac{1}{|A|}A^{t}{}_{c}$

(xii) $(A^{-1})^{-1} = A$, provided the inverse exists.

　　　Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $|A| = ad - bc \neq 0$

　　　Now $A_c = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$, $A^{t}{}_{c} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

　　　$A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

　　　Thus the inverse of a 2 × 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be written instantly as

$\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ provided $ad - bc \neq 0$

Example 3

Find the inverse of A $= \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}$, if it exists.

Solution :

　　　$|A| = \begin{vmatrix} 5 & 3 \\ 4 & 2 \end{vmatrix} = -2 \neq 0 \therefore A^{-1}$ exists.

　　　$A^{-1} = \dfrac{1}{-2}\begin{pmatrix} 2 & -3 \\ -4 & 5 \end{pmatrix} = -\dfrac{1}{2}\begin{pmatrix} 2 & -3 \\ -4 & 5 \end{pmatrix}$

Example 4

Show that the inverses of the following do not exist:

(i) $A = \begin{pmatrix} -2 & 6 \\ 3 & -9 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 3 & 1 & -2 \\ 2 & 7 & 3 \\ 6 & 2 & -4 \end{pmatrix}$

Solution :

(i) $|A| = \begin{vmatrix} -2 & 6 \\ 3 & -9 \end{vmatrix} = 0$ $\therefore A^{-1}$ does not exist.

(ii) $|A| = \begin{vmatrix} 3 & 1 & -2 \\ 2 & 7 & 3 \\ 6 & 2 & -4 \end{vmatrix} = 0$ $\therefore A^{-1}$ does not exist.

Example 5

Find the inverse of $A = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$, if it exists.

Solution :

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 15 \neq 0 \quad \therefore A^{-1} \text{ exists.}$$

We have, $A^{-1} = \dfrac{1}{|A|} A^t_c$

Now, the cofactors are

$$C_{11} = \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -5, \quad C_{12} = -\begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} = 7, \quad C_{13} = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1,$$

$$C_{21} = -\begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} = 10, \quad C_{22} = \begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} = -8, \quad C_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1,$$

$$C_{31} = \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -5, \quad C_{32} = -\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} = 10, \quad C_{33} = \begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} = -5,$$

Hence

$$A_c = \begin{pmatrix} -5 & 7 & 1 \\ 10 & -8 & 1 \\ -5 & 10 & -5 \end{pmatrix}, \quad A^t_c = \begin{pmatrix} -5 & 10 & -5 \\ 7 & -8 & 10 \\ 1 & 1 & -5 \end{pmatrix} \quad \therefore A^{-1} = \frac{1}{15}\begin{pmatrix} -5 & 10 & -5 \\ 7 & -8 & 10 \\ 1 & 1 & -5 \end{pmatrix}$$

Example 6

Show that A = $\begin{pmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{pmatrix}$ **and B =** $\begin{pmatrix} \dfrac{1}{17} & \dfrac{5}{17} & \dfrac{1}{17} \\ \dfrac{8}{17} & \dfrac{6}{17} & -\dfrac{9}{17} \\ \dfrac{10}{17} & -\dfrac{1}{17} & -\dfrac{7}{17} \end{pmatrix}$ **are inverse of each other.**

Solution :

$$AB = \begin{pmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{pmatrix} \begin{pmatrix} \dfrac{1}{17} & \dfrac{5}{17} & \dfrac{1}{17} \\ \dfrac{8}{17} & \dfrac{6}{17} & -\dfrac{9}{17} \\ \dfrac{10}{17} & -\dfrac{1}{17} & -\dfrac{7}{17} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{pmatrix} \dfrac{1}{17} \begin{pmatrix} 1 & 5 & 1 \\ 8 & 6 & -9 \\ 10 & -1 & -7 \end{pmatrix} = \dfrac{1}{17} \begin{pmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Since A and B are square matrices and AB = I, A and B are inverse of each other.

EXERCISE 1.1

1. Find the Adjoint of the matrix $\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}$.

2. Find the Adjoint of the matrix $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$.

3. Show that the Adjoint of the matrix A = $\begin{pmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{pmatrix}$ is A itself.

4. If A = $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$, verify that A (Adj A) = (Adj A) A = | A | I.

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5. Given $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$, verify that Adj (AB) = (Adj B) (Adj A).

6. In the second order matrix $A = (a_{ij})$, given that $a_{ij} = i + j$, write out the matrix A and verify that $|\text{Adj } A| = |A|$.

7. Given $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & -1 \end{pmatrix}$, verify that $|\text{ Adj } A| = |A|^2$.

8. Write the inverse of $A = \begin{pmatrix} 2 & 4 \\ -3 & 2 \end{pmatrix}$.

9. Find the inverse of $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$.

10. Find the inverse of $A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ and verify that $AA^{-1} = I$.

11. If $A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ and none of the a's are zero, find A^{-1}.

12. If $A = \begin{pmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{pmatrix}$, show that the inverse of A is itself.

13. If $A^{-1} = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$, find A.

14. Show that $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} \dfrac{1}{18} & -\dfrac{5}{18} & \dfrac{7}{18} \\ \dfrac{7}{18} & \dfrac{1}{18} & -\dfrac{5}{18} \\ -\dfrac{5}{18} & \dfrac{7}{18} & \dfrac{1}{18} \end{pmatrix}$ are inverse of each other.

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15. If $A = \begin{pmatrix} 2 & -3 \\ -4 & 8 \end{pmatrix}$, compute A^{-1} and show that $4A^{-1} = 10\,I - A$.

16. If $A = \begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix}$ verify that $(A^{-1})^{-1} = A$.

17. Verify $(AB)^{-1} = B^{-1}A^{-1}$, when $A = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -6 & 0 \\ 0 & 9 \end{pmatrix}$.

18. Find λ if the matrix $\begin{pmatrix} 6 & 7 & -1 \\ 3 & \lambda & 5 \\ 9 & 11 & \lambda \end{pmatrix}$ has no inverse.

19. If $X = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & p & q \end{pmatrix}$ find p, q such that $Y = X^{-1}$.

20. If $\begin{pmatrix} 4 & -3 \\ 5 & 2 \end{pmatrix} X = \begin{pmatrix} 14 \\ 29 \end{pmatrix}$, find the matrix X.

1.2 SYSTEMS OF LINEAR EQUATIONS

1.2.1 Submatrices and minors of a matrix

Matrices obtained from a given matrix A by omitting some of its rows and columns are called sub matrices of A.

e.g. If $A = \begin{pmatrix} 3 & 2 & 4 & 1 & 5 \\ 2 & 0 & 1 & -1 & 4 \\ 2 & 1 & 0 & 4 & 2 \\ 3 & 1 & 4 & 1 & 2 \end{pmatrix}$, some of the submatrices of A are:

$$\begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 4 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 2 \\ 1 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & -1 \\ 4 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 2 & 4 & 5 \\ 2 & 0 & 1 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

The determinants of the square submatrices are called minors of the matrix.

Some of the minors of A are :

$$\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 3 & 5 \\ 3 & 2 \end{vmatrix}, \begin{vmatrix} 3 & 4 & 1 \\ 2 & 1 & -1 \\ 2 & 0 & 4 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & -1 & 4 \\ 0 & 4 & 2 \\ 4 & 1 & 2 \end{vmatrix}$$

1.2.2 Rank of a matrix

A positive integer 'r' is said to be the **rank** of a non zero matrix A, denoted by $\rho(A)$, if

(i)   there is at least one minor of A of order 'r' which is not zero and

(ii)   every minor of A of order greater than 'r' is zero.

Note

(i)   The rank of a matrix A is the order of the largest non zero minor of A.

(ii)   If A is a matrix of order $m \times n$ then $\rho(A) \le$ minimum (m, n)

(iii)   The rank of a zero matrix is taken to be 0.

(iv)   For non zero matrices, the least value of the rank is 1.

(v)   The rank of a non singular matrix of order $n \times n$ is n.

(vi) $\rho(A) = \rho(A^t)$

(vii) $\rho(I_2) = 2, \rho(I_3) = 3$

Example 7

Find the rank of the matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 2 \\ 0 & 1 & 5 \end{pmatrix}$

Solution :

Order of A is 3×3. $\therefore \rho(A) \le 3$

Consider the only third order minor

$$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 0 & 2 \\ 0 & 1 & 5 \end{vmatrix} = -2 \ne 0.$$

There is a minor of order 3 which is not zero. $\therefore \rho(A) = 3$.

Example 8

Find the rank of the matrix $A = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix}$

Solution :

Order of A is 3×3. ∴ $\rho(A) \le 3$

Consider the only third order minor

$$\begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

The only minor of order 3 is zero. ∴ $\rho(A) \le 2$

Consider the second order minors.

We find, $\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} = 3 \ne 0$

There is a minor of order 2 which is non zero. ∴ $\rho(A) = 2$

Example 9

Find the rank of the matrix $A = \begin{pmatrix} 2 & 4 & 5 \\ 4 & 8 & 10 \\ -6 & -12 & -15 \end{pmatrix}$.

Solution :

Order of A is 3×3. ∴ $\rho(A) \le 3$

Consider the only third order minor

$$\begin{vmatrix} 2 & 4 & 5 \\ 4 & 8 & 10 \\ -6 & -12 & -15 \end{vmatrix} = 0 \qquad\qquad (R1 \propto R2)$$

The only minor of order 3 is zero. ∴ $\rho(A) \le 2$

Consider the second order minors. Obviously they are all zero.

∴ $\rho(A) \le 1$ Since A is a non zero matrix. $\rho(A) = 1$

Example 10

Find the rank of the matrix $A = \begin{pmatrix} 1 & -3 & 4 & 7 \\ 9 & 1 & 2 & 0 \end{pmatrix}$.

Solution :

Order of A is 2×4. $\quad \therefore \rho(A) \leq 2$

Consider the second order minors.

We find,

$\begin{vmatrix} 1 & -3 \\ 9 & 1 \end{vmatrix} = 28 \neq 0$

There is a minor of order 2 which is not zero.

$\therefore \rho(A) = 2$

Example 11

Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & -4 & 5 \\ 2 & -1 & 3 & 6 \\ 8 & 1 & 9 & 7 \end{pmatrix}$

Solution :

Order of A is 3×4. $\therefore \rho(A) \leq 3$

Consider the third order minors.

We find,

$\begin{vmatrix} 1 & 2 & -4 \\ 2 & -1 & 3 \\ 8 & 1 & 9 \end{vmatrix} = -40 \neq 0$

There is a minor of order 3 which is not zero. $\therefore \rho(A) = 3$.

1.2.3 Elementary operations and equivalent matrices.

The process of finding the values of a number of minors in our endeavour to find the rank of a matrix becomes laborious unless by a stroke of luck we get a non zero minor at an early stage. To get over this difficulty, we introduce many zeros in the matrix by what are called **elementary operations** so that the evaluation of the minors is rendered easier. It can be proved that the elementary operations do not alter the rank of a matrix.

The following are the elementary operations :

 (i) *The interchange of two rows.*

(ii) *The multiplication of a row by a non zero number.*

(iii) *The addition of a multiple of one row to another row.*

If a matrix B is obtained from a matrix A by a finite number of elementary operations then we say that the matrices A and B are **equivalent matrices** and we write A ~ B.

Also, while introducing many zeros in the given matrix, it would be desirable (but not necessary) to reduce it to a **triangular form**.

A matrix $A = (a_{ij})$ is said to be in a triangular form if $a_{ij} = 0$ whenever $i > j$.

e.g. The matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 7 & 3 & 0 \\ 0 & 0 & 2 & 9 \end{pmatrix}$ is in a triangular form.

Example 12

Find the rank of the matrix $A = \begin{pmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix}$

Solution :

Order of A is 3×4. $\therefore \rho(A) \leq 3$

Let us reduce the matrix A to a triangular form.

$$A = \begin{pmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix}$$

Applying $R_1 \leftrightarrow R_3$

$$A \sim \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{pmatrix}$$

Applying $R_3 \to R_3 - 5R_1$

$$A \sim \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{pmatrix}$$

Applying $R_3 \to R_3 - 8R_2$

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$$A \sim \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{pmatrix}$$

This is now in a triangular form

We find,

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{vmatrix} = -12 \neq 0$$

There is a minor of order 3 which is not zero. $\therefore \rho(A) = 3$

Example 13

Find the rank of the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \end{pmatrix}$

Solution :

Order of A is 3×4. $\therefore \rho(A) \leq 3$

Let us reduce the matrix A to a triangular form.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \end{pmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - 2R_1$

$$A \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & -2 & -5 & 0 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 + R_2$

13

$$A \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & -8 & 0 \end{pmatrix}$$

This is now in a triangular form

We find,

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & -8 \end{vmatrix} = -16 \neq 0$$

There is a minor of order 3 which is not zero. $\therefore \rho(A) = 3$

Example 14

Find the rank of the matrix $A = \begin{pmatrix} 4 & 5 & 2 & 2 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 8 & 0 \end{pmatrix}$.

Solution :

Order of A is 3×4. $\therefore \rho(A) \leq 3$

Let us reduce the matrix A to a triangular form.

$$A = \begin{pmatrix} 4 & 5 & 2 & 2 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 8 & 0 \end{pmatrix}$$

Applying $R_3 \rightarrow \dfrac{R_3}{4}$

$$A \sim \begin{pmatrix} 4 & 5 & 2 & 2 \\ 3 & 2 & 1 & 6 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

Applying $R_1 \leftrightarrow R_3$

$$A \sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 3 & 2 & 1 & 6 \\ 4 & 5 & 2 & 2 \end{pmatrix}$$

Applying $R_2 \rightarrow R_2 - 3R_1, \ R_3 \rightarrow R_3 - 4R_1$

$$A \sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & -5 & 6 \\ 0 & 1 & -6 & 2 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 + R_2$

$$A \sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & -5 & 6 \\ 0 & 0 & -11 & 8 \end{pmatrix}$$

This is in a triangular form

We find,

$$\begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & -5 \\ 0 & 0 & -11 \end{vmatrix} = 11 \neq 0$$

There is a minor of order 3 which is not zero. $\therefore \rho(A) = 3$

1.2.4 Systems of linear equations.

A system of (simultaneous) equations in which the variables (ie. the unknowns) occur only in the first degree is said to be linear.

A system of linear equations can be represented in the form $AX = B$. For example, the equations $x - 3y + z = -1$, $2x + y - 4z = -1$, $6x - 7y + 8z = 7$ can be written in the matrix form as

$$\begin{pmatrix} 1 & -3 & 1 \\ 2 & 1 & -4 \\ 6 & -7 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 7 \end{pmatrix}$$

$$A \qquad X \; = \; B$$

A is called the **coefficient matrix.** If the matrix A is augmented with the column matrix B, at the end, we get the **augmented matrix**,

$$\begin{pmatrix} 1 & -3 & 1 & \vdots & -1 \\ 2 & 1 & -4 & \vdots & -1 \\ 6 & -7 & 8 & \vdots & 7 \end{pmatrix} \text{ denoted by } (A, B)$$

A system of (simultaneous) linear equations is said to be **homogeneous** if the constant term in each of the equations is zero. A system of linear homogeneous equations can be

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represented in the form AX = O. For example, the equations $3x + 4y - 2z = 0$, $5x + 2y = 0$, $3x - y + z = 0$ can be written in the matrix form as

$$\begin{pmatrix} 3 & 4 & -2 \\ 5 & 2 & 0 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{A} \qquad \text{X} = \text{O}$$

1.2.5 Consistency of equations

A system of equations is said to be **consistent** if it has at least one set of solution. Otherwise it is said to be **inconsistent**.

Consistent equations may have

(i) unique solution (that is, only one set of solution) or

(ii) infinite sets of solution.

By way of illustration, consider first the case of linear equations in two variables.

The equations $4x - y = 8$, $2x + y = 10$ represent two straight lines intersecting at (3, 4). They are consistent and have the unique solution $x = 3$, $y = 4$. (Fig. 1.1)

Consistent ;

Unique solution

Fig. 1.1
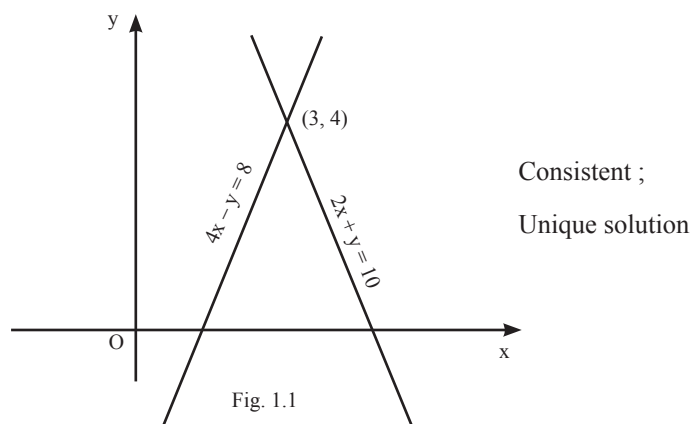
The equations $5x - y = 15$, $10x - 2y = 30$ represent two coincident lines. We find that any point on the line is a solution. The equations are consistent and have infinite sets of solution such as $x = 1$, $y = -10$; $x = 3$, $y = 0$; $x = 4$, $y = 5$ and so on (Fig. 1.2) Such equations are called dependent equations.

Consistent ;

Infinite sets of solution

Fig. 1.2

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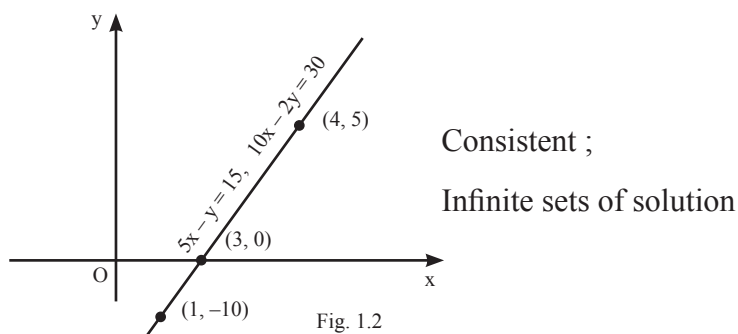
The equations $4x - y = 4$, $8x - 2y = 5$ represent two parallel straight lines. The equations are inconsistent and have no solution. (Fig. 1.3)

Inconsistent ;

No solution

Fig. 1.3
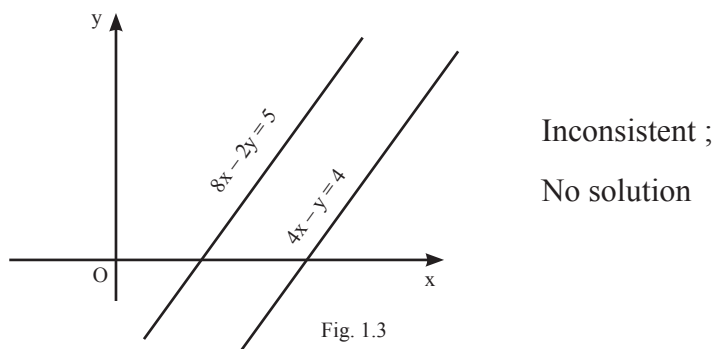
Now consider the case of linear equations in three variables. The equations $2x + 4y + z = 5$, $x + y + z = 6$, $2x + 3y + z = 6$ are consistent and have only one set of unique solution viz. $x = 2$, $y = -1$, $z = 5$. On the other hand, the equations $x + y + z = 1$, $x + 2y + 4z = 1$, $x + 4y + 10z = 1$ are consistent and have infinite sets of solution such as $x = 1$, $y = 0$, $z = 0$; $x = 3$, $y = -3$, $z = 1$; and so on. All these solutions are included in $x = 1 + 2k$, $y = -3k$, $z = k$ where k is a parameter.

The equations $x + y + z = -3$, $3x + y - 2z = -2$, $2x + 4y + 7z = 7$ do not have even a single set of solution. They are inconsistent.

All homogeneous equations do have the trivial solution $x = 0$, $y = 0$, $z = 0$. Hence the homogeneous equations are all consistent and the question of their being consistent or otherwise does not arise at all.

The homogeneous equations may or may not have solutions other than the trivial solution. For example, the equations $x + 2y + 2z = 0$, $x - 3y - 3z = 0$, $2x + y - z = 0$ have only the trivial solution viz., $x = 0$, $y = 0$, $z = 0$. On the other hand the equations $x + y - z = 0$, $x - 2y + z = 0$, $3x + 6y - 5z = 0$ have infinite sets of solution such as $x = 1$, $y = 2$, $z = 3$; $x = 3$, $y = 6$, $z = 9$ and so on. All these non trivial solutions are included in $x = t$, $y = 2t$, $z = 3t$ where t is a parameter.

1.2.6 Testing the consistency of equations by rank method

Consider the equations AX = B in 'n' unknowns

1. If $\rho(A, B) = \rho(A)$, then the equations are consistent.

2. If $\rho(A, B) \neq \rho(A)$, then the equations are inconsistent.

3. If $\rho(A, B) = \rho(A) = n$, then the equations are consistent and have unique solution.

4. If $\rho(A, B) = \rho(A) < n$, then the equations are consistent and have infinite sets of solution.

Consider the equations AX = 0 in 'n' unkowns

1. If $\rho(A) = n$ then equations have the trivial solution only.

2. If $\rho(A) < n$ then equations have the non trivial solutions also.

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Example 15

Show that the equations $2x - y + z = 7$, $3x + y - 5z = 13$, $x + y + z = 5$ are consistent and have unique solution.

Solution :

The equations take the matrix form as

$$\begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 13 \\ 5 \end{pmatrix}$$

$$\quad A \qquad X \; = \; B$$

Now $(A, B) = \begin{pmatrix} 2 & -1 & 1 & \vdots & 7 \\ 3 & 1 & -5 & \vdots & 13 \\ 1 & 1 & 1 & \vdots & 5 \end{pmatrix}$

Applying $R_1 \leftrightarrow R_3$

$(A, B) \sim \begin{pmatrix} 1 & 1 & 1 & \vdots & 5 \\ 3 & 1 & -5 & \vdots & 13 \\ 2 & -1 & 1 & \vdots & 7 \end{pmatrix}$

Applying $R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 2R_1$

$(A, B) \sim \begin{pmatrix} 1 & 1 & 1 & \vdots & 5 \\ 0 & -2 & -8 & \vdots & -2 \\ 0 & -3 & -1 & \vdots & -3 \end{pmatrix}$

Applying $R_3 \to R_3 - \dfrac{3}{2} R_2$

$(A, B) \sim \begin{pmatrix} 1 & 1 & 1 & \vdots & 5 \\ 0 & -2 & -8 & \vdots & -2 \\ 0 & 0 & 11 & \vdots & 0 \end{pmatrix}$

Obviously,

$\rho(A, B) = 3, \; \rho(A) = 3$

The number of unknown is 3.

Hence $\rho(A, B) = \rho(A) =$ the number of unknowns.

$\therefore$ The equations are consistent and have unique solution.

Example 16

Show that the equations $x + 2y = 3$, $y - z = 2$, $x + y + z = 1$ are consistent and have infinite sets of solution.

Solution :

The equations take the matrix form as

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$A \quad X = B$$

Now $(A, B) = \begin{pmatrix} 1 & 2 & 0 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & 2 \\ 1 & 1 & 1 & \vdots & 1 \end{pmatrix}$

Applying $R_3 \rightarrow R_3 - R_1$

$(A, B) \sim \begin{pmatrix} 1 & 2 & 0 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & -1 & 1 & \vdots & -2 \end{pmatrix}$

Applying $R_3 \rightarrow R_3 + R_2$

$(A, B) \sim \begin{pmatrix} 1 & 2 & 0 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$

Obviously,

$\rho(A, B) = 2$, $\rho(A) = 2$.

The number of unknown is 3.

Hence $\rho(A, B) = \rho(A) <$ the number of unknowns.

$\therefore$ The equations are consistent and have infinite sets of solution.

Example 17

Show that the equations $x - 3y + 4z = 3$, $2x - 5y + 7z = 6$, $3x - 8y + 11z = 1$ are inconsistent.

Solution :

The equations take the matrix form as

$$\begin{pmatrix} 1 & -3 & 4 \\ 2 & -5 & 7 \\ 3 & -8 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix}$$

$$A \qquad X = B$$

Now

$$(A, B) = \begin{pmatrix} 1 & -3 & 4 & \vdots & 3 \\ 2 & -5 & 7 & \vdots & 6 \\ 3 & -8 & 11 & \vdots & 1 \end{pmatrix}$$

Applying $R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - 3R_1$

$$(A, B) \sim \begin{pmatrix} 1 & -3 & 4 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & -8 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 - R_2$

$$(A, B) \sim \begin{pmatrix} 1 & -3 & 4 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & -8 \end{pmatrix}$$

Obviously,
$\rho(A, B) = 3, \rho(A) = 2$.
Hence $\rho(A, B) \neq \rho(A)$
∴ The equations are inconsistent.

Example 18

Show that the equations $x + y + z = 0$, $2x + y - z = 0$, $x - 2y + z = 0$ have only the trivial solution.

Solution :

The matrix form of the equations is

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A \qquad X = O$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

Applying $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - R_1$

$$A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & -3 & 0 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 - 3R_2$

$$A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 9 \end{pmatrix}$$

Obviously,

$\rho(A) = 3$

The number of unknowns is 3.

Hence $\rho(A)$ = the number of unknowns.

∴ The equations have only the trivial solution.

Example 19

Show that the equations $3x + y + 9z = 0$, $3x + 2y + 12z = 0$, $2x + y + 7z = 0$ have non trivial solutions also.

Solution :

The matrix form of the equations is

$$\begin{pmatrix} 3 & 1 & 9 \\ 3 & 2 & 12 \\ 2 & 1 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\quad A \quad\quad X = O$$

$$A = \begin{pmatrix} 3 & 1 & 9 \\ 3 & 2 & 12 \\ 2 & 1 & 7 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 3 & 1 & 9 \\ 3 & 2 & 12 \\ 2 & 1 & 7 \end{vmatrix} = 0, \quad \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} = 3 \neq 0$$

$$\therefore \rho(A) = 2$$

The number of unknows is 3.

Hence $\rho(A) <$ the number of unknowns.

$\therefore$ The equations have non trivial solutions also.

Example 20

Find k if the equations $2x + 3y - z = 5$, $3x - y + 4z = 2$, $x + 7y - 6z = k$ are consistent.

Solution :

$$(A,B) = \begin{pmatrix} 2 & 3 & -1 & \vdots & 5 \\ 3 & -1 & 4 & \vdots & 2 \\ 1 & 7 & -6 & \vdots & k \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 3 & -1 \\ 3 & -1 & 4 \\ 1 & 7 & -6 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 3 & -1 & 4 \\ 1 & 7 & -6 \end{vmatrix} = 0, \qquad \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -11 \neq 0$$

Obviously $\rho(A) = 2$.

For the equations to be consistent, $\rho(A, B)$ should also be 2.

Hence every minor of (A, B) of order 3 should be zero.

$$\therefore \begin{vmatrix} 3 & -1 & 5 \\ -1 & 4 & 2 \\ 7 & -6 & k \end{vmatrix} = 0$$

Expanding and simplifying, we get $k = 8$.

Exampe 21

Find k if the the equations $x + y + z = 3$, $x + 3y + 2z = 6$, $x + 5y + 3z = k$ are inconsistent.

Solution :

$$(A, B) = \begin{pmatrix} 1 & 1 & 1 & \vdots & 3 \\ 1 & 3 & 2 & \vdots & 6 \\ 1 & 5 & 3 & \vdots & k \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 5 & 3 \end{pmatrix}$$

We find,

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 5 & 3 \end{vmatrix} = 0, \qquad \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2 \neq 0$$

Obviously $\rho(A) = 2$.

For the equations to be inconsistent, $\rho(A, B)$ should not be 2.

$$(A, B) = \begin{pmatrix} 1 & 1 & 1 & \vdots & 3 \\ 1 & 3 & 2 & \vdots & 6 \\ 1 & 5 & 3 & \vdots & k \end{pmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1, \quad R_3 \rightarrow R_3 - R_1$

$$(A, B) \sim \begin{pmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 2 & 1 & \vdots & 3 \\ 0 & 4 & 2 & \vdots & k-3 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 - 2R_2$

$$(A, B) \sim \begin{pmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 2 & 1 & \vdots & 3 \\ 0 & 0 & 0 & \vdots & k-9 \end{pmatrix}$$

$\rho(A, B) \neq 2$ only when $k \neq 9$

$\therefore$ The equations are inconsistent when k assumes any real value other than 9.

Example 22

Find the value of k for the equations $kx + 3y + z = 0$, $3x - 4y + 4z = 0$, $kx - 2y + 3z = 0$ to have non trivial solution.

Solution :

$$A = \begin{pmatrix} k & 3 & 1 \\ 3 & -4 & 4 \\ k & -2 & 3 \end{pmatrix}$$

For the homogeneous equations to have non trivial solution, $\rho(A)$ should be less than the number of unknowns viz., 3.

$\therefore \rho(A) \neq 3$

$$\text{Hence} \begin{vmatrix} k & 3 & 1 \\ 3 & -4 & 4 \\ k & -2 & 3 \end{vmatrix} = 0$$

Expanding and simplifying, we get $k = \dfrac{11}{4}$.

Example 23

Find *k* if the equations $x + 2y + 2z = 0$, $x - 3y - 3z = 0$, $2x + y + kz = 0$ have only the trivial solution.

Solution :

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & 1 & k \end{pmatrix}$$

For the homogeneous equations to have only the trivial solution, $\rho(A)$ should be equal to the number of unknowns viz., 3.

$$\therefore \begin{vmatrix} 1 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & 1 & k \end{vmatrix} \neq 0, \quad k \neq 1.$$

The equations have only the trivial solution when k assumes any real value other than 1.

EXERCISE 1.2

1) Find the rank of each of the following matrices

(i) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 4 & 2 & 5 \end{pmatrix}$
(ii) $\begin{pmatrix} 3 & 2 & 1 \\ 0 & 4 & 5 \\ 3 & 6 & 6 \end{pmatrix}$
(iii) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$

(iv) $\begin{pmatrix} -2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 4 & 7 \end{pmatrix}$
(v) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -2 & -4 \end{pmatrix}$
(vi) $\begin{pmatrix} -2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ -1 & -3 & -4 & -7 \end{pmatrix}$

(vii) $\begin{pmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 3 \end{pmatrix}$
(viii) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
(ix) $\begin{pmatrix} 9 & 6 \\ -6 & 4 \end{pmatrix}$

2) Find the ranks of A + B and AB where

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{pmatrix}$$

3) Prove that the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if the rank of the

matrix $\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$ is less than 3.

4) Show that the equations $2x + 8y + 5z = 5$, $x + y + z = -2$, $x + 2y - z = 2$ are consistent and have unique solution.

5) Show that the equations $x - 3y - 8z = -10$, $3x + y - 4z = 0$, $2x + 5y + 6z = 13$ are consistent and have infinite sets of solution.

6) Test the system of equations $4x - 5y - 2z = 2$, $5x - 4y + 2z = -2$, $2x + 2y + 8z = -1$ for consistency.

7) Show that the equations $4x - 2y = 3$, $6x - 3y = 5$ are inconsistent.

8) Show that the equations $x + y + z = -3$, $3x + y - 2z = -2$, $2x + 4y + 7z = 7$ are not consistent.

9) Show that the equations $x + 2y + 2z = 0$, $x - 3y - 3z = 0$, $2x + y - z = 0$ have no other solution other than $x = 0$, $y = 0$ and $z = 0$.

10) Show that the equations $x + y - z = 0$, $x - 2y + z = 0$, $3x + 6y - 5z = 0$ have non trivial solutions also.

11) Find k if the equations $x + 2y - 3z = -2$, $3x - y - 2z = 1$, $2x + 3y - 5z = k$ are consistent.

12) Find k if the equations $x + y + z = 1$, $3x - y - z = 4$, $x + 5y + 5z = k$ are inconsistent.

13) Find the value of k for the equations $2x - 3y + z = 0$, $x + 2y - 3z = 0$, $4x - y + kz = 0$ to have non trivial solutions.

14) Find k for which the equations $x + 2y + 3z = 0$, $2x + 3y + 4z = 0$ and $7x + ky + 9z = 0$ have no non trivial solutions.

1.3 SOLUTION OF LINEAR EQUATIONS

1.3.1 Solution by Matrix method

When $|A| \neq 0$, the equations $AX = B$ have the unique solution given by $X = A^{-1}B$.

Example 24

Solve the equations $2x - y = 3$, $5x + y = 4$ using matrices.

Solution :

The equations can be written in matrix form as

$$\begin{pmatrix} 2 & -1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\quad A \qquad X \quad = B$$

$$|A| = \begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} = 7 \neq 0$$

$\therefore$ The unique solution is given by

$X = A^{-1}B$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 1 & 1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 7 \\ -7 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \therefore x = 1, \ y = -1$$

cofactors		
$+(-1-2),$	$-(-1-1),$	$+(2-1)$
$-(-8-10),$	$+(-2-5),$	$-(4-8)$
$+(8-5),$	$-(2-5),$	$+(2-8)$

Example 25

Solve the equations $2x + 8y + 5z = 5$, $x + y + z = -2$, $x + 2y - z = 2$ by using matrix method.

Solution :

The equations can be written in matrix form as

$$\begin{pmatrix} 2 & 8 & 5 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}$$

$$\quad A \qquad\quad X \ = \ B$$

$$|A| = \begin{vmatrix} 2 & 8 & 5 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 15 \neq 0$$

The unique solution is given by

$X = A^{-1}B.$

We now find A^{-1}.

$$A_c = \begin{pmatrix} -3 & 2 & 1 \\ 18 & -7 & 4 \\ 3 & 3 & -6 \end{pmatrix}$$

cofactors		
$+(-35-0),$	$-(-32-0),$	$+(0-35)$
$-(-1-0),$	$+(-1-1),$	$-(0-1)$
$+(36-35),$	$-(36-32),$	$+(35-32)$

$$A_c^t = \begin{pmatrix} -3 & 18 & 3 \\ 2 & -7 & 3 \\ 1 & 4 & -6 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} A_c^t = \frac{1}{15} \begin{pmatrix} -3 & 18 & 3 \\ 2 & -7 & 3 \\ 1 & 4 & -6 \end{pmatrix}$$

Now

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{15} \begin{pmatrix} -3 & 18 & 3 \\ 2 & -7 & 3 \\ 1 & 4 & -6 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{15} \begin{pmatrix} -45 \\ 30 \\ -15 \end{pmatrix} \text{ ie., } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$$

$$x = -3, \quad y = 2, \quad z = -1$$

Example 26

A woman invested different amounts at 8%, $8\frac{3}{4}$% and 9%, all at simple interest. Altogether she invested Rs. 40,000 and earns Rs. 3,455 per year. How much does she have invested at each rate if she has Rs. 4,000 more invested at 9% than at 8%? Solve by using matrices.

Solution :

Let *x, y, z* be the amounts in Rs. invested at 8%, $8\frac{3}{4}$% and 9% respectively.

According to the problem,

$x + y + z = 40{,}000$

$\dfrac{x \times 8 \times 1}{100} + \dfrac{35 \times y \times 1}{400} + \dfrac{9 \times z \times 1}{100} = 3{,}455$ and

$z - x = 4{,}000$

$\Rightarrow \quad x + y + z = 40{,}000$

$32x + 35y + 36z = 13{,}82{,}000$

$x - z = -4{,}000$

The equations can be written in matrix form as

$$\begin{pmatrix} 1 & 1 & 1 \\ 32 & 35 & 36 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 40,000 \\ 13,82,000 \\ -4,000 \end{pmatrix}$$

$$\qquad A \qquad X = \qquad B$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 32 & 35 & 36 \\ 1 & 0 & -1 \end{vmatrix} = -2 \neq 0$$

$\therefore$ The unique solution is given by $X = A^{-1}B$

We now find A^{-1}.

$$A_c = \begin{pmatrix} -35 & 68 & -35 \\ 1 & -2 & 1 \\ 1 & -4 & 3 \end{pmatrix}$$

$$A^t{}_c = \begin{pmatrix} -35 & 1 & 1 \\ 68 & -2 & -4 \\ -35 & 1 & 3 \end{pmatrix}$$

Now,

$$A^{-1} = \frac{1}{|A|} A^t{}_c = \frac{1}{-2} \begin{pmatrix} -35 & 1 & 1 \\ 68 & -2 & -4 \\ -35 & 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -35 & 1 & 1 \\ 68 & -2 & -4 \\ -35 & 1 & 3 \end{pmatrix} \begin{pmatrix} 40,000 \\ 13,82,000 \\ -4,000 \end{pmatrix}$$

$$\Rightarrow \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -22,000 \\ -28,000 \\ -30,000 \end{pmatrix}$$

$$\Rightarrow \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11,000 \\ 14,000 \\ 15,000 \end{pmatrix}$$

Now

Hence the amounts invested at 8%, $8\frac{3}{4}$% and 9% are Rs. 11,000, Rs. 14,000 and Rs. 15,000 respectively.

1.3.2 Solution by Determinant method (Cramer's rule)

Let the equations be

$$a_1x + b_1y + c_1z = d_1,$$

$$a_2x + b_2y + c_2z = d_2,$$

$$a_3x + b_3y + c_3z = d_3.$$

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \qquad \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

When $\Delta \neq 0$, the unique solution is given by

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}, \quad z = \frac{\Delta_z}{\Delta}$$

Example 27

Solve the equations $x + 2y + 5z = 23$, $3x + y + 4z = 26$, $6x + y + 7z = 47$ by determinant method.

Solution :

The equations are

$$x + 2y + 5z = 23$$

$$3x + y + 4z = 26$$

$$6x + y + 7z = 47$$

$$\Delta = \begin{vmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 6 & 1 & 7 \end{vmatrix} = -6 \neq 0 ; \qquad \Delta_x = \begin{vmatrix} 23 & 2 & 5 \\ 26 & 1 & 4 \\ 47 & 1 & 7 \end{vmatrix} = -24$$

$$\Delta_y = \begin{vmatrix} 1 & 23 & 5 \\ 3 & 26 & 4 \\ 6 & 47 & 7 \end{vmatrix} = -12 ; \qquad \Delta_z = \begin{vmatrix} 1 & 2 & 23 \\ 3 & 1 & 26 \\ 6 & 1 & 47 \end{vmatrix} = -18$$

By Cramer's rule

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{-24}{-6} = 4 \quad ; \quad y = \frac{\Delta_y}{\Delta} = \frac{-12}{-6} = 2$$

$$z = \frac{\Delta_z}{\Delta} = \frac{-18}{-6} = 3 \quad ; \quad \Rightarrow x = 4, \ y = 2, \ z = 3$$

Example 28

Solve the equations 2x – 3y – 1 = 0, 5x + 2y – 12 = 0 by Cramer's rule.

Solution :

The equations are $2x - 3y = 1$, $5x + 2y = 12$

$$\Delta = \begin{vmatrix} 2 & -3 \\ 5 & 2 \end{vmatrix} = 19 \neq 0 \quad ; \quad \Delta_x = \begin{vmatrix} 1 & -3 \\ 12 & 2 \end{vmatrix} = 38$$

$$\Delta_y = \begin{vmatrix} 2 & 1 \\ 5 & 12 \end{vmatrix} = 19$$

By Cramer's rule

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{38}{19} = 2, \quad y = \frac{\Delta_y}{\Delta} = \frac{19}{19} = 1$$

$$\Rightarrow x = 2, \quad y = 1$$

Example 29

A salesman has the following record of sales during three months for three items A, B and C which have different rates of commission.

Months	Sales of units			Total commission drawn (in Rs.)
	A	B	C	
January	90	100	20	800
February	130	50	40	900
March	60	100	30	850

Find out the rates of commission on the items A, B and C. Solve by Cramer's rule.

Solution :

Let x, y and z be the rates of commission in Rs. per unit for A, B and C items respectively.

According to the problem,

$$90x + 100y + 20z = 800$$

$$130x + 50y + 40z = 900$$

$$60x + 100y + 30z = 850$$

Dividing each of the equations by 10 throughout,

$$9x + 10y + 2z = 80$$

$$13x + 5y + 4z = 90$$

$$6x + 10y + 3z = 85$$

Now,

$$\Delta = \begin{vmatrix} 9 & 10 & 2 \\ 13 & 5 & 4 \\ 6 & 10 & 3 \end{vmatrix} = -175 \neq 0 \; ; \qquad \Delta_x = \begin{vmatrix} 80 & 10 & 2 \\ 90 & 5 & 4 \\ 85 & 10 & 3 \end{vmatrix} = -350$$

$$\Delta_y = \begin{vmatrix} 9 & 80 & 2 \\ 13 & 90 & 4 \\ 6 & 85 & 3 \end{vmatrix} = -700 \; ; \qquad \Delta_z = \begin{vmatrix} 9 & 10 & 80 \\ 13 & 5 & 90 \\ 6 & 10 & 85 \end{vmatrix} = -1925$$

By Cramer's rule

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{-350}{-175} = 2 \qquad ; \quad y = \frac{\Delta_y}{\Delta} = \frac{-700}{-175} = 4$$

$$z = \frac{\Delta_z}{\Delta} = \frac{-1925}{-175} = 11$$

Hence the rates of commission for A, B and C are Rs. 2, Rs. 4 and Rs. 11 respectively.

EXERCISE 1.3

1) Solve by matrix method the equations $2x + 3y = 7$, $2x + y = 5$.

2) Solve by matrix method the equations

$x - 2y + 3z = 1$, $3x - y + 4z = 3$, $2x + y - 2z = -1$

3) Solve by Cramer's rule the equations $6x - 7y = 16$, $9x - 5y = 35$.

4) Solve by determinant method the equations

$2x + 2y - z - 1 = 0$, $x + y - z = 0$, $3x + 2y - 3z = 1$.

5) Solve by Cramer's rule : $x + y = 2$, $y + z = 6$, $z + x = 4$.

6) Two types of radio valves A, B are available and two types of radios P and Q are assembled in a small factory. The factory uses 2 valves of type A and 3 valves of type B for the type of radio P, and for the radio Q it uses 3 valves of type A and 4 valves of type B. If the number of valves of type A and B used by the factory are 130 and 180 respectively, find out the number of radios assembled. Use matrix method.

7) The cost of 2kg. of wheat and 1kg. of sugar is Rs. 7. The cost of 1kg. wheat and 1kg. of rice is Rs. 7. The cost of 3kg. of wheat, 2kg. of sugar and 1kg. of rice is Rs. 17. Find the cost of each per kg., using matrix method.

8) There are three commodities X, Y and Z which are bought and sold by three dealers A, B andC. Dealer A purchases 2 units of X and 5 units of Z and sells 3 units of Y, dealer

B purchases 5 units of X, 2 units of Y and sells 7 units of Z and dealer C purchases 3 units of Y, 1 unit of Z and sells 4 units of X. In the process A earns Rs. 11 and C Rs. 5 but B loses Rs. 12. Find the price of each of the commodities X, Y and Z, by using determinants.

9) A company produces three products everyday. The total production on a certain day is 45 tons. It is found that the production of the third product exceeds the production of the first product by 8 tons while the total production of the first and third product is twice the production of second product. Determine the production level of each product by using Cramer's rule.

1.4 INPUT - OUTPUT ANALYSIS

Consider a simple economic model consisting of two industries A_1 and A_2, where each produces only one type of product. Assume that each industry consumes part of its own output and rest from the other industry for its operation. The industries are thus interdependent. Further assume that whatever is produced is consumed. That is the total output of each industry must be such as to meet its own demand, the demand of the other industry and the external demand that is the final demand.

Our aim is to determine the output levels of each of the two industries in order to meet a change in final demand, based on a knowledge of the current outputs of the two industries, of course under the assumption that the structure of the economy does not change.

Let a_{ij} be the rupee value of the output of A_i consumed by A_j, i,j = 1, 2.

Let x_1 and x_2 be the rupee value of the current outputs of A_1 and A_2 respectively.

Let d_1 and d_2 be the rupee value of the final demands for the outputs of A_1 and A_2 respectively.

These assumptions lead us to frame the two equations

$$a_{11} + a_{12} + d_1 = x_1$$
$$a_{21} + a_{22} + d_2 = x_2 \qquad\qquad — (1)$$

$$\text{Let } b_{ij} = \frac{a_{ij}}{x_j}, i,\ j = 1, 2$$

That is

$$b_{11} = \frac{a_{11}}{x_1},\ b_{12} = \frac{a_{12}}{x_2},\ b_{21} = \frac{a_{21}}{x_1},\ b_{22} = \frac{a_{22}}{x_2},$$

Then equations (1) take the form

$$b_{11} x_1 + b_{12} x_2 + d_1 = x_1$$
$$b_{21} x_1 + b_{22} x_2 + d_2 = x_2$$

These can be rearranged as

$$(1 - b_{11}) x_1 - b_{12} x_2 = d_1$$
$$- b_{21} x_1 + (1 - b_{22}) x_2 = d_2.$$

This takes the matrix form

$$\begin{pmatrix} 1 - b_{11} & -b_{12} \\ -b_{21} & 1 - b_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

That is $(I - B) X = D$

$$\text{Where } B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

Solving this

$$X = (I - B)^{-1} D.$$

The matrix B is known as the **technology matrix.**

Hawkins-Simon conditions ensure the viability of the system. If B is the technology matrix then Hawkins – Simon conditions are

(i) the main diagonal elements in $I - B$ must be positive and

(ii) $| I - B |$ must be positive.

Example 30

The data below are about an economy of two industries P and Q. The values are in lakhs of rupees.

Producer	User		Final Demand	Total Output
	P	Q		
P	16	12	12	40
Q	12	8	4	24

Find the technology matrix and test whether the system is viable as per Hawkins - Simon conditions.

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Solution :

With the usual notation we have,

$a_{11} = 16,$ $a_{12} = 12,$ $x_1 = 40$

$a_{21} = 12,$ $a_{22} = 8,$ $x_2 = 24$

Now

$$b_{11} = \frac{a_{11}}{x_1} = \frac{16}{40} = \frac{2}{5}, \quad b_{12} = \frac{a_{12}}{x_2} = \frac{12}{24} = \frac{1}{2},$$

$$b_{21} = \frac{a_{21}}{x_1} = \frac{12}{40} = \frac{3}{10}, \quad b_{22} = \frac{a_{22}}{x_2} = \frac{8}{24} = \frac{1}{3}.$$

The technology matrix is

$$B = \begin{pmatrix} \frac{2}{5} & \frac{1}{2} \\ \frac{3}{10} & \frac{1}{3} \end{pmatrix}$$

$$I - B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{2}{5} & \frac{1}{2} \\ \frac{3}{10} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{1}{2} \\ -\frac{3}{10} & \frac{2}{3} \end{pmatrix}$$

The main diagonal elements $I - B$ viz., $\frac{3}{5}$ and $\frac{2}{3}$ are positive. Also

$$|I - B| = \begin{vmatrix} \frac{3}{5} & -\frac{1}{2} \\ -\frac{3}{10} & \frac{2}{3} \end{vmatrix} = \frac{1}{4}. \qquad |I - B| \text{ is positive.}$$

$\therefore$ The two Hawkins-Simon conditions are satisfied. Hence the system is viable.

Example 31

    In an economy there are two industries P and Q and the following table gives the supply and demand positions in crores of rupees.

Producer	User		Final Demand	Total Output
	P	Q		
P	10	25	15	50
Q	20	30	10	60

Determine the outputs when the final demand changes to 35 for P and 42 for Q.

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Solution :

With the usual notation we have,

$$a_{11} = 10, \qquad a_{12} = 25, \qquad x_1 = 50$$
$$a_{21} = 20, \qquad a_{22} = 30, \qquad x_2 = 60$$

Now

$$b_{11} = \frac{a_{11}}{x_1} = \frac{10}{50} = \frac{1}{5}, \quad b_{12} = \frac{a_{12}}{x_2} = \frac{25}{60} = \frac{5}{12},$$

$$b_{21} = \frac{a_{21}}{x_1} = \frac{20}{50} = \frac{2}{5}, \quad b_{22} = \frac{a_{22}}{x_2} = \frac{30}{60} = \frac{1}{2}$$

$\therefore$ The technology matrix is

$$B \quad = \begin{pmatrix} \frac{1}{5} & \frac{5}{12} \\ \frac{2}{5} & \frac{1}{2} \end{pmatrix}$$

$$I - B \quad = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{5} & \frac{5}{12} \\ \frac{2}{5} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & -\frac{5}{12} \\ -\frac{2}{5} & \frac{1}{2} \end{pmatrix}$$

$$|I - B| \quad = \begin{vmatrix} \frac{4}{5} & -\frac{5}{12} \\ -\frac{2}{5} & \frac{1}{2} \end{vmatrix} = \frac{7}{30}$$

$$(I - B)^{-1} = \frac{1}{\frac{7}{30}} \begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \frac{30}{7} \begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 15 & \frac{25}{2} \\ 12 & 24 \end{pmatrix}$$

Now,

$$X = (I - B)^{-1} D$$

$$= \frac{1}{7} \begin{pmatrix} 15 & \frac{25}{2} \\ 12 & 24 \end{pmatrix} \begin{pmatrix} 35 \\ 42 \end{pmatrix} = \begin{pmatrix} 15 & \frac{25}{2} \\ 12 & 24 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 150 \\ 204 \end{pmatrix}$$

The output of the industry P should be Rs.150 crores and that of Q should be Rs. 204 crores.

EXERCISE 1.4

1) The technology matrix of an economic system of two industries is $\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{2}{5} & \frac{2}{3} \end{pmatrix}$. Test whether the system is viable as per Hawkins Simon conditions.

2) The technology matrix of an economic system of two industries is $\begin{pmatrix} \frac{3}{5} & \frac{9}{10} \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$. Test whether the system is viable as per Hawkins Simon conditions.

3) The technology matrix of an economic system of two industries is $\begin{pmatrix} \frac{2}{5} & \frac{1}{10} \\ \frac{7}{10} & \frac{3}{5} \end{pmatrix}$. Find the output levels when the final demand changes to 34 and 51 units.

4) The data below are about an economy of two industries P and Q. The values are in millions of rupees.

Producer	User		Final Demand	Total Output
	P	Q		
P	14	6	8	28
Q	7	18	11	36

Determine the outputs if the final demand changes to 20 for P and 30 for Q.

5) Suppose the inter-relationship between the production of two industries P and Q in a year (in lakhs of rupees) is

Producer	User		Final Demand	Total Output
	P	Q		
P	15	10	10	35
Q	20	30	15	65

Find the outputs when the final demand changes to (i) 12 for P and 18 for Q (ii) 8 for P and 12 for Q.

6) In an economy of two industries P and Q the following table gives the supply and demand positions in millions of rupees.

Producer	User		Final Demand	Total Output
	P	Q		
P	16	20	4	40
Q	8	40	32	80

Find the outputs when the final demand changes to 18 for P and 44 for Q.

7)   The data below are about an economy of two industries P and Q. The values are in crores of rupees.

Producer	User		Final Demand	Total Output
	P	Q		
P	50	75	75	200
Q	100	50	50	200

Find the outputs when the final demand changes to 300 for P and 600 for Q.

8)   The inter - relationship between the production of two industries P and Q in crores of rupees is given below.

Producer	User		Total Output
	P	Q	
P	300	800	2,400
Q	600	200	4,000

If the level of final demand for the output of the two industries is 5,000 for P and 4,000 for Q, at what level of output should the two industries operate?

1.5 TRANSITION PROBABILITY MATRICES

These are matrices in which the individual elements are the probabilities of transition from one state to another of an event. The probabilities of the various changes applied to the initial state by matrix multiplication gives a forecast of the succeeding state.

The following examples illustrate the method.

Example 32

Two products A and B currently share the market with shares 60% and 40 % each respectively. Each week some brand switching takes place. Of those who bought A the previous week, 70% buy it again whereas 30% switch over to B. Of those who bought B the previous week, 80% buy it again whereas 20% switch over to A. Find their shares after one week and after two weeks. If the price war continues, when is the equilibrium reached?

Solution :

Transition Probability matrix

$$T = \begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} \end{array}$$

Shares after one week

$$\begin{array}{cc} & A \quad B \\ \begin{array}{c} A \quad B \\ (0.6 \quad 0.4) \end{array} & \begin{array}{c} A \\ B \end{array}\begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} = \begin{array}{c} A \quad B \\ (0.5 \quad 0.5) \end{array} \end{array}$$

A = 50%,   B = 50%

Shares after two weeks

$$\begin{array}{cc} & A \quad B \\ \begin{array}{c} A \quad B \\ (0.5 \quad 0.5) \end{array} & \begin{array}{c} A \\ B \end{array}\begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} = \begin{array}{c} A \quad B \\ (0.45 \quad 0.55) \end{array} \end{array}$$

A = 45%,   B = 55%

Equilibrium

At equilibrium we must have

(A   B) T = (A B) where A + B = 1

$$\Rightarrow (A \ B) \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} \qquad = (A \ B)$$

$\Rightarrow 0.7\,A + 0.2\,B \qquad\qquad = A$

$\Rightarrow 0.7\,A + 0.2\,(1 - A) = A$

Simplifying, we get A   = 0.4

∴ Equilibrium is reached when A's share is 40% and B's share is 60%

Example 33

A new transit system has just gone into operation in a city. Of those who use the transit system this year, 10% will switch over to using their own car next year and 90% will continue to use the transit system. Of those who use their cars this year, 80% will continue to use their cars next year and 20% will switch over to the transit system. Suppose the population of the city remains constant and that 50% of the commuters use the transit system and 50% of the commuters use their own car this year,

(i)   What percent of commuters will be using the transit system after one year?

(ii)   What percent of commuters will be using the transit system in the long run?

Solution :

Transition Probability Matrix

$$T = \begin{array}{c} \\ S \\ C \end{array}\begin{array}{c} S \quad\ C \\ \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} \end{array}$$

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Percentage after one year

$$\begin{array}{cc} & S \quad C \\ \begin{array}{cc} S & C \\ (0.5 & 0.5) \end{array} & \begin{array}{c} S \\ C \end{array}\begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} = \begin{array}{cc} A & B \\ (0.55 & 0.45) \end{array} \end{array}$$

S = 55%,   C = 45%

Equilibrium will be reached in the long run,

At equilibrium we must have

$(S \quad C) \, T = (S \quad C)$ where $S + C = 1$

$$\Rightarrow (S \quad C) \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} = (S \quad C)$$

$\Rightarrow 0.9\, S + 0.2\, C \qquad\qquad = S$

$\Rightarrow 0.9\, S + 0.2\, (1 - S) = S$

Simplifying, we get $S = 0.67$

$\therefore$ 67% of the commuters will be using the transit system the long run.

EXERCISE 1.5

1) Two products P and Q share the market currently with shares 70% and 30% each respectively. Each week some brand switching takes place. Of those who bought P the previous week, 80% buy it again whereas 20% switch over to Q. Of those who bought Q the previous week, 40% buy it again whereas 60% switch over to P. Find their shares after two weeks. If the price war continues, when is the equilibrium reached?

2) The subscription department of a magazine sends out a letter to a large mailing list inviting subscriptions for the magazine. Some of the people receiving this letter already subscribe to the magazine while others do not. From this mailing list, 60% of those who already subscribe will subscribe again while 25% of those who do not now subscribe will subscribe. On the last letter it was found that 40% of those receiving it ordered a subscription. What percent of those receiving the current letter can be expected to order a subscription?

3) Two newspapers A and B are published in a city. Their present market shares are 15% for A and 85% for B. Of those who bought A the previous year, 65% continue to buy it again while 35% switch over to B. Of those who bought B the previous year, 55% buy it again and 45% switch over to A. Find their market shares after two years.

EXERCISE 1.6

Choose the correct answer

1) If the minor of a_{23} equals the cofactor of a_{23} in $|a_{ij}|$ then the minor of a_{23} is

    a) 1   b) 2   c) 0   d) 3

2) The Adjoint of $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ is

    a) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ d) $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

3) The Adjoint of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is

    a) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ b) $\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$ c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ d) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

4) If $AB = BA = |A|\, I$ then the matrix B is

    a) the inverse of A   b) the transpose of A

    c) the Adjoint of A   d) 2A

5) If A is a square matrix of order 3 then $|\,\text{Adj } A\,|$ is

    a) $|A|^2$ b) $|A|$ c) $|A|^3$ d) $|A|^4$

6) If $|A| = 0$ then $|\text{Adj } A|$ is

    a) 0   b) 1   c) -1 d) ± 1

7) The inverse of $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ is

    a) $\begin{pmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ c) $\begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$ d) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

8)   If $A = \begin{pmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{pmatrix}$ then A^{-1} is

a) $\begin{pmatrix} -0.8 & 0.6 \\ -0.6 & 0.8 \end{pmatrix}$
b) $\begin{pmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{pmatrix}$
c) $\begin{pmatrix} 0.8 & 0.6 \\ 0.6 & 0.8 \end{pmatrix}$
d) $\begin{pmatrix} 0.2 & 0.4 \\ -0.4 & 0.2 \end{pmatrix}$

9)   For what value of k the matrix A, where $A = \begin{pmatrix} 2 & k \\ 3 & 5 \end{pmatrix}$ has no inverse?

a) $\dfrac{3}{10}$
b) $\dfrac{10}{3}$
c) 3
d) 10

10)   If $A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$ then $A^{-1} A$ is

a) 0
b) A
c) I
d) A^2

11)   The rank of an $n \times n$ matrix each of whose elements is 1 is

a) 1
b) 2
c) n
d) n^2

12)   The rank of an $n \times n$ matrix each of whose elements is 2 is

a) 1
b) 2
c) n
d) n^2

13)   The rank of a zero matrix is

a) 0
b) 1
c) -1
d) ∞

14)   The rank of a non singular matrix of order $n \times n$ is

a) n
b) n^2
c) 0
d) 1

15)   A system of linear homogeneous equations has at least

a) one solution
b) two solutions
c) three solutions
d) four solutions

16)   The equations $AX = B$ can be solved by Cramer's rule only when

a) $|A| = 0$
b) $|A| \neq 0$
c) $A = B$
d) $A \neq B$

17)   The number of Hawkins - Simon conditions for the viability of an input-output model is

a) 1
b) 3
c) 4
d) 2

18)   If $T = \begin{array}{c} \\ A \\ B \end{array}\begin{pmatrix} \overset{A}{0.7} & \overset{B}{0.3} \\ x & 0.8 \end{pmatrix}$ is a transition probability matrix, then the value of x is

a) 0.3
b) 0.2
c) 0.3
d) 0.7

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ANALYTICAL GEOMETRY 　2

_____

2.1 CONICS

Intersections of cone by a plane

The parabola, ellipse and hyperbola are all members of a class of curves called conics.

The above three curves can be obtained by cutting a cone with a plane and so they are called conics (Fig. 2.1).
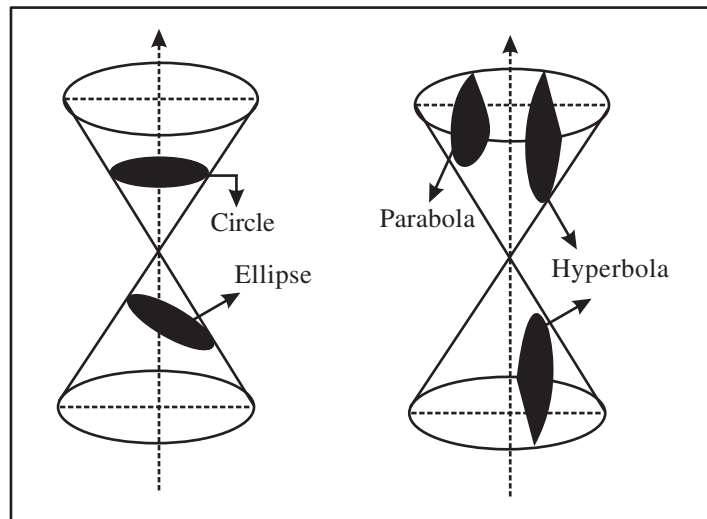
Fig. 2.1

A **conic** is the locus of a point which moves in a plane such that its distance from a fixed point in the plane bears a constant ratio to its distance from a fixed straight line in that plane.

Focus, Directrix, Eccentricity: In the above definition of a conic, the fixed point is called the **focus,** the fixed line the **directrix** and the constant ratio, the **eccentricity** of the conic.

The eccentricity is usually denoted by the letter 'e'.

In the Fig. 2.2, S is the focus, the line LM is the directrix and

$$\frac{SP}{PM} = e$$

The conic is

　　a parabola if $e = 1$,

　　an ellipse if $e < 1$ and

　　a hyperbola if $e > 1$.
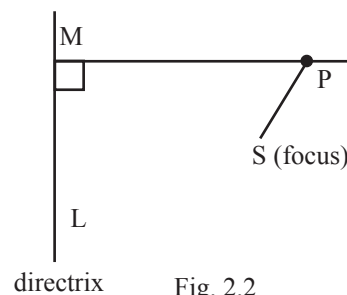
Fig. 2.2

Note:

The derivations of standard equations of conics and tracings of conics (2.2.1, 2.3.1, 2.4.1, 2.4.5, 2.2.2, 2.3.2, 2.4.2) are exempted from the examination and they are to be explained for better understanding.

2.1.1 The general equation of a conic

We know that a conic is the locus of a point moving such that its distance from the focus bears a constant ratio to its distance from the directrix.

Let the focus be $S(x_1, y_1)$ and the directrix be $Ax + By + C = 0$

Let the eccentricity of the conic be e and P(x, y) be any point on it.

Then $SP = \sqrt{(x - x_1)^2 + (y - y_1)^2}$

Perpendicular distance of P from $Ax + By + C = 0$ is

$$PM = \pm \frac{Ax + By + C}{\sqrt{A^2 + B^2}}$$

$$\frac{SP}{PM} = e$$

$$\Rightarrow \frac{\sqrt{(x - x_1)^2 + (y - y_1)^2}}{\pm \dfrac{Ax + By + C}{\sqrt{A^2 + B^2}}} = e$$

or $(x - x_1)^2 + (y - y_1)^2 = e^2 \left[\dfrac{(Ax + By + C)^2}{(A^2 + B^2)} \right]$

Simplifying, we get an equation of the second degree in x and y of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

This is the general equation of a conic.

Remarks :

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents,

(i)   a pair of straight lines if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

(ii)   a circle if a = b and h = 0

If the above two conditions are not satisfied, then

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents,

(iii)   a parabola if $h^2 - ab = 0$

(iv)   an ellipse if $h^2 - ab < 0$

(v)   a hyperbola if $h^2 - ab > 0$

Example 1

The equation $4x^2 + 4xy + y^2 + 4x + 32y + 16 = 0$ represents a conic. Identify the conic.

Solution :

Comparing the given equation,

$4x^2 + 4xy + y^2 + 4x + 32y + 16 = 0$ with the general second degree equation in x and y

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

we get $\quad a = 4, 2h = 4, b = 1$

$\therefore h^2 - ab = (2)^2 - 4(1) = 0$

$\therefore$ The given conic is a parabola.

Example 2

Identify the conic represented by $16x^2 + 25y^2 - 118x - 150y - 534 = 0.$

Solution:

Here, $a = 16, 2h = 0, b = 25$

$\therefore h^2 - ab = 0 - 16 \times 25 = -400 < 0$

$\therefore$ The conic is an ellipse.

EXERCISE 2.1

Identify the conics represented by the following equations:

1) $\quad x^2 - 6xy + 9y^2 + 26x - 38y + 49 = 0$

2) $\quad 7x^2 + 12xy - 2y^2 + 22x + 16y - 7 = 0$

3) $\quad 7x^2 + 2xy + 7y^2 - 60x - 4y + 44 = 0$

2.2 PARABOLA

2.2.1 Standard Equation of parabola
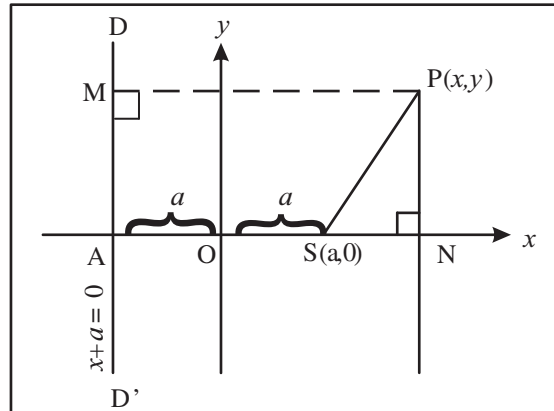
Fig. 2.3

Let S be the focus and the line DD′ be the directrix. Draw SA perpendicular to DD′ cutting DD′ at A. Let SA = 2a. Take AS as the x axis and Oy perpendicular to AS through the middle point O of AS as the y axis.

Then S is (a,0) and the directrix DD′ is the line x + a = 0.

Let P(x, y) be any point on the parabola. Draw PM ⊥ DD′ and PN ⊥ Ox

PM = NA = NO + OA = x + a.

$SP^2 = (x - a)^2 + y^2$

Then $\dfrac{SP}{PM} = e$ [P is a point on the parabola]

or, $SP^2 = e^2(PM)^2$

or, $(x - a)^2 + y^2 = (x + a)^2$ (here $e = 1$)

or, $y^2 = 4ax$

This is the standard equation of the parabola.

Note:

(i) In any parabola the line which passes through the focus and is perpendicular to the directrix is called the **axis** of the parabola, and the point of intersection of the curve and its axis is called the **vertex.**

(ii) The chord which passes through the focus and is perpendicular to the axis is called the **latus rectum.**

2.2.2 Tracing of the parabola y² = 4ax

1)   (a)   Putting $y = 0$, the only value of x we get is zero.

∴ The curve cuts the x - axis at (0,0) only.

(b)   If $x < 0$, y is imaginary. Hence the curve does not exist for negative values of x.

(c)   The equation of the parabola is unaltered if y is replaced *by – y*. Hence the curve is symmetrical about x-axis.

(d)   As x increases, y also increases. As $x \to \infty$, $y \to \pm \infty$. Hence the curve diverges and assumes the form as shown in Fig. 2.4.
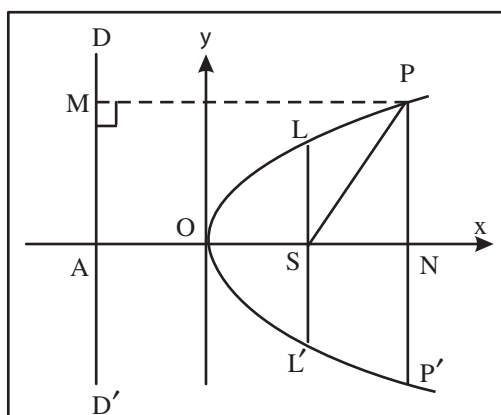
Fig. 2.4

2)   **Directrix :** The directrix is a line parallel to the y-axis and its equation is

$x = - a$ or, $x + a = 0$.

3)   The x-axis is the **axis** of the parabola and the y-axis is the **tangent at the vertex.**

4)   **Latus rectum:** Through S, LSL′ be drawn $\perp$ to AS. Corresponding to $x = a$, $y^2 = 4a^2$ or, $y = \pm 2a$. $\therefore$ SL = SL′ = 2a.

So, LL′ = 4a, LL′ is called Latus rectum of the parabola.

SL (or SL′) is the Semi-latus rectum.

$OS = \dfrac{1}{4} LL' = a$

$y^2 = - 4ax$ $x^2 = 4ay$ $x^2 = - 4ay$
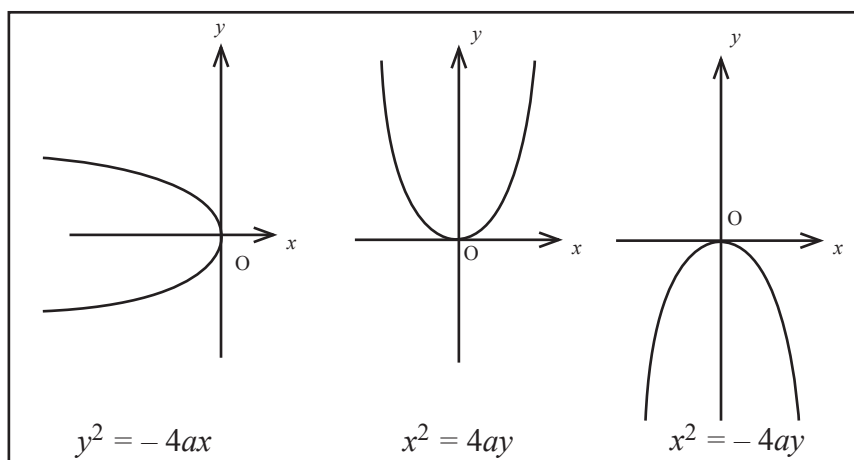
Fig. 2.5

Note

In Fig 2.5, $y^2 = -4ax$ is a parabola which lies only on the negative side of the x-axis. $x^2 = 4ay$ is a parabola whose axis of symmetry is the y-axis and it lies on the positive side of y-axis. $x^2 = -4ay$ lies on the negative side of y-axis.

Equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Latus rectum	$4a$	$4a$	$4a$	$4a$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$

Example 3

Find the equation of the parabola whose focus is the point (2, 1) and whose directrix is the straight line $2x + y + 1 = 0$

Solution :

Let P (x, y) be a point on the parabola.

If PM is drawn perpendicular to the directrix.

$\dfrac{SP}{PM} = 1$ where S is the focus of the parabola.

$$\therefore \qquad SP^2 = PM^2$$

or, $\qquad (x-2)^2 + (y-1)^2 = \left(\dfrac{2x+y+1}{\sqrt{2^2+1^2}}\right)^2$

$x^2 - 4x + 4 + y^2 - 2y + 1 = \dfrac{(2x+y+1)^2}{5}$

$5x^2 + 5y^2 - 20x - 10y + 25 = 4x^2 + y^2 + 1 + 4xy + 2y + 4x$

$x^2 - 4xy + 4y^2 - 24x - 12y + 24 = 0$

This is the required equation.

Example 4

Find the focus, latus rectum, vertex and directrix of the parabola $y^2 - 8x - 2y + 17 = 0.$

Solution :

The given equation can be written as

$y^2 - 2y = 8x - 17$

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$y^2 - 2y + 1 \quad = 8x - 16$ (completing the square)

$(y - 1)^2 \quad = 8(x - 2)$

Changing the origin to the point (2,1) by putting $y - 1 = Y$, $x - 2 = X$, the equation of the parabola is $Y^2 = 8X$.

$\therefore$ The vertex is the new origin and latus rectum is 8. So focus is the point (2,0) in the new coordinates.

So, with respect to the original axes the focus is the point (4,1) and the directrix is the line $X + 2 = 0$

or, $x - 2 + 2 = 0$ or $x = 0$

The vertex is $(X = 0, Y = 0) = (x = 2, y = 1) = (2, 1)$

Example 5

Find the vertex, focus, axis, directrix and length of semilatus rectum of the parabola $4y^2 + 12x - 20y + 67 = 0$

Solution :

$$4y^2 + 12x - 20y + 67 = 0$$

or $\quad 4y^2 - 20y = -12x - 67$

which can be rearranged and written as $4(y^2 - 5y) = -12x - 67$

$$4\left\{ y^2 - 5y + \frac{25}{4} - \frac{25}{4} \right\} = -12x - 67$$

$$4\left[\left(y - \frac{5}{2} \right)^2 - \frac{25}{4} \right] = -12x - 67$$

$$4\left(y - \frac{5}{2} \right)^2 = 25 - 12x - 67$$

$$= -12\left(x + \frac{7}{2} \right)$$

$$\Rightarrow \quad \left(y - \frac{5}{2} \right)^2 = 3\left(-x - \frac{7}{2} \right)$$

To bring it to the form $Y^2 = 4aX$,

set $X = -x - \dfrac{7}{2}$ and $Y = y - \dfrac{5}{2}$

We now get $Y^2 = 3X$. Here $4a = 3$ and so $a = \dfrac{3}{4}$

We now tabulate the results.

Referred to (X, Y)	Referred to (x, y) $$x = -X - \frac{7}{2}, y = Y + \frac{5}{2}$$
Vertex (0, 0)	$\left(0 - \frac{7}{2}, 0 + \frac{5}{2}\right) = \left(-\frac{7}{2}, \frac{5}{2}\right)$
Axis Y=0 (X-axis)	$y - \frac{5}{2} = 0$ or $y = \frac{5}{2}$
Focus $(a,0) = \left(\frac{3}{4}, 0\right)$	$\left(-\frac{3}{4} - \frac{7}{2}, 0 + \frac{5}{2}\right) = \left(-\frac{17}{4}, \frac{5}{2}\right)$
Directrix $X = -a$ i.e $X = \frac{-3}{4}$	$-x - \frac{7}{2} = \frac{-3}{4}$ or $x = -\frac{11}{4}$
Semi latus rectum $2a = 2 \times \frac{3}{4} = \frac{3}{2}$	$\frac{3}{2}$

Note

We could also have taken a transformation $X = x + \frac{7}{2}$ and $Y = y - \frac{5}{2}$ so that we would have got $Y^2 = -3X$ and compared it with $y^2 = -4ax$.

Example 6

The average cost y of a monthly output x kgs. of a firm producing a metal is Rs. ($\frac{1}{10}x^2 - 3x + 50$). Show that the average variable cost curve is a parabola. Find the output and average cost at the vertex of the parabola.

Solution:

The average variable cost curve is

$$y = \frac{1}{10}x^2 - 3x + 50$$
$$10y = x^2 - 30x + 500$$
$$10y = (x - 15)^2 + 275$$
$$(x - 15)^2 = 10y - 275$$
$$(x - 15)^2 = 10(y - 27.5)$$
$$X^2 = 10Y \text{ where}$$

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X = x – 15, Y = y – 27.5 and

4a = 10   ⟹   a = 2.5

Thus we get the average variable cost curve as a parabola whose vertex is

(X = 0, Y = 0)

or   (x = 15, y = 27.5)

or   (15, 27.5)

Hence the output and average cost at the vertex are 15 kgs. and Rs. 27.50 respectively.

Example 7

The supply of a commodity is related to the price by the relation $x = 5\sqrt{2p-10}$ Show that the supply curve is a parabola. Find its vertex and the price below which supply is 0?

Solution:

The supply price relation is given by,

$x^2 = 25\,(2p – 10) \Rightarrow x^2 = 50\,(p – 5)$

⟹ $X^2 = 4a\text{P}$ where $X = x$

and $\text{P} = p – 5$

∴   the supply curve is a parabola whose vertex is (X = 0, P = 0).

