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STANDARD NINE

TERM II

VOLUME 2

MATHEMATICS

NOT FOR SALE

Untouchability is Inhuman and a Crime

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Preface

The Trimester pattern has been introduced in Tamil Nadu as a milestone in the history of school education. In this method, a voluminous textbook has been divided into three small booklets, one for each term for easy understanding of concepts. The first term Mathematics textbook has already been prepared and distributed to all schools.The units such as Set Theory, Real Number System, Algebra, Geometry, Coordinate Geometry and Practical Geometry have been included for the first term. Among these, Set theory and Coordinate Geometry have been discussed completely. The remaining portions are learnt in the second and third term also as a continuation of the first term.

The Mathematics textbook for the second term includes topics such as Algebra, Trigonometry, Statistics, and Practical Geometry. As the concepts of Algebra and Practical Geometry in the second term are in continuation of the Term I, the teachers should enable the students to recall the concepts learnt in the first term while teaching higher level concepts in the second term.

At the end of each unit, FA (a) activities have been suggested. The teachers need to select the appropriate activity for explaining and reinforcing the concepts. The same activities can be used for conducting Formative Assessment (a) also. The teachers are free either to use them as such or they can design their own new activities that are appropriate for their students and school setting with the objective of making them learn and enjoy the beauty of mathematics.

-Textbook team

SYMBOLS

$=$	equal to
$\neq$	not equal to
$<$	less than
$\leq$	less than or equal to
$>$	greater than
$\geq$	greater than or equal to
$\approx$	equivalent to
$\cup$	union
$\cap$	intersection
$\mathbb{U}$	universal Set
$\in$	belongs to
$\notin$	does not belong to
$\subset$	proper subset of
$\subseteq$	subset of or is contained in
$\not\subset$	not a proper subset of
$\not\subseteq$	not a subset of or is not contained in
A' (or) A^c	complement of A
$\varnothing$ (or) $\{\ \}$	empty set or null set or void set
$n(A)$	number of elements in the set A
$P(A)$	power set of A
$\|\|\|^{ly}$	similarly
$\mathrm{P}(A)$	probability of the event A

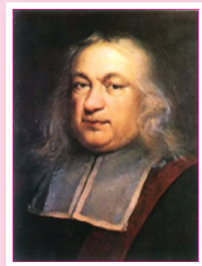
Δ	symmetric difference
$\mathbb{N}$	natural numbers
$\mathbb{W}$	whole numbers
$\mathbb{Z}$	integers
$\mathbb{R}$	real numbers
$\triangle$	triangle
$\angle$	angle
$\perp$	perpendicular to
$\parallel$	parallel to
$\Longrightarrow$	implies
$\therefore$	therefore
$\because$	since (or) because
$\vert \quad \vert$	absolute value
$\simeq$	approximately equal to
$\vert$ (or) :	such that
$\equiv$ (or) $\cong$	congruent
$\equiv$	identically equal to
π	pi
$\pm$	plus or minus
$\blacksquare$	end of the proof

CONTENTS

1 ALGEBRA

Mathematics is as much an aspect of
culture as it is a collection of algorithms

\- CARL BOYER



PIERRE DE FERMAT

Pierre de Fermat was the most brilliant mathematician of his era and, along with Déscartes, one of the most influential. Although mathematics was just his hobby (Fermat was a government lawyer), Fermat practically founded Number Theory, and also played key roles in the discoveries of Analytic Geometry and Calculus. He was also an excellent geometer (e.g. discovering a triangle's Fermat point), and (in collaboration with Blaise Pascal) discovered probability theory. Fermat was also the first European to find the integration formula for the general polynomial; he used his calculus to find centers of gravity, etc.

Main Targets

- To classify polynomials.
- To use algebraic identities.
- To factorize a polynomial.
- To solve linear equations in two variables.
- To solve linear inequation in one variable.

1.1 Introduction

In the first term we have learnt about the types of polynomials, zeros of polynomials, roots of polynomial equations, division of polynomials, remainder theorem, factor theorem and their applications. In this term, we extend the identities studied in class VIII to trinomials and third degree expansions. Also, we will learn factorization of polynomials, solving a pair of linear equations in two variables using substitution method and solving linear inequations in one variable.

1.2 Algebraic Identities

Key Concept — **Algebraic Identities**

An identity is an equality that remains true regardless of the values of any variables that appear within it.

We have learnt the following identities in class VIII. Using these identities let us solve some problems and extend the identities to trinomials and third degree expansions.

$(a+b)^2 \equiv a^2 + 2ab + b^2$	$(a+b)(a-b) \equiv a^2 - b^2$
$(a-b)^2 \equiv a^2 - 2ab + b^2$	$(x+a)(x+b) \equiv x^2 + (a+b)x + ab$

Example 1.1

Expand the following using identities

(i) $(2a+3b)^2$ (ii) $(3x-4y)^2$ (iii) $(4x+5y)(4x-5y)$ (iv) $(y+7)(y+5)$

Solution

(i) $(2a+3b)^2 = (2a)^2 + 2(2a)(3b) + (3b)^2$

$= 4a^2 + 12ab + 9b^2$

(ii) $(3x-4y)^2 = (3x)^2 - 2(3x)(4y) + (4y)^2$

$= 9x^2 - 24xy + 16y^2$

(iii) $(4x+5y)(4x-5y) = (4x)^2 - (5y)^2$

$= 16x^2 - 25y^2$

(iv) $(y+7)(y+5) = y^2 + (7+5)y + (7)(5)$

$= y^2 + 12y + 35$

1.2.1 Expansion of the Trinomial $(x \pm y \pm z)^2$

$$
\begin{aligned}
(x+y+z)^2 &= (x+y+z)(x+y+z) \\
&= x(x+y+z) + y(x+y+z) + z(x+y+z) \\
&= x^2 + xy + xz + yx + y^2 + yz + zx + zy + z^2 \\
&= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx
\end{aligned}
$$

$$\boxed{(x+y+z)^2 \equiv x^2 + y^2 + z^2 + 2xy + 2yz + 2zx}$$

(ii) $(x-y+z)^2 = [x+(-y)+z]^2$

$= x^2 + (-y)^2 + z^2 + 2(x)(-y) + 2(-y)(z) + 2(z)(x)$

$$= x^2 + y^2 + z^2 - 2xy - 2yz + 2zx$$

$$(x - y + z)^2 \equiv x^2 + y^2 + z^2 - 2xy - 2yz + 2zx$$

In the same manner we get the expansion for the following

(iii) $(x + y - z)^2 \equiv x^2 + y^2 + z^2 + 2xy - 2yz - 2zx$

(iv) $(x - y - z)^2 \equiv x^2 + y^2 + z^2 - 2xy + 2yz - 2zx$

Example 1.2

Expand (i) $(2x + 3y + 5z)^2$ (ii) $(3a - 7b + 4c)^2$ (iii) $(3p + 5q - 2r)^2$

(iv) $(7l - 9m - 6n)^2$

Solution

(i) $(2x + 3y + 5z)^2 = (2x)^2 + (3y)^2 + (5z)^2 + 2(2x)(3y) + 2(3y)(5z) + 2(5z)(2x)$

$= 4x^2 + 9y^2 + 25z^2 + 12xy + 30yz + 20zx$

(ii) $(3a - 7b + 4c)^2$

$= (3a)^2 + (-7b)^2 + (4c)^2 + 2(3a)(-7b) + 2(-7b)(4c) + 2(4c)(3a)$

$= 9a^2 + 49b^2 + 16c^2 - 42ab - 56bc + 24ca$

(iii) $(3p + 5q - 2r)^2$

$= (3p)^2 + (5q)^2 + (-2r)^2 + 2(3p)(5q) + 2(5q)(-2r) + 2(-2r)(3p)$

$= 9p^2 + 25q^2 + 4r^2 + 30pq - 20qr - 12rp$

(iv) $(7l - 9m - 6n)^2$

$= (7l)^2 + (-9m)^2 + (-6n)^2 + 2(7l)(-9m) + 2(-9m)(-6n) + 2(-6n)(7l)$

$= 49l^2 + 81m^2 + 36n^2 - 126lm + 108mn - 84nl$

1.2.2 Identities Involving Product of Binomials $(x+a)(x+b)(x+c)$

$$\begin{aligned}(x + a)(x + b)(x + c) &= [(x + a)(x + b)](x + c)\\ &= [x^2 + (a + b)x + ab](x + c)\\ &= x^3 + (a + b)x^2 + abx + cx^2 + c(a + b)x + abc\\ &= x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc\end{aligned}$$

$$\boxed{(x + a)(x + b)(x + c) \equiv x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc}$$

1.2.3 Expansion of $(x \pm y)^3$

In the above identity by substituting $a = b = c = y$, we get

$(x + y)(x + y)(x + y) = x^3 + (y + y + y)x^2 + [(y)(y) + (y)(y) + (y)(y)]x + (y)(y)(y)$

$$(x+y)^3 = x^3 + (3y)x^2 + (3y^2)x + y^3$$
$$= x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^3 \equiv x^3 + 3x^2y + 3xy^2 + y^3$$
(or) $$(x+y)^3 \equiv x^3 + y^3 + 3xy(x+y)$$

Replacing y by $-y$ in the above identity, we get

$$(x-y)^3 \equiv x^3 - 3x^2y + 3xy^2 - y^3$$
(or) $$(x-y)^3 \equiv x^3 - y^3 - 3xy(x-y)$$

Using these identities of 1.2.2 and 1.2.3, let us solve the following problems.

Example 1.3

Find the product of

(i) $(x+2)(x+5)(x+7)$ (ii) $(a-3)(a-5)(a-7)$ (iii) $(2a-5)(2a+5)(2a-3)$

Solution

(i) $(x+2)(x+5)(x+7)$

$$= x^3 + (2+5+7)x^2 + [(2)(5)+(5)(7)+(7)(2)]x + (2)(5)(7)$$
$$= x^3 + 14x^2 + (10+35+14)x + 70$$
$$= x^3 + 14x^2 + 59x + 70$$

(ii) $(a-3)(a-5)(a-7) = [a+(-3)][a+(-5)][a+(-7)]$

$$= a^3 + (-3-5-7)a^2 + [(-3)(-5)+(-5)(-7)+(-7)(-3)]a + (-3)(-5)(-7)$$
$$= a^3 - 15a^2 + (15+35+21)a - 105$$
$$= a^3 - 15a^2 + 71a - 105$$

(iii) $(2a-5)(2a+5)(2a-3) = [2a+(-5)][2a+5][2a+(-3)]$

$$= (2a)^3 + (-5+5-3)(2a)^2 + [(-5)(5)+(5)(-3)+(-3)(-5)](2a) + (-5)(5)(-3)$$
$$= 8a^3 + (-3)4a^2 + (-25-15+15)2a + 75$$
$$= 8a^3 - 12a^2 - 50a + 75$$

Example 1.4

If $a+b+c = 15$, $ab+bc+ca = 25$ find $a^2+b^2+c^2$.

Solution We have $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$. So,

$$15^2 = a^2+b^2+c^2+2(25)$$

$$225 = a^2 + b^2 + c^2 + 50$$

$$\therefore a^2 + b^2 + c^2 = 225 - 50 = 175$$

Example 1.5

Expand (i) $(3a + 4b)^3$ (ii) $(2x - 3y)^3$

Solution

(i) $(3a + 4b)^3 = (3a)^3 + 3(3a)^2(4b) + 3(3a)(4b)^2 + (4b)^3$

$= 27a^3 + 108a^2b + 144ab^2 + 64b^3$

(ii) $(2x - 3y)^3 = (2x)^3 - 3(2x)^2(3y) + 3(2x)(3y)^2 - (3y)^3$

$= 8x^3 - 36x^2y + 54xy^2 - 27y^3$

Example 1.6

Evaluate each of the following using suitable identities.

(i) $(105)^3$ (ii) $(999)^3$

Solution

(i) $(105)^3 = (100 + 5)^3$

$= (100)^3 + (5)^3 + 3(100)(5)(100 + 5)$ $(\because (x + y)^3 = x^3 + y^3 + 3xy(x + y))$

$= 1000000 + 125 + 1500(105)$

$= 1000000 + 125 + 157500 = 1157625$

(ii) $(999)^3 = (1000 - 1)^3$

$= (1000)^3 - (1)^3 - 3(1000)(1)(1000 - 1)$

$(\because (x - y)^3 = x^3 - y^3 - 3xy(x - y))$

$= 1000000000 - 1 - 3000(999)$

$= 1000000000 - 1 - 2997000 = 997002999$

Some Useful Identities involving sum ,difference and product of x and y

$$x^3 + y^3 \equiv (x + y)^3 - 3xy(x + y)$$

$$x^3 - y^3 \equiv (x - y)^3 + 3xy(x - y)$$

Let us solve some problems involving above identities.

Example 1.7

Find $x^3 + y^3$ if $x + y = 4$ and $xy = 5$

Solution We know that $x^3 + y^3 = (x + y)^3 - 3xy(x + y)$

$\therefore$ $x^3 + y^3 = (4)^3 - 3(5)(4) = 64 - 60 = 4$

Example 1.8

Find $x^3 - y^3$ if $x - y = 5$ and $xy = 16$

Solution We know that $x^3 - y^3 = (x-y)^3 + 3xy(x-y)$

$$\therefore \quad x^3 - y^3 = (5)^3 + 3(16)(5) = 125 + 240 = 365$$

Example 1.9

If $x + \frac{1}{x} = 5$, find the value of $x^3 + \frac{1}{x^3}$

Solution We know that $x^3 + y^3 = (x+y)^3 - 3xy(x+y)$

Put $y = \frac{1}{x}$, $x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right)$

$$= (5)^3 - 3(5) = 125 - 15 = 110$$

Example 1.10

If $y - \frac{1}{y} = 9$, find the value of $y^3 - \frac{1}{y^3}$

Solution We know that, $x^3 - y^3 = (x-y)^3 + 3xy(x-y)$

Put $x = y$ *and* $y = \frac{1}{y}$, $y^3 - \frac{1}{y^3} = \left(y - \frac{1}{y}\right)^3 + 3\left(y - \frac{1}{y}\right)$

$$= (9)^3 + 3(9) = 729 + 27 = 756$$

The following identity is frequently used in higher studies

$$x^3 + y^3 + z^3 - 3xyz \equiv (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

Note If $x + y + z = 0$ then $x^3 + y^3 + z^3 = 3xyz$

Example 1.11

Simplify $(x + 2y + 3z)(x^2 + 4y^2 + 9z^2 - 2xy - 6yz - 3zx)$

Solution We know that, $(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx) = x^3 + y^3 + z^3 - 3xyz$

$$\therefore \ (x + 2y + 3z)(x^2 + 4y^2 + 9z^2 - 2xy - 6yz - 3zx)$$

$$= (x + 2y + 3z)[x^2 + (2y)^2 + (3z)^2 - (x)(2y) - (2y)(3z) - (3z)(x)]$$

$$= (x)^3 + (2y)^3 + (3z)^3 - 3(x)(2y)(3z)$$

$$= x^3 + 8y^3 + 27z^3 - 18xyz$$

Example 1.12

Evaluate $12^3 + 13^3 - 25^3$

Solution Let $x = 12$, $y = 13$, $z = -25$. Then

$$x + y + z = 12 + 13 - 25 = 0$$

If $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

$$\therefore\ 12^3 + 13^3 - 25^3 = 12^3 + 13^3 + (-25)^3 = 3(12)(13)(-25) = -11700$$

Exercise 1.1

1. Expand the following

   (i) $(5x + 2y + 3z)^2$ (ii) $(2a + 3b - c)^2$ (iii) $(x - 2y - 4z)^2$ (iv) $(p - 2q + r)^2$

2. Find the expansion of

   (i) $(x + 1)(x + 4)(x + 7)$ (ii) $(p + 2)(p - 4)(p + 6)$

   (iii) $(x + 5)(x - 3)(x - 1)$ (iv) $(x - a)(x - 2a)(x - 4a)$

   (v) $(3x + 1)(3x + 2)(3x + 5)$ (vi) $(2x + 3)(2x - 5)(2x - 7)$

3. Using algebraic identities find the coefficients of x^2 term, x term and constant term.

   (i) $(x + 7)(x + 3)(x + 9)$ (ii) $(x - 5)(x - 4)(x + 2)$

   (iii) $(2x + 3)(2x + 5)(2x + 7)$ (iv) $(5x + 2)(1 - 5x)(5x + 3)$

4. If $(x + a)(x + b)(x + c) \equiv x^3 - 10x^2 + 45x - 15$ find $a + b + c$, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ and $a^2 + b^2 + c^2$.

5. Expand : (i) $(3a + 5b)^3$ (ii) $(4x - 3y)^3$ (iii) $\left(2y - \frac{3}{y}\right)^3$

6. Evaluate : (i) 99^3 (ii) 101^3 (iii) 98^3 (iv) 102^3 (v) 1002^3

7. Find $8x^3 + 27y^3$ if $2x + 3y = 13$ and $xy = 6$.

8. If $x - y = -6$ and $xy = 4$, find the value of $x^3 - y^3$.

9. If $x + \frac{1}{x} = 4$, find the value of $x^3 + \frac{1}{x^3}$.

10. If $x - \frac{1}{x} = 3$, find the value of $x^3 - \frac{1}{x^3}$.

11. Simplify : (i) $(2x + y + 4z)(4x^2 + y^2 + 16z^2 - 2xy - 4yz - 8zx)$

    (ii) $(x - 3y - 5z)(x^2 + 9y^2 + 25z^2 + 3xy - 15yz + 5zx)$

12. Evaluate using identities : (i) $6^3 - 9^3 + 3^3$ (ii) $16^3 - 6^3 - 10^3$

1.3 Factorization of Polynomials

We have seen how the distributive property may be used to expand a product of algebraic expressions into sum or difference of expressions.

For example,

(i) $x(x + y) = x^2 + xy$ (ii) $x(y - z) = xy - xz$

(iii) $a(a^2 - 2a + 1) = a^3 - 2a^2 + a$

Now, we will learn how to convert a sum or difference of expressions into a product of expressions.

Now, consider $ab + ac$. Using the distributive law, $a(b+c) = ab + ac$, by writing in the reverse direction $ab + ac$ is $a(b+c)$. This process of expressing $ab + ac$ into $a(b+c)$ is known as factorization. In both the terms, ab and ac 'a' is the common factor. Similarly,

$5m + 15 = 5(m) + 5(3) = 5(m+3)$.

In $b(b-5) + g(b-5)$ clearly $(b-5)$ is a common factor.

$b(b-5) + g(b-5) = (b-5)(b+g)$

Example 1.13

Factorize the following

(i) $pq + pr - 3ps$ (ii) $4a - 8b + 5ax - 10bx$ (iii) $2a^3 + 4a^2$ (iv) $6a^5 - 18a^3 + 42a^2$

Solution

(i) $pq + pr - 3ps = p(q + r - 3s)$

(ii) $4a - 8b + 5ax - 10bx = (4a - 8b) + (5ax - 10bx)$

$= 4(a-2b) + 5x(a-2b) = (a-2b)(4+5x)$

(iii) $2a^3 + 4a^2$

Highest common factor is $2a^2$

$\therefore 2a^3 + 4a^2 = 2a^2(a+2)$.

(iv) $6a^5 - 18a^3 + 42a^2$

Highest common factor is $6a^2$

$\therefore 6a^5 - 18a^3 + 42a^2 = 6a^2(a^3 - 3a + 7)$

1.3.1 Factorization Using Identities

(i) $a^2 + 2ab + b^2 \equiv (a+b)^2$

(ii) $a^2 - 2ab + b^2 \equiv (a-b)^2$ (or) $a^2 - 2ab + b^2 \equiv (-a+b)^2$

(iii) $a^2 - b^2 \equiv (a+b)(a-b)$

(iv) $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \equiv (a+b+c)^2$

Example 1.14

Factorize (i) $4x^2 + 12xy + 9y^2$ (ii) $16a^2 - 8a + 1$ (iii) $9a^2 - 16b^2$

(iv) $(a+b)^2 - (a-b)^2$ (v) $25(a+2b-3c)^2 - 9(2a-b-c)^2$ (vi) $x^5 - x$

Solution

(i) $4x^2 + 12xy + 9y^2 = (2x)^2 + 2(2x)(3y) + (3y)^2 = (2x+3y)^2$

(ii) $16a^2 - 8a + 1 = (4a)^2 - 2(4a)(1) + (1)^2 = (4a-1)^2$ or $(1-4a)^2$

(iii) $9a^2 - 16b^2 = (3a)^2 - (4b)^2 = (3a+4b)(3a-4b)$

(iv) $(a+b)^2-(a-b)^2 = [(a+b)+(a-b)][(a+b)-(a-b)]$

$= (a+b+a-b)(a+b-a+b) = (2a)(2b) = (4)(a)(b)$

(v) $25(a+2b-3c)^2-9(2a-b-c)^2 = [5(a+2b-3c)]^2-[3(2a-b-c)]^2$

$= [5(a+2b-3c)+3(2a-b-c)][5(a+2b-3c)-3(2a-b-c)]$

$= (5a+10b-15c+6a-3b-3c)(5a+10b-15c-6a+3b+3c)$

$= (11a+7b-18c)(-a+13b-12c)$

(vi) $x^5-x = x(x^4-1) = x[(x^2)^2-(1)^2]$

$= x(x^2+1)(x^2-1) = x(x^2+1)[(x)^2-(1)^2]$

$= x(x^2+1)(x+1)(x-1)$

1.3.2 Factorization Using the Identity

$a^2+b^2+c^2+2ab+2bc+2ca \equiv (a+b+c)^2$

Example 1.15

Factorize $a^2+4b^2+36-4ab-24b+12a$

Solution $a^2+4b^2+36-4ab-24b+12a$ can be written as

$(a)^2+(-2b)^2+(6)^2+2(a)(-2b)+2(-2b)(6)+2(6)(a) = (a-2b+6)^2$ or

$(-a)^2+(2b)^2+(-6)^2+2(-a)(2b)+2(2b)(-6)+2(-6)(-a) = (-a+2b-6)^2$

That is $(a-2b+6)^2 = [(-1)(-a+2b-6)^2] = (-1)^2(-a+2b-6)^2 = (-a+2b-6)^2$

Example 1.16

Factorize $4x^2+y^2+9z^2-4xy+6yz-12zx$

Solution $4x^2+y^2+9z^2-4xy+6yz-12zx$

$= (2x)^2+(-y)^2+(-3z)^2+2(2x)(-y)+2(-y)(-3z)+2(-3z)(2x)$

$= (2x-y-3z)^2$ or $(-2x+y+3z)^2$

1.3.3 Factorization of x^3+y^3 and x^3-y^3

We have $x^3+3x^2y+3xy^2+y^3 = (x+y)^3$. So,

$x^3+y^3+3xy(x+y) = (x+y)^3$

$\Longrightarrow x^3+y^3 = (x+y)^3-3xy(x+y)$

$= (x+y)[(x+y)^2-3xy]$

$= (x+y)(x^2+2xy+y^2-3xy)$

$= (x+y)(x^2-xy+y^2)$

$$\boxed{x^3+y^3 \equiv (x+y)(x^2-xy+y^2)}$$

We have $x^3 - 3x^2y + 3xy^2 - y^3 = (x-y)^3$. So,

$$x^3 - y^3 - 3xy(x-y) = (x-y)^3$$

$$\begin{aligned} \implies x^3 - y^3 &= (x-y)^3 + 3xy(x-y) \\ &= (x-y)\left[(x-y)^2 + 3xy\right] \\ &= (x-y)(x^2 - 2xy + y^2 + 3xy) \\ &= (x-y)(x^2 + xy + y^2) \end{aligned}$$

$$\boxed{x^3 - y^3 \equiv (x-y)(x^2 + xy + y^2)}$$

Using the above identities let us factorize the following expressions.

Example 1.17

Factorize (i) $8x^3 + 125y^3$ (ii) $27x^3 - 64y^3$

Solution

(i)
$$\begin{aligned} 8x^3 + 125y^3 &= (2x)^3 + (5y)^3 \\ &= (2x+5y)[(2x)^2 - (2x)(5y) + (5y)^2] \\ &= (2x+5y)(4x^2 - 10xy + 25y^2) \end{aligned}$$

(ii)
$$\begin{aligned} 27x^3 - 64y^3 &= (3x)^3 - (4y)^3 \\ &= (3x-4y)[(3x)^2 + (3x)(4y) + (4y)^2] \\ &= (3x-4y)(9x^2 + 12xy + 16y^2) \end{aligned}$$

Exercise 1.2

1. Factorize the following expressions:

   (i) $2a^3 - 3a^2b + 2a^2c$ (ii) $16x + 64x^2y$ (iii) $10x^3 - 25x^4y$

   (iv) $xy - xz + ay - az$ (v) $p^2 + pq + pr + qr$

2. Factorize the following expressions:

   (i) $x^2 + 2x + 1$ (ii) $9x^2 - 24xy + 16y^2$

   (iii) $b^2 - 4$ (iv) $1 - 36x^2$

3. Factorize the following expressions:

   (i) $p^2 + q^2 + r^2 + 2pq + 2qr + 2rp$ (ii) $a^2 + 4b^2 + 36 - 4ab + 24b - 12a$

(iii) $9x^2 + y^2 + 1 - 6xy + 6x - 2y$ (iv) $4a^2 + b^2 + 9c^2 - 4ab - 6bc + 12ca$

(v) $25x^2 + 4y^2 + 9z^2 - 20xy + 12yz - 30zx$

4. Factorize the following expressions:

(i) $27x^3 + 64y^3$ (ii) $m^3 + 8$ (iii) $a^3 + 125$

(iv) $8x^3 - 27y^3$ (v) $x^3 - 8y^3$

1.3.4 Factorization of the Quadratic Polynomials of the type $ax^2 + bx + c$; $a \neq 0$

So far we have used the identities to factorize certain types of polynomials. In this section we will learn, without identities how to resolve quadratic polynomials into two linear polynomials when (i) $a = 1$ and (ii) $a \neq 1$

(i) Factorizing the quadratic polynomials of the type $x^2 + bx + c$.

suppose $(x + p)$ and $(x + q)$ are the two factors of $x^2 + bx + c$. Then we have

$$\begin{aligned} x^2 + bx + c &= (x + p)(x + q) \\ &= x(x + p) + q(x + p) \\ &= x^2 + px + qx + pq \\ &= x^2 + (p + q)x + pq \end{aligned}$$

This implies that the two numbers p and q are chosen in such way that $c = pq$ and $b = p + q$ to get $x^2 + bx + c = (x + p)(x + q)$

We use this basic idea to factorize the following problems

For example,

(1) $x^2 + 8x + 15 = (x + 3)(x + 5)$

here $c = 15 = 3 \times 5$ and $3 + 5 = 8 = b$

(2) $x^2 - 5x + 6 = (x - 2)(x - 3)$

here $c = 6 = (-2) \times (-3)$ and $(-2) + (-3) = -5 = b$

(3) $x^2 + x - 2 = (x + 2)(x - 1)$

here $c = -2 = (+2) \times (-1)$ and $(+2) + (-1) = 1 = b$

(4) $x^2 - 4x - 12 = (x - 6)(x + 2)$

here $c = -12 = (-6) \times (+2)$ and $(-6) + (+2) = -4 = b$

In the above examples the constant term is split into two factors such that their sum is equal to the coefficients of x.

Example 1.18

Factorize the following.

(i) $x^2 + 9x + 14$ (ii) $x^2 - 9x + 14$ (iii) $x^2 + 2x - 15$ (iv) $x^2 - 2x - 15$

Solution

(i) $x^2 + 9x + 14$

To factorize we have to find p and q, such that $pq = 14$ and $p + q = 9$.

$$\begin{aligned} x^2 + 9x + 14 &= x^2 + 2x + 7x + 14 \\ &= x(x+2) + 7(x+2) \\ &= (x+2)(x+7) \\ \therefore\ x^2 + 9x + 14 &= (x+7)(x+2) \end{aligned}$$

Factors of 14	Sum of factors
1, 14	15
2, 7	9
The required factors are 2, 7	

(ii) $x^2 - 9x + 14$

To factorize we have to find p and q such that $pq = 14$ and $p + q = -9$

$$\begin{aligned} x^2 - 9x + 14 &= x^2 - 2x - 7x + 14 \\ &= x(x-2) - 7(x-2) \\ &= (x-2)(x-7) \\ \therefore\ x^2 - 9x + 14 &= (x-2)(x-7) \end{aligned}$$

Factors of 14	Sum of factors
–1, –14	–15
–2, –7	–9
The required factors are –2, –7	

(iii) $x^2 + 2x - 15$

To factorize we have to find p and q, such that $pq = -15$ and $p + q = 2$

$$\begin{aligned} x^2 + 2x - 15 &= x^2 - 3x + 5x - 15 \\ &= x(x-3) + 5(x-3) \\ &= (x-3)(x+5) \\ \therefore\ x^2 + 2x - 15 &= (x-3)(x+5) \end{aligned}$$

Factors of –15	Sum of factors
–1, 15	14
–3, 5	2
The required factors are –3, 5	

(iv) $x^2 - 2x - 15$

To factorize we have to find p and q, such that $pq = -15$ and $p + q = -2$

$$\begin{aligned} x^2 - 2x - 15 &= x^2 + 3x - 5x - 15 \\ &= x(x+3) - 5(x+3) \\ &= (x+3)(x-5) \\ \therefore\ x^2 - 2x - 15 &= (x+3)(x-5) \end{aligned}$$

Factors of –15	Sum of factors
1, –15	–14
3, –5	–2
The required factors are 3, –5	

(ii) Factorizing the quadratic polynomials of the type $ax^2 + bx + c$.

Since a is different from 1, the linear factors of $ax^2 + bx + c$ will be of the form $(rx + p)$ and $(sx + q)$.

Then, $ax^2 + bx + c \;=\; (rx + p)(sx + q)$

$= rsx^2 + (ps + qr)x + pq$

Comparing the coefficients of x^2, we get $a = rs$. Similarly, comparing the coefficients of x, we get $b = ps + qr$. And, on comparing the constant terms, we get $c = pq$.

This shows us that b is the sum of two numbers ps and qr, whose product is $(ps)\times(qr) = (pr)\times(sq) = ac$

Therefore, to factorize $ax^2 + bx + c$, we have to write b as the sum of two numbers whose product is ac.

The following steps to be followed to factorize $ax^2 + bx + c$

Step1 : Multiply the coefficient of x^2 and constant term.

Step2 : Split this product into two factors such that their sum is equal to the coefficient of x.

Step3 : The terms are grouped into two pairs and factorize.

Example 1.19

Factorize the following

(i) $2x^2 + 15x + 27$ (ii) $2x^2 - 15x + 27$

(iii) $2x^2 + 15x - 27$ (iv) $2x^2 - 15x - 27$

Solution

(i) $2x^2 + 15x + 27$

Coefficient of $x^2 = 2$; constant term = 27

Their product $= 2 \times 27 = 54$

Coefficient of $x = 15$

$\therefore$ product = 54; sum = 15

Factors of 54	Sum of factors
1, 54	55
2, 27	29
3, 18	21
6, 9	15
The required factors are 6, 9	

$2x^2 + 15x + 27 = 2x^2 + 6x + 9x + 27$

$= 2x(x + 3) + 9(x + 3)$

$= (x + 3)(2x + 9)$

$\therefore\ 2x^2 + 15x + 27 = (x + 3)(2x + 9)$

(ii) $2x^2 - 15x + 27$

Coefficient of $x^2 = 2$; constant term = 27

Their product= $2 \times 27 = 54$

Coefficient of $x = -15$

$\therefore$ product= 54 ; sum = – 15

Factors of 54	Sum of factors
–1, –54	–55
–2, –27	–29
–3, –18	–21
–6, –9	–15
The required factors are –6, –9	

$$2x^2 - 15x + 27 = 2x^2 - 6x - 9x + 27$$
$$= 2x(x-3) - 9(x-3)$$
$$= (x-3)(2x-9)$$
$$\therefore\ 2x^2 - 15x + 27 = (x-3)(2x-9)$$

(iii) $2x^2 + 15x - 27$

Coefficient of $x^2 = 2$; constant term = – 27

Their product $= 2 \times - 27 = -54$

Coefficient of $x = 15$

$\therefore$ product = –54; sum = 15

Factors of –54	Sum of factors
–1, 54	53
–2, 27	25
–3, 18	15
The required factors are –3, 18	

$$2x^2 + 15x - 27 = 2x^2 - 3x + 18x - 27$$
$$= x(2x-3) + 9(2x-3)$$
$$= (2x-3)(x+9)$$
$$\therefore\ 2x^2 + 15x - 27 = (2x-3)(x+9)$$

(iv) $2x^2 - 15x - 27$

Coefficient of $x^2 = 2$; constant term = – 27

Their product $= 2 \times - 27 = -54$

Coefficient of $x = -15$

$\therefore$ product = –54; sum = –15

Factors of –54	Sum of factors
1, –54	–53
2, –27	–25
3, –18	–15
The required factors are 3, –18	

$$2x^2 - 15x - 27 = 2x^2 + 3x - 18x - 27$$
$$= x(2x+3) - 9(2x+3)$$
$$= (2x+3)(x-9)$$
$$\therefore\ 2x^2 - 15x - 27 = (2x+3)(x-9)$$

Example 1.20

Factorize $(x+y)^2+9(x+y)+8$

Solution Let $x+y=p$

Then the equation is p^2+9p+8

Coefficient of $p^2=1$; constant term $=8$

Their product$=1\times 8=8$

Coefficient of $p=9$

Factors of 8	Sum of factors
1, 8	9
The required factors are 1, 8	

$\therefore$ product = 8; sum = 9

$$p^2+9p+8 = p^2+p+8p+8$$
$$= p(p+1)+8(p+1)$$
$$= (p+1)(p+8)$$

substituting, $p=x+y$

$\therefore\ (x+y)^2+9(x+y)+8=(x+y+1)(x+y+8)$

Example 1.21

Factorize : (i) x^3-2x^2-x+2 (ii) x^3+3x^2-x-3

Solution

(i) Let $p(x)=x^3-2x^2-x+2$

$p(x)$ is a cubic polynomial, so it may have three linear factors.

The constant term is 2. The factors of 2 are –1, 1, –2 and 2.

$p(-1)=(-1)^3-2(-1)^2-(-1)+2=-1-2+1+2=0$

$\therefore\ (x+1)$ is a factor of $p(x)$.

$p(1)=(1)^3-2(1)^2-1+2=1-2-1+2=0$

$\therefore\ (x-1)$ is a factor of $p(x)$.

$p(-2)=(-2)^3-2(-2)^2-(-2)+2=-8-8+2+2=-12\neq 0$

$\therefore\ (x+2)$ is not a factor of $p(x)$.

$p(2)=(2)^3-2(2)^2-2+2=8-8-2+2=0$

$\therefore\ (x-2)$ is a factor of $p(x)$.

The three factors of $p(x)$ are $(x+1),(x-1)$ and $(x-2)$

$\therefore\ x^3-2x^2-x+2=(x+1)(x-1)(x-2)$.

Another method

$$x^3-2x^2-x+2=x^2(x-2)-1(x-2)$$

$$= (x-2)(x^2-1)$$

$$= (x-2)(x+1)(x-1) \qquad [(\because a^2-b^2=(a+b)(a-b)]$$

(ii) Let $p(x) = x^3+3x^2-x-3$

$p(x)$ is a cubic polynomial, so it may have three linear factors.

The constant term is –3. The factors of –3 are –1, 1,–3 and 3.

$p(-1) = (-1)^3+3(-1)^2-(-1)-3 = -1+3+1-3 = 0$

$\therefore (x+1)$ is a factor of $p(x)$.

$p(1) = (1)^3+3(1)^2-1-3 = 1+3-1-3 = 0$

$\therefore (x-1)$ is a factor of $p(x)$.

$p(-3) = (-3)^3+3(-3)^2-(-3)-3 = -27+27+3-3 = 0$

$\therefore (x+3)$ is a factor of $p(x)$.

The three factors of $p(x)$ are $(x+1), (x-1)$ and $(x+3)$

$\therefore x^3+3x^2-x-3 = (x+1)(x-1)(x+3)$.

Exercise 1.3

1. Factorize each of the following.

(i) $x^2+15x+14$ (ii) $x^2+13x+30$ (iii) $y^2+7y+12$

(iv) $x^2-14x+24$ (v) $y^2-16y+60$ (vi) $t^2-17t+72$

(vii) $x^2+14x-15$ (viii) $x^2+9x-22$ (ix) $y^2+5y-36$

(x) $x^2-2x-99$ (xi) $m^2-10m-144$ (xii) y^2-y-20

2. Factorize each of the following.

(i) $3x^2+19x+6$ (ii) $5x^2+22x+8$ (iii) $2x^2+9x+10$

(iv) $14x^2+31x+6$ (v) $5y^2-29y+20$ (vi) $9y^2-16y+7$

(vii) $6x^2-5x+1$ (viii) $3x^2-10x+8$ (ix) $3x^2+5x-2$

(x) $2a^2+17a-30$ (xi) $11+5x-6x^2$ (xii) $8x^2+29x-12$

(xiii) $2x^2-3x-14$ (xiv) $18x^2-x-4$ (xv) $10-7x-3x^2$

3. Factorize the following

(i) $(a+b)^2+9(a+b)+14$ (ii) $(p-q)^2-7(p-q)-18$

4. Factorize the following

(i) x^3+2x^2-x-2 (ii) x^3-3x^2-x+3

(iii) x^3+x^2-4x-4 (iv) x^3+5x^2-x-5

1.4 Linear Equations

Recall the linear equations in one variable is of the form $ax + b = 0$, where a, b are constants and $a \neq 0$.

For example, solving $3x + 2 = 8$

$\Longrightarrow 3x = 8 - 2 \Longrightarrow 3x = 6 \Longrightarrow x = \frac{6}{3} \Longrightarrow x = 2$

In fact a linear equation in one variable has a unique solution.

1.4.1 Pair of Linear Equations in Two Variables

In general linear equation in two variables x and y is of the form $ax + by = c$ where a, b and c are constants and $a \neq 0$, $b \neq 0$.

Let us consider a pair of linear equations in two variables x and y.

$$a_1x + b_1y = c_1 \quad (1)$$
$$a_2x + b_2y = c_2 \quad (2)$$

Where a_1, a_2, b_1, b_2, c_1 and c_2 are constants and $a_1 \neq 0$, $b_1 \neq 0$, $a_2 \neq 0$ and $b_2 \neq 0$.

If an ordered pair (x_0, y_0) satisfies both the equations, then (x_0, y_0) is called a solution of these equations. Hence, solving these equations involves the method of finding the ordered pair (x_0, y_0) that satisfies both the equations.

The substitution method, the elimination method and the cross-multiplication method are some of the methods commonly used to solve the system of equations.

In this chapter we consider only the substitution method to solve the linear equations in two variables.

Substitution method

In this method, one of the two variables is expressed interms of the other, using either of the equations. It is then substituted in the other equation and solved.

Example 1.22

Solve the following pair of equations by substitution method.

$$2x + 5y = 2 \text{ and } x + 2y = 3$$

Solution We have

$$2x + 5y = 2 \quad (1)$$
$$x + 2y = 3 \quad (2)$$

Equation (2) becomes, $x = 3 - 2y$ (3)

Substituting x in (1) we get, $2(3 - 2y) + 5y = 2$

$$\Longrightarrow 6 - 4y + 5y = 2$$
$$-4y + 5y = 2 - 6$$
$$\therefore y = -4$$

Substituting $y = -4$ in (3), we get, $x = 3 - 2(-4) = 3 + 8 = 11$

$\therefore$ The solution is $x = 11$ and $y = -4$

Example 1.23

Solve $x + 3y = 16,\ 2x - y = 4$ by using substitution method.

Solution

We have $x + 3y = 16 \quad (1)$

$$2x - y = 4 \quad (2)$$

Equation (1) becomes, $x = 16 - 3y \quad (3)$

Substituting x in (2) we get, $2(16 - 3y) - y = 4$

$$\Longrightarrow 32 - 6y - y = 4$$

$$-6y - y = 4 - 32$$

$$-7y = -28$$

$$y = \frac{-28}{-7} = 4$$

Substituting $y = 4$ in (3) we get, $x = 16 - 3(4)$

$$= 16 - 12 = 4$$

$\therefore$ The solution is $x = 4$ and $y = 4$.