⟹ $(x = 0, p = 5) \Rightarrow (0, 5)$

and supply is zero below $p = 5$.
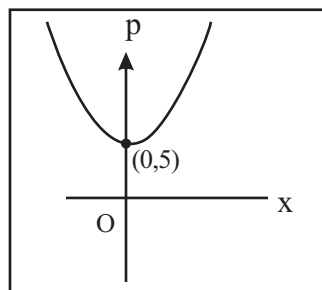
Fig.2.6

Example 8

The girder of railway bridge is a parabola with its vertex at the highest point, which is 15 metres above the span of length 150 metres. Find its height 30 metres from the mid point.

Solution :

Let the parabola be $x^2 = -4ay$.

This passes through A $(75, -15)$

$\Rightarrow (-4a)(-15) = (75)^2$

$4a = \dfrac{(75)^2}{15} = 375$

Hence the parabola is $x^2 = -375\,y$

Now B $(30, -\lambda)$ lies on the parabola

$\Rightarrow \qquad (-375)(-\lambda) = 30^2$

or $\qquad \lambda = \dfrac{900}{375} = \dfrac{12}{5} = 2.4$

Fig.2.7

$\therefore$ Required height $= 15 - 2.4 = 12.6$m
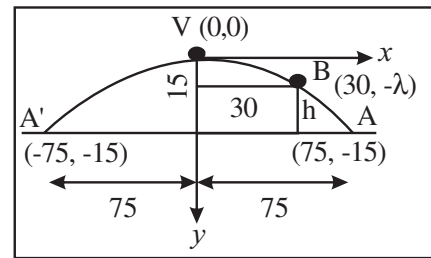
Example 9

The profit Rs. *y* accumulated in lakhs in *x* months is given by $y = -4x^2 + 28x - 40$. Find the best time to end the project.

Solution :

$$4x^2 - 28x = -40 - y$$

$$4(x^2 - 7x) = -40 - y$$

$$4\left(x^2 - 7x + \dfrac{49}{4}\right) = -40 - y + 49$$

$$\left(x - \dfrac{7}{2}\right)^2 = \dfrac{1}{4}(9 - y)$$

$$\left(x - \dfrac{7}{2}\right)^2 = -\dfrac{1}{4}(y - 9)$$

Required time $= \dfrac{7}{2} = 3\dfrac{1}{2}$ months

EXERCISE 2.2

1) Find the equations of the parabolas with the following foci and directrices:

(i) $(1, 2)$; $x + y - 2 = 0$ (ii) $(1, -1)$; $x - y = 0$

(iii) $(0, 0)$; $x - 2y + 2 = 0$ (iv) $(3,4)$, $x - y + 5 = 0$

2)   Find the vertex, axis, focus and directrix of the following parabola:

(i) $x^2 = 100y$ (ii) $y^2 = 20x$ (iii) $y^2 = -28x$ (iv) $x^2 = -60y$

3)   Find the foci, latus recta, vertices and directrices of the following parabolas:

(i) $y^2 + 4x - 2y + 3 = 0$ (ii) $y^2 - 4x + 2y - 3 = 0$

(iii) $y^2 - 8x - 9 = 0$ (iv) $x^2 - 3y + 3 = 0$

4)   The average variable cost of a monthly output of x tonnes of a firm producing a valuable metal is Rs. $\frac{1}{10}x^2 - 3x + 62.5$. Show that the average variable cost curve is a parabola . Find also the ouput and the average cost at the vertex of the parabola.

Note: The point (x_1, y_1) lies outside, on or inside the parabola according as $y_1{}^2 - 4ax_1$ is greater than, equal to or less than zero.

2.3 ELLIPSE

2.3.1 Standard Equation of ellipse:
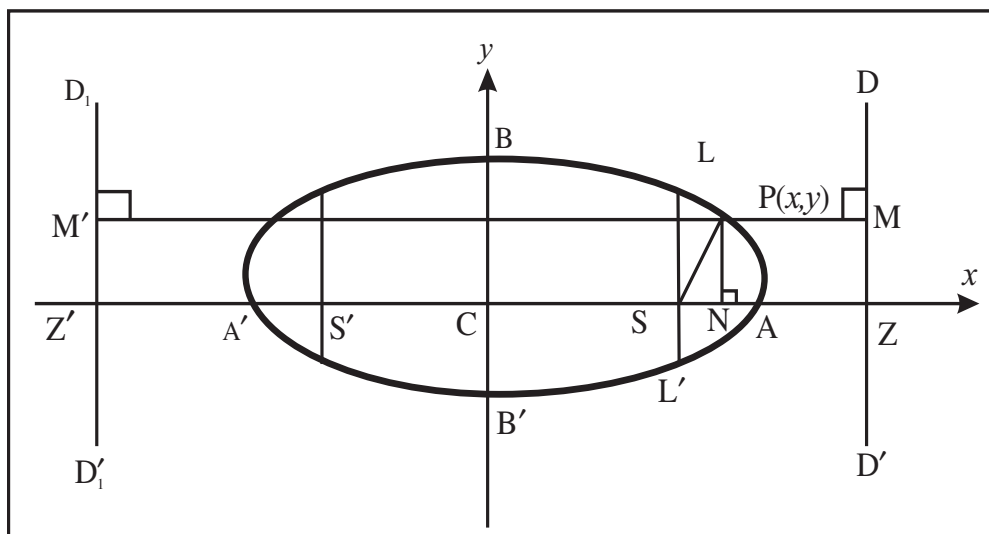
Fig.2.8

Let S be the focus and DD' be the directrix.

Draw SZ perpendicular to DD'. Let A, A' divide SZ internally and externally respectively in the ratio e:1 where e is the eccentricity. Then A, A' are points on the ellipse.

Let C be the mid-point of AA' and let AA' = 2a. Take CA to be the x axis and Cy which is $\perp$ to CA to be the y axis. C is the origin.

$$\frac{SA}{AZ} = e, \frac{SA'}{A'Z} = e$$

$$\therefore \quad SA = e(AZ) \qquad \qquad ....(1)$$

$$A'S = e(A'Z) \qquad \qquad ....(2)$$

$$(1)+(2) \Rightarrow SA + A'S = e(AZ + A'Z)$$

$$AA' = e(CZ - CA + A'C + CZ)$$

$$2a = e(2CZ) \qquad \qquad (\text{since } CA = CA')$$

$$\Rightarrow CZ = \frac{a}{e}$$

$$(2)-(1) \quad \Rightarrow A'S - SA = e\,(A'Z - AZ)$$

$$A'C + CS - (CA - CS) = e\,(AA')$$

$$\text{or} \quad 2CS = e\,.\,2a$$

$$\Rightarrow \quad CS = ae$$

So, S is the point $(ae, 0)$.

Let P (x, y) be any point on the ellipse, draw PM $\perp$ DD' and PN $\perp$ CZ.

$$\Rightarrow \quad PM = NZ = CZ - CN$$

$$= \frac{a}{e} - x$$

$$\frac{SP}{PM} = e \text{ (since the point P lies on the ellipse)}$$

$$SP^2 = e^2\,PM^2$$

$$(x - ae)^2 + y^2 = e^2 \left(\frac{a}{e} - x\right)^2 = (a - ex)^2$$

$$x^2 - 2aex + a^2 e^2 + y^2 = a^2 - 2aex + e^2 x^2$$

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

$$\text{Put } b^2 = a^2(1 - e^2)$$

$$\text{Hence } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (a > b)$$

This is the equation of ellipse in the standard form.

2.3.2 Tracing of the Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(i) The curve does not pass through the origin. When $y = 0$, $x = \pm a$. Therefore it meets the x axis at the points $(\pm a, 0)$. Similarly it meets the y axis at the points $(0, \pm b)$.

(ii) The curve is symmetrical about both the axes of coordinates since both the powers of x and y are even. If (x, y) be a point on the curve, so also are $(-x, y)$, $(x, -y)$ and $(-x, -y)$.

(iii) We may put the equation of the ellipse in the form

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

If $|x| > a$, i.e. if either $x > a$ or $x < -a$, $a^2 - x^2$ becomes negative so that $\sqrt{a^2 - x^2}$ is imaginary. Hence there is no point of the curve lying either to the right of the line $x = a$ or to the left of the line $x = -a$.

If $|x| \le a$ i.e. if $-a \le x \le a$, the expression under the radical sign is positive and we get two equal and opposite values of y.

The curve lies entirely between these two lines. Note that $x = a$ is a tangent at A $(a, 0)$ and $x = -a$ is a tangent at A' $(-a, 0)$.

(iv) Similarly writing the equation in the form

$$x = \pm \frac{a}{b}\sqrt{b^2 - y^2}$$

we find that the curve does not extend above the line $y=b$ and below the line $y=-b$.

In fact the curve lies entirely between the lines $y = b$, and $y = -b$ which are respectively tangents to the curve at B and B'.

(v) If x increases from 0 to a, y decreases from b to 0. Similarly if y increases from 0 to b, x decreases from a to 0.

(vi) **Latus rectum:** Through S, LSL' be drawn perpendicular to AS. Corresponding to $x = ae$, we have

$$\frac{a^2 e^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ or } y^2 = b^2(1 - e^2) = b^2\frac{b^2}{a^2} = \frac{b^4}{a^2}$$

or $y = \pm \dfrac{b^2}{a} \Rightarrow \text{SL} = \text{SL}' = \dfrac{b^2}{a}$. So, $\text{LL}' = \dfrac{2b^2}{a}$ is the latus rectum of the ellipse.

These information about the curve are sufficient to enable us to find the shape of the curve as given in the Fig. 2.9. Unlike the parabola, the ellipse is a closed curve.
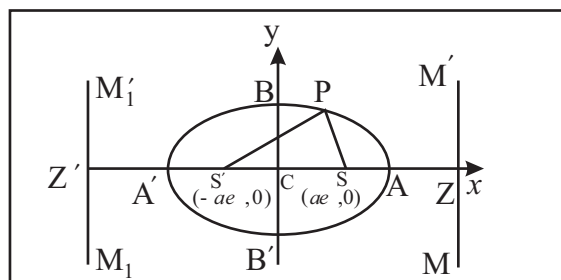
Fig. 2.9

An important property

If P is any point on the ellipse whose foci are S and S', then SP + S′P = 2a where 2a is the length of the major axis.

2.3.3 Centre, vertices, foci, axes and directrices for the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ (a > b)

(i) Centre

We have seen that if (x , y) be a point on the curve, then $(- x , - y)$ is also a point on the curve. Again if $(x, - y)$ is a point on the curve, $(- x , y)$ is also a point on the curve. This shows that every line through C meets the curve at two points equidistant from C. Thus every chord through C is bisected at C. The point C is therefore called the centre of the ellipse. C(0,0) is the middle point of AA′.

(ii) Vertices

The points A and A′ where the line joining the foci S and S' meets the curve are called the vertices of the ellipse. A is the point $(a, 0)$ and A′ is the point $(- a, 0)$.

(iii) Foci

The points S(ae, 0) and S' $(- ae , 0)$ are the foci of the ellipse.

(iv) Axes

The two lines AA′ and BB′ with respect to which the curve is symmetrical are called respectively the major axis and the minor axis of the ellipse.

Since e < 1, $1 - e^2$ is also less than unity.

Therefore, $b^2 = a^2 (1 - e^2)$ is less than a^2 so that $b < a$.

Thus BB′ < AA′. AA′ is called the **major axis** and BB′ the **minor axis** . The segment CA $= a$ is called the semi-major axis and the segment CB $= b$ the semi minor axis.

(v) Directrices

Equation to the directrix MZ in the Fig. 2.9 is $x = \dfrac{a}{e}$

Equation to the directrix M_1'Z' in the Fig. 2.9 is $x = -\dfrac{a}{e}$

Due to symmetry, there are two directrices.

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(vi) $b^2 = a^2 (1 - e^2)$

$$\therefore e = \sqrt{1 - \frac{b^2}{a^2}}$$

Example 10

Find the equation of the ellipse whose eccentricity is $\frac{1}{2}$, one of the foci is (-1, 1) and the corresponding directrix is $x - y + 3 = 0$.

Solution :

Given the focus is S($-$ 1,1), directrix is $x - y + 3 = 0$ and $e = \frac{1}{2}$

Let P(x_1, y_1) be any point on the ellipse. Then $SP^2 = e^2 \, PM^2$ where PM is the perpendicular distance of $x - y + 3 = 0$ from P.

$$(x_1 + 1)^2 + (y_1 - 1)^2 = \frac{1}{4}\left(\frac{x_1 - y_1 + 3}{\sqrt{1+1}}\right)^2$$

$8(x_1 + 1)^2 + 8 \, (y_1 - 1)^2 = (x_1 - y_1 + 3)^2$

$7x_1^2 + 2x_1 y_1 + 7y_1^2 + 10x_1 - 10y_1 + 7 = 0$

Locus of (x_1, y_1), i.e., the equation of the ellipse is

$7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$

Example 11

Find the equation of the ellipse whose foci are (2,0) and ($-$ 2, 0) and eccentricity is $\frac{1}{2}$.

Solution :

We know that S (ae, 0) and S' ($- ae$,0) are the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$

Now the foci are (2, 0) and ($-$ 2, 0) and $e = \frac{1}{2}$

$\Rightarrow ae = 2$ and $e = \frac{1}{2}$

$\Rightarrow a = 4$ or $a^2 = 16$

The centre C is the mid-point of SS' and hence C is (0,0). S and S' are on the x axis.

So the equation of the ellipse is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

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$$b^2 = a^2(1-e^2) = 16\left(1 - \frac{1}{4}\right) = 12$$

∴ the equation of the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

Example 12

Find the eccentricity, foci and latus rectum of the ellipse $9x^2 + 16y^2 = 144$.

Solution :

The equation of the ellipse is

$$9x^2 + 16y^2 = 144 \qquad \therefore \frac{x^2}{16} + \frac{y^2}{9} = 1$$

The given equation is of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Then $a = 4$ and $b = 3$.

$$\therefore \quad e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$$

The foci are S $(ae, 0)$ and S′ $(-ae, 0)$.

or, S $(\sqrt{7}, 0)$ and S′ $(-\sqrt{7}, 0)$

The lactus rectum $= \dfrac{2b^2}{a} = \dfrac{2(3^2)}{4} = \dfrac{9}{2}$

Example 13

Find the centre, eccentricity, foci and directrices of the ellipse $3x^2 + 4y^2 - 6x + 8y - 5 = 0$.

Solution :

The given equation can be written as

$(3x^2 - 6x) + (4y^2 + 8y) = 5$

$3(x^2 - 2x) + 4(y^2 + 2y) = 5$

⇒ $\quad 3(x^2 - 2x + 1 - 1) + 4(y^2 + 2y + 1 - 1) = 5$

⇒ $\quad 3(x-1)^2 + 4(y+1)^2 = 5 + 3 + 4 = 12$

⇒ $\quad \dfrac{(x-1)^2}{4} + \dfrac{(y+1)^2}{3} = 1$

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If we put $X = x - 1$ and $Y = y + 1$ in the above equation

we get, $\dfrac{X^2}{4} + \dfrac{Y^2}{3} = 1$

which can be compared with the standard equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

$b^2 = a^2(1 - e^2)$ gives $3 = 4(1 - e^2)$ or

$e^2 = 1 - \dfrac{3}{4} = \dfrac{1}{4} \Rightarrow e = \dfrac{1}{2}$

Now let us tabulate the results:

	Referred to (X, Y)	**Referred to (x, y)** $x = X + 1, y = Y - 1$
Centre	$(0, 0)$	$(0 + 1, 0 - 1) = (1, -1)$
Foci	$(\pm ae, 0) = (1, 0)$ and $(-1, 0)$	$(2, -1)$ and $(0, -1)$
Directrices	$X = \pm \dfrac{a}{e}$ or $X = \pm 4$	$x - 1 = \pm 4$ or $x = 5$ and $x = -3$

EXERCISES 2.3

1) Find the equation of the ellipse whose

(i) focus is $(1, 2)$ directrix is $2x - 3y + 6 = 0$ and eccentricity is $\dfrac{2}{3}$

(ii) focus is $(0, 0)$ directrix is $3x + 4y - 1 = 0$ and eccentricity is $\dfrac{5}{6}$

(iii) focus is $(1, -2)$ directrix is $3x - 2y + 1 = 0$ and $e = \dfrac{1}{\sqrt{2}}$.

2) Find the equation of the ellipse whose

(i) foci are $(4, 0)$ and $(-4, 0)$ and $e = \dfrac{1}{3}$

(ii) foci are $(3, 0)$ and $(-3, 0)$ and $e = \sqrt{\dfrac{3}{8}}$

(iii) the vertices are $(0, \pm 5)$ and foci are $(0, \pm 4)$.

3) Find the centre, vertices, eccentricity, foci and latus rectum and directrices of the ellipse.

(i) $9x^2 + 4y^2 = 36$

(ii) $7x^2 + 4y^2 - 14x + 40y + 79 = 0$

(iii) $9x^2 + 16y^2 + 36x - 32y - 92 = 0$

2.4 HYPERBOLA

2.4.1 Standard Equation of Hyperbola
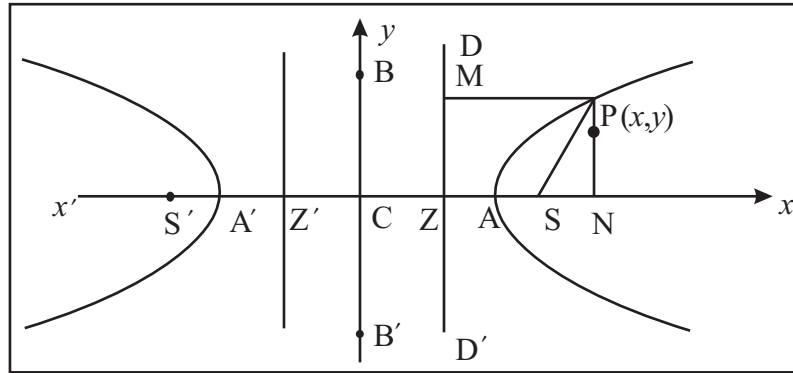
Fig. 2.10

Let S be the focus and DD′ be the directrix. Draw SZ ⊥ DD′. Let A, A′ divide SZ internally and externally respectively in the ratio e:1 where e is the eccentricity; then A, A′ are points on the hyperbola. Take C, the mid-point of AA′ as origin, CZ as *x*-axis and Cy perpendicular to CZ as y-axis.

Let AA′ = 2*a*.   Now $\dfrac{SA}{AZ} = e$, $\dfrac{SA'}{A'Z} = e$.

So   SA $= e\,(AZ)$ ....(1)

and   SA′ $= e\,(A'Z)$ ....(2)

$(1) + (2) \Rightarrow SA + SA' = e\,(AZ + A'Z)$

or, $CS - CA + CS + CA' = e \cdot AA'$

2CS $= e \cdot 2a$

CS $= ae$

$(2) - (1) \Rightarrow SA' - SA = e\,(A'Z - AZ)$

or   AA′ $= e\,(CZ + CA' - CA + CZ)$

2*a* $= e \cdot 2CZ$

So,   CZ $= \dfrac{a}{e}$

Let P (*x, y*) be any point on the hyperbola, and let PM ⊥ DD′ and PN ⊥ CA.

Then $\dfrac{SP}{PM} = e$ or, $SP^2 = e^2 PM^2$

$\Rightarrow \qquad (x - ae)^2 + (y - 0)^2 = e^2 [CN - CZ]^2$

$$= e^2 \left(x - \dfrac{a}{e} \right)^2 = (xe - a)^2$$

$\Rightarrow \qquad x^2 (e^2 - 1) - y^2 = a^2 e^2 - a^2$

$\Rightarrow \qquad x^2 (e^2 - 1) - y^2 = a^2 (e^2 - 1)$

$\Rightarrow \qquad \dfrac{x^2}{a^2} - \dfrac{y^2}{a^2 (e^2 - 1)} = 1$

Put $\qquad b^2 = a^2 (e^2 - 1)$

Hence $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

This is the standard equation of hyperbola.

The line AA′ is the **transverse axis** and the line through C which is perpendicular to AA′ is the **conjugate axis** of the hyperbola.

2.4.2 Tracing of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

(i)   The curve does not pass through the origin. When $y = 0$, $x = \pm a$. Therefore it meets the x axis at the points $(\pm a, 0)$. Thus the points A and A′ in which the curve cuts the x axis are equidistant from the centre.

We have $CA = CA′ = a$ and $AA′ = 2a$.

Similarly when $x = 0$, y becomes imaginary. Therefore the curve does not cut the y-axis.

Let us take two points B and B′ on the y-axis such that $CB = CB′ = b$. Then $BB′ = 2b$.

(ii)   The curve is symmetrical about both x axis and y axis since both the powers of x and y are even. If (x, y) be a point on the curve, so also are the points $(-x, y)$, $(x, -y)$ and $(-x, -y)$.

(iii)   We write the equation of the hyperbola in the form

$$y = \pm \dfrac{b}{a} \sqrt{x^2 - a^2}$$

If $|x| \geq a$, i.e. if either $x \geq a$ or $x \leq -a$, $x^2 - a^2 \geq 0$ and we get two equal and opposite values of y. In this case, as the value of x numerically increases, the corresponding two values of y increase numerically.

The curve therefore consists of two branches each extending to infinity in two directions as shown in the Fig.2.11.

If $|x| < a$ or $-a < x < a$, then $x^2 - a^2$ is a negative quantity. Therefore y imaginary and there is no point of the curve between the lines $x = -a$ and $x = a$. The curve lies to the left of the line $x = -a$ and to the right of $x = a$.

(iv)   Similarly by writing the equation in the form

$$x = \pm \frac{a}{b} \sqrt{y^2 + b^2}$$

We find that y can have any real value without limitation and that for each value of y we get two equal and opposite values of x.

These information about the curve are sufficient to enable us to find the shape of the curve. The curve drawn is therefore as shown in the Fig.2.11.

(v)   **Latus rectum:** Through S, LSL′ be drawn perpendicular to AS. Corresponding to $x = ae$, we have

$$\frac{a^2 e^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ or } y^2 = b^2(e^2 - 1) = b^2 \frac{b^2}{a^2} = \frac{b^4}{a^2}$$

or $y = \pm \dfrac{b^2}{a} \Rightarrow$ SL = SL′ $= \dfrac{b^2}{a}$. So, LL′ $= \dfrac{2b^2}{a}$ is the latus rectum of the hyperbola.

Note that the hyperbola is not a closed curve. The curve consists of two parts detached from each other.
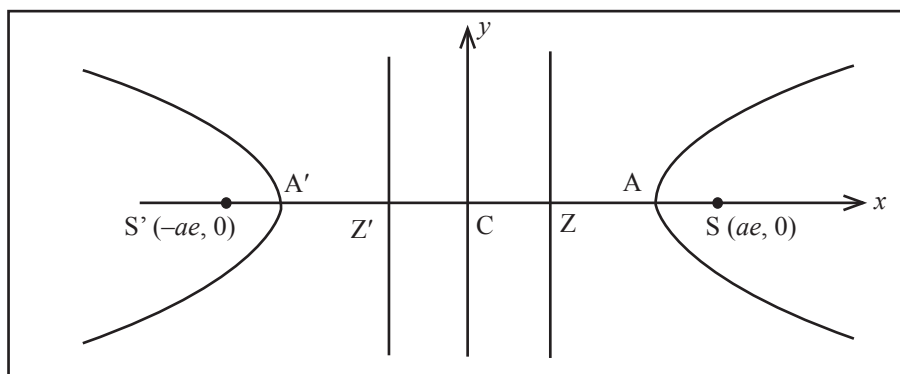
Fig. 2.11

An important property:

The difference between the focal distances of any point on a hyperbola is constant and equal to the length of the transverse axis of the hyperbola. i.e., SP~S′P = 2a.

2.4.3 Asymptote of a curve

A straight line which touches a curve at infinity but does not lie altogether at infinity is called an **asymptote** of that curve.

Note

$ax^2 + bx + c = 0$ has both roots equal to zero if $b = c = 0$ and both roots infinite if $a = b = 0$.

The asymptotes of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

The coordinates of the point of intersection of the line $y = mx + c$ and the hyperbola are given by $\dfrac{x^2}{a^2} - \dfrac{(mx+c)^2}{b^2} = 1$.

$$x^2\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right) - \frac{2mc}{b^2}x - \frac{c^2}{b^2} - 1 = 0$$

$$x^2(b^2 - a^2 m^2) - 2ma^2 cx - a^2 c^2 - a^2 b^2 = 0$$

Now if $y = mx + c$ is an asymptote, both the roots of this equation are infinite.

∴ Coefficient of $x = 0$ and coefficient of $x^2 = 0$.

$$\Rightarrow \quad -2ma^2 c = 0 \text{ and } b^2 - a^2 m^2 = 0. \quad \therefore c = 0, \ m = \pm\frac{b}{a}$$

Hency there are two asymptotes, namely

$$y = \frac{b}{a}x \qquad \text{and} \qquad y = -\frac{b}{a}x$$

$$or, \quad \frac{x}{a} - \frac{y}{b} = 0 \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} = 0$$

Their combined equation is

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 0 \ \text{ or, } \ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

Note

(i)   The asymptotes evidently pass through the centre $C(0,0)$ of the hyperbola (Fig 2.12).

(ii)   The slopes of the asymptotes are $\dfrac{b}{a}$ and $-\dfrac{b}{a}$. Hence the asymptotes are equally inclined to the transverse axis. That is, the transverse and conjugate axes bisect the angles between the asymptotes (Fig 2.12).

(iii)   If 2α is the angle between the asymptotes then $\tan \alpha = \dfrac{b}{a}$

∴ Angle between the asymptotes $= 2 \tan^{-1}\left(\dfrac{b}{a}\right)$

(iv)   The combined equation of the asymptotes differs from that of the hyperbola by a constant only.
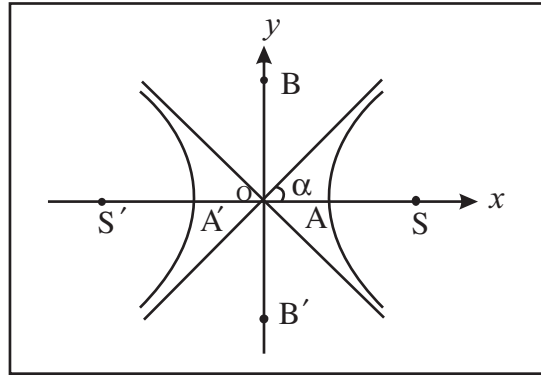
Fig.2.12

Example 14

Find the equation of the hyperbola in standard form whose eccentricity is $\sqrt{2}$ and the distance between the foci is 16

Solution :

Given $e = \sqrt{2}$

Let   S and S′ be the foci, then S′S = 16

But   S′S = $2ae$ $\therefore$ $2ae = 16$

Thus we have $(2a)(\sqrt{2}) = 16$, $\Rightarrow a = 4\sqrt{2}$

Also $b^2 = a^2 (e^2 - 1)$

$$= (4\sqrt{2})^2 (2 - 1) = 32$$

The equation of the hyperbola is

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{32} - \frac{y^2}{32} = 1$$
$$\Rightarrow x^2 - y^2 = 32$$

2.4.4 Rectangular Hyperbola (R.H.)

A hyperbola is said to be rectangular if the asymptotes are at right angles.

If 2α is the angle between the asymptotes, then $\tan \alpha = \dfrac{b}{a}$

If the hyperbola is R.H., then $2\alpha = 90^\circ$

$\therefore$ $\alpha = 45^\circ$ $\Rightarrow$ a = b.

$\therefore$ Equation of the rectangular hyperbola is $x^2 - y^2 = a^2$.

$\Rightarrow$ A hyperbola is also said to be rectangular when its transverse and conjugate axes are equal in length. i.e. $a = b$

$\therefore$ $b^2 = a^2 (e^2 - 1)$ gives $a^2 = a^2 (e^2 - 1)$. or $e = \sqrt{2}$

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2.4.5 Standard equation of rectangular hyperbola

Let the asymptotes of a rectangular hyperbola be taken as the coordinate axes.

The equations of the asymptotes are $x = 0$ and $y = 0$. Their combined equation is $xy = 0$.

Since the equation of the hyperbola differs from the equation of asymptotes by a constant, the equation of the hyperbola is

$$xy = k \qquad \text{------------- (1)}$$

where k is a consant.

Let the transverse axis, $AA' = 2a$.

Draw AM perpendicular to one asymptote, the x-axis.

$\angle ACM = 45^{\text{o}}$ where C is the centre

So, $CM = CA \cos 45^{\text{o}} = \dfrac{a}{\sqrt{2}}$

and $MA = CA \sin 45^{\text{o}} = \dfrac{a}{\sqrt{2}}$

$\therefore$ The coordinates of A are $\left(\dfrac{a}{\sqrt{2}}, \dfrac{a}{\sqrt{2}} \right)$

It lies on the rectangular hyperbola. Therefore,

$$k = \dfrac{a}{\sqrt{2}} \dfrac{a}{\sqrt{2}}$$

or, $k = \dfrac{a^2}{2}$

The equation for the rectangular hyperbola is

$$xy = \dfrac{a^2}{2}$$

Let $c^2 = \dfrac{a^2}{2}$

Hence $xy = c^2$

Fig.2.13

This is the standard form of the equation of the rectangular hyperbola.
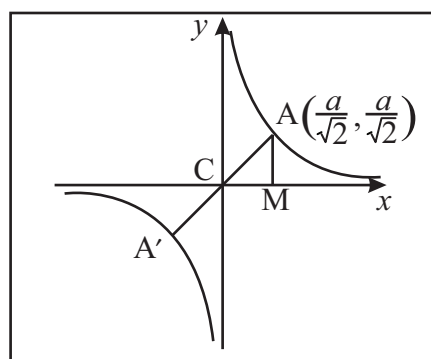
Example 15

Find the equation of the hyperbola whose eccentricity is $\sqrt{3}$, focus is (1, 2) and the corresponding directrix is $2x + y = 1.$

Solution :

Focus S is (1, 2), directrix is $2x + y = 1$ and $e = \sqrt{3}$.

If P (x_1, y_1) is any point on the hyperbola, then $SP^2 = e^2PM^2$, where PM is the perpendicular to the directrix $2x + y = 1$.

$$(x_1 - 1)^2 + (y_1 - 2)^2 = 3\frac{(2x_1 + y_1 - 1)^2}{5}$$

$$5(x_1^2 - 2x_1 + 1 + y_1^2 - 4y_1 + 4) = 3(2x_1 + y_1 - 1)^2$$

$$7x_1^2 + 12x_1y_1 - 2y_1^2 - 2x_1 + 14y_1 - 22 = 0$$

∴   Locus of (x_1, y_1) is

$$7x^2 + 12xy - 2y^2 - 2x + 14y - 22 = 0$$

Example 16

Find the equations of the asymptotes of the hyperbola $2x^2+5xy+2y^2-11x-7y-4=0.$

Solution :

The combined equation of the asymptotes differs from the equation of the hyperbola only by a constant.

So, the combined equation of the asymptotes is

$$2x^2 + 5xy + 2y^2 - 11x - 7y + k = 0 \qquad \text{............... (1)}$$

which is a pair of straight lines staisfying the condition

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \qquad \text{............... (2)}$$

In the given equation

$$a = 2, h = \frac{5}{2}, b = 2, f = \frac{-7}{2}, g = -\frac{11}{2}, c = k$$

Substituting in (2) we get $k = 5$.

So the combined equation of the asymptotes is

$$2x^2 + 5xy + 2y^2 - 11x - 7y + 5 = 0$$

⟹ $(2x^2 + 5xy + 2y^2) - 11x - 7y + 5 = 0$

⟹ $(2x + y) (x + 2y) - 11x - 7y + 5 = 0$

⟹ $(2x + y + l) (x + 2y + m) = 0$

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$\Rightarrow \quad l + 2m = -11$ (comparing the coefficients of x)

and $\quad 2l + m = -7$ (comparing the coefficients of y)

$\Rightarrow \quad l = -1, m = -5$

The equations of the asymptotes are

$\quad 2x + y - 1 = 0$ and $x + 2y - 5 = 0$.

Example 17

Find the centre, eccentricity, foci and latus rectum of the hyperbola $9x^2 - 16y^2 - 18x - 64y - 199 = 0$.

Solution :

The given equation can be written as

$$9(x^2 - 2x) - 16(y^2 + 4y) = 199$$

Completing squares in x and y, we get

$$9(x-1)^2 - 16(y+2)^2 = 199 + 9 - 64$$

$$= 144$$

$$\frac{(x-1)^2}{16} - \frac{(y+2)^2}{9} = 1$$

To bring this to the standard form, set $X = x - 1$ and $Y = y + 2$; we get,

$$\frac{X^2}{16} - \frac{Y^2}{9} = 1$$

$$b^2 = a^2(e^2 - 1)$$

$$\Rightarrow e^2 = \frac{9}{16} + 1 = \frac{25}{16} \text{ or, } e = \frac{5}{4}$$

Now we can tabulate the results:

	Referred to (X, Y)	Referred to (x, y) $x = X + 1, y = Y - 2$
Centre	(0, 0)	(0 + 1, 0 − 2) = (1, − 2)
Foci	(± ae, 0) $= (5, 0)$ and $(-5, 0)$	(5 + 1, 0 − 2) and (− 5 + 1 , (0 − 2) (6, − 2) and (− 4, − 2)
Latus rectum	$= \dfrac{2b^2}{a} = \dfrac{2 \times 9}{4} = \dfrac{9}{2}$	$\dfrac{9}{2}$

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Example 18

Find the equation to the hyperbola which has the line $x + 4y - 5 = 0$ and $2x - 3y + 1 = 0$ for its asymptotes and which passes through the point (1, 2).

Solution :

The combined equation of the asymptotes is

$$(x + 4y - 5) (2x - 3y + 1) = 0$$

The equation of the hyperbola differs from this combined equation of asymptotes only by a constant. Thus the hyperbola is

$$(x + 4y - 5) (2x - 3y + 1) = k, \text{ where } k \text{ is a constant}$$

Since the hyperbola passes through (1, 2)

$$[1 + 4(2) - 5] [2(1) - 3(2) + 1] = k \quad \text{i.e. } k = -12$$

$\Rightarrow$ The equation of the hyperbola is $(x + 4y - 5) (2x - 3y + 1) = -12$

or $\quad 2x^2 + 5xy - 12y^2 - 9x + 19y + 7 = 0$

Example 19

The cost of production of a commodity is Rs.12 less per unit at a place A than it is at a place B and distance between A and B is 100km. Assuming that the route of delivery of the commodity is along a striaght line and that the delivery cost is 20 paise per unit per km, find the curve, at any point of which the commodity can be supplied from either A or B at the same total cost.

Solution:

Choose the midpoint of AB as the origin O(0,0).

Let P be a point on the required curve so that the commodity supplied from either A or B at the same total cost.

Let the cost per unit at $B = C$

$\therefore$ the cost per unit at $A = C - 12$

Delivery cost per unit from A to $P = \dfrac{20}{100}$ AP
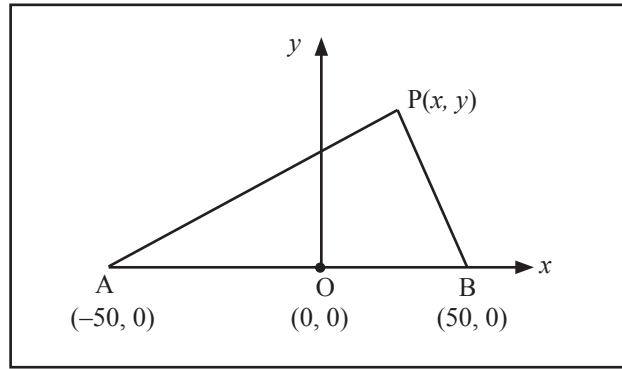
Fig.2.14

Delivery cost per unit from B to P $= \dfrac{20}{100}$ BP

Total cost is same whether the commodity is delivered from either A or B.

$\therefore$ $(C-12) + \dfrac{20}{100}(AP) = C + \dfrac{20}{100}(BP)$

$\therefore$ $\dfrac{AP}{5} - \dfrac{BP}{5} = 12$ i.e. $AP - BP = 60$

$\sqrt{(x+50)^2 + y^2} - \sqrt{(x-50)^2 + y^2} = 60$

$\sqrt{x^2 + y^2 + 100x + 2500} - \sqrt{x^2 + y^2 - 100x + 2500} = 60$

Simplifying we get,

$6400x^2 - 3600y^2 = 5760000$

$\therefore$ $16x^2 - 9y^2 = 14400$

$\dfrac{x^2}{900} - \dfrac{y^2}{1600} = 1$ $\therefore \dfrac{x^2}{(30)^2} - \dfrac{y^2}{(40)^2} = 1$

Thus we get the required curve as hyperbola.

Example 20

A machine sells at Rs.p and the demand, x (in hundreds) machines per year is given by $x = \dfrac{90}{p+5} - 6$. What type of demand curve corresponds to the above demand's law? At what price does the demand tend to vanish?

Solution:

The demand curve is

$x + 6 = \dfrac{90}{p+5}$ $\Rightarrow (x+6)\,(p+5) = 90$

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$\Rightarrow$ XP = 90 where X = x + 6, P = p + 5

$\Rightarrow$ The demand curve is a rectangular hyperbola.

Demand = 0 $\Rightarrow$ 6 (p + 5) = 90

$\Rightarrow$ p = Rs.10.

EXERCISE 2.4

1)   Find the equation of the hyperbola with

(i) focus (2, 2), eccentricity $\dfrac{3}{2}$ and directrix $3x - 4y = 1$.

(ii) focus (0, 0), eccentricity $\dfrac{5}{4}$ and directirx $x \cos \alpha + y \sin \alpha = p$.

2)   Find the equation of the hyperbola whose foci are (6, 4) and (-4, 4) and eccentricity 2.

3)   Find the equation of the hyperbola whose

(i) centre is (1, 0), one focus is (6, 0) and transverse axis 6.

(ii) centre is (3, 2), one focus is (5, 2) and one vertex is (4, 2).

(iii) centre is (6, 2), one focus is (4, 2) and e = 2.

4)   Find the centre, eccentricity, foci and directrices for the following hyperbolas

(i) $9x^2 - 16y^2 = 144$ (ii) $\dfrac{(x+2)^2}{9} - \dfrac{(y+4)^2}{7} = 1$ (iii) $12x^2 - 4y^2 - 24x + 32y - 127 = 0$

5)   Find the equation to the asymptotes of the hyperbola

(i) $3x^2 - 5xy - 2y^2 + 17x + y + 14 = 0$ (ii) $8x^2 + 10xy - 3y^2 - 2x + 4y - 2 = 0$

6)   Find the equation to the hyperbola which passes through (2,3) and has for its asymptotes the lines $4x + 3y - 7 = 0$ and $x - 2y = 1$.

7)   Find the equation to the hyperbola which has $3x - 4y + 7 = 0$ and $4x + 3y + 1 = 0$ for asymptotes and which passes through the origin.

EXERCISE 2.5

Choose the correct answer

1) The eccentricity of a parabola is

   (a) 1   (b) 0   (c)2   (d) -1

2) The eccentricity of a conic is $\dfrac{1}{\sqrt{2}}$. The conic is

   (a) a parabola   (b) an ellipse   (c) a circle   (d) a hyperbola

3) Latus rectum of $y^2 = 4ax$ is

   (a) $2a$ (b) $3a$ (c) $4a$ (d) a

4) Focus of $y^2 = -4ax$ is

   (a) $(a, 0)$ (b) $(0, a)$ (c) $(0, -a)$ (d) $(-a, 0)$

5) Equation of the directrix of $x^2 = 4ay$ is

   (a) $x + a = 0$ (b) $x - a = 0$ (c) $y + a = 0$ (d) $y - a = 0$

6) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ represents an ellipse $(a > b)$ if

   (a) $b^2 = a^2(1 - e^2)$ (b) $b^2 = -a^2(1 - e^2)$

   (c) $b^2 = \dfrac{a^2}{1 - e^2}$ (d) $b^2 = \dfrac{1 - e^2}{a^2}$

7) Latus rectum of an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $(a > b)$ is

   (a) $\dfrac{2a^2}{b}$ (b) $\dfrac{a^2}{2b}$ (c) $\dfrac{2b^2}{a}$ (d) $\dfrac{b^2}{2a}$

8) Focus of $y^2 = 16x$ is

   (a) $(2,0)$ (b) $(4,0)$ (c) $(8,0)$ (d) $(2,4)$

9) Equation of the directrix of $y^2 = -8x$ is

   (a) $x + 2 = 0$ (b) $x - 2 = 0$ (c) $y + 2 = 0$ (d) $y - 2 = 0$

10) The length of the latus rectum of $3x^2 + 8y = 0$, is

   (a) $\dfrac{8}{3}$ (b) $\dfrac{2}{3}$ (c) 8   (d) $\dfrac{3}{8}$

11) The parabola $x^2 + 16y = 0$ is completely

   (a) above x-axis   (b) below x-axis   (c) left of y-axis   (d) right of y-axis

12) The semi major and semi minor axes of $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$ is

    (a) (4, 5)   (b) (8, 10)   (c) (5, 4)   (d) (10, 8)

13) The length of latus rectum of $4x^2 + 9y^2 = 36$ is

    (a) $\dfrac{4}{3}$ (b) $\dfrac{8}{3}$ (c) $\dfrac{4}{9}$ (d) $\dfrac{8}{9}$

14) In an ellipse $e = \dfrac{3}{5}$, the length of semi minor axis is 2. The length of major axis is

    (a) 4   (b) 5   (c) 8   (d) 10

15) Eccentricity of the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$ is

    (a) $\dfrac{3}{2}$ (b) $\dfrac{9}{4}$ (c) $\dfrac{5}{4}$ (d) 4

16) The sum of focal distances of any point on the ellipse is equal to length of its

    (a) minor axis   (b) semi minor axis   (c) major axis   (d) semi major axis

17) The difference between the focal distances of any point on the hyperbola is equal to length of its

    (a) transverse axis   (b) semi transverse axis

    (c) conjugate axis   (d) semi conjugate axis.

18) Asymptotes of a hyperbola pass through

    (a) one of the foci   (b) one of the vertices

    (c) the centre of the hyperbola   (d) one end of its latus rectum.

19) Eccentricity of the rectangular hyperbola is

    (a) 2   (b) $\dfrac{1}{2}$ (c) $\sqrt{2}$ (d) $\dfrac{1}{\sqrt{2}}$

20) If a is the length of the semi transverse axis of rectangular hyperbola $xy = c^2$ then the value of c^2 is

    (a) a^2 (b) $2a^2$ (c) $\dfrac{a^2}{2}$ (d) $\dfrac{a^2}{4}$

APPLICATIONS OF DIFFERENTIATION - I   3

Differentiation plays a vital role in Economics and Commerce. Before we embark on demonstrating applications of differentiation in these fields, we introduce a few Economic terminologies with usual notations.

3.1 FUNCTIONS IN ECONOMICS AND COMMERCE

3.1.1 Demand Function

Let q be the demand (quantity) of a commodity and p the price of that commodity. The demand function is defined as $q = f(p)$ where p and q are positive. Generally, p and q are inversely related.

Observe the graph of the demand function $q = f(p)$
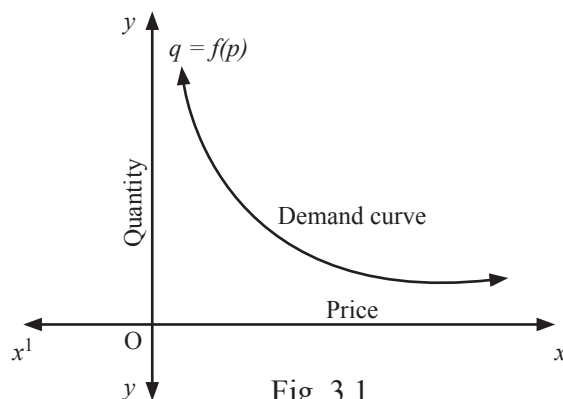
Fig. 3.1

Following observations can be made from the graph (Fig 3.1)

(i) only the first quadrant portion of the graph of the demand function is shown since p and q are positive.

(ii) slope of the demand curve is negative.

3.1.2 Supply Function

Let x denotes amount of a particular commodity that sellers offer in the market at various price p, then the supply function is given by $x = f(p)$ where x and p are positive.

Generally x and p are directly related

Observe the graph of the supply function, $x = f(p)$
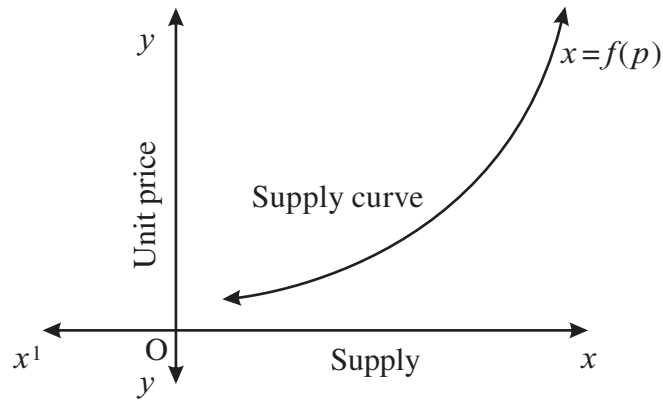
Fig. 3.2

Following observations can be made from the graph (Fig 3.2)

(i) only the first quadrant portion of the graph of the supply function is shown since the function has meaning only for nonnegative values of q and p.

(ii) slope of the supply function is positive.

3.1.3 Cost Function

Normally total cost consists of two parts.

(i) Variable cost and (ii) fixed cost. Variable cost is a single - valued function of output, but fixed cost is independent of the level of output.

Let $f(x)$ be the variable cost and k be the fixed cost when the output is x units. The total cost function is defined as $C(x) = f(x) + k$, where x is positive.

Note that $f(x)$ does not contain constant term.

We define Average Cost (AC), Average Variable Cost (AVC), Average Fixed Cost (AFC), Marginal Cost (MC), and Marginal Average Cost (MAC) as follows.

(i)   Average Cost $(AC) = \dfrac{f(x)+k}{x} = \dfrac{\text{Total Cost}}{\text{Output}}$

(ii)   Average Variable Cost $(AVC) = \dfrac{f(x)}{x} = \dfrac{\text{Variable Cost}}{\text{Output}}$

(iii)   Average Fixed Cost $(AFC) = \dfrac{k}{x} = \dfrac{\text{Fixed Cost}}{\text{Output}}$

(iv)   Marginal Cost $(MC) = \dfrac{d}{dx} C(x) = C'(x)$

(v)   Marginal Average Cost $(MAC) = \dfrac{d}{dx} (AC)$

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Note

If C(x) is the total cost of producing x units of some product then its derivative C′(x) is the marginal cost which is the approximate cost of producing 1 more unit when the production level is x units. The graphical representation is shown here (Fig 3.3).
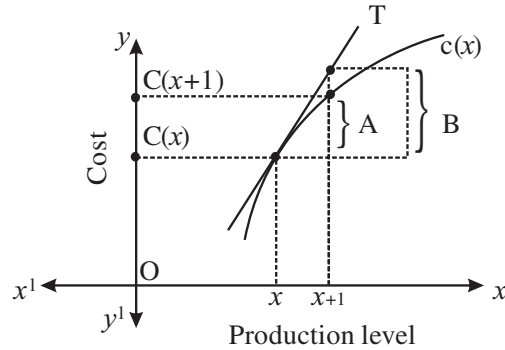
Fig. 3.3

$$A = C(x + 1) - C(x)$$
$$B = C'(x) = \text{Marginal Cost}$$

3.1.4 Revenue Function

Let x units be sold at Rs. p per unit. Then the total revenue R(x) is defined as R(x) = px, where p and x are positive.

Average revenue (AR) = $\dfrac{\text{Total revenue}}{\text{quantity sold}} = \dfrac{px}{x} = p$

(i.e Average revenue and price are the same)

Marginal revenue (MR) = $\dfrac{d}{dx}$ (R) = R′(x)

Note

If R(x) be the total revenue gained from selling x units of some product, then its derivative, R′(x) is the marginal revenue, which is approximate revenue gained from selling 1 more unit when the sales level is x units. The graphical representaion is shown here (Fig 3.4)
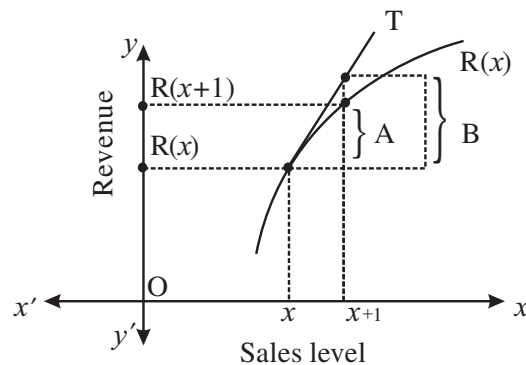
Fig. 3.4

$$A = R(x+1) - R(x)$$
$$B = R'(x) = \text{Marginal Revenue}$$

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3.1.5 Profit Function

The profit function $P(x)$ is defined as the difference between the total revenue and the total cost. i.e. $P(x) = R(x) - C(x)$.

3.1.6. Elasticity

The elasticity of a function $y = f(x)$, with respect to x, is defined as

$$\eta = \frac{Ey}{Ex} = \underset{\Delta x \to 0}{Lt} \frac{\dfrac{\Delta y}{y}}{\dfrac{\Delta x}{x}}$$

$$= \frac{x}{y} \frac{dy}{dx}$$

Thus the elasticity of y with respect to x is the limit of the ratio of the relative increment in y to the relative increment in x, as the increment in x tends to zero. The elasticity is a pure number, independent of the units in x and y.

3.1.7 Elasticity of Demand

Let $q = f(p)$ be the demand function, where q is the demand and p is the price. Then the elasticity of demand is $\eta_d = \dfrac{p}{q} \dfrac{dq}{dp}$ (Fig.3.5)
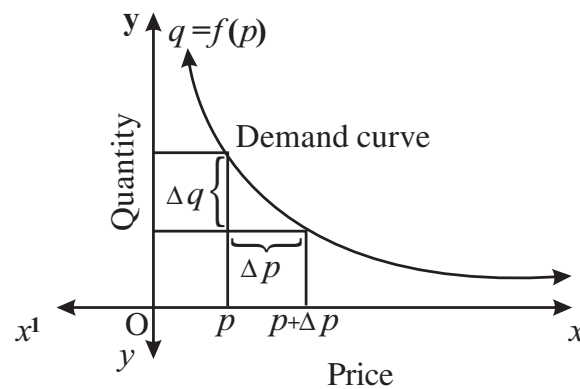
Fig. 3.5

$$\text{Elasticity of demand} = \underset{\Delta p \to 0}{Lt} \frac{\dfrac{\Delta q}{q}}{\dfrac{\Delta p}{p}} = \frac{p}{q} \frac{dq}{dp}$$

Since the slope of the demand curve is negative and elasticity is a positive quantity the elasticity of demand is given by

$$\eta_d = -\frac{p}{q} \frac{dq}{dp}$$

3.1.8. Elasticity of Supply

Let $x = f(p)$ be the supply function, where x is the supply and p is the price. The elasticity of supply is defined as

$$\eta_s = \frac{p}{x} \frac{dx}{dp}$$

3.1.9 Equilibrium Price

The price at which quantity demanded is equal to quantity supplied is called equilibrium price.

3.1.10 Equilibrium Quantity

The quantity obtained by substituting the value of equilibrium price in any one of the given demand or supply functions is called equilibrium quantity.

3.1.11 Relation between Marginal Revenue and Elasticity of Demand

Let q units be demanded at unit price p so that $p = f(q)$ where f is differentiable. The revenue is given by

$$R(q) = qp$$

$$R(q) = q f(q) \qquad\qquad\qquad [p = f(q)]$$

Marginal revenue is obtained by differentiating R(q) with respect to q.

$$\therefore R'(q) = q\, f'(q) + f(q)$$

$$= q\frac{dp}{dq} + p \qquad\qquad \left[here\ \frac{dp}{dq} = f'(q) \right]$$

$$R'(q) = p\left(1 + \frac{q}{p}\frac{dp}{dq}\right)$$

$$= p\left[1 + \frac{1}{\dfrac{p}{q}\dfrac{dq}{dp}} \right]$$

$$= p\left[1 + \left\{ \frac{-1}{-\dfrac{p}{q}\dfrac{dq}{dp}} \right\} \right]$$

Since $\eta_d = -\dfrac{p}{q}\dfrac{dq}{dp}$,

$$\text{Marginal Revenue} = R^1(q) = p\left[1 - \frac{1}{\eta_d} \right]$$

Example 1

A firm produces x tonnes of output at a total cost $C(x) = \dfrac{1}{10}x^3 - 4x^2 + 20x + 5$.

Find   **(i) Average cost**   **(ii) Average Variable Cost**   **(iii) Average Fixed Cost**

   (iv) Marginal Cost and   **(v) Marginal Average Cost.**

Solution :

$$C(x) = \frac{1}{10}x^3 - 4x^2 + 20x + 5$$

(i)   Average Cost (AC) $= \dfrac{\text{Total Cost}}{\text{Output}}$

$$= \left(\frac{1}{10}x^2 - 4x + 20 + \frac{5}{x} \right)$$

(ii)   Average Variable Cost (AVC) $= \dfrac{\text{Variable Cost}}{\text{Output}}$

$$= \frac{1}{10}x^2 - 4x + 20$$

(iii)   Average Fixed Cost (AFC) $= \dfrac{\text{Fixed Cost}}{\text{Output}} = \dfrac{5}{x}$

(iv)   Marginal Cost (MC) $= \dfrac{d}{dx}\, C(x)$

$$= \frac{d}{dx}\left(\frac{1}{10}x^3 - 4x^2 + 20x + 5 \right)$$

$$= \left(\frac{3}{10}x^2 - 8x + 20 \right)$$

(v)   Marginal Average Cost (MAC) $= \dfrac{d}{dx}\, (AC)$

$$= \frac{d}{dx}\left(\frac{1}{10}x^2 - 4x + 20 + \frac{5}{x} \right)$$

$$= \left(\frac{1}{5}x - 4 - \frac{5}{x^2} \right)$$

Example 2

The total cost C of making x units of product is $C = 0.00005x^3 - 0.06x^2 + 10x + 20,000$. Find the marginal cost at 1000 units of output.

Solution :

$$C = 0.00005x^3 - 0.06x^2 + 10x + 20,000$$

$$\text{Marginal Cost } \frac{dC}{dx} = (0.00005)(3x^2) - (0.06)\,2x + 10$$

$$= 0.00015\,x^2 - 0.12x + 10$$

$$\text{when } x = 1000$$

$$\frac{dC}{dx} = (0.00015)(1000)^2 - (0.12)(1000) + 10$$

$$= 150 - 120 + 10 = 40$$

At $x = 1000$ units, Marginal Cost is Rs. 40

Example 3

Find the elasticity of demand for the functions $x = 100 - p - p^2$ when $p = 5$.

Solution :

$$x = 100 - p - p^2$$

$$\frac{dx}{dp} = -1 - 2p.$$

$$\text{Elasticity of demand } \eta_d = -\frac{p}{x}\frac{dx}{dp}$$

$$= -\frac{p(-1-2p)}{100 - p - p^2} = \frac{p + 2p^2}{100 - p - p^2}$$

$$\text{When } p = 5, \ \eta_d = \frac{5 + 50}{100 - 5 - 25}$$

$$= \frac{55}{70} = \frac{11}{14}$$

Example 4

Find the elasticity of supply for the supply function $x = 2p^2 + 8p + 10$.

Solution :

$$x = 2p^2 + 8p + 10$$

$$\frac{dx}{dp} = 4p + 8$$

Elasticity of supply $\eta_s = \dfrac{p}{x} \dfrac{dx}{dp}$

$$= \dfrac{4p^2 + 8p}{2p^2 + 8p + 10} = \dfrac{2p^2 + 4p}{p^2 + 4p + 5}$$

Example 5

For the function $y = 4x - 8$ find the elasticity and also obtain the value when $x = 6$.

Solution :

$y = 4x - 8$

$\dfrac{dy}{dx} = 4$

Elasticity $\eta = \dfrac{x}{y} \dfrac{dy}{dx}$

$$\eta = \dfrac{x}{4x - 8}(4) = \dfrac{x}{x - 2}$$

when $x = 6$, $\eta = \dfrac{6}{6 - 2} = \dfrac{3}{2}$

Example 6

If $y = \dfrac{1-2x}{2+3x}$ find $\dfrac{Ey}{Ex}$. Obtain the values of η when $x = 0$ and $x = 2$.

Solution :

We have $y = \dfrac{1 - 2x}{2 + 3x}$

Differentiating with respect to x, we get

$\dfrac{dy}{dx} = \dfrac{(2 + 3x)(-2) - (1 - 2x)(3)}{(2 + 3x)^2}$

$$= \dfrac{-4 - 6x - 3 + 6x}{(2 + 3x)^2} = \dfrac{-7}{(2 + 3x)^2}$$

$\eta = \dfrac{Ey}{Ex} = \dfrac{x}{y} \dfrac{dy}{dx}$

$$= \dfrac{x(2 + 3x)}{(1 - 2x)} \times \dfrac{-7}{(2 + 3x)^2}$$

$\eta = \dfrac{-7x}{(1 - 2x)(2 + 3x)}$

when $x = 0$, $\eta = 0$

when $x = 2$, $\eta = \dfrac{7}{12}$

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Example 7

A demand function is given by $xp^n = k$, where n and k are constants. Calculate price elasticity of demand.

Solution :

Given $xp^n = k$

$\Rightarrow \quad x = k\,p^{-n}$

$\dfrac{dx}{dp} = -nk\ p^{-n-1}$

Elasticity of demand $\eta_d = -\dfrac{p}{x}\dfrac{dx}{dp}$

$$= -\dfrac{p}{kp^{-n}}(-nk\ p^{-n-1})$$

$= n,$ which is a constant.

Example 8

Show that the elasticity of demand at all points on the curve $xy^2 = c$ (c is constant), where y represents price will be numerically equal to 2.

Solution :

We have $\qquad xy^2 = c$

$$x = \dfrac{c}{y^2}$$

Differentiating with respect to y,

$\dfrac{dx}{dy} = -\dfrac{2c}{y^3}$

Elasticity of demand $\eta_d = -\dfrac{y}{x}\dfrac{dx}{dy} = \dfrac{-y}{\dfrac{c}{y^2}}\left(\dfrac{-2c}{y^3}\right) = 2$

Example 9

The demand curve for a monopolist is given by $x = 100 - 4p$

(i) Find the total revenue, average revenue and marginal revenue.

(ii) At what value of x, the marginal revenue is equal to zero?

Solution :

We have $\qquad x = 100 - 4p$

$$p = \dfrac{100 - x}{4}$$

Total revenue $R = px$

$$= \left(\frac{100 - x}{4}\right) x = \frac{100x - x^2}{4}$$

Average revenue $= p = \dfrac{100 - x}{4}$

Marginal revenue $= \dfrac{d}{dx}(R)$

$$= \frac{d}{dx}\left(\frac{100x - x^2}{4}\right)$$

$$= \frac{1}{4}[100 - 2x] = \frac{50 - x}{2}$$

(ii)   Marginal revenue is zero implies

$$\frac{50 - x}{2} = 0 \Rightarrow x = 50$$

∴   Marginal revenue is zero when $x = 50$.

Example 10

If AR and MR denote the average and marginal revenue at any output level, show that elasticity of demand is equal to $\dfrac{AR}{AR - MR}$. Verify this for the linear demand law $p = a + bx$, where p is price and x is the quantity.

Solution :

Total revenue $R = px$

Average Revenue $AR = p$

Marginal Revenue $MR = \dfrac{d}{dx}(R) = \dfrac{d}{dx}(px)$

$$= p + x\frac{dp}{dx}$$

Now,

$$\frac{AR}{(AR - MR)} = \frac{p}{p - \left(p + x\dfrac{dp}{dx}\right)}$$

$$= -\frac{p}{x}\frac{dx}{dp}$$

$$= \text{Elasticity of demand } \eta_d$$

$$\therefore \frac{AR}{(AR - MR)} = \eta_d$$

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Given $p = a + bx$

Differentiating with respect to x,

$$\frac{dp}{dx} = b$$

$$R = px = ax + bx^2$$

$$AR = a + bx \qquad (AR = price)$$

$$MR = \frac{d}{dx}(ax + bx^2)$$

$$= a + 2bx$$

$$\therefore \frac{AR}{(AR - MR)} = \frac{a + bx}{a + bx - a - 2bx} = -\frac{(a + bx)}{bx} \qquad ....(1)$$

$$\eta_d = -\frac{p}{x}\frac{dx}{dp}$$

$$= -\frac{(a + bx)}{x}\frac{1}{b} = -\frac{(a + bx)}{bx} \qquad ......(2)$$

from (1) and (2) we get that $\dfrac{AR}{(AR - MR)} = \eta_d$.

Example 11

Find the equilibrium price and equilibrium quantity for the following demand and supply functions, $Q_d = 4 - 0.06p$ and $Q_s = 0.6 + 0.11p$.

Solution :

At the equilibrium price

$$Q_d = Q_s$$

$$\Rightarrow 4 - 0.06p = 0.6 + 0.11p$$

$$\Rightarrow \quad 0.17p = 3.4$$

$$\Rightarrow \quad p = \frac{3.4}{0.17}$$

$$p = 20$$

when $p = 20$, $Q_d = 4 - (0.06)(20)$

$$= 4 - 1.2 = 2.8$$

$\therefore$ Equilibrium price = 20 and Equilibrium quantity = 2.8

Example 12

The demand for a given commodity is given by $q = \dfrac{p}{p-5}$ (p > 5), where p is the unit price. Find the elasticity of demand when p = 7. Interpret the result.

Solution :

Demand function $q = \dfrac{p}{p-5}$

Differentiating with respect to p, we get

$$\frac{dq}{dp} = \frac{(p-5)(1) - p(1)}{(p-5)^2} = \frac{-5}{(p-5)^2}$$

Elasticity of demand $\eta_d = -\dfrac{p}{q}\dfrac{dq}{dp} = \dfrac{-p(p-5)}{p}\left\{-\dfrac{5}{(p-5)^2}\right\}$

$$= \frac{5}{p-5}$$

when $p = 7$, $\qquad \eta_d = \dfrac{5}{7-5} = 2.5$

This means that if the price increases by 1% when p = 7, the quantity demanded will decrease by approximately 2.5%. Also if the price decreases by 1% when p = 7, the quantity demanded will increase by approximately 2.5%.

Example 13

The demand for a given commodity is $q = -60p + 480$, (0 < p < 7) where p is the price. Find the elasticity of demand and marginal revenue when p = 6.

Solution :

Demand function $q = -60p + 480$

Differentiating with respect to p, we get

$$\frac{dq}{dp} = -60$$

Elasticity of demand $\eta_d = -\dfrac{p}{q}\dfrac{dq}{dp} = \dfrac{-p}{-60p + 480}(-60) = -\dfrac{p}{p-8}$

when $p = 6$, $\qquad \eta_d = \dfrac{-6}{6-8} = 3$

Marginal revenue $= p\left(1 - \dfrac{1}{\eta_d}\right) = 6\left(1 - \dfrac{1}{3}\right) = 4$

$\therefore$ Marginal revenue = Rs. 4

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EXERCISE 3.1

1) A firm produces x tonnes of output at a total cost C(x) =Rs. ($\frac{1}{2}x^3 - 4x^2 + 25x + 8$). Find (i) Average Cost (ii) Average Variable Cost and (iii) Average Fixed Cost. Also find the value of each of the above when the output level is 10 tonnes.

2) The total cost C of making x units of product is $C(x) = 25 + 3x^2 + \sqrt{x}$. Find the marginal cost at output level of 100 units.

3) The total cost of making x units is given by $C(x) = 50 + 5x + 2\sqrt{x}$. What is the marginal cost at 100 units of output?

4) If the cost of making x units is C = $\frac{1}{2}x + 26\sqrt{x+4}$. Find the marginal cost at output of 96 units.

5) If the total cost C of making x tonnes of a product is $C = 10 + 30\sqrt{x}$. Find the marginal cost at 100 tonnes output and find the level of output at which the marginal cost is Rs. 0.40 per ton.

6) The cost function for the production of x units of an item is given by C = $\frac{1}{10}x^3 - 4x^2 + 8x + 4$. Find (i) the average cost (ii) the marginal cost and (iii) the marginal average cost.

7) If the total cost C of making x units is $C = 50 + 10x + 5x^2$. Find the average cost and marginal cost when $x = 1.3$.

8) The total cost C of producing x units is $C = 0.00004x^3 - 0.002x^2 + 3x + 10,000$. Find the marginal cost of 1000 units output.

9) Show that the elasticity of demand at all points on the curve $xy = c^2$ (y being price, and c is the constant) will be numerically equal to one.

10) Find the elasticity of demand when the demand is $q = \frac{20}{p+1}$ and $p = 3$. Interpret the result.

11) Given the demand function $q = 165 - 3p - 2p^2$, find the elasticity of demand at the price $p = 5$. Interpret the result.

12) Show that the elasticity of demand function $p = \frac{100}{q}$ is unity for every value of q.

13) Find the elasticity of demand with respect to the price for the following demand functions.

    (i) $p = \sqrt{a-bx}$, a and b are constants   (ii) $x = \frac{8}{p^{3/2}}$

14) A demand curve is $xp^m = b$ where m and b are constants. Calculate the price elasticity of demand.

15) Find the elasticity of supply for the supply function $x = 2p^2 + 5$.

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16) The supply of certain items is given by the supply function $x = a\sqrt{p-b}$, where p is the price, a and b are positive constants. $(p > b)$. Find an expression for elasticity of supply η_s. Show that it becomes unity when the price is $2b$.

17) For the demand function $p = 550 - 3x - 6x^2$ where x is the quantity demanded and p is the price per unit, find the average revenue and marginal revenue.

18) The sales S, for the product with price x is given by $S = 20{,}000\, e^{-0.6x}$.

Find (i) total sales revenue R, where $R = xS$ (ii) Marginal revenue

19) The price and quantity x of a commodity are related by the equation $x = 30 - 4p - p^2$. Find the elasticity of demand and marginal revenue.

20) Find the equilibrium price and equilibrium quantity for the following demand and supply functions.

$q_d = 4 - 0.05p$ and $q_s = 0.8 + 0.11p$

21) Find the marginal revenue for the revenue function $R(x) = 100x + \dfrac{x^2}{2}$ where $x = 10$.

22) The price and quantity q of a commodity are related by the equation $q = 32 - 4p - p^2$. Find the elasticity of demand and marginal revenue when $p = 3$.

3.2 DERIVATIVE AS A RATE OF CHANGE

Let a relation between two variables 'x' and 'y' be denoted by $y = f(x)$. Let Δx be the small change in x and Δy be the corresponding change in y. Then we define average rate of change of y with respect to x is $\dfrac{\Delta y}{\Delta x}$, where $\Delta y = f(x + \Delta x) - f(x)$ and $\underset{\Delta x \to 0}{\mathrm{Lt}}\, \dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx}$

= Instantaneous rate of change of y with respect to x.

3.2.1 Rate of change of a quantity

Let the two quantities x and y be connected by the relation $y = f(x)$. Then $f'(x_0)$ represents the rate of change of y with respect to x at $x = x_0$.

3.2.2 Related rates of change

We will find the solution to the problems which involve equations with two or more variables that are implicit functions of time. Since such variables will not usually be defined explicitly in terms of time, we will have to differentiate implicitly with respect to time to determine the relation between their time-rates of change.

Example 14

If $y = \dfrac{300}{x}$ find the average rate of change of y with respect to x when x increases from 10 to 10.5. Find also the instantaneous rate of change of y at $x = 10$.

Solution :

(i)   Average rate of change of y with respect to x at $x = x_0$ is $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$

Here $f(x) = \dfrac{300}{x}$, $x = 10$ and $\Delta x = 0.5$

∴ Average rate of change of y with respect to x is

$$\dfrac{f(10.5) - f(10)}{0.5} = \dfrac{28.57 - 30}{0.5} = \dfrac{-1.43}{0.5}$$

$= -2.86$ units per unit change in x.

The negative sign indicates that y decreases per unit increase in x.

(ii)   The instantaneous rate of change of y is $\dfrac{dy}{dx}$

$$y = \dfrac{300}{x}$$

$$\therefore \dfrac{dy}{dx} = \dfrac{-300}{x^2}$$

At $x = 10, \dfrac{dy}{dx} = \dfrac{-300}{(10)^2} = -3$

⇒   The instantaneous rate of change at $x = 10$ is -3 units per unit change in x.

The negative sign indicates the decrease rate of change with respect to x.

Example 15

A point moves along the graph of xy = 35 in such a way that its abscissa is increasing at the rate of 3 units per second when the point is (5, 7). Find the rate of change of y - coordinate at that moment.

Solution :

Here x and y are functions of time 't'.

$$x\,y = 35$$

Differentiating with respect to 't' we get

$$\frac{d}{dt}(xy) = \frac{d}{dt}(35)$$

⇒ $$x\frac{dy}{dt} + y\frac{dx}{dt} = 0$$

⇒ $$\frac{dy}{dt} = -\frac{y}{x}\frac{dx}{dt}$$

We need the value of $\dfrac{dy}{dt}$ when $x = 5$, $y = 7$ and $\dfrac{dx}{dt} = 3$.

$\therefore \quad \dfrac{dy}{dt} = -\dfrac{7}{5} \times 3$

$= -4.2$ units per second.

i.e. y co-ordinate is decreasing at the rate of 4.2 units per second.

Example 16

The unit price, p of a product is related to the number of units sold, x, by the demand equation $p = 400 - \dfrac{x}{1000}$. The cost of producing x units is given by C(x) = $50x + 16,000$. The number of units produced and sold, x is increasing at a rate of 200 units per week. When the number of units produced and sold is 10,000, determine the instantaneous rate of change with respect to time, t (in weeks) of (i) Revenue (ii) Cost (iii) Profit.

Solution :

(i)   Revenue $R = px$

$$= \left(400 - \dfrac{x}{1000} \right) x$$

$$R = 400x - \dfrac{x^2}{1000}$$

$$\dfrac{d}{dt}(R) = \dfrac{d}{dt}(400x) - \dfrac{d}{dt}\left(\dfrac{x^2}{1000} \right)$$

$$\dfrac{dR}{dt} = \left(400 - \dfrac{x}{500} \right) \dfrac{dx}{dt}$$

when $x = 10,000$, and $\dfrac{dx}{dt} = 200$

$$\dfrac{dR}{dt} = \left(400 - \dfrac{10,000}{500} \right)(200)$$

$= $ Rs.76,000 per week.

i.e. Revenue is increasing at a rate of Rs.76,000 per week.

(ii) $C(x) = 50x + 16,000.$

$$\frac{d}{dt}(C) = \frac{d}{dt}(50x) + \frac{d}{dt}(16,000)$$

$$\frac{dC}{dt} = 50\frac{dx}{dt} + 0$$

$$= 50\frac{dx}{dt}$$

when $\frac{dx}{dt} = 200,$ $\frac{dC}{dt} = 50 \times 200$

$$= \text{Rs. } 10,000 \text{ per week.}$$

i.e. Cost is increasing at a rate of Rs.10,000 per week.

(iii)   Profit $P = R - C$

$$\frac{dP}{dt} = \frac{dR}{dt} - \frac{dC}{dt}$$

$$= 76,000 - 10,000$$

$$= \text{Rs.}66,000 \text{ per week.}$$

i.e. Profit is increasing at a rate of Rs.66,000 per week.

Example 17

If the perimeter of a circle increases at a constant rate, prove that the rate of increase of the area varies as the radius of the circle.

Solution :

Let P be the perimeter and A be the area of the circle of radius r.

Then $P = 2\pi r$ and $A = \pi r^2$

$$\frac{dP}{dt} = 2\pi\frac{dr}{dt} \qquad\qquad ....(1)$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} \qquad\qquad ....(2)$$

Using (1) and (2) we get,

$$\frac{dA}{dt} = r\frac{dP}{dt}$$

Since perimeter P increases at a constant rate $\frac{dP}{dt}$ is constant.

$\therefore$ $\frac{dA}{dt} \propto r$ i.e. the rate of increase of A is proportional to the radius.

Example 18

A metal cylinder is heated and expands so that its radius increases at a rate of 0.4 cm per minute and its height increases at a rate of 0.3 cm per minute retaining its shape. Determine the rate of change of the surface area of the cylinder when its radius is 20 cms. and height is 40 cms.

Solution :

The surface area of the cylinder is

$A = 2\pi rh.$

Differentiating both sides with respect to '*t*'

$$\frac{dA}{dt} = 2\pi \left[r\frac{dh}{dt} + h\frac{dr}{dt} \right]$$

Given $r = 20,\ h = 40,\ \frac{dr}{dt} = 0.4,\ \frac{dh}{dt} = 0.3$

$$\therefore \frac{dA}{dt} = 2\pi\, [20 \times 0.30 + 40 \times 0.40]$$

$$= 2\pi[6 + 16]$$

$$= 44\pi \ \text{cm}^2 \, / \, \text{minute}$$

Example 19

For the function y = x³ + 21, what are the values of x, when y increases 75 times as fast as x ?

Solution :

$y = x^3 + 21$

Differentiating both sides with respect to '*t*'

$$\frac{dy}{dt} = 3x^2 \frac{dx}{dt} + 0$$

$$\frac{dy}{dt} = 3x^2 \frac{dx}{dt}$$

Given $\frac{dy}{dt} = 75\frac{dx}{dt}$

$$3x^2 \frac{dx}{dt} = 75\frac{dx}{dt}$$

$$3x^2 = 75$$

$$x^2 = 25$$

$$x = \pm 5$$

Example 20

The demand y for a commodity is $y = \dfrac{12}{x}$, where x is the price. Find the rate at which the demand changes when the price is Rs. 4.

Solution :

The rate of change of the demand y with respect to the price is $\dfrac{dy}{dx}$.

We have $y = \dfrac{12}{x}$

Differentiating with respect to x, we get $\dfrac{dy}{dx} = -\dfrac{12}{x^2}$

∴ The rate of change of demand with respect to price x is $-\dfrac{12}{x^2}$ When the price is Rs. 4 the rate of change of demand is $\dfrac{-12}{16} = -\dfrac{3}{4}$.

This means that when the price is Rs.4, an increase in price by 1% will result in the fall of demand by 0.75%.