Example 1.24

Solve by substitution method $\frac{1}{x} + \frac{1}{y} = 4$ and $\frac{2}{x} + \frac{3}{y} = 7, \quad x \neq 0, y \neq 0$

Solution

Let $\frac{1}{x} = a$ and $\frac{1}{y} = b$

The given equations become

$$a + b = 4 \quad (1)$$

$$2a + 3b = 7 \quad (2)$$

Equation (1) becomes $b = 4 - a \quad (3)$

Substituting b in (2) we get, $2a + 3(4 - a) = 7$

$$\Longrightarrow 2a + 12 - 3a = 7$$

$$2a - 3a = 7 - 12$$

$$-a = -5 \Longrightarrow a = 5$$

Substituting $a = 5$ in (3) we get, $b = 4 - 5 = -1$

But $\frac{1}{x} = a \Rightarrow x = \frac{1}{a} = \frac{1}{5}$

$$\frac{1}{y} = b \Rightarrow y = \frac{1}{b} = \frac{1}{-1} = -1$$

$\therefore$ The solution is $x = \frac{1}{5}$, $y = -1$

Example 1.25

The cost of a pen and a note book is ₹ 60. The cost of a pen is ₹ 10 less than that of a notebook. Find the cost of each.

Solution

Let the cost of a pen = ₹ x

Let the cost of a note book = ₹ y

From given data we have

$x + y = 60 \qquad (1)$

$x = y - 10 \qquad (2)$

Substituting x in (1) we get, $y - 10 + y = 60$

$\Longrightarrow y + y = 60 + 10 \Longrightarrow 2y = 70$

$\therefore y = \frac{70}{2} = 35$

Substituting $y = 35$ in (2) we get, $x = 35 - 10 = 25$

$\therefore$ The cost of a pen is ₹ 25.

The cost of a note book is ₹ 35.

Example 1.26

The cost of three mathematics books and four science books is ₹ 216. The cost of three mathematics books is the same as that of four science books. Find the cost of each book.

Solution

Let the cost of a mathematics book be ₹ x and cost of a science book be ₹ y.

By given data,

$3x + 4y = 216 \qquad (1)$

$3x = 4y \qquad (2)$

The equation (2) becomes, $x = \frac{4y}{3} \qquad (3)$

Substituting x in (1) we get, $3\left(\frac{4y}{3}\right) + 4y = 216$

$\Longrightarrow 4y + 4y = 216 \Longrightarrow 8y = 216$

$$\therefore\ y = \frac{216}{8} = 27$$

substituting y =27 in (3) we get, $x = \dfrac{4(27)}{3} = 36$

$\therefore$ The cost of one mathematics book = ₹ 36.

The cost of one science book = ₹ 27.

Example 1.27

From Dharmapuri bus stand if we buy 2 tickets to Palacode and 3 tickets to Karimangalam the total cost is ₹ 32, but if we buy 3 tickets to Palacode and one ticket to Karimangalam the total cost is ₹ 27. Find the fares from Dharmapuri to Palacode and to Karimangalam.

Solution

Let the fare from Dharmapuri to Palacode be ₹ x and to Karimangalam be ₹ y.

From the given data, we have

$2x + 3y = 32$ (1)

$3x + y = 27$ (2)

Equation (2) becomes, $y = 27 - 3x$ (3)

Substituting y in (1) we get, $2x + 3(27 - 3x) = 32$

$$\Longrightarrow 2x + 81 - 9x = 32$$

$$2x - 9x = 32 - 81$$

$$-7x = -49$$

$$\therefore x = \frac{-49}{-7} = 7$$

Substituting $x = 7$ in (3) we get, $y = 27 - 3(7) = 27 - 21 = 6$

$\therefore$ The fare from Dharmapuri to Palacode is ₹ 7 and to Karimangalam is ₹ 6.

Example 1.28

The sum of two numbers is 55 and their difference is 7. Find the numbers .

Solution

Let the two numbers be x and y, where $x > y$

By the given data, $x + y = 55$ (1)

$x - y = 7$ (2)

Equation (2) becomes, $x = 7 + y$ (3)

Substituting x in (1) we get, $7 + y + y = 55$

$$\implies 2y = 55 - 7 = 48$$

$$\therefore y = \frac{48}{2} = 24$$

Substituting $y = 24$ in (3) we get, $x = 7 + 24 = 31$.

$\therefore$ The required two numbers are 31 and 24.

Example 1.29

A number consist of two digits whose sum is 11. The number formed by reversing the digits is 9 less than the original number. Find the number.

Solution

Let the tens digit be x and the units digit be y. Then the number is $10x + y$.

Sum of the digits is $x + y = 11$ (1)

The number formed by reversing the digits is $10y + x$.

Given data, $(10x + y) - 9 = 10y + x$

$$\implies 10x + y - 10y - x = 9$$

$$9x - 9y = 9$$

Dividing by 9 on both sides, $x - y = 1$ (2)

Equation (2) becomes $x = 1 + y$ (3)

Substituting x in (1) we get, $1 + y + y = 11$

$$\implies 2y + 1 = 11$$

$$2y = 11 - 1 = 10$$

$$\therefore y = \frac{10}{2} = 5$$

Substituting $y = 5$ in (3) we get, $x = 1 + 5 = 6$

$\therefore$ The number is $10x + y = 10(6) + 5 = 65$

1.5 Linear Inequations in One Variable

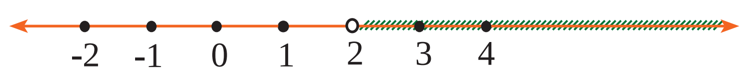
We know that $x + 4 = 6$ is a linear equation in one variable. Solving we get $x = 2$. There is only one such value for x in a linear equation in one variable.

Let us consider,

$x + 4 > 6$

ie $x > 6 - 4$

$x > 2$

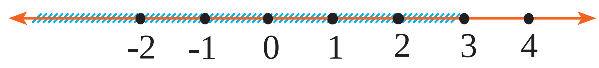


So any real number greater than 2 will satisfy this inequation. We represent those real numbers in the number line.

Unshaded circle indicates that point is not included in the solution set.

Example 1.30

Solve $4(x-1) \leq 8$



Solution

$4(x-1) \leq 8$

Dividing by 4 on both sides,

$x - 1 \leq 2$

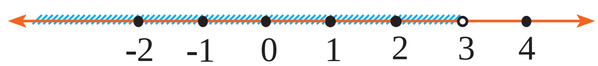
$\Longrightarrow \quad x \leq 2 + 1 \Longrightarrow x \leq 3$

The real numbers less than or equal to 3 are solutions of given inequation.

Shaded circle indicates that point is included in the solution set.

Example 1.31

Solve $3(5-x) > 6$



Solution We have, $3(5-x) > 6$

Dividing by 3 on both sides, $5 - x > 2$

$\Longrightarrow -x > 2 - 5 \Longrightarrow -x > -3$

$\therefore\ x < 3$ (See remark given below)

The real numbers less than 3 are solutions of given inequation.

Remark (i) $-a > -b \Longrightarrow a < b$ (ii) $a < b \Longrightarrow \frac{1}{a} > \frac{1}{b}$ where $a \neq 0,\ b \neq 0$

(iii) $a < b \Longrightarrow ka < kb$ for $k > 0$ (iv) $a < b \Longrightarrow ka > kb$ for $k < 0$

Example 1.32

Solve $3 - 5x \leq 9$

Solution We have, $3 - 5x \leq 9$

$\Longrightarrow -5x \leq 9 - 3 \Longrightarrow -5x \leq 6$

$\Longrightarrow 5x \geq -6 \Longrightarrow x \geq -\frac{6}{5} \Longrightarrow x \geq -1.2$

The real numbers greater than or equal to -1.2 are solutions of given inequation.

Exercise 1.4

1. Solve the following equations by substitution method .

(i) $x + 3y = 10;\ 2x + y = 5$ (ii) $2x + y = 1;\ 3x - 4y = 18$

(iii) $5x + 3y = 21;\ 2x - y = 4$ (iv) $\frac{1}{x} + \frac{2}{y} = 9;\ \frac{2}{x} + \frac{1}{y} = 12 \quad (x \neq 0, y \neq 0)$

(v) $\frac{3}{x} + \frac{1}{y} = 7;\ \frac{5}{x} - \frac{4}{y} = 6\ (x \neq 0, y \neq 0)$

2. Find two numbers whose sum is 24 and difference is 8.

3. A number consists of two digits whose sum is 9. The number formed by reversing the digits exceeds twice the original number by 18. Find the original number.

4 Kavi and Kural each had a number of apples . Kavi said to Kural “If you give me 4 of your apples, my number will be thrice yours”. Kural replied “If you give me 26, my number will be twice yours”. How many did each have with them?.

5. Solve the following inequations.

(i) $2x + 7 > 15$ (ii) $2(x - 2) < 3$ (iii) $2(x + 7) \leq 9$ (iv) $3x + 14 \geq 8$

Exercise 1.5

Choose the Correct Answer

1. The expansion of $(x + 2)(x - 1)$ is

(A) $x^2 - x - 2$ (B) $x^2 + x + 2$ (C) $x^2 + x - 2$ (D) $x^2 - x + 2$

2. The expansion of $(x + 1)(x - 2)(x + 3)$ is

(A) $x^3 + 2x^2 - 5x - 6$ (B) $x^3 - 2x^2 + 5x - 6$

(C) $x^3 + 2x^2 + 5x - 6$ (D) $x^3 + 2x^2 + 5x + 6$

3. $(x - y)(x^2 + xy + y^2)$ is equal to

(A) $x^3 + y^3$ (B) $x^2 + y^2$ (C) $x^2 - y^2$ (D) $x^3 - y^3$

4. Factorization of $x^2 + 2x - 8$ is

(A) $(x + 4)(x - 2)$ (B) $(x - 4)(x + 2)$ (C) $(x + 4)(x + 2)$ (D) $(x - 4)(x - 2)$

5. If one of the factors of $x^2 - 6x - 16$ is $(x + 2)$ then other factor is

(A) $x + 5$ (B) $x - 5$ (C) $x + 8$ (D) $x - 8$

6. If $(2x + 1)$ and $(x - 3)$ are the factors of $ax^2 - 5x + c$, then the values of a and c are respectively

(A) 2,3 (B) –2,3 (C) 2,–3 (D) 1,–3

7. If $x + y = 10$ and $x - y = 2$, then value of x is

(A) 4 (B) –6 (C) –4 (D) 6

8. The solution of $2 - x < 5$ is

(A) $x > -3$ (B) $x < -3$ (C) $x > 3$ (D) $x < 3$

Points to Remember

- $(x + y + z)^2 \equiv x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$
- $(x + y)^3 \equiv x^3 + y^3 + 3xy(x + y)$ $\quad x^3 + y^3 \equiv (x + y)(x^2 - xy + y^2)$
- $(x - y)^3 \equiv x^3 - y^3 - 3xy(x - y)$ $\quad x^3 - y^3 \equiv (x - y)(x^2 + xy + y^2)$
- $x^3 + y^3 + z^3 - 3xyz \equiv (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$
- $(x + a)(x + b)(x + c) \equiv x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$

Activity 1

Objective : To explain the identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ through geometrical representation.

Required materials : Unit cubes.

Procedures : Let us take $a = 3$, $b = 1$

Step 1 :

To represent a^3 make a cube of dimension $a \times a \times a$. ie., $3 \times 3 \times 3$ cubic units

Step 2 :

To represent b^3 make a cube of dimension $b \times b \times b$ ie., $1 \times 1 \times 1$ cubic units

Step 3 :

To represent $a^3 - b^3$ in the following manner

–

=

Step 4 :

To represent $(a-b)a^2$ make a cuboid of dimension $(a–b) \times a \times a$. ie., $2 \times 3 \times 3$ cubic units.

Step 5 :

To represent $(a–b)ab$, make a cuboid of dimension $(a–b) \times a \times b$. ie., $2 \times 3 \times 1$ cubic units

Step 6 :

To represent $(a-b)b^2$ make a cuboid of dimension $(a–b) \times b \times b$. ie., $2 \times 1 \times 1$ cubic units.

Mathematics

Step 7 :

To represent $(a-b)a^2+(a-b)ab+(a-b)b^2=(a-b)(a^2+ab+b^2)$, join all the cuboids from the step 4, 5 and 6.

 +  +  = 

from the observation,

(i) The number of unit cubes in a^3 is 27.

(ii) The number of unit cubes in b^3 is 1.

(iii) The number of unit cubes in a^3-b^3 is 26.

(iv) The number of unit cubes in $(a-b)a^2$ is 18.

(v) The number of unit cubes in $(a-b)ab$ is 6.

(vi) The number of unit cubes in $(a-b)b^2$ is 2.

(vii) The number of unit cubes in $(a-b)a^2+(a-b)ab+(a-b)b^2$ is 18+6+2 = 26.

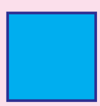
Learning outcomes

It is observed that the number of unit cubes in a^3-b^3 is equal to the number of unit cubes in $(a-b)a^2+(a-b)ab+(a-b)b^2=(a-b)(a^2+ab+b^2)$.

Activity 2

Objective : To know the factorisation of polynomials using paper cuttings.

Required materials : Cut out a paper into three types of sheets as given below.

Type 1: Square sheets of area x^2 sq. units. 

Type 2: Rectangular sheets of area x sq. units whose length is x units and breadth is 1 unit. 

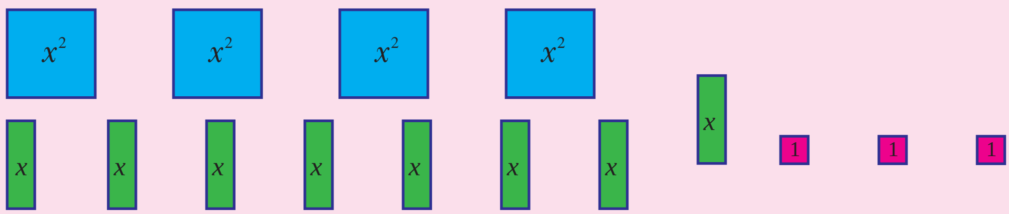
Type 3: Square sheets of area 1 sq. units. 

Procedures :

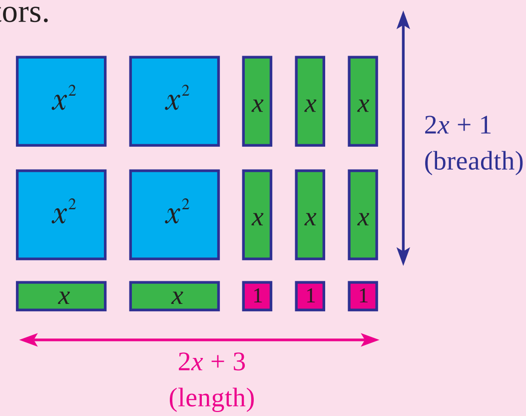
For example,

To factorise $4x^2 + 8x + 3$, the students need to take four x^2 sheets, eight x sheets and three unit sheets.

The sheets selected are given below.



The selected sheets are to be placed such that they form a rectangle. The length and breadth are the required factors.



The sides of the rectangle are $(2x + 3)$ and $(2x + 1)$.

$\therefore\ 4x^2 + 8x + 3 = (2x + 3)(2x + 1)$

Learning outcome

To factorise the polynomials using paper cuttings and also to form a rectangle by joining the sheets whose length $(2x+3)$ and breadth $(2x+1)$ are the required factors of the polynomial.

Factorise the following polynomials using paper cuttings

(i) $2x^2 + 5x + 3$ (ii) $3x^2 + 4x + 1$ (iii) $x^2 + 5x + 6$

Mathematics

Activity 3

Complete the following table by using the given example.

Sl. No.	Quadratic Polynomial $p(x)$	Factors of the quadratic Polynomial		Quadratic equation $p(x) = 0$	Solutions of the quadratic equation
1	$x^2 - 6x - 27$	$x-9$	$x + 3$	$x^2 - 6x - 27 = 0$	9, –3
2	$x^2 + 11x + 24$	$x + 8$			
3	$2x^2 + 7x + 6$		$x + 2$		
4					–3, –2
5	$6x^2 - 31x + 35$		$2x-7$		
6	$x^2 + 19x - 70$	$x - 5$			
7					4, –6
8	$x^2 - 3x - 18$	$x + 3$			
9	$10x^2 - 19x - 117$		$5x + 13$		

Activity 4

Identify the correct factor.

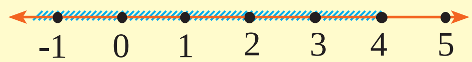
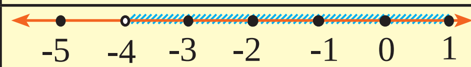
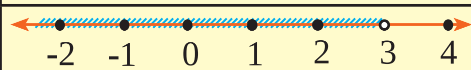
Put (✓) mark if the factors given against the polynomial are correct.

Sl. No.	Polynomial $p(x)$	Factors of $p(x)$					
		$x-1$	$x+1$	$x-2$	$x+2$	$x-3$	$x+3$
1	$x^3 - 2x^2 - x + 2$						
2	$x^3 + x^2 - 4x - 4$						
3	$x^3 + 3x^2 - 3x - 3$						
4	$x^3 - 3x^2 - 4x + 12$						
5	$x^3 + 2x^2 - 5x - 6$						
6	$x^3 - 2x^2 - 5x - 6$						
7	$x^3 + 2x^2 - x - 2$						
8	$x^3 - x^2 - 4x + 4$						
9	$x^3 - 3x^2 - x + 3$						
10	$x^3 + 3x^2 - 4x - 12$						

Activity 5

Writing and Graphing inequalities

Complete the table (the first two rows have been done for you)

Verbal description	Inequality	Graph
x is less than or equal to 4	$x \leq 4$	
$-3x$ less than 12	$-3x < 12$ ie., $x > -4$	
$3-4x$ is greater than or equal to 11		
	$-2x-6 > -8$	
		
	$x \geq 0$	

Exercise 1.1

1. (i) $25x^2 + 4y^2 + 9z^2 + 20xy + 12yz + 30zx$ (ii) $4a^2 + 9b^2 + c^2 + 12ab - 6bc - 4ca$

(iii) $x^2 + 4y^2 + 16z^2 - 4xy + 16yz - 8zx$ (iv) $p^2 + 4q^2 + r^2 - 4pq - 4qr + 2rp$

2. (i) $x^3 + 12x^2 + 39x + 28$ (ii) $p^3 + 4p^2 - 20p - 48$ (iii) $x^3 + x^2 - 17x + 15$

(iv) $x^3 - 7ax^2 + 14a^2x - 8a^3$ (v) $27x^3 + 72x^2 + 51x + 10$ (vi) $8x^3 - 36x^2 - 2x + 105$

3. (i) 19, 111, 189 (ii) –7, 2, 40 (iii) 60, 142, 105 (iv) –100, –5, 6 **4.** –10, –3, 10

5. (i) $27a^3 + 135a^2b + 225ab^2 + 125b^3$ (ii) $64x^3 - 144x^2y + 108xy^2 - 27y^3$

(iii) $8y^3 - 36y + \frac{54}{y} - \frac{27}{y^3}$ **6.** (i) 970299 (ii) 1030301 (iii) 941192 (iv) 1061208

(v) 1006012008 **7.** 793 **8.** –288 **9.** 52 **10.** 36

11. (i) $8x^3 + y^3 + 64z^3 - 24xyz$ (ii) $x^3 - 27y^3 - 125z^3 - 45xyz$ **12.** (i) –486 (ii) 2880

Exercise 1.2

1. (i) $a^2(2a-3b+2c)$ (ii) $16x(1+4xy)$ (iii) $5x^3(2-5xy)$
(iv) $(y-z)(x+a)$ (v) $(p+q)(p+r)$

2. (i) $(x+1)^2$ (ii) $(3x-4y)^2$ (iii) $(b+2)(b-2)$ (iv) $(1+6x)(1-6x)$

3. (i) $(p+q+r)^2$ (ii) $(a-2b-6)^2$ (iii) $(3x-y+1)^2$
(iv) $(2a-b+3c)^2$ (v) $(5x-2y-3z)^2$

4. (i) $(3x+4y)(9x^2-12xy+16y^2)$ (ii) $(m+2)(m^2-2m+4)$
(iii) $(a+5)(a^2-5a+25)$ (iv) $(2x-3y)(4x^2+6xy+9y^2)$
(v) $(x-2y)(x^2+2xy+4y^2)$

Exercise 1.3

1. (i) $(x+1)(x+14)$ (ii) $(x+3)(x+10)$ (iii) $(y+3)(y+4)$
(iv) $(x-2)(x-12)$ (v) $(y-6)(y-10)$ (vi) $(t-8)(t-9)$
(vii) $(x-1)(x+15)$ (viii) $(x-2)(x+11)$ (ix) $(y-4)(y+9)$
(x) $(x+9)(x-11)$ (xi) $(m+8)(m-18)$ (xii) $(y+4)(y-5)$

2. (i) $(3x+1)(x+6)$ (ii) $(5x+2)(x+4)$ (iii) $(x+2)(2x+5)$
(iv) $(14x+3)(x+2)$ (v) $(5y-4)(y-5)$ (vi) $(9y-7)(y-1)$
(vii) $(3x-1)(2x-1)$ (viii) $(3x-4)(x-2)$ (ix) $(3x-1)(x+2)$
(x) $(2a-3)(a+10)$ (xi) $(x+1)(11-6x)$ (xii) $(8x-3)(x+4)$
(xiii) $(x+2)(2x-7)$ (xiv) $(9x+4)(2x-1)$ (xv) $(1-x)(3x+10)$

3. (i) $(a+b+2)(a+b+7)$ (ii) $(p-q+2)(p-q-9)$

4. (i) $(x+1)(x-1)(x+2)$ (ii) $(x+1)(x-1)(x-3)$
(iii) $(x+1)(x+2)(x-2)$ (vi) $(x+1)(x-1)(x+5)$

Exercise 1.4

1. (i) $x=1, y=3$ (ii) $x=2, y=-3$ (iii) $x=3, y=2$ (iv) $x=\frac{1}{5}, y=\frac{1}{2}$
(v) $x=\frac{1}{2}, y=1$ **2.** 16, 8 **3.** 27 **4.** 50, 22

5. (i) $x>4$ (ii) $x<3.5$ (iii) $x\leq -2.5$ (iv) $x\geq -2$

Exercise 1.5

1. C **2.** A **3.** D **4.** A **5.** D **6.** C **7.** D **8.** A

Mathematics

2 TRIGONOMETRY

There is perhaps nothing which so occupies the middle position of mathematics as trigonometry.