EXERCISE 3.2

1) If $y = \dfrac{500}{x}$ find the average rate of change of y with respect to x, when x increases from 20 to 20.5 units. Find also the instantaneous rate of change of y at $x = 20$.

2) A point moves on the graph of $xy = 8$ in such a manner that its y-coordinate is increasing at a rate of 2 units per second, when the point is at (2, 4). Find the rate of change of the x-coordinate at that moment.

3) A point moves on the curve $4x^2 + 2y^2 = 18$ in such a way that its x co-ordinate is decreasing at a rate of 3 units per second when the point is at (2,1). Find the rate of change of the y-coordinate at that moment.

4) A point moves along the curve $y^2 = 12x$ in such a way that its x - coordinate is increasing at the rate of $5\sqrt{2}$ units per second when the point is at (3, 6). Show that the y - coordinate increases at the same rate as that of x - coordinate.

5) Given are the following revenue, cost and profit equations $R = 800x - \dfrac{x^2}{10}$, $C = 40x + 5,000$, $P = R - C$, where x denotes the number of units produced and sold (per month). When the production is at 2000 units and increasing at the rate of 100 units per month, determine the instantaneous rate of change with respect to time, t (in months), of (i) Revenue (ii) Cost and (iii) Profit

6) The unit price, p of some product is related to the number of units sold, x, by the demand function $p = 200 - \dfrac{x}{1000}$. The cost of producing x units of this product is given by $C = 40x + 12,000$. The number of units produced and sold x is increasing at

the rate of 300 units per week. When the number of units produced and sold is 20,000 determine the instantaneous rate of change with respect to time, t (in weeks) of (i) Revenue (ii) Cost (iii) Profit.

7) Using derivative as a rate measure prove the following statement : "If the area of a circle increases at a uniform rate, then the rate of increase of the perimeter varies inversely as the radius of the circle".

8) The radius of a circular plate is increasing at the rate of 0.2 cm per second. At what rate is the area increasing when the radius of the plate is 25 cm.?

9) A metal cylinder when heated, expands in such a way that its radius r, increases at the rate of 0.2 cm. per minute and its height h increases at a rate of 0.15 cm per minute. retaining its shape. Determine the rate of change of the volume of the cylinder when its radius is 10 cms and its height is 25 cms.

10) For what values of x, is the rate of increase of $x^3 - 5x^2 + 5x + 8$ is twice the rate of increase of x ?

3.3 DERIVATIVE AS MEASURE OF SLOPE

3.3.1 Slope of the Tangent Line

Geometrically $\dfrac{dy}{dx}$ represents the slope or gradient of the tangent line to the curve $y = f(x)$ at the point P(x, y). If θ is the inclination of the tangent line with the positive direction of x - axis, then slope of the line (Fig. 3.6).

$$m = \tan\theta = \frac{dy}{dx} \text{ at P } (x, y).$$
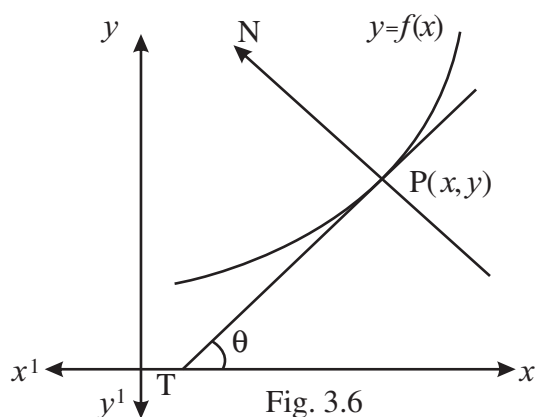
Fig. 3.6

Note

(i) If the tangent to the curve is parallel to the x-axis, then $\theta = 0$ which implies $\tan\theta = 0$.

$$\therefore \ \frac{dy}{dx} = 0 \text{ at that point.}$$

(ii)   If the tangent to the curve is parallel to the y-axis, then $\theta = 90^\circ$ which implies $\tan \theta = \infty$.

$$\therefore \frac{dy}{dx} = \infty \text{ or } \frac{dx}{dy} = 0 \text{ at that point.}$$

3.3.2 Equation of the Tangent

From Analytical Geometry, the equation of the tangent to the curve $y = f(x)$ at $P(x_1, y_1)$ is

$$y - y_1 = \frac{dy}{dx}(x - x_1) \text{ where } \frac{dy}{dx} \text{ is the slope of the tangent at P.}$$

or $y - y_1 = m(x - x_1)$ where $m = \dfrac{dy}{dx}$ at P.

The point P is called the point of contact.

Note

Two tangents to the curve $y = f(x)$ will be

(i)   parallel if the slopes are equal and

(ii)   perpendicular to each other if the product of their slopes is -1.

3.3.3 Equation of the Normal

The line which is perpendicular to the tangent at the point of contact $P(x, y)$ is called normal.

$\therefore$ The equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{1}{\dfrac{dy}{dx}}(x - x_1), \text{ provided } \frac{dy}{dx} \neq 0 \text{ at } (x_1, y_1)$$

$$\text{or } y - y_1 = -\frac{1}{m}(x - x_1) \text{ where } m = \left(\frac{dy}{dx}\right) \text{ at } (x_1, y_1).$$

Example 21

Find the slope of the curve $y = \dfrac{x^2 - 12}{x - 4}$ $(x \neq 4)$ at the point $(0, 3)$ and determine the points where the tangent is parallel to the axis of x.

Solution :

We have $y = \dfrac{x^2 - 12}{x - 4}$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{(x-4)(2x) - (x^2 - 12)(1)}{(x-4)^2}$$

$$= \frac{x^2 - 8x + 12}{(x-4)^2}$$

$\therefore$ The slope of the curve at $(0, 3) = \dfrac{dy}{dx}$ at $(0, 3)$

$$= \frac{3}{4}$$

The points at which tangents are parallel to the x-axis are given by $\dfrac{dy}{dx} = 0$.

$\Rightarrow \qquad x^2 - 8x + 12 = 0$

$\Rightarrow \qquad (x - 2)(x - 6) = 0 \qquad\qquad \therefore x = 2, 6$

when $x = 2$, $y = 4$

when $x = 6$, $y = 12$

$\therefore$ The points at which the tangents are parallel to the x - axis are $(2, 4)$ and $(6, 12)$.

Example 22

Determine the values of *l* and *m* so that the curve, $y = lx^2 + 3x + m$ **may pass through the point (0, 1) and have its tangent parallel to the** x**-axis at** $x = 0.75$.

Solution :

We have $y = l\,x^2 + 3x + m$.

Differentiating with respect to x, we get

$$\frac{dy}{dx} = 2lx + 3.$$

At $\quad x = 0.75$, $\dfrac{dy}{dx} = 2l\,(0.75) + 3$

$$= 1.5l + 3.$$

The tangent at $x = 0.75$ is parallel to the x-axis

$\therefore \qquad$ At $x = 0.75$, $\dfrac{dy}{dx} = 0$

$\Rightarrow \quad 1.5l + 3 = 0$

$\Rightarrow \quad l = -\dfrac{3}{1.5} = -2.$

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Since the curve is passing through the point (0, 1) we get that

$1 = l(0)^2 + 3(0) + m \Rightarrow m = 1.$

∴ $l = -2$ and $m = 1.$

Example 23

For the cost function $y = 2x \left(\dfrac{x+4}{x+3} \right) + 3$, prove that the marginal cost falls continuously as the output x increases.

Solution :

We have $y = 2x \left(\dfrac{x+4}{x+3} \right) + 3$

$$y = \frac{2x^2 + 8x}{x+3} + 3 \qquad\qquad ....(1)$$

Marginal cost is $\dfrac{dy}{dx}$

∴   Differentiating (1) with respect to x, we get

$$\frac{dy}{dx} = \frac{(x+3)(4x+8) - (2x^2 + 8x)(1)}{(x+3)^2} + 0$$

$$= \frac{2(x^2 + 6x + 12)}{(x+3)^2} = \frac{2(x^2 + 6x + 9 + 3)}{(x+3)^2}$$

$$= 2\left(\frac{(x+3)^2 + 3}{(x+3)^2} \right) = 2\left(1 + \frac{3}{(x+3)^2} \right)$$

This shows that as x increases, the marginal cost $\dfrac{dy}{dx}$ decreases.

Example 24

Prove that for the cost function $C = 100 + x + 2x^2$, where x is the output, the slope of AC curve $= \dfrac{1}{x} (MC - AC)$.

(MC is the marginal cost and AC is the average cost)

Solution :

Cost function is $C = 100 + x + 2x^2$

Average cost (AC) $= \dfrac{100 + x + 2x^2}{x}$

$$= \frac{100}{x} + 1 + 2x$$

Slope of AC $= \frac{d}{dx}(AC)$

$$= \frac{d}{dx}\left(\frac{100}{x} + 1 + 2x\right) = -\frac{100}{x^2} + 2 \qquad \text{......(1)}$$

Marginal cost MC $= \frac{d}{dx}(C)$

$$= \frac{d}{dx}(100 + x + 2x^2) = 1 + 4x$$

$$MC - AC = (1 + 4x) - \left(\frac{100}{x} + 1 + 2x\right)$$

$$= -\frac{100}{x} + 2x$$

$$\frac{1}{x}(MC - AC) = \frac{1}{x}\left(-\frac{100}{x} + 2x\right)$$

$$= -\frac{100}{x^2} + 2 \qquad \text{........(2)}$$

From (1) and (2) we get

Slope of AC $\quad = \frac{1}{x}(MC - AC)$

Example 25

Find the equations of the tangent and normal at the point (a cos θ, b sin θ) on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution :

We have $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Differentiating with respect to x, we get

$$\frac{1}{a^2}(2x) + \frac{1}{b^2}2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

At $(a\cos\theta, b\sin\theta) \dfrac{dy}{dx} = -\dfrac{b\cos\theta}{a\sin\theta} = m.$

Equation of the tangent is

$$y - y_1 = m\,(x - x_1)$$

$$\Rightarrow \quad y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$\Rightarrow \quad ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$$

$$\Rightarrow \quad bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta) = ab$$

Dividing both sides by 'ab', we get

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

$\therefore$ Equation of the tangent is $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$

Equation of the normal is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$\Rightarrow \quad y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$$

$$\Rightarrow \quad by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$$

$$\Rightarrow \quad ax\sin\theta - by\cos\theta = \sin\theta\cos\theta(a^2 - b^2)$$

When $\sin\theta\cos\theta \neq 0$, dividing by '$\sin\theta\cos\theta$' we get

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

$\therefore$ The equation of the normal is $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$

Example 26

Find the equation of the tangent and normal to the demand curve $y = 10 - 3x^2$ at (1, 7).

Solution :

Demand curve $y = 10 - 3x^2$

Differentiating both sides with respect x, we have

$$\frac{dy}{dx} = -6x$$

At $(1,7)$ $\dfrac{dy}{dx} = -6 = m.$

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Equation of the tangent is

$$y - y_1 = m (x - x_1)$$

$$\Rightarrow \quad y - 7 = -6 (x - 1)$$

$$\Rightarrow \quad 6x + y - 13 = 0$$

Equation of the normal is

$$y - y_1 = -\frac{1}{m} (x - x_1)$$

$$\Rightarrow \quad y - 7 = -\frac{1}{-6} (x - 1)$$

$$y - 7 = \frac{1}{6} (x - 1)$$

$$6y - 42 = x - 1$$

$$\Rightarrow \quad x - 6y + 41 = 0$$

Example 27

Find the points on the curve $y = (x - 1) (x - 2)$ at which the tangent makes an angle 135º with the positive direction of the x-axis.

Solution :

We have $y = (x - 1) (x - 2)$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = (x - 1) (1) + (x - 2) (1)$$

$$= 2x - 3 \qquad\qquad\qquad .... (1)$$

Also the tangent is making 135º with the x-axis.

$m = \dfrac{dy}{dx} = \tan\theta$	$\tan 135^o = \tan(180^o - 45^o)$
$= \tan 135^o$	$= -\tan 45^o$
$= -1 \qquad ......(2)$	$= -1$

Equating (1) and (2) we get

$$2x - 3 = -1 \text{ or } 2x = 2$$

$$x = 1$$

When $x = 1$, $y = (1 - 1) (1 - 2) = 0$

$\therefore$ The required point is (1, 0).

EXERCISE 3.3

1) Find the slope of the tangent line at the point (0, 5) of the curve $y = \frac{1}{3}(x^2 + 10x - 15)$. At what point of the curve the slope of the tangent line is $\frac{8}{5}$?

2) Determine the coefficients a and b so that the curve $y = ax^2 - 6x + b$ may pass through the point (0, 2) and have its tangent parallel to the x-axis at $x = 1.5$.

3) For the cost function $y = 3x\left(\frac{x+7}{x+5}\right) + 5$, prove that the marginal cost falls continuously as the output x increases.

4) Find the equations of the tangents and normals to the following curves

    (i) $y^2 = 4x$ at (1, 2)   (ii) $y = \sin 2x$ at $x = \frac{\pi}{6}$

    (iii) $x^2 + y^2 = 13$ at (-3, -2)   (iv) $xy = 9$ at $x = 4$.

    (v) $y = x^2 \log x$ at $x = e$ (vi) $x = a \cos \theta$, $y = b \sin \theta$ at $\theta = \frac{\pi}{4}$

5) Find the equation of the tangent and normal to the supply curve $y = x^2 + x + 2$ when $x = 6$.

6) Find the equation of the tangent and normal to the demand curve $y = 36 - x^2$ when $y = 11$.

7) At what points on the curve $3y = x^3$ the tangents are inclined at 45° to the x-axis.

8) Prove that $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = b\,e^{-x/a}$ at the point where the curve cuts the y-axis.

9) Find the equation of the tangent and normal to the curve $y(x - 2)(x - 3) - x + 7 = 0$ at the point where it cuts the x axis.

10) Prove that the curves $y = x^2 - 3x + 1$ and $x(y + 3) = 4$ intersect at right angles at the point $(2, -1)$.

11) Find the equation of the tangent and normal at the point $(a \sec \theta, b \tan \theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

12) At what points on the circle $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangent is parallel to (i) x-axis (ii) y-axis.

EXERCISE 3.4

Choose the correct answer

1) The average fixed cost of the function $C = 2x^3 - 3x^2 + 4x + 8$ is

   a) $\dfrac{2}{x}$
   b) $\dfrac{4}{x}$
   c) $\dfrac{-3}{x}$
   d) $\dfrac{8}{x}$

2) If 60 units of some product cost Rs. 1400 and 40 units cost Rs. 1200 to manufacture, then the variable cost per unit is

   a) Rs. 100
   b) Rs. 2600
   c) Rs. 10
   d) Rs. 5

3) If 20 units of some product cost Rs. 2500 and 50 units cost Rs. 3400 to produce, the linear cost function is

   a) $y = 30x + 1900$
   b) $y = 20x + 5900$

   c) $y = 50x + 3400$
   d) $y = 10x + 900$

4) Variable cost per unit is Rs. 40, fixed cost is Rs. 900 and unit selling price is Rs. 70. Then the profit equation is

   a) $P = 30x - 900$
   b) $P = 15x - 70$

   c) $P = 40x - 900$
   d) $P = 70x + 3600$

5) For the cost function $c = \dfrac{1}{10}e^{2x}$, the marginal cost is

   a) $\dfrac{1}{10}$
   b) $\dfrac{1}{5}e^{2x}$
   c) $\dfrac{1}{10}e^{2x}$
   d) $\dfrac{1}{10}e^{x}$

6) Given the demand equation $p = -x + 10$; $(0 \le x \le 10)$ where p denotes the unit selling price and x denotes the number of units demanded of some product. Then the marginal revenue at $x = 3$ units is

   a) Rs. 5
   b) Rs. 10
   c) Rs. 4
   d) Rs. 30

7) The demand for some commodity is given by $q = -3p + 15$ $(0 < p < 5)$ where p is the unit price. The elasticity of demand is

   a) $\dfrac{9p^2 + 15}{p}$
   b) $\dfrac{9p - 45}{p}$
   c) $\dfrac{15p - 9}{p}$
   d) $\dfrac{p}{-p + 5}$

8) For the function $y = 3x + 2$ the average rate of change of y when x increases from 1.5 to 1.6 is

a) 1   b) 0.5   c) 0.6   d) 3

9) If $y = 2x^2 + 3x$, the instantaneous rate of change of y at $x = 4$ is

a) 16   b) 19   c) 30   d) 4

10) If the rate of change of y with respect to x is 6 and x is changing at 4 units/sec, then the rate of change of y per sec is

a) 24 units/sec   b) 10 units/sec   c) 2 units/sec   d) 22 units/sec

11) The weekly profit P, in rupees of a corporation is determined by the number x of shirts produced per week according to the formula $P = 2000x - 0.03x^2 - 1000$. Find the rate at which the profit is changing when the production level x is 1000 shirts per week.

a) Rs.140   b) Rs. 2000   c) Rs.1500   d) Rs. 1940

12) The bottom of a rectangular swimming tank is $25m$ by $40m$. Water is pumped into the tank at the rate of $500m^3$/min. Find the rate at which the level of the water in the tank is rising?

a) 0.5m/min   b) 0.2m/min   c) 0.05m/min   d) 0.1m/min

13) The slope of the tangent at (2, 8) on the curve $y = x^3$ is

a) 3   b)12   c) 6   d) 8

14) The slope of the normal to the curve $\sqrt{x} + \sqrt{y} = 5$ at (9, 4) is

a) $\dfrac{2}{3}$ b) $-\dfrac{2}{3}$ c) $\dfrac{3}{2}$ d) $-\dfrac{3}{2}$

15) For the curve $y = 1 + ax - x^2$ the tangent at (1, $-$ 2) is parallel to x-axis. The value of 'a' is

a) $-$ 2   b) 2   c) 1   d) -1

16) The slope of the tangent to the curve $y = \cos t$, $x = \sin t$ at $t = \dfrac{\pi}{4}$ is

a) 1   b) 0   c) $\dfrac{1}{\sqrt{2}}$ d) $-$ 1

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17) The point at which the tangent to the curve $y^2 = x$ makes an angle $\dfrac{\pi}{4}$ with the x-axis is

     a) $\left(\dfrac{1}{2}, \dfrac{1}{4}\right)$
         b) $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$
         c) $\left(\dfrac{1}{4}, \dfrac{1}{2}\right)$
         d) $(1, -1)$

18) The tangent to the curve $y = 2x^2 - x + 1$ at $(1,2)$ is parallel to the line

     a) $y = 3x$
         b) $y = 2x + 4$
         c) $2x + y + 7 = 0$
         d) $y = 5x - 7$

19) The slope of the tangent to the curve $y = x^2 - \log x$ at $x = 2$ is

     a) $\dfrac{7}{2}$
         b) $\dfrac{2}{7}$
         c) $-\dfrac{7}{2}$
         d) $-\dfrac{2}{7}$

20) The slope of the curve $x = y^2 - 6y$ at the point where it crosses the y axis is

     a) 5
         b) -5
         c) $\dfrac{1}{6}$
         d) $-\dfrac{1}{16}$

APPLICATIONS OF DIFFERENTIATION - II   **4**

The concept of maxima and minima is applied in Economics to study profit maximisation, inventory control and economic order quantity.

We also learn what a partial derivative is and how to calculate it. Application of partial derivatives are also discussed with the production function, marginal productivities of labour and capital and with partial elasticities of demand.

4.1 MAXIMA AND MINIMA

4.1.1 Increasing and Decreasing Functions

A function $y = f(x)$ is said to be an increasing function of x in an interval, say $a \leq x \leq b$, if y increases as x increases. i.e. if $a \leq x_1 \leq x_2 \leq b$, then $f(x_1) \leq f(x_2)$.

A function $y = f(x)$ is said to be a decreasing function of x in an interval, say $a \leq x \leq b$, if y decreases as x increases. i.e. if $a \leq x_1 \leq x_2 \leq b$, then $f(x_1) \geq f(x_2)$.

4.1.2. Sign of the derivative

Let f be an increasing function defined in a closed interval $[a,b]$. Then for any two values x_1 and x_2 in $[a, b]$ with $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$.

$\therefore f(x_1) \leq f(x_2)$ and $x_2 - x_1 > 0$

$\Rightarrow \quad \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0$

$\Rightarrow \quad \underset{x_2 \to x_1}{\text{Lt}} \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0$, if this limit exists.

$\Rightarrow \quad f'(x) \geq 0$ for all $x \in [a,b]$

Similarly, if f is decreasing on $[a, b]$ then $f'(x) \leq 0$, if the derivative exists.

The converse holds with the additional condition, that f is continuous on $[a, b]$.

Note

Let f be continuous on $[a, b]$ and has derivative at each point of the open inverval (a, b), then

(i)   If $f'(x) > 0$ for every $x \in (a, b)$, then f is strictly increasing on $[a, b]$.

(ii)   If $f'(x) < 0$ for every $x \in (a, b)$, then f is strictly decreasing on $[a, b]$.

(iii)   If $f'(x) = 0$ for every $x \in (a, b)$, then f is a constant function on $[a, b]$.

(iv)   If $f'(x) \geq 0$ for every $x \in (a, b)$, then f is increasing on $[a, b]$.

(v)   If $f'(x) \leq 0$ for every $x \in (a, b)$, then f is decreasing on $[a, b]$.

The above results are used to test whether a given function is increasing or decreasing.

4.1.3 Stationary Value of a Function

A function $y = f(x)$ may neither be an increasing function nor be a decreasing function of x at some point of the interval $[a, b]$. In such a case, $y = f(x)$ is called stationary at that point. At a stationary point $f'(x) = 0$ and the tangent is parallel to the x-axis.

Example 1

If $y = x - \dfrac{1}{x}$, prove that y is a strictly increasing function for all real values of x. $(x \neq 0)$

Solution :

We have $y = x - \dfrac{1}{x}$

Differentiating with respect to x, we get

$\dfrac{dy}{dx} = 1 + \dfrac{1}{x^2} > 0$ for all values of x, except $x = 0$.

$\therefore y$ is a strictly increasing function for all real values of x. $(x \neq 0)$

Example 2

If $y = 1 + \dfrac{1}{x}$, show that y is a strictly decreasing function for all real values of x. $(x \neq 0)$

Solution :

We have $y = 1 + \dfrac{1}{x}$

$\dfrac{dy}{dx} = 0 - \dfrac{1}{x^2} < 0$ for all values of x, $(x \neq 0)$

$\therefore y$ is a strictly decreasing function for all real values of x. $(x \neq 0)$

Example 3

Find the ranges of values of x in which $2x^3 - 9x^2 + 12x + 4$ is strictly increasing and strictly decreasing.

Solution :

Let $y = 2x^3 - 9x^2 + 12x + 4$

$$\frac{dy}{dx} = 6x^2 - 18x + 12$$

$$= 6(x^2 - 3x + 2)$$

$$= 6(x - 2)(x - 1)$$

$\frac{dy}{dx} > 0$ when $x < 1$ or $x > 2$

x lies outside the interval $(1, 2)$

$\frac{dy}{dx} < 0$ when $1 < x < 2$

∴ The function is strictly increasing outside the interval $[1, 2]$ and strictly decreasing in the interval $(1, 2)$

Example 4

Find the stationary points and the stationary values of the function $f(x) = x^3 - 3x^2 - 9x + 5$.

Solution :

Let $y = x^3 - 3x^2 - 9x + 5$

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

At stationary points, $\frac{dy}{dx} = 0$

∴ $3x^2 - 6x - 9 = 0$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x + 1)(x - 3) = 0$

The stationary points are obtained when $x = -1$ and $x = 3$

when $x = -1$, $y = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = 10$

when $x = 3$, $y = (3)^3 - 3(3)^2 - 9(3) + 5 = -22$

∴ The stationary values are 10 and -22

The stationary points are $(-1, 10)$ and $(3, -22)$

Example 5

For the cost function $C = 2000 + 1800x - 75x^2 + x^3$ find when the total cost (C) is increasing and when it is decreasing. Also discuss the behaviour of the marginal cost (MC)

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Solution :

Cost function $C = 2000 + 1800x - 75x^2 + x^3$

$$\frac{dC}{dx} = 1800 - 150x + 3x^2$$

$\frac{dC}{dx} = 0 \qquad \Rightarrow 1800 - 150x + 3x^2 = 0$

$\qquad\qquad\qquad \Rightarrow 3x^2 - 150x + 1800 = 0$

$\qquad\qquad\qquad \Rightarrow x^2 - 50x + 600 = 0$

$\qquad\qquad\qquad \Rightarrow (x - 20)(x - 30) = 0$

$\qquad\qquad\qquad \Rightarrow x = 20 \text{ or } x = 30$
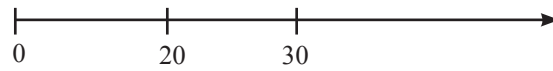
For

(i) $0 < x < 20$, $\frac{dC}{dx} > 0$ | (i) $x = 10$ then $\frac{dC}{dx}$ s= $600 > 0$

(ii) $20 < x < 30$, $\frac{dC}{dx} < 0$ | (ii) $x = 25$ then $\frac{dC}{dx} = -75 < 0$

(iii) $x > 30$; $\frac{dC}{dx} > 0$ | (iii) $x = 40$ then $\frac{dC}{dx} = 600 > 0$

∴ C is increasing for $0 < x < 20$ and for $x > 30$.

C is decreasing for $20 < x < 30$

$MC = \frac{d}{dx}(C)$

∴ $\quad MC = 1800 - 150x + 3x^2$

$\frac{d}{dx}(MC) = -150 + 6x$

$\frac{d}{dx}(MC) = 0 \quad \Rightarrow 6x = 150$

$\qquad\qquad\qquad \Rightarrow x = 25.$
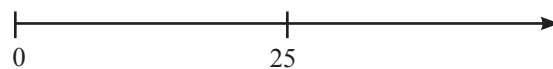
For,

(i) $0 < x < 25$, $\dfrac{d}{dx}$ (MC) < 0 (i) $x = 10$ then $\dfrac{d}{dx}$ (MC) $= -90 < 0$

(ii) $x > 25$, $\dfrac{d}{dx}$ (MC) > 0 (ii) $x = 30$ then $\dfrac{d}{dx}$ (MC) $= 30 > 0$

$\therefore$ MC is decreasing for $x < 25$ and increasing for $x > 25$.

4.1.4 Maximum and Minimum Values

Let f be a function defined on $[a, b]$ and c an interior point of $[a, b]$ (i.e.) c is in the open interval (a, b). Then

(i) $f(c)$ is said to be a maximum or relative maximum of the function f at $x = c$ if there is a neighbourhood $(c - \delta , c + \delta)$ of c such that for all $x \in (c - \delta, c + \delta)$ other than $c, f(c) > f(x)$

(ii) $f(c)$ is said to be a minimum or relative minimum of the function f at $x = c$ if there is a neighbourhood $(c - \delta , c + \delta)$ of c such that for all $x \in (c - \delta , c + \delta)$ other than c, $f(c) < f(x)$.

(iii) $f(c)$ is said to be an extreme value of f or extremum at c if it is either a maximum or minimum.

4.1.5 Local and Global Maxima and Minima

Consider the graph (Fig. 4.1) of the function $y = f(x)$.
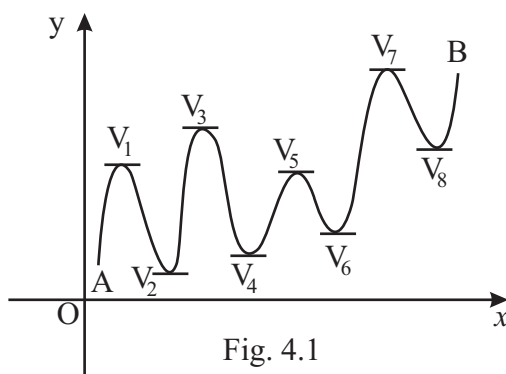
Fig. 4.1

The function $y = f(x)$ has several maximum and minimum points. At the points V_1, V_2, ... V_8, $\dfrac{dy}{dx} = 0$. In fact the function has maxima at V_1, V_3, V_5, V_7 and minima at V_2, V_4, V_6, V_8. Note that maximum value at V_5 is less than the minimum value at V_8. These maxima and minima are called local or relative maxima and minima. If we consider the part of the curve between A and B then the function has absolute maximum or global maximum at V_7 and absolute minimum or global minimum at V_2.

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Note

By the terminology maximum or minimum we mean local maximum or local minimum respectively.

4.1.6 Criteria for Maxima and Minima

	Maximum	**Minimum**
Necessary condition	$\dfrac{dy}{dx} = 0$	$\dfrac{dy}{dx} = 0$
Sufficient condition	$\dfrac{dy}{dx} = 0$; $\dfrac{d^2 y}{dx^2} < 0$	$\dfrac{dy}{dx} = 0$; $\dfrac{d^2 y}{dx^2} > 0$

4.1.7 Concavity and Convexity

Consider the graph (Fig. 4.2) of the function $y = f(x)$.

Let PT be the tangent to the curve $y = f(x)$ at the point P. The curve (or an arc of the curve) which lies above the tangent line PT is said to be concave upward or convex downward.

Fig. 4.2
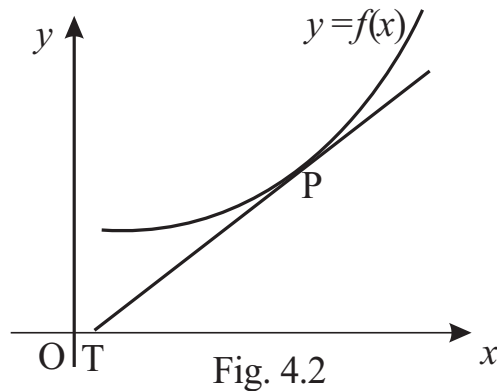
The curve (or an arc of the curve) which lies below the tangent line PT (Fig. 4.3) is said to be convex upward or concave downward.

Fig. 4.3

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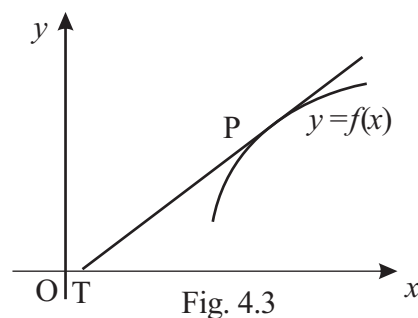
4.1.8 Conditions for Concavity and Convexity.

Let $f(x)$ be twice differentiable. Then the curve $y = f(x)$ is

(i)   concave upward on any interval if $f''(x) > 0$

(ii)   convex upward on any interval if $f''(x) < 0$

4.1.9 Point of Inflection

A point on a curve $y = f(x)$, where the concavity changes from upto down or vice versa is called a Point of Inflection.

For example, in $y = x^{\frac{1}{3}}$ (Fig. 4.4) has a point of inflection at $x = 0$
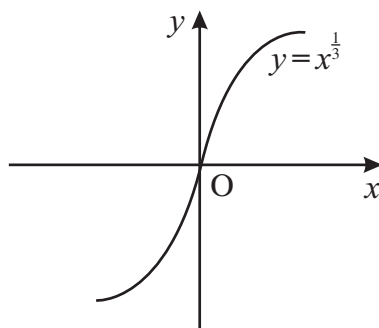
Fig. 4.4

4.1.10 Conditions for point of inflection

A point $(c, f(c))$ on a curve $y = f(x)$ is a point of inflection (i) if $f''(c) = 0$ or $f''(c)$ is not defined and (ii) if $f''(x)$ changes sign as x increases through c i.e. $f'''(c) \neq 0$ when $f'''(x)$ exists.

Example 6

Investigate the maxima and minima of the function $2x^3 + 3x^2 - 36x + 10$.

Solution :

Let y $= 2x^3 + 3x^2 - 36x + 10$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = 6x^2 + 6x - 36 \qquad\qquad ....(1)$$

$$\frac{dy}{dx} = 0 \Rightarrow 6x^2 + 6x - 36 = 0$$

$$\Rightarrow x^2 + x - 6 = 0$$

$$\Rightarrow (x + 3)(x - 2) = 0$$

$$\Rightarrow x = -3, 2$$

Again differentiating (1) with respect to x, we get

$$\frac{d^2 y}{dx^2} = 12x + 6$$

when $x = -3$, $\frac{d^2 y}{dx^2} = 12(-3) + 6 = -30 < 0$

∴ It attains maximum at $x = -3$

∴ Maximum value is $y = 2(-3)^3 + 3(-3)^2 - 36(-3) + 10 = 91$

when $x = 2$, $\frac{d^2 y}{dx^2} = 12(2) + 6 = 30 > 0$

∴ It attains minimum at $x = 2$

∴ Minimum value is $y = 2(2)^3 + 3(2)^2 - 36(2) + 10 = -34$

Example 7

Find the absolute (global) maximum and minimum value of the function $f(x) = 3x^5 - 25x^3 + 60x + 1$ in the interval $[-2, 1]$.

Solution :

Given $f(x) = 3x^5 - 25x^3 + 60x + 1$

$f'(x) = 15x^4 - 75x^2 + 60$

The necessary condition for maximum and minimum is

$f'(x) = 0$

$\Rightarrow 15x^4 - 75x^2 + 60 = 0$

$\Rightarrow x^4 - 5x^2 + 4 = 0$

$\Rightarrow x^4 - 4x^2 - x^2 + 4 = 0$

$\Rightarrow (x^2 - 1)(x^2 - 4) = 0$

∴ $x = \pm 1, -2$, $(2 \notin [-2, 1])$

$f''(x) = 60x^3 - 150x$

$f''(-2) = 60(-2)^3 - 150(-2) = -180 < 0$

∴ $f(x)$ is maximum.

$f''(-1) = 60(-1)^3 - 150(-1) = 90 > 0$

∴ $f(x)$ is minimum.

$f''(1) = 60(1)^3 - 150(1) = -90 < 0$

∴ $f(x)$ is maximum.

The maximum value when $x = -2$ is

$$f(-2) = 3(-2)^5 - 25(-2)^3 + 60(-2) + 1 = -15$$

The minimum value when $x = -1$ is

$$f(-1) = 3(-1)^5 - 25(-1)^3 + 60(-1) + 1 = -37$$

The maximum value when $x = 1$ is

$$f(1) = 3(1)^5 - 25(1)^3 + 60(1) + 1 = 39$$

∴ Absolute maximum value = 39

and Absolute minimum value -37

Example 8

What is the maximum slope of the tangent to the curve $y = -x^3 + 3x^2 + 9x - 27$ and at what point is it?

Solution :

Given $y = -x^3 + 3x^2 + 9x - 27$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -3x^2 + 6x + 9$$

Slope of the tangent is $-3x^2 + 6x + 9$

Let $M = -3x^2 + 6x + 9$

Differentiating with respect to x, we get

$$\frac{dM}{dx} = -6x + 6 \qquad ....(1)$$

Slope is maximum when $\frac{dM}{dx} = 0$ and $\frac{d^2M}{dx^2} < 0$

$$\frac{dM}{dx} = 0 \Rightarrow -6x + 6 = 0$$

$$\Rightarrow x = 1$$

Again differentiating (1) with respect to x, we get

$$\frac{d^2M}{dx^2} = -6 < 0, \text{ M is maximum at } x = 1$$

∴ Maximum value of M when $x = 1$ is

$$M = -3(1)^2 + 6(1) + 9 = 12$$

When $x = 1$; $y = -(1)^3 + 3(1)^2 + 9(1) - 27 = -16$

∴ Maximum slope = 12

The required point is $(1, -16)$

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Example 9

Find the points of inflection of the curve $y = 2x^4 - 4x^3 + 3$.

Solution :

Given $y = 2x^4 - 4x^3 + 3$

Differentiate with respect to x, we get

$$\frac{dy}{dx} = 8x^3 - 12x^2$$

$$\frac{d^2y}{dx^2} = 24x^2 - 24x$$

$$\frac{d^2y}{dx^2} = 0 \quad \Rightarrow 24x(x-1) = 0$$

$$\Rightarrow x = 0, 1$$

$$\frac{d^3y}{dx^3} = 48x - 24$$

when $x = 0, 1 \quad \frac{d^3y}{dx^3} \neq 0.$

∴   points of inflection exist.

when $x = 0, \quad y = 2(0)^4 - 4(0)^3 + 3 = 3$

when $x = 1, \quad y = 2(1)^4 - 4(1)^3 + 3 = 1$

∴   The points of inflection are $(0, 3)$ and $(1, 1)$.

Example 10

Find the intervals on which the curve $f(x) = x^3 - 6x^2 + 9x - 8$ is convex upward and convex downward.

Solution :

We have $f(x) = x^3 - 6x^2 + 9x - 8$

Differentiating with respect to x,

$$f'(x) = 3x^2 - 12x + 9$$

$$f''(x) = 6x - 12$$

$$f''(x) = 0 \Rightarrow 6(x-2) = 0 \qquad \therefore x = 2$$

For

(i) $-\infty < x < 2, f''(x) < 0$ | (i) $x = 0$ then $f''(x) = -12 < 0$

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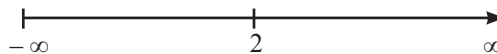
(ii) $2 < x < \infty, f''(x) > 0$ |(ii) $x = 3$ then $f''(x) = 6 > 0$

$\therefore$ The curve is convex upward in the interval $(-\infty, 2)$

The curve is convex downward in the interval $(2, \infty)$

EXERCISE 4.1

1) Show that the function $x^3 + 3x^2 + 3x + 7$ is an increasing function for all real values of x.

2) Prove that $75 - 12x + 6x^2 - x^3$ always decreases as x increases.

3) Separate the intervals in which the function $x^3 + 8x^2 + 5x - 2$ is increasing or decreasing.

4) Find the stationary points and the stationary values of the function $f(x) = 2x^3 + 3x^2 - 12x + 7$.

5) For the following total revenue functions, find when the total revenue (R) is increasing and when it is decreasing. Also discuss the behaviour of marginal revenue (MR).

   (i) $R = -90 + 6x^2 - x^3$ (ii) $R = -105x + 60x^2 - 5x^3$

6) For the following cost functions, find when the total cost (C) is increasing and when it is decreasing. Also discuss the behaviour of marginal cost (MC).

   (i) $C = 2000 + 600x - 45x^2 + x^3$ (ii) $C = 200 + 40x - \frac{1}{2}x^2$.

7) Find the maximum and minimum values of the function

   (i) $x^3 - 6x^2 + 7$ (ii) $2x^3 - 15x^2 + 24x - 15$ (iii) $x^2 + \frac{16}{x}$ (iv) $x^3 - 6x^2 + 9x + 15$

8) Find the absolute (global) maximum and minimum values of the function $f(x) = 3x^5 - 25x^3 + 60x + 15$ in the interval $\left[-\frac{3}{2}, 3\right]$.

9) Find the points of inflection of the curve $y = x^4 - 4x^3 + 2x + 3$.

10) Show that the maximum value of the function $f(x) = x^3 - 27x + 108$ is 108 more than the minimum value.

11) Find the intervals in which the curve $y = x^4 - 3x^3 + 3x^2 + 5x + 1$ is convex upward and convex downward.

12) Determine the value of output q at which the cost function $C = q^2 - 6q + 120$ is minimum.

13) Find the maximum and minimum values of the function $x^5 - 5x^4 + 5x^3 - 1$. Discuss its nature at $x = 0$.

14) Show that the function $f(x) = x^2 + \dfrac{250}{x}$ has a minimum value at $x = 5$.

15) The total revenue (TR) for commodity x is TR $= 12x + \dfrac{x^2}{2} - \dfrac{x^3}{3}$. Show that at the highest point of average revenue (AR), AR = MR (where MR = Marginal Revenue).

4.2 APPLICATION OF MAXIMA AND MINIMA

The concept of zero slope helps us to determine the maximum value of profit functions and the minimum value of cost functions. In this section we will analyse the practical application of Maxima and Minima in commerce.

Example 11

A firm produces x tonnes of output at a total cost C $= (\dfrac{1}{10}x^3 - 5x^2 + 10x + 5)$. At what level of output will the marginal cost and the average variable cost attain their respective minimum?

Solution :

Cost C(x) $= $ Rs.$(\dfrac{1}{10}x^3 - 5x^2 + 10x + 5)$

Marginal Cost $= \dfrac{d}{dx}$ (C)

MC $= \dfrac{3}{10}x^2 - 10x + 10$

Average variable cost $= \dfrac{\text{Variable cost}}{x}$

AVC $= (\dfrac{1}{10}x^2 - 5x + 10)$

(i) Let $y = $ MC $= \dfrac{3}{10}x^2 - 10x + 10$

Differentiating with respect to x, we get

$\dfrac{dy}{dx} = \dfrac{3}{5}x - 10$

Marginal cost is minimum when $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} > 0$

$\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{3}{5}x - 10 = 0$ or $x = \dfrac{50}{3}$

when $x = \dfrac{50}{3}$, $\dfrac{d^2y}{dx^2} = \dfrac{3}{5} > 0$ $\therefore$ MC is mnimum.

$\therefore$ Marginal cost attains its minimum at $x = \dfrac{50}{3}$ units.

(ii)   Let $z = \text{AVC} = (\dfrac{1}{10}x^2 - 5x + 10)$

Differentiating with respect to x, we get

$\dfrac{dz}{dx} = \dfrac{1}{5}x - 5$

AVC is minimum when $\dfrac{dz}{dx} = 0$, and $\dfrac{d^2z}{dx^2} > 0$

$\dfrac{dz}{dx} = 0 \Rightarrow \dfrac{1}{5}x - 5 = 0 \Rightarrow x = 25$

when $x = 25$, $\dfrac{d^2z}{dx^2} = \dfrac{1}{5} > 0$ $\therefore$ AVC is minimum at $x = 25$ units.

$\therefore$ Average variable cost attains minimum at $x = 25$ units.

Example 12

A certain manufacturing concern has total cost function $C = 15 + 9x - 6x^2 + x^3$. Find x, when the total cost is minimum.

Solution :

Cost $C = 15 + 9x - 6x^2 + x^3$

Differentiating with respect to x, we get

$\dfrac{dC}{dx} = 9 - 12x + 3x^2$ ....(1)

Cost is minimum when $\dfrac{dC}{dx} = 0$ and $\dfrac{d^2C}{dx^2} > 0$

$\dfrac{dC}{dx} = 0 \Rightarrow 3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$\Rightarrow x = 3, \quad x = 1$

Differentiating (1) with respect to x we get

$$\frac{d^2C}{dx^2} = -12 + 6x$$

when $x = 1$; $\frac{d^2C}{dx^2} = -12 + 6 = -6 < 0$ ∴C is maximum

when $x = 3$; $\frac{d^2C}{dx^2} = -12 + 18 = 6 > 0$ ∴C is minimum

∴ when $x = 3$, the total cost is minimum

Example 13

The relationship between profit P and advertising cost x is given by $P = \dfrac{4000x}{500 + x} - x$. Find x which maximises P.

Solution :

Profit $P = \dfrac{4000x}{500 + x} - x$

Differentiating with respect to x we get

$$\frac{dP}{dx} = \frac{(500 + x)4000 - (4000x)(1)}{(500 + x)^2} - 1$$

$$= \frac{2000000}{(500 + x^2)} - 1 \qquad \text{...............(1)}$$

Profit is maximum when $\dfrac{dP}{dx} = 0$ and $\dfrac{d^2P}{dx^2} < 0$

$$\frac{dP}{dx} = 0 \Rightarrow \frac{20000000}{(500 + x)^2} - 1 = 0$$

$$\Rightarrow 2000000 \qquad = (500 + x)^2$$

$$\Rightarrow 1000 \times \sqrt{2} \qquad = 500 + x$$

$$1000 \times 1\,414 \ = 500 + x$$

$$x = 914$$

Differentiating (1) with respect to x we get

$$\frac{d^2P}{dx^2} = -\frac{4000000}{(500 + x)^3}$$

∴ when $x = 914$; $\dfrac{d^2P}{dx^2} < 0$ ∴ Profit is maximum.

Example 14

The total cost and total revenue of a firm are given by $C = x^3 - 12x^2 + 48x + 11$ and $R = 83x - 4x^2 - 21$. Find the output (i) when the revenue is maximum (ii) when profit is maximum.

Solution :

(i)   Revenue $R = 83x - 4x^2 - 21$

Differentiating with respect to x,

$$\frac{dR}{dx} = 83 - 8x$$

$$\frac{d^2R}{dx^2} = -8$$

Revenue is maximum when $\frac{dR}{dx} = 0$ and $\frac{d^2R}{dx^2} < 0$

$$\frac{dR}{dx} = 0 \Rightarrow 83 - 8x = 0 \quad \therefore x = \frac{83}{8}$$

Also $\frac{d^2R}{dx^2} = -8 < 0.$ $\therefore$ R is maximum.

$\therefore$ When the output $x = \frac{83}{8}$ units, revenue is maximum

(ii)   Profit $P = R - C$

$$= (83x - 4x^2 - 21) - (x^3 - 12x^2 + 48x + 11)$$

$$= -x^3 + 8x^2 + 35x - 32$$

Differentiating with respect to x,

$$\frac{dP}{dx} = -3x^2 + 16x + 35$$

$$\frac{d^2P}{dx^2} = -6x + 16$$

Profit is maximum when $\frac{dP}{dx} = 0$ and $\frac{d^2P}{dx^2} < 0$

$$\therefore \frac{dP}{dx} = 0 \qquad \Rightarrow -3x^2 + 16x + 35 = 0$$

$$\Rightarrow 3x^2 - 16x - 35 = 0$$

$$\Rightarrow (3x + 5)(x - 7) = 0$$

$$\Rightarrow x = \frac{-5}{3} \text{ or } x = 7$$

when $x = \dfrac{-5}{3}$, $\dfrac{d^2 P}{dx^2} = -6\left(\dfrac{-5}{3}\right) + 16 = 26 > 0$ $\therefore$P is minimum

when $x = 7$, $\dfrac{d^2 P}{dx^2} = -6(7) + 16 = -26 < 0$ $\therefore$P is maximum

$\therefore$ when $x = 7$ units, profit is maximum.

Example 15

The total cost function of a firm is C $= \dfrac{1}{3} x^3 - 5x^2 + 28x + 10$ where x is the output. A tax at Rs.2 per unit of output is imposed and the producer adds it to his cost. If the market demand function is given by p $= 2530 - 5x$, where Rs. p is the price per unit of output, find the profit maximising output and price.

Solution :

Total Revenue (R) $= px$

$$= (2530 - 5x)x = 2530x - 5x^2$$

Total cost after the imposition of tax is

$$C + 2x = \frac{1}{3} x^3 - 5x^2 + 28x + 10 + 2x$$

$$= \frac{1}{3} x^3 - 5x^2 + 30x + 10$$

Profit $=$ Revenue $-$ Cost

$$= (2530x - 5x^2) - (\frac{1}{3} x^3 - 5x^2 + 30x + 10)$$

$$P = -\frac{1}{3} x^3 + 2500x - 10$$

Differentiating P with respect to x,

$$\frac{dP}{dx} = -x^2 + 2500 \qquad\qquad\qquad \text{---------(1)}$$

Conditions for maximum profit are

$$\frac{dP}{dx} = 0 \text{ and } \frac{d^2 P}{dx^2} < 0$$

$$\frac{dP}{dx} = 0 \qquad \Rightarrow 2500 - x^2 = 0$$

$$\Rightarrow x^2 = 2500 \text{ or } x = 50$$

Differentiating (1) with respect to x

$$\frac{d^2 P}{dx^2} = -2x$$

When $x = 50$, $\dfrac{d^2 P}{dx^2} = -100 < 0$ $\therefore$ P is maximum

$\therefore$ Profit maximising output is 50 units

When $x = 50$, price $p = 2530 - (5 \times 50)$

$$= 2530 - 250$$

$$= \text{Rs. } 2280$$

4.2.1 Inventory Control

Inventory is defined as the stock of goods. In practice raw materials are stored upto a capacity for smooth and efficient running of business.

4.2.2 Costs Involved in Inventory Problems

(i)   **Holding cost or storage cost or inventory carrying cost. (C_1)**

The cost associated with carrying or holding the goods in stock is known as holding cost per unit per unit time.

(ii)   **Shortage cost (C_2)**

The penalty costs that are incurred as a result of running out of stock are known as shortage cost.

iii)   **Set up cost or ordering cost or procurement cost : (C_3)**

This is the cost incurred with the placement of order or with the initial preparation of production facility such as resetting the equipment for production.

4.2.3 Economic Order Quantity (EOQ)

Economic order quantity is that size of order which minimises total annual cost of carrying inventory and the cost of ordering under the assumed conditions of certainty with the annual demands known. Economic order quantity is also called Economic lot size formula.

4.2.4 Wilson's Economic Order Quantity Formula

Note :

The derivation of this formula is given for better understanding and is exempted from examination.

The formula is to determine the optimum quantity ordered (or produced) and the optimum interval between successive orders, if the demand is known and uniform with no shortages.

Let us have the following assumptions.

(i)   Let R be the uniform demand per unit time.

(ii)   Supply or production of items to the inventory is instantaneous.

(iii)   Holding cost is Rs. C_1 per unit time.

(iv)   Let there be n orders (cycles) per year, each time q units are ordered (produced).

(v)   Let Rs C_3 be the ordering (set up) cost per order (cycle). Let t be the time taken between each order.

Diagramatic representation of this model is given below :
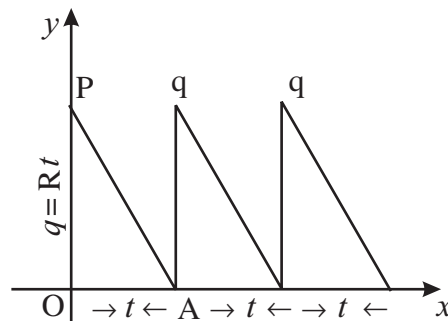
Fig. 4.5

If a production run is made at intervals t, a quantity q = Rt must be produced in each run. Since the stock in small time dt is $Rt\, dt$, the stock in period t is

$$\int_0^t Rt\ dt = \frac{1}{2}Rt^2$$

$$= \frac{1}{2}qt \qquad (Rt = q)$$

= Area of the inventory triangle OAP (Fig. 4.5).

Cost of holding inventory per production run $= \dfrac{1}{2}\,C_1 Rt^2$.

Set up cost per production run $= C_3$.

Total cost per production run $= \dfrac{1}{2}\,C_1 Rt^2 + C_3$

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Average total cost per unit time

$$C(t) = \frac{1}{2} C_1 R t + \frac{C_3}{t} \qquad \text{..............(1)}$$

C(t) is minimum if $\frac{d}{dt} C(t) = 0$ and $\frac{d^2}{dt^2} C(t) > 0$

Differentiating (1) with respect to t we get

$$\frac{d}{dt} C(t) = \frac{1}{2} C_1 R - \frac{C_3}{t^2} \qquad \text{...........(2)}$$

$$\frac{d}{dt} C(t) = 0 \quad \Rightarrow \frac{1}{2} C_1 R - \frac{C_3}{t^2} = 0$$

$$\Rightarrow \quad t = \sqrt{\frac{2C_3}{C_1 R}}$$

Differentiating (2) with respect to t, we get

$$\frac{d^2}{dt^2} C(t) = \frac{2C_3}{t^3} > 0, \text{ when } t = \sqrt{\frac{2C_3}{C_1 R}}$$

Thus C(t) is minimum for optimum time interval

$$t_0 = \sqrt{\frac{2C_3}{C_1 R}}$$

Optimum quantity q_0 to be produced during each production run,

$$\text{EOQ} = q_0 = R t_0 = \sqrt{\frac{2C_3 R}{C_1}}$$

This is known as the Optimal Lot – size formula due to Wilson.

Note: (i) Optimum number of orders per year

$$n_0 = \frac{\text{demand}}{\text{EOQ}} = R \sqrt{\frac{C_1}{2C_3 R}} = \sqrt{\frac{RC_1}{2C_3}} = \frac{1}{t_0}$$

(ii)   Minimum average cost per unit time, $C_0 = \sqrt{2C_1 C_3 R}$

(iii)   Carrying cost = $\frac{q_0}{2} \times C_1$, Ordering cost = $\frac{R}{q_0} \times C_3$

(iv)   At EOQ, Ordering cost = Carrying cost.

Example 16

A manufacturer has to supply 12,000 units of a product per year to his customer. The demand is fixed and known and no shortages are allowed. The inventory holding cost is 20 paise per unit per month and the set up cost per run is Rs.350. Determine (i) the optimum run size q_0 (ii) optimum scheduling period t_0 (iii) minimum total variable yearly cost.

Solution :

Supply rate R $= \dfrac{12,000}{12} = 1,000$ units / month.

$C_1 = 20$ paise per unit per month

$C_3 = $ Rs.350 per run.

(i) $q_0 = \sqrt{\dfrac{2C_3R}{C_1}} = \sqrt{\dfrac{2 \times 350 \times 1000}{0.20}}$

$= 1,870$ units / run.

(ii) $t_0 = \sqrt{\dfrac{2C_3}{C_1R}} = \sqrt{\dfrac{2 \times 350}{0.20 \times 1000}} = 56$ days

(iii) $C_0 = \sqrt{2C_1C_3R} = \sqrt{2 \times 0.20 \times 12 \times 350 \times (1000 \times 12)}$

$= $ Rs. 4, 490 per year.

Example 17

A company uses annually 24,000 units of raw materials which costs Rs. 1.25 per unit, placing each order costs Rs. 22.50 and the holding cost is 5.4% per year of the average inventory. Find the EOQ, time between each order, total number of orders per year. Also verify that at EOQ carrying cost is equal to ordering cost.

Solution :

Requirement $= 24,000$ units / year

Ordering Cost (C_3) $= $ Rs.22.50

Holding cost (C_1) $= 5.4\%$ of the value of each unit.

$= \dfrac{5.4}{100} \times 1.25$

$= $ Rs.0.0675 per unit per year.

EOQ $= \sqrt{\dfrac{2RC_3}{C_1}} = \sqrt{\dfrac{2 \times 24000 \times 22.5}{0.0675}} = 4000$ units.

$$\text{Time between each order} = t_0 = \frac{q_0}{R} = \frac{4000}{24000} = \frac{1}{6} \text{ year}$$

$$\text{Number of order per year} = \frac{R}{q_0} = \frac{24000}{4000} = 6$$

$$\text{At EOQ carrying cost} = \frac{q_0}{2} \times C_1 = \frac{4000}{2} \times 0.0675 = \text{Rs.135}$$

$$\text{Ordering cost} = \frac{R}{q_0} \times C_3 = \frac{24000}{4000} \times 22.50 = \text{Rs.135}$$

Example 18

A manufacturing company purchases 9000 parts of a machine for its annual requirements. Each part costs Rs.20. The ordering cost per order is Rs.15 and carrying charges are 15% of the average inventory per year.

Find   (i) economic order quantity

(ii) time between each order

(iii) minimum average cost

Solution :

$$\text{Requirement R} = 9{,}000 \text{ parts per year}$$

$$C_1 = 15\% \text{ unit cost}$$

$$= \frac{15}{100} \times 20 = \text{Rs.3 each part per year.}$$

$$C_3 = \text{Rs.15 per order}$$

$$\text{EOQ} = \sqrt{\frac{2C_3 R}{C_1}} = \sqrt{\frac{2 \times 15 \times 9000}{3}}$$

$$= 300 \text{ units.}$$

$$t_0 = \frac{q_0}{R} = \frac{300}{9000} = \frac{1}{30} \text{ year}$$

$$= \frac{365}{30} = 12 \text{ days (approximately).}$$

$$\text{Minimum Average cost} = \sqrt{2C_1 C_3 R}$$

$$= \sqrt{2 \times 3 \times 15 \times 9000} = \text{Rs.900}$$

EXERCISE 4.2

1) A certain manufacturing concern has the total cost function $C = \frac{1}{5}x^2 - 6x + 100$. Find when the total cost is minimum.

2) A firm produces an output of x tons of a certain product at a total cost given by $C = 300x - 10x^2 + \frac{1}{3}x^3$. Find the output at which the average cost is least and the corresponding value of the aveage cost.

3) The cost function, when the output is x, is given by $C = x(2e^x + e^{-x})$. Show that the minimum average cost is $2\sqrt{2}$.

4) A firm produces x tons of a valuable metal per month at a total cost C given by C = Rs. $(\frac{1}{3}x^3 - 5x^2 + 75x + 10)$. Find at what level of output, the marginal cost attains its minimum.

5) A firm produces x units of output per week at a total cost of Rs. $(\frac{1}{3}x^3 - x^2 + 5x + 3)$. Find the level at which the marginal cost and the average variable cost attain their respective minimum.

6) It is known that in a mill the number of labourers x and the total cost C are related by $C = \frac{3}{2(x-4)} + \frac{3}{32}x$. What value of x will minimise the cost?

7) $R = 21x - x^2$ and $C = \frac{x^3}{3} - 3x^2 + 9x + 16$ are respectively the sales revenue and cost function of x units sold.

   Find (i) At what output the revenue is maximum? What is the total revenue at this point?

   (ii) What is the marginal cost at a minimum?

   (iii) What output will maximise the profit?

8) A firm has revenue function R = 8x and a production cost function C = 150000 + 60 $\left(\frac{x^2}{900}\right)$. Find the total profit function and the number of units to be sold to get the maximum profit.

9) A radio manufacturer finds that he can sell x radios per week at Rs p each, where p = $2(100 - \frac{x}{4})$. His cost of production of x radios per week is Rs. $(120x + \frac{x^2}{2})$. Show that his profit is maximum when the production is 40 radios per week. Find also his maximum profit per week.

10) A manufacturer can sell x items per week at a price of $p = 600 - 4x$ rupees. Production cost of x items works out to Rs. C where C = 40x + 2000. How much production will yield maximum profit?

11) Find the optimum output of a firm whose total revenue and total cost functions are given by R = 30x – x^2 and C = 20 + 4x, x being the output of the firm.

12) Find EOQ for the data given below. Also verify that carrying costs is equal to ordering costs at EOQ.

Item	Monthly Requirements	Ordering cost per order	Carrying cost per unit
A	9000	Rs. 200	Rs. 3.60
B	25000	Rs. 648	Rs. 10.00
C	8000	Rs. 100	Rs. 0.60

13) Calculate the EOQ in units and total variable cost for the following items, assuming an ordering cost of Rs.5 and a holding cost of 10%

Item	Annual demand	Unit price (Rs.)
A	460 Units	1.00
B	392 Units	8.60
C	800 Units	0.02
D	1500 Units	0.52

14) A manufacturer has to supply his customer with 600 units of his products per year. Shortages are not allowed and storage cost amounts to 60 paise per unit per year. When the set up cost is Rs. 80 find,

(i) the economic order quantity.

(ii) the minimum average yearly cost

(iii) the optimum number of orders per year

(iv) the optimum period of supply per optimum order.

15) The annual demand for an item is 3200 units. The unit cost is Rs.6 and inventory carrying charges 25% per annum. If the cost of one procurement is Rs.150, determine (i) Economic order quantity. (ii) Time between two consecutive orders (iii) Number of orders per year (iv) minimum average yearly cost.

4.3 PARTIAL DERIVATIVES

In differential calculus, so far we have discussed functions of one variable of the form $y = f(x)$. Further one variable may be expressed as a function of several variables. For example, production may be treated as a function of labour and capital and price may be a function of supply and demand. In general, the cost or profit depends upon a number of independent variables, for example, prices of raw materials, wages on labour, market conditions and so on. Thus a dependent variable y depends on a number of independent variables $x_1, x_2, x_3 .. x_n$. It is denoted by $y = f(x_1, x_2, x_3 .. x_n)$ and is called a function of n variables. In this section, we will restrict the study to functions of two or three variables and their derivatives only.

4.3.1. Definition

Let $u = f(x, y)$ be a function of two independent variables x and y. The derivative of $f(x, y)$ with respect to x, keeping y constant, is called partial derivative of u with respect to x and is denoted by $\dfrac{\partial u}{\partial x}$ or $\dfrac{\partial f}{\partial x}$ or f_x or u_x. Similarly we can define partial derivative of f with respect to y.

Thus we have

$$\frac{\partial f}{\partial x} = \underset{\Delta x \to 0}{\text{Lt}} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

provided the limit exists.

(Here y is fixed and Δx is the increment of x)

Also $\dfrac{\partial f}{\partial y} = \underset{\Delta y \to 0}{\text{Lt}} = \dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$

provided the limit exists.

(Here x is fixed and Δy is the increment of y)

4.3.2 Successive Partial Derivatives.

The partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are in general functions of x and y. So we can differentiate functions $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ partially with respect to x and y. These derivatives are called second order partial derivatives of $f(x, y)$. Second order partial derivatives are denoted

by $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2} = f_{xx}$

$\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2} = f_{yy}$

$\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y} = f_{xy}$

$\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x} = f_{yx}$

Note

If f, f_x, f_y are continuous then $f_{xy} = f_{yx}$

4.3.3 Homogeneous Function

A function $f(x, y)$ of two independent variable x and y is said to be homogeneous in x and y of degree n if $f(tx, ty) = t^n f(x, y)$ for $t > 0$.

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4.3.4 Euler's Theorem on Homogeneous Function

Theorem : Let f be a homogeneous function in x and y of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n \ f.$$

Corrollary : In general if $f\ (x_1,\ x_2,\ x_3\ ..\ x_m)$ is a homogeneous function of degree n in variables $x_1, x_2, x_3\ ..\ x_m$, then,

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + x_3\frac{\partial f}{\partial x_3} + ... + x_m\frac{\partial f}{\partial x_m} = n\,f.$$

Example 19

It $u(x, y) = 1000 - x^3 - y^2 + 4x^3y^6 + 8y$, find each of the following.

(i) $\dfrac{\partial u}{\partial x}$ (ii) $\dfrac{\partial u}{\partial y}$ (iii) $\dfrac{\partial^2 u}{\partial x^2}$ (iv) $\dfrac{\partial^2 u}{\partial y^2}$ (v) $\dfrac{\partial^2 u}{\partial x \partial y}$ (vi) $\dfrac{\partial^2 u}{\partial y \partial x}$

Solution:

$u\ (x,\ y) = 1000 - x^3 - y^2 + 4x^3y^6 + 8y$

(i)

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(1000 - x^3 - y^2 + 4x^3 y^6 + 8y)$$

$$= 0 - 3x^2 - 0 + 4(3x^2)y^6 + 0$$

$$= -3x^2 + 12x^2 y^6$$

(ii)

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(1000 - x^3 - y^2 + 4x^3 y^6 + 8y)$$

$$= 0 - 0 - 2y + 4x^3(6y^5) + 8$$

$$= -2y + 24x^3 y^5 + 8$$

(iii)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(-3x^2 + 12x^2 y^6)$$

$$= -6x + 12(2x)y^6$$

$$= -6x + 24xy^6$$

(iv)

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(-2y + 24x^3 y^5 + 8)$$

$$= -2 + 24x^3(5y^4) + 0$$

$$= -2 + 120x^3 y^4$$

(v) $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial}\right)$

$$= \dfrac{\partial}{\partial x}(-2y + 24x^3 y^5 + 8)$$

$$= 0 + 24(3x^2)y^5 + 0$$

$$= 72x^2 y^5$$

(vi) $\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$

$$= \dfrac{\partial}{\partial y}(-3x^2 + 12x^2 y^6)$$

$$= 0 + 12x^2(6y^5) = 72x^2 y^5$$

Example 20

If $f(x, y) = 3x^2 + 4y^3 + 6xy - x^2y^3 + 5$ find (i) $f_x(1, -1)$ (ii) $f_{yy}(1, 1)$ (iii) $f_{xy}(2, 1)$

Solution :

(i) $f(x, y) = 3x^2 + 4y^3 + 6xy - x^2y^3 + 5$

$$f_x = \dfrac{\partial}{\partial x}(f) = \dfrac{\partial}{\partial x}(3x^2 + 4y^3 + 6xy - x^2y^3 + 5)$$

$$= 6x + 0 + 6(1)y - (2x)y^3 + 0$$

$$= 6x + 6y - 2xy^3$$

$$f_x(1,-1) = 6(1) + 6(-1) - 2(1)(-1)^3 = 2$$

(ii) $f_y = \dfrac{\partial}{\partial y}(f) = \dfrac{\partial}{\partial y}(3x^2 + 4y^3 + 6xy - x^2y^3 + 5)$

$$= 12y^2 + 6x - 3x^2 y^2$$

$$f_{yy} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$$

$$= \dfrac{\partial}{\partial y}(12y^2 + 6x - 3x^2 y^2)$$

$$= 24y - 6x^2 y$$

$$f_{yy}(1,1) = 18$$

(iii) $f_{xy} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial x}(12y^2 + 6x - 3x^2 y^2)$

$$= 6 - 6xy^2$$

$$f_{yy}(2,1) = -6$$

Example 21

If u = log $\sqrt{x^2 + y^2 + z^2}$, then prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{x^2 + y^2 + z^2}$$

Solution :

We have $\quad u = \frac{1}{2} \log(x^2 + y^2 + z^2) \quad\quad\quad$ ....(1)

Differentiating (1) partially with respect to x,

$$\frac{\partial u}{\partial x} = \frac{1}{2} \frac{2x}{x^2 + y^2 + z^2} = \frac{x}{x^2 + y^2 + z^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2 + z^2}\right) = \frac{(x^2 + y^2 + z^2)(1) - x(2x)}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{x^2 + y^2 + z^2 - 2x^2}{(x^2 + y^2 + z^2)^2} = \frac{-x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2}$$

Differentiating (1) partially with respect to y we get,

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2 + z^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2 + z^2)(1) - y(2y)}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{-y^2 + z^2 + x^2}{(x^2 + y^2 + z^2)^2}$$

Differntiating (1) partially with respect to z we get,

$$\frac{\partial u}{\partial z} = \frac{z}{x^2 + y^2 + z^2}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{(x^2 + y^2 + z^2)(1) - z(2z)}{(x^2 + y^2 + z^2)^2} = \frac{-z^2 + x^2 + y^2}{(x^2 + y^2 + z^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-x^2 + y^2 + z^2 - y^2 + z^2 + x^2 - z^2 + x^2 + y^2}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} = \frac{1}{x^2 + y^2 + z^2}$$

Example 22

Using Euler's theorem if $u = \log \dfrac{x^4 + y^4}{x - y}$ show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$.

Solution :

$$u = \log \frac{x^4 + y^4}{x - y}$$

$$\Rightarrow e^u = \frac{x^4 + y^4}{x - y}$$

This is a homogeneous function of degree 3 in x and y

∴ By Euler's theorem,

$$x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = 3e^u$$

$$x\, e^u\, \frac{\partial u}{\partial x} + y e^u\, \frac{\partial u}{\partial y} = 3e^u$$

dividing by e^u we get $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$

Example 23

The revenue derived from selling x calculators and y adding machines is given by $R(x, y) = -x^2 + 8x - 2y^2 + 6y + 2xy + 50$. If 4 calculators and 3 adding machines are sold, find the marginal revenue of selling (i) one more calculator (ii) one more adding machine.

Solution :

(i)   The marginal revenue of selling one more calculator is R_x.

$$R_x = \frac{\partial}{\partial x}(R) = \frac{\partial}{\partial x}(-x^2 + 8x - 2y^2 + 6y + 2xy + 50)$$

$$= -2x + 8 - 0 + 0 + 2(1)(y)$$

$$R_x(4,3) = -2(4) + 8 + 2(3) = 6$$

∴ At (4,3), revenue is increasing at the rate of Rs.6 per caculator sold.

∴ Marginal revenue is Rs. 6.

(ii)   Marginal Revenue of selling one more adding machine is R_y

$$R_y = \frac{\partial}{\partial y}(R) = \frac{\partial}{\partial y}(-x^2 + 8x - 2y^2 + 6y + 2xy + 50)$$

$$= 0 + 0 - 4y + 6 + 2x(1)$$

$$= -4y + 6 + 2x$$

$$R_y(4,3) = -4(3) + 6 + 2(4) = 2$$

Thus at (4, 3) revenue is increasing at the rate of approximately Rs.2 per adding machine.

Hence Marginal revenue is Rs.2.

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EXERCISE 4.3

1) If $u = 4x^2 - 3y^2 + 6xy$, find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$.

2) If $u = x^3 + y^3 + z^3 - 3xyz$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 3u$.

3) If $z = 4x^6 - 8x^3 - 7x + 6xy + 8y + x^3y^5$, find each of the following

   (i) $\dfrac{\partial u}{\partial x}$ (ii) $\dfrac{\partial u}{\partial y}$ (iii) $\dfrac{\partial^2 z}{\partial x^2}$ (iv) $\dfrac{\partial^2 z}{\partial y^2}$ (v) $\dfrac{\partial^2 z}{\partial x \partial y}$ (vi) $\dfrac{\partial^2 z}{\partial y \partial x}$

4) If $f(x, y) = 4x^2 - 8y^3 + 6x^5y^2 + 4x + 6y + 9$, evaluate the following.

   (i) f_x (ii) $f_x(2, 1)$ (iii) f_y (iv) $f_y(0, 2)$ (v) f_{xx}

   (vi) $f_{xx}(2, 1)$ (vii) f_{yy} (viii) $f_{yy}(1, 0)$ (ix) f_{xy} (x) $f_{xy}(2, 3)$

   (xi) $f_{yx}(2, 3)$

5) If $u = x^2y + y^2z + z^2x$, show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = (x + y + z)^2$.

6) If $u = \log\sqrt{x^2 + y^2}$, show that $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 = \dfrac{1}{x^2 + y^2}$

7) If $u = x^3 + 3xy^2 + y^3$, prove that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$

8) If $u = \log(x^2 + y^2 + z^2)$ prove that $x\dfrac{\partial^2 u}{\partial y \partial z} = y\dfrac{\partial^2 u}{\partial z \partial x} = z\dfrac{\partial^2 u}{\partial x \partial y}$.

9) Use Euler's theorem to prove the following

   (i) If $u = \dfrac{x^2 + y^2}{\sqrt{x + y}}$ then prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{3}{2}u$

   (ii) If $z = e^{x^3 + y^3}$ then prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 3z\log z$.

   (iii) If $f = \log\left(\dfrac{x^2 + y^2}{x + y}\right)$ then show that $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 1$.

   (iv) If $u = \tan^{-1}\left(\dfrac{x^2 + y^2}{x - y}\right)$ then prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\sin 2u$.

10) The cost of producing x washers and y dryers is given by $C(x, y) = 40x + 200y + 10xy + 500$. Presently, 50 washers and 90 dryers are being produced. Find the marginal cost of producing (i) one more washer (ii) one more dryer.

11) The revenue derived from selling x pens and y note books is given by $R(x, y) = 2x^2 + y^2 + 4x + 5y + 800$.

At present, the retailer is selling 30 pens and 50 notebooks. Which of these two product lines should be expanded in order to yield the greater increase in revenue?

12) The annual profit of a certain hotel is given by $P(x, y) = 100x^2 + 4y^2 + 2x + 5y + 10000$. Where x is the number of rooms available for rent and y is the monthly advertising expenditures. Presently, the hotel has 90 rooms available and is spending Rs.1000 per month on advertising.

(i) If an additional room is constructed, how will this affect annual profit?

(ii) If an additional rupee is spent on monthly advertising expenditures, how will this affect annual profit?

4.4 APPLICATIONS OF PARTIAL DERIVATIVES

In this section we learn how the concept of partial derivatives are used in the field of Commerce and Economics.

4.4.1 Production Function

Production P of a firm depends upon several economic factors like investment or capital (K), labour (L), raw material (R), etc. Thus $P = f(K, L, R, ...)$. If P depends only on labour (L) and capital (K), then we write $P = f(L, K)$.

4.4.2 Marginal Productivities

Let $P = f(L, K)$ represent a production function of two variables L and K.

$\dfrac{\partial P}{\partial L}$ is called the 'Marginal Productivity of Labour' and $\dfrac{\partial P}{\partial K}$ is the 'Marginal Productivity of Capital'.

4.4.3 Partial Elasticities of Demand

Let $q_1 = f(p_1, p_2)$ be the demand for commodity A which depends upon the prices p_1 and p_2 of commodities A and B respectively.

The partial elasticity of demand q_1 with respect to p_1 is defined as

$$-\frac{p_1}{q_1} \frac{\partial q_1}{\partial p_1} = \frac{Eq_1}{Ep_1}$$

Similarly the partial elasticity of demand of q_1 with respect to price p_2 is -

$$-\frac{p_2}{q_1} \frac{\partial q_1}{\partial p_2} = \frac{Eq_1}{Ep_2}$$

Example 24

Find the marginal productivities of capital (K) and labour (L), if P = 10K – K² + KL, when K = 2 and L = 6.

Solution :

We have $P = 10K - K^2 + KL$ ....(1)

The marginal productivity of capital is $\dfrac{\partial P}{\partial K}$.

∴ Differentiating (1) partially with respect to K we get

$$\dfrac{\partial P}{\partial K} = 10 - 2K + (1)\, L$$
$$= 10 - 2K + L$$

when K = 2, and L = 6, $\dfrac{\partial P}{\partial K} = 10 - 2(2) + 6 = 12$

The marginal productivity of labour is $\dfrac{\partial P}{\partial L}$

∴ Differentiating (1) partially with respect to L we get

$$\dfrac{\partial P}{\partial L} = K$$

when K = 2, and L = 6 $\dfrac{\partial P}{\partial L} = 2$.

∴ Marginal productivity of capital = 12 units

∴ Marginal productivity of labour = 2 units

Example 25

For some firm, the number of units produced when using x units of labour and y units of capital is given by the production function $f(x, y) = 80\ x^{\frac{1}{4}}\ y^{\frac{3}{4}}$. Find (i) the equations for both marginal productivities. (ii) Evaluate and interpret the results when 625 units of labour and 81 units of capital are used.

Solution :

Given $f(x, y) = 80\ x^{\frac{1}{4}}\ y^{\frac{3}{4}}$ .... (1)

Marginal productivity of labour is $f_x(x, y)$.

∴ Differentiating (1) partially with respect to x, we get

$$f_x = 80 \dfrac{1}{4} x^{-\frac{3}{4}} y^{\frac{3}{4}} = 20 x^{-\frac{3}{4}} y^{\frac{3}{4}}$$

Marginal productivity of capital is $f_y(x, y)$

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$\therefore$ Differentiating (1) partially with resepect to y we get

$$f_y = 80\ x^{\frac{1}{4}}\left(\frac{3}{4}\right)y^{-\frac{1}{4}} = 60x^{\frac{1}{4}}y^{-\frac{1}{4}}$$

(ii) $f_x(625, 81) = 20(625)^{-\frac{3}{4}}(81)^{\frac{3}{4}}$

$$= 20\left(\frac{1}{125}\right)(27) = 4.32$$

i.e. when 625 units of labour and 81 units of capital are used, one more unit of labour results in 4.32 more units of production.

$$f_y(625,81) = 60(625)^{\frac{1}{4}}(81)^{-\frac{1}{4}}$$

$$= 60(5)\left(\frac{1}{3}\right) = 100$$

(i.e.) when 625 units of labour and 81 units of capital are used, one more unit of capital results in 100 more units of production.