– J.F. HERBART

Main Targets

- To understand Trigonometric Ratios
- To understand Trigonometric Ratios of Complementary Angles
- Method of Using Trigonometric Table

ARYABHATTA
(A.D. 476 – 550)

The first use of the idea of 'sine' in the way we use it today was in the work Aryabhatiyam by Aryabhatta, in A.D. 500. Aryabhatta was the first of great Indian Mathematicians. He lived at Kusumapura or Pataliputra in ancient Magadha or modern Patna in Bihar State. The time of birth of Aryabhatta may be fixed at Mesa-Sankranti on March 21, A.D 476. At the age of 23 years Aryabhatta wrote at least two books on astronomy (1) Aryabhatta (2) Aryabhatta-Siddhanta. The Aryabhatta deals with both mathematics and astronomy.

2.1 Introduction

The word *trigonometry* is a derivation from the Greek language and means *measurement of triangles.* This is because trigonometry was initially used to study relationships between different sides of a given triangle. **Hipparchus**, a Greek astronomer and mathematician developed the subject trigonometry and the first trigonometric table was compiled by him. He is now known as "**the Father of Trigonometry**". Trigonometry is an ancient mathematical tool with many applications, even in our modern world. Ancient civilizations used right triangle trigonometry for the purpose of measuring angles and distances in surveying land and astronomy. Trigonometry can be applied in the fields of navigation, planetary motion, and vibrations (sound waves, guitar strings), to name a few.

2.2 Trigonometric Ratios

2.2.1 Angle

We begin this section with the definition of an angle, which will involve several terms and their definitions.

Key Concept **Angle**

An angle is a portion of the 2-dimensional plane which resides between two different directed line segments. The starting position of the angle is known as the *initial side* and the ending position of the angle is known as the *terminal side*. The point from which both of the directed line segments originate is known as the *vertex* of the angle.

See the Fig. 2.1 below for a visual example of an angle.

Here the ray OA is rotated about the point O to the position OB to generate the angle AOB denoted by $\angle AOB$. OA is the initial side, OB is the terminal side and O is the vertex of the angle. We will often use Greek letters to denote angles, such as θ, α, β, etc.,

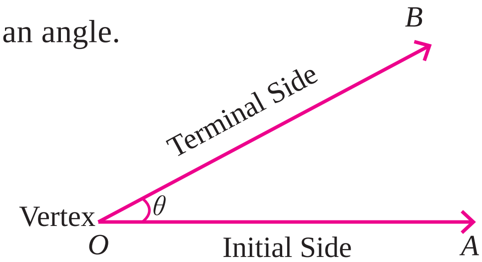


Fig. 2.1

A more common unit of measurement for an angle is the *degree*. This unit was used by the Babylonians as early as 1000 B.C. One degree (written 1°) is the measure of an angle generated by $\frac{1}{360}$ of one revolution.

2.2.2 Pythagoras Theorem

The Pythagoras theorem is a tool to solve for unknown values on right triangle.

Pythagoras Theorem: The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

This relationship is useful in solving many problems and in developing trigonometric concepts.

2.2.3 Trigonometric Ratios

Consider the right triangle in the Fig. 2.2. In the right triangle, we refer to the lengths of the three sides according to how they are placed in relation to the angle θ

- The side that is opposite to the right angle is called the *Hypotenuse*. This is the longest side in a right triangle.
- The side that is opposite to the angle θ is called the *Opposite side*.
- The side that runs alongside the angle θ and which is not the Hypotenuse is called the *Adjacent side*.

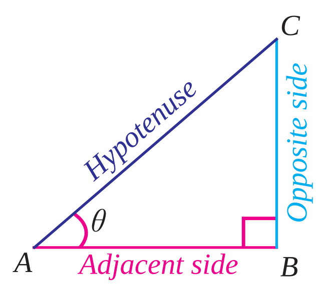
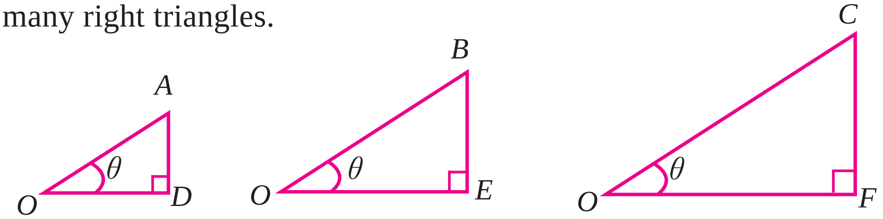


Fig. 2.2

When trigonometry was first developed it was based on *similar right triangles*. All right triangles that have a common acute angle are similar. So, for a given acute angle θ, we have many right triangles.



For each triangle above, the ratios of the corresponding sides are equal.

For example,

$$\frac{AD}{OA} = \frac{BE}{OB} = \frac{CF}{OC}; \qquad \frac{OD}{OA} = \frac{OE}{OB} = \frac{OF}{OC}$$

That is, the ratios depend only on the size of θ and not on the particular right triangle used to compute the ratios. We can form six ratios with the sides of a right triangle. Long ago these ratios were given names.

The ratio $\frac{\textit{Opposite side}}{\textit{Hypotenuse}}$ is called *sine* of angle θ and is denoted by $\sin\theta$

The ratio $\frac{\textit{Adjacent side}}{\textit{Hypotenuse}}$ is called *cosine* of angle θ and is denoted by $\cos\theta$

The ratio $\frac{\textit{Opposite side}}{\textit{Adjacent side}}$ is called *tangent* of angle θ and is denoted by $\tan\theta$

The ratio $\frac{\textit{Hypotenuse}}{\textit{Opposite side}}$ is called *cosecant* of angle θ and is denoted by $\operatorname{cosec}\theta$

The ratio $\frac{\textit{Hypotenuse}}{\textit{Adjacent side}}$ is called *secant* of angle θ and is denoted by $\sec\theta$

The ratio $\frac{\textit{Adjacent side}}{\textit{Opposite side}}$ is called *cotangent* of angle θ and is denoted by $\cot\theta$

Key Concept **Trigonometric Ratios**

Let θ be an acute angle of a right triangle. Then the six trigonometric ratios of θ are as follows

$$\sin\theta = \frac{\textit{Opposite side}}{\textit{Hypotenuse}} \qquad \operatorname{cosec}\theta = \frac{\textit{Hypotenuse}}{\textit{Opposite side}}$$

$$\cos\theta = \frac{\textit{Adjacent side}}{\textit{Hypotenuse}} \qquad \sec\theta = \frac{\textit{Hypotenuse}}{\textit{Adjacent side}}$$

$$\tan\theta = \frac{\textit{Opposite side}}{\textit{Adjacent side}} \qquad \cot\theta = \frac{\textit{Adjacent side}}{\textit{Opposite side}}$$

Reciprocal Relations

The trigonometric ratios $\operatorname{cosec}\theta$, $\sec\theta$ and $\cot\theta$ are reciprocals of $\sin\theta$, $\cos\theta$ and $\tan\theta$ respectively.

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Remark

1. The basic trigonometric ratios $\sin\theta$, $\cos\theta$ and $\tan\theta$ are connected by the relation $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
2. When calculating the trigonometric ratios of an acute angle θ, you may use any right triangle which has θ as one of the angles.
3. Since we defined the trigonometric ratios in terms of ratios of sides, you can think of the units of measurement for those sides as cancelling out in those ratios. This means that the values of the trigonometric functions are unitless numbers.

Example 2.1

Find the six trigonometric ratios of the angle θ in the right triangle *ABC*, as shown at right.

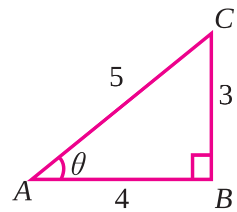


Fig. 2.3

Solution From the Fig. 2.3, the opposite side = 3, the adjacent side = 4 and the hypotenuse = 5.

$$\sin\theta = \frac{BC}{AC} = \frac{3}{5} \qquad \operatorname{cosec}\theta = \frac{AC}{BC} = \frac{5}{3}$$

$$\cos\theta = \frac{AB}{AC} = \frac{4}{5} \qquad \sec\theta = \frac{AC}{AB} = \frac{5}{4}$$

$$\tan\theta = \frac{BC}{AB} = \frac{3}{4} \qquad \cot\theta = \frac{AB}{BC} = \frac{4}{3}$$

Example 2.2

In the right triangle *PQR* as shown at right, find the six trigonometric ratios of the angle θ.

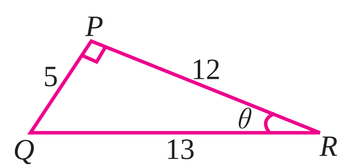


Fig. 2.4

Solution From the Fig. 2.4 the opposite side = 5, the adjacent side = 12 and the hypotenuse = 13.

$$\sin\theta = \frac{PQ}{RQ} = \frac{5}{13} \qquad \operatorname{cosec}\theta = \frac{RQ}{PQ} = \frac{13}{5}$$

$$\cos\theta = \frac{PR}{RQ} = \frac{12}{13} \qquad \sec\theta = \frac{RQ}{PR} = \frac{13}{12}$$

$$\tan\theta = \frac{PQ}{PR} = \frac{5}{12} \qquad \cot\theta = \frac{PR}{PQ} = \frac{12}{5}$$

Example 2.3

From the Fig. 2.5, find the six trigonometric ratios of the angle θ

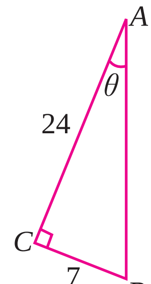


Fig. 2.5

Solution From the Fig. 2.5, $AC = 24$ and $BC = 7$. By Pythagoras theorem

$$AB^2 = BC^2 + CA^2 = 7^2 + 24^2 = 49 + 576 = 625$$

$$\therefore AB = \sqrt{625} = 25$$

We now use the three sides find the six trigonometric ratios of angle θ

$$\sin\theta = \frac{BC}{AB} = \frac{7}{25} \qquad \operatorname{cosec}\theta = \frac{AB}{BC} = \frac{25}{7}$$

$$\cos\theta = \frac{AC}{AB} = \frac{24}{25} \qquad \sec\theta = \frac{AB}{AC} = \frac{25}{24}$$

$$\tan\theta = \frac{BC}{AC} = \frac{7}{24} \qquad \cot\theta = \frac{AC}{BC} = \frac{24}{7}$$

Example 2.4

In ΔABC, right angled at B, $15\sin A = 12$. Find the other five trigonometric ratios of the angle A. Also find the six ratios of the angle C

Solution Given that $15\sin A = 12$, so $\sin A = \frac{12}{15}$. Let us consider ΔABC (see Fig. 2.6), right angled at B, with $BC = 12$ and $AC = 15$. By Pythagoras theorem

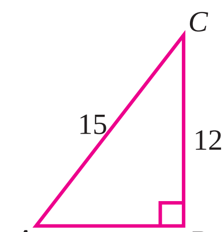


Fig. 2.6

$$AC^2 = AB^2 + BC^2$$

$$15^2 = AB^2 + 12^2$$

$$AB^2 = 15^2 - 12^2 = 225 - 144 = 81$$

$$\therefore AB = \sqrt{81} = 9$$

We now use the three sides to find the six trigonometric ratios of angle A and angle C.

$$\cos A = \frac{AB}{AC} = \frac{9}{15} = \frac{3}{5} \qquad \sin C = \frac{AB}{AC} = \frac{9}{15} = \frac{3}{5}$$

$$\tan A = \frac{BC}{AB} = \frac{12}{9} = \frac{4}{3} \qquad \cos C = \frac{BC}{AC} = \frac{12}{15} = \frac{4}{5}$$

$$\operatorname{cosec} A = \frac{AC}{BC} = \frac{15}{12} = \frac{5}{4} \qquad \tan C = \frac{AB}{BC} = \frac{9}{12} = \frac{3}{4}$$

$$\sec A = \frac{AC}{AB} = \frac{15}{9} = \frac{5}{3} \qquad \operatorname{cosec} C = \frac{AC}{AB} = \frac{15}{9} = \frac{5}{3}$$

$$\cot A = \frac{AB}{BC} = \frac{9}{12} = \frac{3}{4} \qquad \sec C = \frac{AC}{BC} = \frac{15}{12} = \frac{5}{4}$$

$$\cot C = \frac{BC}{AB} = \frac{12}{9} = \frac{4}{3}$$

Example 2.5

In $\triangle PQR$, right angled at Q, PQ=8 and PR=17. Find the six trigonometric ratios of the angle P

Solution Given that PQR is a right triangle, right angled at Q, (see Fig. 2.7), PQ=8 and PR=17. By Pythagoras theorem,

$$\begin{aligned} PR^2 &= PQ^2 + QR^2 \\ 17^2 &= 8^2 + QR^2 \\ QR^2 &= 17^2 - 8^2 \\ &= 289 - 64 = 225 \\ \therefore \quad QR &= \sqrt{225} = 15 \end{aligned}$$

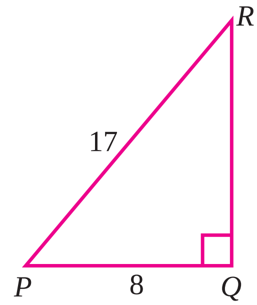


Fig. 2.7

We now use the lengths of the three sides to find the six trigonometric ratios of angle P

$$\sin P = \frac{RQ}{PR} = \frac{15}{17} \qquad \operatorname{cosec} P = \frac{PR}{RQ} = \frac{17}{15}$$

$$\cos P = \frac{PQ}{PR} = \frac{8}{17} \qquad \sec P = \frac{PR}{PQ} = \frac{17}{8}$$

$$\tan P = \frac{RQ}{PQ} = \frac{15}{8} \qquad \cot P = \frac{PQ}{RQ} = \frac{8}{15}$$

Example 2.6

If $\cos A = \frac{35}{37}$, find $\frac{\sec A + \tan A}{\sec A - \tan A}$.

Solution Given that $\cos A = \frac{35}{37}$. Let us consider $\triangle ABC$ (see Fig. 2.8), $\angle B = 90^\circ$, with $AB = 35$ and $AC = 37$. By Pythagoras theorem, we have

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ 37^2 &= 35^2 + BC^2 \\ BC^2 &= 37^2 - 35^2 \\ &= 1369 - 1225 = 144 \\ \therefore BC &= \sqrt{144} = 12 \end{aligned}$$

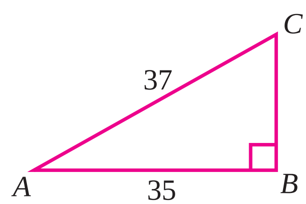


Fig. 2.8

$$\sec A = \frac{AC}{AB} = \frac{37}{35}, \qquad \tan A = \frac{BC}{AB} = \frac{12}{35}$$

Now, $\sec A + \tan A = \frac{37}{35} + \frac{12}{35} = \frac{49}{35}$, $\qquad \sec A - \tan A = \frac{37}{35} - \frac{12}{35} = \frac{25}{35}$

$$\therefore \frac{\sec A + \tan A}{\sec A - \tan A} = \frac{\frac{49}{35}}{\frac{25}{35}} = \frac{49}{35} \times \frac{35}{25} = \frac{49}{25}$$

Example 2.7

If $\tan\theta = \frac{20}{21}$, show that $\frac{1-\sin\theta+\cos\theta}{1+\sin\theta+\cos\theta} = \frac{3}{7}$.

Solution Given that $\tan\theta = \frac{20}{21}$. Let us consider the right triangle *ABC* (see Fig. 2.9), with *AB* = 21 and *BC* = 20. By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2 = 20^2 + 21^2 = 400 + 441 = 841.$$

$$\therefore\ AC = \sqrt{841} = 29.$$

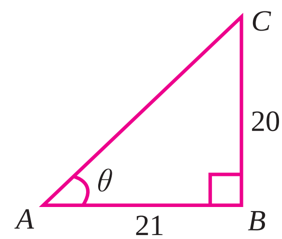


Fig. 2.9

$$\sin\theta = \frac{BC}{AC} = \frac{20}{29}, \qquad \cos\theta = \frac{AB}{AC} = \frac{21}{29}$$

$$1-\sin\theta+\cos\theta = 1-\frac{20}{29}+\frac{21}{29} = \frac{29-20+21}{29} = \frac{30}{29}$$

$$1+\sin\theta+\cos\theta = 1+\frac{20}{29}+\frac{21}{29} = \frac{29+20+21}{29} = \frac{70}{29}$$

$$\frac{1-\sin\theta+\cos\theta}{1+\sin\theta+\cos\theta} = \frac{\frac{30}{29}}{\frac{70}{29}} = \frac{30}{29}\times\frac{29}{70} = \frac{30}{70} = \frac{3}{7}$$

2.3 Trigonometric Ratios of Some Special Angles

For certain special angles such as $30^\circ, 45^\circ$ and 60°, which are frequently seen in applications, we can use geometry to determine the trigonometric ratios.