Example 26

The demand for a commodity A is $q_1 = 240 - p_1^2 + 6p_2 - p_1 p_2$. Find the partial Elasticities $\dfrac{Eq_1}{Ep_1}$ and $\dfrac{Eq_1}{Ep_2}$ when $p_1 = 5$ and $p_2 = 4$.

Solution :

Given $q_1 = 240 - p^2_1 + 6p_2 - p_1p_2$

$$\frac{\partial q_1}{\partial p_1} = -2p_1 - p_2$$

$$\frac{\partial q_1}{\partial p_2} = 6 - p_1$$

$$\frac{Eq_1}{Ep_1} = -\frac{p_1}{q_1}\frac{\partial q_1}{\partial p_1}$$

$$= \frac{-p_1}{240 - p_1{}^2 + 6p_2 - p_1p_2}(-2p_1 - p_2)$$

when $p_1 = 5$ and $p_2 = 4$

$$\left(\frac{Eq_1}{Ep_1}\right) = \frac{-(5)\,(-10 - 4)}{240 - 25 + 24 - 20} = \frac{70}{219}$$

$$\frac{Eq_1}{Ep_2} = \frac{-p_2}{q_1}\frac{\partial q_1}{\partial p_2}$$

$$= \frac{-p_2(6 - p_1)}{240 - p_1^2 + 6p_2 - p_1 p_2}$$

when $p_1 = 5$ and $p_2 = 4$

$$\left(\frac{Eq_1}{Ep_2}\right) = \frac{-4(6 - 5)}{240 - 25 + 24 - 20} = \frac{-4}{219}$$

EXERCISE 4.4

1) The production function of a commodity is $P = 10L + 5K - L^2 - 2K^2 + 3KL$.

   Find (i) the marginal productivity of labour

   (ii) the marginal productivity of capital

   (iii) the two marginal productivities when $L = 1$ and $K = 2$.

2) If the production of a firm is given by $P = 3K^2L^2 - 2L^4 - K^4$,
   prove that $L \dfrac{\partial P}{\partial L} + K \dfrac{\partial P}{\partial K} = 4P$.

3) If the production function is $Z = y^2 - xy + x^2$ where x is the labour and y is the capital find the marginal productivities of x and y when $x = 2$ and $y = 3$.

4) For some firm, the number of units produced when using x_1 units of labour and y units of capital is given by the production function $f(x, y) = 100 x^{\frac{2}{5}} y^{\frac{3}{5}}$. Find

   (i) both marginal productivities.

   (ii) interpret the results when 243 units of labour and 32 units of capital are used.

5) For the production function $p = 5(L)^{0.7} (K)^{0.3}$ find the marginal productivities of labour (L) and capital (K) when $L = 10$ and $K = 3$.

6) For the production function $P = C(L)^{\alpha} (K)^{\beta}$ where C is a positive constant and if $\alpha + \beta = 1$ show that

   $$K \frac{\partial P}{\partial K} + L \frac{\partial P}{\partial L} = P.$$

7) The demand for a quantity A is $q_1 = 16 - 3p_1 - 2p_2^2$. Find (i) the partial elasticities $\dfrac{Eq_1}{Ep_1}, \dfrac{Eq_1}{Ep_2}$ (ii) the partial elasticities for $p_1 = 2$ and $p_2 = 1$.

8) The demand for a commodity A is $q_1 = 10 - 3p_1 - 2p_2$. Find the partial elasticities when $p_1 = p_2 = 1$.

9) The demand for a commodity X is $q_1 = 15 - p_1^2 - 3p_2$. Find the partial elasticities when $p_1 = 3$ and $p_2 = 1$.

10) The demand function for a commodity Y is $q_1 = 12 - p_1^2 + p_1p_2$. Find the partial elasticities when $p_1 = 10$ and $p_2 = 4$.

EXERCISE 4.5

Choose the correct answer

1) The stationary value of x for $f(x) = 3(x - 1)(x - 2)$ is

    a) 3   b) $\dfrac{3}{2}$ c) $\dfrac{2}{3}$ d) $\dfrac{-3}{2}$

2) The maximum value of $f(x) = \cos x$ is

    a) 0   b) $\dfrac{\sqrt{3}}{2}$ c) $\dfrac{1}{2}$ d) 1

3) $y = x^3$ is always

    a) an increasing function of x b) decreasing function of x

    c) a constant function   d) none of these.

4) The curve $y = 4 - 2x - x^2$ is

    a) concave upward   b) concave downward

    c) straight line   d) none of these.

5) If $u = e^{x^2 + y^2}$, then $\dfrac{\partial u}{\partial x}$ is equal to

    a) y^2u b) x^2u c) $2xu$ d) $2yu$

6) If $u = \log(e^x + e^y)$ then $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}$ is equal to

    a) $\dfrac{1}{e^x + e^y}$ b) $\dfrac{e^x}{e^x + e^y}$ c) 1   d) $e^x + e^y$

7) If $u = x^y$ $(x > 0)$ then $\dfrac{\partial u}{\partial y}$ is equal to

    a) $x^y \log x$ b) $\log x$ c) $y^x \log x$ d) $\log y^x$

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8) $f(x, y) = \dfrac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}$ is a homogeneous function of degree

a) $\dfrac{1}{2}$ b) $\dfrac{1}{3}$ c) $\dfrac{1}{6}$ d) $\dfrac{1}{5}$

9) If $f(x, y) = 2x + ye^{-x}$, then $f_y(1, 0)$ is equal to

a) e   b) $\dfrac{1}{e}$ c) e^2 d) $\dfrac{1}{e^2}$

10) If $f(x, y) = x^3 + y^3 + 3xy$ then f_{xy} is

a) 6x   b) 6y   c) 2   d) 3

11) If marginal revenue is Rs.25 and the elasticitiy of demand with respect to price is 2, then average revenue is

a) Rs.50   b) Rs.25   c) Rs.27   d) Rs.12.50

12) The elasticity of demand when marginal revenue is zero, is

a) 1   b) 2   c) – 5   d) 0

13) The marginal revenue is Rs.40 and the average revenue is Rs.60. The elasticity of demand with respect to price is

a) 1   b) 0   c) 2   d) 3

14) If $u = x^2 - 4xy + y^2$ then $\dfrac{\partial^2 u}{\partial y^2}$ is

a) 2   b) 2xy   c) $2x^2 y$ d) $2xy^2$

15) If $z = x^3 + 3xy^2 + y^3$ then the marginal productivity of x is

a) $x^2 + y^2$ b) $6xy + 3y^2$ c) $3(x^2 + y^2)$ d) $(x^2 + y^2)^2$

16) If $q_1 = 2000 + 8p_1 - p_2$ then $\dfrac{\partial q_1}{\partial p_1}$ is

a) 8   b) – 1   c) 2000   d) 0

17) The marginal productivity of labour (L) for the production function $P = 15K - L^2 + 2KL$ when $L = 3$ and $K = 4$ is

a) 21   b) 12   c) 2   d) 3

18) The production function for a firm is $P = 3L^2 - 5KL + 2k^2$. The marginal productivity of capital (K) when L = 2 and K = 3 is

a) 5   b) 3   c) 6   d) 2

19) The cost function $y = 40 - 4x + x^2$ is minimum when x

a) $x = 2$ b) $x = -2$ c) $x = 4$ d) $x = -4$

20) If R = 5000 units / year, C_1 = 20 paise, C_3 = Rs.20 then EOQ is

a) 1000   b) 5000   c) 200   d) 100

APPLICATIONS OF INTEGRATION $\quad$ **5**

_____

In the present chapter we give some properties of definite integral, geometrical interpretation of definite integral and applications of integration in finding total and average functions from the given marginal functions. We further find demand function when the price and elasticity of demand are known. Finally we discuss a few problems under consumers' surplus and producers' surplus.

5.1 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

We state below an important theorem which enables us to evaluate definite integrals by making use of antiderivative.

Theorem :

Let f be a continuous function defined on the closed interval [a, b]. Let F be an antiderivative of f. Then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

5.1.1 Properties of definite integrals

1) $\quad \displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

Proof :

Let F(x) be the antiderivative of f(x). Then we have,

$$\int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a) = -[F(a) - F(b)] = -\int_b^a f(x)\, dx$$

2) $\quad \displaystyle\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ for $a < c < b.$

Proof :

Let *a, b, c* be three real numbers such that $a < c < b$.

$$\text{L.H.S.} = \int_a^b f(x)\, dx = F(b) - F(a) \qquad\qquad .... (1)$$

$$\text{R.H.S.} = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx = F(c) - F(a) + F(b) - F(c) = F(b) - F(a) \qquad .... (2)$$

From (1) and (2), $\displaystyle\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$

3) $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Proof :

Let $a+b-x = t$ $\qquad\qquad \therefore -dx = dt$

$\qquad x = a \implies t = b$

$\qquad x = b \implies t = a$

Thus when x varies from a to b,

t varies from b to a.

$\therefore \displaystyle\int_a^b f(x)\,dx = -\int_b^a f(a+b-t)\,dt$

$\qquad\qquad = \displaystyle\int_a^b f(a+b-t)\,dt \qquad\qquad$ [by property (1)]

$\qquad\qquad = \displaystyle\int_a^b f(a+b-x)\,dx \quad [\text{since } \int_a^b f(x)\,dx = \int_a^b f(t)\,dt]$

4) $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Proof :

Let $a-x = t \quad -dx = dt$

$x = 0 \implies t = a$

$x = a \implies t = 0$

$\therefore \displaystyle\int_0^a f(x)\,dx = \int_a^0 f(a-t)(-dt) = \int_0^a f(a-t)\,dt$

$\qquad\qquad\qquad = \displaystyle\int_0^a f(a-x)\,dx$

5) (i) $\displaystyle\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is an even function.

(ii) $\displaystyle\int_{-a}^a f(x)\,dx = 0$ if $f(x)$ is an odd function.

Proof :

(i) If $f(x)$ is an even function, then $f(-x) = f(x)$.

$\displaystyle\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx \qquad\qquad$ [by property (2)]

Put $t = -x$ in the first integral then, $dt = -dx$

$x = -a \implies t = a$

$x = 0 \implies t = 0$

$\therefore \displaystyle\int_{-a}^a f(x)\,dx = -\int_a^0 f(-t)\,dt + \int_0^a f(x)\,dx$

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$$= \int_0^a f(-x)\, dx + \int_0^a f(x)\, dx$$

$$= \int_0^a f(x)\, dx + \int_0^a f(x)\, dx \qquad (f(x) \text{ is an even function})$$

$$= 2 \int_0^a f(x)\, dx$$

(ii)   If f(x) is an odd function then

$$f(-x) = -f(x)$$

$$\therefore \int_{-a}^a f(x)\, dx = \int_{-a}^0 f(x)\, dx + \int_0^a f(x)\, dx$$

Put t = – x in the first integral. Then $dt = - dx$

$$x = -a \implies t = a$$
$$x = 0 \implies t = 0$$

$$\therefore \int_{-a}^a f(x)\, dx = -\int_a^0 f(-t)\, dt + \int_0^a f(x)\, dx$$

$$= \int_0^a f(-x)\, dx + \int_0^a f(x)\, dx$$

$$= -\int_0^a f(x)\, dx + \int_0^a f(x)\, dx \qquad (\text{since } f(x) \text{ is an odd function})$$

$$= 0$$

Example 1

Evaluate $\int_{-1}^1 (x^3 + x)\, dx$

Solution :

$f(x) = x^3 + x$ is an odd function.

$$\implies \int_{-1}^1 (x^3 + x)\, dx = 0 \qquad \text{[by property 5(ii)]}$$

Example 2

Evaluate $\int_{-2}^2 (x^4 + x^2)\, dx$

Solution :

$f(x) = x^4 + x^2$ is an even function

$$\Rightarrow \int_{-2}^{2}(x^4+x^2)\,dx = 2\int_{0}^{2}(x^4+x^2)\,dx \qquad\qquad \text{[by property 5(i)]}$$

$$= 2\left[\frac{x^5}{5}+\frac{x^3}{3}\right]_{0}^{2}$$

$$= 2\left[\frac{2^5}{5}+\frac{2^3}{3}\right] = \frac{272}{15}$$

Example 3

Evaluate $\displaystyle\int_{0}^{\frac{\pi}{2}}\frac{\sqrt{\sin^3 x}}{\sqrt{\sin^3 x}+\sqrt{\cos^3 x}}\,dx$

Solution :

Let $\displaystyle I = \int_{0}^{\frac{\pi}{2}}\frac{\sqrt{\sin^3 x}}{\sqrt{\sin^3 x}+\sqrt{\cos^3 x}}\,dx$ -------(1)

By property (4), $\displaystyle\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx$

Again $\displaystyle I = \int_{0}^{\frac{\pi}{2}}\frac{\sqrt{\sin^3\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin^3\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos^3\left(\frac{\pi}{2}-x\right)}}\,dx$

$$= \int_{0}^{\frac{\pi}{2}}\frac{\sqrt{\cos^3 x}}{\sqrt{\cos^3 x}+\sqrt{\sin^3 x}}\,dx \qquad\text{-------(2)}$$

Adding (1) and (2) we get,

$$2I = \int_{0}^{\frac{\pi}{2}}\frac{\sqrt{\sin^3 x}+\sqrt{\cos^3 x}}{\sqrt{\sin^3 x}+\sqrt{\cos^3 x}}\,dx$$

$$= \int_{0}^{\frac{\pi}{2}}dx = [x]_{0}^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$\therefore\ I = \frac{\pi}{4}$$

$$\Rightarrow \int_{0}^{\frac{\pi}{2}}\frac{\sqrt{\sin^3 x}}{\sqrt{\sin^3 x}+\sqrt{\cos^3 x}}\,dx = \frac{\pi}{4}$$

Example 4

Evaluate $\int_0^1 x(1-x)^5 dx$

Solution :

By property (4), $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$\therefore \int_0^1 x(1-x^5)\,dx = \int_0^1 (1-x)(1-1+x)^5\,dx = \int_0^1 (1-x)\,x^5\,dx$

$$= \int_0^1 (x^5 - x^6)\,dx = \left[\dfrac{x^6}{6} - \dfrac{x^7}{7}\right]_0^1 = \dfrac{1}{42}$$

$\therefore \int_0^1 x(1-x)^5\,dx = \dfrac{1}{42}$

Example 5

Evaluate $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \dfrac{dx}{1+\sqrt{\tan x}}$

Solution :

Let $I = \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \dfrac{dx}{1+\sqrt{\tan x}}$

$= \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \dfrac{\sqrt{\cos x}\,dx}{\sqrt{\sin x}+\sqrt{\cos x}}$ --------(1)

By property (3), $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$\therefore I = \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \dfrac{\sqrt{\cos(\frac{\pi}{6}+\frac{\pi}{3}-x)}}{\sqrt{\sin(\frac{\pi}{6}+\frac{\pi}{3}-x)}+\sqrt{\cos(\frac{\pi}{6}+\frac{\pi}{3}-x)}}\,dx$

$= \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \dfrac{\sqrt{\cos(\frac{\pi}{2}-x)}}{\sqrt{\sin(\frac{\pi}{2}-x)}+\sqrt{\cos(\frac{\pi}{2}-x)}}\,dx$

$= \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \dfrac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$ --------(2)

Adding (1) and (2) we get

$$2\,I \;=\; \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \;=\; \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx \;=\; [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \;=\; \frac{\pi}{6}$$

$$\therefore\, I \;=\; \frac{\pi}{12} \qquad \therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}} \;=\; \frac{\pi}{12}$$

EXERCISE 5.1

Evaluate the following using the properties of definite integral:

1) $\displaystyle\int_{-10}^{10}\left(4x^5 + 6x^3 + \frac{2}{3}\,x\right)dx$ 2) $\displaystyle\int_{-2}^{2}(3x^2 + 5x^4)\,dx$

3) $\displaystyle\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \; dx$ 4) $\displaystyle\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$ 5) $\displaystyle\int_{0}^{2} x\sqrt{2-x}\;dx$

6) $\displaystyle\int_{0}^{1} x(1-x)^3 dx$ 7) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\cot x}}$ 8) $\displaystyle\int_{0}^{2} \frac{\sqrt{x}\,dx}{\sqrt{x}+\sqrt{2-x}}$

9) $\displaystyle\int_{0}^{\pi} x\sin^2 x \; dx$ 10) $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{a\sin x + b\cos x}{\sin x + \cos x}\,dx$

5.2 GEOMETRICAL INTERPRETATION OF DEFINITE INTEGRAL AS AREA UNDER A CURVE

The area A of the region bounded by the curve $y = f(x)$, the x - axis and the ordinates at $x = a$ and $x = b$ is given by,

$$\text{Area, } A = \int_{a}^{b} y \; dx$$

$$= \int_{a}^{b} f(x) \; dx$$
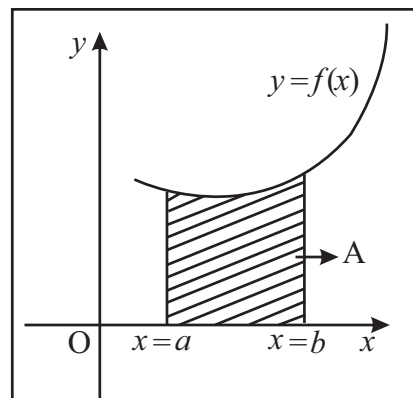
Fig 5.1

Note

The graph of $y = f(x)$ must not cross the x -axis between $x = a$ and $x = b$.

Similarly the area A of the region bounded by the curve $x = g(y)$, the y - axis and the

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abscissa $y = c$ and $y = d$ is given by

Area, A = $\int\limits_{c}^{d} x \ dy$

= $\int\limits_{c}^{d} g(y) \ dy$
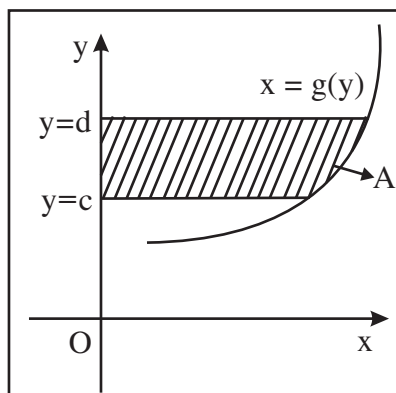
Fig. 5.2

Note

The graph of $x = g(y)$ must not cross the axis of y between $y = c$ and $y = d$.

Example 6

Find the area enclosed by the parabola $y^2 = 4x$, $x = 1$, $x = 4$ and the x - axis.

Solution :

The area under the curve is

$A = \int\limits_{a}^{b} y \ dx$

$= \int\limits_{1}^{4} \sqrt{4x} \ dx$

$= 2 \int\limits_{1}^{4} \sqrt{x} \ dx = 2 \left[\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_{1}^{4}$

$= 2 \times \dfrac{2}{3} \left(4^{\frac{3}{2}} - 1^{\frac{3}{2}} \right)$

$= \dfrac{28}{3}$ sq. units.
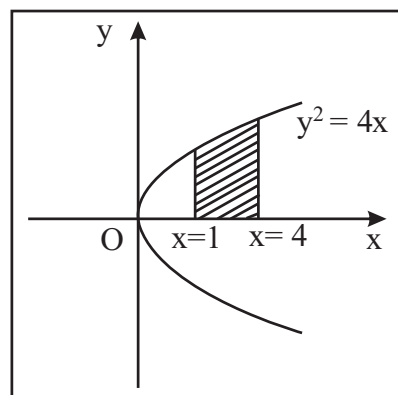
Fig.5.3

Example 7

Find the area of the region bounded by the parabola $x^2 = 4y$, $y = 2$, $y = 4$ and the y - axis.

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Solution :

The area under the curve is,

$$A = \int_c^d x \, dy$$

$$= \int_2^4 \sqrt{4y} \, dy$$

$$= 2\int_2^4 \sqrt{y} \, dy = 2\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$

$$= 2 \times \frac{2}{3}\left(4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right) = \frac{32 - 8\sqrt{2}}{3} \; sq.\,units.$$
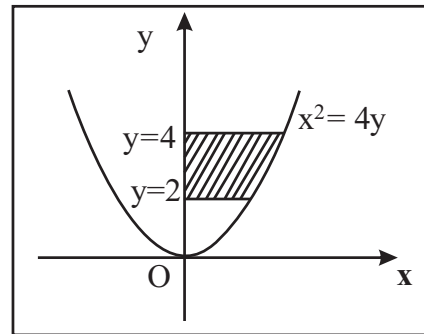
Fig.5.4

Example 8

Find the area under the curve $y = 4x^2 - 8x + 6$ bounded by the y - axis x - axis and the ordinate at $x = 2$.

Solution :

The y-axis is the ordinate at $x = 0$. The area bounded by the ordinates at $x = 0$, $x = 2$ and the given curve is

$$A = \int_a^b y \, dx$$

$$= \int_0^2 (4x^2 - 8x + 6) \, dx$$

$$= \left[4\frac{x^3}{3} - \frac{8x^2}{2} + 6x\right]_0^2$$

$$= \frac{4}{3}(2)^3 - 4(2)^2 + 6(2) - 0$$

$$= \frac{20}{3} \; sq.units.$$
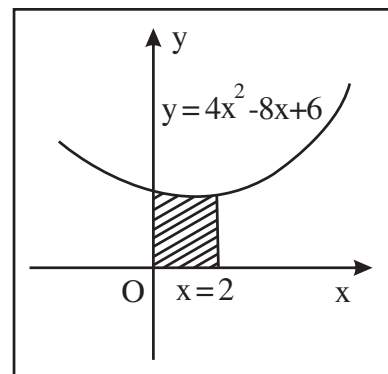
Fig.5.5

Example 9

Find the area bounded by the semi cubical parabola $y^2 = x^3$ and the lines $x = 0$, $y = 1$ and $y = 2$.

Solution :

$$\text{Area, } A = \int_c^d x \, dy$$

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$$= \int_1^2 y^{\frac{2}{3}} \, dy = \left[\frac{y^{\frac{5}{3}}}{\frac{5}{3}} \right]_1^2$$

$$= \frac{3}{5} \left[2^{\frac{5}{3}} - 1 \right] \text{ sq. units.}$$
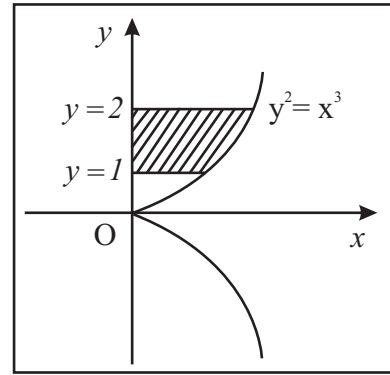
Fig.5.6

Example 10

Find the area bounded by one arch of the curve $y = \sin ax$ and the x - axis.

Solution :

The limits for one arch of the curve $y = \sin ax$ are $x = 0$ and $x = \dfrac{\pi}{a}$

Area, $A = \int_a^b y \, dx$

$$= \int_0^{\frac{\pi}{a}} \sin ax \, dx$$

$$= \left[-\frac{\cos ax}{a} \right]_0^{\frac{\pi}{a}}$$

$$= -\frac{1}{a} [\cos \pi - \cos 0]$$

$$= \frac{2}{a} \text{ sq. units}$$
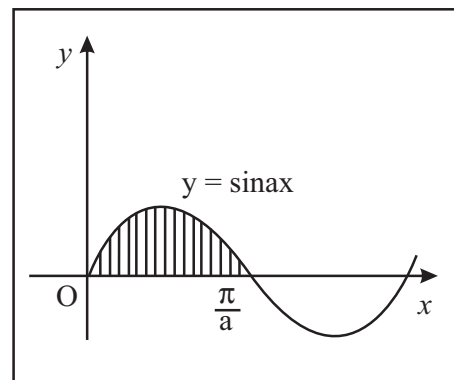
Fig.5.7

Example 11

Find the area of one loop of the curve $y^2 = x^2 (4 - x^2)$ between $x = 0$ and $x = 2$.

Solution :

Equation of the curve is $y^2 = x^2 (4 - x^2)$

$\therefore \ y = \pm x \sqrt{4 - x^2}$

Area, $A = \int_a^b y \, dx$

$\qquad = 2 \times$ Area in the I quadrant

Fig.5.8

$$= 2 \int_0^2 x \sqrt{4 - x^2} \, dx \qquad\qquad (\ y > 0 \text{ in the I quadrant})$$

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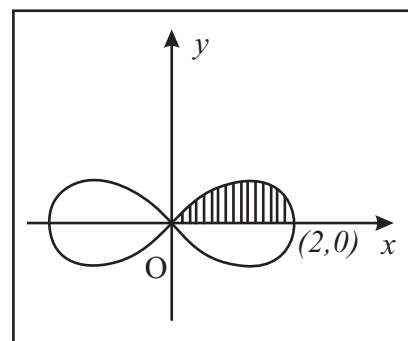
$$\therefore A = 2 \int_{4}^{0} \sqrt{t} \, \frac{(-dt)}{2} = \int_{0}^{4} \sqrt{t} \, dt$$

$$\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right]_{0}^{4}$$

$$= \frac{16}{3} \text{ sq.units.}$$

Put $t = 4 - x^2$

$dt = -2x \, dx$

$-\dfrac{dt}{2} = x \, dx$

when $x = 0$, $t = 4$

when $x = 2$, $t = 0$

EXERCISE 5.2

1) Find the area under the curve $y = 4x - x^2$ included between $x = 0$, $x = 3$ and the x - axis.

2) Find the area of the region bounded by the curve $y = 3x^2 - 4x + 5$, the x - axis and the lines $x = 1$ and $x = 2$.

3) Find the area under the curve $y = \dfrac{1}{1+x^2}$, x-axis, $x = -1$, and $x = 1$.

4) Find the area contained between the x - axis and one arch of the curve $y = \cos x$ bounded between $x = -\dfrac{\pi}{2}$ and $x = \dfrac{\pi}{2}$.

5) Find the area of one loop of the curve $y^2 = x^2 (1 - x^2)$ between $x = 0$ and $x = 1$.

6) Find the area under the demand curve $xy = 1$ bounded by the ordinates $x = 3$, $x = 9$ and x - axis.

7) Find the area cut off from the parabola $y^2 = 4ax$ by its latus rectum.

8) Find the area bounded by the curve $x = 3y^2 - 9$ and the lines $x = 0$, $y = 0$ and $y = 1$.

9) Find the area above the axis of x bounded by $y = \dfrac{4}{x}$, $x = 1$ and $x = 4$.

10) Find the area of the circle of radius 'a' using integration.

11) Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

5.3 APPLICATIONS OF INTEGRATION IN ECONOMICS AND COMMERCE

We learnt already that the marginal function is obtained by differentiating the total function. We were given the total cost, total revenue or demand function and we obtained the marginal cost, marginal revenue or elasticity of demand.

Now we shall obtain the total function when marginal function is given.

5.3.1 The cost function and average cost function from marginal cost function:

If C is the cost of producing an output x then marginal cost function, MC $= \dfrac{dC}{dx}$. Using integration as reverse process of differentiation we obtain,

Cost function, $C = \int (MC) \, dx + k$

where k is the constant of integration which can be evaluated if the fixed cost is known. If the fixed cost is not known, then $k = 0$.

Average cost function, AC $= \dfrac{C}{x}$ x $\neq$ 0

Example 12

The marginal cost function of manufacturing x units of a commodity is $6 + 10x - 6x^2$. Find the total cost and average cost, given that the total cost of producing 1 unit is 15.

Solution :

Given that,

$$MC = 6 + 10x - 6x^2$$

$$C = \int (MC)\, dx + k$$

$$= \int (6 + 10x - 6x^2)\, dx + k$$

$$= 6x + \frac{10x^2}{2} - \frac{6x^3}{3} + k$$

$$= 6x + 5x^2 - 2x^3 + k \qquad \dots\dots\dots (1)$$

Given, when $x = 1$, $C = 15$

$$\therefore (1) \Rightarrow 15 = 6 + 5 - 2 + k$$

$$\Rightarrow k = 6$$

$\therefore$ Total Cost function, $C = 6x + 5x^2 - 2x^3 + 6$

Average Cost function, $AC = \dfrac{C}{x}$, $x \neq 0$

$$= 6 + 5x - 2x^2 + \frac{6}{x}$$

Example 13

The marginal cost function of manufacturing x units of a commodity is $3x^2 - 2x + 8$. If there is no fixed cost find the total cost and average cost functions.

Solution :

Given that,

$$MC = 3x^2 - 2x + 8$$

$$C = \int (MC)\, dx + k$$

$$= \int (3x^2 - 2x + 8)\, dx + k$$

$$= x^3 - x^2 + 8x + k$$

No fixed cost $\Rightarrow k = 0$

Total cost, $C = x^3 - x^2 + 8x$

Average cost, $AC = \dfrac{C}{x}$, $x \neq 0$

$$= x^2 - x + 8$$

Example 14

The marginal cost function of manufacturing x units of a commodity is $3 - 2x - x^2$. If the fixed cost is 200, find the total cost and average cost functions.

Solution :

Given that,

MC $= 3 - 2x - x^2$

$$C = \int (MC)\, dx + k$$

$$= \int (3 - 2x - x^2)\, dx + k$$

$$= 3x - x^2 - \frac{x^3}{3} + k \qquad \text{............ (1)}$$

Given that fixed cost C = 200

$\therefore$ (1) $\Rightarrow$ k = 200

$$\therefore \quad C = 3x - x^2 - \frac{x^3}{3} + 200$$

$$AC = \frac{C}{x} , x \neq 0$$

$$= 3 - x - \frac{x^3}{3} + \frac{200}{x}$$

5.3.2 The revenue function and demand function from marginal revenue function

If R is the total revenue function when the output is x, then marginal revenue

MR $= \dfrac{dR}{dx}$. Integrating with respect to 'x' we get

Revenue function, R $= \int (MR)\, dx + k$

where 'k' is the constant of integration which can be evaluated under given conditions. when $x = 0$, the total revenue R $= 0$,

Demand function, $p = \dfrac{R}{x} , x \neq 0$

Example 15

If the marginal revenue for a commodity is MR $= 9 - 6x^2 + 2x$, find the total revenue and demand function.

Solution:

Given that, MR $= 9 - 6x^2 + 2x$

$$R = \int (MR)\, dx + k$$

$$= \int (9 - 6x^2 + 2x)\, dx + k$$

$$= 9x - 2x^3 + x^2 + k$$

since R $= 0$ when $x = 0$, $k = 0$

R $= 9x - 2x^3 + x^2$

$p = \dfrac{R}{x}, x \neq 0 \Rightarrow p = 9 - 2x^2 + x$

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Example 16

For the marginal revenue function $MR = 3 - 2x - x^2$, find the revenue function and demand function.

Solution :

Given that,

$$MR = 3 - 2x - x^2$$

$$\therefore R = \int (MR)\, dx + k$$

$$= \int (3 - 2x - x^2)\, dx + k$$

$$= 3x - x^2 - \frac{x^3}{3} + k$$

Since $R = 0$ when $x = 0$, $k = 0$

$$\therefore \quad R = 3x - x^2 - \frac{x^3}{3}$$

$$p = \frac{R}{x}, \; x \neq 0$$

$$p = 3 - x - \frac{x^2}{3}$$

Example 17

The marginal revenue for a commodity is $MR = \dfrac{e^x}{100} + x + x^2$, find the revenue function.

Solution :

Given that,

$$MR = \frac{e^x}{100} + x + x^2$$

$$\therefore R = \int (MR)\, dx + k$$

$$= \int \left(\frac{e^x}{100} + x + x^2 \right) dx + k$$

$$= \frac{e^x}{100} + \frac{x^2}{2} + \frac{x^3}{3} + k$$

when no product is sold, revenue is zero.
when $x = 0$, $R = 0$.

$$\therefore \; 0 = \frac{e^0}{100} + 0 + 0 + k \quad \therefore k = -\frac{1}{100}$$

$$\therefore \; \text{Revenue, } R = \frac{e^x}{100} + \frac{x^2}{2} + \frac{x^3}{3} - \frac{1}{100}$$

5.3.3 The demand function when the elasticity of demand is given

We know that,

Elasticity of demand $\eta_d = \dfrac{-p}{x}\dfrac{dx}{dp}$

$$\Rightarrow \frac{-dp}{p} = \frac{dx}{x}\frac{1}{\eta_d}$$

Integrating both sides

$$-\int\frac{dp}{p} = \frac{1}{\eta_d}\ \int\frac{dx}{x}$$

This equation yields the demand function 'p' as a function of 'x'.

The revenue function can be found out by using the relation, $R = px$.

Example 18

The elasticity of demand with respect to price p for a commodity is $\dfrac{x-5}{x}$, $x > 5$ when the demand is 'x'. Find the demand fuunction if the price is 2 when demand is 7. Also find the revenue function.

Solution :

Given that,

Elasticity of demand, $\eta_d = \dfrac{x-5}{x}$

i.e. $-\dfrac{p}{x}\dfrac{dx}{dp} = \dfrac{x-5}{x}$

$$\Rightarrow \frac{dx}{x-5} = -\frac{dp}{p}$$

Integrating both sides,

$$\int\frac{dx}{x-5} = -\int\frac{dp}{p} + \log k$$

$\Rightarrow \log(x-5) = -\log p + \log k$

$\Rightarrow \log(x-5) + \log p = \log k$

$\Rightarrow \log p\,(x-5) = \log k$

$\Rightarrow p\,(x-5) = k$ .... (1)

when $p = 2,\ x = 7$,

$k = 4$

$\therefore$ The demand function is,

$$p = \frac{4}{x-5}$$

Revenue, $R = px$ or $R = \dfrac{4x}{x-5}, x > 5$

Example 19

The elasticity of demand with respect to price for a commodity is a constant and is equal to 2. Find the demand function and hence the total revenue function, given that when the price is 1, the demand is 4.

Solution :

Given that,

Elasticity of demand , $\eta_d = 2$

$\Rightarrow \quad -\dfrac{p}{x}\dfrac{dx}{dp} = 2$

$\Rightarrow \quad \dfrac{dx}{x} = -2\dfrac{dp}{p}$

Integrating both sides,

$\Rightarrow \quad \displaystyle\int\dfrac{dx}{x} = -2\int\dfrac{dp}{p} + \log k$

$\log x = -2 \log p + \log k$

$\log x + \log p^2 = \log k$

$p^2 x = k \qquad \text{................ (1)}$

Given, when $x = 4, \quad p = 1$

From (1) we get $k = 4$

$(1)) \Rightarrow xp^2 = 4 \qquad \text{or} \quad p^2 = \dfrac{4}{x}$

Demand function $p = \dfrac{2}{\sqrt{x}}$, ; Revenue $R = px = 2\sqrt{x}$

Example 20

The marginal cost and marginal revenue with respect to a commodity of a firm are given by $C'(x) = 4 + 0.08\,x$ and $R'(x) = 12$. Find the total profit, given that the total cost at zero output is zero.

Solution :

Given that,

$\text{MC} \quad = 4 + 0.08\,x$

$C(x) \quad = \displaystyle\int(\text{MC})\,dx + k_1$

$\qquad = \displaystyle\int(4 + 0.08\,x)\,dx + k_1$

$\qquad = 4x + 0.08\,\dfrac{x^2}{2} + k_1$

$\qquad = 4x + 0.04\,x^2 + k_1 \qquad \text{............... (1)}$

But given when $x = 0, C = 0$

$\therefore \ (1) \Rightarrow 0 = 0 + 0 + k_1$

$\therefore \qquad k_1 = 0$

$\qquad C(x) = 4x + 0.04x^2 \qquad \text{................ (2)}$

Given that,

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$$MR = 12$$

$$\therefore \quad R(x) = \int MR\ dx + k_2$$

$$= \int 12\ dx + k_2$$

$$= 12x + k_2$$

Revenue $= 0$ when $x = 0$

$$\therefore \quad k_2 = 0$$

$$\therefore \quad R(x) = 12x \qquad\qquad .... (3)$$

Total profit function, $P(x) = R(x) - C(x)$

$$= 12x - 4x - 0.04x^2$$

$$= 8x - 0.04x^2$$

Example 21

The marginal revenue function (in thousands of rupees) of a commodity is $7 + e^{-0.05x}$ where x is the number of units sold. Find the total revenue from the sale of 100 units $(e^{-5} = 0.0067)$

Solution :

Given that,

Marginal revenue, $R'(x) = 7 + e^{-0.05}\,x$

$\therefore$ Total revenue from sale of 100 units is

$$R = \int_0^{100} (7 + e^{-0.05x})\ dx$$

$$= \left[7x + \frac{e^{-0.05\,x}}{-0.05} \right]_0^{100}$$

$$= 700 - \frac{100}{5}\,(e^{-5} - 1)$$

$$= 700 - 20\,(0.0067 - 1)$$

$$= 700 + 20 - 0.134$$

$$= (720 - 0.134) \qquad \text{thousands}$$

$$= 719.866 \times 1000$$

Revenue, $R = $ Rs.$7,19,866$.

Example 22

The marginal cost $C'(x)$ and marginal revenue $R'(x)$ are given by $C'(x) = 20 + \dfrac{x}{20}$ and $R'(x) = 30$ The fixed cost is Rs.200. Determine the maximum profit.

Solution :

Given $C'(x) = 20 + \dfrac{x}{20}$

$$\therefore C(x) = \int C'(x)\,dx + k_1$$

$$= \int \left(20 + \frac{x}{20}\right) dx + k_1$$

$$= 20x + \frac{x^2}{40} + k_1 \qquad \text{............ (1)}$$

When quantity produced is zero, the fixed cost is Rs.200.

i.e. when $x = 0$, $C = 200$,

$$\Rightarrow k_1 = 200$$

Cost function is $C(x) = 20x + \dfrac{x^2}{40} + 200$

The revenue $R'(x)$ $= 30$

$$\therefore R(x) = \int R'(x)\,dx + k_2$$

$$= \int 30\,dx + k_2$$

$$= 30x + k_2$$

When no product is sold, revenue $= 0$ .... (2)

i.e. when $x = 0$, $R = 0$

$$\therefore \text{Revenue, } R(x) = 30x$$

Profit, P = Total revenue – Total cost

$$= 30x - 20x - \frac{x^2}{40} - 200 = 10x - \frac{x^2}{40} - 200$$

$$\frac{dP}{dx} = 10 - \frac{x}{20} \; ; \frac{dP}{dx} = 0 \Rightarrow x = 200$$

$$\frac{d^2P}{dx^2} = \frac{-1}{20} < 0$$

$\therefore$ Profit is maximum when $x = 200$

$\therefore$ Maximum profit is $P = 2000 - \dfrac{40000}{40} - 200$

Profit = Rs.800

Example 23

A company determines that the marginal cost of producing x units is $C'(x) = 10.6x$. The fixed cost is Rs.50. The selling price per unit is Rs.5. Find (i) Total cost function (ii) Total revenue function (iii) Profit function.

Solution :

$C'(x) = 10.6\,x$

$$\therefore C(x) = \int C'(x)\,dx + k = \int 10.6\,x\,dx + k$$

$$= 10.6\,\frac{x^2}{2} + k$$

$$= 5.3x^2 + k \qquad \text{............ (1)}$$

Give fixed cost = Rs.50

(i.e.) when $x = 0$, $C = 50$ $k = 50$

Hence Cost function, $C = 5.3\, x^2 + 50$

(ii)   Total revenue = number of units sold × price per unit

       Let x be the number of units sold. Given that selling price per unit is Rs.5.

       $\therefore$ Revenue $R(x) = 5x$.

(iii)   Profit, P = Total revenue – Total cost

$$= 5x - (5.3\, x^2 + 50)$$
$$= 5x - 5.3\, x^2 - 50$$

Example 24

Determine the cost of producing 3000 units of commodity if the marginal cost in rupees per unit is $C'(x) = \dfrac{x}{3000} + 2.50$.

Solution :

Given, Marginal cost, $C'(x) = \dfrac{x}{3000} + 2.50$

$\therefore C(x) = \int C'(x)\, dx + k = \int \left(\dfrac{x}{3000} + 2.50 \right) dx + k$

$$= \dfrac{x^2}{6000} + 2.50\, x + k$$

When $x = 0$, $C = 0$ $\therefore$ $k = 0$, $\Rightarrow$ $C(x) = \dfrac{x^2}{6000} + 2.50\, x$

When $x = 3000$,

       Cost of production, $C(x) = $ Rs.9000.

Example 25

The marginal cost at a production level of x units is given by $C'(x) = 85 + \dfrac{375}{x^2}$. Find the cost of producing 10 incremental units after 15 units have been produced.

Solution :

Given, $C'(x) = 85 + \dfrac{375}{x^2}$ $C(x) = \int C'(x)\, dx + k$

The cost of producing 10 incremental units after 15 units have been produced

$$= \int_{15}^{25} C'(x)\, dx = \int_{15}^{25} \left(85 + \dfrac{375}{x^2} \right) dx$$

$$= \left[85\, x - \dfrac{375}{x} \right]_{15}^{25} = \text{Rs.860}$$

$\therefore$ Required cost = Rs.860

EXERCISE 5.3

1) The marginal cost function of production x units, is $MC = 10 + 24x - 3x^2$ and the total cost of producing one unit is Rs.25. Find the total cost function and the average cost function.

2) The marginal cost function is $MC = \dfrac{100}{x}$. Find the cost function $C(x)$ if $C(16) = 100$. Also find the average cost function.

3) The marginal cost of manufacturing x units of product is $MC = 3x^2 - 10x + 3$. The total cost of producing one unit of the product is Rs.7. Find the total cost and average cost function.

4) For the marginal cost function $MC = 5 - 6x + 3x^2$, x is the output. If the cost of producing 10 items is Rs.850, find the total cost and average cost function.

5) The marginal cost function is $MC = 20 - 0.04x + 0.003 x^2$ where x is the number of units produced. The fixed cost of production is Rs.7,000. Find the total cost and the average cost.

6) If the marginal revenue function is $R'(x) = 15 - 9x - 3x^2$, find the revenue function and average revenue function.

7) If the marginal revenue of a commodity is given by $MR = 9 - 2x + 4x^2$, find the demand function and revenue function.

8) Find the total revenue function and the demand function for the marginal revenue function $MR = 100 - 9x^2$.

9) Find the revenue function and the demand function if the marginal revenue for x units is $MR = 2 + 4x - x^2$.

10) The marginal revenue of a commodity is given by $MR = 4 - 3x$. Find the revenue function and the demand function

11) The elasticity of demand with respect to price 'p' is $\dfrac{3-x}{x}$, x < 3. Find the demand function and the revenue function when the price is 2 and the demand is 1.

12) The elasticity of demand with respect to price p for a commodity is $\dfrac{p}{x^2}$ when the demand is x. Find the demand function and revenue function if the demand is 2 when the price is 3.

13) Find the demand function for which the elasticity of demand is 1.

14) The marginal cost function of a commodity in a firm is $2 + 3e^{3x}$ where x is the output. Find the total cost and average cost function if the fixed cost is Rs.500.

15) The marginal revenue function is given by $R'(x) = \dfrac{3}{x^2} - \dfrac{2}{x}$ Find the revenue function and demand function if $R(1) = 6$.

16) The marginal revenue is $R'(x) = 16 - x^2$. Find the revenue and demand function.

17) The marginal cost of production of a firm is given by $C'(x) = 5 + 0.13x$. The marginal revenue is given by $R'(x) = 18$. The fixed cost is Rs.120. Find the profit function.

18) The marginal revenue (in thousands of rupees) of a commodity is $R'(x) = 4 + e^{-0.03x}$ where x denotes the number of units sold. Determine the total revenue from the sale of 100 units of the commodity ($e^{-3} = 0.05$).

5.4 CONSUMERS' SURPLUS

A demand curve for a commodity shows the amount of the commodity that will be bought by people at any given price p. Suppose that the prevailing market price is p_0. At this price an amount x_0 of the commodity determined by the demand curve will be sold. However there are buyers who would be willing to pay a price higher than p_0. All such buyers will gain from the fact that the prevailing market price is only p_0. This gain is called **Consumers' Surplus.** It is represented by the area below the demand curve $p = f(x)$ and above the line $p = p_0$.

Thus Consumers' Surplus, CS = [Total area under the demand function bounded by $x = 0$, $x = x_0$ and x-axis - Area of the rectangle OAPB]

$$\therefore \ \text{CS} = \int_0^{x_0} f(x)\, dx - p_0 x_0$$
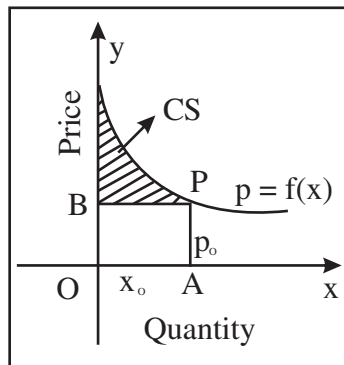
Fig. 5.9

Example 26

Find the consumers' surplus for the demand function $p = 25 - x - x^2$ when $p_0 = 19$.

Solution :

Given that,

The demand function is $p = 25 - x - x^2$

$$p_0 = 19$$

$$\therefore \qquad 19 = 25 - x - x^2$$
$$\Rightarrow \qquad x^2 + x - 6 = 0$$
$$\Rightarrow \qquad (x + 3)(x - 2) = 0$$
$$\Rightarrow \qquad x = 2 \ (\text{or}) \ x = -3$$
$$\therefore \qquad x_0 = 2 \qquad\qquad \text{[demand cannot be negative]}$$
$$\therefore \qquad p_0 x_0 = 19 \times 2 = 38$$

$$\text{CS} = \int_0^{x_0} f(x)\, dx - p_0 x_0$$

$$= \int_0^2 (25 - x - x^2)\, dx - 38$$

$$= \left[25x - \frac{x^2}{2} - \frac{x^3}{3} \right]_0^2 - 38$$

$$= \left[25(2) - 2 - \frac{8}{3} \right] - 38 = \frac{22}{3} \text{ units}$$

Example 27

The demand of a commodity is $p = 28 - x^2$ Find the consumers' surplus when demand $x_0 = 5$

Solution :

Given that,

The demand function, $p = 28 - x^2$

$$\text{when } x_0 = 5$$

$$p_0 = 28 - 25$$

$$= 3$$

$$\therefore p_0 x_0 = 15$$

$$CS = \int_0^{x_0} f(x)\, dx - p_0 x_0$$

$$= \int_0^5 (28 - x^2)\, dx - 15$$

$$= \left[28x - \frac{x^3}{3} \right]_0^5 - 15$$

$$= \left[28 \times 5 - \frac{125}{3} \right] - 15 = \frac{250}{3} \text{ units}$$

Example 28

The demand function for a commodity is $p = \dfrac{12}{x+3}$. Find the consumers' surplus when the prevailing market price is 2.

Solution :

Given that, Demand function, $p = \dfrac{12}{x+3}$

$$p_0 = 2 \Rightarrow 2 = \frac{12}{x+3}$$

or $\quad 2x + 6 = 12$ or $x = 3 \quad \therefore x_0 = 3 \Rightarrow p_0 x_0 = 6$

$$CS = \int_0^{x_0} f(x)\, dx - p_0 x_0 \quad = \int_0^3 \frac{12}{x+3}\, dx - 6$$

$$= 12 \left[\log(x+3) \right]_0^3 - 6$$

$$= 12 [\log 6 - \log 3] - 6 = 12 \log \frac{6}{3} - 6 = 12 \log 2 - 6$$

5.5 PRODUCERS' SURPLUS

A supply curve for a commodity shows the amount of the commodity that will be brought into the market at any given price p. Suppose the prevailing market price is p_0. At this price an amount x_0 of the commodity, determined by the supply curve, will be offered to buyers. However, there are producers who are willing to supply the commodity at a price lower than p_0. All such producers will gain from the fact that the prevailing market price is only p_0. This gain is called 'Producers' Surplus'. It is represented by the area above the supply curve $p = g(x)$ and below the line $p = p_0$.

Thus Producers' Surplus, PS = [Area of the whole rectangle OAPB - Area under the supply curve bounded by $x = 0$, $x = x_0$ and x - axis]

$$\therefore \text{ PS } = p_0 x_0 - \int_0^{x_0} g(x)\, dx$$
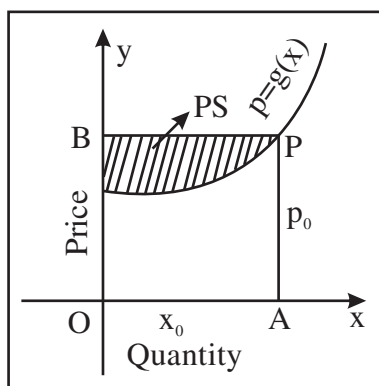
Fig. 5.10

Example 29

The supply function for a commodity is $p = x^2 + 4x + 5$ where x denotes supply. Find the producers' surplus when the price is 10.

Solution :

Given that,

Supply function, $p = x^2 + 4x + 5$

For $p_0 = 10$,

$10 = x^2 + 4x + 5 \qquad \Rightarrow x^2 + 4x - 5 = 0$

$\Rightarrow \qquad (x + 5)(x - 1) = 0 \Rightarrow x = -5$ or $x = 1$

Since supply cannot be negative, $x = -5$ is not possible.

$\therefore \quad x = 1$

$\therefore p_0 = 10$ and $x_0 = 1 \qquad \Rightarrow \quad p_0 x_0 = 10$

Producers' Surplus,

$$\text{PS } = p_0 x_0 - \int_0^{x_0} g(x)\, dx$$

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$$= 10 - \int_0^1 (x^2 + 4x + 5)\, dx$$

$$= 10 - \left[\frac{x^3}{3} + \frac{4x^2}{2} + 5x \right]_0^1$$

$$= 10 - \left[\frac{1}{3} + 2 + 5 \right] = \frac{8}{3} \text{ units}$$

Example 30

Find the producer's surplus for the supply function $p = x^2 + x + 3$ when $x_0 = 4$.

Solution :

Given that,

Supply function, $p = x^2 + x + 3$

when $x_0 = 4$, $p_0 = 4^2 + 4 + 3 = 23$

$p_0 x_0 = 92$

Producers' Surplus

$$\text{PS} = p_0 x_0 - \int_0^{x_0} g(x)\, dx = 92 - \int_0^4 (x^2 + x + 3)\, dx$$

$$= 92 - \left[\frac{x^3}{3} + \frac{x^2}{2} + 3x \right]_0^4$$

$$= 92 - \left[\frac{64}{3} + \frac{16}{2} + 12 \right] = \frac{152}{3} \text{ units.}$$

Example 31

Find the producer's surplus for the supply function $p = 3 + x^2$ when the price is 12.

Solution :

Given that,

Supply function, $p = 3 + x^2$. When $p_0 = 12$,

$12 = 3 + x^2$ or $x^2 = 9$ or $x = \pm 3$

Since supply cannot be negative,

$x = 3$, i.e. $x_0 = 3$,

$\therefore \ p_0 x_0 = 36$.

Producer's Surplus,

$$\text{PS} = p_0 x_0 - \int_0^{x_0} g(x)\, dx$$

$$= 36 - \int_0^3 (3 + x^2)\, dx \;=\; 36 - \left[3x + \frac{x^3}{3} \right]_0^3$$

$$= 36 - \left[9 + \frac{27}{3} - 0 \right] = 18 \text{ units.}$$

Example 32

The demand and supply functions under pure competition are $p_d = 16 - x^2$ and $p_s = 2x^2 + 4$. Find the consumers' surplus and producers' surplus at the market equilibrium price.

Solution :

For market equilibrium,

Quantity demanded = Quantity supplied

$\Rightarrow \qquad 16 - x^2 = 2x^2 + 4 \Rightarrow 3x^2 = 12$

$\Rightarrow \qquad x^2 = 4 \qquad\qquad x = \pm 2$ But $x = -2$ is in admissible.

$\therefore \qquad x = 2 \quad$ (i.e.) $x_0 = 2$

$\therefore \qquad p_0 = 16 - (2)^2 \;= 12$

$\therefore \qquad p_0 x_0 = 12 \times 2 = 24$

Consumer's Surplus,

$$CS \;=\; \int_0^{x_0} f(x)\, dx - p_0 x_0$$

$$= \int_0^2 (16 - x^2)\, dx - 24$$

$$= \left[16x - \frac{x^3}{3} \right]_0^2 - 24 = 32 - \frac{8}{3} - 24 = \frac{16}{3} \text{ units.}$$

Producers' Surplus

$$PS \;=\; p_0 x_0 \;-\; \int_0^{x_0} g(x)\, dx$$

$$= 24 - \int_0^2 (2x^2 + 4)\, dx = 24 - \left[\frac{2x^3}{3} + 4x \right]_0^2$$

$$= 24 - \frac{2 \times 8}{3} - 8 \;=\; \frac{32}{3} \text{ units.}$$

EXERCISE 5.4

1) If the demand function is $p = 35 - 2x - x^2$ and the demand x_0 is 3, find the consumers' surplus.

2) If the demand function for a commodity is $p = 36 - x^2$ find the consumers' surplus for $p_0 = 11$.

3) The demand function for a commodity is $p = 10 - 2x$. Find the consumers' surplus for (i) $p = 2$ (ii) $p = 6$.

4) The demand function for a commodity is $p = 80 - 4x - x^2$. Find the consumers' surplus for $p = 20$.

5) If the supply function is $p = 3x^2 + 10$ and $x_0 = 4$, find the producers' surplus.

6) If the supply law is $p = 4 - x + x^2$, find the producers' surplus when the price is 6.

7) The supply function for a commodity is $p = 3 + x$. Find the producers' surplus when (i) $x_0 = 3$. (ii) $x_0 = 6$.

8) For a commodity, the supply law is $p = \dfrac{x^2}{2} + 3$. Find the producers' surplus when the price is 5.

9) The demand and supply function for a commodity are $p_d = 16 - 2x$ and $p_s = x^2 + 1$. Find the consumers' surplus and producers' surplus at the market equilibrium price.

10) The demand and supply law under a pure competion are given by $p_d = 23 - x^2$ and $p_s = 2x^2 - 4$. Find the consumers' surplus and producers' surplus at the market equilibrium price.

11) Under pure competition the demand and supply laws for commodity and $p_d = 56 - x^2$ and $p_s = 8 + \dfrac{x^2}{3}$. Find the consumers' surplus and producers' surplus at the equilibrium price.

12) Find the consumers' surplus and the producers' surplus under market equilibrium if the demand function is $p_d = 20 - 3x - x^2$ and the supply function is $p_s = x - 1$.

13) In a perfect competition the demand and supply curves of a commodity are given by $p_d = 40 - x^2$ and $p_s = 3x^2 + 8x + 8$. Find the consumers' surplus and producers' surplus at the market equilibrium price.

14) The demand and supply function for a commodity are given by $p_d = 15 - x$ and $p_s = 0.3x + 2$. Find the consumers' surplus and producers' surplus at the market equlibrium price.

15) The demand and supply curves are given by $p_d = \dfrac{16}{x+4}$ and $p_s = \dfrac{x}{2}$. Find the consumers' surplus and producers' surplus at the market equilibrium price.

EXERCISE 5.5

Choose the correct answer

1) If $f(x)$ is an odd function then $\int_{-a}^{a} f(x)\, dx$ is

   a) 1   b) 2a   c) 0   d) a

2) If $f(x)$ is an even function then $\int_{-a}^{a} f(x)\, dx$ is

   a) $2\int_{0}^{a} f(x)\, dx$ b) $\int_{0}^{a} f(x)\, dx$ c) $-2a$ d) 2a

3) $\int_{-3}^{3} x\, dx$ is

   a) 0   b) 2   c) 1   d) -1

4) $\int_{-2}^{2} x^4\, dx$ is

   a) $\dfrac{32}{5}$ b) $\dfrac{64}{5}$ c) $\dfrac{16}{5}$ d) $\dfrac{8}{5}$

5) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\, dx$ is

   a) 0   b) -1 c) 1   d) $\dfrac{\ne}{2}$

6) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx$ is

   a) 2   b) -2 c) -1 d) 1

7) The area under the curve $y = f(x)$, the x-axis and the ordinates at $x = a$ and $x = b$ is

   a) $\int_{a}^{b} y\, dx$ b) $\int_{a}^{b} y\, dy$ c) $\int_{a}^{b} x\, dy$ d) $\int_{a}^{b} x\, dx$

8) The area under the curve $x = g(y)$, the y - axis and the lines $y = c$ and $y = d$ is

   a) $\int_{c}^{d} y\, dy$ b) $\int_{c}^{d} x\, dy$ c) $\int_{c}^{d} y\, dx$ d) $\int_{c}^{d} x\, dx$

9) The area bounded by the curve $y = e^x$, the x - axis and the lines $x = 0$ and $x = 2$ is

   a) $e^2 - 1$ b) $e^2 + 1$ c) e^2 d) $e^2 - 2$

10) The area bounded by $y = x$, y - axis and $y = 1$ is

a) 1   b) $\dfrac{1}{2}$ c) log 2   d) 2

11) The area of the region bounded by $y = x + 1$ the x - axis and the lines $x = 0$ and $x = 1$ is

a) $\dfrac{1}{2}$ b) 2   c) $\dfrac{3}{2}$ d) 1

12) The area bounded by the demand curve $xy = 1$, the x - axis, $x = 1$ and $x = 2$ is

a) log 2   b) log $\dfrac{1}{2}$ c) 2 log 2   d) $\dfrac{1}{2}$ log 2

13) If the marginal cost function MC = $3e^{3x}$, then the cost function is

a) $\dfrac{e^{3x}}{3}$ b) $e^{3x} + k$ c) $9e^{3x}$ d) $3e^{3x}$

14) If the marginal cost function MC = 2 - 4x , then the cost function is

a) $2x - 2x^2 + k$ b) $2 - 4x^2$ c) $\dfrac{2}{x} - 4$ d) $2x - 4x^2$

15) The marginal revenue of a firm is MR = 15 - 8x. Then the revenue function is

a) $15x - 4x^2 + k$ b) $\dfrac{15}{x} - 8$ c) $- 8$ d) $15x - 8$

16) The marginal revenue R' $(x) = \dfrac{1}{x+1}$ then the revenue function is

a) log $|x + 1| + k$ b) $-\dfrac{1}{(x+1)}$ c) $\dfrac{1}{(x+1)^2}$ d) log $\dfrac{1}{x+1}$

17) The consumers' surplus for the demand function $p = f(x)$ for the quantity x_0 and price p_0 is

a) $\displaystyle\int_0^{x_0} f(x)\,dx - p_0 x_0$ b) $\displaystyle\int_0^{x_0} f(x)\,dx$

c) $p_0 x_0 - \displaystyle\int_0^{x_0} f(x)\,dx$ d) $\displaystyle\int_0^{p_0} f(x)\,dx$

18) The producers' surplus for the supply function $p = g(x)$ for the quantity x_0 and price p_0 is

a) $\displaystyle\int_0^{x_0} g(x)\,dx - p_0 x_0$ b) $p_0 x_0 - \displaystyle\int_0^{x_0} g(x)\,dx$

c) $\displaystyle\int_0^{x_0} g(x)\,dx$ d) $\displaystyle\int_0^{p_0} g(x)\,dx$

DIFFERENTIAL EQUATIONS **6**

Many physical problems, when formulated in mathematical forms, lead to **differential equations**. Differential equations enter naturally as models for many phenomena in economics, commerce, engineering etc. Many of these phenomena are complex in nature and very difficult to understand. But when they are described by differential equations, it is easy to analyse them. For example, if the rate of change of cost for x outputs is directly proportional to the cost, then this phenomenon is described by the differential equation,

$\dfrac{dC}{dx}= k$ C, where C is the cost and k is constant. The solution of this differential equation is

$C = C_0\, e^{kx}$ where $C = C_0$ when $x = 0$.

6.1 FORMATION OF DIFFERENTIAL EQUATIONS

A **Differential Equation** is one which involves one or more independent variables, a dependent variable and one or more of their differential coefficients.

There are two types of differential equations:

(i) Ordinary differential equations involving only one independent variable and derivatives of the dependent variable with respect to the independent variable.

(ii) Partial differential equations which involve more than one independent variable and partial derivatives of the dependent variable with respect to the independent variables.

The following are a few examples for differential equations:

(1) $\left(\dfrac{dy}{dx}\right)^2 - 3\dfrac{dy}{dx} + 2y = e^x$

(2) $\dfrac{d^2 y}{dx^2} - 5\dfrac{dy}{dx} + 3y = 0$

(3) $\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = k\dfrac{d^2 y}{dx^2}$

(4) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$

(5) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

(6) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = x + y$

In the above equations,

(1), (2) and (3) are ordinary differential equations and

(4), (5) and (6) are partial differential equations.

In this chapter we shall study ordinary differential equations only.

6.1.1 Order and Degree of a Differential Equation

The order of the derivative of the highest order present in a differential equation is called the **order** of the differential equation.

For example, consider the differential equation

$$x^2 \left(\frac{d^2 y}{dx^2} \right)^3 + 3 \left(\frac{d^3 y}{dx^3} \right)^2 + 7 \frac{dy}{dx} - 4y = 0$$

The orders of $\dfrac{d^3 y}{dx^3}, \dfrac{d^2 y}{dx^2}$ and $\dfrac{dy}{dx}$ are 3, 2, and 1 respectively. So the highest order is 3. Thus the order of the differential equation is 3.

The degree of the derivative of the highest order present in a differential equation is called the **degree** of the differential equation. Here the differential coefficients should be free from the radicals and fractional exponents.

Thus the degree of

$$x^2 \left(\frac{d^2 y}{dx^2} \right)^3 + 3 \left(\frac{d^3 y}{dx^3} \right)^2 + 7 \frac{dy}{dx} - 4y = 0 \text{ is } 2.$$

Example 1

Write down the order and degree of the following differential equations.

(i) $\left(\dfrac{dy}{dx} \right)^3 - 4 \left(\dfrac{dy}{dx} \right) + y = 3e^x$

(ii) $\left(\dfrac{d^2 y}{dx^2} \right)^3 + 7 \left(\dfrac{dy}{dx} \right)^4 = 3 \sin x$

(iii) $\dfrac{d^2 x}{dy^2} + a^2 x = 0$

(iv) $\left(\dfrac{dy}{dx} \right)^2 - 3 \dfrac{d^3 y}{dx^3} + 7 \dfrac{d^2 y}{dx^2} + 4 \left(\dfrac{dy}{dx} \right) - \log x = 0$

(v) $\sqrt{1 + \left(\dfrac{dy}{dx} \right)^2} = 4x$

(vi) $\left[1 + \left(\dfrac{dy}{dx} \right)^2 \right]^{\frac{2}{3}} = \dfrac{d^2 y}{dx^2}$

(vii) $\dfrac{d^2 y}{dx^2} - \sqrt{\dfrac{dy}{dx}} = 0$

(viii) $\sqrt{1 + x^2} = \dfrac{dy}{dx}$

Solution :

The order and the degree respectively are,

(i) 1 ; 3   (ii) 2 ; 3   (iii) 2 ; 1   (iv) 3 ; 1

(v) 1 ; 2   (vi) 2 ; 3   (vii) 2 ; 2   (viii) 1 ; 1

Note

Before ascertaining the order and degree in (v), (vi) & (vii) we made the differential coefficients free from radicals and fractional exponents.

6.1.2 Family of curves

Sometimes a family of curves can be represented by a single equation. In such a case the equation contains an arbitrary constant c. By assigning different values for c, we get a family of curves. In this case c is called the **parameter** or **arbitrary constant** of the family.

Examples

(i) $y = mx$ represents the equation of a family of straight lines through the origin , where m is the parameter.

(ii) $x^2 + y^2 = a^2$ represents the equation of family of concentric circles having the origin as centre, where a is the parameter.

(iii) $y = mx + c$ represents the equation of a family of straight lines in a plane, where m and c are parameters.

6.1.3 Formation of Ordinary Differential Equation

Consider the equation $y = mx + \lambda$ ----(1)

where m is a constant and λ is the parameter.

This represents one parameter family of parallel straight lines having same slope m.

Differentiating (1) with respect to x, we get, $\dfrac{dy}{dx} = m$

This is the differential equation representing the above family of straight lines.

Similarly for the equation $y = Ae^{5x}$, we form the differential equation $\dfrac{dy}{dx} = 5y$ by eliminating the arbitrary constant A.

The above functions represent one-parameter families. Each family has a differential equation. To obtain this differential equation differentiate the equation of the family with respect to x, treating the parameter as a constant. If the derived equation is free from parameter then the derived equation is the differential equation of the family.

Note

(i)   The differential equation of a two parameter family is obtained by differentiating the equation of the family twice and by eliminating the parameters.

(ii)   In general, the order of the differential equation to be formed is equal to the number of arbitrary constants present in the equation of the family of curves.

Example 2

Form the differential equation of the family of curves $y = A \cos 5x + B \sin 5x$ where **A and B are parameters.**

Solution :

Given $y = A \cos 5x + B \sin 5x$

$$\frac{dy}{dx} = -5A \sin 5x + 5B \cos 5x$$

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$$\frac{d^2y}{dx^2} = -25(A\cos 5x) - 25(B\sin 5x) = -25y$$

$$\therefore \quad \frac{d^2y}{dx^2} + 25y = 0$$

Example 3

Form the differential equation of the family of curves $y = ae^{3x} + be^{x}$ where **a** and **b** are parameters.

Solution :

$$y = ae^{3x} + be^x \qquad\qquad ......(1)$$

$$\frac{dy}{dx} = 3ae^{3x} + be^x \qquad\qquad ......(2)$$

$$\frac{d^2y}{dx^2} = 9ae^{3x} + be^x \qquad\qquad ......(3)$$

$$(2)-(1) \Rightarrow \frac{dy}{dx} - y = 2ae^{3x} \qquad\qquad ......(4)$$

$$(3)-(2) \Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = 6ae^{3x} = 3\left(\frac{dy}{dx} - y\right) \qquad [\text{using } (4)]$$

$$\Rightarrow \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

Example 4

Find the differential equation of a family of curves given by $y = a\cos(mx + b)$, **a** and **b** being arbitrary constants.

Solution :

$$y = a\cos(mx + b) \qquad\qquad ......(1)$$

$$\frac{dy}{dx} = -ma\sin(mx + b)$$

$$\frac{d^2y}{dx^2} = -m^2 a\cos(mx + b) = -m^2 y \qquad [\text{using } (1)]$$

$$\therefore \quad \frac{d^2y}{dx^2} + m^2 y = 0 \text{ is the required differential equation}$$

Example 5

Find the differnetial equation by eliminating the arbitrary constants **a** and **b** from $y = a\tan x + b\sec x$.

Solution :

$$y = a\tan x + b\sec x$$

Multiplying both sides by cos x we get,

$$y \cos x = a \sin x + b$$

Differentiating with respect to x we get

$$y(-\sin x) + \frac{dy}{dx} \cos x = a \cos x$$

$$\Rightarrow \quad -y \tan x + \frac{dy}{dx} = a \qquad \qquad ......(1)$$

Differentiating (1) with respect to x, we get

$$\frac{d^2 y}{dx^2} - \frac{dy}{dx} \tan x - y \sec^2 x = 0$$

EXERCISE 6.1

1) Find the order and degree of the following:

(i) $x^2 \dfrac{d^2 y}{dx^2} - 3\dfrac{dy}{dx} + y = \cos x$

(ii) $\dfrac{d^3 y}{dx^3} - 3\left(\dfrac{d^2 y}{dx^2}\right)^2 + 5\dfrac{dy}{dx} = 0$

(iii) $\dfrac{d^2 y}{dx^2} - \sqrt{\dfrac{dy}{dx}} = 0$

(iv) $\left(1 + \dfrac{d^2 y}{dx^2}\right)^{\frac{1}{2}} = \dfrac{dy}{dx}$

(v) $\left(1 + \dfrac{dy}{dx}\right)^{\frac{1}{3}} = \dfrac{d^2 y}{dx^2}$

(vi) $\sqrt{1 + \dfrac{d^2 y}{dx^2}} = x\dfrac{dy}{dx}$

(vii) $\left(\dfrac{d^2 y}{dx^2}\right)^{\frac{3}{2}} = \left(\dfrac{dy}{dx}\right)^2$

(viii) $3\dfrac{d^2 y}{dx^2} + 5\left(\dfrac{dy}{dx}\right)^3 - 3y = e^x$

(ix) $\dfrac{d^2 y}{dx^2} = 0$

(x) $\left(\dfrac{d^2 y}{dx^2} + 1\right)^{\frac{2}{3}} = \left(\dfrac{dy}{dx}\right)^{\frac{1}{3}}$

2) Find the differential equation of the following

(i) $y = mx$ 　　　　　　　(ii) $y = cx - c + c^2$

(iii) $y = mx + \dfrac{a}{m}$, where m is arbitrary constant.

(iv) $y = mx + c$ where m and c are arbitrary constants.

3) Form the differential equation of family of rectangular hyperbolas whose asymptotes are the coordinate axes.

4) Find the differential equation of all circles $x^2 + y^2 + 2gx = 0$ which pass through the origin and whose centres are on the x-axis.

5) Form the differnetial equation $y^2 = 4a (x + a)$, where a is the parameter.

6)   Form the differential equation for $y = a\ cos\ 3x + b\ sin\ 3x$ where a and b are parameters.

7)   Find the differential equation for the family of concentric circles $x^2 + y^2 = a^2$, a is the parameter.

6.2 FIRST ORDER DIFFERENTIAL EQUATIONS

6.2.1 Solution of a differential equation

     A **solution** of a differential equation is an explicit or implicit relation between the variables which satisfies the given differential equation and does not contain any derivatives.

     If the solution of a differential equation contains as many arbitrary constants of integration as its order, then the solution is said to be the **general solution** of the differential equation.

     The solution obtained from the general solution by assigning particular values for the arbitrary constants, is said to be a **particular solution** of the differential equation.

For example,

Differential equation	General equation	Particular solution
(i) $\dfrac{dy}{dx} = \sec^2 x$	$y = \tan x + c$ (c is arbitrary constant)	$y = \tan x - 5$
(ii) $\dfrac{dy}{dx} = x^2 + 2x$	$y = \dfrac{x^3}{3} + x^2 + c$	$y = \dfrac{x^3}{3} + x^2 + 8$
(iii) $\dfrac{d^2 y}{dx^2} - 9y = 0$	$y = Ae^{3x} + Be^{-3x}$	$y = 5e^{3x} - 7e^{-3x}$

6.2.2 Variables Separable

     If it is possible to re-arrange the terms of the first order and first degree differential equation in two groups, each containing only one variable, the variables are said to be separable.

     When variables are separated, the differential equation takes the form $f(x)\ dx + g(y)\ dy = 0$ in which $f(x)$ is a function of x only and $g(y)$ is a function of y only.

     Then the general solution is

$$\int f(x)\ dx + \int g\ (y)\ dy = c \ (c \text{ is a constant of integration})$$

For example, consider $x\dfrac{dy}{dx} - y = 0$

$$x\frac{dy}{dx} = y \implies \frac{dy}{y} = \frac{dx}{x} \qquad \text{(separating the variables)}$$

$$\implies \int \frac{dy}{y} = \int \frac{dx}{x} + k \ \text{ where } k \text{ is a constant of integration}$$

$$\implies \log y = \log x + k.$$

The value of k varies from $-\infty$ to ∞.

This general solution can be expressed in a more convenient form by assuming the constant of integration to be $\log c$. This is possible because $\log c$ also can take all values between $-\infty$ and ∞ as k does. By this assumption, the general solution takes the form

$$\log y - \log x = \log c \Rightarrow \log\left(\frac{y}{x}\right) = \log c$$

i.e. $\dfrac{y}{x} = c \Rightarrow y = cx$

which is an elegant form of the solution of the differential equation.

Note

(i) When y is absent, the general form of first order linear differential equation reduces to $\dfrac{dy}{dx} = f(x)$ and therefore the solution is $y = \int f(x)\, dx + c$.

(ii) When x is absent, it reduces to $\dfrac{dy}{dx} = g(y)$

and in this case, the solution is $\displaystyle\int \frac{dy}{g(y)} = \int dx + c$

Example 6

Solve the differential equation $xdy + ydx = 0$.

Solution :

$xdy + ydx = 0$, dividing by xy we get

$$\frac{dy}{y} + \frac{dx}{x} = 0. \text{ Then } \int \frac{dy}{y} + \int \frac{dx}{x} = c_1$$

$\therefore \qquad \log y + \log x = \log c \Rightarrow xy = c$

Note

(i) $xdy + ydx = 0 \Rightarrow d(xy) = 0 \Rightarrow xy = c,$ a constant.

(ii) $d\left(\dfrac{x}{y}\right) = \dfrac{ydx - xdy}{y^2} \quad \therefore \displaystyle\int \frac{ydx - xdy}{y^2} = \int d\left(\frac{x}{y}\right) + c = \frac{x}{y} + c$

Example 7

Solve $\dfrac{dy}{dx} = e^{3x+y}$

Solution :

$$\frac{dy}{dx} = e^{3x}e^y \Rightarrow \frac{dy}{e^y} = e^{3x}dx$$

$$\int e^{-y}dy = \int e^{3x}dx + c$$

$$\Rightarrow -e^{-y} = \frac{e^{3x}}{3} + c \Rightarrow \frac{e^{3x}}{3} + e^{-y} = c$$

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Example 8

Solve $(x^2 - ay)\, dx = (ax - y^2)\, dy$

Solution :

Writing the equation as

$$x^2 dx + y^2 dy = a(x dy + y dx)$$

$$\Rightarrow \quad x^2 dx + y^2 dy = a\, d\,(xy)$$

$$\therefore \quad \int x^2 dx + \int y^2 dy = a \int d\,(xy) + c$$

$$\Rightarrow \quad \frac{x^3}{3} + \frac{y^3}{3} = a(xy) + c$$

Hence the general solution is $x^3 + y^3 = 3axy + c$.

Example 9

Solve : $(\sin x + \cos x)\, dy + (\cos x - \sin x)\, dx = 0$

Solution :

The given equation can be written as

$$dy + \frac{\cos x - \sin x}{\sin x + \cos x} dx = 0$$

$$\Rightarrow \quad \int dy + \int \frac{\cos x - \sin x}{\sin x + \cos x} dx = c$$

$$\Rightarrow \quad y + \log(\sin x + \cos x) = c$$

Put $1 + y^2 = t$
$2y\,dy = dt$
Put $1 + x^2 = u$
$2x\,dx = du$

Example 10

Solve : $x\, \dfrac{dy}{dx} + \cos y = 0$, given $y = \dfrac{\pi}{4}$ when $x = \sqrt{2}$

Solution :

$$x\,dy = -\cos\, y\,dx$$

$$\therefore \quad \int \sec y\, dy = -\int \frac{dx}{x} + k, \text{ when k is a constant of integration.}$$

$$\log(\sec y + \tan y) + \log x = \log c, \text{ where } \quad k = \log c$$

or $\quad x(\sec y + \tan y) = c.$

When $x = \sqrt{2}, y = \dfrac{\pi}{4}$, we have

$$\sqrt{2}\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) = c \text{ or } c = \sqrt{2}(\sqrt{2} + 1) = 2 + \sqrt{2}$$

$\therefore \quad$ The particular solution is $x(\sec y + \tan y) = 2 + \sqrt{2}$

Example 11

The relationship between the cost of operating a warehouse and the number of units of items stored in it is given by $\dfrac{dC}{dx} = ax + b$, where C is the monthly cost of operating the warehouse and x is the number of units of items in storage. Find C as a function of x if $C = C_0$ when $x = 0$.

Solution :

Given $\dfrac{dC}{dx} = ax + b \qquad \therefore dC = (ax + b)\, dx$

$\displaystyle \int dC = \int (ax + b)\, dx + k, \ (k \text{ is a constant})$

$\Rightarrow \qquad C = \dfrac{ax^2}{2} + bx + k,$

when $x = 0$, $C = C_0 \therefore (1) \Rightarrow C_0 = \dfrac{a}{2}(0) + b(0) + k$

$\Rightarrow \qquad k = C_0$

Hence the cost function is given by

$C = \dfrac{a}{2}x^2 + bx + c$

Example 12

The slope of a curve at any point is the reciprocal of twice the ordinate of the point. The curve also passes through the point (4, 3). Find the equation of the curve.

Solution :

Slope of the curve at any point $P(x, y)$ is the slope of the tangent at $P(x, y)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{2y} \quad \Rightarrow 2y\,dy = dx$

$\displaystyle \int 2y\ dy = \int dx + c \ \Rightarrow y^2 = x + c$

Since the curve passes through (4, 3), we have

$9 = 4 + c \Rightarrow c = 5$

$\therefore$ Equation of the curve is $y^2 = x + 5$.

EXERCISE 6.2

1) Solve :

(i) $\dfrac{dy}{dx} + \sqrt{\dfrac{1 - y^2}{1 - x^2}} = 0$

(ii) $\dfrac{dy}{dx} = \dfrac{1 + y^2}{1 + x^2}$

(iii) $\dfrac{dy}{dx} = \dfrac{y + 2}{x - 1}$

2) Solve : $(1 - e^x)\sec^2 y\ dy + 3e^x \tan y\ dx = 0$

3) Solve : (i) $\dfrac{dy}{dx} = 2xy + 2ax$ \qquad (ii) $x\,(y^2 + 1)\,dx + y\,(x^2 + 1)\,dy = 0$

4) Find the equation of the curve whose slope at the point (x, y) is $3x^2 + 2$, if the curve passes through the point $(1, -1)$

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5) The gradient of the curve at any point (x, y) is proportional to its abscissa. Find the equation of the curve if it passes through the points $(0, 0)$ and $(1, 1)$

6) The marginal cost function for producing x units is MC $= e^{3x + 7}$. Find the total cost function and the average cost function, given that the cost is zero when there is no production.

6.2.3 Homogeneous differential equations

A differential equation in x and y is said to be **homogeneous** if it can be defined in the form

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)} \text{ where } f(x, y) \text{ and } g(x, y) \text{ are}$$

Homogeneous functions of the same degree in x and y.

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}, \frac{dy}{dx} = \frac{x^2 + y^2}{2xy}, \frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$$

$$\text{and} \quad \frac{dy}{dx} = \frac{\sqrt{x^2 - y^2} + y}{x}$$

are some examples of first order homogeneous differnetial equations.

6.2.4 Solving first order homogeneous differential equations

If we put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and the differential equation reduces to variables separable form. The solution is got by replacing $\frac{y}{x}$ for v after the integration is over.

Example 13

Solve the differential equation $(x^2 + y^2) \, dx = 2xy dy.$

Solution :

The given differential equation can be written as

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \qquad \text{.... (1)}$$

This is a homogeneous differential equation

Put, $y = vx$ $\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$ $\qquad$ ..... (2)

Substituting (2) in (1) we get,

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2 x^2}{2x(vx)} = \frac{1 + v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1 + v^2}{2v} - v \Rightarrow x\frac{dv}{dx} = \frac{1 - v^2}{dv}$$

Now, separating the variables,

$$\frac{2v}{1 - v^2} dv = \frac{dx}{x} \text{ or } \int \frac{-2v}{1 - v^2} dv = \int \frac{-dx}{x} + c_1$$

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$$\log(1-v^2) = -\log x + \log c \qquad \left[\because \int \frac{f'(x)}{f(x)} dx = \log f(x)\right]$$

or $\log (1 - v^2) + \log x = \log c$

$\Rightarrow (1 - v^2) x = c$

Replacing v by $\frac{y}{x}$, we get

$$\left(1 - \frac{y^2}{x^2}\right) x = c \text{ or } x^2 - y^2 = cx$$

Example 14

Solve $(x + y) \, dy + (x - y) \, dx = 0$

Solution :

The equation is $\dfrac{dy}{dx} = -\left(\dfrac{x-y}{x+y}\right)$ .... (1)

Put $y = vx$ $\therefore$ $\dfrac{dy}{dx} = v + x\dfrac{dy}{dx}$

we get $v + x\dfrac{dv}{dx} = -\dfrac{x-vx}{x+vx}$ or $v + x\dfrac{dv}{dx} = -\dfrac{1-v}{1+v}$

i.e. $x\dfrac{dv}{dx} = -\left(\dfrac{1-v}{1+v} + v\right)$ or

$$x\dfrac{dv}{dx} = \dfrac{-(1-v+v+v^2)}{1+v}$$

$\therefore$ $\dfrac{1+v}{1+v^2} dv = -\dfrac{1}{x} dx$ or

$$\int \dfrac{dv}{1+v^2} dv + \dfrac{1}{2}\int \dfrac{2v}{1+v^2} dv = \int -\dfrac{1}{x} dx + c$$

$$\tan^{-1} v + \dfrac{1}{2}\log(1+v^2) = -\log x + c$$

i.e. $\tan^{-1}\left(\dfrac{y}{x}\right) + \dfrac{1}{2}\log\left(\dfrac{x^2+y^2}{x^2}\right) = -\log x + c$

$$\tan^{-1}\left(\dfrac{y}{x}\right) + \dfrac{1}{2}\log(x^2+y^2) - \dfrac{1}{2}\log x^2 = -\log x + c$$

i.e. $\tan^{-1}\left(\dfrac{y}{x}\right) + \dfrac{1}{2}\log(x^2+y^2) = c$

Example 15

The net profit p and quantity x satisfy the differential equation $\dfrac{dp}{dx} = \dfrac{2p^3 - x^3}{3xp^2}$.

Find the relationship between the net profit and demand given that $p = 20$ when $x = 10$.

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Solution :

$$\frac{dp}{dx} = \frac{2p^3 - x^3}{3xp^2} \qquad \qquad .....(1)$$

is a differential equation in *x* and *p* of homogeneous type

Put $\quad p = vx \quad \therefore \quad \frac{dp}{dx} = v + x\frac{dv}{dx}$

(1) $\quad \Rightarrow v + x\frac{dv}{dx} = \frac{2v^3 - 1}{3v^2} \quad \Rightarrow x\frac{dv}{dx} = \frac{2v^3 - 1}{3v^2} - v$

$$\Rightarrow x\frac{dv}{dx} = -\left[\frac{1 + v^3}{3v^2}\right]$$

$$\frac{3v^2}{1 + v^3}dv = -\frac{dx}{x} \quad \therefore \int\frac{3v^2}{1 + v^3}dv = -\int\frac{dx}{x} = k$$

$$\Rightarrow \log(1 + v^3) = -\log x + \log k, \text{ where } k \text{ is a constant}$$

$$\log(1 + v^3) = \log\frac{k}{x} \quad \text{i.e. } 1 + v^3 = \frac{k}{x}$$

Replacing $\quad v \text{ by } \frac{p}{x}$, we get

$$\Rightarrow x^3 + p^3 = kx^2$$

Put when $x = 10$, it is given that $p = 20$

$$\therefore \; (10)^3 + (20)^3 = k(10)^2 \Rightarrow k = 90$$

$$\therefore \; x^3 + p^3 = 90x^2$$

$$p^3 = x^2(90 - x) \text{ is the required relationship.}$$

Example 16

The rate of increase in the cost C of ordering and holding as the size *q* of the order increases is given by the differential equation

$$\frac{dC}{dq} = \frac{C^2 + 2Cq}{q^2}. \text{ Find the relationship between C and } q \text{ if } C = 1 \text{ when } q = 1.$$

Solution :

$$\frac{dC}{dq} = \frac{C^2 + 2Cq}{q^2} \qquad \qquad ....(1)$$

This is a homogeneous equation in C and *q*

Put $\quad C = vq \quad \therefore \quad \frac{dC}{dq} = v + q\frac{dv}{dq}$

(1) $\Rightarrow v + q\frac{dv}{dq} = \frac{v^2q^2 + 2vq^2}{q^2} = v^2 + 2v$

$$\Rightarrow \quad q\frac{dv}{dq} = v^2 + v = v(v + 1) \Rightarrow \frac{dv}{v(v + 1)} = \frac{dq}{q}$$

$$\Rightarrow \quad \int\frac{(v + 1) - v}{v(v + 1)}dv = \int\frac{dq}{q} + k, \, k \text{ is a constant}$$

$$\Rightarrow \quad \int \frac{dv}{v} - \int \frac{dv}{v+1} = \int \frac{dq}{q} + \log k,$$

$$\Rightarrow \quad \log v - \log(v+1) = \log q + \log k$$

$$\Rightarrow \quad \log \frac{v}{v+1} = \log qk \quad \text{or} \quad \frac{v}{v+1} = kq$$

Replacing v by $\dfrac{C}{q}$ we get, $C = kq(C + q)$

when $C = 1$ and $q = 1$

$$C = kq(C + q) \implies k = \frac{1}{2}$$

$\therefore \quad C = \dfrac{q(C+q)}{2}$ is the relation between C and q.

Example 17

The total cost of production y and the level of output x are related to the marginal cost of production by the equation $(6x^2 + 2y^2)\, dx - (x^2 + 4xy)\, dy = 0$. What is the relation between total cost and output if $y = 2$ when $x = 1$?

Solution :

Given $(6x^2 + 2y^2)\, dx = (x^2 + 4xy)\, dy$

$$\therefore \frac{dy}{dx} = \frac{6x^2 + 2y^2}{x^2 + 4xy} \qquad\qquad ....(1)$$

is a homogeneous equation in x and y.

Put $\quad y = vx \quad \therefore \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$(1) \Rightarrow \quad v + x\dfrac{dv}{dx} = \dfrac{6x^2 + 2y^2}{x^2 + 4xy} \quad$ or $\quad \dfrac{1+4v}{6-v-2v^2}dv = \dfrac{1}{x}dx$

$\therefore \quad -\displaystyle\int \frac{-1-4v}{6-v-2v^2}\,dv = \int \frac{1}{x}dx + k, \text{where } k \text{ is a constant}$

$\Rightarrow \quad -\log(6 - v - 2v^2) = \log x + \log k = \log kx$

$\Rightarrow \quad \dfrac{1}{6-v-2v^2} = kx$

$\Rightarrow \quad x = c(6x^2 - xy - 2y^2) \text{ where } c = \dfrac{1}{k} \text{ and } v = \dfrac{y}{x}$

when $\quad x = 1$ and $y = 2, \qquad 1 = c(6 - 2 - 8) \Rightarrow c = -\dfrac{1}{4}$

$\Rightarrow \quad 4x = (2y^2 + xy - 6x^2)$

EXERCISE 6.3

1) Solve the following differential equations

(i) $\dfrac{dy}{dx} = \dfrac{y}{x} - \dfrac{y^2}{x^2}$
$\qquad\qquad\qquad\qquad$ (ii) $2\dfrac{dy}{dx} = \dfrac{y}{x} - \dfrac{y^2}{x^2}$

(iii) $\dfrac{dy}{dx} = \dfrac{y^2 - 2xy}{x^2 - 2xy}$
$\qquad\qquad\qquad$ (iv) $\dfrac{dy}{dx} = \dfrac{xy}{x^2 - y^2}$

2) The rate of increase in the cost C of ordering and holding as the size q of the order increases is given by the differential equation $\dfrac{dC}{dq} = \dfrac{C^2 + q^2}{2Cq}$. Find the relationship between C and q if C = 4 when $q = 2$.

3) The total cost of production y and the level of output x are related to the marginal cost of production by the equation $\dfrac{dy}{dx} = \dfrac{24x^2 - y^2}{xy}$. What is the total cost function if $y = 4$ when $x = 2$?

6.2.5 First order linear differential equation

A first order differential equation is said to be **linear** when the dependent variable and its derivatives occur only in first degree and no product of these occur.

An equation of the form $\dfrac{dy}{dx} + Py = Q$,

where P and Q are functions of x only, is called a **first order linear differential equation.**

For example,

(i) $\dfrac{dy}{dx} + 3y = x^3$; here P $= 3$, Q $= x^3$

(ii) $\dfrac{dy}{dx} + y\tan x = \cos x$, $\qquad$ P $= \tan x$, Q $= \cos x$

(iii) $\dfrac{dy}{dx}x - 3y = xe^x$, $\qquad\qquad$ P $= -\dfrac{3}{x}$, Q $= e^x$

(iv) $(1 + x^2)\dfrac{dy}{dx} + xy = (1 + x^2)^3$, P $= \dfrac{x}{1 + x^2}$, Q $= (1 + x^2)^2$

are first order linear differential equations.

6.2.6 Integrating factor (I.F)

A given differential equation may not be integrable as such. But it may become integrable when it is multiplied by a function. Such a function is called the **integrating factor (I.F).** Hence an integrating factor is one which changes a differential equation into one which is directly integrable.

Let us show that $e^{\int P dx}$ is the integrating factor

for $\dfrac{dy}{dx} + Py = Q$ $\qquad\qquad$ .....(1)

where P and Q are function of x,

Now, $\dfrac{d}{dx}\left(ye^{\int Pdx}\right) = \dfrac{dy}{dx}e^{\int Pdx} + y\dfrac{d}{dx}\left(e^{\int Pdx}\right)$

$$= \dfrac{dy}{dx}e^{\int Pdx} + ye^{\int Pdx}\dfrac{d}{dx}\int Pdx$$

$$= \dfrac{dy}{dx}e^{\int Pdx} + ye^{\int Pdx}\;\; P = \left(\dfrac{dy}{dx} + Py\right)e^{\int Pdx}$$

When (1) is multiplied by $e^{\int Pdx}$,

it becomes $\left(\dfrac{dy}{dx} + Py\right)e^{\int Pdx} = Qe^{\int Pdx}$

$$\Rightarrow \dfrac{d}{dx}\left(ye^{\int Pdx}\right) = Q\,e^{\int Pdx}$$

Integrating this, we have

$$y\,e^{\int Pdx} = \int Q e^{\int Pdx}\quad dx + c \qquad\qquad ....(2)$$

So $e^{\int Pdx}$ is the integrating factor of the differential equation.

Note

(i) $e^{\log f(x)} = f(x)$ when $f(x) > 0$

(ii)   If $Q = 0$ in $\dfrac{dy}{dx} + Py = Q$, then the general solution is $y\,(\text{I.F}) = c$, where c is a constant.

(iii)   For the differential equation $\dfrac{dx}{dy} + Px = Q$ where P and Q are functions of y alone, the (I.F) is $e^{\int Pdy}$ and the solution is
$x\,(\text{I.F}) = \int Q(\text{I.F})\;dy + c$

Example 18

Solve the equation $(1 - x^2)\,\dfrac{dy}{dx} - xy = 1$.

Solution :

The given equation is $(1 - x)^2\,\dfrac{dy}{dx} - xy = 1$

$$\Rightarrow \dfrac{dy}{dx} = \dfrac{x}{1-x^2}\,y = \dfrac{1}{1-x^2}$$

This is of the form $\dfrac{dy}{dx} + Py = Q,$

where $P = \dfrac{-x}{1-x^2};\;\; Q = \dfrac{1}{1-x^2}$

I.F $= e^{\int Pdx} = e^{\int \frac{-x}{1-x^2}dx} = \sqrt{1-x^2}$

The general solution is,

$$y(\text{I.F}) = \int Q\ (\text{I.F})\ dx + c$$

$$y\sqrt{1-x^2} = \int \frac{1}{1-x^2}\sqrt{1-x^2}\ dx + c$$

$$= \int \frac{dx}{\sqrt{1-x^2}} + c$$

$$y\sqrt{1-x^2} = \sin^{-1} x + c$$

Example 19

Solve $\dfrac{dy}{dx} + ay = e^x$ **(where a ≠ – 1)**

Solution :

The given equation is of the form $\dfrac{dy}{dx} + Py = Q$

Here $\quad P = a \quad ; Q = e^x$

$$\therefore \text{I.F} = e^{\int P dx} = e^{ax}$$

The general solution is

$$y(\text{I.F}) = \int Q\ (\text{I.F})\ dx + c$$

$$\Rightarrow \quad y\ e^{ax} = \int e^x e^{ax} dx + c = \int e^{(a+1)x} dx + c$$

$$y\ e^{ax} = \frac{e^{(a+1)x}}{a+1} + c$$

Example 20

Solve $\cos x\ \dfrac{dy}{dx} + y \sin x = 1$

Solution :

The given equation can be reduced to

$$\frac{dy}{dx} + y\frac{\sin x}{\cos x} = \frac{1}{\cos x} \quad \text{or} \quad \frac{dy}{dx} + y \tan x = \sec x$$

Here $\quad P = \tan x \quad ; \quad Q = \sec x$

$$\text{I.F} = e^{\int \tan x\ dx} = e^{\log \sec x} = \sec x$$

The general solution is

$$y(\text{I.F}) = \int Q\ (\text{I. F})\ dx + c$$

$$y \sec x = \int \sec^2 x\ dx + c$$

$$\therefore \quad y \sec x = \tan x + c$$

Example 21

A bank pays interest by treating the annual interest as the instantaneous rate of change of the principal. A man invests Rs.50,000 in the bank deposit which accrues interest, 6.5% per year compounded continuously. How much will he get after 10 years? (Given : $e^{.65} = 1.9155$)

Solution :

Let $P(t)$ denotes the amount of money in the account at time t. Then the differential equation governing the growth of money is

$$\frac{dP}{dt} = \frac{6.5}{100} P = 0.065P \quad \Rightarrow \quad \int \frac{dP}{P} = \int (0.065)\ dt + c$$

$$\log_e P = 0.065t + c \qquad \therefore\ \ P = e^{0.065t}\ e^c$$

$$P = c_1\ e^{0.065t} \qquad\qquad\qquad ....(1)$$

$$t = 0, \quad P = 50000.$$

$$(1) \Rightarrow \quad 50000\ = c_1\, e^0 \ \ \text{or} \ \ c_1 = 50000$$

$$\therefore \qquad\qquad P\ = 50000\ e^{0.065t}$$

$$\text{At} \qquad t = 10,\, P = 50000\ e^{0.065 \times 10}\ \ = 50000\ e^{0.65}$$

$$= 50000 \times (1.9155)\ = \text{Rs.95,775}.$$

Example 22

Solve $\dfrac{dy}{dx} + y \cos x = \dfrac{1}{2} \sin 2x$

Solution :

Here $\quad P = \cos x \qquad ; \quad Q = \dfrac{1}{2} \sin 2x$

$$\int P\, dx = \int \cos x\ dx = \sin x$$

$$\text{I.F} = e^{\int P dx} = e^{\sin x}$$

The general solution is

$$y\,(\text{I.F}) = \int Q\,(\text{I.F})\, dx + c$$

$$= \int \frac{1}{2} \sin 2x\ .\ e^{\sin x} dx + c$$

$$= \int \sin x \cos x\ .\ e^{\sin x} dx + c$$

$$= \int t\ e^t\ dt + c = e^t\,(t-1) + c$$

$$= e^{\sin x}\,(\sin x - 1) + c$$

Example 23

A manufacturing company has found that the cost C of operating and maintaining the equipment is related to the length m of intervals between overhauls by the equation

$m^2 \dfrac{dC}{dm} + 2mC = 2$ and C = 4 and when m = 2. Find the relationship between C and m.

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Solution :

Given $m^2 \dfrac{dC}{dm} + 2mC = 2$ or $\dfrac{dC}{dm} + \dfrac{2C}{m} = \dfrac{2}{m^2}$

This is a first order linear differential equation of the form

$\dfrac{dy}{dx} + Py = Q$, where $P = \dfrac{2}{m}$; $Q = \dfrac{2}{m^2}$

$\text{I.F} = e^{\int p\,dm} = e^{\int \frac{2}{m} dm} = e^{\log m^2} = m^2$

General solution is

$C\,(\text{I. F}) = \int Q\,(\text{I.F})\,dm + k$ where k is a constant

$Cm^2 = \int \dfrac{2}{m^2} m^2\,dm + k$

$Cm^2 = 2m + k$

When $C = 4$ and $m = 2$, we have

$16 = 4 + k$ $\Rightarrow$ $k = 12$

The relationship between C and m is

$Cm^2 = 2m + 12 = 2\,(m + 6)$

Example 24

Equipment maintenance and operating costs C are related to the overhaul interval x by the equation $x^2 \dfrac{dC}{dx} - 10xC = -10$ with $C = C_0$ when $x = x_0$. Find C as a function of x.

Solution :

$x^2 \dfrac{dC}{dx} - 10xC = -10$ or $\dfrac{dC}{dx} - \dfrac{10C}{x} = -\dfrac{10}{x^2}$

This is a first order linear differential equation.

$P = -\dfrac{10}{x}$ and $Q = -\dfrac{10}{x^2}$

$\int P\,dx = \int -\dfrac{10}{x}\,dx = -10\log x = \log\left(\dfrac{1}{x^{10}}\right)$

$\text{I.F} = e^{\int P\,dx} = e^{\log\left(\frac{1}{x^{10}}\right)} = \dfrac{1}{x^{10}}$

General solution is

$C\,(\text{I.F}) = \int Q\,(\text{I.F})\,dx + k$, where k is a constant.

$\dfrac{C}{x^{10}} = \int \dfrac{-10}{x^2}\left(\dfrac{1}{x^{10}}\right)dx + k$ or

$\dfrac{C}{x^{10}} = \dfrac{10}{11}\left(\dfrac{1}{x^{11}}\right) + k$

when $C = C_0$ $x = x_0$

$$\frac{C_0}{x_0{}^{10}} = \frac{10}{11}\left(\frac{1}{x_0{}^{11}}\right) + k \implies k = \frac{C}{x_0{}^{10}} - \frac{10}{11x_0{}^{11}}$$

∴ The solution is

$$\frac{C}{x^{10}} = \frac{10}{11}\left(\frac{1}{x^{11}}\right) + \left[\frac{C}{x_0{}^{10}} - \frac{10}{11x_0{}^{11}}\right]$$

$$\implies \frac{C}{x^{10}} - \frac{C}{x_0{}^{10}} = \frac{10}{11}\left(\frac{1}{x^{11}} - \frac{1}{x_0{}^{11}}\right)$$

EXERCISE 6.4

1) Solve the following differential equations

(i) $\dfrac{dy}{dx} + y \cot x = \operatorname{cosec} x$

(ii) $\dfrac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x$, if $y = 0$ when $x = \dfrac{\pi}{2}$

(iii) $x\dfrac{dy}{dx} - 3y = x^2$

(iv) $\dfrac{dy}{dx} + \dfrac{2xy}{1+x^2} = \dfrac{1}{(1+x^2)^2}$ given that $y = 0$ when $x = 1$

(v) $\dfrac{dy}{dx} - y \tan x = e^x \sec x$

(vi) $\log x \dfrac{dy}{dx} + \dfrac{y}{x} = \sin 2x$

2) A man plans to invest some amount in a small saving scheme with a guaranteed compound interest compounded continuously at the rate of 12 percent for 5 years. How much should he invest if he wants an amount of Rs.25000 at the end of 5 year period. ($e^{-0.6} = 0.5488$)

3) The change in the cost of ordering and holding C as quantity q is given by $\dfrac{dC}{dq} = a - \dfrac{C}{q}$ where a is a constant.
Find C as a function of q if $C = C_0$ when $q = q_0$.

6.3 SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The general form of linear and second order differential equation with constant coefficients is

$$a\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + cy = f(x).$$

We shall consider the cases where

(i) $f(x) = 0$ and $f(x) = Ke^{\lambda x}$

| Let $\sin x = t$,
| then $\cos x \, dx = dt$

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For example,

(i) $3\dfrac{d^2 y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$ (or) $3y - 5y + 6y = 0$

(ii) $\dfrac{d^2 y}{dx^2} - 4\dfrac{dy}{dx} + 3y = e^{5x}$ (or) $(D^2 - 4D + 3)y = e^{5x}$

(iii) $\dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} - y = 7$ (or) $(D^2 + D - 1)\, y = 7$

are second order linear differential equation

6.3.1 Auxiliary equations and Complementary functions

For the differential equation, $a\dfrac{d^2 y}{dx^2} + b\dfrac{dy}{dx} + cy = f(x)$, $am^2 + bm + c = 0$ is said to be the **auxiliary equation**. This is a quadratic equation in m. According to the nature of the roots m_1 and m_2 of auxiliary equation we write the complementary function (C.F) as follows.

Nature of roots	Complementary function
(i) Real and unequal $(m_1 \neq m_2)$	$Ae^{m_1 x} + Be^{m_2 x}$
(ii) Real and equal $(m_1 = m_2 = m$ say$)$	$(Ax + B)\, e^{mx}$
(iii) Complex roots $(\alpha \pm i\beta)$	$e^{\alpha x}\,(A \cos \beta x + B \sin \beta x)$
(In all the cases, A and B are arbitrary constants)	

6.3.2 Particular Integral (P.I)

Consider $(aD^2 + bD + c)\, y = e^{\lambda x}$

Let $f(D) = aD^2 + bD + c$

Case 1 : If $f(\lambda) \neq 0$ then λ is not a root of the auxiliary equation $f(m) = 0$.

Rule : $P.I = \dfrac{1}{f(D)} e^{\lambda x} = \dfrac{1}{f(\lambda)} e^{\lambda x}$

Case 2 : If $f(\lambda) = 0$, λ satisfies the auxiliary equation $f(m) = 0$. Then we proceed as follows.

(i)   Let the auxiliary equation have two distinct roots m_1 and m_2 and let $\lambda = m_1$.
      Then $f(m) = a\,(m - m_1)\,(m - m_2) = a\,(m - \lambda)\,(m - m_2)$

Rule : $P.I = \dfrac{1}{a(D - \lambda)\,(D - m_2)} e^{\lambda x} = \dfrac{1}{a(\lambda - m_2)}\, xe^{\lambda x}$

(ii) Let the auxiliary equation have two equal roots each equal to λ. i.e. $m_1 = m_2 = \lambda$.
      $\therefore f(m) = a\,(m - \lambda)^2$

Rule : $\therefore\ P.I = \dfrac{1}{a(D - \lambda)^2} e^{\lambda x} = \dfrac{1}{a}\dfrac{x^2}{2!} e^{\lambda x}$

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6.3.3 The General solution

The general solution of a second order linear differential equation is $y = $ Complementary function (C.F) + Particular integral (P.I)

Example 25

Solve $3\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 2y = 0$

Solution :

The auxiliary equation is $3m^2 - 5m + 2 = 0$

$\Rightarrow (3m - 2)(m - 1) = 0$

The roots are $m_1 = \dfrac{2}{3}$ and $m_2 = 1$ (Real and distinct)

$\therefore$ The complementary function is

$$\text{C.F} = Ae^{\frac{2}{3}x} + Be^x$$

The general solution is

$$y = Ae^{\frac{2}{3}x} + Be^x$$

Example 26

Solve $(16D^2 - 24D + 9)\, y = 0$

Solution :

The auxiliary equation is $16m^2 - 24m + 9 = 0$.

$(4m - 3)^2 = 0 \Rightarrow m = \dfrac{3}{4}, \dfrac{3}{4}$

The roots are real and equal

The C.F is $(Ax + B)\, e^{\frac{3}{4}x}$

The general solution is $y = (Ax + B)\, e^{\frac{3}{4}x}$

Example 27

Solve $(D^2 - 6D + 25)\, y = 0.$

Solution :

The auxiliary equation is $m^2 - 6m + 25 = 0$

$\Rightarrow \quad m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{6 \pm \sqrt{36 - 100}}{2} = \dfrac{6 \pm 8i}{2} = 3 \pm 4i$

The roots are complex and is of the form

$\alpha \pm i\beta$ with $\alpha = 3$ and $\beta = 4$

$$C.F = e^{\alpha x} (A \cos \beta x + B \sin \beta x)$$

$$= e^{3x} (A \cos 4x + B \sin 4x)$$

The general solution is

$$y = e^{3x} (A \cos 4x + B \sin 4x)$$

Example 28

Solve $\dfrac{d^2 y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{5x}$

Solution :

The auxiliary equation is $m^2 - 5m + 6 = 0 \Rightarrow m = 3, 2$

∴ Complementary function $C.F = Ae^{3x} + Be^{2x}$

$$P.I = \frac{1}{D^2 - 5D + 6} e^{5x} = \frac{1}{6} e^{5x}$$

∴ The general solution is

$$y = C.F + P.I$$

$$y = Ae^{3x} + Be^{2x} + \frac{e^{5x}}{6}$$

Example 29

Solve $\dfrac{d^2 y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 2e^{-3x}$

Solution :

The auxiliary equation is $m^2 + 4m + 4 = 0 \Rightarrow m = -2, -2$

∴ Complementary function is $C.F = (Ax + B) e^{-2x}$

$$P.I = \frac{1}{D^2 + 4D + 4} 2e^{-3x}$$

$$= \frac{1}{(-3)^2 + 4(-3) + 4} 2e^{-3x} = 2e^{-3x}$$

∴ The general solution is

$$y = C.F + P.I$$

$$y = (Ax + B) e^{-2x} + 2e^{-3x}$$

Example 30

Solve $\dfrac{d^2 y}{dx^2} - 2\dfrac{dy}{dx} + 4y = 5 + 3e^{-x}$

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Solution :

The auxiliary equation is $m^2 - 2m + 4 = 0$

$$\Rightarrow \quad m = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm i2\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$

$$\text{C.F} = e^x(A\cos\sqrt{3}x + B\sin\sqrt{3}x)$$

$$\text{P. I}_1 = \frac{1}{D^2 - 2D + 4} 5\, e^{0x} = \frac{1}{4} 5\, e^{0x} = \frac{5}{4}$$

$$\text{P. I}_2 = \frac{1}{D^2 - 2D + 4} 3\, e^{-x}$$

$$= \frac{1}{(-1)^2 - 2(-1) + 4} 3e^{-x} = \frac{3e^{-x}}{7}$$

∴ The general solution is

$$y = \text{C. F} + \text{P. I}_1 + \text{P. I}_2$$

$$y = e^x(A\cos\sqrt{3}x + B\sin\sqrt{3}x) + \frac{5}{4} + \frac{3}{7}e^{-x}$$

Example 31

Solve $(4D^2 - 8D + 3)\, y = e^{\frac{1}{2}x}$

Solution :

The auxiliary equation is $4m^2 - 8m + 3 = 0$

$$m_1 = \frac{3}{2},\ m_2 = \frac{1}{2}$$

$$\text{C.F} = A\, e^{\frac{3}{2}x} + B\, e^{\frac{1}{2}x}$$

$$\text{P. I} = \frac{1}{4D^2 - 8D + 3} e^{\frac{1}{2}x} = \frac{1}{4\left(D - \frac{3}{2}\right)\left(D - \frac{1}{2}\right)} e^{\frac{1}{2}x}$$

$$= \frac{1}{4\left(\frac{1}{2} - \frac{3}{2}\right)\left(D - \frac{1}{2}\right)} e^{\frac{1}{2}x} = \frac{x}{-4} e^{\frac{1}{2}x}$$

∴   The general solution is

$$y = \text{C.F} + \text{P. I}$$

$$y = Ae^{\frac{3}{2}x} + Be^{\frac{1}{2}x} - \frac{x}{4}e^{\frac{x}{2}}$$

Example 32

Solve : $(D^2 + 10D + 25)\, y = \frac{5}{2} + e^{-5x}$

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Solution :

The auxiliary equation is $m^2 + 10m + 25 = 0$

$\Rightarrow (m + 5)^2 = 0$

$\Rightarrow m = -5, -5$

$\therefore$ C.F $= (Ax + B)\, e^{-5x}$

$$\text{P. I}_1 = \frac{1}{D^2 + 10D + 25} \frac{5}{2} e^{0x} = \frac{1}{25}\left(\frac{5}{2}\right) = \frac{1}{10}$$

$$\text{P. I}_2 = \frac{1}{D^2 + 10D + 25} e^{-5x} = \frac{1}{(D+5)^2} e^{-5x}$$

$$= \frac{x}{2!} e^{-5x} = \frac{x^2}{2}(e^{-5x})$$

$\therefore$ The general solution is

$y = $ C.F $+$ P. I$_1$ $+$ P. I$_2$

$$y = (Ax + B)e^{-5x} + \frac{1}{10} + \frac{x^2}{2}\, e^{-5x}$$

Example 33

Suppose that the quantity demanded $Q_d = 42 - 4p - 4\dfrac{dp}{dt} + \dfrac{d^2 p}{dt^2}$ and quantity supplied $Q_s = -6 + 8p$ where p is the price. Find the equilibrium price for market clearance.

Solution :

For market clearance, the required condition is $Q_d = Q_s$.

$$\Rightarrow \quad 42 - 4p - 4\frac{dp}{dt} + \frac{d^2 p}{dt^2} = -6 + 8p$$

$$\Rightarrow \quad 48 - 12p - 4\frac{dp}{dt} + \frac{d^2 p}{dt^2} = 0$$

$$\Rightarrow \quad \frac{d^2 p}{dt^2} - 4\frac{dp}{dt} - 12p = -48$$

The auxiliarly equation is $m^2 - 4m - 12 = 0$

$$\Rightarrow \quad m = 6, -2$$

$$\text{C.F} = Ae^{6t} + Be^{-2t}$$

$$\text{P. I} = \frac{1}{D^2 - 4D - 12}(-48)e^{0t} = \frac{1}{-12}(-48) = 4$$

$\therefore$ The general solution is

$p = $ C.F $+$ P. I

$p = Ae^{6t} + Be^{-2t} + 4$

EXERCISE 6.5

1) Solve :

   (i) $\dfrac{d^2 y}{dx^2} - 10\dfrac{dy}{dx} + 24y = 0$ (ii) $\dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} = 0$

   (iii) $\dfrac{d^2 y}{dx^2} + 4y = 0$ (iv) $\dfrac{d^2 y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$

2) Solve :

   (i) $(3D^2 + 7D - 6)\, y = 0$ (ii) $(4D^2 - 12D + 9)\, y = 0$ (iii) $(3D^2 - D + 1)\, y = 0$

3) Solve :

   (i) $(D^2 - 13D + 12)\, y = e^{-2x} + 5e^x$

   (ii) $(D^2 - 5D + 6)\, y = e^{-x} + 3e^{-2x}$

   (iii) $(D^2 - 14D + 49)\, y = 3 + e^{7x}$

   (iv) $(15D^2 - 2D - 1)\, y = e^{\frac{x}{3}}$

4)   Suppose that $Q_d = 30 - 5P + 2\dfrac{dP}{dt} + \dfrac{d^2 P}{dt^2}$ and $Q_s = 6 + 3P$. Find the equilibrium price for market clearance.

EXERCISE 6.6

Choose the correct answer

1) The differential equation of straight lines passing through the origin is

   (a) $x\,\dfrac{dy}{dx} = y$ (b) $\dfrac{dy}{dx} = \dfrac{x}{y}$ (c) $\dfrac{dy}{dx} = 0$ (d) $x\,\dfrac{dy}{dx} = \dfrac{1}{y}$

2) The degree and order of the differential equation $\dfrac{d^2 y}{dx^2} - 6\sqrt{\dfrac{dy}{dx}} = 0$ are

   (a) 2 and 1   (b) 1 and 2   (c) 2 and 2   (d) 1 and 1

3) The order and degree of the differential equation

$$\left(\dfrac{dy}{dx}\right)^2 - 3\dfrac{d^3 y}{dx^3} + 7\dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} = x + \log x \ \ \text{are}$$

   (a) 1 and 3   (b) 3 and 1   (c) 2 and 3   (d) 3 and 2

4) The order and degree of $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{2}{3}} = \dfrac{d^2 y}{dx^2}$ are

   (a) 3 and 2   (b) 2 and 3   (c) 3 and 3   (d) 2 and 2

5) The solution of $x\, dy + y\, dx = 0$ is

(a) $x + y = c$ (b) $x^2 + y^2 = c$ (c) $xy = c$ (d) $y = cx$

6) The solution of $x\, dx + y\, dy = 0$ is

(a) $x^2 + y^2 = c$ (b) $\dfrac{x}{y} = c$ (c) $x^2 - y^2 = c$ (d) $xy = c$

7) The solution of $\dfrac{dy}{dx} = e^{x-y}$ is

(a) $e^y\, e^x = c$ (b) $y = \log ce^x$ (c) $y = \log (e^x + c)$ (d) $e^{x+y} = c$

8) The solution of $\dfrac{dP}{dt} = ke^{-t}$ (k is a constant) is

(a) $c - \dfrac{k}{e^t} = p$ (b) $p = ke^t + c$ (c) $t = \log \dfrac{c-p}{k}$ (d) $t = \log_e p$

9) In the differential equation $(x^2 - y^2)\, dy = 2xy\, dx$, if we make the subistitution $y = vx$ then the equation is transformed into

(a) $\dfrac{1 + v^2}{v + v^3}\, dv = \dfrac{dx}{x}$ (b) $\dfrac{1 - v^2}{v(1 + v^2)}\, dv = \dfrac{dx}{x}$

(c) $\dfrac{dv}{v^2 - 1} = \dfrac{dx}{x}$ (d) $\dfrac{dv}{1 + v^2} = \dfrac{dx}{x}$

10) When $y = vx$ the differential equation $x\dfrac{dy}{dx} = y + \sqrt{x^2 + y^2}$ reduces to

(a) $\dfrac{dv}{\sqrt{v^2 - 1}} = \dfrac{dx}{x}$ (b) $\dfrac{v\, dv}{\sqrt{v^2 + 1}} = \dfrac{dx}{x}$

(c) $\dfrac{dv}{\sqrt{v^2 + 1}} = \dfrac{dx}{x}$ (d) $\dfrac{v\, dv}{\sqrt{1 - v^2}} = \dfrac{dx}{x}$

11) The solution of the equation of the type $\dfrac{dy}{dx} + Py = 0$, (P is a function of x) is given by

(a) $y\, e^{\int P dx} = c$ (b) $y \int P dx = c$ (c) $x\, e^{\int P dx} = y$ (d) $y = cx$

12) The solution of the equation of the type $\dfrac{dx}{dy} + Px = Q$ (P and Q are functions of y) is

(a) $y = \int Q\, e^{\int P dx}\, dy + c$ (b) $ye^{\int P dx} = \int Q\, e^{\int P dx}\, dx + c$

(c) $xe^{\int P dy} = \int Q\, e^{\int P dy}\, dy + c$ (d) $x\, e^{\int P dy} = \int Q\, e^{\int P dx}\, dx + c$

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13) The integrating factor of $x\dfrac{dy}{dx} - y = e^x$ is

(a) $\log x$ 　　　　 (b) $e^{\frac{-1}{x}}$ 　　　　 (c) $\dfrac{1}{x}$ 　　　　 (d) $\dfrac{-1}{x}$

14) The integrating factor of $(1 + x^2)\dfrac{dy}{dx} + xy = (1 + x^2)^3$ is

(a) $\sqrt{1+x^2}$ 　　　 (b) $\log(1 + x^2)$ 　　　 (c) $e^{\tan^{-1}x}$ 　　　 (d) $\log^{(\tan^{-1}x)}$

15) The integrating factor of $\dfrac{dy}{dx} + \dfrac{2y}{x} = x^3$ is

(a) $2\log x$ 　　　　 (b) e^{x^2} 　　　　 (c) $3\log(x^2)$ 　　　　 (d) x^2

16) The complementary function of the differential equations $(D^2 - D)y = e^x$ is

(a) $A + B\, e^x$ 　　　 (b) $(Ax + B)\, e^x$ 　　　 (c) $A + Be^{-x}$ 　　　 (d) $(A + Bx)\, e^{-x}$

17) The complementary function of the differential equation $(D^2 - 2D + 1)y = e^{2x}$ is

(a) $Ae^x + Be^{-x}$ 　　　 (b) $A + Be^x$ 　　　 (c) $(Ax + B)e^x$ 　　　 (d) $A + Be^{-x}$

18) The particular integral of the differential equation $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{5x}$ is

(a) $\dfrac{e^{5x}}{6}$ 　　　 (b) $\dfrac{xe^{5x}}{2!}$ 　　　 (c) $6e^{5x}$ 　　　 (d) $\dfrac{e^{5x}}{25}$

19) The particular integral of the differential equation $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = e^{3x}$ is

(a) $\dfrac{e^{3x}}{2!}$ 　　　 (b) $\dfrac{x^2e^{3x}}{2!}$ 　　　 (c) $\dfrac{xe^{3x}}{2!}$ 　　　 (d) $9e^{3x}$

20) The solution of $\dfrac{d^2y}{dx^2} - y = 0$ is

(a) $(A + B)\, e^x$ 　　 (b) $(Ax + B)\, e^{-x}$ 　　 (c) $Ae^x + \dfrac{B}{e^x}$ 　　 (d) $(A + Bx)\, e^{-x}$

INTERPOLATION AND FITTING A STRAIGHT LINE   **7**

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7.1 INTERPOLATION

Interpolation is the art of reading between the lines in a table. It means insertion or filling up intermediate values of a function from a given set of values of the function. The following table represents the population of a town in the decennial census.

Year :	1910	1920	1930	1940	1950
Population : (in thousands)	12	15	20	27	39

Then the process of finding the population for the year 1914, 1923, 1939, 1947 etc. with the help of the above data is called **interpolation.** The process of finding the population for the year 1955, 1960 etc. is known as extrapolation.

The following assumptions are to be kept in mind for interpolation :

(i)   The value of functions should be either in increasing order or in decreasing order.

(ii)   The rise or fall in the values should be uniform. In other words that there are no sudden jumps or falls in the value of function during the period under consideration.

The following methods are used in interpolation :

1) Graphic method,   2) Algebraic method

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7.1.1 Graphic method of interpolation

Let $y = f(x)$, then we can plot a graph between different values of x and corresponding values of y. From the graph we can find the value of y for given x.

Example 1

From the following data, estimate the population for the year 1986 graphically.

Year :	1960	1970	1980	1990	2000
Population : (in thousands)	12	15	20	26	33

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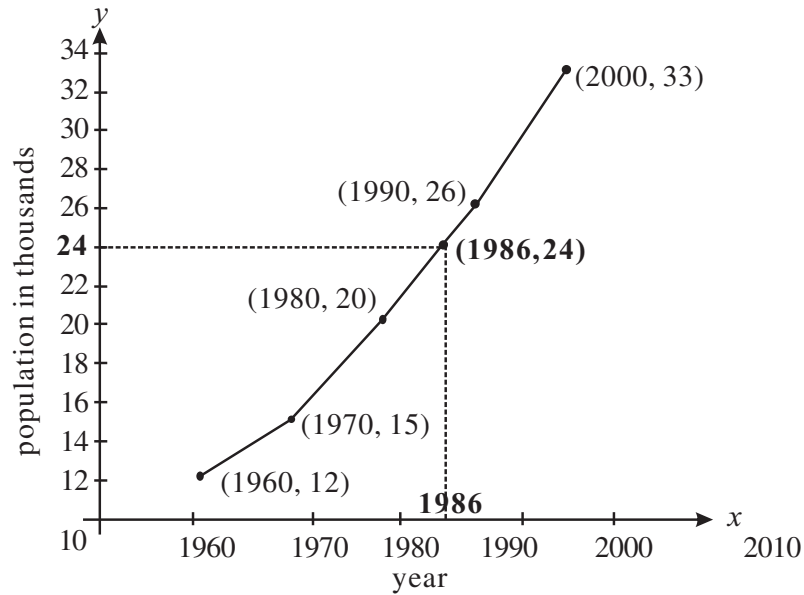
Solution :

From the graph, it is found that the population for 1986 was 24 thousands

Example 2

Using graphic method, find the value of *y* when *x* = 27, from the following data.

x :	10	15	20	25	30
y :	35	32	29	26	23

Solution :
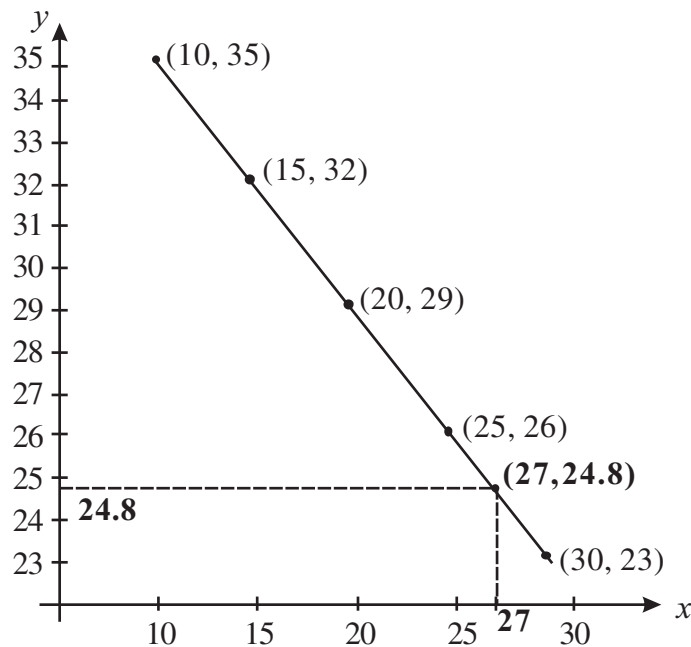
The value of y when x = 27 is 24.8

7.1.2 Algebraic methods of interpolation

The mathematical methods of interpolation are many. Of these we are going to study the following methods:

(i)   Finite differences

(ii)   Gregory-Newton's formula

(iii)   Lagrange's formula

7.1.3 Finite differences

Consider the arguments x_0, x_1, x_2, ... x_n and the entries y_0, y_1, y_2, ..., y_n. Here $y = f(x)$ is a function used in interpolation.

Let us assume that the x-values are in the increasing order and equally spaced with a space-length h.

Then the values of x may be taken to be x_0, $x_0 + h$, $x_0 + 2h$, ... $x_0 + nh$ and the function assumes the values $f(x_0), f(x_0 + h), f(x_0 + 2h), ..., f(x_0 + nh)$

Forward difference operator

For any value of x, the forward difference operator Δ(delta) is defined by

$$\Delta f(x) = f(x + h) - f(x).$$

In particular, $\Delta y_0 = \Delta f(x_0) = f(x_0 + h) - f(x_0) = y_1 - y_0$

$\Delta f(x)$, $\Delta[f(x + h)]$, $\Delta[f(x + 2h)]$, ... are the first order differences of $f(x)$.

Consider $\Delta^2 f(x) = \Delta[\Delta\{f(x)\}]$

$$= \Delta[f(x + h) - f(x)]$$

$$= \Delta[f(x + h)] - \Delta[f(x)]$$

$$= [f(x + 2h) - f(x + h)] - [f(x + h) - f(x)]$$

$$= f(x + 2h) - 2f(x + h) + f(x).$$

$\Delta^2 f(x)$, $\Delta^2[f(x + h)]$, $\Delta^2[f(x + 2h))]$ ... are the second order differences of $f(x)$.

In a similar manner, the higher order differences $\Delta^3 f(x)$, $\Delta^4 f(x)$,...$\Delta^n f(x)$, ... are all defined.

Backward difference operator

For any value of x, the backward difference operator ∇(nabla) is defined by

$$\nabla f(x) = f(x) - f(x - h)$$

In particular, $\nabla y_n = \nabla f(x_n) = f(x_n) - f(x_n - h) = y_n - y_{n-1}$

$\nabla f(x)$, $\nabla[f(x+h)]$, $\nabla[f(x+2h)]$, ... are the first order differences of $f(x)$.

Consider $\nabla^2 f(x) = \nabla[\nabla\{f(x)\}] = \nabla[f(x) - f(x-h)]$

$$= \nabla[f(x)] - \nabla[f(x-h)]$$

$$= f(x) - 2f(x-h) + f(x-2h)$$

$\nabla^2 f(x)$, $\nabla^2[f(x+h)]$, $\nabla^2[f(x+2h)]$ ... are the second order differences of $f(x)$.

In a similar manner the higher order backward differences $\nabla^3 f(x)$, $\nabla^4 f(x)$,...$\nabla^n f(x)$, ... are all defined.

Shifting operator

For any value of x, the shifting operator E is defined by

$$E[f(x)] = f(x+h)$$

In particular, $E(y_0) = E[f(x_0)] = f(x_0 + h) = y_1$

Further, $E^2[f(x)] = E[E\{f(x)\}] = E[f(x+h))] = f(x+2h)$

Similarly $E^3[f(x)] = f(x+3h)$

In general $E^n[f(x)] = f(x+nh)$

The relation between Δ and E

We have $\Delta f(x) = f(x+h) - f(x)$

$$= Ef(x) - f(x)$$

$$\Delta f(x) = (E-1)f(x)$$

$\Rightarrow \quad \Delta \quad = E - 1$

i.e. $\quad E \quad = 1 + \Delta$

Results

1. The differences of constant function are zero.

2. If $f(x)$ is a polynomial of the nth degree in x, then the nth difference of $f(x)$ is constant and $\Delta^{n+1} f(x) = 0$.

Example 3

Find the missing term from the following data.

x :	1	2	3	4
$f(x)$:	100	–	126	157

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Solution :

Since three values of $f(x)$ are given, we assume that the polynomial is of degree two.

Hence third order differences are zeros.

$\Rightarrow \qquad \Delta^3 [f(x_0)] = 0$

or $\qquad \Delta^3(y_0) \qquad = 0$

$\therefore \qquad (E-1)^3 y_0 = 0 \qquad\qquad\qquad (\Delta = E - 1)$

$\qquad (E^3 - 3E^2 + 3E - 1) y_0 \qquad = 0$

$\Rightarrow \qquad y_3 - 3y_2 + 3y_1 - y_0 \qquad = 0$

$\qquad 157 - 3(126) + 3y_1 - 100 \quad = 0$

$\qquad\qquad\qquad\qquad \therefore \qquad y_1 \quad = 107$

i.e. the missing term is 107

Example 4

Estimate the production for 1962 and 1965 from the following data.

Year :	1961	1962	1963	1964	1965	1966	1967
Production : **(in tons)**	**200**	**–**	**260**	**306**	**–**	**390**	**430**

Solution :

Since five values of $f(x)$ are given, we assume that polynomial is of degree four.

Hence fifth order diferences are zeros.

$\therefore \qquad \Delta^5 [f(x_0)] \qquad = 0$

i.e. $\qquad \Delta^5 (y_0) \qquad = 0$

$\therefore \qquad (E-1)^5 (y_0) \ = 0$

i.e. $\qquad (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) y_0 = 0$

$\qquad y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 \qquad = 0$

$\qquad 390 - 5y_4 + 10(306) - 10(260) + 5y_1 - 200 = 0$

$\Rightarrow \qquad y_1 - y_4 = -130 \qquad\qquad\qquad$ --------------(1)

Since fifth order differences are zeros, we also have

$\qquad \Delta^5 [f(x_1)] \qquad = 0$

i.e. $\qquad \Delta^5 (y_1) \qquad = 0$

$\therefore \qquad (E-1)^5 (y_1) \ = 0$

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$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) y_1 = 0$

$y_6 - 5y_5 + 10y_4 - 10y_3 + 5y_2 - y_1 \quad = 0$

$430 - 5(390) + 10y_4 - 10(306) + 5(260) - y_1 = 0$

$\Rightarrow \quad 10y_4 - y_1 = 3280 \qquad\qquad$ --------------(2)

By solving the equations (1) and (2) we get,

$\quad y_1 = 220$ and $y_4 = 350$

∴ The productions for 1962 and 1965 are 220 tons and 350 tons respectively.

7.1.4 Derivation of Gregory - Newton's forward formula

Note :

The derivation of this formula is given for better understanding and is exempted from examination

Let the function $y = f(x)$ be a polynomial of degree n which assumes $(n + 1)$ values $f(x_0)$, $f(x_1), f(x_2)... f(x_n)$, where $x_0, x_1, x_2, ... x_n$ are in the increasing order and are equally spaced.

Let $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = ... = x_n - x_{n-1} = h$ (a positive quantity)

Here $f(x_0) = y_0, f(x_1) = y_1, ... f(x_n) = y_n$

Now f(x) can be written as,

$f(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0) (x - x_1) + ...$

$\qquad\qquad + a_n (x - x_0) (x - x_1) ... (x - x_{n-1}) \qquad$ ----------(1)

When $x = x_0$, (1) implies

$\quad f(x_0) \quad = a_0 \quad$ or $\quad a_0 = y_0$

When $x = x_1$, (1) $\Rightarrow$

$\qquad\qquad f(x_1) = a_0 + a_1(x_1 - x_0)$

i.e. $\quad y_1 \quad = y_0 + a_1 h$

∴ $\quad a_1 \quad = \dfrac{y_1 - y_0}{h} \quad \Rightarrow \quad a_1 = \dfrac{\Delta y_0}{h}$

When $x = x_2$, (1) $\Rightarrow$

$\qquad f(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$

$\qquad y_2 = y_0 + \dfrac{\Delta y_0}{h}(2h) + a_2(2h) (h)$

$2h^2 a_2 = y_2 - y_0 - 2\Delta y_0$

$\qquad\qquad = y_2 - y_0 - 2(y_1 - y_0)$

$\qquad\qquad = y_2 - 2y_1 + y_0 = \Delta^2 y_0$

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$$a_2 = \frac{\Delta^2 y_0}{2! \, h^2}$$

In the same way we can obtain

$$a_3 = \frac{\Delta^3 y_0}{3! \, h^3},$$

$$a_4 = \frac{\Delta^4 y_0}{4! \, h^4}, \ ....., \ a_n = \frac{\Delta^n y_0}{n! \, h^n}$$

substituting the values of $a_0, a_1, ..., a_n$ in (1) we get

$$f(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2! \, h^2}(x - x_0)(x - x_1) + ...$$

$$+ \frac{\Delta^n y_0}{n! \, h^n}(x - x_0)(x - x_1)....(x - x_{n-1}) \qquad ......(2)$$

Denoting $\frac{x - x_0}{h}$ by u, we get

$$x - x_0 = hu$$
$$x - x_1 = (x - x_0) - (x_1 - x_0) = hu - h = h(u - 1)$$
$$x - x_2 = (x - x_0) - (x_2 - x_0) = hu - 2h = h(u - 2)$$
$$x - x_3 = h(u - 3)$$

In general

$$x - x_{n-1} = h \{u - (n - 1)\}$$

Thus (2) becomes,

$$\boxed{\begin{aligned} f(x) &= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + ... \\ &\quad + \frac{u(u-1)(u-2)....(u - \overline{n-1})}{n!}\Delta^n y_0 \\ &\text{where } u = \frac{x - x_0}{h}. \end{aligned}}$$

This is the Gregory-Newton's forward formula.

Example 5

Find y when $x = 0.2$ given that

x :	0	1	2	3	4
y :	176	185	194	202	212

Solution :

0.2 lies in the first interval (x_0, x_1) i.e. (0, 1). So we can use Gregory-Newton's forward interpolation formula. Since five values are given, the interpolation formula is

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$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 \text{ where } u = \frac{x-x_0}{h}$$

Here $h = 1$, $x_0 = 0$ and $x = 0.2$

$$\therefore u = \frac{0.2 - 0}{1} = 0.2$$

The forward difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	**176**				
		9			
1	185		**0**		
		9		**-1**	
2	194		−1		**4**
		8		3	
3	202		2		
		10			
4	212				

$$\therefore \quad y = 176 + \frac{0.2}{1!}(9) + \frac{0.2(0.2-1)}{2!}(0)$$

$$+ \frac{(0.2)(0.2-1)(0.2-2)}{3!}(-1) + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)}{4!}(4)$$

$$= 176 + 1.8 - 0.048 - 0.1344$$

$$= 177.6176$$

i.e. when $x = 0.2$, $y = 177.6176$

Example 6

If $y_{75} = 2459$, $y_{80} = 2018$, $y_{85} = 1180$ and $y_{90} = 402$ find y_{82}.

Solution :

We can write the given data as follows:

x :	75	80	85	90
y :	2459	2018	1180	402

82 lies in the interval (80, 85). So we can use Gregory- Newton's forward interpolation formula. Since four values are given, the interpolation formula is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

where $u = \frac{x-x_0}{h}$

Here $h = 5$, $x_0 = 75$ $x = 82$

$$\therefore u = \frac{82 - 75}{5} = \frac{7}{5} = 1.4$$

The forward difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
75	**2459**			
		-441		
80	2018		-397	
		-838		
85	1180		60	**457**
		-778		
90	402			

$$\therefore \quad y = 2459 + \frac{1.4}{1!}(-441) + \frac{1.4(1.4-1)}{2!}(-397) + \frac{1.4(1.4-1)(1.4-2)}{3!}(457)$$

$$= 2459 - 617.4 - 111.6 - 25.592$$

$$y = 1704.408 \quad \text{when} \quad x = 82$$

Example 7

From the following data calculate the value of $e^{1.75}$

x :	1.7	1.8	1.9	2.0	2.1
e^x :	5.474	6.050	6.686	7.389	8.166

Solution :

Since five values are given, the interpolation formula is

$$y_x = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

where $u = \dfrac{x - x_0}{h}$

Here $h = 0.1$, $x_0 = 1.7$ $x = 1.75$

$$\therefore u = \frac{1.75 - 1.7}{0.1} = \frac{0.05}{0.1} = 0.5$$

The forward difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.7	**5.474**				
		0.576			
1.8	6.050		**0.060**		
		0.636		**0.007**	
1.9	6.686		0.067		**0**
		0.703		0.007	
2.0	7.389		0.074		
		0.777			
2.1	8.166				

$$\therefore \quad y = 5.474 + \frac{0.5}{1!}(0.576) + \frac{0.5(0.5-1)}{2!}(0.06) + \frac{0.5(0.5-1)(0.5-2)}{3!}(0.007)$$

$$= 5.474 + 0.288 - 0.0075 + 0.0004375$$

$$\therefore \quad y = 5.7549375 \quad \text{when} \quad x = 1.75$$

Example 8

From the data, find the number of students whose height is between 80cm. and 90cm.

Height in cms x :	40-60	60-80	80-100	100-120	120-140
No. of students y :	250	120	100	70	50

Solution :

The difference table

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below	60	**250**				
			120			
Below	80	370		-20		
			100		-10	
Below	100	470		-30		**20**
			70		10	
Below	120	540		-20		
			50			
Below	140	590				

Let us calculate the number of students whose height is less than 90cm.

Here $\quad x = 90 \qquad u = \dfrac{x - x_0}{h} = \dfrac{90 - 60}{20} = 1.5$

$$y(90) = 250 + (1.5)(120) + \frac{(1.5)(1.5-1)}{2!}(-20) + \frac{(1.5)(1.5-1)(1.5-2)}{3!}(-10)$$
$$+ \frac{(1.5)(1.5-1)(1.5-2)(1.5-3)}{4!}(20)$$

$$= 250 + 180 - 7.5 + 0.625 + 0.46875$$
$$= 423.59 \simeq 424$$

Therefore number of students whose height is between 80cm. and 90cm. is $y(90) - y(80)$

ie.. $424 - 370 = 54$.

Example 9

Find the number of men getting wages between Rs.30 and Rs.35 from the following table

Wages x :	20-30	30-40	40-50	50-60
No. of men y :	9	30	35	42

Solution :

The difference table

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
Under	30	**9**			
			30		
Under	40	39		5	
			35		2
Under	50	74		7	
			42		
Under	60	116			

Let us calculate the number of men whose wages is less than Rs.35.

For $\quad x = 35, \qquad u = \dfrac{x - x_0}{h} = \dfrac{35 - 30}{10} = 0.5$

By Newton's forward formula,

$$y(35) = 9 + \frac{(0.5)}{1!}(30) + \frac{(0.5)(0.5-1)}{2!}(5) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}(2)$$

$$= 9 + 15 - 0.6 + 0.1$$

$$= 24 \text{ (approximately)}$$

Therefore number of men getting wages between
Rs.30 and Rs.35 is $y(35) - y(30)$ i.e. $24 - 9 = 15$.

7.1.5 Gregory-Newton's backward formula

Note :

The derivation of this formula is given for better understanding and is exempted from examination.

Let the function $y = f(x)$ be a polynomial of degree n which assumes $(n + 1)$ values $f(x_0)$, $f(x_1), f(x_2), \dots f(x_n)$ where $x_0, x_1, x_2, \dots\dots x_n$ are in the increasing order and are equally spaced.

Let $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \dots x_n - x_{n-1} = h$ (a positive quantity)

Here $f(x)$ can be written as

$$f(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \dots\dots$$

$$+ a_n(x - x_n)(x - x_{n-1})\dots(x - x_1) \qquad \dots\dots(1)$$

When $x = x_n$, $(1) \Rightarrow$

$$f(x_n) = a_0 \text{ or } a_0 = y_n$$

When $x = x_{n-1}$, $(1) \Rightarrow$

$$f(x_{n-1}) = a_0 + a_1(x_{n-1} - x_n)$$

or $y_{n-1} = y_n + a_1(-h)$

or $a_1 = \dfrac{y_n - y_{n-1}}{h} \Rightarrow a_1 = \dfrac{\nabla y_n}{h}$

When $x = x_{n-2}$, $(1) \Rightarrow$

$$f(x_{n-2}) = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$y_{n-2} = y_n + \frac{\nabla y_n}{h}(-2h) + a_2(-2h)(-h)$$

$$2h^2 a_2 = (y_{n-2} - y_n) + 2\nabla y_n$$

$$= y_{n-2} - y_n + 2(y_n - y_{n-1})$$

$$= y_{n-2} - 2y_{n-1} + y_n = \nabla^2 y_n$$

$$\therefore a_2 = \frac{\nabla^2 y_n}{2! h^2}$$

In the same way we can obtain

$$a_3 = \frac{\nabla^3 y_n}{3! h^3}, \quad a_4 = \frac{\nabla^4 y_n}{4! h^4} \, ... a_n = \frac{\nabla^n y_n}{n!}$$

$$\therefore \quad f(x) = y_n + \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{2! h^2}(x - x_n)(x - x_{n-1}) + ...$$

$$+ \frac{\nabla^n y_n}{n!}(x - x_n)(x - x_{n-1})...(x - x_1) \qquad .....(2)$$

Further, denoting $\dfrac{x - x_n}{h}$ by u, we get

$$x - x_n \quad = h_u$$
$$x - x_{n-1} = (x - x_n)(x_n - x_{n-1}) = hu + h = h(u+1)$$
$$x - x_{n-2} = (x - x_n)(x_n - x_{n-2}) = hu + 2h = h(u+2)$$
$$x - x_{n-3} = h(u+3)$$

In general,

$$x - x_{n-k} = h(u + k)$$

Thus (2) becomes,

$$\boxed{\begin{aligned} f(x) &= y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + ... \\ &\quad + \frac{u(u+1)....\{u+(n-1)\}}{n!}\nabla^n y_n \text{ where } u = \frac{x - x_n}{h} \end{aligned}}$$

This is the Gregory-Newton's backward formula.

Example 10

Using Gregory-Newton's formula estimate the population of town for the year 1995.

Year x :	1961	1971	1981	1991	2001
Population y : in thousands)	46	66	81	93	101

Solution :

1995 lies in the interval (1991, 2001). Hence we can use Gregory-Newton's backward interpolation formula. Since five values are given, the interpolation formula is

$$y = y_4 + \frac{u}{1!}\nabla y_4 + \frac{u(u+1)}{2!}\nabla^2 y_4 + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_4$$

$$+ \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_4 \text{ where } u = \frac{x - x_4}{h}$$

Here $h = 10$, $x_4 = 2001$ $x = 1995$

$$\therefore \quad u = \frac{1995 - 2001}{10} = -0.6$$

The backward difference table :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1961	46				
		20			
1971	66		−5		
		15		2	
1981	81		−3		−3
		12		−1	
1991	93		−4		
		8			
2001	**101**				

$$\therefore y = 101 + \frac{(-0.6)}{1!}(8) + \frac{(-0.6)(-0.6+1)}{2!}(-4) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!}(-1) +$$

$$\frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{4!}(-3)$$

$$= 101 - 4.8 + 0.48 + 0.056 + 0.1008 \qquad \therefore y = 96.8368$$

i.e. the population for the year 1995 is 96.837 thousands.

Example 11

From the following table, estimate the premium for a policy maturing at the age of 58

Age x :	40	45	50	55	60
Premium y :	114.84	96.16	83.32	74.48	68.48

Solution :

Since five values are given, the interpolation formula is

$$y = y_4 + \frac{u}{1!}\nabla y_4 + ... + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_4$$

where $u = \dfrac{58-60}{5} = -0.4$

The backward difference table :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
40	114.84				
		− 18.68			
45	96.16		5.84		
		− 12.84		− 1.84	
50	83.32		4.00		0.68
		− 8.84		− 1.16	
55	74.48		2.84		
		− 6.00			
60	68.48				

$$\therefore y = 68.48 + \frac{(-0.4)}{1!}(-6) + \frac{(-0.4)(0.6)}{2}(2.84) + \frac{(-0.4)(0.6)(1.6)}{6}(-1.16)$$

$$+ \frac{(-0.4)(0.6)(1.6)(2.6)}{24}(0.68)$$

$$= 68.48 + 2.4 - 0.3408 + 0.07424 - 0.028288$$

$$\therefore \quad y = 70.5851052 \quad \text{i.e. } y \approx 70.59$$

$\therefore$ Premium for a policy maturing at the age of 58 is 70.59

Example 12

From the following data, find y when $x = 4.5$

x :	1	2	3	4	5
y :	1	8	27	64	125

Solution :

Since five values are given, the interpolation formula is

$$y = y_4 + \frac{u}{1!} \nabla y_4 + \dots + \frac{u(u+1)(u+2)(u+3)}{4!} \nabla^4 y_4$$

where $u = \dfrac{x - x_4}{h}$

Here $u = \dfrac{4.5 - 1.5}{1} = -0.5$

The backward difference table :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		
		61			
5	125				

$$\therefore y = 125 + \frac{(-0.5)}{1}(61) + \frac{(-0.5)(0.5)}{2}(24) + \frac{(-0.5)(0.5)(1.5)}{6}(6)$$

$$\therefore y = 91.125 \quad \text{when } x = 4.5$$

7.1.6 Lagrange's formula

Let the function $y = f(x)$ be a polynomial of degree n which assumes $(n + 1)$ values $f(x_0)$, $f(x_1)$, $f(x_2)$, ... $f(x_n)$ corresponding to the arguments x_0, x_1, x_2, ... x_n (not necessarily equally spaced).

Here $f(x_0) = y_0, f(x_1) = y_1, ..., f(x_n) = y_n$.

Then the Largange's formula is

$$f(x) = y_0 \frac{(x - x_1)(x - x_2)\dots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)}$$

$$+ y_1 \frac{(x - x_0)(x - x_2)\dots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\dots(x_1 - x_n)}$$

$$+ \dots + y_n \frac{(x - x_0)(x - x_1)\dots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\dots(x_n - x_{n-1})}$$

Example 13

Using Lagrange's formula find the value of y when $x = 42$ from the following table

x :	40	50	60	70
y :	31	73	124	159

Solution :

By data we have

$$x_0 = 40, x_1 = 50, x_2 = 60, \quad x_3 = 70 \text{ and } x = 42$$

$$y_0 = 31, y_1 = 73, y_2 = 124, y_3 = 159$$

Using Lagrange's formula, we get

$$y = y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$+ y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+ y_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$+ y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$y(42) = 31 \frac{(-8)(-18)(-28)}{(-10)(-20)(-30)} + 73 \frac{(2)(-18)(-28)}{(10)(-10)(-20)}$$

$$+ 124 \frac{(2)(-8)(-28)}{(20)(10)(-10)} + 159 \frac{(2)(-8)(-18)}{(30)(20)(10)}$$

$$= 20.832 + 36.792 - 27.776 + 7.632$$

$$y = 37.48$$

Example 14

Using Lagrange's formula find the value of y when $x = 4$ from the following table

x :	0	3	5	6	8
y :	276	460	414	343	110

Solution :

Given

$$x_0 = 0, x_1 = 3, x_2 = 5, \quad x_3 = 6, x_4 = 8 \text{ and } x = 4$$

$$y_0 = 276, y_1 = 460, y_2 = 414, y_3 = 343, y_4 = 110$$

Using Lagrange's formula,

$$y = y_0 \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}$$

$$+ y_1 \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}$$

$$+ y_2 \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}$$

$$+ y_3 \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}$$

$$+ y_4 \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

$$= 276 \frac{(1)(-1)(-2)(-4)}{(-3)(-5)(-6)(-8)} + 460 \frac{(4)(-1)(-2)(-4)}{(3)(-2)(-3)(-5)} + 414 \frac{(4)(1)(-2)(-4)}{(5)(2)(-1)(-3)}$$

$$+ 343 \frac{(4)(1)(-1)(-4)}{(6)(3)(1)(-2)} + 110 \frac{(4)(1)(-1)(-2)}{(8)(5)(3)(2)}$$

$$= -3.066 + 163.555 + 441.6 - 152.44 + 3.666$$

$$y = 453.311$$

Example 15

Using Lagrange's formula and y(11) from the following table

x :	6	7	10	12
y :	13	14	15	17

Solution :

Given

$$x_0 = 6, \quad x_1 = 7, \quad x_2 = 10, \quad x_3 = 12 \text{ and } x = 11$$

$$y_0 = 13, y_1 = 14, y_2 = 15, \quad y_3 = 17$$

Using Lagrange's formula

$$= 13 \frac{(4)(1)(-1)}{(-1)(-4)(-6)} + 14 \frac{(5)(1)(-1)}{(1)(-3)(-5)} + 15 \frac{(5)(4)(-1)}{(4)(3)(-2)} + 17 \frac{(5)(4)(1)}{(6)(5)(2)}$$

$$= 2.1666 - 4.6666 + 12.5 + 5.6666$$

$$y = 15.6666$$

EXERCISE 7.1

1) Using Graphic method, find the value of y when x = 42, from the following data.

x :	20	30	40	50
y :	51	43	34	24

2) The population of a town is as follows.

Year	x :	1940	1950	1960	1970	1980	1990
Population (in lakhs)	y :	20	24	29	36	46	50

Estimate the population for the year 1976 graphically

3) From the following data, find f(3)

x :	1	2	3	4	5
f(x) :	2	5	-	14	32

4) Find the missing term from the following data.

x :	0	5	10	15	20	25
y :	7	11	14	-	24	32

5) From the following data estimate the export for the year 2000

Year	x :	1999	2000	2001	2002	2003
Export (in tons)	y :	443	-	369	397	467

6) Using Gregory-Newton's formula, find y when $x = 145$ given that

x :	140	150	160	170	180
y :	46	66	81	93	101

7) Using Gregory-Newton's formula, find $y(8)$ from the following data.

x :	0	5	10	15	20	25
y :	7	11	14	18	24	32

8) Using Gregory-Newton's formula, calculate the population for the year 1975

Year	1961	1971	1981	1991	2001
Population	98572	132285	168076	198690	246050

9) From the following data find the area of a circle of diameter 96 by using Gregory-Newton's formula

Diameter	x :	80	85	90	95	100
Area	y :	5026	5674	6362	7088	7854

10) Using Gregory-Newton's formula, find y when $x = 85$

x :	50	60	70	80	90	100
y :	184	204	226	250	276	304

11) Using Gregory-Newton's formula, find y (22.4)

x :	19	20	21	22	23
y :	91	100	110	120	131

12) From the following data find $y(25)$ by using Lagrange's formula

x :	20	30	40	50
y :	512	439	346	243

13) If $f(0) = 5, f(1) = 6, f(3) = 50, f(4) = 105$, find $f(2)$ by using Lagrange's formula.

14) Apply Lagrange's formula to find y when $x = 5$ given that

x :	1	2	3	4	7
y :	2	4	8	16	128

7.2 FITTING A STRAIGHT LINE

A commonly occurring problem in many fields is the necessity of studying the relationship between two (or more) variables.

For example the weight of a baby is related to its age ; the price of a commodity is related to its demand ; the maintenance cost of a car is related to its age.

7.2.1 Scatter diagram

This is the simplest method by which we can represent diagramatically a bivariate data. Suppose x and y denote respectively the age and weight of an adult male, then consider a sample of n individuals with ages $x_1, x_2, x_3, \ldots x_n$ and the corresponding weights as $y_1, y_2, y_3, \ldots y_n$. Plot the points $(x_1, y_1), (x_2, y_2), (x_3, y_3) \ldots (x_n, y_n)$ on a rectangular co-ordinate system. The resulting set of points in a graph is called a **scatter diagram.**

Fig. 7.1
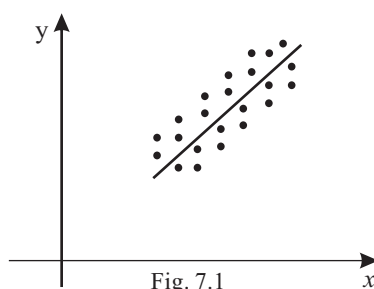
From the scatter diagram it is often possible to visualize a smooth curve approximating the data. Such a curve is called an approximating curve. In the above figure, the data appears to be well approximated by a straight line and we say that a linear relationship exists between the two variables.