2.3.1 Trigonometric Ratios of 30° and 60°

Let $\triangle ABC$ be an equilateral triangle whose sides have length *a* (see Fig. 2.10). Draw $AD \perp BC$, then *D* bisects the side *BC*. So, $BD = DC = \frac{a}{2}$ and $\angle BAD = \angle DAC = 30^\circ$. Now, in right triangle *ADB*, $\angle BAD = 30^\circ$ and $BD = \frac{a}{2}$. So,

$$AB^2 = AD^2 + BD^2$$

$$a^2 = AD^2 + \left[\frac{a}{2}\right]^2$$

$$AD^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$$

$$\therefore\ AD = \frac{\sqrt{3}}{2}a$$

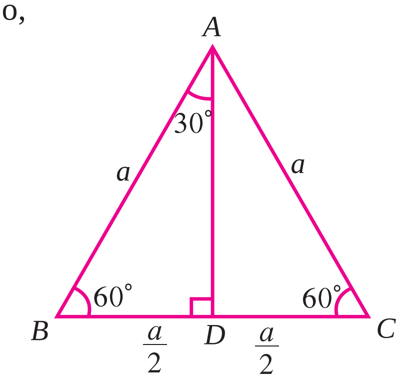


Fig. 2.10

Hence, we can find the trigonometric ratios of angle 30° from the right triangle BAD

$$\sin 30^\circ = \frac{BD}{AB} = \frac{\frac{a}{2}}{a} = \frac{1}{2} \qquad \operatorname{cosec} 30^\circ = \frac{1}{\sin 30^\circ} = 2$$

$$\cos 30^\circ = \frac{AD}{AB} = \frac{\frac{\sqrt{3}}{2}a}{a} = \frac{\sqrt{3}}{2} \qquad \sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}}$$

$$\tan 30^\circ = \frac{BD}{AD} = \frac{\frac{a}{2}}{\frac{\sqrt{3}}{2}a} = \frac{1}{\sqrt{3}} \qquad \cot 30^\circ = \frac{1}{\tan 30^\circ} = \sqrt{3}$$

In $\triangle ABD$, $\angle ABD = 60^\circ$. So, we can determine the trigonometric ratios of angle 60°

$$\sin 60^\circ = \frac{AD}{AB} = \frac{\frac{\sqrt{3}}{2}a}{a} = \frac{\sqrt{3}}{2} \qquad \operatorname{cosec} 60^\circ = \frac{1}{\sin 60^\circ} = \frac{2}{\sqrt{3}}$$

$$\cos 60^\circ = \frac{BD}{AB} = \frac{\frac{a}{2}}{a} = \frac{1}{2} \qquad \sec 60^\circ = \frac{1}{\cos 60^\circ} = 2$$

$$\tan 60^\circ = \frac{AD}{BD} = \frac{\frac{\sqrt{3}}{2}a}{\frac{a}{2}} = \sqrt{3} \qquad \cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}}$$

2.3.2 Trigonometric Ratio of 45°

If an acute angle of a right triangle is 45°, then the other acute angle is also 45°. Thus the triangle is isosceles. Let us consider the triangle ABC with $\angle B = 90^\circ$, $\angle A = \angle C = 45^\circ$. Then $AB = BC$. Let $AB = BC = a$. By Pythagoras theorem,

$$AC^2 = AB^2 + BC^2$$

$$= a^2 + a^2 = 2a^2$$

$$\therefore AC = a\sqrt{2}$$

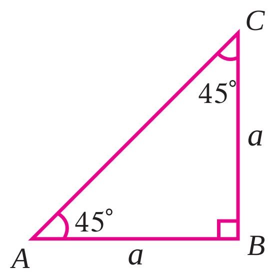


Fig. 2.11

From Fig. 2.11, we can easily determine the trigonometric ratios of 45°

$\sin 45^\circ = \dfrac{BC}{AC} = \dfrac{a}{a\sqrt{2}} = \dfrac{1}{\sqrt{2}}$ $\qquad$ $\operatorname{cosec} 45^\circ = \dfrac{1}{\sin 45^\circ} = \sqrt{2}$

$\cos 45^\circ = \dfrac{AB}{AC} = \dfrac{a}{a\sqrt{2}} = \dfrac{1}{\sqrt{2}}$ $\qquad$ $\sec 45^\circ = \dfrac{1}{\cos 45^\circ} = \sqrt{2}$

$\tan 45^\circ = \dfrac{BC}{AB} = \dfrac{a}{a} = 1$ $\qquad$ $\cot 45^\circ = \dfrac{1}{\tan 45^\circ} = 1$

2.3.3 Trigonometric Ratios of 0°and 90°

Consider Fig. 2.12 which shows a circle of radius 1 unit centered at the orgin. Let P be a point on the circle in the first quadrant with coordinates (x, y).

We drop a perpendicular PQ from P to the x-axis in order to form the right triangle OPQ. Let $\angle POQ = \theta$, then

$$\sin\theta = \frac{PQ}{OP} = \frac{y}{1} = y \text{ (} y \text{ coordinate of } P\text{)}$$

$$\cos\theta = \frac{OQ}{OP} = \frac{x}{1} = x \text{ (} x \text{ coordinate of } P\text{)}$$

$$\tan\theta = \frac{PQ}{OQ} = \frac{y}{x}$$

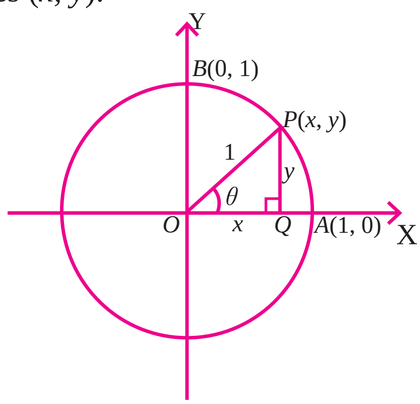


Fig. 2.12

If OP coincides with OA, then angle $\theta = 0^\circ$. Since the coordinates of A are (1, 0), we have

$\sin 0^\circ = 0$ (y coordinate of A) $\qquad$ $\operatorname{cosec} 0^\circ$ is not defined

$\cos 0^\circ = 1$ (x coordinate of A) $\qquad$ $\sec 0^\circ = 1$

$\tan 0^\circ = \dfrac{\sin 0^\circ}{\cos 0^\circ} = \dfrac{0}{1} = 0.$ $\qquad$ $\cot 0^\circ$ is not defined

If OP coincides with OB, then angle $\theta = 90^\circ$. Since the coordinates of B are (0, 1), we have

$\sin 90^\circ = 1$ (y coordinate of B) $\qquad$ $\operatorname{cosec} 90^\circ = 1$

$\cos 90^\circ = 0$ (x coordinate of B) $\qquad$ $\sec 90^\circ$ is not defined

$\tan 90^\circ = \dfrac{\sin 90^\circ}{\cos 90^\circ} = \dfrac{1}{0}$ is not defined. $\qquad$ $\cot 90^\circ = 0$

The six trigonometric ratios of angles $0^\circ, 30^\circ, 45^\circ, 60^\circ$ and 90° are provided in the following table.

angle θ / ratio	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined
$\operatorname{cosec}\theta$	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined
$\cot\theta$	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Example 2.8

Evaluate $\sin^2 45^\circ + \tan^2 45^\circ + \cos^2 45^\circ$.

Solution We know, $\sin 45^\circ = \frac{1}{\sqrt{2}}$, $\tan 45^\circ = 1$ and $\cos 45^\circ = \frac{1}{\sqrt{2}}$

$$\therefore\ \sin^2 45^\circ + \tan^2 45^\circ + \cos^2 45^\circ = \left(\frac{1}{\sqrt{2}}\right)^2 + (1)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{2} + 1 + \frac{1}{2} = 2$$

Note We write $(\sin\theta)^2$ as $\sin^2\theta$

Example 2.9

Evaluate $\dfrac{12\cos^2 30^\circ - 2\tan^2 60^\circ}{4\sec^2 45^\circ}$.

Solution We know, $\cos 30^\circ = \frac{\sqrt{3}}{2}$, $\tan 60^\circ = \sqrt{3}$ and $\sec 45^\circ = \sqrt{2}$

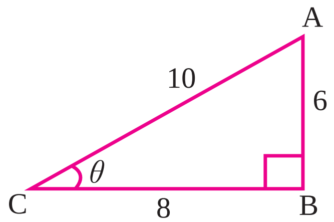
$$\therefore\ \frac{12\cos^2 30^\circ - 2\tan^2 60^\circ}{4\sec^2 45^\circ} = \frac{\left(12 \times \left(\frac{\sqrt{3}}{2}\right)^2\right) - \left(2 \times (\sqrt{3})^2\right)}{4 \times (\sqrt{2})^2}$$

$$= \frac{\left(12 \times \frac{3}{4}\right) - (2 \times 3)}{4 \times 2}$$

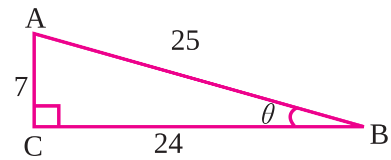
$$= \frac{9-6}{8} = \frac{3}{8}$$

Exercise 2.1

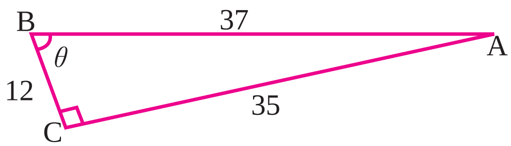
1. From the following diagrams, find the trigonometric ratios of the angle θ



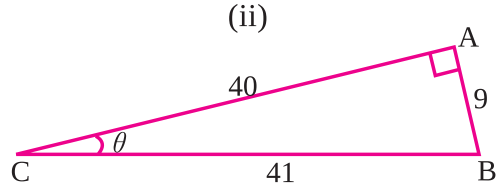
(i)



(ii)



(iii)



(iv)

2. Find the other trigonometric ratios of the following

(i) $\sin A = \frac{9}{15}$ (ii) $\cos A = \frac{15}{17}$ (iii) $\tan P = \frac{5}{12}$

(iv) $\sec\theta = \frac{17}{8}$ (v) $\operatorname{cosec}\theta = \frac{61}{60}$ (vi) $\sin\theta = \frac{x}{y}$.

3. Find the value of θ, if

(i) $\sin\theta = \frac{1}{\sqrt{2}}$ (ii) $\sin\theta = 0$ (iii) $\tan\theta = \sqrt{3}$ (iv) $\cos\theta = \frac{\sqrt{3}}{2}$.

4. In $\triangle ABC$, right angled at B, $AB = 10$ and $AC = 26$. Find the six trigonometric ratios of the angles A and C.

5. If $5\cos\theta - 12\sin\theta = 0$, find $\frac{\sin\theta + \cos\theta}{2\cos\theta - \sin\theta}$.

6. If $29\cos\theta = 20$, find $\sec^2\theta - \tan^2\theta$.

7. If $\sec\theta = \frac{26}{10}$, find $\frac{3\cos\theta + 4\sin\theta}{4\cos\theta - 2\sin\theta}$.

8. If $\tan\theta = \frac{a}{b}$, find $\sin^2\theta + \cos^2\theta$.

9. If $\cot\theta = \frac{15}{8}$, evaluate $\frac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)}$.

10. In triangle PQR, right angled at Q, if $\tan P = \frac{1}{\sqrt{3}}$ find the value of

(i) $\sin P\cos R + \cos P\sin R$ (ii) $\cos P\cos R - \sin P\sin R$.

11. If $\sec\theta = \frac{13}{5}$, show that $\frac{2\sin\theta - 3\cos\theta}{4\sin\theta - 9\cos\theta} = 3$.

12. If $\sec A = \frac{17}{8}$, prove that $1 - 2\sin^2 A = 2\cos^2 - 1$.

13. Evaluate.

(i) $\sin 45^\circ + \cos 45^\circ$

(ii) $\sin 60^\circ \tan 30^\circ$

(iii) $\dfrac{\tan 45^\circ}{\tan 30^\circ + \tan 60^\circ}$

(iv) $\cos^2 60^\circ \sin^2 30^\circ + \tan^2 30^\circ \cot^2 60^\circ$

(v) $6\cos^2 90^\circ + 3\sin^2 90^\circ + 4\tan^2 45^\circ$

(vi) $\dfrac{4\cot^2 60^\circ + \sec^2 30^\circ - 2\sin^2 45^\circ}{\sin^2 60^\circ + \cos^2 45^\circ}$

(vii) $\dfrac{\tan^2 60^\circ + 4\cos^2 45^\circ + 3\sec^2 30^\circ + 5\cos^2 90^\circ}{\operatorname{cosec} 30^\circ + \sec 60^\circ - \cot^2 30^\circ}$

(viii) $4(\sin^4 30^\circ + \cos^4 60^\circ) - 3(\cos^2 45^\circ - \sin^2 90^\circ)$.

14. Verify the following equalities.

(i) $\sin^2 30^\circ + \cos^2 30^\circ = 1$

(ii) $1 + \tan^2 45^\circ = \sec^2 45^\circ$

(iii) $\cos 60^\circ = 1 - 2\sin^2 30^\circ = 2\cos^2 30^\circ - 1$

(iv) $\cos 90^\circ = 1 - 2\sin^2 45^\circ = 2\cos^2 45^\circ - 1$

(v) $\dfrac{\cos 60^\circ}{1 + \sin 60^\circ} = \dfrac{1}{\sec 60^\circ + \tan 60^\circ}$

(vi) $\dfrac{1 - \tan^2 60^\circ}{1 + \tan^2 60^\circ} = 2\cos^2 60^\circ - 1$

(vii) $\dfrac{\sec 30^\circ + \tan 30^\circ}{\sec 30^\circ - \tan 30^\circ} = \dfrac{1 + \sin 30^\circ}{1 - \sin 30^\circ}$

(viii) $\tan^2 60^\circ - 2\tan^2 45^\circ - \cot^2 30^\circ + 2\sin^2 30^\circ + \frac{3}{4}\operatorname{cosec}^2 45^\circ = 0$

(ix) $4\cot^2 45^\circ - \sec^2 60^\circ + \sin^2 60^\circ + \cos^2 60^\circ = 1$

(x) $\sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ = \sin 90^\circ$.

2.4 Trigonometric Ratios for Complementary Angles

Two acute angles are complementary to each other if their sums are equal to 90°. In a right triangle the sum of the two acute angles is equal to 90°. So, the two acute angles of a right triangle are always complementary to each other.

Let ABC be a right triangle, right angled at B (see Fig. 6.13). If $\angle ACB = \theta$, then $\angle BAC = 90^\circ - \theta$ and hence the angles $\angle BAC$ and $\angle ACB$ are complementary.

We have

$$\left.\begin{array}{llll} \sin\theta = \dfrac{AB}{AC} & & \operatorname{cosec}\theta = \dfrac{AC}{AB} \\ \cos\theta = \dfrac{BC}{AC} & & \sec\theta = \dfrac{AC}{BC} \\ \tan\theta = \dfrac{AB}{BC} & & \cot\theta = \dfrac{BC}{AB} \end{array}\right\} \quad (1)$$

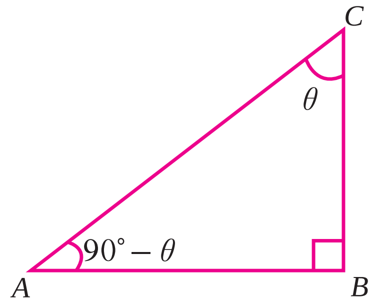


Fig. 2.13

Similarly, for the angle $(90^\circ - \theta)$, we have

$$\left.\begin{array}{llll} \sin(90^\circ - \theta) = \dfrac{BC}{AC} & & \operatorname{cosec}(90^\circ - \theta) = \dfrac{AC}{BC} \\ \cos(90^\circ - \theta) = \dfrac{AB}{AC} & & \sec(90^\circ - \theta) = \dfrac{AC}{AB} \\ \tan(90^\circ - \theta) = \dfrac{BC}{AB} & & \cot(90^\circ - \theta) = \dfrac{AB}{BC} \end{array}\right\} \quad (2)$$

Comparing the equations in (1) and (2) we get,

$$\begin{array}{ll} \sin\theta = \dfrac{AB}{AC} = \cos(90^\circ - \theta) & \operatorname{cosec}\theta = \dfrac{AC}{AB} = \sec(90^\circ - \theta) \\ \cos\theta = \dfrac{BC}{AC} = \sin(90^\circ - \theta) & \sec\theta = \dfrac{AC}{BC} = \operatorname{cosec}(90^\circ - \theta) \\ \tan\theta = \dfrac{AB}{BC} = \cot(90^\circ - \theta) & \cot\theta = \dfrac{BC}{AB} = \tan(90^\circ - \theta) \end{array}$$

Key Concept **Trigonometric Ratios of Complementary Angles**

Let θ be an acute angle of a right triangle. Then we have the following identities for trigonometric ratios of complementary angles.

$$\begin{array}{ll} \sin\theta = \cos(90^\circ - \theta) & \operatorname{cosec}\theta = \sec(90^\circ - \theta) \\ \cos\theta = \sin(90^\circ - \theta) & \sec\theta = \operatorname{cosec}(90^\circ - \theta) \\ \tan\theta = \cot(90^\circ - \theta) & \cot\theta = \tan(90^\circ - \theta) \end{array}$$

Example 2.10

Evaluate $\dfrac{\cos 56^\circ}{\sin 34^\circ}$.

Solution The angles 56° and $34^\circ$are complementary. So, using trigonometric ratios of complementary angles $\cos 56^\circ = \cos(90^\circ - 34^\circ) = \sin 34^\circ$. Hence $\dfrac{\cos 56^\circ}{\sin 34^\circ} = \dfrac{\sin 34^\circ}{\sin 34^\circ} = 1$

Example 2.11

Evaluate $\dfrac{\tan 25^\circ}{\cot 65^\circ}$

Solution We write $\tan 25^\circ = \tan(90^\circ - 65^\circ) = \cot 65^\circ$. Hence,

$$\frac{\tan 25^\circ}{\cot 65^\circ} = \frac{\cot 65^\circ}{\cot 65^\circ} = 1$$

Example 2.12

Evaluate $\dfrac{\cos 65^\circ \sin 18^\circ \cos 58^\circ}{\cos 72^\circ \sin 25^\circ \sin 32^\circ}$.

Solution Using trigonometric ratios of complementary angles, we get

$$\cos 65^\circ = \cos(90^\circ - 25^\circ) = \sin 25^\circ,$$
$$\sin 18^\circ = \sin(90^\circ - 72^\circ) = \cos 72^\circ$$
$$\cos 58^\circ = \cos(90^\circ - 32^\circ) = \sin 32^\circ.$$
$$\therefore \frac{\cos 65^\circ \sin 18^\circ \cos 58^\circ}{\cos 72^\circ \sin^\circ 25^\circ \sin 32^\circ} = \frac{\sin 25^\circ \cos 72^\circ \sin 32^\circ}{\cos 72^\circ \sin 25^\circ \sin 32^\circ} = 1$$

Example 2.13

Show that $\tan 35^\circ \tan 60^\circ \tan 55^\circ \tan 30^\circ = 1$.

Solution We have $\tan 35^\circ = \tan(90^\circ - 55^\circ) = \cot 55^\circ$

$$\tan 60^\circ = \tan(90^\circ - 30^\circ) = \cot 30^\circ$$
$$\therefore \tan 35^\circ \tan 60^\circ \tan 55^\circ \tan 30^\circ = \cot 55^\circ \cot 30^\circ \tan 55^\circ \tan 30^\circ$$
$$= \frac{1}{\tan 55^\circ} \times \frac{1}{\tan 30^\circ} \times \tan 55^\circ \times \tan 30^\circ = 1$$

Example 2.14

If $\operatorname{cosec} A = \sec 25^\circ$, find A.

Solution We have $\operatorname{cosec} A = \sec(90^\circ - A)$. So,

$$\sec(90^\circ - A) = \sec 25^\circ \implies 90^\circ - A = 25^\circ$$
$$\therefore A = 90^\circ - 25^\circ = 65^\circ$$

Note In Example 2.14, the value of A is obtained not by cancelling sec on both sides but using uniqueness of trigonometric ratios for acute angles. That is, if α and β are acute angles,

$$\sin \alpha = \sin \beta \implies \alpha = \beta$$
$$\cos \alpha = \cos \beta \implies \alpha = \beta, \text{ etc.}$$

Example 2.15

If $\sin A = \cos 33^\circ$ find A

Solution We have $\sin A = \cos(90^\circ - A)$. So,

$$\cos(90^\circ - A) = \cos 33^\circ \implies 90^\circ - A = 33^\circ$$
$$\therefore A = 90^\circ - 33^\circ = 57^\circ$$

Exercise 2.2

1. Evaluate

(i) $\dfrac{\sin 36^\circ}{\cos 54^\circ}$ (ii) $\dfrac{\operatorname{cosec} 10^\circ}{\sec 80^\circ}$ (iii) $\sin\theta \sec(90^\circ - \theta)$

(iv) $\dfrac{\sec 20^\circ}{\operatorname{cosec} 70^\circ}$ (v) $\dfrac{\sin 17^\circ}{\cos 73^\circ}$ (vi) $\dfrac{\tan 46^\circ}{\cot 44^\circ}$.

2. Simplify

(i) $\cos 38^\circ \cos 52^\circ - \sin 38^\circ \sin 52^\circ$ (ii) $\dfrac{\cos 80^\circ}{\sin 10^\circ} + \cos 59^\circ \operatorname{cosec} 31^\circ$

(iii) $\dfrac{\sin 36^\circ}{\cos 54^\circ} - \dfrac{\tan 54^\circ}{\cot 36^\circ}$ (iv) $3\dfrac{\tan 67^\circ}{\cot 23^\circ} + \dfrac{1}{2}\dfrac{\sin 42^\circ}{\cos 48^\circ} + \dfrac{5}{2}\dfrac{\operatorname{cosec} 61^\circ}{\sec 29^\circ}$

(v) $\dfrac{\cos 37^\circ}{\sin 53^\circ} \times \dfrac{\sin 18^\circ}{\cos 72^\circ}$ (vi) $2\dfrac{\sec(90^\circ - \theta)}{\operatorname{cosec}\theta} + 7\dfrac{\cos(90^\circ - \theta)}{\sin\theta}$

(vii) $\dfrac{\sec(90^\circ - \theta)}{\sin(90^\circ - \theta)} \times \dfrac{\cos\theta}{\tan(90^\circ - \theta)} - \sec\theta$ (viii) $\dfrac{\sin 35^\circ}{\cos 55^\circ} + \dfrac{\cos 55^\circ}{\sin 35^\circ} - 2\cos^2 60^\circ$

(ix) $\cot 12^\circ \cot 38^\circ \cot 52^\circ \cot 60^\circ \cot 78^\circ$.

3. Find A if

(i) $\sin A = \cos 30^\circ$ (ii) $\tan 49^\circ = \cot A$ (iii) $\tan A \tan 35^\circ = 1$

(iv) $\sec 35^\circ = \operatorname{cosec} A$ (v) $\operatorname{cosec} A \cos 43^\circ = 1$ (vi) $\sin 20^\circ \tan A \sec 70^\circ = \sqrt{3}$.

4. Show that

(i) $\cos 48^\circ - \sin 42^\circ = 0$ (ii) $\cos 20^\circ \cos 70^\circ - \sin 70^\circ \sin 20^\circ = 0$

(iii) $\sin(90^\circ - \theta)\tan\theta = \sin\theta$ (iv) $\dfrac{\cos(90^\circ - \theta)\tan(90^\circ - \theta)}{\cos\theta} = 1$.

2.5 Method of Using Trigonometric Table

We have computed the trigonometric ratios for angles 0°, 30°, 45°, 60° and 90°. In our daily life, we come across situations, wherein we need to solve right triangles which have angles different from 0°, 30°, 45°, 60° and 90°. To apply the results of trigonometric ratios to these situations, we need to know the values of trigonometric ratios of all the acute angles. Trigonometrical tables indicating approximate values of sines, cosines and tangents of all the acute angles have been provided at the end of the book.

To express fractions of degrees, One degree is divided into 60 minutes and One minute is divided into 60 seconds. One minute is denoted by $1'$and One second is denoted by $1''$.

Therefore, $1^\circ = 60'$ and $1' = 60''$

The trigonometrical tables give the values, correct to four places of decimals of all the three tirgonometric ratios for angles from 0° to 90° spaced at intervals of $6'$. A trigonometric table consists of three parts.

(i) A column on the extreme left which contains degrees from 0° to 90°

(ii) Ten columns headed by $0', 6', 12', 18', 24', 30', 36', 42', 48'$ and $54'$ respectively

(iii) Five columns under the head *Mean difference* and these five columns are headed by $1'$, $2'$, $3'$, $4'$ and $5'$

The ten columns mentioned in (ii) provide the values for sine, cosine and tangent of angles in multiple of $6'$. For angles containing other numbers of minutes, the appropriate adjustment is obtained from the mean difference columns.

The mean difference is to be added in the case of sine and tangent, while it is to be subtracted in the case of cosine.

Example 2.16

Find the value of $\sin 46^\circ 51'$

Solution The relevant part of the sine table is given below.

	$0'$	$6'$	$12'$	$18'$	$24'$	$30'$	$36'$	$42'$	$48'$	$54'$	Mean Diff.
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1 2 3 4 5
46°									0.7290		6

Write $46^\circ 51' = 46^\circ 48' + 3'$. From the table we have

$$\sin 46^\circ 48' = 0.7290$$

Mean Difference for $3' = 0.0006$

$\therefore \sin 46^\circ 51' = 0.7290 + 0.0006 = 0.7296$

Example 2.17

Find the value of $\cos 37^\circ 16'$

Solution

	$0'$	$6'$	$12'$	$18'$	$24'$	$30'$	$36'$	$42'$	$48'$	$54'$	Mean Diff.
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1 2 3 4 5
37°			0.7965								7

Write $37^\circ 16' = 37^\circ 12' + 4'$. From the table

$$\cos 37^\circ 12' = 0.7965$$

Mean Difference for $4' = 0.0007$

Since $\cos\theta$ decreases from 1 to 0 as θ increases from 0° to 90°, we must subtract the Mean Difference.

$\therefore \cos 37^\circ 16' = 0.7965 - 0.0007 = 0.7958$

Example 2.18

Find the value of $\tan 25^\circ 15'$

Solution

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Diff.
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1 2 3 4 5
25°			0.4706								11

Write $25^\circ 15' = 25^\circ 12' + 3'$. From the table

$$\tan 25^\circ 12' = 0.4706$$

$$\text{Mean Difference for } 3' = 0.0011$$

$$\therefore \tan 25^\circ 15' = 0.4706 + 0.011 = 0.4717$$

Example 2.19

If $\sin\theta = 0.0958$, find the angle θ.

Solution From the sine table, we find 0.0958 is corresponding to $\sin 5^\circ 30'$.

$$\Longrightarrow \sin 5^\circ 30' = 0.0958$$

$$\therefore \theta = 5^\circ 30'$$

Example 2.20

If $\sin\theta = 0.0987$, find the angle θ.

Solution From the sine table, we find the value 0.0993 is corresponding to $\sin 5^\circ 42'$ and 0.0006 is corresponding to $2'$. So,

$$\sin\theta = 0.0987$$

$$= 0.0993 - 0.0006$$

$$= \sin 5^\circ 42' - \text{Mean Difference for } 2'$$

$$\sin\theta = \sin 5^\circ 40'$$

$$\therefore \theta = 5^\circ 40'$$

Example 2.21

Find the angle θ if $\tan\theta = 0.4040$

Solution From the tangent table, we find the value 0.4040 is corresponding to $\tan 22^\circ 0'$.

$$\Longrightarrow \tan 22^\circ = 0.4040$$

$$\therefore \theta = 22^\circ$$

Mathematics

Example 2.22

Simplify $\sin 30^\circ 30' + \cos 5^\circ 33'$.

Solution From the sine table $\sin 30^\circ 30' = 0.5075$. And from the cosine table $\cos 5^\circ 30' = 0.9954$ and Mean Difference for $3' = 0.0001$. So,

$$\cos 5^\circ 33' = 0.9954 - 0.0001 = 0.9953$$

$$\therefore \sin 30^\circ 30' + \cos 5^\circ 33' = 0.5075 + 0.9953 = 1.5028$$

Example 2.23

Simplify $\cos 70^\circ 12' + \tan 48^\circ 54'$.

Solution From the cosine and tangent tables, we find

$$\cos 70^\circ 12' = 0.3387, \quad \tan 48^\circ 54' = 1.1463$$

$$\therefore \cos 70^\circ 12' + \tan 48^\circ 54' = 0.3387 + 1.1463 = 1.4850$$

Example 2.24

Find the area of the right triangle given in Fig. 2.14.

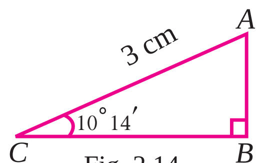


Fig. 2.14

Solution From the Fig. 2.14., $\sin\theta = \dfrac{AB}{AC} \Rightarrow \sin 10^\circ 14' = \dfrac{AB}{3}$

From the sine table, $\sin 10^\circ 12' = 0.1771$ and Mean Difference for $2' = 0.0006$

$$\therefore \sin 10^\circ 14' = 0.1777 \Longrightarrow 0.1777 = \frac{AB}{3}$$

$$\therefore AB = 0.1777 \times 3 = 0.5331 \text{ cm}$$

$$\cos\theta = \frac{BC}{AC} \Longrightarrow \cos 10^\circ 14' = \frac{BC}{3}$$

From the cosine table, $\cos 10^\circ 12' = 0.9842$ and Mean Difference for $2' = 0.0001$

$$\therefore \cos 10^\circ 14' = 0.9842 - 0.0001 = 0.9841$$

$$0.9841 = \frac{BC}{3}$$

$$\therefore BC = 0.9841 \times 3 = 2.9523 \text{ cm}$$

Area of the right rriangle $= \dfrac{1}{2}bh = \dfrac{1}{2} \times 2.9523 \times 0.5331$

$= 0.786935565$

$\therefore$ Area of the triangle is 0.7869 cm^2 (approximately)

Example 2.25

Find the length of the chord of a circle of radius 6 cm subtending an angle of 165° at the centre.

Solution Let AB be the chord of a circle of radius 6 cm with O as centre. Draw $OC \perp AB$. Therefore C is the mid point of AB and $\angle AOB = 165^\circ$. Then

$$\angle AOC = \frac{165^\circ}{2} = 82^\circ 30'$$

In the right triangle OCA,

$$\sin 82^\circ 30' = \frac{AC}{OA} \Longrightarrow AC = \sin 82^\circ 30' \times OA$$

$$AC = 0.9914 \times 6 = 5.9484 \text{ cm}$$

$\therefore$ Length of the chord AB = AC $\times 2 = 5.9484 \times 2 = 11.8968$ cm

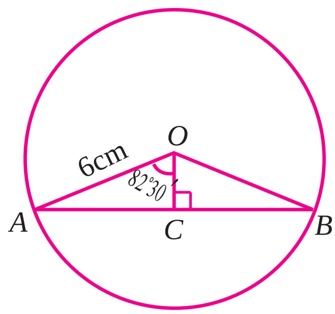


Fig. 2.15

Example 2.26

Find the length of the side of a regular polygon of 9 sides inscribed in a circle of radius 8 units.

Solution Let AB be a side of the regular polygon with 9 sides in the circle of radius 8 units. If O is a centre of the circle, then $\angle AOB = \dfrac{360^\circ}{9} = 40^\circ$. Draw $OC \perp AB$ then

$$\angle AOC = \frac{40^\circ}{2} = 20^\circ$$

$$\sin 20^\circ = \frac{AC}{OA} = \frac{AC}{8}$$

$$\text{i.e., } 0.3420 = \frac{AC}{8}$$

$$AC = 0.3420 \times 8 = 2.736$$

$\therefore$ Length of the side $AB = 2 \times AC = 2 \times 2.736 = 5.472$ units

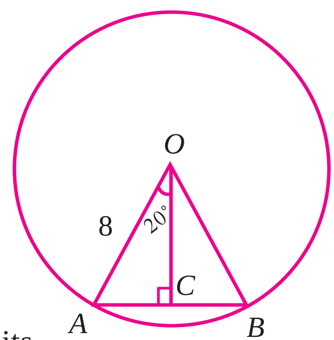


Fig. 2.16

Example 2.27

Find the radius of the incircle of a regular hexagon of side 6 cm.

Solution Let AB be the side of the regular hexagon and let O be the centre of the incircle. Draw $OC \perp AB$. If r is the radius of the circle, then $OC = r$. So,

$$\angle AOB = \frac{360^\circ}{6} = 60^\circ$$

$$\therefore \angle AOC = \frac{60^\circ}{2} = 30^\circ$$

$$\therefore \tan 30^\circ = \frac{AC}{r}$$

$$\text{i.e., } \frac{1}{\sqrt{3}} = \frac{3}{r}$$

$$\therefore r = 3 \times 1.732 = 5.196 \text{ cm}$$

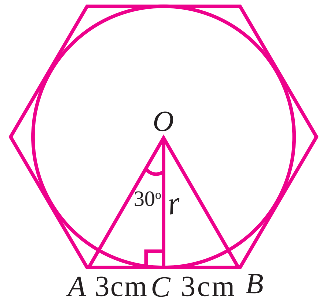


Fig. 2.17

Hence, radius of incircle is 5.196 cm

Exercise 2.3

1. Find the value of the following.

   (i) $\sin 26^\circ$ (ii) $\cos 72^\circ$ (iii) $\tan 35^\circ$ (iv) $\sin 75^\circ 15'$

   (v) $\sin 12^\circ 12'$ (vi) $\cos 12^\circ 35'$ (vii) $\cos 40^\circ 20'$ (viii) $\tan 10^\circ 26'$

   (ix) $\cot 20^\circ$ (x) $\cot 40^\circ 20'$

2. Find the value of θ, if

   (i) $\sin\theta = 0.7009$ (ii) $\cos\theta = 0.9664$ (iii) $\tan\theta = 0.3679$

   (iv) $\cot\theta = 0.2334$ (v) $\tan\theta = 63.6567$

3. Simplify, using trigonometric tables

   (i) $\sin 30^\circ 30' + \cos 40^\circ 20'$ (ii) $\tan 45^\circ 27' + \sin 20^\circ$

   (iii) $\tan 63^\circ 12' - \cos 12^\circ 42'$ (iv) $\sin 50^\circ 26' + \cos 18^\circ + \tan 70^\circ 12'$

   (v) $\tan 72^\circ + \cot 30^\circ$

4. Find the area of the right triangle with hypotenuse 20 cm and one of the acute angle is 48°

5. Find the area of the right triangle with hypotenuse 8 cm and one of the acute angle is 57°

6. Find the area of the isosceles triangle with base 16 cm and vertical angle $60^\circ 40'$

7. Find the area of the isosceles triangle with base 15 cm and vertical angle 80°

8. A ladder makes an angle 30° with the floor and its lower end is 12 m away from the wall. Find the length of the ladder.

9. Find the angle made by a ladder of length 4 m with the ground if its one end is 2 m away from the wall and the other end is on the wall.

10. Find the length of the chord of a circle of radius 5 cm subtending an angle of 108° at the centre.

11. Find the length of the side of regular polygon of 12 sides inscribed in a circle of radius 6 cm

12. Find the radius of the incircle of a regular hexagon of side 24 cm.

Exercise 2.4

Choose the Correct Answer

1. The value of $\sin^2 60^\circ + \cos^2 60^\circ$ is equal to

   (A) $\sin^2 45^\circ + \cos^2 45^\circ$ (B) $\tan^2 45^\circ + \cot^2 45^\circ$

   (C) $\sec^2 90^\circ$ (D) 0

2. If $x = \dfrac{2\tan 30^\circ}{1 - \tan^2 30^\circ}$, then the value of x is

   (A) $\tan 45^\circ$ (B) $\tan 30^\circ$ (C) $\tan 60^\circ$ (D) $\tan 90^\circ$

3. The value of $\sec^2 45^\circ - \tan^2 45^\circ$ is equal to

   (A) $\sin^2 60^\circ - \cos^2 60^\circ$ (B) $\sin^2 45^\circ + \cos^2 60^\circ$

   (C)$\sec^2 60^\circ - \tan^2 60^\circ$ (D) 0

4. The value of $2\sin 30^\circ \cos 30^\circ$ is equal to

   (A) $\tan 30^\circ$ (B) $\cos 60^\circ$ (C) $\sin 60^\circ$ (D) $\cot 60^\circ$

5. The value of $\operatorname{cosec}^2 60^\circ - 1$ is equal to

   (A) $\cos^2 60^\circ$ (B) $\cot^2 60^\circ$ (C) $\sec^2 60^\circ$ (D) $\tan^2 60^\circ$

6. $\cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ$ is equal to

   (A) $\cos 90^\circ$ (B) $\operatorname{cosec} 90^\circ$ (C) $\sin 30^\circ + \cos 30^\circ$ (D) $\tan 90^\circ$

7. The value of $\dfrac{\sin 27^\circ}{\cos 63^\circ}$ is

   (A) 0 (B) 1 (C) $\tan 27^\circ$ (D) $\cot 63^\circ$

8. If $\cos x = \sin 43^\circ$, then the value of x is

(A) 57° (B) 43° (C) 47° (D) 90°

9. The value of $\sec 29^\circ - \operatorname{cosec} 61^\circ$ is

(A) 1 (B) 0 (C) $\sec 60^\circ$ (D) $\operatorname{cosec} 29^\circ$

10. If $3x \operatorname{cosec} 36^\circ = \sec 54^\circ$, then the value of x is

(A) 0 (B) 1 (C) $\frac{1}{3}$ (D) $\frac{3}{4}$

11. The value of $\sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ$ is equal to

(A) $\sec 90^\circ$ (B) $\tan 90^\circ$ (C) $\cos 60^\circ$ (D) $\sin 90^\circ$

12. If $\cos A \cos 30^\circ = \frac{\sqrt{3}}{4}$, then the measure of A is

(A) 90° (B) 60° (C) 45° (D) 30°

13. The value of $\tan 26^\circ \cot 64^\circ$ is

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 0 (D) 1

14. The value of $\sin 60^\circ - \cos 30^\circ$ is

(A) 0 (B) $\frac{1}{\sqrt{2}}$ (C) $\frac{\sqrt{3}}{2}$ (D) 1

15. The value of $\cos^2 30^\circ - \sin^2 30^\circ$ is

(A) $\cos 60^\circ$ (B) $\sin 60^\circ$ (C) 0 (D) 1

Points to Remember

- **Pythagoras Theorem:**

  The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

- **Trigonometric Ratios:**

  Let θ be an acute angle of a right triangle. Then the six trigonometric ratios of θ are as follows.

  $$\sin\theta = \frac{\textit{Opposite side}}{\textit{Hypotenuse}} \qquad \operatorname{cosec}\theta = \frac{\textit{Hypotenuse}}{\textit{Opposite side}}$$

  $$\cos\theta = \frac{\textit{Adjacent side}}{\textit{Hypotenuse}} \qquad \sec\theta = \frac{\textit{Hypotenuse}}{\textit{Adjacent side}}$$

  $$\tan\theta = \frac{\textit{Opposite side}}{\textit{Adjacent side}} \qquad \cot\theta = \frac{\textit{Adjacent side}}{\textit{Opposite side}}$$

- **Reciprocal Relations:**

  $$\sin\theta = \frac{1}{\operatorname{cosec}\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

  $$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

- **Trigonometric Ratios of Complementary Angles:**

  Let θ be an acute angle of a right triangle. Then we have the following identities for trigonometric ratios of complementary angles.

  $$\sin\theta = \cos(90^\circ - \theta) \qquad \operatorname{cosec}\theta = \sec(90^\circ - \theta)$$

  $$\cos\theta = \sin(90^\circ - \theta) \qquad \sec\theta = \operatorname{cosec}(90^\circ - \theta)$$

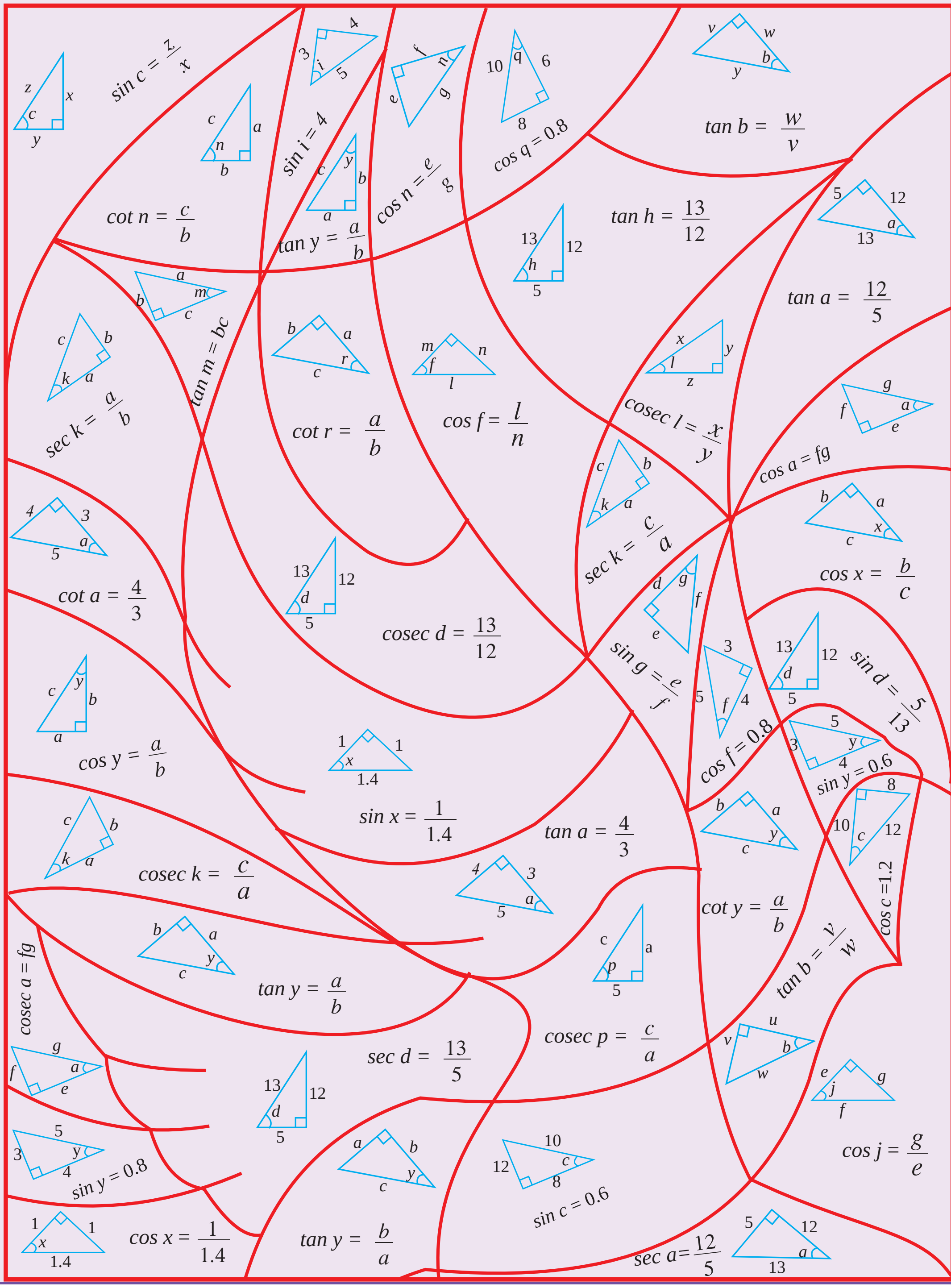
  $$\tan\theta = \cot(90^\circ - \theta) \qquad \cot\theta = \tan(90^\circ - \theta)$$

Activity 1

To find the hidden figure

Objective : To find and verify the trigonometric ratios.