7.2.2 Principle of least squares

Generally more than one curve of a given type will appear to fit a set of data. In constructing lines it is necessary to agree on a definition of a "best fitting line".

Consider the data points $(x_1, y_1), (x_2, y_2), (x_3, y_3) \dots (x_n, y_n)$. For a given value of x, say x_1, in general there will be a difference between the value y, and the corresponding value as determined from the curve C (in Fig. 7.2)
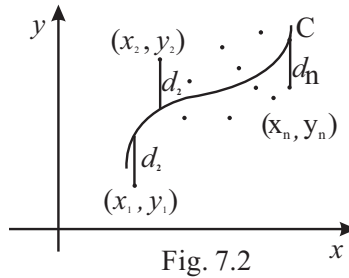
Fig. 7.2

We denote this difference by d_1, which is referred to as a deviation or error. Here d_1 may be positive, negative or zero. Similarly corresponding to the values $x_2, x_3, \dots x_n$ we obtain the deviations $d_2, d_3, \dots d_n$.

A measure of the "goodness of fit" of the curve to the set of data is provided by the quantities $d_1^2, d_2^2, \dots d_n^2$.

Of all the curves approximating a given set of data points, the curve having the property that $d_1^2 + d_2^2 + d_3^2 + \dots + d_n^2$ is a minimum is the best fitting curve. If the approximating curve is a straight line then such a line is called the "line of best fit".

7.2.3 Derivation of normal equations by the principle of least squares

Let us consider the fitting of a straight line

$$y = ax + b \qquad \text{------------(1)}$$

to set of n points $(x_1, y_1), (x_2, y_2), \dots (x_n, y_n)$.

For the different values of a and b equation (1) represents a family of straight lines. Our aim is to determine a and b so that the line (1) is the line of best fit.
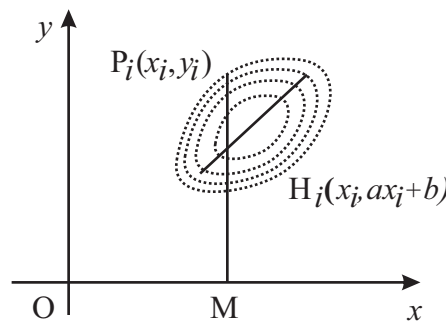
Fig. 7.3

Now a and b are determined by applying principle of least squares.

Let $P_i (x_i, y_i)$ be any point in the scatter diagram. Draw P_iM perpendicular to x-axis meeting the line (1) in H_i. The x-coordinate of H_i is x_i. The ordinate of H_i is $ax_i + b$.

$$P_iH_i = P_iM - H_iM$$
$$= y_i - (ax_i + b) \text{ is the deviation for } y_i.$$

According to the principle of least squares, we have to find a and b so that

$$E = \sum_{i=1}^{n} P_i H_i^2 = \sum_{i=1}^{n} [y_i - (ax_i + b)]^2 \text{ is minimum.}$$

For maxima or minima, the partial derivatives of E with respect to a and b should vanish separately.

$$\therefore \frac{\partial E}{\partial a} = 0 \implies -2\sum_{i=1}^{n} x_i[y_i - (ax_i + b)] = 0$$

$$a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i \qquad\qquad ...(2)$$

$$\therefore \frac{\partial E}{\partial b} = 0 \implies -2\sum_{i=1}^{n} [y_i - (ax_i + b)] = 0$$

i.e., $\Sigma y_i - a\Sigma x_i - nb = 0$

$$\implies a\sum_{i=1}^{n} x_i + nb = \sum_{i=1}^{n} y_i \qquad\qquad ...(3)$$

(2) and (3) are known as the **normal equations**. Solving the normal equations we get a and b.

Note

The normal equations for the line of best fit of the form

$y = a + bx$ are

$na + b\Sigma x_i = \Sigma y_i$

$a\Sigma x_i + b\Sigma x_i^2 = \Sigma x_i y_i$

Example 16

Fit a straight line to the following

$\Sigma x = 10, \Sigma y = 19, \Sigma x^2 = 30, \Sigma xy = 53$ **and** $n = 5$.

Solution :

The line of best fit is $y = ax + b$

$\Sigma y = a\Sigma x + nb$

$\Sigma xy = a\Sigma x^2 + b\Sigma x$

$\implies$ $10a + 5b = 19$ ... (1)

$30a + 10b = 53$ ... (2)

Solving (1) and (2) we get, $a = 1.5$ and $b = 0.8$

The line of best fit is $y = 1.5x + 0.8$

Example 17

In a straight line of best fit find x-intercept when

$\Sigma x = 10$, $\Sigma y = 16.9$, $\Sigma x^2 = 30$, $\Sigma xy = 47.4$ and $n = 7$.

Solution :

The line of best fit is $y = ax + b$

The normal equation are

$\Sigma y = a\Sigma x + nb$

$\Sigma xy = a\Sigma x^2 + b\Sigma x$

$\Rightarrow$ $10a + 7b = 16.9$ ... (1)

$30a + 10b = 47.4$ ... (2)

Solving (1) and (2) we get,

$a = 1.48$ and $b = 0.3$

The line of best fit is $y = 1.48x + 0.3$

$\therefore$ The *x*-intercept of the line of best fit is $-\dfrac{0.3}{1.48}$

Example 18

Fit a straight line for the following data.

x :	0	1	2	3	4
y :	1	1	3	4	6

Solution :

The line of best fit is $y = ax + b$

The normal equation are

$a\Sigma x + nb = \Sigma y$ ... (1)

$a\Sigma x^2 + b\Sigma x = \Sigma xy$ ... (2)

Now from the data

x	y	x^2	xy
0	1	0	0
1	1	1	1
2	3	4	6
3	4	9	12
4	6	16	24
10	**15**	**30**	**43**

By substituting these values in (1) and (2) we get,

$$10a + 5b = 15 \qquad \text{... (3)}$$

$$30a + 10b = 47.4 \qquad \text{... (4)}$$

Solving (3) and (4) we get, $a = 1.3$ and $b = 0.4$

The line of best fit is $y = 1.3x + 0.4$.

Example 19

Fit a straight line for the following data.

x :	4	8	12	16	20	24
y :	7	9	13	17	21	25

Solution :

Take the origin at $\dfrac{12+16}{2} = 14$

Let $u_i = \dfrac{x_i - 14}{2}$ Here $n = 6$

The line of best fit is $y = au + b$

The normal equations are
$$a\Sigma u + nb \quad = \Sigma y \qquad \text{... (1)}$$
$$a\Sigma u^2 + b\Sigma u \quad = \Sigma uy \qquad \text{... (2)}$$

x	y	u	u^2	uy
4	7	-5	25	-35
8	9	-3	9	-27
12	13	-1	1	-13
16	17	1	1	17
20	21	3	9	63
24	25	5	25	125
Total	**92**	**0**	**70**	**130**

On substituting the values in the normal equations (1) and (2)

$a = 1.86$ and $b = 15.33$

The line of best fit is $y = 1.86 \left(\dfrac{x-14}{2} \right) + 15.33 = 0.93x + 2.31$

Example 20

Fit a straight line for the following data.

x :	100	200	300	400	500	600
y :	90.2	92.3	94.2	96.3	98.2	100.3

Solution :

Let $u = \dfrac{x_i - 350}{50}$ and $v_i = y_i - 94.2$ Here $n = 6$

The line of best fit is $v = au + b$

The normal equations are

$a\Sigma u + nb = \Sigma v$ ... (1)

$a\Sigma u^2 + b\Sigma u = \Sigma uv$ ... (2)

x	y	u	v	u^2	uv
100	90.2	-5	-4	25	20
200	92.3	-3	-1.9	9	5.7
300	94.2	-1	0	1	0
400	96.3	1	2.1	1	2.1
500	98.2	3	4	9	12
600	100.3	5	6.1	25	30.5
Total		**0**	**6.3**	**70**	**70.3**

Substituting the values in (1) and (2) we get

$a = 1.0043$ and $b = 1.05$

The line of best fit is $v = 1.0043\, u + 1.05$

$\Rightarrow$ $y = 0.02x + 88.25$

EXERCISE 7.2

1) Define a scatter diagram.

2) State the principle of least squares.

3) Fit the line of best fit if $\Sigma x = 75$, $\Sigma y = 115$, $\Sigma x^2 = 1375$, $\Sigma xy = 1875$, and $n = 6$.

4) In a line of best fit find the slope and the y intercept if $\Sigma x = 10$, $\Sigma y = 25$, $\Sigma x^2 = 30$, $\Sigma xy = 90$, and $n = 5$.

5) Fit a straight line $y = ax + b$ to the following data by the method of least squares

x :	0	1	3	6	8
y :	1	3	2	5	4

6) A group of 5 students took tests before and after training and obtained the following scores.

Scores before training	3	4	4	6	8
Scores after training	4	5	6	8	10

Find by the method of least squares the straight line of best fit

7) By the method of least squares find the best fitting straight line to the data given below:

x :	100	120	140	160	180	200
y :	0.45	0.55	0.60	0.70	0.80	0.85

8) Fit a straight line to the data given below. Also estimate the value y at $x = 3.5$

x :	0	1	2	3	4
y :	1	1.8	3.3	4.5	6.3

9) Find by the method of least squares, the line of best fit for the following data.

Depth of water applied x : (in cm)	0	12	24	36	48
Average yield y : (tons / acre)	35	55	65	80	90

10) The following data show the advertising expenses (expressed as a percentage of total expenses) and the net operating profits (expressed as a percentage of total sales) in a random sample of six drug stores.

Advertising expenses	0.4	1.0	1.3	1.5	2.0	2.8
Net operating profits	1.90	2.8	2.9	3.6	4.3	5.4

Fit a line of best fit.

11) The following data is the number of hours which ten students studied for English and the scores obtained by them in the examinations.

Hours studied x :	4	9	10	12	14	22
Test score y :	31	58	65	68	73	91

(i) Fit a straight line $y = ax + b$

(ii) Predict the score of the student who studied for 17 hours.

EXERCISE 7.3

Choose the correct answer

1) $\Delta f(x) =$

(a) $f(x + h)$ (b) $f(x) - f(x + h)$ (c) $f(x + h) - f(x)$ (d) $f(x) - f(x - h)$

2) $E^2 f(x) =$

(a) $f(x + h)$ (b) $f(x + 2h)$ (c) $f(2h)$ (d) $f(2x)$

3) $E =$

(a) $1 + \Delta$ (b) $1 - \Delta$ (c) $\nabla + 1$ (d) $\nabla - 1$

4) $\nabla f(x + 3h) =$

(a) $f(x + 2h)$ (b) $f(x + 3h) - f(x + 2h)$

(c) $f(x + 3h)$ (d) $f(x + 2h) - f(x - 3h)$

5) When $h = 1$, $\Delta(x^2) =$

(a) $2x$ (b) $2x - 1$ (c) $2x + 1$ (d) 1

6) The normal equations for estimating a and b so that the line $y = ax + b$ may be the line of best fit are

(a) $a\sum x_i^2 + b\sum x_i = \sum x_i y_i$ and $a\sum x_i + nb = \sum y_i$

(b) $a\sum x_i + b\sum x_i^2 = \sum x_i y_i$ and $a\sum x_i^2 + nb = \sum y_i$

(c) $a\sum x_i + nb = \sum x_i y_i$ and $a\sum x_i^2 + b\sum x_i = \sum y_i$

(d) $a\sum x_i^2 + nb = \sum x_i y_i$ and $a\sum x_i + b\sum x_i = \sum y_i$

7) In a line of best fit $y = 5.8 \, (x - 1994) + 41.6$ the value of y when $x = 1997$ is

(a) 50   (b) 54   (c) 59   (d) 60

8) Five data relating to x and y are to be fit in a straight line. It is found that $\sum x = 0$ and $\sum y = 15$. Then the y-intercept of the line of best fit is

(a) 1   (b) 2   (c) 3   (d) 4

9) The normal equations of fitting a straight line $y = ax + b$ are $10a + 5b = 15$ and $30a + 10b = 43$. The slope of the line of best fit is

(a) 1.2   (b) 1.3   (c) 13   (d) 12

10) The normal equations obtained in fitting a straight line $y = ax + b$ by the method of least squares over n points (x, y) are $4 = 4a + b$ and $\sum xy = 120a + 24b$. Then $n =$

(a) 30   (b) 5   (c) 6   (d) 4

PROBABILITY DISTRIBUTIONS   **8**

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8.1 RANDOM VARIABLE AND PROBABILITY FUNCTION

Random variable

A **random variable** is a real valued function defined on a sample space S and taking values in $(-\infty, \infty)$

8.1.1 Discrete Random Variable

A random variable X is said to be discrete if it assumes only a finite or an infinite but countable number of values.

Examples

(i) Consider the experiment of tossing a coin twice. The sample points of this experiment are $s_1 = $ (H, H), $s_2 = $ (H, T), $s_3 = $ (T, H) and $s_4 = $ (T, T).

Random variable X denotes the number of heads obtained in the two tosses.

$$\text{Then} \quad X(s_1) = 2 \qquad\qquad X(s_2) = 1$$
$$X(s_3) = 1 \qquad\qquad X(s_4) = 0$$
$$R_X = \{0, 1, 2\}$$

where *s* is the typical element of the sample space, X(*s*) represents the real number which the random variable X associates with the outcomes.

R_X, the set of all possible values of X, is called the range space X.

(ii) Consider the experiment of rolling a pair of fair dice once.

Then sample space

S = {(1, 1) (1, 2) .... (1, 6)

.   .   .

.   .   .

.   .   .

(6, 1) (6, 2) .... (6, 6)}

Let the random variable X denote the sum of the scores on the two dice. Then $R_X = \{2, 3, 4, ..., 12\}$.

(iii)   Consider the experiment of tossing of 3 coins simultaneously.

Let the random variable X be Number of heads obtained in this experiment.

Then

$$S = \{HHH, HHT, HTT, TTT, TTH, THH, HTH, THT\}$$

$$R_X = \{0, 1, 2, 3\}$$

(iv)   Suppose a random experiment consists of throwing 4 coins and recording the number of heads.

Then $R_X = \{0, 1, 2, 3, 4\}$

The number of printing mistakes in each page of a book and the number of telephone calls received by the telephone operator of a firm, are some other examples of discrete random variable.

8.1.2 Probability function and Probability distribution of a Discrete random variable

Let X be a discrete random variable assuming values $x_1, x_2, x_3...$ If there exists a function p denoted by $p(x_i) = P[X = x_i]$ such that

(i) $p(x_i) \geq 0$ for $i = 1, 2, ...$

(ii) $\sum_i p(x_i) = 1$

then p is called as the **probability function or probability mass function (p.m.f)** of X.

The collection of all pairs $(x_i, p(x_i))$ is called the probability distribution of X.

Example 1

Consider the experiment of tossing two coins. Let X be a random variable denoting the number of heads obtained.

X :	0	1	2
p (x_i) :	$\dfrac{1}{4}$	$\dfrac{1}{2}$	$\dfrac{1}{4}$

Is p(x_i) a p.m.f?

Solution :

(i) $p(x_i) > 0$ for all i

(ii) $\sum p(x_i) = p(0) + p(1) + p(2)$

$$= \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$$

Hence $p(x_i)$ is a p.m.f.

Example 2

Consider the discrete random variable X as the sum of the numbers that appear, when a pair of dice is thrown. The probability distribution of X is

X :	2	3	4	5	6	7	8	9	10	11	12
$p(x_i)$:	$\dfrac{1}{36}$	$\dfrac{2}{36}$	$\dfrac{3}{36}$	$\dfrac{4}{36}$	$\dfrac{5}{36}$	$\dfrac{6}{36}$	$\dfrac{5}{36}$	$\dfrac{4}{36}$	$\dfrac{3}{36}$	$\dfrac{2}{36}$	$\dfrac{1}{36}$

Is $p(x_i)$ a p.m.f?

Solution :

$p(x_i) > 0$ for all i

(ii) $\quad \Sigma\, p(x_i) = \dfrac{1}{36} + \dfrac{2}{36} + \dfrac{3}{36} + \ldots\ldots\ldots + \dfrac{1}{36} = 1$

Hence $p(x_i)$ is a p.m.f.

8.1.3 Cumulative Distribution function : (c.d.f.)

Let X be a discrete random variable. The function F(x) is said to be the cumulative distribution function (c.d.f.) of the random variable X if

$\qquad$ F(x) $\;= P(X \le x)$

$\qquad\qquad = \Sigma\, p(x_i)$ where the sum is taken over i such that $x_i \le x$.

Remark : P $(a < X \le b) = F(b) - F(a)$

Example 3

A random variable X has the following probability function :

Values of X, x :	– 2	– 1	0	1	2	3
$p(x)$:	0.1	k	0.2	$2k$	0.3	k

(i) Find the value of k.

(ii) Construct the c.d.f. of X

Solution :

(i) $\quad$ Since $\Sigma_i\, p(x_i) = 1$,

$\qquad p\,(-2) + p\,(-1) + p\,(0) + p\,(1) + p\,(2) + p\,(3) = 1$

$\qquad 0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$\qquad 0.6 + 4k = 1 \Rightarrow 4k = 1 - 0.6$

$\qquad 4k = 0.4 \qquad\qquad\qquad \therefore k = \dfrac{.4}{4} = 0.1$

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Hence the given probability function becomes,

x :	-2	-1	0	1	2	3
$p(y)$:	0.1	0.1	0.2	0.2	0.3	0.1

(ii)   Cumulative distribution function $F(x) = P (X \le x)$

x	$F(x) = P(X \le x)$
-2	$F(-2) = P(X \le -2) = 0.1$
-1	$F(-1) = P(X \le -1) = P(X = -2) + P(X = -1)$
	$= 0.1 + 0.1 = 0.2$
0	$F(0) = P(X \le -2) = P(X = -2) + P(X = -1) + P(X = 0)$
	$= 0.1 + 0.1 + 0.2 = 0.4$
1	
2	$F(1) = P(X \le 1) = 0.6$
3	$F(2) = P(X \le 2) = 0.9$
	$F(3) = P(X \le 3) = 1$

$$F(x) = 0 \text{ if } x < -2$$
$$= .1 \text{ if } -2 \le x < -1$$
$$= .2 \text{ if } -1 \le x < 0$$
$$= .4 \text{ if } 0 \le x < 1$$
$$= .6 \text{ if } 1 \le x < 2$$
$$= 0.9, \text{ if } 2 \le x < 3$$
$$= 1 \text{ if } x \ge 3$$

Example 4

For the following probability distribution of X

X :	0	1	2	3
p (x) :	$\dfrac{1}{6}$	$\dfrac{1}{2}$	$\dfrac{3}{10}$	$\dfrac{1}{30}$

Find (i) P (X ≤ 1)   (ii) P (X ≤ 2)   (iii) P (0 < X < 2)

Solution :

(i) $P (X \le 1) = P (X = 0) + P(X = 1)$
$$= p(0) + p(1)$$
$$= \frac{1}{6} + \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$$

(ii) $P (X \le 2) = P (X = 0) + P(X = 1) + P (X = 2)$
$$= \frac{1}{6} + \frac{1}{2} + \frac{3}{10} = \frac{29}{30}$$

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Aliter $P(X \leq 2)$ can also be obtained as

$$P(X \leq 2) = 1 - P(X > 2)$$

$$= 1 - P(X = 3) = 1 - \frac{1}{30} = \frac{29}{30}$$

(iii) $P(0 < X < 2) = P(X = 1) = \frac{1}{2}$

8.1.4 Continuous Random Variable

A random variable X is said to be continuous if it takes a continuum of values. i.e. if it takes all possible values between certain defined limits.

For example,

(i)   The amount of rainfall on a rainy day.

(ii)   The height of individuals. (iii) The weight of individuals.

8.1.5 Probability function

A function f is said to be the probability density function (p.d.f) of a continuous random variable X if the following conditions are satisfied

(i) $f(x) \geq 0$ for all x (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Remark :

(i)   The probability that the random variable X lies in the interval (a, b) is given by

$$P(a < X < b) = \int_{a}^{b} f(x)\,dx.$$

(ii) $P(X = a) = \int_{a}^{a} f(x)\,dx = 0$

(iii) $P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < X < b)$

8.1.6 Continuous Distribution function

If X is a continuous random variable with p.d.f. $f(x)$, then the function

$$F_X(x) = P(X \leq x)$$

$$= \int_{-\infty}^{x} f(t)\,dt$$

is called the distribution function (d.f.) or cumulative distribution function (c.d.f) of the random variable X.

Properties : The cumulative distribution function has the following properties.

(i) $\underset{x \to -\infty}{\text{Lt}} F(x) = 0$ i.e. $F(-\infty) = 0$

(ii) $\underset{x \to \infty}{\text{Lt}} F(x) = 1$ i.e. $F(\infty) = 1$

(iii) Let F be the c.d.f of a continuous random variable X with p.d.f f. Then $f(x) = \dfrac{d}{dx} F(x)$ for all x at which F is differentiable.

Example 5

A continuous random variable X has the following p.d.f

$$f(x) = \begin{cases} k(2-x) & \text{for } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Determine the value of k.

Solution :

If $f(x)$ be the p.d.f., then $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \qquad 0 + \int_{0}^{2} f(x)\,dx + 0 = 1$$

$$\Rightarrow \qquad \int_{0}^{2} k(2-x)\,dx = 1$$

$$k\left(\int_{0}^{2} 2\,dx - x\,dx\right) = 1 \qquad \therefore \quad k = \frac{1}{2}$$

Hence $f(x) = \begin{cases} \dfrac{1}{2}(2-x) & \text{for } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

Example 6

Verify that

$$f(x) = \begin{cases} 3x^2 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

is a p.d.f and evaluate the following probabilities

(i) $P(X \le \frac{1}{3})$ (ii) $P(\frac{1}{3} \le X \le \frac{1}{2})$

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Solution :

Clearly $f(x) \geq 0$ for all x and hence one of the conditions for p.d.f is satisfied.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{1} f(x)\,dx = \int_{0}^{1} 3x^2\,dx = 1$$

∴ The other condition for p.d.f is also satisfied.

Hence the given function is a p.d.f

(i) $P\left(X \leq \dfrac{1}{3}\right) \quad = \int_{-\infty}^{\frac{1}{3}} f(x)\,dx \qquad\qquad P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$

$$= \int_{0}^{\frac{1}{3}} 3x^2\,dx = \frac{1}{27}$$

(ii) $P\left(\dfrac{1}{3} \leq X \leq \dfrac{1}{2}\right) = \int_{\frac{1}{3}}^{\frac{1}{2}} f(x)\,dx$

$$= \int_{\frac{1}{3}}^{\frac{1}{2}} 3x^2\,dx = \frac{1}{8} - \frac{1}{27} = \frac{19}{216}$$

Example 7

Given the p.d.f of a continuous random variable X as follows

$$f(x) = \begin{cases} kx(1-x) & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find k and c.d.f

Solution :

If X is a continuous random variable with p.d.f $f(x)$ then

$$\int_{-\infty}^{\infty} f(x)\,dx \;\; = 1$$

$$\int_{0}^{1} kx(1-x)\,dx = 1$$

$$k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{0}^{1} = 1 \qquad \therefore\; k = 6$$

Hence the given p.d.f becomes

$$f(x) = \begin{cases} 6x(1-x) & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

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To find c.d.f F(x)

$$F(x) = 0 \qquad \text{for } x \le 0$$

$$F(x) = P (X \le x) = \int_{-\infty}^{x} f(x) \, dx$$

$$= \int_{0}^{x} 6x(1-x) \, dx = 3x^2 - 2x^3 \quad \text{for } 0 < x < 1$$

$$F(x) = 1 \qquad \text{for } x \ge 1$$

∴ The c.d.f of X is as follows.

$$
\begin{aligned}
F(x) &= 0 && \text{for } x \le 0 \\
&= 3x^2 - 2x^3 && \text{for } 0 < x < 1. \\
&= 1 && \text{for } x \ge 1
\end{aligned}
$$

Example 8

Suppose that the life in hours of a certain part of radio tube is a continuous random variable X with p.d.f is given by

$$f(x) = \begin{cases} \dfrac{100}{x^2} & \text{when } x \ge 100 \\ 0 & \text{elsewhere} \end{cases}$$

(i) What is the probability that all of three such tubes in a given radio set will have to be replaced during the first of 150 hours of operation?

(ii) What is the probability that none of three of the original tubes will have to be replaced during that first 150 hours of operation?

Solution :

(i) A tube in a radio set will have to be replaced during the first 150 hours if its life is < 150 hours. Hence, the required probability '*p*' that a tube is replaced during the first 150 hours is,

$$p = P(X \le 150) = \int_{100}^{150} f(x) \, dx$$

$$= \int_{100}^{150} \frac{100}{x^2} \, dx = \frac{1}{3}$$

∴ The probability that all three of the original tubes will have to be replaced during the first

150 hours $= p^3 = \left(\dfrac{1}{3}\right)^3 = \dfrac{1}{27}.$

(ii) The probability that a tube is not replaced during the first 150 hours of operation is given by

$$P(X > 150) = 1 - P (X \le 150) = 1 - \frac{1}{3} = \frac{2}{3}$$

$\therefore$ the probability that none of the three tubes will be replaced during the 150 hours of

operation $= \left(\dfrac{2}{3}\right)^3 = \dfrac{8}{27}$

EXERCISE 8.1

1) Which of the following set of functions define a probability space on $S = [x_1, x_2, x_3]$?

(i) $p(x_1) = \dfrac{1}{3}$, $p(x_2) = \dfrac{1}{2}$ $p(x_3) = \dfrac{1}{4}$

(ii) $p(x_1) = \dfrac{1}{3}$ $p(x_2) = \dfrac{1}{6}$ $p(x_3) = \dfrac{1}{2}$

(iii) $p(x_1) = 0$ $p(x_2) = \dfrac{1}{3}$ $p(x_3) = \dfrac{2}{3}$

(iv) $p(x_1) = p(x_2) = \dfrac{2}{3}$ $p(x_3) = \dfrac{1}{3}$

2) Consider the experiment of throwing a single die. The random variable X represents the score on the upper face and assumes the values as follows:

X	:	1	2	3	4	5	6	
$p(x_i)$	:	$\dfrac{1}{6}$	$\dfrac{1}{6}$	$\dfrac{1}{6}$	$\dfrac{1}{6}$	$\dfrac{1}{6}$	$\dfrac{1}{6}$	Is $p(x_i)$ a p.m.f.

3) A random variable X has the following probability distribution. Values of

X, x	:	0	1	2	3	4	5	6	7	8
$p(x)$	:	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Determine the value of a

(ii) Find $P(X < 3)$, $P(X > 3)$ and $P(0 < X < 5)$

4) The following function is a probability mass function - Verify.

$$p(x) = \begin{cases} \dfrac{1}{3} & \text{for } x = 1 \\ \dfrac{2}{3} & \text{for } x = 2 \\ 0 & \text{otherwise} \end{cases}$$ Hence find the c.d.f

5) Find k if the following function is a probability mass function

$$p(x) = \begin{cases} \dfrac{k}{6} & \text{for } x = 0 \\[2mm] \dfrac{k}{3} & \text{for } x = 2 \\[2mm] \dfrac{k}{2} & \text{for } x = 4 \\[2mm] 0 & \text{otherwise} \end{cases}$$

6) A random variable X has the following probability distribution values of X

x	:	-2	0	5
$p(x)$	:	$\dfrac{1}{4}$	$\dfrac{1}{4}$	$\dfrac{1}{2}$

Evaluate the following probabilities

(a) $P(X \le 0)$ (b) $P(X < 0)$ (c) $P(0 \le X \le 10)$

7) A random variable X has the following probability function

Values of X, x :	0	1	2	3
$p(x)$:	$\dfrac{1}{16}$	$\dfrac{3}{8}$	k	$\dfrac{5}{16}$

(i) Find the values of k (ii) Construct the c.d.f of X

8) A continuous random variable has the following p.d.f

$$f(x) = kx^2, \qquad 0 \le x \le 10$$

$$= 0 \text{ otherwise.}$$

Determine k and evaluate (i) $P(.2 \le X \le 0.5)$ (ii) $P(X \le 3)$

9) If the function $f(x)$ is defined by

$$f(x) = ce^{-x}, \qquad\qquad 0 \le x < \infty. \qquad \text{Find the value of } c.$$

10) Let X be a continuous random variable with p.d.f.

$$f(x) = \begin{cases} ax, & 0 < x \le 1 \\ a, & 1 \le x \le 2 \\ -ax + 3a, & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(i) Determine the constant a

(ii) Compute $P(X \le 1.5)$

11) Let X be the life length of a certain type of light bulbs in hours. Determine 'a' so that the function

$$f(x) = \frac{a}{x^2}, \qquad 1000 \leq x \leq 2000$$

$$= 0 \quad \text{otherwise.}$$

may be the probability density function.

12) The kms. X in thousands which car owners get with a certain kind of tyre is a random variable having p.d.f.

$$f(x) = \frac{1}{20} e^{-\frac{x}{20}}, \quad \text{for } x > 0$$

$$= 0 \qquad \text{for } x \leq 0$$

Find the probabilities that one of these tyres will last

(i) atmost 10,000 kms

(ii) anywhere from 16,000 to 24,000 kms

(iii) atleast 30,000 kms.

8.2 MATHEMATICAL EXPECTATION

The concept of Mathematical expectation plays a vital role in statistics. Expected value of a random variable is a weighted average of all the possible outcomes of an experiment.

If X is a discrete random value which can assume values $x_1, x_2, \ldots x_n$ with respective probabilities $p(x_i) = P[X = x_i]$; i = 1, 2 ... n then its **mathematical expectation** is defined as

$$E(X) = \sum_{i=1}^{n} x_i \, p(x_i), \qquad \left(\text{Here } \sum_{i=1}^{n} p(x_i) = 1\right)$$

If X is a continuous random variable with probability density function $f(x)$, then

$$E(X) = \int_{-\infty}^{\infty} x \, f(x) \, dx$$

Note

E(X) is also known as the mean of the random variable X.

Properties

1) $E(c) = c$ where c is constant
2) $E(X + Y) = E(X) + E(Y)$
3) $E(aX + b) = aE(X) + b$ where a and b are constants.
4) $E(XY) = E(X) E(Y)$ if X and Y are independent

Note

The above properties holds good for both discrete and continous random variables.

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Variance

Let X be a random variable. Then the **Variance of X**, denoted by Var (X) or σ^2_x is

$$\text{Var (X)} = \sigma^2_x = E\,[X - E(X)]2$$
$$= E(X^2) - [E(X)]^2$$

The positive square root of Var(X) is called the **Standard Deviation of X** and is denoted by σ_x.

Example 9

A multinational bank is concerned about the waiting time (in minutes) of its customer before they would use ATM for their transaction. A study of a random sample of 500 customers reveals the following probability distribution.

X	:	0	1	2	3	4	5	6	7	8
$p(x)$	:	.20	.18	.16	.12	.10	.09	.08	.04	.03

Calculate the expected value of waiting time, X, of the customer.

Solution :

Let X denote the waiting time (in minutes) per customer.

X	:	0	1	2	3	4	5	6	7	8
$p(x)$	:	.20	.18	.16	.12	.10	.09	.08	.04	.03

Then $E(X) = \sum x\, p(x)$

$$= (0 \times .2) + (1 \times 0.18) + \text{........} + (8 \times 0.03) = 2.71$$

The expected value of X is equal to 2.71 minutes. Thus the average waiting time of a customer before getting access to ATM is 2.71 minutes.

Example 10

Find the expected value of the number of heads appearing when two fair coins are tossed.

Solution :

Let X be the random variable denoting the number of heads.

Possible values of X	:	0	1	2
Probabilities $p(x_i)$	:	$\dfrac{1}{4}$	$\dfrac{1}{2}$	$\dfrac{1}{4}$

The Expected value of X is

$$E(X) = x_1 p(x_1) + x_2 p(x_2) + x_3 p(x_3)$$

$$= 0\left(\frac{1}{4}\right) + 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) = 1$$

Therefore, the expected number of heads appearing in the experiment of tossing 2 fair coins is 1.

Example 11

The probability that a man fishing at a particular place will catch 1, 2, 3, 4 fish are 0.4, 0.3, 0.2 and 0.1 respectively. What is the expected number of fish caught?

Solution :

Possible values of X   :   1   2   3   4

Probabilities $p(x_i)$:   0.4   0.3   0.2   0.1

$\therefore$ $E(X) = \sum_i x_i p(x_i)$

$= x_1 p(x_1) + x_2 p(x_2) + x_2 p(x_3) + x_4 p(x_4)$

$= 1(.4) + 2(.3) + 3(.2) + 4(.1)$

$= .4 + .6 + .6 + .4 = 2$

Example 12

A person receives a sum of rupees equal to the square of the number that appears on the face when a balance die is tossed. How much money can he expect to receive?

Solution :

Random variable X: as square of the number that can appear on the face of a die. Thus

Possible values of X   : 1^2 2^2 3^2 4^2 5^2 6^2

Probabilities $p(x_i)$: $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$

The Expected amount that he receives,

$$E(X) = 1^2\left(\frac{1}{6}\right) + 2^2\left(\frac{1}{6}\right) + \dots\dots + 6^2\left(\frac{1}{6}\right)$$

$$= Rs.\frac{91}{6}$$

Example 13

A player tosses two fair coins. He wins Rs.5 if two heads appear, Rs. 2 if 1 head appears and Rs.1 if no head occurs. Find his expected amount of gain.

Solution :

Consider the experiment of tossing two fair coins. There are four sample points in the sample space of this experiment.

i.e. S = {HH, HT, TH, TT}

Let X be the random variable denoting the amount that a player wins associated with the sample point.

Thus,

Possible values of X :   5   2   1

Probabilities $p(x_i)$: $\dfrac{1}{4}$ $\dfrac{1}{2}$ $\dfrac{1}{4}$

$$E(X) = 5\left(\frac{1}{4}\right) + 2\left(\frac{1}{2}\right) + 1\left(\frac{1}{4}\right)$$

$$= \frac{5}{4} + 1 + \frac{1}{4} = \frac{10}{4} = \frac{5}{2}$$

$$= Rs.\,2.50$$

Hence expected amount of winning is Rs.2.50

Example 14

A random variable X has the probability function as follows :

values of X :   – 1   0   1

Probabilities :   0.2   0.3   0.5

Evaluate (i) E(3X +1)   (ii) E(X^2)   (iii) Var(X)

Solution :

X   :   – 1   0   1

$p(x_i)$:   0.2   0.3   0.5

(i)   E(3X + 1) = 3E(X) + 1

Now $E(X) = -1 \times 0.2 + 0 \times 0.3 + 1 \times 0.5$

$$= -1 \times 0.2 + 0 + 0.5 = 0.3$$

E(3X + 1) = 3(0.3) + 1 = 1.9

(ii) $E(X^2) = \Sigma x^2\, p(x)$

$$= (-1)^2 \times 0.2 + (0)^2 \times 0.3 + (1)^2 \times 0.5$$

$$= 0.2 + 0 + 0.5 = 0.7$$

(iii)   Var(X) $= E(X^2) - [E(X)]^2$

$$= .7 - (.3)^2 = .61$$

Example 15

Find the mean, variance and the standard deviation for the following probability distribution

Values of X	:	1	2	3	4
Probability, $p(x)$	:	0.1	0.3	0.4	0.2

Solution :

$$\text{Mean } = E(X) = \Sigma x \, p(x)$$

$$= 1(0.1) + 2(0.3) + 3(0.4) + 4(0.2) = 2.7$$

$$\text{Variance } = E(X^2) - [E(X)]^2$$

Now $\quad E(X^2) = \Sigma x^2 \, p(x)$

$$= 1^2(0.1) + 2^2(0.3) + 3^2(0.4) + 4^2(0.2) = 8.1$$

$\therefore \quad$ Variance $= 8.1 - (2.7)^2$

$$= 8.1 - 7.29 = .81$$

Standard Deviation $= \sqrt{0.81} = 0.9$

Example 16

Let X be a continuous random variable with p.d.f.

$$f(x) = \begin{cases} \dfrac{1}{2} & \text{for } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) E(X) (ii) E(X²)   (iii) Var(X)

Solution :

(i) $\quad E(X) = \displaystyle\int_{-\infty}^{\infty} x f(x) \, dx \qquad\qquad$ (by definition)

$$= \frac{1}{2} \int_{-1}^{1} x \, dx = \frac{1}{2} \left[\frac{x^2}{2} \right]_{-1}^{1} = 0$$

(ii) $\quad E(X^2) = \displaystyle\int_{-1}^{1} x^2 f(x) \, dx$

$$= \int_{-1}^{1} x^2 \frac{1}{2} \, dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} \right]_{-1}^{1} = \frac{1}{3}$$

(iii) $\mathrm{Var(X) = E(X)^2 - [E(X)]^2}$

$$= \frac{1}{3} - 0 = \frac{1}{3}$$

EXERCISE 8.2

1)   A balanced die is rolled. A person recieves Rs. 10 when the number 1 or 3 or 5 occurs and loses Rs. 5 when 2 or 4 or 6 occurs. How much money can he expect on the average per roll in the long run?

2)   Two unbiased dice are thrown. Find the expected value of the sum of the points thrown.

3)   A player tossed two coins. If two heads show he wins Rs. 4. If one head shows he wins Rs. 2, but if two tails show he must pay Rs. 3 as penalty. Calculate the expected value of the sum won by him.

4)   The following represents the probability distribution of D, the daily demand of a certain product. Evaluate E(D).

D	:	1	2	3	4	5
$P[D = d]$	:	0.1	0.1	0.3	0.3	0.2

5)   Find E (2X − 7) and E (4X + 5) for the following probability distribution.

X	:	− 3	− 2	− 1	0	1	2	3
$p(x)$	:	.05	.1	.3	0	.3	.15	.1

6)   Find the mean, variance and standard deviation of the following probability distribution.

Values of X :	− 3	− 2	− 1	0	1	2	3
Probability $p(x)$:	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$

7)   Find the mean and variance for the following probability distribution.

$$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0 & , & x < 0 \end{cases}$$

8.3 DISCRETE DISTRIBUTIONS

We know that the frequency distributions are based on observed data derived from the collected sample information. For example, we may study the marks of the students of a class and formulate a frequency distribution as follows:

Marks	No. of students
0 - 20	10
20 - 40	12
40 - 60	25
60 - 80	15
80 - 100	18
Total	**80**

The above example clearly shows that the observed frequency distributions are obtained by grouping. Measures like averages, dispersion, correlation, etc. generally provide us a consolidated view of the whole observed data. This may very well be used in formulating certain ideas (inference) about the characteristics of the whole set of data.

Another type of distribution in which variables are distributed according to some definite probability law which can be expressed mathematically are called theoretical probability distribution.

The probability distribution is a total listing of the various values the random variable can take along with the corresponding probabilities of each value. For example; consider the pattern of distribution of machine breakdown in a manufacturing unit. The random variable would be the various values the machine breakdown could assume. The probability corresponding to each value of the breakdown as the relative frequency of occurence of the breakdown. This probability distribution is constructed by the actual breakdown pattern discussed over a period of time.

Theoretical probability distributions are basically of two types

(i) Discrete and   (ii) Continuous

In this section, we will discuss theoretical discrete distributions namely, Binomial and Poisson distributions.

8.3.1 Binomial Distribution

It is a distribution associated with repetition of independent trials of an experiment. Each trial has two possible outcomes, generally called success and failure. Such a trial is known as **Bernoulli trial**.

Some examples of Bernoulli trials are :

(i) a toss of a coin (Head or tail)

(ii) the throw of a die (even or odd number)

An experiment consisiting of a repeated number of Bernoulli trials is called a **binomial experiment**. A binomial experiment must possess the following properties:

(i)   there must be a fixed number of trials.

(ii)   all trials must have identical probabilities of success (p) i.e. if we call one of the two outcomes as "success" and the other as "failure", then the probability p of success remains as constant throughout the experiment.

(iii)   the trials must be independent of each other i.e. the result of any trial must not be affected by any of the preceeding trial.

Let X denote the number of successes in 'n' trials of a binomial experiment. Then X follows a binomial distribution with parameters n and p and is denoted by X~B(n, p).

A random variable X is said to follow **Binomial distribution** with parameters n and p, if it assumes only non-negative values and its probability mass function is given by

$$P[X = x] = p(x) = {}^n C_x p^x q^{n-x}\; ; x = 0, 1, 2, ..n\; ; q = 1 - p$$

Remark

(i) $\displaystyle\sum_{x=0}^{n} p(x) = \sum_{x=0}^{n} {}^n C_x p^x q^{n-x} = (q + p)^n = 1$

(ii) $\displaystyle {}^n C_r = \frac{n(n-1)...(n - \overline{r-1})}{1.2.3.... \, r}$

Mean and Variance

For the binomial distribution

Mean $= np$

Variance $= npq$; Standard Deviation $= \sqrt{npq}$

Example 17

What is the probability of getting exactly 3 heads in 8 tosses of a fair coin.

Solution :

Let p denote the probability of getting head in a toss.

Let X be the number of heads in 8 tosses.

Then $p = \dfrac{1}{2},\quad q = \dfrac{1}{2}\quad$ and $n = 8$

Probability of getting exactly 3 heads is

$$P\,(X = 3) = {}^8 C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^5$$

$$= \frac{8 \times 7 \times 6}{1 \times 2 \times 3} \left(\frac{1}{2}\right)^8 = \frac{7}{32}$$

Example 18

Write down the Binomial distribution whose mean is 20 and variance being 4.

Solution :

Given mean, $np = 20$; variance, $npq = 4$

Now $\quad q = \dfrac{npq}{np} = \dfrac{4}{20} = \dfrac{1}{5} \quad \therefore \quad p = 1 - q = \dfrac{4}{5}$

From $\quad np = 20$, we have

$$n = \dfrac{20}{p} = \dfrac{20}{\dfrac{4}{5}} = 25$$

Hence the binomial distribution is

$$p(x) = {}^n C_x p^x q^{n-x} = {}^{25} C_x \left(\dfrac{4}{5}\right)^x \left(\dfrac{1}{5}\right)^{n-x}, \quad x = 0, 1, 2, ..., 25$$

Example 19

On an average if one vessel in every ten is wrecked, find the probability that out of five vessels expected to arrive, atleast four will arrive safely.

Solution :

Let the probability that a vessel will arrive safely, $p = \dfrac{9}{10}$

Then probability that a vessel will be wrecked, $q = 1 - p = \dfrac{1}{10}$

No. of vessels, $n = 5$

$\therefore \quad$ The probability that atleast 4 out of 5 vessels to arrive safely is

$P(X \geq 4) = P(X = 4) + P(X = 5)$

$$= {}^5 C_4 \left(\dfrac{9}{10}\right)^4 \dfrac{1}{10} + {}^5 C_5 \left(\dfrac{9}{10}\right)^5$$

$$= 5(.9)^4(.1) + (.9)^5 = .91854$$

Example 20

For a binomial distribution with parameters $n = 5$ and $p = .3$ find the probabilities of getting (i) atleast 3 successes (ii) atmost 3 successes.

Solution :

Given $n = 5, \; p = .3 \qquad \therefore q = .7$

(i)   The probability of atleast 3 successes

$P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5)$

$$= {}^5 C_3 (0.3)^3 (0.7)^2 + {}^5 C_4 (0.3)^4 (0.7) + {}^5 C_5 (.3)^5 (0.7)^0$$

$$= .1631$$

(ii)   The probability of atmost 3 successes

$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$$= (.7)^5 + {}^5C_1 (.7)^4 (.3) + {}^5C_2 (.7)^3(.3)^2 + {}^5C_3 (.7)^2 (.3)^3$$

$$= .9692$$

8.3.2 Poisson distribution

Poisson distribution is also a discrete probability distribution and is widely used in statistics. Poisson distribuition occurs when there are events which do not occur as outcomes of a definite number of trials of an experiment but which occur at random points of time and space wherein our interest lies only in the number of occurences of the event, not in its non-occurances. This distribution is used to describe the behaviour of rare events such as

(i)   number of accidents on road

(ii)   number of printing mistakes in a book

(iii)   number of suicides reported in a particular city.

Poisson distribution is an approximation of binomial distribution when n (number of trials) is large and *p,* the probability of success is very close to zero with np as constant.

A random variable X is said to follow a **Poisson distribution** with parameter $\lambda > 0$ if it assumes only non-negative values and its probability mass function is given by

$$P[X = x] = p(x) = \frac{e^{-\lambda}\lambda^x}{x!}; \quad x = 0, 1, 2, ...$$

Remark

It should be noted that

$$\sum_{x=0}^{\infty} P[X = x] = \sum_{x=0}^{\infty} p(x) = 1$$

Mean and Variance

For the of poisson distribution

Mean, $E(X) = \lambda$, Variance, $Var(X) = \lambda$, $S.D = \sqrt{\lambda}$

Note

For poission distribution mean and variance are equal.

Example 21

Find the probability that atmost 5 defective fuses will be found in a box of 200 fuses if experience shows that 2 percent of such fuses are defective. ($e^{-4} = 0.0183$)

Solution :

p = probability that a fuse is defective = $\dfrac{2}{100}$

$n = 200$

$$\therefore \lambda = np = \frac{2}{100} \times 200 = 4$$

Let X denote the number of defective fuses found in a box.

Then the distribution is given by

$$P[X = x] = p(x) = \frac{e^{-4} 4^x}{x!}$$

So, probability that atmost 5 defective fuses will be found in a box of 200 fuses
= P (X ≤ 5)
= P (X = 0) + P (X = 1) + P (X = 2) + P (X = 3) + P (X = 4) + P (X = 5)

$$= e^{-4} + \frac{e^{-4} 4}{1!} + \frac{e^{-4} 4^2}{2!} + \frac{e^{-4} 4^3}{3!} + \frac{e^{-4} 4^4}{4!} + \frac{e^{-4} 4^5}{5!}$$

$$= e^{-4} (1 + \frac{4}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!})$$

$$= 0.0183 \times \frac{643}{15} = 0.785$$

Example 22

Suppose on an average 1 house in 1000 in a certain district has a fire during a year. If there are 2000 houses in that district, what is the probability that exactly 5 houses will have fire during the year? ($e^{-2} = 0.13534$)

Solution :

$$p = \text{probability that a house catches fire} = \frac{1}{1000}$$

Let *x* denote the number of houses that has a fire.

Here $n = 2000$ $\therefore \lambda = np = 2000 \times \frac{1}{1000} = 2$

Then the distribution is given by $P[X = x] = \frac{e^{-2} 2^x}{x!}$, $x = 0, 1, 2, ...$

Probability that exactly 5 houses will have a fire during the year is

$$P(X = 5) = \frac{e^{-2} 2^5}{5!}$$

$$= \frac{.13534 \times 32}{120} = .0361$$

Example 23

The number of accidents in a year attributed to taxi drivers in a city follows poisson distribution with mean 3. Out of 1000 taxi drivers, find the approximate number of drivers with

(i)   **no accident in a year**

(ii)   **more than 3 accidents in a year**

Solution :

Here $\lambda = np = 3$

$$N = 1000$$

Then the distribution is

$P[X = x] = \dfrac{e^{-3}3^x}{x!}$ where X denotes the number accidents.

(i)   P (no accidents in a year) = P (X = 0)

$$= e^{-3} = 0.05$$

$\therefore$ Number of drivers with no accident = $1000 \times 0.05 = 50$

(ii)   P (that more than 3 accident in a year) = P (X > 3)

$$= 1 - P(X \le 3)$$

$$= 1 - \left[e^{-3} + \dfrac{e^{-3}3^1}{1!} + \dfrac{e^{-3}3^2}{2!} + \dfrac{e^{-3}3^3}{3!} \right]$$

$$= 1 - e^{-3}[1 + 3 + 4.5 + 4.5]$$

$$= 1 - e^3(13) = 1 - .65 = .35$$

$\therefore$ Number of drivers with more than 3 accidents

$$= 1000 \times 0.35 = 350$$

EXERCISE 8.3

1)   Ten coins are thrown simultaneously. Find the probability of getting atleast 7 heads.

2)   In a binomial distribution consisting of 5 independent trials, probabilities of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the parameter '*p*' of the distribution.

3)   For a binomial distribution, the mean is 6 and the standard deviation is $\sqrt{2}$. Write down all the terms of the distribution.

4)   The average percentage of failure in a certain examination is 40. What is the probability that out of a group of 6 candidates atleast 4 passed in the examination?

5)   An unbiased coin is tossed six times. What is the probability of obtaining four or more heads?

6)   It is stated that 2% of razor blades supplied by a manufacturer are defective. A random sample of 200 blades is drawn from a lot. Find the probability that 3 or more blades are defective. (e^{-4} = .01832)

7)   Find the probability that atmost 5 defective bolts will be found in a box of 200 bolts, if

it is known that 2% of such bolts are expected to be defective ($e^{-4} = 0.01832$)

8) An insurance company insures 4,000 people against loss of both eyes in car accidents. Based on previous data, the rates were computed on the assumption that on the average 10 persons in 1,00,000 will have car accidents each year that result in this type of injury. What is the probability that more than 3 of the injured will collect on their policy in a given year? ($e^{-0.4} = 0.6703$)

9) It is given that 3% of the electric bulbs manufactured by a company are defective. Find the probability that a sample of 100 bulbs will contain (i) no defective (ii) exactly one defective. ($e^{-3} = 0.0498$) .

10) Suppose the probability that an item produced by particular machine is defective equals 0.2. If 10 items produced from this machine are selected at random, what is the probability that not more than one defective is found? ($e^{-2} = .13534$)

8.4 CONTINUOUS DISTRIBUTIONS

The binomial and Poisson distributions discussed in the previous section are the most useful theoretical distributions. In order to have mathematical distribution suitable for dealing with quantities whose magnitudes vary continuously like heights and weights of individuals, a continuous distribution is needed. Normal distribution is one of the most widely used continuous distributions.

8.4.1 Normal Distribution

Normal Distribution is considered to be the most important and powerful of all the distributions in statistics. It was first introduced by De Moivre in 1733 in the development of probability. Laplace (1749 - 1827) and Gauss (1827 - 1855) were also associated with the development of Normal distribution.

A random variable X is said to follow a **Normal Distribution** with mean μ and variance σ^2 denoted by X ~ N(μ, σ^2), if its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} , \quad -\infty < x < \infty, \ -\infty < \mu < \infty, \ \sigma > 0$$

Remark

The parameters μ and σ^2 completely describe the normal distribution. Normal distribution could be also considered as limiting form of binomial distribution under the following conditions:

(i) n, the number of trials is indefinitely large i.e. n → ∞
(ii)   neither p nor q is very small.

The graph of the p.d.f of the normal distribution is called the **Normal curve**, and it is given below.

$x = \mu$

Normal probability curve
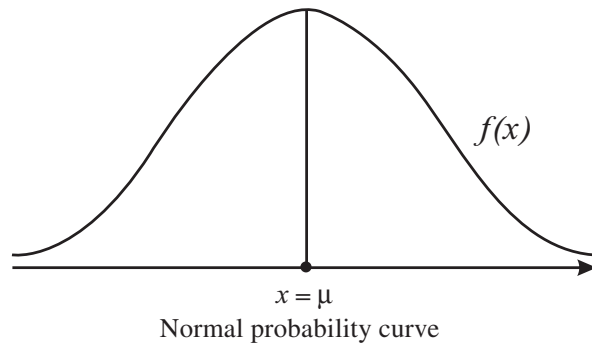
8.4.2 Properties of Normal Distribution

The following are some of the important properties of the normal curve and the normal distribution.

(i)   The curve is "bell - shaped" and symmetric about $x = \mu$

(ii)   Mean, Median and Mode of the distribution coincide.

(iii)   There is one maximum point of the normal curve which occurs at the mean (μ). The height of the curve declines as we go in either direction from the mean.

(iv)   The two tails of the curve extend infinitely and never touch the horizontal (x) axis.

(v)   Since there is only one maximum point, the normal curve is unimodal i.e. it has only one mode.

(vi)   Since $f(x)$ being the probability, it can never be negative and hence no portion of the curve lies below the x - axis.

(vii)   The points of inflection are given by $x = \mu \pm \sigma$

(viii) Mean Deviation about mean

$$\sqrt{\frac{2}{\pi}}\sigma = \frac{4}{5}\sigma$$

(ix)   Its mathematical equation is completely determined if the mean and S.D are known i.e. for a given mean μ and S.D σ, there is only one Normal distribution.

(x)   Area Property : For a normal distribution with mean μ and S.D σ, the total area under normal curve is 1, and

(a) $P(\mu - \sigma < X < \mu + \sigma) = 0.6826$

i.e. (mean) $\pm$ 1σ covers 68.27%;

(b) $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$

i.e. (mean) $\pm$ 2σ covers 95.45% area

(c) $P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$

i.e. (mean) $\pm$ 3σ covers 99.73% area

10.4.3 Standard Normal Distribution

A random variable which has a normal distribution with a mean $\mu = 0$ and a standard deviation $\sigma = 1$ is referred to as **Standard Normal Distribution.**

Remark

(i) If $X \sim N(\mu, \sigma^2)$, then $Z = \dfrac{X - \mu}{\sigma}$ is a standard normal variate with $E(Z) = 0$ and $var(Z) = 1$ i.e. $Z \sim N(0, 1)$.

(ii) It is to be noted that the standard normal distribution has the same shape as the normal distribution but with the special properties of $\mu = 0$ and $\sigma = 1$.

$$Z = 0 \qquad\qquad\qquad\qquad Z$$
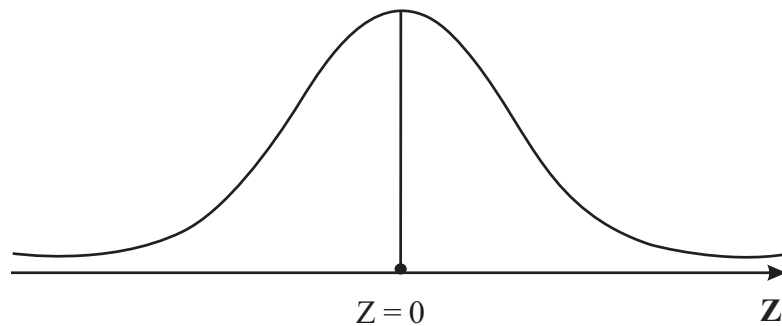
A random variable Z is said to have a **standard normal distribution** if its probability density function is given by

$$\varphi(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}}, \quad -\infty < z < \infty$$

Example 24

What is the probability that Z
(a) lies between 0 and 1.83
(b) is greater than 1.54
(c) is greater than – 0.86
(d) lies between 0.43 and 1.12
(e) is less than 0.77

Solution :

(a) **Z lies between 0 and 1.83.**

$$Z = 0 \quad 1.83 \qquad\qquad Z$$
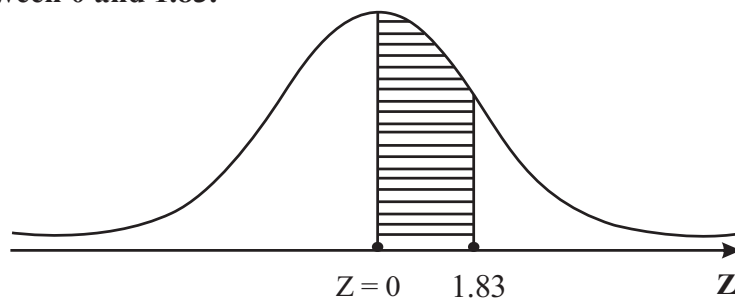
$P(0 \le Z \le 1.83) = 0.4664$ (obtained from the tables directly)

(b) **Z is greater than 1.54 i.e. P(Z ≥ 1.54)**
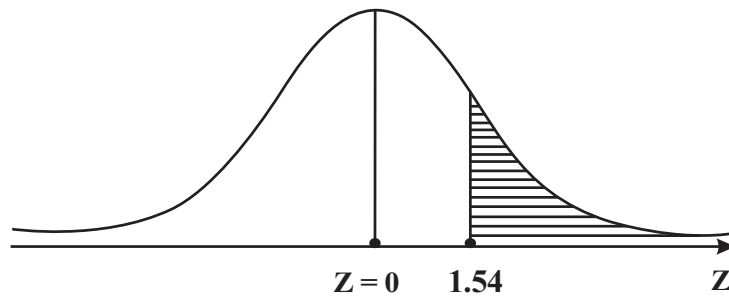
Since the total area to the right of $Z = 0$ is 0.5 and area between $Z = 0$ and 1.54 (from tables) is 0.4382

$$P(Z \geq 1.54) = 0.5 - P(0 \leq Z \leq 1.54)$$

$$= 0.5 - .4382 = .0618$$

(c) Z is greater than -0.86 i.e. P(Z ≥ -0.86)
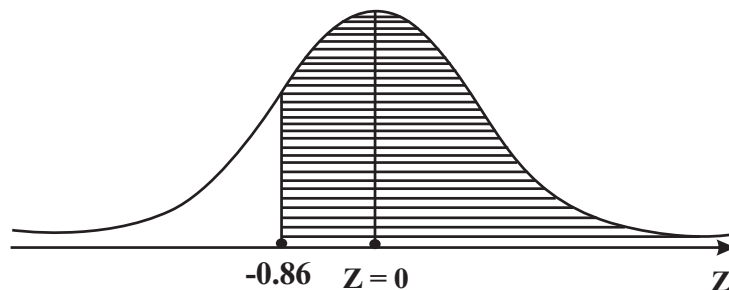
Here the area of interest $P(Z \geq -0.86)$ is represented by the two components.

(i)   Area between $Z = -0.86$ and $Z = 0$, which is equal to 0.3051 (from tables)

(ii) $Z > 0$, which is 0.5

$$\therefore P(Z \geq -0.86) = 0.3051 + 0.5 = 0.8051$$

(d) Z lies between 0.43 and 1.12
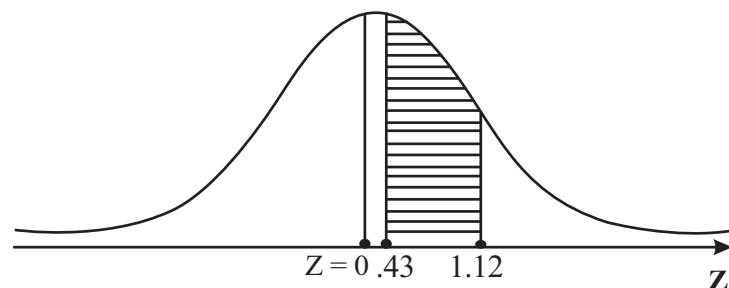
$$\therefore P(0.43 \leq Z \leq 1.12) = P(0 \leq Z \leq 1.12) - P(0 \leq Z \leq 0.43)$$

$$= 0.3686 - 0.1664 \text{ (from tables)}$$

$$= 0.2022.$$

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(e) Z is less than 0.77
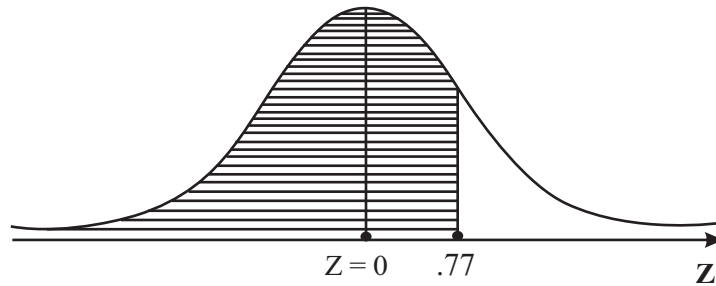
$$P(\ Z \le 0.77) = 0.5 + P(0 \le Z \le 0.77)$$

$$= 0.5 + .2794 = .7794 \text{ (from tables)}$$

Example 25

If X is a normal random variable with mean 100 and variance 36 find (i) P(X > 112) (ii) P(X < 106) (iii) P(94 < X < 106)

Solution :

Mean, $\mu = 100$; Variance, $\sigma^2 = 36$; S.D, $\sigma = 6$

Then the standard normal variate Z is given by

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 100}{6}$$

When X = 112, then $Z = \frac{112 - 100}{6} = 2.$

$$\therefore P(X > 112) = P(Z > 2)$$
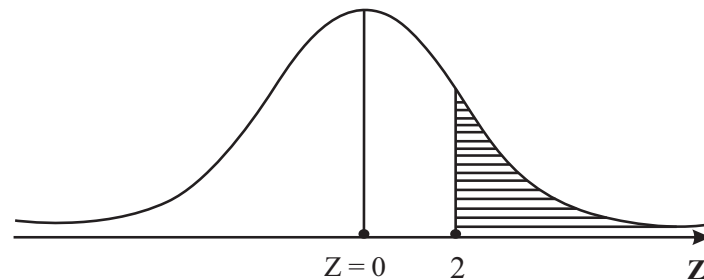
$$= P(0 \le Z < \infty) - P(0 \le Z \le 2)$$

$$= 0.5 - 0.4772 = 0.0228 \text{ (from tables)}$$

(ii)   For a given value X = 106, $Z = \frac{106 - 100}{6} = 1$

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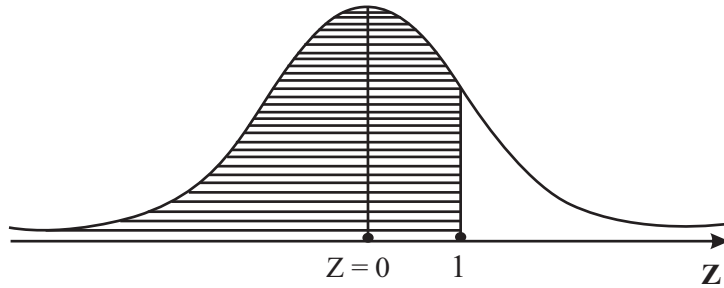
$$P(X < 106) = P (Z < 1)$$

$$= P (- \infty \le Z \le 0) + P(0 \le Z \le 1)$$

$$= 0.5 + 0.3413 = 0.8413 \text{ (from tables)}$$

When X = 94, $\quad Z = \dfrac{94 - 100}{6} = -1$

X = 106, $\quad Z = \dfrac{106 - 100}{6} = +1$

(iii)
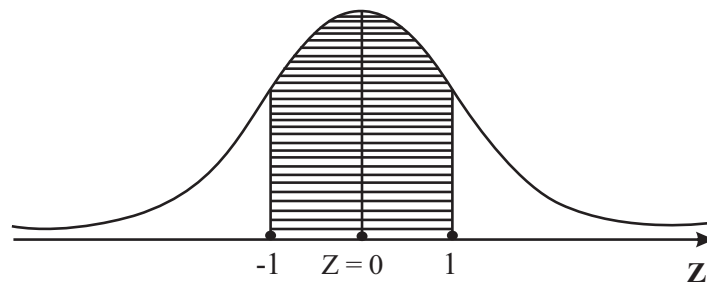
∴ $\quad$ P(94 < X < 106) = P(-1 < Z < 1)

$$= P(-1 < Z < 0) + P(0 < Z < 1)$$

$$= 2 \, P(0 < Z < 1) \text{ (by symmetry)}$$

$$= 2 \, (0.3413)$$

$$= 0.6826$$

Example 26

In a sample of 1000 candidates the mean of certain test is 45 and S.D 15. Assuming the normality of the distrbution find the following:

(i) How many candidates score between 40 and 60?

(ii) How many candidates score above 50?

(iii) How many candidates score below 30?

Solution :

Mean = μ = 45 and S.D. = σ = 15

$$\text{Then } Z = \frac{X - \mu}{\sigma} = \frac{X - 45}{15}$$

(i) $P(40 < X < 60) = P\left(\dfrac{40 - 45}{15} < Z < \dfrac{60 - 45}{15}\right)$

$$= P\left(-\frac{1}{3} < Z < 1\right)$$
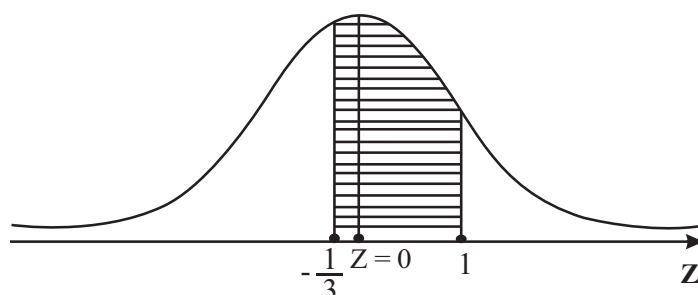
$$= P\left(-\frac{1}{3} \le Z \le 0\right) + P(0 \le Z \le 1)$$
$$= P(0 \le Z \le 0.33) + P(0 \le Z \le 1)$$
$$= 0.1293 + 0.3413 \text{ (from tables)}$$

$$P(40 < X < 60) = 0.4706$$

Hence number of candidates scoring between 40 and 60

$$= 1000 \text{ x } 0.4706 = 470.6 \simeq 471$$

(ii) $P(X > 50) = P\left(Z > \dfrac{1}{3}\right)$
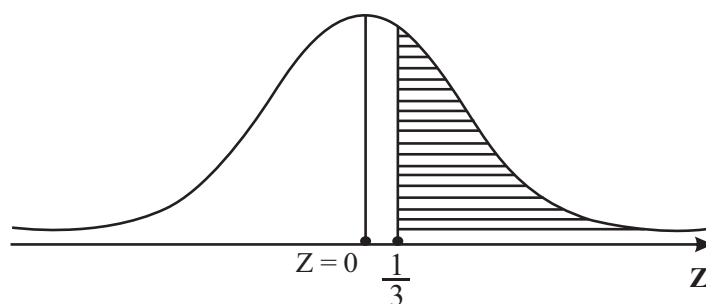
$$= 0.5 - P\left(0 < Z < \frac{1}{3}\right) = 0.5 - P(0 < Z < 0.33)$$

$$= 0.5 - 0.1293 = 0.3707 \text{ (from tables)}$$

Hence number of candidates scoring above 50

$$= 1000 \times 0.3707 = 371.$$

(iii) $P(X < 30) = P(Z < -1)$
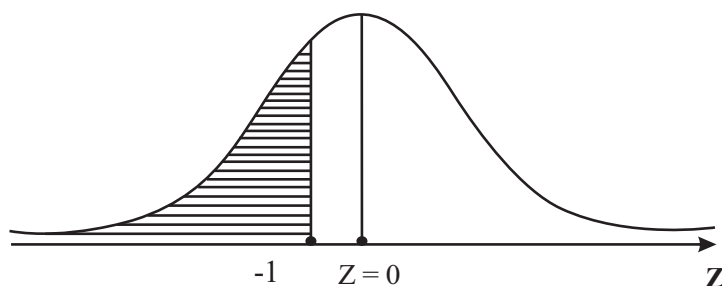
$$= 0.5 - P(-1 < Z < 0)$$

$$= 0.5 - P(0 < Z < 1) \quad \therefore \text{ Symmetry}$$

$$= 0.5 - 0.3413 = 0.1587 \text{ (from tables)}$$

$\therefore$ Number of candidates scoring less than 30

$$= 1000 \times 0.1587 = 159$$

Example 27

The I.Q (intelligence quotient) of a group of 1000 school children has mean 96 and the standard deviation 12. Assuming that the distribution of I.Q among school children is normal, find approximately the number of school children having I.Q.

(i) less than 72 (ii) between 80 and 120

Solution :

Given N = 1000, $\mu = 96$ and $\sigma = 12$

Then $Z = \dfrac{X - \mu}{\sigma} = \dfrac{X - 96}{12}$

(i) $P(X < 72) = P(Z < -2)$
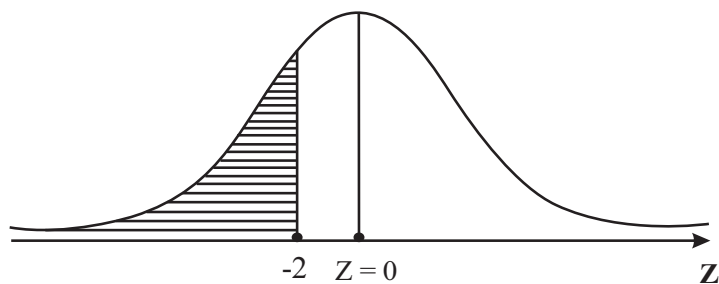
$$= P(-\infty < Z \le 0) - P(-2 \le Z \le 0)$$

$$= P(0 \le Z < \infty) - P(0 \le Z \le 2) \text{ (By symmetry)}$$

$$= 0.5 - 0.4772 \text{ (from tables)} = 0.0228.$$

$\therefore$ Number of school children having I.Q less than 72

$$= 1000 \times 0.0228 = 22.8 \sim 23$$

(ii) $P(80 < X < 120) = P(-1.33 < Z < 2)$
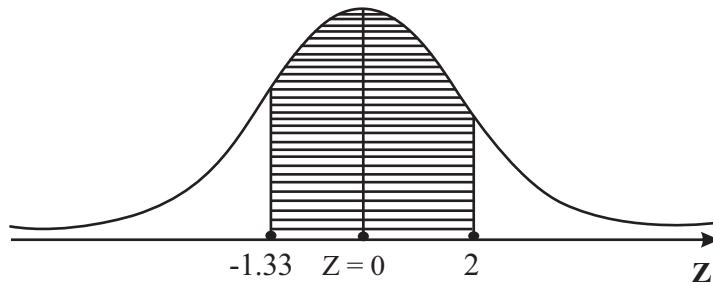
$$= P(-1.33 \leq Z \leq 0) + P(0 \leq Z \leq 2)$$

$$= P(0 \leq Z \leq 1.33) + P(0 \leq Z \leq 2)$$

$$= .4082 + .4772 \text{ (from tables)}$$

$$= 0.8854$$

∴ Number of school children having I.Q. between 80 and 120

$$= 1000 \times .8854 = 885.$$

Example 28

In a normal distribution 20% of the items are less than 100 and 30% are over 200. Find the mean and S.D of the distribution.

Solution :

Representing the given data diagramtically,
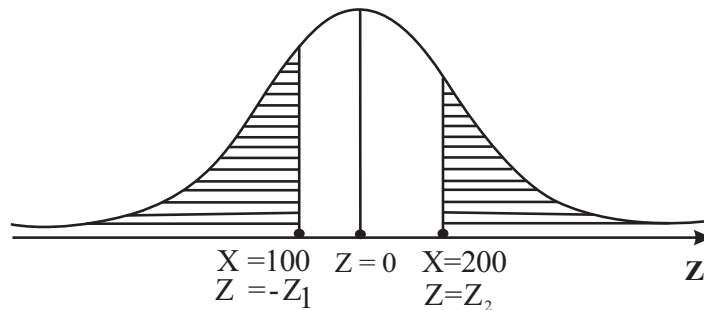
From the diagram

$$P(-Z_1 < Z < 0) = 0.3$$

i.e. $P(0 < Z < Z_1) = 0.3$

∴ $Z1 = 0.84$ (from the normal table)

Hence $-0.84 = \dfrac{100 - \mu}{\sigma}$

i.e. $100 - \mu = -0.84\ \sigma$ ...... (1)

$P(0 < Z < Z_2) = 0.2$

$\therefore$ $Z_2 = 0.525$ (from the normal table)

Hence $0.525 = \dfrac{200 - \mu}{\sigma}$

i.e. $200 - \mu = 0.525\ \sigma$ .... (2)

Solving (1) and (2), $\mu = 161.53$

$\sigma = 73.26$

EXERCISE 8.4

1)   Find the area under the standard normal curve which lies

      (i)   to the right of $Z = 2.70$

      (ii)   to the left of $Z = 1.73$

2)   Find the area under the standard normal curve which lies

      (i)   between $Z = 1.25$ and $Z = 1.67$

      (ii)   between $Z = -0.90$ and $Z = -1.85$

3)   The distribution of marks obtained by a group of students may be assumed to be normal with mean 50 marks and standard deviation 15 marks. Estimate the proportion of students with marks below 35.

4)   The marks in Economics obtained by the students in Public examination is assumed to be approximately normally distributed with mean 45 and S.D 3. A student taking this subject is chosen at random. What is the probability that his mark is above 70?

5)   Assuming the mean height of soldiers to be 68.22 inches with a variance 10.8 inches. How many soldiers in a regiment of 1000 would you expect to be over 6 feet tall ?

6)   The mean yield for one-acre plot is 663 kgs with a S.D 32 kgs. Assuming normal distribution, how many one-acre plot in a batch of 1000 plots would you expect to have yield (i) over 700 kgs (ii) below 650 kgs.

7)   A large number of measurements is normally distributed with a mean of 65.5" and S.D of 6.2". Find the percentage of measurements that fall between 54.8" and 68.8".

8)   The diameter of shafts produced in a factory conforms to normal distribution. 31% of the shafts have a diameter less than 45mm. and 8% have more than 64mm. Find the mean and standard deviation of the diameter of shafts.

9) The results of a particular examination are given below in a summary form.

Result	Percentage of candidates
1. passed with distinction	10
2. passed	60
3. failed	30

It is known that a candidate gets plucked if he obtained less than 40 marks out of 100 while he must obtain atleast 75 marks in order to pass with distinction. Determine the mean and the standard deviation of the distribution assuming this to be normal.