Procedure : Find the hidden figure by colouring all region which display correct trigonometric ratios



Mathematics

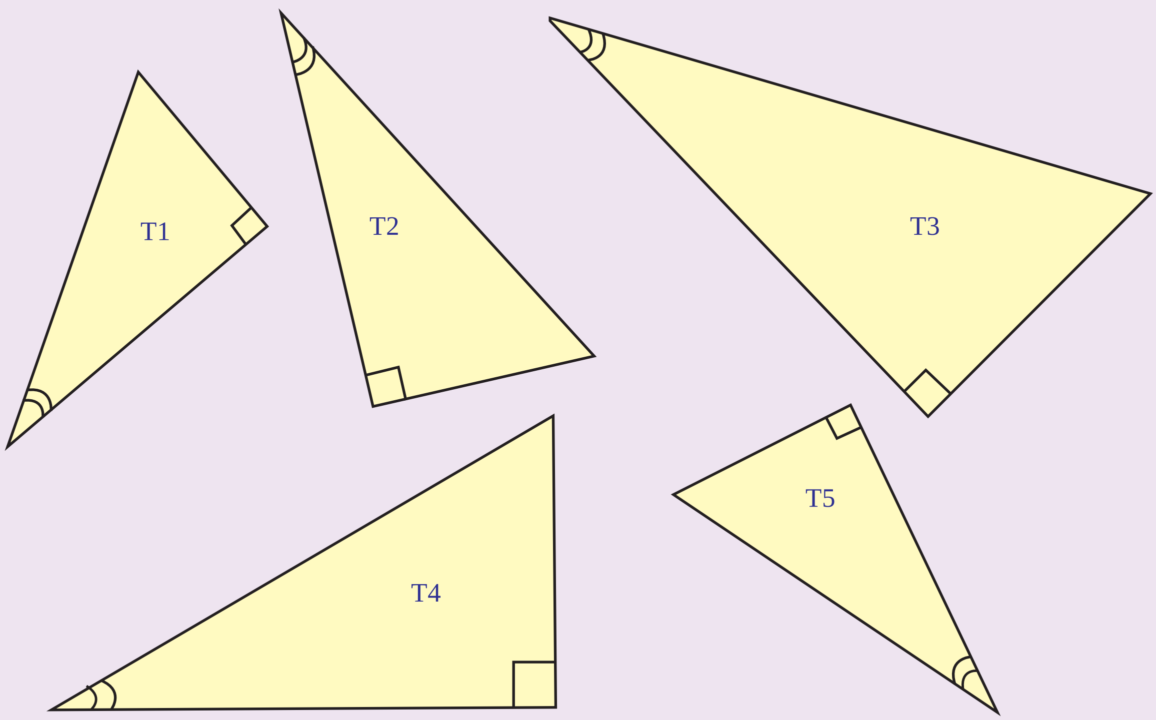
Activity 2

Trigonometric ratios for triangles

Calculating ratios for right angled triangles with angles of 30°

- In the following triangles, identify the angles measuring 90°, 30°. What is the measure of the third angle?

- Label the side opposite to 90° as "hyp" with respect to the angle measuring 30° and label the other sides as "adj" for adjacent and "opp" for opposite.



Complete the table below.

Marked Angle 30° / Sides	Opposite side in cm	Adjacent side in cm	Hypotenuse in cm	$\frac{\text{Opposite side}}{\text{Hypotenuse}}$ (for angle$=30^\circ$)		$\frac{\text{Adjacent side}}{\text{Hypotenuse}}$ (for angle$=30^\circ$)		$\frac{\text{Opposite side}}{\text{Adjacent side}}$ (for angle$=30^\circ$)	
				fraction	decimal	fraction	decimal	fraction	decimal
T1									
T2									
T3									
T4									
T5									
Mean Value (correct to 2 decimal places)									

Mathematics

Activity 3

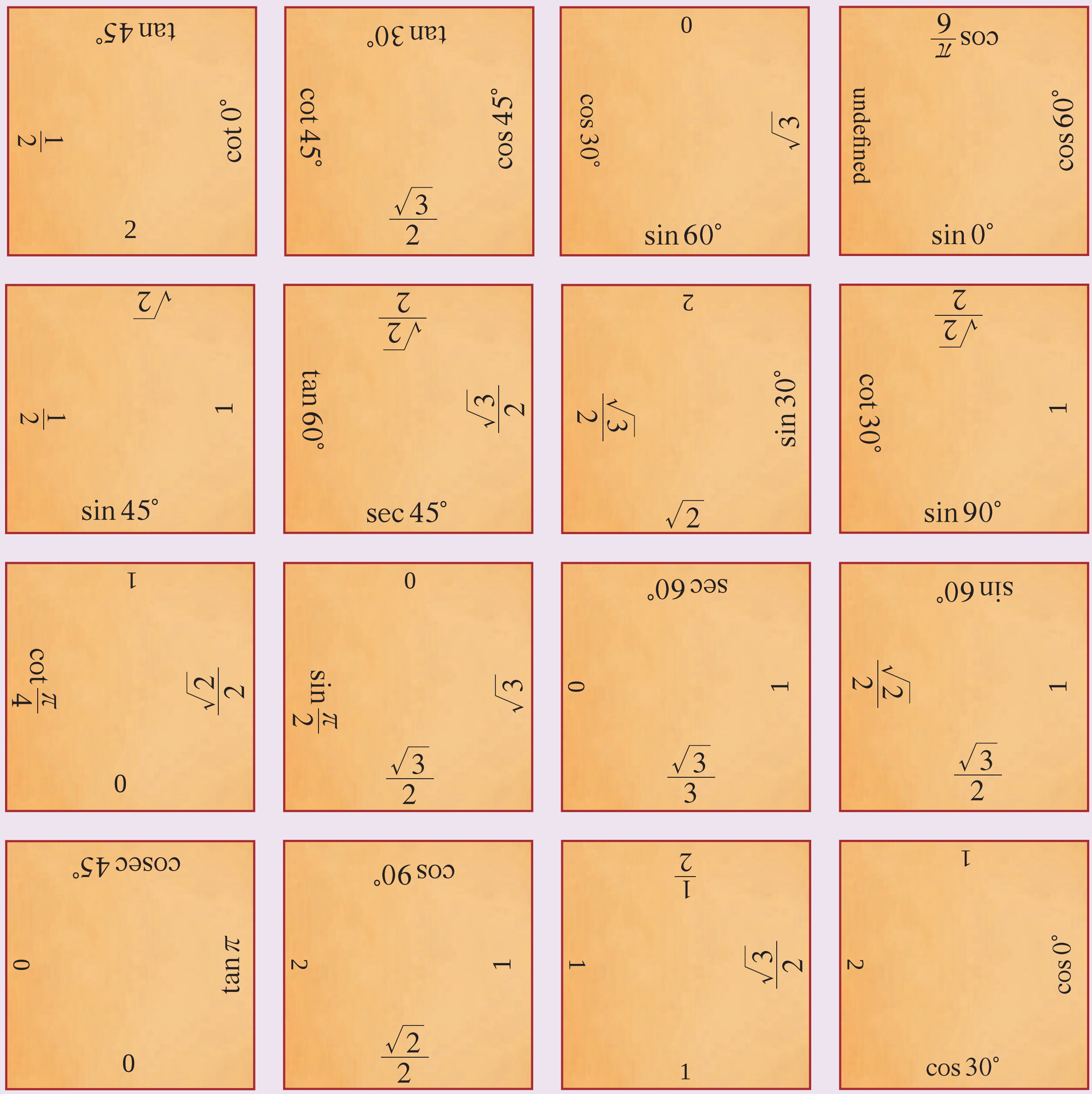
Trigonometric square

Using the trigonometric ratios and its values, match the sides of the given squares and form a single trigonometric square.

Objective : To identify the trigonometric ratios and their values.

Procedure : ● Take 16 squares of equal sizes as given below.

● Arrange the squares using trigonometric ratios and its values in such a way that a side with a particular ratio must coincide with the side of another square having its value.



Mathematics

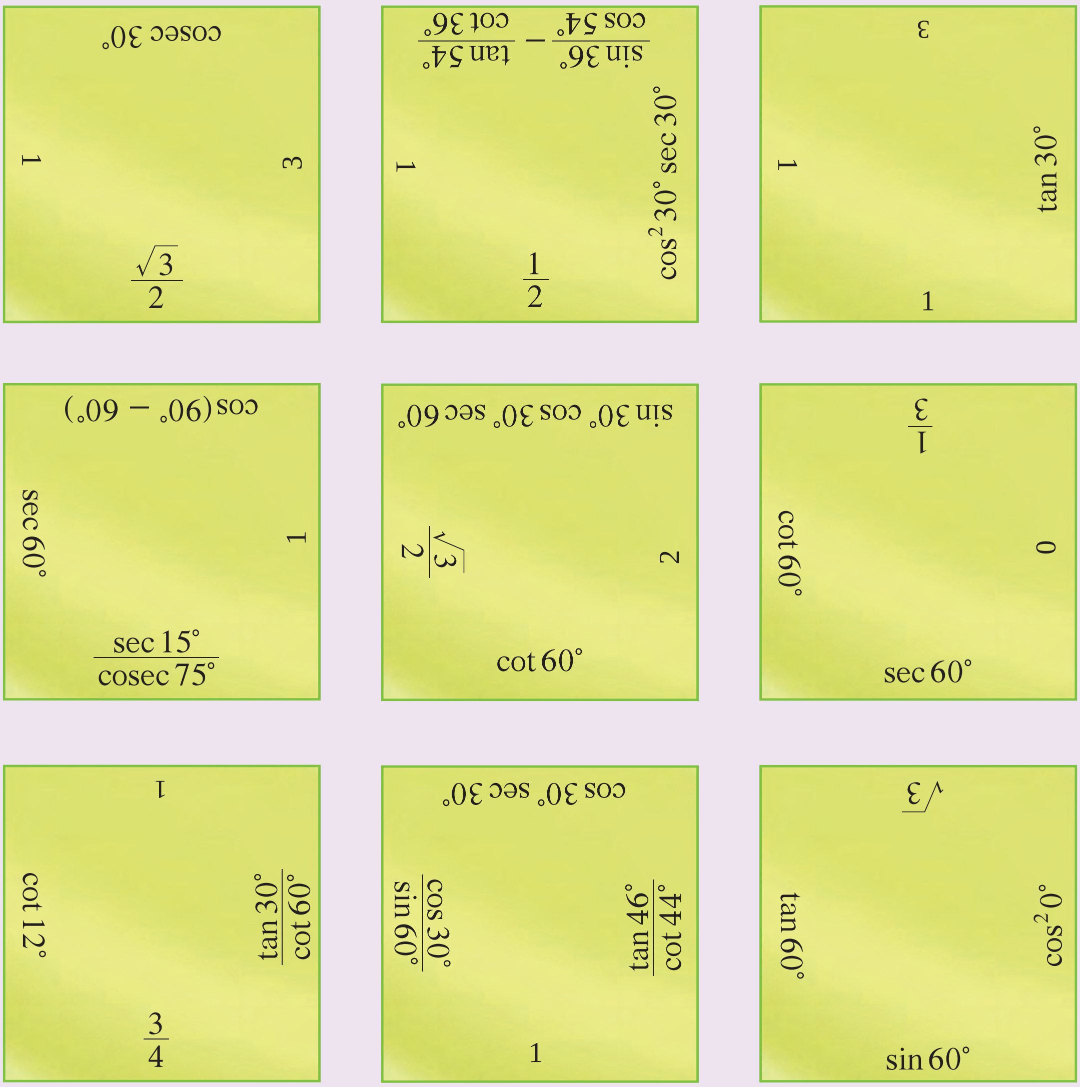
Activity 4

Using the trigonometric ratios of complementary angles, match the sides of the given squares and form a single square.

Objective : To identify the trigonometric ratios of complementary angles.

Procedure : ● Take 9 squares of equal sizes as given below.

● Arrange the squares using trigonometric ratios of complementary angles in such a way that a side with a particular ratio must coincide with other equal ratio in the side of another square having its value.



NATURAL SINES

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
0	0.0000	0.0017	0.0035	0.0052	0.0070	0.0087	0.0105	0.0122	0.0140	0.0157	3	6	9	12	15
1	0.0175	0.0192	0.0209	0.0227	0.0244	0.0262	0.0279	0.0297	0.0314	0.0332	3	6	9	12	15
2	0.0349	0.0366	0.0384	0.0401	0.0419	0.0436	0.0454	0.0471	0.0488	0.0506	3	6	9	12	15
3	0.0523	0.0541	0.0558	0.0576	0.0593	0.0610	0.0628	0.0645	0.0663	0.0680	3	6	9	12	15
4	0.0698	0.0715	0.0732	0.0750	0.0767	0.0785	0.0802	0.0819	0.0837	0.0854	3	6	9	12	15
5	0.0872	0.0889	0.0906	0.0924	0.0941	0.0958	0.0976	0.0993	0.1011	0.1028	3	6	9	12	14
6	0.1045	0.1063	0.1080	0.1097	0.1115	0.1132	0.1149	0.1167	0.1184	0.1201	3	6	9	12	14
7	0.1219	0.1236	0.1253	0.1271	0.1288	0.1305	0.1323	0.1340	0.1357	0.1374	3	6	9	12	14
8	0.1392	0.1409	0.1426	0.1444	0.1461	0.1478	0.1495	0.1513	0.1530	0.1547	3	6	9	12	14
9	0.1564	0.1582	0.1599	0.1616	0.1633	0.1650	0.1668	0.1685	0.1702	0.1719	3	6	9	12	14
10	0.1736	0.1754	0.1771	0.1788	0.1805	0.1822	0.1840	0.1857	0.1874	0.1891	3	6	9	12	14
11	0.1908	0.1925	0.1942	0.1959	0.1977	0.1994	0.2011	0.2028	0.2045	0.2062	3	6	9	11	14
12	0.2079	0.2096	0.2113	0.2130	0.2147	0.2164	0.2181	0.2198	0.2215	0.2233	3	6	9	11	14
13	0.2250	0.2267	0.2284	0.2300	0.2317	0.2334	0.2351	0.2368	0.2385	0.2402	3	6	8	11	14
14	0.2419	0.2436	0.2453	0.2470	0.2487	0.2504	0.2521	0.2538	0.2554	0.2571	3	6	8	11	14
15	0.2588	0.2605	0.2622	0.2639	0.2656	0.2672	0.2689	0.2706	0.2723	0.2740	3	6	8	11	14
16	0.2756	0.2773	0.2790	0.2807	0.2823	0.2840	0.2857	0.2874	0.2890	0.2907	3	6	8	11	14
17	0.2924	0.2940	0.2957	0.2974	0.2990	0.3007	0.3024	0.3040	0.3057	0.3074	3	6	8	11	14
18	0.3090	0.3107	0.3123	0.3140	0.3156	0.3173	0.3190	0.3206	0.3223	0.3239	3	6	8	11	14
19	0.3256	0.3272	0.3289	0.3305	0.3322	0.3338	0.3355	0.3371	0.3387	0.3404	3	5	8	11	14
20	0.3420	0.3437	0.3453	0.3469	0.3486	0.3502	0.3518	0.3535	0.3551	0.3567	3	5	8	11	14
21	0.3584	0.3600	0.3616	0.3633	0.3649	0.3665	0.3681	0.3697	0.3714	0.3730	3	5	8	11	14
22	0.3746	0.3762	0.3778	0.3795	0.3811	0.3827	0.3843	0.3859	0.3875	0.3891	3	5	8	11	14
23	0.3907	0.3923	0.3939	0.3955	0.3971	0.3987	0.4003	0.4019	0.4035	0.4051	3	5	8	11	14
24	0.4067	0.4083	0.4099	0.4115	0.4131	0.4147	0.4163	0.4179	0.4195	0.4210	3	5	8	11	13
25	0.4226	0.4242	0.4258	0.4274	0.4289	0.4305	0.4321	0.4337	0.4352	0.4368	3	5	8	11	13
26	0.4384	0.4399	0.4415	0.4431	0.4446	0.4462	0.4478	0.4493	0.4509	0.4524	3	5	8	10	13
27	0.4540	0.4555	0.4571	0.4586	0.4602	0.4617	0.4633	0.4648	0.4664	0.4679	3	5	8	10	13
28	0.4695	0.4710	0.4726	0.4741	0.4756	0.4772	0.4787	0.4802	0.4818	0.4833	3	5	8	10	13
29	0.4848	0.4863	0.4879	0.4894	0.4909	0.4924	0.4939	0.4955	0.4970	0.4985	3	5	8	10	13
30	0.5000	0.5015	0.5030	0.5045	0.5060	0.5075	0.5090	0.5105	0.5120	0.5135	3	5	8	10	13
31	0.5150	0.5165	0.5180	0.5195	0.5210	0.5225	0.5240	0.5255	0.5270	0.5284	2	5	7	10	12
32	0.5299	0.5314	0.5329	0.5344	0.5358	0.5373	0.5388	0.5402	0.5417	0.5432	2	5	7	10	12
33	0.5446	0.5461	0.5476	0.5490	0.5505	0.5519	0.5534	0.5548	0.5563	0.5577	2	5	7	10	12
34	0.5592	0.5606	0.5621	0.5635	0.5650	0.5664	0.5678	0.5693	0.5707	0.5721	2	5	7	10	12
35	0.5736	0.5750	0.5764	0.5779	0.5793	0.5807	0.5821	0.5835	0.5850	0.5864	2	5	7	10	12
36	0.5878	0.5892	0.5906	0.5920	0.5934	0.5948	0.5962	0.5976	0.5990	0.6004	2	5	7	9	12
37	0.6018	0.6032	0.6046	0.6060	0.6074	0.6088	0.6101	0.6115	0.6129	0.6143	2	5	7	9	12
38	0.6157	0.6170	0.6184	0.6198	0.6211	0.6225	0.6239	0.6252	0.6266	0.6280	2	5	7	9	11
39	0.6293	0.6307	0.6320	0.6334	0.6347	0.6361	0.6374	0.6388	0.6401	0.6414	2	4	7	9	11
40	0.6428	0.6441	0.6455	0.6468	0.6481	0.6494	0.6508	0.6521	0.6534	0.6547	2	4	7	9	11
41	0.6561	0.6574	0.6587	0.6600	0.6613	0.6626	0.6639	0.6652	0.6665	0.6678	2	4	7	9	11
42	0.6691	0.6704	0.6717	0.6730	0.6743	0.6756	0.6769	0.6782	0.6794	0.6807	2	4	6	9	11
43	0.6820	0.6833	0.6845	0.6858	0.6871	0.6884	0.6896	0.6909	0.6921	0.6934	2	4	6	8	11
44	0.6947	0.6959	0.6972	0.6984	0.6997	0.7009	0.7022	0.7034	0.7046	0.7059	2	4	6	8	10