EXERCISE 8.5

Choose the correct answer

1) If a fair coin is tossed three times the probability function p(x) of the number of heads x is

(a)
x	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$

(b)
x	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(c)
x	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$

(d) none of these

2) If a discrete random variable has the probability mass function as

x	0	1	2	3
$p(x)$	k	$2k$	$3k$	$5k$

then the value of k is

(a) $\dfrac{1}{11}$ (b) $\dfrac{2}{11}$ (c) $\dfrac{3}{11}$ (d) $\dfrac{4}{11}$

3) If the probability density function of a variable X is defined as $f(x) = Cx (2 - x)$, $0 < x < 2$ then the value of C is

(a) $\dfrac{4}{3}$ (b) $\dfrac{6}{4}$ (c) $\dfrac{3}{4}$ (d) $\dfrac{3}{5}$

4) The mean and variance of a binomial distribution are

(a) np, npq (b) pq, npq (c) $np, \sqrt{npq}$ (d) np, nq

5) If X~N (μ, σ), the standard Normal variate is distributed as

(a) N(0, 0) (b) N(1, 0) (c) N(0, 1) (d) N(1, 1)

6) The normal distribution curve is

(a) Bimodal (b) Unimodal (c) Skewed (d) none of these

7) If X is a poission variate with $P(X = 1) = P(X = 2)$, the mean of the Poisson variate is equal to

(a) 1 (b) 2 (c) – 2 (d) 3

8) The standard deviation of a Poissson variate is 2, the mean of the poisson variate is

(a) 2 (b) 4 (c) $\sqrt{2}$ (d) $\dfrac{1}{\sqrt{2}}$

9) The random variables X and Y are independent if

(a) $E(X\,Y) = 1$ (b) $E(XY) = 0$

(c) $E(X\,Y) = E(X)\,E(Y)$ (d) $E(X+Y) = E(X) + E(Y)$

10) The mean and variance of a binomial distribution are 8 and 4 respectively. Then $P(X = 1)$ is equal to

(a) $\dfrac{1}{2^{12}}$ (b) $\dfrac{1}{2^4}$ (c) $\dfrac{1}{2^6}$ (d) $\dfrac{1}{2^{10}}$

11) If X~N (μ, σ²), the points of inflection of normal distribution curve are

(a) $\pm\,\mu$ (b) $\mu \pm \sigma$ (c) $\sigma \pm \mu$ (d) $\mu \pm 2\sigma$

12) If X~N (μ, σ²), the maximum probability at the point of inflection of normal distribution is

(a) $\dfrac{1}{\sqrt{2\pi}}\,e^{\frac{1}{2}}$ (b) $\dfrac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}}$ (c) $\dfrac{1}{\sigma\sqrt{2\pi}}$ (d) $\dfrac{1}{\sqrt{2\pi}}$

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X	−1	−2	1	2
$p(x)$	$\dfrac{1}{3}$	$\dfrac{1}{6}$	$\dfrac{1}{6}$	$\dfrac{1}{3}$

13) If a random variable X has the following probability distribution then the expected value of X is

(a) $\dfrac{3}{2}$
(b) $\dfrac{1}{6}$
(c) $\dfrac{1}{2}$
(d) $\dfrac{1}{3}$

14) If $X \sim N(5, 1)$, the probability density function for the normal variate X is

(a) $\dfrac{1}{5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-1}{5}\right)^2}$

(b) $\dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-1}{5}\right)^2}$

(c) $\dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-5)^2}$

(d) $\dfrac{1}{\sqrt{\pi}} e^{-\frac{1}{2}(x-5)^2}$

15) If $X \sim N(8, 64)$, the standard normal variate Z will be

(a) $\dfrac{X-64}{8}$
(b) $\dfrac{X-8}{64}$
(c) $\dfrac{X-8}{8}$
(d) $\dfrac{X-8}{\sqrt{8}}$

SAMPLING TECHNIQUES AND STATISTICAL INFERENCE

9

9.1 SAMPLING AND TYPES OF ERRORS

Sampling is being used in our everyday life without knowing about it. For examples, a cook tests a small quantity of rice to see whether it has been well cooked and a grain merchant does not examine each grain of what he intends to purchase, but inspects only a small quantity of grains. Most of our decisions are based on the examination of a few items only.

In a statistical investigation, the interest usually lies in the assessment of general magnitude and the study of variation with respect to one or more characteristics relating to individuals belonging to a group. This group of individuals or units under study is called **population** or **universe**. Thus in statistics, population is an aggregate of objects or units under study. The population may be finite or infinite.

9.1.1 Sampling and sample

Sampling is a method of selecting units for analysis such as households, consumers, companies etc. from the respective population under statistical investigation. The theory of sampling is based on the **principle of statistical regularity.** According to this principle, a moderately large number of items chosen at random from a large group are almost sure on an average to possess the characteristics of the larger group.

A smallest non-divisible part of the population is called a unit. A unit should be well defined and should not be ambiguous. For example, if we define unit as a household, then it should be defined that a person should not belong to two households nor should it leave out persons belonging to the population.

A finite subset of a population is called a sample and the number of units in a **sample** is called its **sample size**.

By analysing the data collected from the sample one can draw inference about the population under study.

9.1.2 Parameter and Statistic

The statistical constants of a population like mean (μ), variance (σ^2), proportion (P) are termed as **parameters.** Statistical measures like mean ($\overline{X}$), variance (s^2), proportion (p) computed from the sampled observations are known as **statistics.**

Sampling is employed to throw light on the population parameter. A statistic is an estimate based on sample data to draw inference about the population parameter.

9.1.3 Need for Sampling

Suppose that the raw materials department in a company receives items in lots and issues them to the production department as and when required. Before accepting these items, the inspection department inspects or tests them to make sure that they meet the required specifications. Thus

(i)   it could inspect all items in the lot or

(ii)   it could take a sample and inspect the sample for defectives and then estimate the total number of defectives for the population as a whole.

The first approach is called **complete enumeration (census).** It has two major disadvantages namely, the time consumed and the cost involved in it.

The second approach that uses sampling has two major advantages. (i) It is significantly less expensive. (ii) It takes least possible time with best possible results.

There are situations that involve destruction procedure where sampling is the only answer. A well-designed statistical sampling methodology would give accurate results and at the same time will result in cost reduction and least time. Thus sampling is the best available tool to decision makers.

9.1.4 Elements of Sampling Plan

The main steps involved in the planning and execution of sample survey are :

(i) Objectives

The first task is to lay down in concrete terms the basic objectives of the survey. Failure to define the objective(s) will clearly undermine the purpose of carrying out the survey itself. For example, if a nationalised bank wants to study savings bank account holders perception of the service quality rendered over a period of one year, the objective of the sampling is, here,to analyse the perception of the account holders in the bank.

(ii) Population to be covered

Based on the objectives of the survey, the population should be well defined. The characteristics concerning the population under study should also be clearly defined. For example, to analyse the perception of the savings bank account holders about the service rendered by the bank, all the account holders in the bank constitute the population to be investigated.

(iii) Sampling frame

In order to cover the population decided upon, there should be some list, map or other acceptable material (called the **frame**) which serves as a guide to the population to be covered. The list or map must be examined to be sure that it is reasonably free from defects. The sampling frame will help us in the selection of sample. All the account numbers of the savings bank account holders in the bank are the sampling frame in the analysis of perception of the customers regarding the service rendered by the bank.

(iv) Sampling unit

For the purpose of sample selection, the population should be capable of being divided up into sampling units. The division of the population into sampling units should be unambiguous. Every element of the population should belong to just one sampling unit. Each account holder of the savings bank account in the bank, form a unit of the sample as all the savings bank account holders in the bank constitute the population.

(v) Sample selection

The size of the sample and the manner of selecting the sample should be defined based on the objectives of the statistical investigation. The estimation of population parameter along with their margin of uncertainity are some of the important aspects to be followed in sample selection.

(vi) Collection of data

The method of collecting the information has to be decided, keeping in view the costs involved and the accuracy aimed at. Physical observation, interviewing respondents and collecting data through mail are some of the methods that can be followed in collection of data.

(vii) Analysis of data

The collected data should be properly classified and subjected to an appropriate analysis. The conclusions are drawn based on the results of the analysis.

9.1.5 Types of Sampling
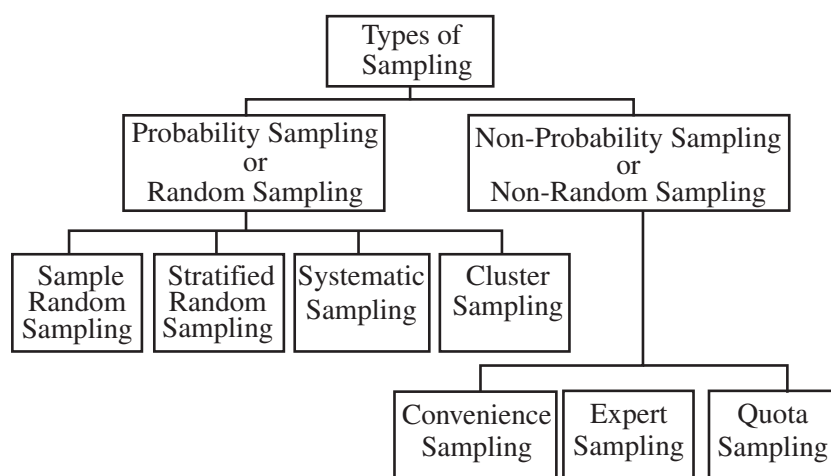
The technique of selecting a sample from a population usually depends on **the nature of the data** and **the type of enquiry.** The procedure of sampling may be broadly classified under the following heads :

(i) Probability sampling or random sampling and

(ii) Non-probability sampling or non-random sampling.

(i) Probability sampling

Probability sampling is a method of sampling that ensures that every unit in the population has a known **non-zero chance** of being included in the sample.

The different methods of random sampling are :

(a) Simple Random Sampling

Simple random sampling is the foundation of probability sampling. It is a special case of probability sampling in which every unit in the population has an **equal chance** of being included in a sample. Simple random sampling also makes the selection of every possible combination of the desired number of units equally likely. Sampling may be done with or without replacement.

It may be noted that when the sampling is with replacement, the units drawn are replaced before the next selection is made. The population size remains constant when the sampling is with replacement.

If one wants to select n units from a population of size N without replacement, then every possible selection of n units must have the same probability. Thus there are ^Nc_n possible ways to pick up n units from the population of size N. Simple random sampling guarantees that a sample of n units, has the same probability $\dfrac{1}{^Nc_n}$ of being selected.

Example

A bank wants to study the Savings Bank account holders perception of the service quality rendered over a period of one year. The bank has to prepare a complete list of savings bank account holders, called as **sampling frame,** say 500. Now the process involves selecting a sample of 50 out of 500 and interviewing them. This could be achieved in many ways. Two common ways are :

(1) **Lottery method :** Select 50 slips from a box containing well shuffled 500 slips of account numbers without replacement. This method can be applied when the population is small enough to handle.

(2) **Random numbers method :** When the population size is very large, the most practical and inexpensive method of selecting a simple random sample is by using the random number tables.

(b) Stratified Random Sampling

Stratified random sampling involves dividing the population into a number of groups called **strata** in such a manner that the units within a stratum are **homogeneous** and the units between the strata are **hetrogeneous.** The next step involves selecting a simple random sample of appropriate size from each stratum. The sample size in each stratum is usually of (a) equal size, (b) proportionate to the number of units in the stratum.

For example, a marketing manager in a consumer product company wants to study the customer's attitude towards a new product in order to improve the sales. Then three typical cities that will influence the sales will be considered as three strata. The customers within a city are similar but between the cities are vastly different. Selection of the customers for the study from each city has to be a random sample to draw meaningful inference on the whole population.

(c) Systematic sampling

Systematic sampling is a convenient way of selecting a sample. It requires less time and cost when compared to simple random sampling.

In this method, the units are selected from the population at a uniform interval. To facilitate this we arrange the items in numerical, alphabetical, geographical or any other order. When a complete list of the population is available, this method is used.

If we want to select a sample of size n from a population of size N under systematic sampling, first select an item j at random such that $1 \leq j \leq k$ where $k = \dfrac{N}{n+1}$ and k is the nearest possible integer. Then $j, j + k, j + 2k, \dots, j + (n-1) k^{th}$ items constitute a systematic random sample.

For example, if we want to select a sample of 9 students out of 105 students numbered as 1, 2, ..., 105 , select a student among 1, 2, ... , 11 at random (say at 3rd position). Here $k = \dfrac{105}{10} = 10.5$ and $\therefore k = 11$. Hence students at the positions 3, 14, 25, 36, 47, 58, 69, 80, 91 form a random sample of size 9.

(d) Cluster sampling

Cluster sampling is used when the population is divided into **groups** or **clusters** such that each cluster is a representative of the population.

If a study has to be done to find out the number of children that each family in Chennai has, then the city can be divided into several clusters and a few clusters can be chosen at random. Every family in the chosen clusters can be a sample unit.

In using cluster sampling the following points should be noted

(a)   For getting precise results clusters should be as small as possible consistent with the

cost and limitations of the survey and

(b)   The number of units in each cluster must be more or less equal.

(ii) Non-Probability Sampling

The fundamental difference between probability sampling and non-probability sampling is that in non-probability sampling procedure, the selection of the sample units does

not ensure a known chance to the units being selected. In other words the units are selected without using the **principle of probability**. Even though the non-probability sampling has advantages such as reduced cost, speed and convenience in implementation, it lacks accuracy in view of the selection bias. Non-probability sampling is suitable for pilot studies and exploratory research

The methods of **non-random sampling** are :

(a) Purposive sampling

In this sampling, the sample is selected with definite purpose in view and the choice of the sampling units depends entirely on the discretion and judgement of the investigator.

For example, if an investigator wants to give the picture that the standard of living has increased in the city of Madurai, he may take the individuals in the sample from the posh localities and ignore the localities where low income group and middle class families live.

(b) Quota sampling

This is a restricted type of purposive sampling. This consists in specifying quotas of the samples to be drawn from different groups and then drawing the required samples from these groups by purposive sampling. Quota sampling is widely used in opinion and market research surveys.

(c) Expert opinion sampling or expert sampling

Expert opinion sampling involves gathering a set of people who have the knowledge and expertise in certain key areas that are crucial to decision making. The advantage of this sampling is that it acts as a support mechanism for some of our decisions in situations where virtually no data are available. The major disadvantage is that even the experts can have prejudices, likes and dislikes that might distort the results.

9.1.6 Sampling and non-sampling errors

The errors involved in the collection of data, processing and analysis of data may be broadly classified as (i) **sampling errors** and (ii) **non-sampling errors**.

(i) Sampling errors

Sampling errors have their origin in sampling and arise due to the fact that only a part of the population has been used to estimate population parameters and draw inference about the population. Increasing in the sample size usually results in decrease in the sampling error.

Sampling errors are primarily due to some of the following reasons :

(a)   Faulty selection of the sample

Some of the bias is introduced by the use of defective sampling technique for the selection of a sample in which the investigator deliberately selects a representative sample to obtain certain results.

(b) Substitution

If difficulty arise in enumerating a particular sampling unit included in the random sample, the investigators usually substitute a convenient member of the population leading to sampling error.

(c) Faulty demarcation of sampling units

Bias due to defective demarcation of sampling units is particularly significant in area surveys such as agricultural experiments. Thus faulty demarcation could cause sampling error.

(ii) Non-sampling errors

The non-sampling errors primarily arise at the stages of observation, classification and analysis of data.

Non-sampling errors can occur at every stage of the planning or execution of census or sample surveys. Some of the more important non-sampling errors arise from the following factors :

(a) Errors due to faulty planning and definitions

Sampling error arises due to improper data specification, error in location of units, measurement of characteristics and lack of trained investigators.

(b) Response errors

These errors occur as a result of the responses furnished by the respondents.

(c) Non-response bias

Non-response biases occur due to incomplete information on all the sampling units.

(d) Errors in coverage

These errors occur in the coverage of sampling units.

(e) Compiling errors

These errors arise due to compilation such as editing and coding of responses.

9.2 ESTIMATION

The technique used for generalising the results of the sample to the population is provided by an important branch of statistics called **statistical inference.** The concept of statistical inference deals with two basic aspects namely (a) **Estimation** and (b) **Testing of hypothesis.**

In statistics, estimation is concerned with making inference about the parameters of the population using information available in the samples. The parameter estimation is very much needed in the decision making process.

The estimation of population parameters such as mean, variance, proportion, etc. from the corresponding sample statistics is an important function of statistical inference.

9.2.1 Estimator

A sample statistic which is used to estimate a population parameter is known as **estimator.**

A good estimator is one which is as close to the true value of population parameter as possible. A good estimator possesses the following properties:

(i) Unbiasedness

As estimate is said to be unbiased if its expected value is equal to its parameter.

The sample mean $\overline{X} = \frac{1}{n} \Sigma x$ is an unbiased estimator of population mean μ. For a sample of size n, drawn from a population of size N, $s^2 = \frac{1}{n-1} \Sigma (x - \overline{x})^2$ is an unbiased estimator of population variance. Hence s^2 is used in **estimation** and in **testing of hypothesis.**

(ii) Consistency

An estimator is said to be consistent if the estimate tends to approach the parameter as the sample size increases.

(iii) Efficiency

If we have two unbiased estimators for the same population parameter, the first estimator is said to be more efficient than the second estimator if the standard error of the first estimator is smaller than that of the second estimator for the same sample size.

(iv) Sufficiency

If an estimator possesses all information regarding the parameter, then the estimator is said to be a sufficient estimator.

9.2.2 Point Estimate and Interval Estimate

It is possible to find two types of estimates for a population parameter. They are **point estimate** and **interval estimate**.

Point Estimate

An estimate of a population parameter given by a single number is called a point estimator of the parameter. Mean ($\overline{x}$) and the sample variance $\left[s^2 = \frac{1}{n-1} \Sigma (x - \overline{x})^2 \right]$ are the examples of point estimates.

A point estimate will rarely coincide with the true population parameter value.

Interval Estimate

An estimate of a population parameter given by two numbers between which the parameter is expected to lie is called an interval estimate of the parameter.

Interval estimate indicates the accuracy of an estimate and is therefore preferable to point estimate. As point estimate provides a single value for the population parameter it may not be suitable in some situation.

For example,

if we say that a distance is measured as 5.28mm, we are giving a point estimate. On the other hand, if we say that the distance is 5.28 ± 0.03 mm i.e. the distance lies between 5.25 and 5.31mm, we are giving an interval estimate.

9.2.3 Confidence Interval for population mean and proportion

The interval within which the unknown value of parameter is expected to lie is called **confidence interval**. The limits so determined are called **confidence limits**.

Confidence intervals indicate the probability that the population parameter lies within a specified range.

Computation of confidence interval

To compute confidence interval we require

(i)   the sample statistic,

(ii)   the standard error (S.E) of sampling distribution of the statistic

(iii)   the degree of accuracy reflected by the Z-value.

If the size of sample is sufficiently large, then the sampling distribution is approximately normal. Therefore, the sample value can be used in estimation of standard error in the place of population value. The Z-distribution is used in case of large samples to estimate the confidence limits.

We give below values of Z corresponding to some confidence levels.

Confidence Levels	99%	98%	96%	95%	80%	50%
Value of Z, Z_c	2.58	2.33	2.05	1.96	1.28	0.674

(i) Confidence interval estimates for means

Let μ and σ be the population mean and standard deviation of the population.

Let $\overline{X}$ and s be the sample mean and standard deviation of the sampling distribution of a statistic.

The confidence limits for μ are given below :

Population size	Sample size	confidence limits for μ
Infinite	n	$\overline{X} \pm (Z_C)\dfrac{s}{\sqrt{n}}$, z_c is the value of Z corresponding to confidence levels.

Population	Sample size	
Finite, N	n	$\overline{X} \pm (Z_C) \dfrac{s}{\sqrt{n}} \sqrt{\dfrac{N-n}{N-1}}$

(ii) Confidence intervals for proportions

If p is the proportion of successes in a sample of size n drawn from a population with P as its proportion of successes, then the confidence intervals for P are given below :

Population	Sample size	Confidence limits for P
Infinite	n	$p \pm (Z_C) \sqrt{\dfrac{pq}{n}}$
Finite, N	n	$p \pm (Z_C) \sqrt{\dfrac{pq}{n}} \sqrt{\dfrac{N-n}{N-1}}$

Example 1

Sensing the downward trend in demand for a leather product, the financial manager was considering shifting his company's resources to a new product area. He selected sample of 10 firms in the leather industry and discovered their earnings (in %) on investment. Find point estimate of the mean and variance of the population from the data given below.

21.0   25.0   20.0   16.0   12.0   10.0   17.0   18.0   13.0   11.0

Solution :

X	$\overline{X}$	$X - \overline{X}$	$(X - \overline{X})^2$
21.0	16.3	4.7	22.09
25.0	16.3	8.7	75.69
20.0	16.3	3.7	13.69
16.0	16.3	-0.3	0.09
12.0	16.3	-4.3	18.49
10.0	16.3	-6.3	39.69
17.0	16.3	0.7	0.49
18.0	16.3	1.7	2.89
13.0	16.3	-3.3	10.89
11.0	16.3	-5.3	28.09
163.0			**212.10**

$$\text{Sample mean, } \overline{X} = \frac{\Sigma X}{n} = \frac{163}{10} = 16.3$$

$$\text{Sample variance, } s^2 = \frac{1}{n-1}\Sigma(X - \overline{X})^2$$

$$= \frac{212.10}{9} = 23.5 \text{ (the sample size is small)}$$

$$\text{Sample standard deviation} = \sqrt{23.5} = 4.85$$

Thus the point estimate of mean and of variance of the population from which the samples are drawn are 16.3 and 23.5 respectively.

Example 2

A sample of 100 students are drawn from a school. The mean weight and variance of the sample are 67.45 kg and 9 kg. respectively. Find (a) 95% and (b) 99% confidence intervals for estimating the mean weight of the students.

Solution :

$$\text{Sample size, } \quad n = 100$$

$$\text{The sample mean, } \quad \overline{X} = 67.45$$

$$\text{The sample variance} \quad s^2 = 9$$

$$\text{The sample standard deviation } \quad s = 3$$

Let μ be the population mean.

(a) The 95% confidence limits for μ are given by

$$\overline{X} \pm (Z_C)\frac{s}{\sqrt{n}}$$

$$\Rightarrow 67.45 \pm (1.96)\frac{3}{\sqrt{100}} \text{ (Here } Z_C = 1.96 \text{ for 95\% confidence level)}$$

$$\Rightarrow \quad 67.45 \pm 0.588$$

Thus the 95% confidence intervals for estimating μ is given by (66.86, 68.04)

b)   The 99% confidence limits for estimating μ are given by

$$\overline{X} \pm (Z_C)\frac{s}{\sqrt{n}}$$

$$\Rightarrow 67.45 \pm (2.58)\frac{3}{\sqrt{100}} \text{ (Here } Z_C = 2.58 \text{ for 99\% confidence level)}$$

$$\Rightarrow \quad 67.45 \pm 0.774$$

Thus the 99% confidence interval for estimating μ is given by (66.67, 68.22)

Example 3

A random sample of size 50 with mean 67.9 is drawn from a normal population. If it is known that the standard error of the sample mean is $\sqrt{0.7}$, find 95% confidence interval for the population mean.

Solution :

$n = 50$, sample mean $\overline{X} = 67.9$

95% confidence limits for population mean μ are :

$\overline{X} \pm (Z_c)\{S.E(\overline{X})\}$

$\Rightarrow \quad 67.9 \pm (1.96)\sqrt{0.7}$

$\Rightarrow \quad 67.9 \pm 1.64$

Thus the 95% confidence intervals for estimating μ is given by (66.2, 69.54)

Example 4

A random sample of 500 apples was taken from large consignment and 45 of them were found to be bad. Find the limits at which the bad apples lie at 99% confidence level.

Solution :

We shall find confidence limits for the proportion of bad apples.

Sample size $n = 500$

Proportion of bad apples in the sample $= \dfrac{45}{500} = 0.09$

$\qquad p = 0.09$

$\therefore$ Proportion of good apples in the sample $q = 1-p = 0.91$.

The confidence limits for the population proportion P of bad apples are given by

$p \pm (Z_c)\left(\sqrt{\dfrac{pq}{n}}\right)$

$\Rightarrow 0.09 \pm (2.58)\sqrt{\dfrac{(0.09)(0.91)}{500}} \quad \Rightarrow \quad 0.09 \pm 0.033$

The required interval is (0.057, 0.123)

Thus, the bad apples in the consignment lie between 5.7% and 12.3%

Example 5

Out of 1000 TV viewers, 320 watched a particular programme. Find 95% confidence limits for TV viewers who watched this programme.

Solution :

Sample size $n = 1000$

Sample proportion of TV viewers $\quad p = \dfrac{x}{n} = \dfrac{320}{1000}$

$$= 0 \cdot 32$$

$$\therefore \quad q \quad = 1 - p = .68$$

$$\text{S.E } (p) \quad = \sqrt{\dfrac{pq}{n}}$$

$$= 0.0147$$

The 95% confidence limits for population propotion P are given by

$$p \pm (1.96) \text{ S.E } (p) = 0.32 \pm 0.028$$

$$\Rightarrow \quad 0.292 \quad \text{and} \quad 0.348$$

$\therefore$ TV viewers of this programme lie between 29.2% and 34.8%

Example 6

Out of 1500 school students, a sample of 150 selected at random to test the accuracy of solving a problem in business mathematics and of them 10 did a mistake. Find the limits within which the number of students who did the problem wrongly in whole universe of 1500 students at 99% confidence level.

Solution :

Population size, $\quad$ N $\quad = 1500$

Sample size, $\quad\quad n \quad = 150$

Sample proportion, $p \quad = \dfrac{10}{150} = 0.07$

$$q \quad = 1 - p = 0.93$$

Standard error of p, SE (p) $= \sqrt{\dfrac{pq}{n}} = 0.02$

The 99% confidence limits for population proportion P are given by

$$p \pm (Z_c)\sqrt{\frac{pq}{n}}\sqrt{\frac{N-n}{N-1}}$$

$$\Rightarrow \quad 0.07 \pm (2.58)(0.02)\sqrt{\frac{1500-150}{1500-1}}$$

$$\Rightarrow \quad 0.07 \pm 0.048$$

∴ The confidence interval for P is (0.022 , 0.118)

∴ The number of students who did the problem wrongly in the population of 1500 lies between .022 × 1500 = 33 and .118 × 1500 = 177.

EXERCISE 9.1

1) A sample of five measurements of the diameter of a sphere were recorded by a scientist as 6.33, 6.37, 6.36, 6.32 and 6.37 mm. Determine the point estimate of (a) mean, (b) variance.

2) Measurements of the weights of a random sample of 200 ball bearings made by a certain machine during one week showed mean of 0.824 newtons and a standard deviation of 0.042 newtons. Find (a) 95% and (b) 99% confidence limits for the mean weight of all the ball bearings.

3) A random sample of 50 branches of State Bank of India out of 200 branches in a district showed a mean annual profit of Rs.75 lakhs and a standard deviation of 10 lakhs. Find the 95% confidence limits for the estimate of mean profit of 200 branches.

4) A random sample of marks in mathematics secured by 50 students out of 200 students showed a mean of 75 and a standard deviation of 10. Find the 95% confidence limits for the estimate of their mean marks.

5) Out of 10000 customer's ledger accounts, a sample of 200 accounts was taken to test the accuracy of posting and balancing wherein 35 mistakes were found. Find 95% confidence limits within which the number of defective cases can be expected to lie.

6) A sample poll of 100 voters chosen at random from all voters in a given district indicated that 55% of them were in favour of a particular candidate. Find (a) 95% confidence limits, (b) 99% confidence limits for the proportion of all voters in favour of this candidate.

9.3 HYPOTHESIS TESTING

There are many problems in which, besides estimating the value of a parameter of the population, we must decide whether a statement concerning a parameter is true or false; that is, we must test a hypothesis about a parameter.

To illustrate the general concepts involved in this kind of decision problems, suppose that a consumer protection agency wants to test a manufacturer's claim that the average life time of electric bulbs produced by him is 200 hours. So it instructs a member of its staff to take 50 electric bulbs from the godown of the company and test them for their lifetime continuously with the intention of rejecting the claim if the mean life time of the bulbs is below 180 hours (say); otherwise it will accept the claim.

Thus **hypothesis** is an assumption that we make about an unknown population parameter. We can collect sample data from the population, arrive at the sample statistic and then test if the hypothesis about the population parameter is true.

9.3.1 Null Hypothesis and Alternative Hypothesis

In hypothesis testing, the statement of the hypothesis or assumed value of the population parameter is always stated before we begin taking the sample for analysis.

A statistical statement about the population parameter assumed before taking the sample for possible rejection on the basis of outcome of sample data is known as a **null hypothesis.**

The null hypothesis asserts that there is no difference between the sample statistic and population parameter and whatever difference is there is attributable to sampling error.

A hypothesis is said to be **alternative hypothesis** when it is complementary to the null hypothesis.

The null hypothesis and alternative hypothesis are usually denoted by H_0 and H_1 respectively.

For example, if we want to test the null hypothesis that the average height of soldiers is 173 cms, then

$$H_0 : \mu = 173 = \mu_0 \text{ (say)}$$

$$H_1 : \mu \neq 173 \neq \mu_0.$$

9.3.2 Types of Error

For testing the hypothesis, we take a sample from the population, and on the basis of the sample result obtained, we decide whether to accept or reject the hypothesis.

Here, two types of **errors** are possible. A null hypothesis could be rejected when it is true. This is called **Type I error** and the probability of committing type I error is denoted by α.

Alternatively, an error could result by accepting a null hypothesis when it is false. This is known as **Type II error** and the probability of committing type II error is denoted by β.

This is illustrated in the following table :

Actual	Decision based on sampling	Error and its Probability
H_0 is True	Rejecting H_0	Type I error ; $\alpha = P\{H_1 / H_0\}$
H_0 is False	Accepting H_0	Type II error ; $\beta = P\{H_0 / H_1\}$

9.3.3 Critical region and level of significance

A region in the sample space which amounts to rejection of null hypothesis (H_0) is called the **critical region**.

After formulating the null and alternative hypotheses about a population parameter, we take a sample from the population and calculate the value of the relevant statistic, and compare it with the hypothesised population parameter.

After doing this, we have to decide the **criteria** for accepting or rejecting the null hypothesis. These criteria are given as a range of values in the form of an interval, say (a, b), so that if the statistic value falls outside the range, we reject the null hypothesis.

If the statistic value falls within the interval (a, b), then we accept H_0. This criterion has to be decided on the basis **of the level of significance**. A 5% level of significance means that 5% of the statistical values arrived at from the samples will fall outside this range (a, b) and 95% of the values will be within the range (a, b).

Thus the level of significance is the probability of Type I error α. The levels of significance usually employed in testing of hypothesis are 5% and 1%.

A high significance level chosen for testing a hypothesis would imply that higher is the probability of rejecting a null hypothesis if it is true.

9.3.4 Test of significance

The tests of significance are (a) Test of significance for **large samples** and (b) Test of significance for **small samples.**

For larger sample size (>30), all the distributions like Binomial, Poisson etc., are approximated by normal distribution. Thus normal probability curve can be used for testing of hypothesis.

For the test statistic Z (standard normal variate), the critical region at 5% level is given by $|Z| \geq 1.96$ and hence the acceptance region is $|Z| < 1.96$. Where as the critical region for Z at 1% level.

The testing hypothesis involves five steps :

(i)   The formulation of null hypothesis and an alternative hypothesis

(ii)   Set up suitable significance level.

(iii)   Setting up the statistical test criteria.

(iv)   Setting up rejection region for the null hypothesis.

(v)   Conclusion.

Example 7

The mean life time of 50 electric bulbs produced by a manufacturing company is estimated to be 825 hours with a standard deviation of 110 hours. If μ is the mean life time of all the bulbs produced by the company, test the hypothesis that μ = 900 hours at 5% level of significance.

Solution :

Null Hypothesis, $H_0 : \mu = 900$

Alternative Hypothesis, $H_1 : \mu \neq 900$

Test statistic , Z is the standard normal variate.

under H_0, $Z = \dfrac{\overline{X} - \mu}{\dfrac{\sigma}{\sqrt{n}}}$ where $\overline{X}$ is the sample mean

σ = s.d. of the population

$= \dfrac{\overline{X} - \mu}{\dfrac{s}{\sqrt{n}}}$ (For large sample, $\sigma = s$)

$= \dfrac{825 - 900}{\dfrac{110}{\sqrt{50}}} = -4.82$

$\therefore \ |Z| = 4.82$

Significant level, $\alpha = 0.05$ or 5%

Critical region is $|Z| \geq 1.96$

Acceptance region is $|Z| < 1.96$

The calculated Z is much greater than 1.96.

Decision : Since the calculated value of $|Z| = 4.82$ falls in the critical region, the value of Z is significant at 5% level.

$\therefore$ the null hypothesis is rejected.

$\therefore$ we conclude that the mean life time of the population of electric bulbs cannot be taken as 900 hours.

Example 8

A company markets car tyres. Their lives are normally distributed with a mean of 50000 kilometers and standard deviation of 2000 kilometers. A test sample of 64 tyres has a mean life of 51250 kms. Can you conclude that the sample mean differs significantly from the population mean? (Test at 5% level)

Solution :

Sample size, n = 64

Sample mean, $\overline{X}$ = 51250

H_0 : population mean μ = 50000

H_1 : $\mu \neq 50000$

under H_0, the test stastistic $Z = \dfrac{\overline{X} - \mu}{\dfrac{\sigma}{\sqrt{n}}} \sim N(0,1)$

$Z = \dfrac{51250 - 50000}{\dfrac{2000}{\sqrt{64}}} = 5$

Since the calculated Z is much greater than 1.96, it is highly significant.

∴ H_0 : μ = 50000 is rejected.

∴   The sample mean differs significantly from the population mean

Example 9

A sample of 400 students is found to have a mean height of 171.38 cms. Can it reasonably be regarded as a sample from a large population with mean height of 171.17 cms and standard deviation of 3.3 cms. (Test at 5% level)

Solution :

Sample size, n = 400

Sample mean, $\overline{X}$ = 171.38

Population mean, μ = 171.17

Sample standard deviation = s.

Population standard deviation, σ = 3.3

Set H_0 : μ = 171.17

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The test statistic, $Z = \dfrac{\overline{X} - \mu}{\dfrac{\sigma}{\sqrt{n}}} \sim N(0,1)$

$= \dfrac{\overline{X} - \mu}{\dfrac{\sigma}{\sqrt{n}}}$ since the sample is large, s = σ

$= \dfrac{171.38 - 171.17}{\dfrac{3.3}{\sqrt{400}}} = 1.273$

Since | Z | = 1.273 < 1.96, we accept the null hypothesis at 5% level of signifiance.

Thus the sample of 400 has come from the population with mean height of 171.17 cms.

EXERCISE 9.2

1) The mean I.Q of a sample of 1600 children was 99. Is it likely that this was a random sample from a population with mean I.Q 100 and standard deviation 15 ? (Test at 5% level of significance)

2) The income distribution of the population of a village has a mean of Rs.6000 and a variance of Rs.32400. Could a sample of 64 persons with a mean income of Rs.5950 belong to this population?

(Test at both 5% and 1% levels of significance)

3) To test the conjecture of the management that 60 percent employees favour a new bonus scheme, a sample of 150 employees was drawn and their opinion was taken whether they favoured it or not. Only 55 employees out of 150 favoured the new bonus scheme Test the conjecture at 1% level of significance.

EXERCISE 9.3

Choose the correct answer

1) The theory of sampling is based on

(a) sample size (b) sample unit

(c) principle of statistical regularity (d) population size

2) A hypothesis complementary to the null hypothesis is called

(a) primary hypothesis (b) statistical statement

(c) alternative hypothesis (d) confidence hypothesis

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3)   The critical region for Z at 1% level is

(a) $|Z| \leq 1.96$ (b) $|Z| \geq 2.58$ (c) $|Z| < 1.96$ (d) $|Z| > 2.58$

4)   The Z-value that is used to establish a 95% confidence interval for the estimation of a population parameter is

(a) 1.28   (b) 1.65   (c) 1.96   (d) 2.58

5)   Probability of rejecting the null hypothesis when it is true is

(a) Type I error   (b) Type II error   (c) Sampling error   (d) Standard error

6)   Which of the following statements is true?

(a) Point estimate gives a range of values

(b) Sampling is done only to estimate a statistic

(c) Sampling is done to estimate the population parameter

(d) Sampling is not possible for an infinite population

7)   The number of ways in which one can select 2 customers out of 10 customers is

(a) 90   (b) 60   (c) 45   (d) 50

APPLIED STATISTICS   **10**

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10.1 LINEAR PROGRAMMING

Linear programming is the general technique of optimum allocation of limited resources such as labour, material, machine, capital etc., to several competing activities such as products, services, jobs, projects, etc., on the basis of given criterion of optimality. The term limited here is used to describe the availability of scarce resources during planning period. The criterion of optimality generally means either performance, return on investment, utility, time, distance etc., The word **linear** stands for the proportional relationship of two or more variables in a model. **Programming** means 'planning' and refers to the process of determining a particular plan of action from amongst several alternatives. It is an extremely useful technique in the decision making process of the management.

10.1.1 Structure of Linear Programming Problem (LPP)

The LP model includes the following three basic elements.

(i)   Decision variables that we seek to determine.

(ii)   Objective (goal) that we aim to optimize (maximize or minimize)

(iii)   Constraints that we need to satisfy.

10.1.2 Formulation of the Linear Programming Problem

The procedure for mathematical formulation of a linear programming consists of the following major steps.

Step 1 : Study the given situation to find the **key decision** to be made

Step 2 : Identify the **variables** involved and designate them by symbols x_j (j = 1, 2 ...)

Step3 : Express the **feasible alternatives** mathematically in terms of variables, which generally are : $x_j \geq 0$ for all j

Step 4 : Identify the **constraints** in the problem and express them as linear inequalities or equations involving the decision variables.

Step 5 : Identify the **objective function** and express it as a linear function of the decision variables.

10.1.3 Applications of Linear programming

Linear programming is used in many areas. Some of them are

(i)   *Transport :* It is used to prepare the distribution plan between source production and destination.

(ii)   *Assignment :* Allocation of the tasks to the persons available so as to get the maximum efficiency.

(iii)   *Marketing :* To find the shortest route for a salesman who has to visit different locations so as to minimize the total cost.

(iv)   Investment : Allocation of capital to differerent activities so as to maximize the return and minimize the risk.

(v)   *Agriculture :* The allotment of land to different groups so as to maximize the output.

10.1.4 Some useful Definitions

A **feasible solution** is a solution which satisfies all the constraints (including non-negativity) of the problem.

A region which contains all feasible solutions is known as **feasible region.**

A feasible solution which optimizes (maximizes or minimizes) the objective function,is called **optimal solution** to the problem.

Note

Optimal solution need not be unique.

Example 1

A furniture manufacturing company plans to make two products, chairs and tables from its available resources, which consists of 400 board feet of mahogany timber and 450 man-hours of labour. It knows that to make a chair requires 5 board feet and 10 man-hours and yields a profit of Rs.45, while each table uses 20 board feet and 15 man - hours and has a profit of Rs.80. How many chairs and tables should the company make to get the maximum profit under the above resource constraints? Formulate the above as an LPP.

Solution :

Mathematical Formulation :

The data of the problem is summarised below:

Products	Raw material (per unit)	Labour (per unit)	Profit (per unit)
Chair	5	10	Rs. 45
Table	20	15	Rs. 80
Total availability	400	450	

Step 1 : The key decision to be made is to determine the number of units of chairs and tables to be produced by the company.

Step 2 : Let x_1 designate the number of chairs and x_2 designate the number of tables, which the company decides to produce.

Step 3 : Since it is not possible to produce negative quantities, feasible alternatives are set of values of x_1 and x_2, such that $x_1 \geq 0$ and $x_2 \geq 0$

Step 4 : The constraints are the limited availability of raw material and labour. One unit of chair requires 5 board feet of timber and one unit of table requires 20 board feet of timber. Since x_1 and x_2 are the quantities of chairs and tables, the total requirement of raw material will be $5x_1 + 20x_2$, which should not exceed the available raw material of 400 board feet timber. So, the raw material constraint becomes,

$$5x_1 + 20x_2 \leq 400$$

Similarly, the labour constraint becomes,

$$10x_1 + 15x_2 \leq 450$$

Step 5 : The objective is to maximize the total profit that the company gets out of selling their products, namely chairs, tables. This is given by the linear function.

$$z = 45x_1 + 80x_2.$$

The linear programming problem can thus be put in the following mathematical form.

$$\text{maximize } z = 45x_1 + 80x_2$$
$$\text{subject to} \quad 5x_1 + 20x_2 \leq 400$$
$$10x_1 + 15x_2 \leq 450$$
$$x_1 \geq 0, x_2 \geq 0$$

Example 2

A firm manufactures headache pills in two sizes A and B. Size A contains 2 mgs. of aspirin. 5 mgs. of bicarbonate and 1 mg. of codeine. Size B contains 1 mg. of aspirin, 8 mgs. of bicarbonate and 6 mgs. of codeine. It is found by users that it requires atleast 12 mgs. of aspirin, 74 mgs. of bicarbonate and 24 mgs. of codeine for providing immediate relief. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem as a standard LPP.

Solution :

The data can be summarised as follows :

Head ache pills	per pill		
	Aspirin	Bicarbonate	Codeine
Size A	2	5	1
Size B	1	8	6
Minimum requirement	12	74	24

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Decision variables :

x_1 = number of pills in size A

x_2 = number of pills in size B

Following the steps as given in (10.1.2) the linear programming problem can be put in the following mathematical format :

Maximize $\quad z = x_1 + x_2$

subject to $\quad 2x_1 + x_2 \geq 12$

$$5x_1 + 8x_2 \geq 74$$

$$x_1 + 6x_2 \geq 24$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

10.1.5 Graphical method

Linear programming problem involving two decision variables can be solved by graphical method. The major steps involved in this method are as follows.

Step 1 : State the problem mathematically.

Step 2 : Plot a graph representing all the constraints of the problem and identify the feasible region (solution space). The feasible region is the intersection of all the regions represented by the constraints of the problem and is restricted to the first quadrant only.

Step 3 : Compute the co-ordinates of all the corner points of the feasible region.

Step 4 : Find out the value of the objective function at each corner point determined in step3.

Step 5 : Select the corner point that optimizes (maximizes or minimizes) the value of the objective function. It gives the optimum feasible solution.

Example 3

A company manufactures two products P_1 and P_2. The company has two types of machines A and B for processing the above products. Product P_1 takes 2 hours on machine A and 4 hours on machine B, whereas product P_2 takes 5 hours on machine A and 2 hours on machine B. The profit realized on sale of one unit of product P_1 is Rs.3 and that of product P_2 is Rs. 4. If machine A and B can operate 24 and16 hours per day respectively, determine the weekly output for each product in order to maximize the profit, through graphical method.

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Solution :

The data of the problem is summarized below.

Product	Hours on		Profit
	Machine A	Machine B	(per unit)
P_1	2	4	3
P_2	5	2	4
Max. hours / week	120	80	

Let x_1 be the number of units of P_1 and x_2 be the number of units of P_2 produced. Then the mathematical formulation of the problem is

$$\text{Maximize } z = 3x_1 + 4x_2$$

$$\text{subject to} \quad 2x_1 + 5x_2 \leq 120$$

$$4x_1 + 2x_2 \leq 80$$

$$x_1, x_2 \geq 0$$

Solution by graphical method

Consider the equation $2x_1 + 5x_2 = 120$, and $4x_1 + 2x_2 = 80$. Clearly $(0, 24)$ and $(60, 0)$ are two points on the line $2x_1 + 5x_2 = 120$. By joining these two points we get the straight line $2x_1 + 5x_2 = 120$. Similarly, by joining the points $(20, 0)$ and $(0, 40)$ we get the sraight line $4x_1 + 2x_2 = 80$. (Fig. 10.1)

Now all the constraints have been represented graphically.

(Fig. 10.1)

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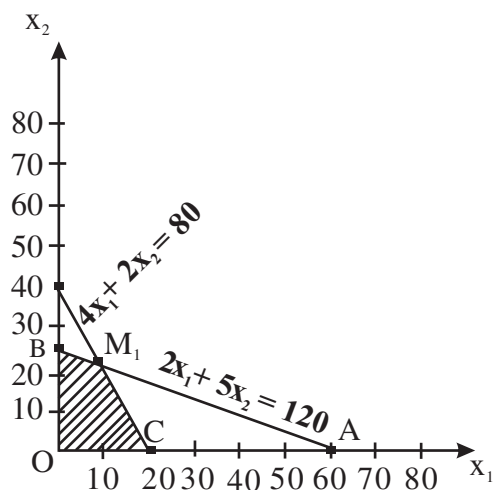
The area bounded by all the constraints called feasible region or solution space is as shown in the Fig. 10.1, by the shaded area OCM_1B

The optimum value of objective function occurs at one of the extreme (corner) points of the feasible region. The coordinates of the extreme points are

$0 = (0, 0)$, $C = (20, 0)$, $M_1 = (10, 20)$, $B = (0, 24)$

We now compute the z-values corresponding to extreme points.

Extreme point	coordinates (x_1, x_2)	$z = 3x_1 + 4x_2$
O	(0, 0)	0
C	(20, 0)	60
M_1	(10, 20)	110
B	(0, 24)	96

The optimum solution is that extreme point for which the objective function has the largest (maximum) value. Thus the optimum solution occurs at the point M_1 i.e. $x_1 = 10$ and $x_2 = 20$.

Hence to maximize profit of Rs.110, the company should produce 10 units of P_1 and 20 units of P_2 per week.

Note

In case of maximization problem, the corner point at which the objective function has a maximum value represent the optimal solution. In case of minimization problem, the corner point at which the objective function has a minimum value represents the optimal solution.

Example 4

Solve graphically :

Minimize $z = 20x_1 + 40x_2$

Subject to $36x_1 + 6x_2 \geq 108$

$3x_1 + 12x_2 \geq 36$

$20x_1 + 10x_2 \geq 100$

$x_1, x_2 \geq 0$

Solution :

A(0, 18) and B(3, 0) ; C(0, 3) and D(12, 0) ; E(0, 10) and F(5, 0) are the points on the lines $36x_1 + 6x_2 = 108$, $3x_1 + 12x_2 = 36$ and $20x_1 + 10x_2 = 100$ respectively. Draw the above lines as Fig. 10.2.
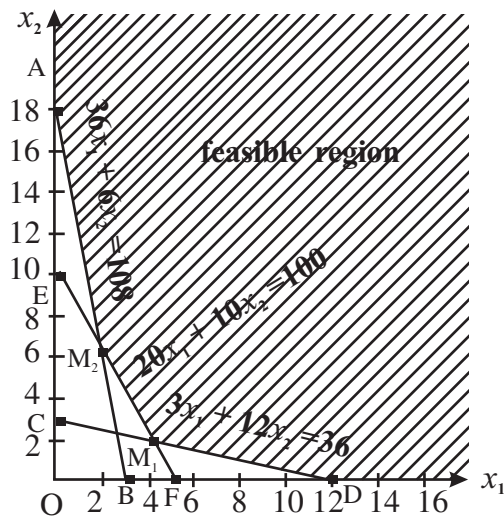
(Fig. 10.2)

Now all the constraints of the given problem have been graphed. The area beyond three lines represents the **feasible region** or **solution space,** as shown in the above figure. Any point from this region would satisfy the constraints.

The coordinates of the extreme points of the feasible region are:

$A = (0, 18)$, $M_2 = (2, 6)$, $M_1 = (4, 2)$, $D = (12, 0)$

Now we compute the z-values corresponding to extreme points.

Extreme point	coordinates	$z = 20x_1 + 40x_2$
	(x_1, x_2)	
A	(0, 18)	720
M_1	(4, 2)	160
M_2	(2, 6)	280
D	(12, 0)	240

The optimum solution is that extreme point for which the objective function has minimum value. Thus optimum solution occurs at the point M_1 i.e. $x_1 = 4$ and $x_2 = 2$ with the objective function value of $z = 160$ ∴ Minimum $z = 160$ at $x_1 = 4$, $x_2 = 2$

Example 5

Maximize $z = x_1 + x_2$ subject to $x_1 + x_2 \leq 1$; $4x_1 + 3x_2 \geq 12$; $x_1, x_2 \geq 0$

Solution :
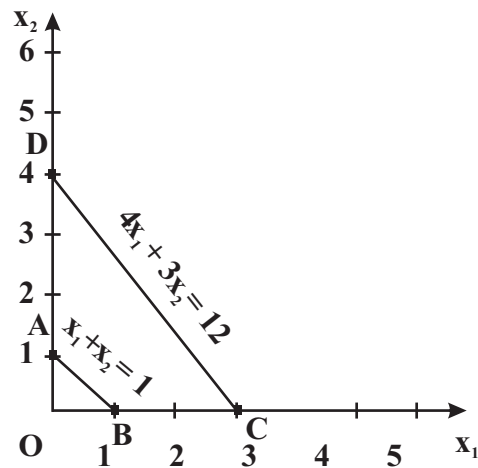
(Fig. 10.3)

From the graph, we see that the given problem has no solution as the feasible region does not exist.

EXERCISE 10.1

1) A company produces two types of products say type A and B. Profits on the two types of product are Rs.30/- and Rs.40/- per kg. respectively. The data on resources required and availability of resources are given below.

	Requirements		Capacity available per month
	Product A	Product B	
Raw materials (kgs)	60	120	12000
Machining hours / piece	8	5	600
Assembling (man hours)	3	4	500

Formulate this problem as a linear programming problem to maximize the profit.

2) A firm manufactures two products A & B on which the profits earned per unit are Rs.3 and Rs.4 respectively. Each product is processed on two machines M_1 and M_2. Product A requires one minute of processing time on M_1 and two minutes on M_2, while B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 7 hrs 30 minutes while M_2 is available for 10 hrs during any working day. Formulate this problem as a linear programming problem to maximize the profit.

3) Solve the following, using graphical method

Maximize $z = 45x_1 + 80x_2$

subject to the constraints

$5x_1 + 20x_2 \leq 400$

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$$10x_1 + 15x_2 \le 450$$

$$x_1, x_2 \ge 0$$

4)   Solve the following, using graphical method

Maximize $z = 3x_1 + 4x_2$

subject to the constraints

$$2x_1 + x_2 \le 40$$

$$2x_1 + 5x_2 \le 180$$

$$x_1, x_2 \ge 0$$

5)   Solve the following, using graphical method

Minimize $z = 3x_1 + 2x_2$

subject to the constraints

$$5x_1 + x_2 \ge 10$$

$$2x_1 + 2x_2 \ge 12$$

$$x_1 + 4x_2 \ge 12$$

$$x_1, x_2 \ge 0$$

10.2 CORRELATION AND REGRESSION

10.2.1 Meaning of Correlation

The term correlation refers to the degree of relationship between two or more variables. If a change in one variable effects a change in the other variable, the variables are said to be correlated. There are basically three types of correlation, namely positive correlation, negative correlation and zero correlatiion.

Positive correlation

If the values of two variables deviate (change) in the same direction i.e. if the increase (or decrease) in one variable results in a corresponding increase (or decrease) in the other, the correlation between them is said to be positive.

Example

(i)   the heights and weights of individuals

(ii)   the income and expenditure

(iii)   experience and salary.

Negative Correlation

If the values of the two variables constantly deviate (change) in the opposite directions i.e. if the increase (or decrease) in one results in corresponding decrease (or increase) in the other, the correlation between them is said to be negative.

Example

(i)   price and demand ,

(ii)   repayment period and EMI

10.2.2 Scatter Diagram

Let $(x_1, y_1), (x_2, y_2) \dots (x_n, y_n)$ be the n pairs of observation of the variables x and y. If we plot the values of x along x-axis and the corresponding values of y along y-axis, the diagram so obtained is called a **scatter diagram.** It gives us an idea of relationship between x and y. The types of scatter diagram under simple linear correlation are given below.

Positive Correlation
(Fig. 10.4)

Negative Correlation
(Fig. 10.5)

No Correlation
(Fig. 10.6)
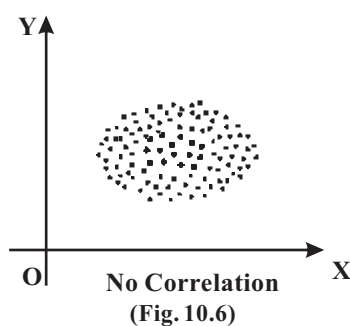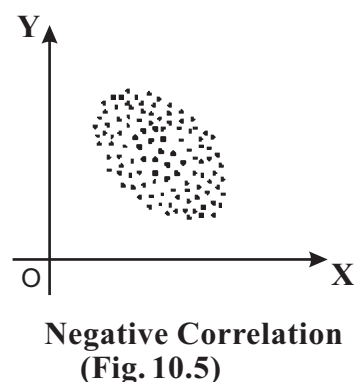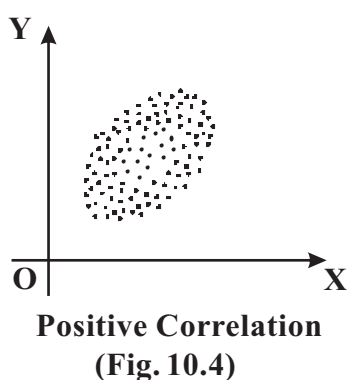
(i)   If the plotted points show an upward trend, the correlation will be positive (Fig. 10.4).

(ii)   If the plotted points show a downward trend, the correlation will be negative (Fig. 10.5).

(iii)   If the plotted point show no trend the variables are said to be uncorrelated (Fig. 10.6).

10.2.3 Co-efficient of Correlation

Karl pearson (1867-1936) a British Biometrician, developed the coefficient of correlation to express the degree of linear relationship between two variables. Correlation co-efficient between two random variables X and Y denoted by r(X, Y), is given by

$$r(X, Y) = \frac{Cov\ (X,\ Y)}{SD\ (X)\ SD\ (Y)}$$

where

$$Cov\ (X, Y) = \frac{1}{n}\sum_i (X_i - \overline{X})\ (Y_i - \overline{Y})\ \text{(covariance between X and Y)}$$

$$SD\ (X) = \sigma_x = \sqrt{\frac{1}{n}\sum_i (X_i - \overline{X})^2}\ \text{(standard deviation of X)}$$

$$SD\ (Y) = \sigma_y = \sqrt{\frac{1}{n}\sum_i (Y_i - \overline{Y})^2}\ \text{(standard deviation of Y)}$$

Hence the formula to compute Karl Pearson correlation co-efficient is

$$r(X, Y) = \frac{\dfrac{1}{n}\sum_i (X_i - \overline{X})\ (Y_i - \overline{Y})}{\sqrt{\dfrac{1}{n}\sum_i (X_i - \overline{X})^2}\ \sqrt{\dfrac{1}{n}\sum_i (Y_i - \overline{Y})^2}}$$

$$= \frac{\sum_i (X_i - \overline{X})\ (Y_i - \overline{Y})}{\sqrt{\sum_i (X_i - \overline{X})^2}\ \sqrt{\sum_i (Y_i - \overline{Y})^2}} = \frac{\sum xy}{\sqrt{\sum x^2}\ \sqrt{\sum y^2}}$$

Note

The following formula may also be used to compute correlation co-efficient between the two variables X and Y.

(i) $r(X, Y) = \dfrac{N\sum XY - \sum X \sum Y}{\sqrt{N\sum X^2 - (\sum X)^2}\ \sqrt{N\sum Y^2 - (\sum Y)^2}}$

(i) $r(X, Y) = \dfrac{N\sum dxdy - \sum dx \sum dy}{\sqrt{N\sum dx^2 - (\sum dx)^2}\ \sqrt{N\sum dy^2 - (\sum dy)^2}}$

where $dx = x - A$; $dy = y - B$ are the deviations from arbitrary values A and B.

10.2.4 Limits for Correlation co-efficient

Correlation co-efficient lies between -1 and +1.

i.e. $-1 \le r(x, y) \le 1$.

(i)   If $r(X, Y) = +1$ the variables X and Y are said to be perfectly possitively correlated.

(ii)   If $r(X, Y) = -1$ the variables X and Y are said to be perfectly negatively correlated.

(iii)   If $r(X, Y) = 0$ the variables X and Y are said to be uncorrelated.

Example 6

Calculate the correlation co-efficient for the following heights (in inches) of fathers(X) and their sons(Y).

X :	65	66	67	67	68	69	70	72
Y :	67	68	65	68	72	72	69	71

Solution :

$$\overline{X} = \frac{\Sigma X}{n} = \frac{544}{8} = 68$$

$$\overline{Y} = \frac{\Sigma Y}{n} = \frac{552}{8} = 69$$

X	Y	$x = X - \overline{X}$	$y = Y - \overline{Y}$	x^2	y^2	xy
65	67	-3	-2	9	4	6
66	68	-2	-1	4	1	2
67	65	-1	-4	1	16	4
67	68	-1	-1	1	1	1
68	72	0	3	0	9	0
69	72	1	3	1	9	3
70	69	2	0	4	0	0
72	71	4	2	16	4	8
544	**552**	**0**	**0**	**36**	**44**	**24**

Karl Pearson Correlation Co-efficient,

$$r(x, y) = \frac{\Sigma xy}{\sqrt{\Sigma x^2} \ \sqrt{\Sigma y^2}} = \frac{24}{\sqrt{36} \ \sqrt{44}} = .603$$

Since $r(x, y) = .603$, the variables X and Y are positively correlated. i.e. heights of fathers and their respective sons are said to be positively correlated.

Example 7

Calculate the correlation co-efficient from the data below:

X :	1	2	3	4	5	6	7	8	9
Y :	9	8	10	12	11	13	14	16	15

Solution :

X	Y	X²	Y²	XY
1	9	1	81	9
2	8	4	64	16
3	10	9	100	30
4	12	16	144	48
5	11	25	121	55
6	13	36	169	78
7	14	49	196	98
8	16	64	256	128
9	15	81	225	135
45	**108**	**285**	**1356**	**597**

$$r(X, Y) = \frac{N\sum XY - \sum X \sum Y}{\sqrt{N\sum X^2 - (\sum X)^2}\ \sqrt{N\sum Y^2 - (\sum Y)^2}}$$

$$= \frac{9(597) - (45)\,(108)}{\sqrt{9(285) - (45)^2}\ \sqrt{9(1356) - (108)^2}} = 0 \cdot 95$$

∴ X and Y are highly positively correlated.

Example 8

Calculate the correlation co-efficient for the ages of husbands (X) and their wives (Y)

X :	23	27	28	29	30	31	33	35	36	39
Y :	18	22	23	24	25	26	28	29	30	32

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Solution :

Let A = 30 and B = 26 then $dx = X - A$ $dy = Y - B$

X	Y	d_x	d_y	d_x^2	d_y^2	$d_x d_y$
23	18	-7	-8	49	64	56
27	22	-3	-4	9	16	12
28	23	-2	-3	4	9	6
29	24	-1	-2	1	4	2
30	25	0	-1	0	1	0
31	26	1	0	1	0	0
33	28	3	2	9	4	6
35	29	5	3	25	9	15
36	30	6	4	36	16	24
39	32	9	6	81	36	54
		11	**− 3**	**215**	**159**	**175**

$$r(x, y) = \frac{N \sum dxdy - \sum dx \sum dy}{\sqrt{N \sum d_x^2 - (\sum dx)^2} \sqrt{N \sum d_y^2 - (\sum dy)^2}}$$

$$= \frac{10(175) - (11)(-3)}{\sqrt{10(215) - (11)^2} \sqrt{10(159) - (-3)^2}}$$

$$= \frac{1783}{1790.8} = 0.99$$

∴ X and Y are highly positively correlated. i.e. the ages of husbands and their wives have a high degree of correlation.

Example 9

Calculate the correlation co-efficient from the following data

N = 25,   ΣX = 125,   ΣY = 100

ΣX² = 650   ΣY² = 436,   ΣXY = 520

Solution :

We know,

$$r(X, Y) = \frac{N \sum XY - \sum X \sum Y}{\sqrt{N \sum X^2 - (\sum X)^2} \; \sqrt{N \sum Y^2 - (\sum Y)^2}}$$

$$= \frac{25(520) - (125)\,(100)}{\sqrt{25(650) - (125)^2} \; \sqrt{25(436) - (100)^2}}$$

$$r = -0.667$$

10.2.5 Regression

Sir Francis Galton (1822 - 1911), a British biometrician, defined **regression** in the context of heriditary characteristics. The literal meaning of the word "regression" is **"Stepping back towards the average".**

Regression is a mathematical measure of the average relationship between two or more variables in terms of the original units of the data.

There are two types of variables considered in regression analysis, namely dependent variable and independent variable(s).

10.2.6 Dependent Variable

The variable whose value is to be predicted for a given independent variable(s) is called dependent variable, denoted by Y. For example, if advertising (X) and sales (Y) are correlated, we could estimate the expected sales (Y) for given advertising expenditure (X). So in this case Y is a dependent variable.

10.2.7 Independent Variable

The variable which is used for prediction is called an independent variable. For example, it is possible to estimate the required amount of expenditure (X) for attaining a given amount of sales (Y), when X and Y are correlated. So in this case Y is independent variable. There can be more than one independent variable in regression.

The line of regression is the line which gives the best estimate to the value of one variable for any specific value of the other variable.

Thus the line of Regression is the **line of best fit** and is obtained by the **principle of least squares.** (Refer pages 219 & 220 of Chapter 7). The equation corresponds to the line of regression is also referred to as regression equation.

10.2.8 Two Regression Lines

For the pair of values of (X, Y), where X is an independent variable and Y is the dependent variable the line of regression of Y on X is given by

$$Y - \overline{Y} = b_{yx} (X - \overline{X})$$

where b_{yx} is the regression co-efficient of Y on X and given by $b_{yx} = r \cdot \dfrac{\sigma_y}{\sigma_x}$, where r is the correlation co-efficient between X and Y and σ_x and σ_y are the standard derivations of X and Y respectively.

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$$\therefore\ b_{yx} = \frac{\Sigma xy}{\Sigma x^2}\ \text{where}\ x = X - \overline{X}\ \text{and}\ y = Y - \overline{Y}$$

Similarly when Y is treated as an independent variable and X as dependent variable, the line of regression of X on Y is given by

$$(X - \overline{X}) = b_{xy}\,(Y - \overline{Y})$$

where $b_{xy} = r\dfrac{\sigma_x}{\sigma_y} = \dfrac{\Sigma xy}{\Sigma y^2}$ here $x = X - \overline{X};\ y = Y - \overline{Y}$

Note

The two regression equations are not reversible or interchangeable because of the simple reason that the basis and assumption for deriving these equations are quite different.

Example 10

Calculate the regression equation of X on Y from the following data.

X :	10	12	13	12	16	15
Y :	40	38	43	45	37	43

Solution :

X	Y	$x = X - \overline{X}$	$y = Y - \overline{Y}$	x^2	y^2	xy
10	40	-3	-1	9	1	3
12	38	-1	-3	1	9	3
13	43	0	2	0	4	0
12	45	-1	4	1	16	-4
16	37	3	-4	9	16	-12
15	43	2	2	4	4	4
78	246	0	0	24	50	6

$$\overline{X} = \frac{\Sigma X}{n} = \frac{78}{6} = 13 \qquad \overline{Y} = \frac{\Sigma Y}{n} = \frac{246}{6} = 41$$

$$b_{xy} = \frac{\Sigma xy}{\Sigma y^2} = \frac{-6}{50} = -0.12$$

Regression equation of X on Y is $(X - \overline{X}) = b_{xy}\,(Y - \overline{Y})$

$X - 13 = -0.12\,(Y - 41) \Rightarrow X = 17.92 - 0.12\,Y$

Example 11

Marks obtained by 10 students in Economics and Statistics are given below.

Marks in Economics : 25 28 35 32 31 36 29 38 34 32
Marks in Statistics : 43 46 49 41 36 32 31 30 33 39

Find (i) the regression equation of Y on X

(ii) estimate the marks in statistics when the marks in Economics is 30.

Solution :

Let the marks in Economics be denoted by X and statistics by Y.

X	Y	$x = X - \overline{X}$	$y = Y - \overline{Y}$	x^2	y^2	xy
25	43	-7	5	49	25	-35
28	46	-4	8	16	64	32
35	49	3	11	9	121	33
32	41	0	3	0	9	0
31	36	-1	-2	1	4	2
36	32	4	-6	16	36	-24
29	31	-3	-7	9	49	21
38	30	6	-8	36	64	-48
34	33	2	-5	4	25	-10
32	39	0	1	0	1	0
320	**380**	**0**	**0**	**140**	**398**	**-93**

$$\overline{X} = \frac{\Sigma X}{n} = \frac{320}{10} = 32 \qquad \overline{Y} = \frac{\Sigma Y}{n} = \frac{380}{10} = 38$$

$$b_{yx} = \frac{\Sigma xy}{\Sigma x^2} = \frac{-93}{140} = -0.664$$

(i)   Regression equation of Y on X is

$$Y - \overline{Y} = b_{yx} (X - \overline{X})$$

$$Y - 38 = -0.664 (X - 32)$$

$\Rightarrow$ $Y = 59.25 - 0.664X$

(ii)   To estimate the marks in statistics (Y) for a given marks in the Economics (X), put X = 30, in the above equation we get,

Y = 59.25 – 0.664 (30)

= 59.25 – 19.92 = 39.33 or 39

Example 12

Obtain the two regression equations for the following data.

X :	4	5	6	8	11
Y :	12	10	8	7	5

Solution :

The above values are small in magnitude. So the following formula may be used to compute the regression co-efficient.

$$b_{xy} = \frac{N \Sigma XY - \Sigma X \Sigma Y}{N \Sigma Y^2 - (\Sigma Y)^2}$$

$$b_{yx} = \frac{N \Sigma XY - \Sigma X \Sigma Y}{N \Sigma X^2 - (\Sigma X)^2}$$

X	Y	X²	Y²	XY
4	12	16	144	48
5	10	25	100	50
6	8	36	64	48
8	7	64	49	56
11	5	121	25	55
34	42	262	382	257

$$\overline{X} = \frac{\Sigma X}{n} = \frac{34}{5} = 6.8 \qquad \overline{Y} = \frac{\Sigma Y}{n} = \frac{42}{5} = 8.4$$

$$b_{xy} = \frac{5(257) - (34)(42)}{5(382) - (42)^2} = -0.98$$

$$b_{yx} = \frac{5(257) - (34)(42)}{5(262) - (34)^2} = -0.93$$

Regression equation of X on Y is

$$(X - \overline{X}) = b_{xy} (Y - \overline{Y})$$

$\Rightarrow$ X – 6.8 = – 0.98 (Y – 8.4)

       X = 15.03 – 0.98Y

Regression equation of Y on X is

$$Y - \overline{Y} = b_{yx} (X - \overline{X})$$

Y – 8.4 = – 0.93 (X – 6.8)

$\Rightarrow$ Y = 14.72 – 0.93X

EXERCISE 10.2

1) Calculate the correlation co-efficient from the following data.

X : 12   9   8   10   11   13   7

Y : 14   8   6   9   11   12   3

2) Find the co-efficient of correlation for the data given below.

X : 10   12   18   24   23   27

Y : 13   18   12   25   30   10

3) From the data given below, find the correlation co-efficient.

X : 46   54   56   56   58   60   62

Y : 36   40   44   54   42   58   54

4) For the data on price (in rupees) and demand (in tonnes) for a commodity, calculate the co-efficient of correlation.

Price (X) :   22   24   26   28   30   32   34   36   38   40

Demand (Y) : 60   58   58   50   48   48   48   42   36   32

5) From the following data, compute the correlation co-efficient.

$N = 11, \Sigma X = 117, \Sigma Y = 260, \Sigma X^2 = 1313$

$\Sigma Y^2 = 6580, \quad \Sigma XY = 2827$

6) Obtain the two regression lines from the following

X :   6   2   10   4   8

Y :   9   11   5   8   7

7) With the help of the regression equation for the data given below calculate the value of X when Y = 20.

X :   10   12   13   17   18

Y :   5   6   7   9   13

8) Price indices of cotton (X) and wool (Y) are given below for the 12 months of a year. Obtain the equations of lines of regression between the indices.

X : 78   77   85   88   87   82   81   77   76   83   97   93

Y : 84   82   82   85   89   90   88   92   83   89   98   99

9)   Find the two regression equations for the data given below.

X :   40   38   35   42   30

Y :   30   35   40   36   29

10.3 TIME SERIES ANALYSIS

Statistical data which relate to successive intervals or points of time, are referred to as "time series".

The following are few examples of time series.

(i) Quarterly production, Half-yearly production, and yearly production for particular commodity.

(ii) Amount of rainfall over 10 years period.

(iii) Price of a commodity at different points of time.

There is a strong notion that the term " time series" usually refer only to Economical data. But it equally applies to data arising in other natural and social sciences. Here the time sequence plays a vital role and it requires special techniques for its analysis. In analysis of time series, we analyse the past in order to understand the future better.

10.3.1 Uses of analysis of Time Series

(i) It helps to study the past conditions, assess the present achievements and to plan for the future.

(ii) It gives reliable forecasts.

(iii) It provides the facility for comparison.

Thus wherever time related data is given in Economics, Business, Research and Planning, the analysis of time series provides the opportunity to study them in proper perspective.

10.3.2 Components of Time Series

A graphical representation of a Time Series data, generally shows the changes (variations) over time. These changes are known as principal components of Time Series . They are

(i) Secular trend   (ii) Seasonal variation

(iii) Cyclical variation   (iv) Irregular variation.

Secular trend (or Trend)

It means the smooth, regular, long-term movement of a series if observed long enough. It is an upward or downward trend. It may increase or decrease over period of time. For example, time series relating to population, price, production, literacy,etc. may show increasing trend and time series relating to birth rate, death rate, poverty may show decreasing trend.

Seasonal Variation

It is a short-term variation. It means a periodic movement in a time series where the period is not longer than one year. A periodic movement in a time series is one which recurs or repeats at regular intervals of time or periods. Following are the examples of seasonal variation.

(i) passenger traffic during 24 hours of a day

(ii) sales in a departmental stores during the seven days of a week.

The factors which cause this type of variation are due to climatic changes of the different season, customs and habits of the people. For example more amount of ice creams will be sold in summer and more number of umbrellas will be sold during rainy seasons.

Cyclical Variation

It is also a short-term variation. It means the oscillatory movement in a time series, the period of oscillation being more than a year. One complete period is called a cycle. Business cycle is the suitable example for cyclical variation. There are many time series relating to Economics and Business, which have certain wave-like movements called business cycle. The four phases in business cycle, (i) prosperity (ii) recession (iii) depression (iv) recovery, recur one after another regularly.
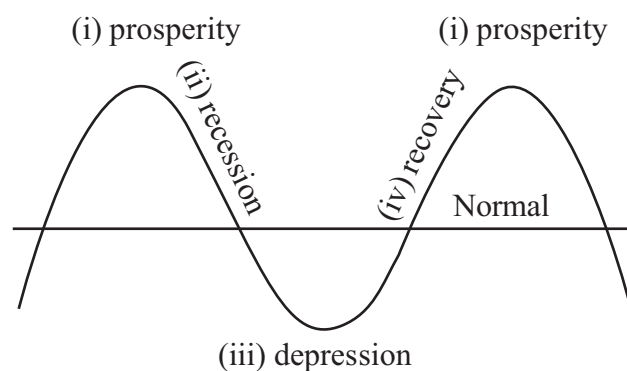
(Fig. 10.7)

Irregular Variation

This type of variation does not follow any regularity. These variations are either totally unaccountable or caused by unforeseen events such as wars, floods, fire, strikes etc. Irregular variation is also called as **Erratic Variation**.

10.3.3 Models

In a given time series, some or all the four components, namely secular trend, seasonal variation, cyclical variation and irregular variation may be present. It is important to separate the different components of times series because either our interest may be on a particular component or we may want to study the series after eliminating the effect of a particular component. Though there exist many models, here we consider only two models.

Multiplicative Model

According to this model, it is assumed that there is a multiplicative relationship among the four components. i.e.,

$$y_t = T_t \times S_t \times C_t \times I_t,$$

Where y_t is the value of the variable at time t, or observed data at time t, T_t is the Secular trend or trend, S_t is the Seasonal variation, C_t is the Cyclical variation and I is the Irregular variation or Erratic variation.

Additive Model

According to this model, it is assumed that y_t be the sum of the four components.

$$y_t = T_t + S_t + C_t + I_t,$$

10.3.4 Measurement of secular trend

The following are the four methods to estimate the secular trend

(i)   Graphic method or free - hand method

(ii)   Method of Semi - Averages

(iii)   Method of Moving Averages

(iv)   Method of least squares.

(i) Graphic Method / Free - hand Method

This is the simplest method of studying the trend procedure. Let us take time on the x - axis, and observed data on the y-axis. Mark a point on a graph sheet, corresponding to each pair of time and observed value. After marking all such possible points, draw a straight line which will best fit to the data according to personal judgement.

It is to be noted that the line should be so drawn that it passes between the plotted points in such a manner that the fluctuations in one direction are approximately equal to those in other directions.

When a trend line is fitted by the free hand method an attention should be paid to conform the following conditions.

(i) The number of points above the line is equal to the number of points below the line, as far as possible.

(ii) The sum of the vertical deviations from the trend of the annual observation above the trend should equal the sum of the vertical deviations from the trend of the observations below the trend.

(iii) The sum of the squares of the vertical deviations of the observations from the trend should be as small as possible.

Example 13

Fit a trend line to the following data by the free hand mehtod.

year	1978	1979	1980	1981	1982	1983	1984	1985	1986
production of steel	20	22	24	21	23	25	23	26	25

Solution :
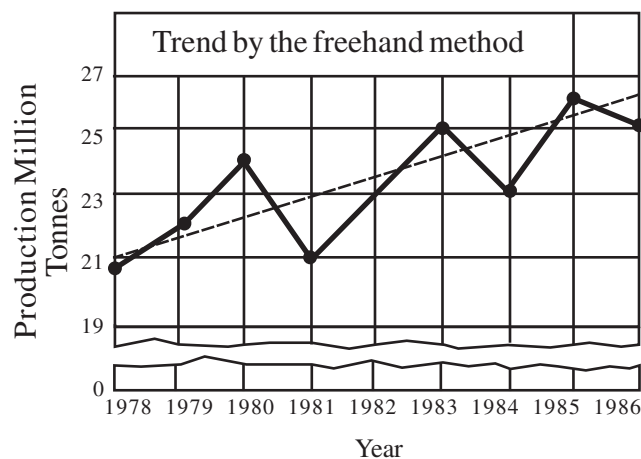
Year
(Fig. 10.8)

Note

(i) The trend line drawn by the free hand method can be extended to predict future values. However, since the free hand curve fitting is too subjective, this method should not be generally used for predictions

(ii) In the above diagram **false base line** (zig-zag) has been used. Generally we use false base line following objectives.

(a) Variations in the data are clearly shown

(b) A large part of the graph is not wasted or space is saved.

(c) The graph provides a better visual communications.

(ii) Method of Semi Averages

This method involves very simple calculations and it is easy to adopt. When this method is used the given data is divided into two equal parts. For example, if we are given data from 1980 to 1999, i.e., over a period of 20 years, the two equal points will be first 10 years, i.e. from 1980 to 1989, and from 1990 to 1999. In case of odd number of years like 7, 11, 13 etc., two equal parts can be made by omitting the middle year. For example, if the data are given for 7 years from 1980 to 1986, the two equal parts would be from 1980 to 1982 and from 1984 to 1986. The middle year 1983 will be omitted.

After dividing the data in two parts, find the arithmetic mean of each part. Thus we get semi averages from which we calculate the annual increase or decrease in the trend.

Example 14

Find trend values to the following data by the method of semi-averages.

Year	1980	1981	1982	1983	1984	1985	1986
Sales	102	105	114	110	108	116	112

Solution :

No. of years = 7 (odd no.) By omitting the middle year (1983) we have

Year	Sales	Semi total	Semi-average
1980	102		
1981	105	321	107
1982	114		
1983	110		
1984	108		
1985	116	336	112
1986	112		

Difference between middle periods = 1985 - 1981 = 4

Difference between semi averages = 112 - 107 = 5

Annual increase in trend = $\dfrac{5}{4}$ = 1.25

Year	1980	1981	1982	1983	1984	1985	1986
Trend	105.75	107	108.25	109.50	110.75	112	113.25

Example 15

The sales in tonnes of a commodity varied from 1994 to 2001 as given below:

Year	1994	1995	1996	1997	1998	1999	2000	2001
Sales	270	240	230	230	220	200	210	200

Find the trend values by the method of semi-average. Estimate the sales in 2005.

Solution :

Year	Sales	Semi total	Semi-average
1994	270		
1995	240		
1996	230	→ 970 →	242.5
1997	230		
1998	220		
1999	200		
2000	210	→ 830 →	207.5
2001	200		

Difference between middle periods $= 1999.5 - 1995.5 = 4$

Decrease in semi averages $= 242.5 - 207.5 = 35$

Annual decrease in trend $= \dfrac{35}{4} = 8.75$

Half yearly decrease in trend $= 4.375$

Year	1994	1995	1996	1997	1998	1999	2000	2001
Sales trend	255.625	246.875	238.125	229.375	220.625	211.875	203.125	194.375

Trend value for the year 2005 $= 194.375 - (8.75 \times 4) = 159.375$

(iii) Method of Moving Averages

This method is simple and flexible algebraic method of measuring trend. The method of Moving Average is a simple device for eliminating fluctuations and obtaining trend values with a fair degree of accuracy. The technique of Moving Average is based on the arithmetic mean but with a distinction. In arithmetic mean we sum all the items and divide the sum by number of items, whereas in Moving Average method there are various averages in one series depending upon the number of years taken in a Moving Average. While applying this method, it is necessary to define a period for Moving Average such as 3 yearly moving average (odd number of years), 4-yearly moving average (even number of years) etc.

Moving Average - Odd number of years (say 3 years)

To find out the trend values by the method of 3-yearly moving averages, the following steps are taken into consideration.

1)   Add up the values of the first three years and place the yearly sum against the median year i.e. the 2nd year.

2)   Leave the first year item and add up the values of the next three years. i.e. from the 2nd year to 4 th year and place the sum (known as moving total) against the 3rd year.

3)   Leave the first two items and add the values of the next three years. i.e. from 3rd year to the 5th year and place the sum (moving total) against the 4th year.

4)   This process must be continued till the value of the last item is taken for calculating the moving average.

5)   Each 3-yearly moving total must be divided by 3 to get the moving averages. This is our required trend values.

Note

The above 5 steps can be applied to get 5-years, 7-years, 9- years etc., Moving Averages.

Moving Average - Even number of years (say 4 years)

1)   Add up the values of the first four years and place the sum against the middle of 2nd and 3rd years.

2)   Leave the first year value and add from the 2nd year onwards to the 5th year and write the sum (moving total) against the middle of the 3rd and the 4th items.

3)   Leave the first two year values and add the values of the next four years i.e. from the 3rd year to the 6th year. Place the sum (moving total) against the middle of the 4th and the 5th items.

4)   This process must be continued till the value of the last item is taken into account.

5)   Add the first two 4-years moving total and write the sum against 3rd year.

6)   Leave the first 4-year moving total and add the next two 4- year moving total. This sum must be placed against 4th year.

7)   This process must be continued till all the four-yearly moving totals are summed up and centred.

8)   Divide the 4-years moving total centred by 8 and write the quotient in a new column. These are our required trend values.

Note

The above steps can be applied to get 6-years, 8-years, 10-years etc., Moving Averages.

Example 16

　　　Calculate the 3-yearly Moving Averages of the production figures (in mat. tonnes) given below

Year	1973	1974	1975	1976	1977	1978	1979	1980	1981
Production	15	21	30	36	42	46	50	56	63

Year	1982	1983	1984	1985	1986	1987
Production	70	74	82	90	95	102

Solution :

　　　Calculation of 3-yearly Moving Averages

Year	Production y	3-yearly Moving total	3-yearly Moving average
1973	15	---	---
1974	21	66	22.00
1975	30	87	29.00
1976	36	108	36.00
1977	42	124	41.33
1978	46	138	46.00
1979	50	152	50.67
1980	56	169	56.33
1981	63	189	63.00
1982	70	207	69.00
1983	74	226	75.33
1984	82	246	82.00
1985	90	267	89.00
1986	95	287	95.67
1987	102	---	---

Example 17

Estimate the trend values using the data given below by taking 4-yearly Moving Average.

Year	1974	1975	1976	1977	1978	1979	1980	1981	1982
Value	12	25	39	54	70	37	105	100	82
Year	1983	1984	1985	1986	1987				
Value	65	49	34	20	7				

Solution :

Year	value	4 year moving total	4 year moving total centered	Two 4-year moving total (Trend values)
1974	12	---	---	---
1975	25		---	---
		→ 130		
1976	39		318	39.75
		→ 188		
1977	54		388	48.50
		→ 200		
1978	70		466	58.25
		→ 266		
1979	37		578	72.25
		→ 312		
1980	105		636	79.50
		→ 324		
1981	100		676	84.50
		→ 352		
1982	82		648	81.00
		→ 296		
1983	65		526	65.75
		→ 230		
1984	49		398	49.75
		→ 168		
1985	34		278	34.75
		→ 110		
1986	20		---	---
1987	7		---	---

Measurement of seasonal variation

Seasonal variation can be measured by the method of simple average.

Method of simple average

This method is the simple method of obtaining a seasonal Index. In this method the following steps are essential to calculate the Index.

(i)   Arrange the data by years, month or quarters as the case may be.

(ii)   Compute the totals of each month or each quarter.

(iii)   Divide each total by the number of years for which the data are given. This gives seasonal averages (monthly or quarterly)

(iv)   Compute average of seasonal averages. This is called grand average.

(v)   Seasonal Index for every season (monthly or quarterly) is calculated as follows

$$\text{Seasonal Index (S. I)} = \frac{\text{Seasonal Average}}{\text{Grand Average}} \times 100$$

Note

(i)   If the data is given monthwise,

$$\text{Seasonal Index} = \frac{\text{Monthly Average}}{\text{Grand Average}} \times 100$$

(ii)   If quarterly data is given,

$$\text{Seasonal Index} = \frac{\text{Quarterly Average}}{\text{Grand Average}} \times 100$$

Example 18

From the data given below calculate Seasonal Indices.

			Year		
Quarter	1984	1985	1986	1987	1988
I	40	42	41	45	44
II	35	37	35	36	38
III	38	39	38	36	38
IV	40	38	40	41	42

Solution :

	Quarters			
Year	I	II	III	IV
1984	40	35	38	40
1985	42	37	39	38
1986	41	35	38	40
1987	45	36	36	41
1988	44	38	38	42
Total	**212**	**181**	**189**	**201**
Average	**42.4**	**36.2**	**37.8**	**40.2**

$$\text{Grand Average} = \frac{42.4 + 36.2 + 37.8 + 40.2}{4} = 39.15$$

$$\text{Seasonal Index (S. I)} = \frac{\text{Quarterly Average}}{\text{Grand Average}} \times 100$$

Hence, S.I for I Quarter $= \dfrac{42.4}{39.15} \times 100 = 108.30$

S.I for II Quarter $= \dfrac{36.2}{39.15} \times 100 = 92.54$

S.I for III Quarter $= \dfrac{37.8}{39.15} \times 100 = 96.55$

S.I for IV Quarter $= \dfrac{40.2}{39.15} \times 100 = 102.68$

Note

Measurement of cyclical variation, and measurement of irregular variation is beyond the scope of this book.

EXERCISE 10.3

1) Draw a trend line by graphic method (free hand)

Year	1995	1996	1997	1998	1999	2000	2001
Production	20	22	25	26	25	27	30

2) Draw a trend line by graphic method

Year	1997	1998	1999	2000	2001
Production	20	24	25	38	60

3) Obtain the trend values by the method of Semi-Avearge.

Year	1987	1988	1989	1990	1991	1992	1993
Production	90	110	130	150	100	150	200

(in tonnes)

4) Obtain the trend values by the method of Semi-Average

Year	1993	1994	1995	1996	1997	1998	1999	2000
Netprofit	38	39	41	43	40	39	35	25

(Rs lakhs)

5) Using three year moving averages determine the trend values for the following data.

Year	1983	1984	1985	1986	1987	1988	1989	1990	1991	1992
Production	21	22	23	25	24	22	25	26	27	26

(in tonnes)

6) Below are given figures of production (in thousand tonnes) of a sugar factory. Obtain the trend values be 3-year moving average.

Year	1980	1981	1982	1983	1984	1985	1986
Production	80	90	92	83	94	99	92

7) Using four yearly moving average calculate the trend values

Year	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990
Production	464	515	518	467	502	540	557	571	586	612

8) Calculate the trend values by four year moving average method.

Year	1978	1979	1980	1981	1982	1983	1984	1985	1986	1987	1988
Production	614	615	652	678	681	655	717	719	708	779	757

9)   Calculate the seasonal indices by the method of simple average for the following data

Year	I quarter	II quarter	III quarter	IV quarter
1985	68	62	61	63
1986	65	58	66	61
1987	68	63	63	67

10)   Calculate the seasonal indices for the following data by the method of simple Average.

	Quarters			
Year	I	II	III	IV
1994	78	66	84	80
1995	76	74	82	78
1996	72	68	80	70
1997	74	70	84	74
1998	76	74	86	82

11)   Calculate the seasonal indices for the following data using average method.

	Quarters			
Year	I	II	III	IV
1982	72	68	80	70
1983	76	70	82	74
1984	74	66	84	80
1985	76	74	84	78
1986	78	74	86	82

10.4 INDEX NUMBERS

"An **Index Number** is a single ratio (usually in percentages) which measures the (combined average) change of several variables between two different times, places and situations" - Alva. M. Tuttle.

Index numbers are the devices for measuring differences in the magnitude of a group of related variables, over two different situations or defined as a measure of the average change in a group of related variables over two different situations. For example, the price of commodities at two different places or two different time periods at the same location. We need Index Numbers to compare the cost of living at different times or in different locations.

10.4.1 Classification of Index Numbers

Index Numbers may be classified in terms of what they measure. They are

(i) Price Index Numbers

(ii) Quantity Index Numbers

(iii) Value Index Numbers

(iv) Special purpose Index Numbers

We shall discuss (i) and (ii).

10.4.2 Uses of Index Numbers

(i)   Index numbers are used to evolve business policies.

(ii)   Index numbers determine the inflation or deflation in economy.

(iii)   Index numbers are used to compare intelligence of students in different locations or for different year.

(iv)   Index numbers serve as economic barometers.

10.4.3 Method of construction of Index Numbers

Index Numbers are broadly divided into two groups

(i) Unweighted Index

(ii) Weighted Index

We confine our attention to weighted index numbers.

10.4.4 Weighted Index Numbers

The method of construction of weighted indices are

(c)   Weighted aggregative method

(d)   Weighted averages of relatives method

Weighted Aggregative Index Numbers

Let p_1 and p_0 be the prices of the current year and the base year respectively. Let q_1 and q_0 be the quantities of the current year and the base year respectively. The formulae for assigning weights to the items are :

(i) Laspeyre's Price Index

$$P_{01}^L = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$ where w = $p_0 q_0$ is the weight assigned to the items and P_{01} is the

price index.

(ii) Paasche's price index

$$P_{01}^P = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$$

Here the weights assigned to the items are the current year quantities i.e. $W = p_0 q_1$

(iii) Fisher's price Index

$$P_{01}^F = \sqrt{P_{01}^L \times P_{01}^P} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100$$

Note

Fisher's price index is the geometric mean of Laspeyre's and Paasche's price index numbers.

Example 19

Compute (i) Laspeyre's (ii) Paasche's and (iii) Fisher's Index Numbers for the 2000 from the following :

Commodity	Price		Quantity	
	1990	2000	1990	2000
A	2	4	8	6
B	5	6	10	5
C	4	5	14	10
D	2	2	19	13

Solution :

Commodity	Price		Quantity					
	Base year	Current year	Base year	Current year				
	p_0	p_1	q_0	q_1	$p_0 q_0$	$p_1 q_0$	$p_0 q_1$	$p_1 q_1$
A	2	4	8	6	16	32	12	24
B	5	6	10	5	50	60	25	30
C	4	5	14	10	56	70	40	50
D	2	2	19	13	38	38	26	26
					160	**200**	**103**	**130**

(i) Laspeyre's Index : $P_{01}^L = \dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$

$$= \frac{200}{160} \times 100 = 125$$

(ii) Paasche's Index : $P_{01}^P = \dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$

$$= \frac{130}{103} \times 100 = 126.21$$

(iii) Fisher's Index : $P_{01}^F = \sqrt{P_{01}^L \times P_{01}^P}$

$$= 125.6$$

Example 20

From the following data, calculate price index number by (a) Laspeyre's method (b) Paasche's method (iii) Fisher's method.

Commodity	Base year		Current year	
	Price	Quantity	Price	Quantity
A	2	40	6	50
B	4	50	8	40
C	6	20	9	30
D	8	10	6	20
E	10	10	5	20

Solution :

Commodity	Base year		Current year					
	Price	Qty	Price	Qty				
	p_0	q_0	p_1	q_1	$p_0 q_0$	$p_1 q_0$	$p_0 q_1$	$p_1 q_1$
A	2	40	6	50	80	240	100	300
B	4	50	8	40	200	400	160	320
C	6	20	9	30	120	180	180	270
D	8	10	6	20	80	60	160	120
E	10	10	5	20	100	50	200	100
					580	930	800	1110

(i) Laspeyre's Price Index : $P_{01}^L = \dfrac{\Sigma\, p_1 q_0}{\Sigma\, p_0 q_0} \times 100$

$$= \frac{930}{580} \times 100 = 160.34$$

(ii) Paasche's Price Index : $P_{01}^P = \dfrac{\Sigma\, p_1 q_1}{\Sigma\, p_0 q_1} \times 100$

$$= \frac{1100}{800} \times 100 = 137.50$$

(iii) Fisher's Index : $P_{01}^F = \sqrt{P_{01}^L \times P_{01}^P} = 148.48$

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10.4.5 Test of adequacy for Index Number

Index Numbers are constructed to study the relative changes in prices, quantities, etc. of one time in comparison with another. Several formulae have been suggested for constructing index numbers and one should select the most appropriate one in a given situation. Following are the tests suggested for choosing an appropriate index.

1) Time reversal test

2) Factor reversal test

Time reversal test

It is a test to determine whether a given method will work both ways in time, forward and backward. When the data for any two years are treated by the same method, but with the bases reversed, the two index numbers secured should be reciprocals of each other so that their product is unity. Symbolically the following relation should be satisfied.

$$\therefore \mathbf{P_{01}} \times \mathbf{P_{10}} = 1$$

(ignoring the factor 100 in each index) where P_{01} is the index for current period 1 on base period 0 and P_{10} is the index for the current period 0 on base period 1.

Factor reversal test

This test holds that the product of a price index and the quantity index should be equal to the corresponding value index. The test is that the change in price multiplied by the change in quantity should be equal to the total change in value.

$$\therefore \mathbf{P_{01}} \times \mathbf{Q_{01}} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} \text{ (ignoring the factor 100 in each index)}$$

P_{01} gives the relative change in price and Q_{01} gives the relative change in quantity. The total value of a given commodity in a given year is the product of the quantity and the price per unit.

$\dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$ is the ratio of the total value in the current period to the total value in the base period and this ratio is called the **true value ratio**.

Fisher's index is known as Ideal Index Number since it is the only index number that satisfies both reversal tests.

Example 21

Calculate Fisher's Ideal Index from the following data and verify that it satisfies both Time Reversal and Factor Reversal test

Commodity	Price		Quantity	
	1985	1986	1985	1986
A	8	20	50	60
B	2	6	15	10
C	1	2	20	25
D	2	5	10	8
E	1	5	40	30

Solution :

Commodity	1985		1986					
	p_0	q_0	p_1	q_1	$p_1 q_0$	$p_0 q_0$	$p_1 q_1$	$p_0 q_1$
A	8	50	20	60	1000	400	1200	480
B	2	15	6	10	90	30	60	20
C	1	20	2	25	40	20	50	25
D	2	10	5	8	50	20	40	16
E	1	40	5	30	200	40	150	30
					1380	510	1500	571

$$\text{Fisher's Ideal Index} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100$$

$$= \sqrt{\frac{1380}{510} \times \frac{1500}{571}} \times 100$$

$$= 2.6661 \times 100 = 266.61$$

Time reversal test

Test is satisfied when $P_{01} \times P_{10} = 1$

$$P_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} = \sqrt{\frac{1380}{510} \times \frac{1500}{571}}$$

$$P_{10} = \sqrt{\frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \times \frac{\Sigma p_0 q_0}{\Sigma p_1 q_1}} = \sqrt{\frac{571}{1500} \times \frac{510}{1380}}$$

$$P_{01} \times P_{10} = \sqrt{\frac{1380}{510} \times \frac{1500}{571} \times \frac{571}{1500} \times \frac{510}{1380}}$$

$$= \sqrt{1} = 1$$

Hence Fisher's Ideal Index satisfies Time reversal test.

Factor reversal test

Test is satisfied when $P_{01} \times Q_{01} = \dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$

$$Q_{01} = \sqrt{\dfrac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \dfrac{\Sigma q_1 p_1}{\Sigma q_0 p_1}} = \sqrt{\dfrac{571}{510} \times \dfrac{1500}{1380}}$$

$$\therefore \ P_{01} \times Q_{01} = \sqrt{\dfrac{1380}{510} \times \dfrac{1500}{571} \times \dfrac{571}{510} \times \dfrac{1500}{1380}}$$

$$= \dfrac{1500}{510} = \dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$$

Hence Fisher's Ideal Index satisfies Factor reversal test.

Example 22

Compute Index Number using Fisher's formula and show that it satisfies time reversal test and factor reversal test.

Commodity	Base year		Current year	
	Price	Quantity	Price	Quantity
A	10	12	12	15
B	7	15	5	20
C	5	24	9	20
D	16	5	14	5

Solution :

Commodity	Base year		Current year					
	p_0	q_0	p_1	q_1	$p_1 q_0$	$p_0 q_0$	$p_1 q_1$	$p_0 q_1$
A	10	12	12	15	144	120	180	150
B	7	15	5	20	75	105	100	140
C	5	24	9	20	216	120	180	100
D	16	5	14	5	70	80	70	80
					505	425	530	470

$$\text{Fisher's Ideal Index} = \sqrt{\dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100$$

$$= \sqrt{\dfrac{505}{425} \times \dfrac{530}{470}} \times 100 = 115.75$$

Time reversal test

Test is satisfied when $P_{01} \times P_{10} = 1$

$$P_{01} = \sqrt{\frac{\Sigma\, p_1 q_0}{\Sigma\, p_0 q_0} \times \frac{\Sigma\, p_1 q_1}{\Sigma\, p_0 q_1}} = \sqrt{\frac{505}{425} \times \frac{530}{470}}$$

$$P_{10} = \sqrt{\frac{\Sigma\, p_0 q_1}{\Sigma\, p_1 q_1} \times \frac{\Sigma\, p_0 q_0}{\Sigma\, p_1 q_0}} = \sqrt{\frac{470}{530} \times \frac{425}{505}}$$

$$P_{01} \times P_{10} = \sqrt{\frac{505}{425} \times \frac{530}{470} \times \frac{470}{530} \times \frac{425}{505}}$$

$$= \sqrt{1} = 1$$

Hence Fisher's Ideal Index satisfies Time Reversal Test.

Factor reversal test

Test is satisfied when $P_{01} \times Q_{01} = \dfrac{\Sigma\, p_1 q_1}{\Sigma\, p_0 q_0}$

$$Q_{01} = \sqrt{\frac{\Sigma\, q_1 p_0}{\Sigma\, q_0 p_0} \times \frac{\Sigma\, q_1 p_1}{\Sigma\, q_0 p_1}} = \sqrt{\frac{470}{530} \times \frac{425}{505}}$$

$$P_{01} \times Q_{01} = \sqrt{\frac{505}{425} \times \frac{530}{470} \times \frac{470}{425} \times \frac{530}{505}}$$

$$= \frac{530}{425} = \frac{\Sigma\, p_1 q_1}{\Sigma\, p_0 q_0}$$

Hence Fisher's Ideal Index satisfies Factor reversal test.

10.4.6 Cost of Living Index (CLI)

Cost of living index numbers are generally designed to represent the average change over time in the prices paid by the ultimate consumer for a specified quantity of goods and services. Cost of living index number is also known as **Consumer price index number**

It is well known that a given change in the level of prices (retail) affects the cost of living of different classes of people in different manners. The general index number fails to reveal this. Therefore it is essential to construct a cost of living index number which helps us in determining the effect of rise and fall in prices on different classes of consumers living in different areas. It is to be noted that the demand for a higher wage is based on the cost of living index. The wages and salaries in most countries are adjusted in accordance with the cost of living index.

10.4.7 Methods of constructing cost of living index

Cost of living index may be constructed by the following methods.

(i)   Aggregate expenditure method or weighted aggregative method

(ii)   Family budget method

Aggregate expenditure method

In this method, the quantities of commodities consumed by the particular group in the base year are used as the weights. On the basis of these weights, aggregate (total) expenditure in current year and base year are calculated and the percentage of change is worked out.

$\therefore$ Cost of Living Index (C.L.I) = $\dfrac{\Sigma\, p_1 q_0}{\Sigma\, p_0 q_0} \times 100$

This method is the most popular method for constructing cost of living index and the method is same as Laspeyre's price index.

Family budget method

In this method, the value weights obtained by multiplying prices by quantities consumed (i.e. $p_0 q_0$) are taken as weights. To get the cost of living index, find the sum of respective products of price relatives and value weights and then divide this sum by the sum of the value weights.

$\therefore$ Cost of Living Index (C.L.I) = $\dfrac{\Sigma\, PV}{\Sigma\, V}$ where

P = $\dfrac{p_1}{p_0}$ × 100 is the price relative and

V = $p_0 q_0$ is the value weight for each item.

This method is same as the Weighted average of price relative method.

10.4.8 Uses of cost of living index number

(i)   The cost of living index number is mainly used in wage negotiations and wage contracts.

(ii)   It is used to calculate the dearness allowance for the employees.

Example 23

Calculate the cost of living index by aggregate expenditure method

Commodity	Quantity 2000	Price (Rs) 2000	Price (Rs) 2003
A	100	8	12.00
B	25	6	7.50
C	10	5	5.25
D	20	48	52.00
E	65	15	16.50
F	30	19	27.00

Solution :

Commodity	Quantity 2000 q_0	Price 2000 p_0	Price 2003 p_1	$p_1 q_0$	$p_0 q_0$
A	100	8	12.00	1200.00	800
B	25	6	7.50	187.50	150
C	10	5	5.25	52.50	50
D	20	48	52.00	1040.00	960
E	65	15	16.50	1072.50	975
F	30	19	27.00	810.00	570
				4362.50	**3505**

$$C.L.I = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

$$= \frac{4362.50}{3505} \times 100 = 124.46$$

Example 24

Construct the cost of living Index Number for 2003 on the basis of 2000 from the following data using family Budget method.

Items	Price		Weights
	2000	2003	
Food	200	280	30
Rent	100	200	20
Clothing	150	120	20
Fuel & lighting	50	100	10
Miscellaneous	100	200	20

Solution :

Calculation of CLI by family budget method

Items	p_0	p_1	weights V	$P = \dfrac{p_1}{p_0} \times 100$	PV
Food	200	280	30	140	4200
Rent	100	200	20	200	4000
Clothing	150	120	20	80	1600
Fuel & Lighting	50	100	10	200	2000
Misc.	100	200	20	200	4000
			100		15800

$$\text{Cost of Living Index (C.L.I)} = \frac{\Sigma \, PV}{\Sigma \, V} = \frac{15800}{100} = 158$$

Hence, there is 58% increase in cost of living in 2003 compared to 2000.

EXERCISE 10.4

1) Compute (i) Laspeyre's (ii) Paasche's and (iii) Fisher's index Numbers

Commodity	Price		Quantity	
	Base year	Current year	Base year	Current year
A	6	10	50	50
B	2	2	100	120
C	4	6	60	60
D	10	12	30	25

2) Construct the price index number from the following data by applying

(i) Laspeyre's (ii) Paasche's (iii) Fisher's method

Commodity	1999		1998	
	Price	Quantity	Price	Quantity
A	4	6	2	8
B	6	5	5	10
C	5	10	4	14
D	2	13	2	19

3) Compute (a) Laspyre's (b) Paasche's (c) Fisher's method of index numbers for 1990 from the following :

Commodity	Price		Quantity	
	1980	1990	1980	1990
A	2	4	8	6
B	5	6	10	5
C	4	5	14	10
D	2	2	19	13

4) From the following data calculate the price index number by

(a) Laspeyre's method (b) paasche's method (c) Fisher's method

Commodity	Base year		Current year	
	Price	Quantity	Price	Quantity
A	5	25	6	30
B	10	5	15	4
C	3	40	2	50
D	6	30	8	35

5) Using the following data, construct Fisher's Ideal index and show that it satisfies Factor Reversal test and Time Reversal test.

Commodity	Price		Quantity	
	Base year	Current year	Base year	Current year
A	6	10	50	56
B	2	2	100	120
C	4	6	60	60
D	10	12	30	24
E	8	12	40	36

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6)   Calculate Fisher's Ideal Index from the following data and show how it satisfies time reversal test and factor reversal test.

Commodity	Base year (1997)		Current year (1998)	
	Price	Quantity	Price	Quantity
A	10	10	12	8
B	8	12	8	13
C	12	12	15	8
D	20	15	25	10
E	5	8	8	8
F	2	10	4	6

7)   Construct cost of living index for 2000 taking 1999 as the base year from the following data using Aggregate Expenditure method.

Commodity	Quantity (Kg.) 1999	Price	
		1999	2000
A	6	5.75	6.00
B	1	5.00	8.00
C	6	6.00	9.00
D	4	8.00	10.00
E	2	2.00	1.80
F	1	20.00	15.00

8)   Calculate the cost of living Index Number using Family Budget method

Commodity	A	B	C	D	E	F	G	H
Quantity in Base year (unit)	20	50	50	20	40	50	60	40
Price in Base year (Rs.)	10	30	40	200	25	100	20	150
Price in current year (Rs.)	12	35	50	300	50	150	25	180

9) Calculate the cost of living index number using Family Budget method for the following data taking the base year as 1995.

Commodity	Weight	Price (per unit)	
		1995	1996
A	40	16.00	20.00
B	25	40.00	60.00
C	5	0.50	0.50
D	20	5.12	6.25
E	10	2.00	1.50

10) From the data given below, construct a cost of living index number by family budget method for 1986 with 1976 as the base year.

Commodity	P	Q	R	S	T	U
Quantity in 1976 Base year	50	25	10	20	30	40
Price per unit in 1976 (Rs.)	10	5	8	7	9	6
Price per unit in 1986 (Rs.)	6	4	3	8	10	12

10.5 STATISTICAL QUALITY CONTROL (SQC)

Every product manufactured is required for a specific purpose. It means that if the product meets the specifications required for its rightful use, it is of good quality and if not, then the quality of the product is considered to be poor.

It is a well known fact that all repetitive process no matter how carefully arranged are not exactly identical and contain some variability. Even in the manufacture of commodities by highly specialised machines it is not unusual to come across differences between various units of production. For example, in the manufacture of corks, bottles etc. eventhough highly efficient machines are used some difference may be noticed in various units. If the differences are not much, it can be ignored and the product can be passed off as within specifications. But if it is beyond certain limits, the article has to be rejected and the cause of such variation has to be investigated.

10.5.1 Causes for variation

The variation occurs due to two types of causes namely

(i) Chance causes (ii) Assignable causes

(i) Chance causes

If the variation occurs due to some inherent pattern of variation and no causes can be assigned to it, it is called **chance** or **random variation**. Chance Variation is tolerable, permissible inevitable and does not materially affect the quality of a product.

(ii) Assignable causes

The causes due to faulty process and procedure are known as assignable causes. the variation due to assignable causes is of non-random nature. Chance causes cannot detected. However assignable causes can be detected and corrected.

10.5.2 Role and advantages of SQC

The role of statistical quality control is to collect and analyse relevant data for the purpose of detecting whether the process is under control or not.

The value of quality control lies in the fact that assignable causes in a process can be quickly detected. Infact the variations are often discovered before the product becomes defective.

SQC is a well accepted and widespread process on the basis of which it is possible to understand the principles and techniques by which decisions are made based on variation.

Statistical quality control is only diagonstic. It tells us whether the standard is being maintained or not. The remedial action rests with the technicians who employ techniques for the maintenance of uniform quality in a continuous flow of manufactured products.

The purpose for which SQC are used are two fold namely, **(a) Process control, (b) Product control.**

In process control an attempt is made to find out if a particular process is within control or not. Process control helps in studying the future performance.

10.5.3 Process and Product control

The main objective in any production process is to control and maintain quality of the manufactured product so that it conforms to specified quality standards. In otherwords, we want to ensure that the proportion of defective items in the manufactured product is not too large. This is called **process control** and is achieved through the technique of control charts.

On the otherhand, by **product control** we mean controlling the quality of the product by critical examination at strategic points and this is achieved through **product control plans** pioneered by Dodge and Romig. Product control aims at guaranteeing a certain quality level to the consumer regardless of what quality level is being maintained by the producer. In otherwords, it attempts to ensure that the product marketed by the sale department does not contain a large number of defective items.

10.5.4 Control Charts

The statistical tool applied in process control is the **control chart**. Control charts are the devices to describe the patterns of variation. The control charts were developed by the physicist, Walter A. Stewart of Bell Telephone Company in 1924. He suggested that the control chart may serve, first to define the goal or standard for the process that the management might strive to attain. Secondly, it may be used as an instrument to attain that

goal and thirdly, it may serve as a means of judging whether the goal is being achieved. Thus, control chart is an instrument to be used in specification, production and inspection and is the core of statistical quality control.

A control chart is essentially a graphic device for presenting data so as to directly reveal the frequency and extent of variations from established standards or goals. Control charts are simple to construct and easy to interpret and they tell the manager at a glance whether or not the process is in control, i.e. within the tolerance limits.

In general a control chart consists of three horizontal lines

(i)   A central line to indicate the desired standard or level of the process (CL)

(ii)   Upper control limit (UCL) and

(iii)   Lower control limit (LCL)

Outline of a control chart

(Fig. 10.9)
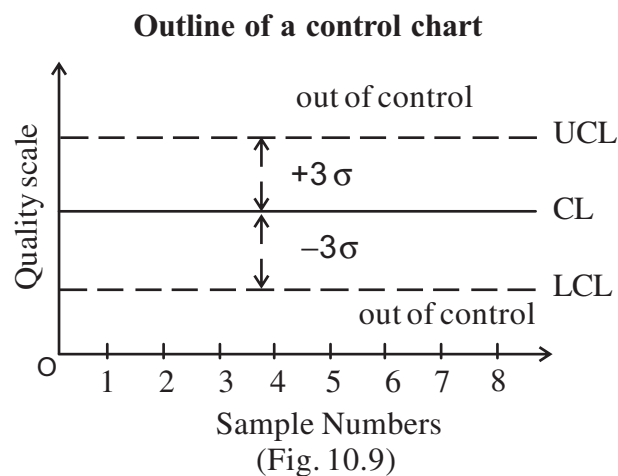
From time to time a sample is taken and the data are plotted on the graph. If all the sample points fall within the upper and lower control limits, it is asumed that the process is "in control" and only chance causes are present. When a sample point falls outside the control limits, it is assumed that variations are due to assignable causes.

Types of Control Charts

Broadly speaking, control charts can be divided under two heads.

(i) Control charts of Variables

(ii) Control charts of Attributes

Control charts of variables concern with measurable data on quality characteristics which are usually continuous in nature. Such type of data utilises $\overline{X}$ and R chart.

Control charts of attributes, namely c, np and p charts concern with the data on quality characteristics, which are not amenable to measurement or attributes (prodcut defective or non defective)

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In this chapter, we consider only the control charts of variables, namely $\overline{X}$ **chart** and **R chart**.

R-Chart (Range chart)

The R chart is used to show the variability or dispersion of the quality produced by a given process. R chart is the companion chart to the $\overline{X}$ chart and both are usually required for adequate analysis of the production process under study. The R chart is generally presented along with the $\overline{X}$ chart. The general procedure for constructing the R chart is similar to that for the $\overline{X}$ chart. The required values for constructing the R chart are :

(i)   The range of each sample, R.

(ii)   The Mean of the sample ranges, $\overline{R}$

(iii)   The control limits are set at

$$U.C.L = D_4\overline{R}$$

$$L.C.L = D_3\overline{R}$$

The values of D_4 and D_3 can be obtained from tables.

$\overline{X}$ Chart

The $\overline{X}$ chart is used to show the quality averages of the samples drawn from a given process. The following values must first be computed before an $\overline{X}$ chart is constructed.

1)   Obtain the mean of each sample $\overline{X}_i : i = 1, 2 \dots n$

2)   Obtain the mean of the sample means

i.e $\overline{\overline{X}} = \dfrac{\overline{X}_1 + \overline{X}_2 + \dots + \overline{X}_n}{n}$

where n is the total number of observations

3)   The control limits are set at

$$U.C.L. = \overline{\overline{X}} + A_2\overline{R}$$

$$LCL = \overline{\overline{X}} - A_2\overline{R}, \text{ where } \overline{R} = \dfrac{\sum\limits_{i=1}^{n} R_i}{n}, \text{ where } R_i \text{ are the sample ranges.}$$

The values of A_2 for different n can be obtained from the tables.

Example 25

The following data relate to the life (in hours) of 10 samples of 6 electric bulbs each drawn at an interval of one hour from a production process. Draw the control chart for $\overline{X}$ and R and comment.

Sample No.	Life time (in hours)					
1	620	687	666	689	738	686
2	501	585	524	585	653	668
3	673	701	686	567	619	660
4	646	626	572	628	631	743
5	494	984	659	643	660	640
6	634	755	625	582	683	555
7	619	710	664	693	770	534
8	630	723	614	535	550	570
9	482	791	533	612	497	499
10	706	524	626	503	661	754

(Given for $n = 6$, $A_2 = 0.483$, $D_3 = 0$, $D_4 = 2.004$)

Solution :

Sample No.	Total	Sample Mean $\overline{X}$	Sample Range R
1	4086	681	118
2	3516	586	167
3	3906	651	134
4	3846	641	171
5	4080	680	490
6	3834	639	200
7	3990	665	236
8	3622	604	188
9	3414	569	309
10	3774	629	251
Total		**6345**	**2264**

Central line $\overline{\overline{X}}$ = mean of the sample means = 634.5

$\overline{R}$ = mean of the sample ranges = 226.4

U.C.L. $= \overline{\overline{X}} + A_2\overline{R}$

$= 634.5 + 0.483 \times 226.4$

$$= 634.5 + 109.35 = 743.85$$

$$\text{L.C.L.} = \overline{\overline{X}} - A_2\overline{R}$$

$$= 634.5 - 0.483 \times 226.4$$

$$= 634.5 - 109.35 = 525.15$$

Central line $\overline{R} = 226.4$

$$\text{U.C.L.} = D_4\overline{R} = 2.004 \times 226.4$$

$$= 453.7056$$

$$\text{L.C.L.} = D_3\overline{R} = 0 \times 226.4 = 0$$

Conclusion :

Since one of the points of the sample range is outside the UCL of R chart, the process is not in control.

Example 26

The following data shows the value of sample mean $\overline{X}$ and the range R for ten samples of size 5 each. Calculate the values for central line and control limits for mean chart and range chart and determine whether the process is in control.

Sample no.	1	2	3	4	5	6	7	8	9	10
Mean $\overline{X}$	11.2	11.8	10.8	11.6	11.0	9.6	10.4	9.6	10.6	10.0
Range (R)	7	4	8	5	7	4	8	4	7	9

(Given for n = 5, A_2 = 0.577, D_3 = 0, D_4 = 2.115)

Solution :

Control limits for $\overline{X}$ chart

$$\overline{\overline{X}} = \frac{1}{n}\Sigma\overline{X}$$

$$= \frac{1}{10}(11.2 + 11.8 + 10.8 + ... + 10.0) = 10.66$$

$$\overline{R} = \frac{1}{n}\Sigma R = \frac{1}{10}(63) = 6.3$$

$$\text{U.C.L} = \overline{\overline{X}} + A_2\overline{R}$$

$$= 10.66 + (0.577 \times 6.3) = 14.295$$

$$\text{L.C.L} = \overline{\overline{X}} - A_2\overline{R}$$

$$= 10.66 - (0.577 \times 6.3) = 7.025$$

$$\text{CL} = \text{Central line} = \overline{\overline{X}} = 10.66$$

Range chart

$$\text{U.C.L} = D_4 \overline{R} = 2.115 \times 6.3 = 13.324$$

$$\text{L.C.L} = D_3 \overline{R} = 0$$

$$\text{C.L} = \overline{R} = 6.3$$

$\overline{X}$ Chart

Sample No.
(Fig. 10.12)

R Chart
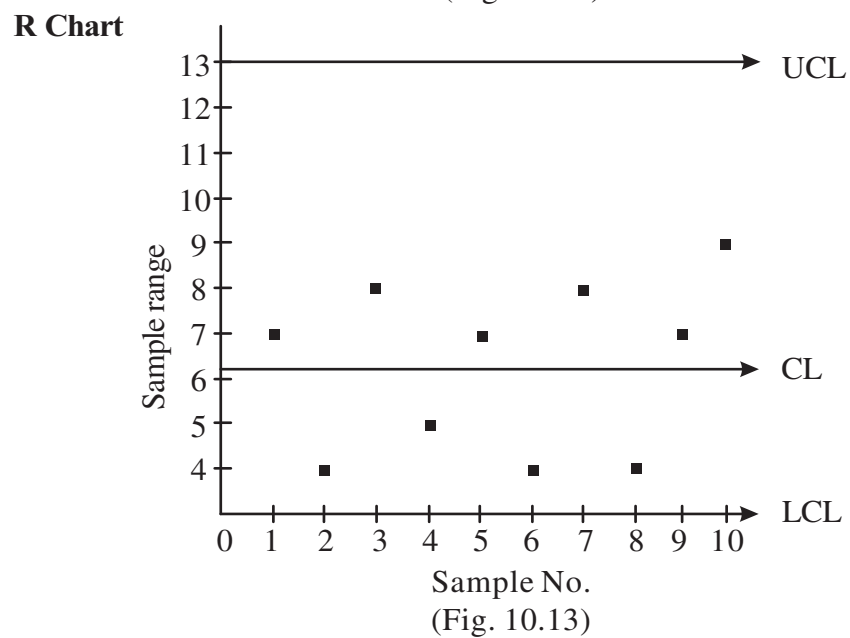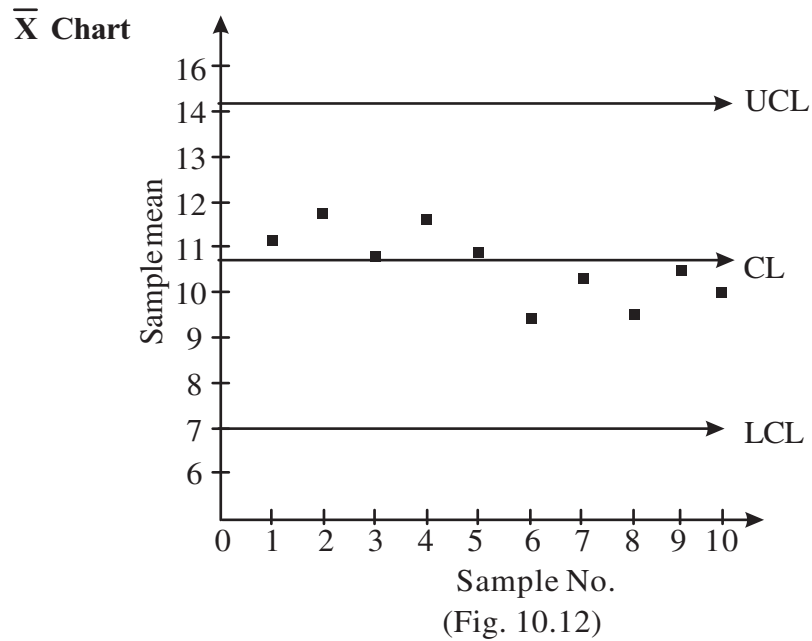
Sample No.
(Fig. 10.13)

Conclusion :

Since all the points of sample mean and range are within the control limits, the process is in control.

EXERCISE 10.5

1) The following are the $\overline{X}$ and R values for 20 samples of 5 readings. Draw $\overline{X}$ chart and R chart and write your conclusion.

Samples	1	2	3	4	5	6	7	8	9	10
$\overline{X}$	34	31.6	30.8	33	35	33.2	33	32.6	33.8	37.8
R	4	4	2	3	5	2	5	13	19	6

Samples	11	12	13	14	15	16	17	18	19	20
$\overline{X}$	35.8	38.4	34	35	38.8	31.6	33	28.2	31.8	35.6
R	4	4	14	4	7	5	5	3	9	6

(Given for $n = 5$, $A_2 = 0.58$, $D_3 = 0$, $D_4 = 2.12$)

EXERCISE 10.6

Choose the correct answer

1) A time series is a set of data recorded

    (a) periodically   (b) at equal time intervals

    (c) at successive points of time   (d) all the above

2) A time series consists of

    (a) two components   (b) three components

    (c) four components   (d) none of these

3) The component of a time series attached to long term variation is termed as

    (a) cyclic variations   (b) secular trend

    (c) irregular variation   (d) all the above

4) The component of a time series which is attached to short term fluctuations is

    (a) seasonal variation   (b) cyclic variation

    (c) irregular variation   (d) all the above

5) Cyclic variations in a time series are casued by

    (a) lock out in a factory   (b) war in a country

    (c) floods in the states   (d) none of these

6) The terms prosperity, recession depression and recovery are in particular attached to

(a) Secular trend   (b) seasonal fluctuation

(c) cyclic movements   (d) irregular variation

7) An additive model of time series with the components T, S, C and I is

(a) $Y = T + S + C - I$ (b) $Y = T + S \times C + I$

(c) $Y = T + S + C + I$ (d) $Y = T + S + C \times I$

8) A decline in the sales of ice cream during November to March is associated with

(a) Seasonal variation   (b) cyclical variation

(c) random variation   (d) secular trend

9) Index number is a

(a) measure of relative changes   (b) a special type of an average

(c) a percentage relative   (d) all the above.

10) Index numbers are expressed

(a) in percentages   (b) in ratios

(c) in terms of absolute value   (d) all the above

11) Most commonly used index numbers are

(a) Diffusion index number   (b) price index number

(c) value index number   (d) none of these

12) Most frequently used index number formulae are

(a) weighted formulae   (b) Unweighted formulae

(c) fixed weighted formulae   (d) none of these

13) Laspeyre's index formula uses the weights of the

(a) base year quantities   (b) current year prices

(c) average of the weights of number of years   (d) none of these

14) The weights used in Paasche's formula belong to

(a) the base period   (b) the current period

(c) to any arbitrary chosen period   (d) none of these

15) Variation in the items produced in a factory may be due to

(a) chance causes   (b) assignable causes

(c) both (a) and (b)   (d) neither (a) or (b)

16) Chance variation in the manufactured product is

(a) controlable   (b) not controlable

(c) both (a) and (b)   (d) none of these

17) The causes leading to vast variation in the specification of a product are usually due to

(a) random process   (b) assignable causes

(c) non-traceable causes   (d) all the above

18) Variation due to assignable causes in the product occur due to

(a) faulty process   (b) carelessness of operators

(c) poor quality of raw material   (d) all the above

19) Control charts in statistical quality consist of

(a) three control lines   (b) upper and lower control limits

(c) the level of process   (d) all the above

20) The range of correlation co-efficient is

(a) 0 to ∞ (b) $-\infty$ to ∞ (c) -1 to 1   (d) none of these

21) If X and Y are two variates, there can be atmost

(a) one regression line   (b) two regression lines

(c) three regression lines   (d) none of these

22) In a regression line of Y on X, the variable X is known as

(a) independent variable   (b) dependent variable

(c) both (a) and (b)   (d) none of these

23) Scatter diagram of the variate values (X, Y) give the idea about

a) functional relationship   (b) regression model

(c) distribution of errors   (d) none of these

24) The lines of regression intersect at the point

(a) (X, Y)   (b) $(\overline{X}, \overline{Y})$ (c) (0, 0)   (d) none of these

25) The term regression was introduced by

(a) R.A.Fisher   (b) Sir Francis Galton

(c) Karl pearson   (d) none of these.

ANSWERS

1. APPLICATIONS OF MATRICES AND DETERMINANTS

Exercise 1.1

1) $\begin{pmatrix} 1 & -3 \\ -2 & -1 \end{pmatrix}$

2) $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$

8) $\dfrac{1}{16} \begin{pmatrix} 2 & -4 \\ 3 & 2 \end{pmatrix}$

9) $\dfrac{1}{3} \begin{pmatrix} 1 & 2 & -2 \\ -4 & -2 & 5 \\ 1 & -1 & 1 \end{pmatrix}$

10) $\begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}$

11) $\begin{pmatrix} \dfrac{1}{a_1} & 0 & 0 \\ 0 & \dfrac{1}{a_2} & 0 \\ 0 & 0 & \dfrac{1}{a_3} \end{pmatrix}$

13) $\begin{pmatrix} 0 & 1 & -2 \\ -1 & -3 & 10 \\ 1 & 2 & -7 \end{pmatrix}$

18) 4, –2   19) –1, 0   20) $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

Exercise 1.2

1) (i) 3 (ii) 2 (iii) 1 (iv) 3 (v) 2 (vi) 3 (vii) 1 (viii) 2 (ix) 2

2) 2, 0.   6) inconsistent

11) $k = -3$ 12) k assumes any real value other than 0

13) $k = -3$ 14) k assumes any real value other than 8

Exercise 1.3

1) 2, 1   2) 0, 1, 1   3) 5, 2   4) 2, -1, 1

5) 0, 2, 4   6) 20, 30   7) Rs.2, Rs.3, Rs.5.

8) Rs.1, Rs.2, Rs.3   9) 11 tons, 15 tons, 19 tons

Exercise 1.4

1) The system is viable   2) The system is not viable

3) 110 units, 320 units   4) Rs.72 millions, Rs.96 millions

5) (i) Rs.42 lakhs, Rs.78 lakhs (ii) Rs.28 lakhs, Rs.52 lakhs

6) Rs. 80 millions, Rs.120 millions   7) Rs.1200 crores, Rs.1600 crores

8) Rs.7104 crores, Rs.6080 crores

Exercise 1.5

1) 74.8%, 25.2% ; 75%, 25%   2) 39%   3) 54.6%, 45.4%

Exercise 1.6

1) c   2) b   3) c   4) c   5) a   6) a   7) b   8) b   9) b   10) c   11) a   12) a

13) a   14) a   15) a   16) b   17) d   18) b

ANALYTICAL GEOMETRY

Exercise 2.1

1) a parabola   2) a hyperbola   3) an ellipse

Exercise 2.2

1)   (i) $x^2 + y^2 - 2xy - 4y + 6 = 0$

           (ii) $x^2 + y^2 + 2xy - 4x + 4y + 4 = 0$

           (iii) $4x^2 + 4xy + y^2 - 4x + 8y - 4 = 0$

           (iv) $x^2 + 2xy + y^2 - 22x - 6y + 25 = 0$

2)   (i) (0, 0), (0, 25), $x = 0$, $y + 25 = 0$

           (ii) (0, 0), (5, 0), $y = 0$, $x + 5 = 0$

           (iii) (0, 0), (–7, 0), $y = 0$, $x - 7 = 0$

           (iv) (0, 0), (0, –15), $x = 0$, $y - 15 = 0$

3)   (i) $\left(-\dfrac{1}{2}, 1\right)$, $\left(-\dfrac{3}{2}, 1\right)$, $2x - 1 = 0$, 4

           (ii) (–1, –1), (0, –1), $x + 2 = 0$, 4

           (iii) $\left(-\dfrac{9}{8}, 0\right)$, $\left(-\dfrac{7}{8}, 0\right)$, $8x + 25 = 0$, 8

           (iv) (0, 1), $\left(0, \dfrac{7}{4}\right)$, $4y - 1 = 0$, 3

4)   Output = 15 tons, and average cost = Rs.40

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Exercise 2.3

1) (i) $101x^2 + 48xy + 81y^2 - 330x - 324y + 441 = 0$

(ii) $27x^2 + 20y^2 - 24xy + 6x + 8y - 1 = 0$

(iii) $17x^2 + 22y^2 + 12xy - 58x + 108y + 129 = 0$

2) (i) $\dfrac{x^2}{144} + \dfrac{y^2}{128} = 1$ (ii) $\dfrac{x^2}{24} + \dfrac{y^2}{15} = 1$

(iii) $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

3) (i) $(0, 0), (0, \pm 3)$; $\dfrac{\sqrt{5}}{3}$; $(0, \pm \sqrt{5})$; $\dfrac{8}{3}$; $y = \pm\dfrac{9}{\sqrt{5}}$

(ii) $(1, -5), (1, \pm \sqrt{7} -5)$; $\dfrac{\sqrt{3}}{\sqrt{7}}$; $(1, \pm \sqrt{3} - 5)$; $\dfrac{8}{\sqrt{7}}$; $y = \dfrac{7}{\sqrt{3}} - 5, y = \dfrac{-7}{\sqrt{3}} - 5,$

(iii) $(-2, 1), (2, 1) (-6, 1))$; $\dfrac{\sqrt{7}}{4}$; $(\pm \sqrt{7} -2, 1)$; $\dfrac{9}{2}$; $x = \dfrac{16}{\sqrt{7}} - 2, x = \dfrac{-16}{\sqrt{7}} - 2,$

Exercise 2.4

1) (i) $19x^2 + 216xy - 44y^2 - 346x - 472y + 791 = 0$

(ii) $16(x^2 + y^2) = 25(x \cos\alpha + y \sin\alpha - p)^2$

2) $12x^2 - 4y^2 - 24x + 32y - 127 = 0$

3) (i) $16x^2 - 9y^2 - 32x - 128 = 0$

(ii) $3x^2 - y^2 - 18x + 4y + 20 = 0$

(iii) $3x^2 - y^2 - 36x + 4y + 101 = 0$

4) (i) $(0, 0)$; $\dfrac{5}{4}$; $(\pm 5, 0)$; $5x \pm 16 = 0$

(ii) $(-2, -4)$; $\dfrac{4}{3}$; $(2, -4) (-6, -4);\ 4x - 1 = 0, 4x + 17 = 0$

(iii) $(1, 4)$; 2 ; $(6, 4) (-4, 4);\ 4x - 9 = 0, 4x + 1 = 0$

5) (i) $3x + y + 2 = 0$ and $x - 2y + 5 = 0$; (ii) $4x - y + 1 = 0$ and $2x + 3y - 1 = 0$

6) $4x^2 - 5xy - 6y^2 - 11x + 11y + 57 = 0$

7) $12x^2 - 7xy - 12y^2 + 31x + 17y = 0$

Exercise 2.5

1) a 2) b 3) c 4) d 5) c 6) a 7) c 8) b 9) b 10) a 11) b 12) c

13) b 14) b 15) a 16) c 17) a 18) c 19) c 20) c

APPLICATIONS OF DIFFERENTIATION - I

Exercise 3.1

1) (i) $\frac{1}{2} x^2 - 4x + 25 + \frac{8}{x}$ (ii) $\frac{1}{2} x^2 - 4x + 25$

(iii) $\frac{8}{x}$, AC = Rs.35.80, AVC = Rs.35, AFC = Rs.0.80

2) Rs.600.05   3) Rs.5.10

4) Rs.1.80   5) Rs.1.50, Rs.1406.25

6) (i) $\frac{1}{10} x^2 - 4x + 8 + \frac{4}{x}$

(ii) $\frac{3}{10} x^2 - 8x + 8$

(iii) $\frac{1}{5} x - 4 - \frac{4}{x^2}$

7) Rs.55, Rs.23   8) Rs.119

10) 0.75   11) 1.15

13) (i) $\frac{2(a-bx)}{bx}$ (ii) $\frac{3}{2}$ 14) m

15) $\frac{4p^2}{2p^2 + 5}$ 16) $\frac{p}{2(p-b)}$

17) AR = p, MR = $550 - 6x - 18x^2$

18) (i) R = $20,000 \, x \, e^{-0.6x}$ (ii) MR = $20,000 \, x \, e^{-0.6x} [1 - 0.6x]$

19) $\frac{4p + 2p^2}{30 - 4p - p^2}, \frac{3p^2 + 8p - 30}{2(p+2)}$

20) 20, 3   21) Rs.110   22) $\frac{30}{11}$, Rs.1.90

Exercise 3.2

1) $-1.22, -1.25$ 2) -1 unit / sec   3) 12 units / sec.

5) (i) revenue is increasing at the rate of Rs.40,000 per month

(ii) cost is increasing at the rate of Rs.4,000 per month

(iii) profit is increasing at the rate of Rs.36,000 per month

6)   (i) revenue is increasing at the rate of Rs.48,000 per week

       (ii) cost is increasing at the rate of Rs.12,000 per week

       (iii) profit is increasing at the rate of Rs.36,000 per week.

8) 10π cm^2 / sec   9) 115π cm^3 / minute   10) $x = \dfrac{1}{3}, 3$

Exercise 3.3

1) $\dfrac{10}{3}, \dfrac{-13}{5}$ 2) $a = 2, b = 2$

4)   (i) $x - y + 1 = 0$, $x + y - 3 = 0$

       (ii) $2x - 2y + \sqrt{3} - \dfrac{\pi}{3} = 0$; $2x + 2y - \sqrt{3} - \dfrac{\pi}{3} = 0$

       (iii) $3x + 2y + 13 = 0$; $2x - 3y = 0$

       (iv) $9x + 16y - 72 = 0$; $64x - 36y - 175 = 0$

       (v) $3ex - y - 2e^2 = 0$; $x + 3ey - 3e^3 - e = 0$

       (vi) $\sqrt{2}\, bx + \sqrt{2}\, ay - 2ab = 0$; $\sqrt{2}\, ax - \sqrt{2}\, by - a^2 + b^2 = 0$

5) $13x - y - 34 = 0$; $x + 13y - 578 = 0$

6) $10x + y - 61 = 0$; $x - 10y + 105 = 0$

7) $\left(1, \dfrac{1}{3}\right), \left(-1, \dfrac{-1}{3}\right)$ 9) $x - 20y - 7 = 0$; $20x + y - 140 = 0$

11) $\dfrac{x}{a}\sec\theta - \dfrac{y}{b}\tan\theta = 1$; $\dfrac{ax}{\sec\theta} + \dfrac{by}{\tan\theta} = a^2 + b^2$

12)   (i) $(1, 0)$ and $(1, 4)$ (ii) $(3, 2)$ and $(-1, 2)$

Exercise 3.4

1) d   2) c   3) a   4) a   5) b   6) c   7) d   8) d   9) b   10) a

11) d 12) a 13) b 14) c 15) b 16) d 17) c 18) a 19) a 20) c

APPLICATIONS OF DIFFERENTIATION-II

Exercise 4.1

3) increasing in $(-\infty, -5)$ and $\left(-\dfrac{1}{3}, \infty\right)$ decreasing in $\left(-5, -\dfrac{1}{3}\right)$

4) $(-2, 27)$, $(1, 0)$

5) (i) R is increasing for $0 < x < 4$ decreasing for $x > 4$, MR is increasing for $0 < x < 2$ and decreasing for $x > 2$.

   (ii) R is increasing for $1 < x < 7$, decreasing for $0 < x < 1$ and $x > 7$. MR is increasing for $0 < x < 4$ and decreasing for $x > 4$.

6) (i) TC is increasing for $0 < x < 10$ and for $x > 20$ and decreasing for $10 < x < 20$. MC is decreasing for $0 < x < 15$ and increasing for $x > 15$.

   (ii) TC is increasing for $0 < x < 40$ and decreasing for $x > 40$. MC is always decreasing.

7) (i) $x = 0$ max. value $= 7$, $x = 4$ min. value $= -25$

   (ii) $x = 1$ max. value $= -4$, $x = 4$ mini. value $= -31$

   (iii) $x = 2$ min. value $= 12$

   (iv) $x = 1$ max. value $= 19$, $x = 3$ mini. value $= 15$

8) $x = 1$ max. value $= 53$, $x = -1$ min. value $= -23$.   9) $(0, 3)$, $(2, -9)$

11) Convex up for $\dfrac{1}{2} < x < 1$ Convex down for $-\infty < x < \dfrac{1}{2}$ and $1 < x < \infty$

12) $q = 3$.

13) $x = 1$ max. value $= 0$

   $x = 3$ mini. value $= -28$

   $x = 0$ point of inflexion exists.

Exercise 4.2

1) $x = 15$ 2) $15, 225$ 4) $x = 5$ 5) $1, \dfrac{3}{2}$

6) $x = 8$ 7) (i) 10.5, Rs.110.25 (ii) $3, 0$ (iii) $x = 6$

8) $x = 60$ 9) Rs.1600 10) $x = 70$ 11) $x = 13$

12) A : 1000, B : 1800, C : 1633

13) A : 214.476, Rs.21.44 B : 67.51 Rs.58.06 C : 2000, Rs.4, D : 537.08, Rs.27.93

14) (i) 400 (ii) Rs.240 (iii) $\dfrac{3}{2}$ orders / year   (iv) $\dfrac{2}{3}$ of a year

15)   (i) 800   (ii) $\frac{1}{4}$ of a year   (iii) 4   (iv) Rs.1200

Exercise 4.3

1) $8x + 6y;\ 6x - 6y$

3)   (i) $24x^5 + 3x^2y^5 - 24x^2 + 6y - 7$

        (ii) $5x^3y^4 + 6x + 8$

        (iii) $120x^4 - 48x + 6xy^5$

        (iv) $20x^3y^3$

        (v) $15x^2y^4 + 6$

        (vi) $15x^2y^4 + 6$

4)   (i) $30x^4y^2 + 8x + 4$ (ii) 500

        (iii) $12x^5y - 24y^2 + 6$ (iv) $- 90$

        (v) $120x^3\,y^2 + 8$ (vi) 968

        (vii) $12x^5 - 48y$ (viii) 12

        (ix) $60x^4y$ (x) 2880   (xi) 2880

14)   (i) 940 (ii) 700

15)   Note Book   (16) (i) Rs.18,002   (ii) Rs.8005

Exercise 4.4

1) (i) 10 - 2L + 3K,   (ii) 5 - 4K + 3L   (iii) 14, 0

3) 1, 4   4) 3.95, 120   5) 2.438, 3.481

7) (i) $\frac{3}{4}$ (ii) $\frac{1}{2}$ 8) $\frac{3}{5}, \frac{2}{5}$ 9) 6, 1   10) $-\frac{10}{3}, \frac{5}{6}$

Exercise 4.5

1) b   2) d   3) a   4) b   5) c   6) c   7) a   8) c   9) b   10) d

11) a   12) a   13) d   14) a   15) c   16) a   17) c   18) d   19) a   20) a

APPLICATIONS OF INTEGRATION

Exercise 5.1

1) 0   2) 80   3) $\dfrac{\pi}{2}$ 4) 2   5) $\dfrac{16}{15}\sqrt{2}$ 6) $\dfrac{1}{20}$ 7) $\dfrac{\pi}{12}$

8) 1   9) $\dfrac{\pi^2}{4}$ 10) $(a+b)\,\dfrac{\pi}{4}$

Exercise 5.2

Answers are in square units.

1) 9   2) 6   3) $\dfrac{\pi}{2}$ 4) 2   5) $\dfrac{2}{3}$ 6) log 3

7) $\dfrac{8a^2}{3}$ 8) 8   9) 4log4   10) πa^2 11) πab

Exercise 5.3

1) $C = 10x + 12x^2 - x^3 + 4$, $AC = 10 + 12x - x^2 + \dfrac{4}{x}$

2) $C = 100\,(\log \dfrac{x}{16} + 1)$, $AC = \dfrac{100}{x}\,(\log \dfrac{x}{16} + 1)$

3) $C = x^3 - 5x^2 + 3x + 8$, $AC = x^2 - 5x + 3 + \dfrac{8}{x}$

4) $C = 5x - 3x^2 + x^3 + 100$, $AC = 5 - 3x + x^2 + \dfrac{100}{x}$

5) $C = 20x - 0.02x^2 + 0.001x^3 + 7000$

$AC = 20 - 0.02x + 0.001x^2 + \dfrac{7000}{x}$

6) $R = 15x - \dfrac{9x^2}{2} - x^3$, $AR = 15 - \dfrac{9x}{2} - x^2$

7) $R = 9x - x^2 + \dfrac{4x^3}{3}$, $p = 9 - x + \dfrac{4x^2}{3}$

8) $R = 100x - 3x^3$, $p = 100 - 3x^2$

9) $R = 2x + 2x^2 - \dfrac{x^3}{3}$, $p = 2 + 2x - \dfrac{x^2}{3}$

10) $R = 4x - \dfrac{3x^2}{2}$, $p = 4 - \dfrac{3x}{2}$

11) $p = 3 - x, \qquad R = 3x - x^2$

12) $p = 5 - \dfrac{x^2}{2}, \qquad R = 5x - \dfrac{x^3}{2}$

13) $p = \dfrac{k}{x}$, k is constant.

14) $C = 2x + e^{3x} + 500, \qquad AC = 2 + \dfrac{e^{3x}}{x} + \dfrac{500}{x}$

15) $R = -\dfrac{3}{x} - \log x^2 + 9, \quad p = -\dfrac{3}{x^2} - \dfrac{\log x^2}{x} + \dfrac{9}{x}$

16) $R = 16\,x - \dfrac{x^3}{3}, \qquad p = 16 - \dfrac{x^2}{3}$

17) $13x - 0.065x^2 - 120$

18) $R = Rs.4,31,667$

Exercise 5.4

1) CS = 27 units

2) CS = $\dfrac{250}{3}$ units

3) (i) CS = 16 units   (ii) CS=4units

4) CS = 216 units

5) PS = 128 units

6) PS = $\dfrac{10}{3}$ units

7) (i) PS $\dfrac{9}{2}$ units (ii) PS = 18 units

8) PS = $\dfrac{8}{3}$ units

9) CS = 9 units ; PS = 18 units

10) CS=18 units ; PS = 36units

11) CS = 144 units PS = 48 units

12) CS = $\dfrac{63}{2}$ units   PS = $\dfrac{9}{2}$ units

13) CS = $\dfrac{16}{3}$ units PS = 32 units

14) CS = 50 units PS = 15 units

15) CS = 16 log2 – 8 units PS = 4 units

Exercise 5.5

1) (c)	2) (a)	3) (a)	4) (b)	5) (a)	6) (a)
7) (a)	8) (b)	9) (a)	10) (b)	11) (c)	12) (a)
13) (b)	14) (a)	15) (a)	16) (a)	17) (a)	18) (b)

DIFFERENTIAL EQUATIONS

Exercise 6.1

1) (i) 2 and 1   (ii) 3 and 1   (iii) 2 and 2   (iv) 2 and 1   (v) 2 and 3

   (vi) 2 and 1   (vii) 2 and 3   (viii) 2 and 1   (ix) 2 and 1   (x) 2 and 2

2) (i) $x\dfrac{dy}{dx} - y = 0$ (ii) $y = x\left(\dfrac{dy}{dx}\right) - \dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^2$ (iii) $x\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{dy}{dx} + a = 0$

   (iv) $\dfrac{d^2y}{dx^2} = 0$

3) $x\dfrac{dy}{dx} + y = 0$ 4) $2xy\dfrac{dy}{dx} + x^2 - y^2 = 0$ 5) $y\left(\dfrac{dy}{dx}\right)^2 + 2x\dfrac{dy}{dx} - y = 0$

6) $\dfrac{d^2y}{dx^2} + 9y = 0$ 7) $y\dfrac{dy}{dx} + x = 0$

Exercise 6.2

1) (i) $\sin^{-1}y + \sin^{-1}x = c$ (ii) $y - x = c\,(1 + xy)$ (iii) $y + 2 = c\,(x - 1)$

2) $\tan y = c\,(1 - e^x)^3$ 3) (i) $\log(y + a) = x^2 + c$ (ii) $(x^2 + 1)(y^2 + 1) = c^2$

4) $y = x^3 + 2x - 4$ 5) $y = x^2$

6) Cost function $C = \dfrac{e^7}{3}\,(e^{3x} - 1)$

   Average cost function $= \dfrac{e^7}{3}\,\dfrac{e^{3x} - 1}{x}$

Exercise 6.3

1) (i) $\dfrac{x}{y} = \log x + c$ (ii) $\dfrac{y + x}{y} = c\sqrt{x}$ (iii) $\dfrac{y}{x}\left(\dfrac{y}{x} - 1\right)x^3 = c$ (iv) $\log y + \dfrac{x^2}{2y^2} = c$

2) $c^2 = q^2 + 6q$ 3) $y^2 = 12x^2 - \dfrac{128}{x^2}$

Exercise 6.4

1) (i) $y \sin x = x + c$ (ii) $y \sin x = 2x^2 - \dfrac{\pi^2}{2}$ (iii) $\dfrac{y}{x^3} = \dfrac{-1}{x} + c$

(iv) $y(1+x^2) = \tan^{-1} x - \dfrac{\pi}{4}$ (v) $y \cos x = e^x + c$ (vi) $y \log x = -\dfrac{\cos 2x}{2} + c$

2) Rs,13,720   3) $cq = \dfrac{a}{2}(q^2 - q_0{}^2) + c_0 q_0$

Exercise 6.5

1) (i) $y = Ae^{4x} + Be^{6x}$ (ii) $y = A + Be^{-x}$

(iii) $y = A \cos 2x + B \sin 2x$ (iv) $y = (Ax + B)e^{-2x}$

2) (i) $y = Ae^{\frac{2}{3}x} + Be^{-3x}$ (ii) $y = (Ax + B)\, e^{\frac{3}{2}x}$

(iii) $y = e^{\frac{1}{6}x}(A \cos \dfrac{\sqrt{11}}{6}x + B \sin \dfrac{\sqrt{11}}{6}x)$

3) (i) $y = Ae^{12x} + Be^x + \dfrac{1}{42}e^{-2x} - \dfrac{3}{20}e^x$ (ii) $y = Ae^{2x} + Be^{3x} + \dfrac{e^{-x}}{12} + \dfrac{5}{11}xe^{-2x}$

(iii) $y = (Ax + B)e^{7x} + \dfrac{3}{49} + \dfrac{x^2}{2}e^{7x}$ (iv) $y = Ae^{\frac{-x}{5}} + e^{\frac{x}{3}}(B + \dfrac{x}{8})$

4) $P = Ae^{-4t} + Be^{2t} + 3$

Exercise 6.6

1) a   2) c   3) b   4) b   5) c   6) a   7) c   8) a   9) b   10) c   11) a   12) c

13) c   14) a   15) d   16) a   17) c   18) a   19) b   20) c

INTERPOLATION AND FITTING A STRAIGHT LINE

Exercise 7.1

1) 33   2) 41 lakhs   3) 7   4) 18   5) 384 tons

6) 56.8672   7) 12.7696   8) 147411.99   9) 7237.87   10) 262.75

11) 124.1568   12) 478.63   13) 19   14) 32.93

Exercise 7.2

3) $y = x + 6.66$ 4) $4 ; - 3$ 5) $y = 0.38\,x + 1.65$

6) $y = 1.19x + 0.66$ 7) $y = 0.0041x + 0.048$ 8) $y = 1.33x + 0.72 ; y = 5.375$

9) $y = 1.125x + 38$ 10) $y = 1.48x + 1.26$

11) $y = 3.14x + 27.34 ; y = 81$ (approximately)

Exercise 7.3

1) c   2) b   3) a   4) b   5) c   6) a   7) c   8) c   9) b   10) c

PROBABILITY DISTRIBUTION

Exercise 8.1

1) (ii) and (iii) 2) yes   3) (i) $\dfrac{1}{81}$;   (ii) $\dfrac{9}{81}, \dfrac{65}{81}, \dfrac{24}{81}$

4) (i) yes

    (ii) $F(x) = 0$ if $x < 1$

$$= \frac{1}{3} \text{ if } 1 \le x < 2$$

$$= 1 \text{ if } x \ge 2$$

5) 1   6) (a) $\dfrac{1}{2}$; (b) $\dfrac{1}{4}$ (c) $\dfrac{3}{4}$

7) (i) $\dfrac{1}{4}$ (ii) $= \dfrac{1}{16}$ if $0 \le x < 1$

$$= \frac{7}{16} \text{ if } 0 \le x < 2$$

$$= \frac{11}{16} \text{ if } 0 \le x < 3$$

$$= 1 \text{ if } x \ge 3$$

8) $k = 0.003$; (i) 0.00012 ; (ii) 0.027   9) 1

10) (i) $\dfrac{1}{2}$; (ii) $\dfrac{1}{2}$ 11) $\dfrac{8}{3} 10^6$

12) (i) 0.3935   (ii) 0.1481   (iii) 0.2231

Exercise 8.2

1) 2.50   2) 7   3) 1.25   4) 3.4   5) – 6.5, 6   6) 0 ; 4 ; 2

7) $\dfrac{1}{2}$; $\dfrac{1}{4}$

Exercise 8.3

1) $\dfrac{176}{1024}$ 2) 0.2   3) $n = 9$, $p = \dfrac{2}{3}$, $q = \dfrac{1}{3}$

4) 0.544320   5) 0.345   6) 0.762   7) 0.264   8) 0.0008

9) (i) 0.04979 (ii) 0.14937   10) 0.4060

Exercise 8.4

1) (i) 0.0035   (ii) 0.9582   2) (i) 0.0581   (ii) 0.1519   3) 0.1587   4) 0   5) 125

6) (i) 123 (ii) 341   7) 66.01   8) $\mu = 50$, $\sigma = 10$ 9) $\mu = 50.09$, $\sigma = 19.4$

Exercise 8.5

1) b   2) a   3) c   4) a   5) c   6) b   7) b   8) b   9) c   10) a   11) b   12) c

13) b   14) c   15) c

SAMPLING TECHNIQUES AND STATISTICAL INFERENCE

Exercise 9.1

1) (a) 6.35 mm ; (b) 0.00055 mm^2 2) (a) (0.818, 0.829) ; (b) (0.816, 0.832)

3) mean profit lies between 72.6 lakhs and 77.4 lakhs

4) mean marks lies between 72.6 and 77.4

5) Number of defective cases lies between 1230 and 2270

6) (a) 45% and 65% ;   (b) 42% and 68%

Exercise 9.2

1) H_0 : sample is from the said population ; $| Z | = 2.67$, H_0 is rejected

2) H_0 : sample is from the said population

  At 5% level, $| Z | = 2.2$; H_0 is rejected

  At 1% level, $| Z | = 2.2$; H_0 is accepted

3) H_0 : $P = 0.60$, $| Z | = 5.9$; H_0 is rejected at 1% level.

Exercise 9.3

1) c   2) c   3) b   4) c   5) a   6) c   7) c

APPLIED STATISTICS

Exercise 10.1

1) Maximize $Z = 30x_1 + 40x_2$ subject to $60x_1 + 120x_2 \leq 12000$, $8x_1 + 5x_2 \leq 600$, $3x_1 + 4x_2 \leq 500$, $x_1 + x_2 \geq 0$

2) Maximize $Z = 3x_1 + 4x_2$ subject to $x_1 + x_2 \leq 450$, $2x_1 + x_2 \leq 600$, $x_1, x_2 \geq 0$

3) $x_1 = 24$, $x_2 = 14$, $Z = 2200$

4) $x_1 = 2.5$, $x_2 = 35$, $Z = 147.5$

5) $x_1 = 1$, $x_2 = 5$, $Z = 13$

Exercise 10.2

1) 0.9485   2) 0.2555   3) 0.7689   4) – 0.9673   5) 0.3566

6) $y = -0.65 x + 11.90$; $x = -1.30y + 16.40$

7) $x = y + 6$, $x = 26$

8) $y = 0.5932x + 38.79$; $x = 0.7954y + 13.34$

9) $y = 0.25x + 24.75$; $x = 0.2683y + 27.88$

Exercise 10.3

3) 100, 110, 120, 130, 140, 150, 160

4) 42.31, 40.94, 39.56, 38.19, 36.81, 35.44, 34.06, 32.69

5) –, 22, 23.33, 24, 23.67, 23.67, 24.33, 26, 26.33, – .

6) – , 87.33, 88.33, 89.67, 92, 95, –.

7) –, –, 495.75, 503.63, 511.63, 529.50, 553, 572.50, –, –.

8) –, –, 648.125, 661.500, 674.625, 687.875, 696.375, 715.250, 735.750, –, –.

9) 105.10, 95.68, 99.35, 99.87

10) 98.4, 92.2, 108.9, 100.5

11) 98.4, 92.14, 108.9, 100.52

Exercise 10.4

1) 136.54, 135.92, 136.23   2) 125, 126.21, 125.6   3) 125, 126.21, 125.6

4) 114.74, 112.73, 113.73   5) 139.79   6) 124.34

7) 119.09   8) 137.27   9) 124.41

10) 101.1

Exercise 10.5

1) $\bar{\bar{X}} = 33.6$, $\bar{R} = 6.2$

$\bar{X}$ chart : UCL = 37.446, LCL = 30.254

$\bar{R}$ chart : LCL = 0, UCL = 13.14

Exercise 10.6

1) d   2) c   3) b   4) d   5) d   6) c   7) c   8) a   9) d   10) a   11) d   12) a

13) a   14) b   15) c   16) d   17) b   18) d   19) a   20) c   21) b   22) a   23) a   24) b

25) b

STANDARD NORMAL DISTRIBUTION - TABLE

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4608	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936

2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.49865	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.49903	.4990	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993129	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995166	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4996631	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998	.4998
3.5	.4997674	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998409	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4998922	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999277	.4999	.4999	.4999	.4999	.4999	.4999	.5000	.5000	.5000
3.9	.4999519	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000