NATURAL SINES

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
45	0.7071	0.7083	0.7096	0.7108	0.7120	0.7133	0.7145	0.7157	0.7169	0.7181	2	4	6	8	10
46	0.7193	0.7206	0.7218	0.7230	0.7242	0.7254	0.7266	0.7278	0.7290	0.7302	2	4	6	8	10
47	0.7314	0.7325	0.7337	0.7349	0.7361	0.7373	0.7385	0.7396	0.7408	0.7420	2	4	6	8	10
48	0.7431	0.7443	0.7455	0.7466	0.7478	0.7490	0.7501	0.7513	0.7524	0.7536	2	4	6	8	10
49	0.7547	0.7559	0.7570	0.7581	0.7593	0.7604	0.7615	0.7627	0.7638	0.7649	2	4	6	8	9
50	0.7660	0.7672	0.7683	0.7694	0.7705	0.7716	0.7727	0.7738	0.7749	0.7760	2	4	6	7	9
51	0.7771	0.7782	0.7793	0.7804	0.7815	0.7826	0.7837	0.7848	0.7859	0.7869	2	4	5	7	9
52	0.7880	0.7891	0.7902	0.7912	0.7923	0.7934	0.7944	0.7955	0.7965	0.7976	2	4	5	7	9
53	0.7986	0.7997	0.8007	0.8018	0.8028	0.8039	0.8049	0.8059	0.8070	0.8080	2	3	5	7	9
54	0.8090	0.8100	0.8111	0.8121	0.8131	0.8141	0.8151	0.8161	0.8171	0.8181	2	3	5	7	8
55	0.8192	0.8202	0.8211	0.8221	0.8231	0.8241	0.8251	0.8261	0.8271	0.8281	2	3	5	7	8
56	0.8290	0.8300	0.8310	0.8320	0.8329	0.8339	0.8348	0.8358	0.8368	0.8377	2	3	5	6	8
57	0.8387	0.8396	0.8406	0.8415	0.8425	0.8434	0.8443	0.8453	0.8462	0.8471	2	3	5	6	8
58	0.8480	0.8490	0.8499	0.8508	0.8517	0.8526	0.8536	0.8545	0.8554	0.8563	2	3	5	6	8
59	0.8572	0.8581	0.8590	0.8599	0.8607	0.8616	0.8625	0.8634	0.8643	0.8652	1	3	4	6	7
60	0.8660	0.8669	0.8678	0.8686	0.8695	0.8704	0.8712	0.8721	0.8729	0.8738	1	3	4	6	7
61	0.8746	0.8755	0.8763	0.8771	0.8780	0.8788	0.8796	0.8805	0.8813	0.8821	1	3	4	6	7
62	0.8829	0.8838	0.8846	0.8854	0.8862	0.8870	0.8878	0.8886	0.8894	0.8902	1	3	4	5	7
63	0.8910	0.8918	0.8926	0.8934	0.8942	0.8949	0.8957	0.8965	0.8973	0.8980	1	3	4	5	6
64	0.8988	0.8996	0.9003	0.9011	0.9018	0.9026	0.9033	0.9041	0.9048	0.9056	1	3	4	5	6
65	0.9063	0.9070	0.9078	0.9085	0.9092	0.9100	0.9107	0.9114	0.9121	0.9128	1	2	4	5	6
66	0.9135	0.9143	0.9150	0.9157	0.9164	0.9171	0.9178	0.9184	0.9191	0.9198	1	2	3	5	6
67	0.9205	0.9212	0.9219	0.9225	0.9232	0.9239	0.9245	0.9252	0.9259	0.9265	1	2	3	4	6
68	0.9272	0.9278	0.9285	0.9291	0.9298	0.9304	0.9311	0.9317	0.9323	0.9330	1	2	3	4	5
69	0.9336	0.9342	0.9348	0.9354	0.9361	0.9367	0.9373	0.9379	0.9385	0.9391	1	2	3	4	5
70	0.9397	0.9403	0.9409	0.9415	0.9421	0.9426	0.9432	0.9438	0.9444	0.9449	1	2	3	4	5
71	0.9455	0.9461	0.9466	0.9472	0.9478	0.9483	0.9489	0.9494	0.9500	0.9505	1	2	3	4	5
72	0.9511	0.9516	0.9521	0.9527	0.9532	0.9537	0.9542	0.9548	0.9553	0.9558	1	2	3	3	4
73	0.9563	0.9568	0.9573	0.9578	0.9583	0.9588	0.9593	0.9598	0.9603	0.9608	1	2	2	3	4
74	0.9613	0.9617	0.9622	0.9627	0.9632	0.9636	0.9641	0.9646	0.9650	0.9655	1	2	2	3	4
75	0.9659	0.9664	0.9668	0.9673	0.9677	0.9681	0.9686	0.9690	0.9694	0.9699	1	1	2	3	4
76	0.9703	0.9707	0.9711	0.9715	0.9720	0.9724	0.9728	0.9732	0.9736	0.9740	1	1	2	3	3
77	0.9744	0.9748	0.9751	0.9755	0.9759	0.9763	0.9767	0.9770	0.9774	0.9778	1	1	2	3	3
78	0.9781	0.9785	0.9789	0.9792	0.9796	0.9799	0.9803	0.9806	0.9810	0.9813	1	1	2	2	3
79	0.9816	0.9820	0.9823	0.9826	0.9829	0.9833	0.9836	0.9839	0.9842	0.9845	1	1	2	2	3
80	0.9848	0.9851	0.9854	0.9857	0.9860	0.9863	0.9866	0.9869	0.9871	0.9874	0	1	1	2	2
81	0.9877	0.9880	0.9882	0.9885	0.9888	0.9890	0.9893	0.9895	0.9898	0.9900	0	1	1	2	2
82	0.9903	0.9905	0.9907	0.9910	0.9912	0.9914	0.9917	0.9919	0.9921	0.9923	0	1	1	2	2
83	0.9925	0.9928	0.9930	0.9932	0.9934	0.9936	0.9938	0.9940	0.9942	0.9943	0	1	1	1	2
84	0.9945	0.9947	0.9949	0.9951	0.9952	0.9954	0.9956	0.9957	0.9959	0.9960	0	1	1	1	2
85	0.9962	0.9963	0.9965	0.9966	0.9968	0.9969	0.9971	0.9972	0.9973	0.9974	0	0	1	1	1
86	0.9976	0.9977	0.9978	0.9979	0.9980	0.9981	0.9982	0.9983	0.9984	0.9985	0	0	1	1	1
87	0.9986	0.9987	0.9988	0.9989	0.9990	0.9990	0.9991	0.9992	0.9993	0.9993	0	0	0	1	1
88	0.9994	0.9995	0.9995	0.9996	0.9996	0.9997	0.9997	0.9997	0.9998	0.9998	0	0	0	0	0
89	0.9998	0.9999	0.9999	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	0	0	0	0	0

NATURAL COSINES

(Numbers in mean difference columns to be subtracted, not added)

Degree	0´	6´	12´	18´	24´	30´	36´	42´	48´	54´	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999	0	0	0	0	0
1	0.9998	0.9998	0.9998	0.9997	0.9997	0.9997	0.9996	0.9996	0.9995	0.9995	0	0	0	0	0
2	0.9994	0.9993	0.9993	0.9992	0.9991	0.9990	0.9990	0.9989	0.9988	0.9987	0	0	0	1	1
3	0.9986	0.9985	0.9984	0.9983	0.9982	0.9981	0.9980	0.9979	0.9978	0.9977	0	0	1	1	1
4	0.9976	0.9974	0.9973	0.9972	0.9971	0.9969	0.9968	0.9966	0.9965	0.9963	0	0	1	1	1
5	0.9962	0.9960	0.9959	0.9957	0.9956	0.9954	0.9952	0.9951	0.9949	0.9947	0	1	1	1	2
6	0.9945	0.9943	0.9942	0.9940	0.9938	0.9936	0.9934	0.9932	0.9930	0.9928	0	1	1	1	2
7	0.9925	0.9923	0.9921	0.9919	0.9917	0.9914	0.9912	0.9910	0.9907	0.9905	0	1	1	2	2
8	0.9903	0.9900	0.9898	0.9895	0.9893	0.9890	0.9888	0.9885	0.9882	0.9880	0	1	1	2	2
9	0.9877	0.9874	0.9871	0.9869	0.9866	0.9863	0.9860	0.9857	0.9854	0.9851	0	1	1	2	2
10	0.9848	0.9845	0.9842	0.9839	0.9836	0.9833	0.9829	0.9826	0.9823	0.9820	1	1	2	2	3
11	0.9816	0.9813	0.9810	0.9806	0.9803	0.9799	0.9796	0.9792	0.9789	0.9785	1	1	2	2	3
12	0.9781	0.9778	0.9774	0.9770	0.9767	0.9763	0.9759	0.9755	0.9751	0.9748	1	1	2	3	3
13	0.9744	0.9740	0.9736	0.9732	0.9728	0.9724	0.9720	0.9715	0.9711	0.9707	1	1	2	3	3
14	0.9703	0.9699	0.9694	0.9690	0.9686	0.9681	0.9677	0.9673	0.9668	0.9664	1	1	2	3	4
15	0.9659	0.9655	0.9650	0.9646	0.9641	0.9636	0.9632	0.9627	0.9622	0.9617	1	2	2	3	4
16	0.9613	0.9608	0.9603	0.9598	0.9593	0.9588	0.9583	0.9578	0.9573	0.9568	1	2	2	3	4
17	0.9563	0.9558	0.9553	0.9548	0.9542	0.9537	0.9532	0.9527	0.9521	0.9516	1	2	3	3	4
18	0.9511	0.9505	0.9500	0.9494	0.9489	0.9483	0.9478	0.9472	0.9466	0.9461	1	2	3	4	5
19	0.9455	0.9449	0.9444	0.9438	0.9432	0.9426	0.9421	0.9415	0.9409	0.9403	1	2	3	4	5
20	0.9397	0.9391	0.9385	0.9379	0.9373	0.9367	0.9361	0.9354	0.9348	0.9342	1	2	3	4	5
21	0.9336	0.9330	0.9323	0.9317	0.9311	0.9304	0.9298	0.9291	0.9285	0.9278	1	2	3	4	5
22	0.9272	0.9265	0.9259	0.9252	0.9245	0.9239	0.9232	0.9225	0.9219	0.9212	1	2	3	4	6
23	0.9205	0.9198	0.9191	0.9184	0.9178	0.9171	0.9164	0.9157	0.9150	0.9143	1	2	3	5	6
24	0.9135	0.9128	0.9121	0.9114	0.9107	0.9100	0.9092	0.9085	0.9078	0.9070	1	2	4	5	6
25	0.9063	0.9056	0.9048	0.9041	0.9033	0.9026	0.9018	0.9011	0.9003	0.8996	1	3	4	5	6
26	0.8988	0.8980	0.8973	0.8965	0.8957	0.8949	0.8942	0.8934	0.8926	0.8918	1	3	4	5	6
27	0.8910	0.8902	0.8894	0.8886	0.8878	0.8870	0.8862	0.8854	0.8846	0.8838	1	3	4	5	7
28	0.8829	0.8821	0.8813	0.8805	0.8796	0.8788	0.8780	0.8771	0.8763	0.8755	1	3	4	6	7
29	0.8746	0.8738	0.8729	0.8721	0.8712	0.8704	0.8695	0.8686	0.8678	0.8669	1	3	4	6	7
30	0.8660	0.8652	0.8643	0.8634	0.8625	0.8616	0.8607	0.8599	0.8590	0.8581	1	3	4	6	7
31	0.8572	0.8563	0.8554	0.8545	0.8536	0.8526	0.8517	0.8508	0.8499	0.8490	2	3	5	6	8
32	0.8480	0.8471	0.8462	0.8453	0.8443	0.8434	0.8425	0.8415	0.8406	0.8396	2	3	5	6	8
33	0.8387	0.8377	0.8368	0.8358	0.8348	0.8339	0.8329	0.8320	0.8310	0.8300	2	3	5	6	8
34	0.8290	0.8281	0.8271	0.8261	0.8251	0.8241	0.8231	0.8221	0.8211	0.8202	2	3	5	7	8
35	0.8192	0.8181	0.8171	0.8161	0.8151	0.8141	0.8131	0.8121	0.8111	0.8100	2	3	5	7	8
36	0.8090	0.8080	0.8070	0.8059	0.8049	0.8039	0.8028	0.8018	0.8007	0.7997	2	3	5	7	9
37	0.7986	0.7976	0.7965	0.7955	0.7944	0.7934	0.7923	0.7912	0.7902	0.7891	2	4	5	7	9
38	0.7880	0.7869	0.7859	0.7848	0.7837	0.7826	0.7815	0.7804	0.7793	0.7782	2	4	5	7	9
39	0.7771	0.7760	0.7749	0.7738	0.7727	0.7716	0.7705	0.7694	0.7683	0.7672	2	4	6	7	9
40	0.7660	0.7649	0.7638	0.7627	0.7615	0.7604	0.7593	0.7581	0.7570	0.7559	2	4	6	8	9
41	0.7547	0.7536	0.7524	0.7513	0.7501	0.7490	0.7478	0.7466	0.7455	0.7443	2	4	6	8	10
42	0.7431	0.7420	0.7408	0.7396	0.7385	0.7373	0.7361	0.7349	0.7337	0.7325	2	4	6	8	10
43	0.7314	0.7302	0.7290	0.7278	0.7266	0.7254	0.7242	0.7230	0.7218	0.7206	2	4	6	8	10
44	0.7193	0.7181	0.7169	0.7157	0.7145	0.7133	0.7120	0.7108	0.7096	0.7083	2	4	6	8	10

NATURAL COSINES

(Numbers in mean difference columns to be subtracted, not added)

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
45	0.7071	0.7059	0.7046	0.7034	0.7022	0.7009	0.6997	0.6984	0.6972	0.6959	2	4	6	8	10
46	0.6947	0.6934	0.6921	0.6909	0.6896	0.6884	0.6871	0.6858	0.6845	0.6833	2	4	6	8	11
47	0.6820	0.6807	0.6794	0.6782	0.6769	0.6756	0.6743	0.6730	0.6717	0.6704	2	4	6	9	11
48	0.6691	0.6678	0.6665	0.6652	0.6639	0.6626	0.6613	0.6600	0.6587	0.6574	2	4	7	9	11
49	0.6561	0.6547	0.6534	0.6521	0.6508	0.6494	0.6481	0.6468	0.6455	0.6441	2	4	7	9	11
50	0.6428	0.6414	0.6401	0.6388	0.6374	0.6361	0.6347	0.6334	0.6320	0.6307	2	4	7	9	11
51	0.6293	0.6280	0.6266	0.6252	0.6239	0.6225	0.6211	0.6198	0.6184	0.6170	2	5	7	9	11
52	0.6157	0.6143	0.6129	0.6115	0.6101	0.6088	0.6074	0.6060	0.6046	0.6032	2	5	7	9	12
53	0.6018	0.6004	0.5990	0.5976	0.5962	0.5948	0.5934	0.5920	0.5906	0.5892	2	5	7	9	12
54	0.5878	0.5864	0.5850	0.5835	0.5821	0.5807	0.5793	0.5779	0.5764	0.5750	2	5	7	9	12
55	0.5736	0.5721	0.5707	0.5693	0.5678	0.5664	0.5650	0.5635	0.5621	0.5606	2	5	7	10	12
56	0.5592	0.5577	0.5563	0.5548	0.5534	0.5519	0.5505	0.5490	0.5476	0.5461	2	5	7	10	12
57	0.5446	0.5432	0.5417	0.5402	0.5388	0.5373	0.5358	0.5344	0.5329	0.5314	2	5	7	10	12
58	0.5299	0.5284	0.5270	0.5255	0.5240	0.5225	0.5210	0.5195	0.5180	0.5165	2	5	7	10	12
59	0.5150	0.5135	0.5120	0.5105	0.5090	0.5075	0.5060	0.5045	0.5030	0.5015	3	5	8	10	13
60	0.5000	0.4985	0.4970	0.4955	0.4939	0.4924	0.4909	0.4894	0.4879	0.4863	3	5	8	10	13
61	0.4848	0.4833	0.4818	0.4802	0.4787	0.4772	0.4756	0.4741	0.4726	0.4710	3	5	8	10	13
62	0.4695	0.4679	0.4664	0.4648	0.4633	0.4617	0.4602	0.4586	0.4571	0.4555	3	5	8	10	13
63	0.4540	0.4524	0.4509	0.4493	0.4478	0.4462	0.4446	0.4431	0.4415	0.4399	3	5	8	10	13
64	0.4384	0.4368	0.4352	0.4337	0.4321	0.4305	0.4289	0.4274	0.4258	0.4242	3	5	8	11	13
65	0.4226	0.4210	0.4195	0.4179	0.4163	0.4147	0.4131	0.4115	0.4099	0.4083	3	5	8	11	13
66	0.4067	0.4051	0.4035	0.4019	0.4003	0.3987	0.3971	0.3955	0.3939	0.3923	3	5	8	11	14
67	0.3907	0.3891	0.3875	0.3859	0.3843	0.3827	0.3811	0.3795	0.3778	0.3762	3	5	8	11	14
68	0.3746	0.3730	0.3714	0.3697	0.3681	0.3665	0.3649	0.3633	0.3616	0.3600	3	5	8	11	14
69	0.3584	0.3567	0.3551	0.3535	0.3518	0.3502	0.3486	0.3469	0.3453	0.3437	3	5	8	11	14
70	0.3420	0.3404	0.3387	0.3371	0.3355	0.3338	0.3322	0.3305	0.3289	0.3272	3	5	8	11	14
71	0.3256	0.3239	0.3223	0.3206	0.3190	0.3173	0.3156	0.3140	0.3123	0.3107	3	6	8	11	14
72	0.3090	0.3074	0.3057	0.3040	0.3024	0.3007	0.2990	0.2974	0.2957	0.2940	3	6	8	11	14
73	0.2924	0.2907	0.2890	0.2874	0.2857	0.2840	0.2823	0.2807	0.2790	0.2773	3	6	8	11	14
74	0.2756	0.2740	0.2723	0.2706	0.2689	0.2672	0.2656	0.2639	0.2622	0.2605	3	6	8	11	14
75	0.2588	0.2571	0.2554	0.2538	0.2521	0.2504	0.2487	0.2470	0.2453	0.2436	3	6	8	11	14
76	0.2419	0.2402	0.2385	0.2368	0.2351	0.2334	0.2317	0.2300	0.2284	0.2267	3	6	8	11	14
77	0.2250	0.2233	0.2215	0.2198	0.2181	0.2164	0.2147	0.2130	0.2113	0.2096	3	6	9	11	14
78	0.2079	0.2062	0.2045	0.2028	0.2011	0.1994	0.1977	0.1959	0.1942	0.1925	3	6	9	11	14
79	0.1908	0.1891	0.1874	0.1857	0.1840	0.1822	0.1805	0.1788	0.1771	0.1754	3	6	9	11	14
80	0.1736	0.1719	0.1702	0.1685	0.1668	0.1650	0.1633	0.1616	0.1599	0.1582	3	6	9	12	14
81	0.1564	0.1547	0.1530	0.1513	0.1495	0.1478	0.1461	0.1444	0.1426	0.1409	3	6	9	12	14
82	0.1392	0.1374	0.1357	0.1340	0.1323	0.1305	0.1288	0.1271	0.1253	0.1236	3	6	9	12	14
83	0.1219	0.1201	0.1184	0.1167	0.1149	0.1132	0.1115	0.1097	0.1080	0.1063	3	6	9	12	14
84	0.1045	0.1028	0.1011	0.0993	0.0976	0.0958	0.0941	0.0924	0.0906	0.0889	3	6	9	12	14
85	0.0872	0.0854	0.0837	0.0819	0.0802	0.0785	0.0767	0.0750	0.0732	0.0715	3	6	9	12	15
86	0.0698	0.0680	0.0663	0.0645	0.0628	0.0610	0.0593	0.0576	0.0558	0.0541	3	6	9	12	15
87	0.0523	0.0506	0.0488	0.0471	0.0454	0.0436	0.0419	0.0401	0.0384	0.0366	3	6	9	12	15
88	0.0349	0.0332	0.0314	0.0297	0.0279	0.0262	0.0244	0.0227	0.0209	0.0192	3	6	9	12	15
89	0.0175	0.0157	0.0140	0.0122	0.0105	0.0087	0.0070	0.0052	0.0035	0.0017	3	6	9	12	15

NATURAL TANGENTS

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
0	0.0000	0.0017	0.0035	0.0052	0.0070	0.0087	0.0105	0.0122	0.0140	0.0157	3	6	9	12	15
1	0.0175	0.0192	0.0209	0.0227	0.0244	0.0262	0.0279	0.0297	0.0314	0.0332	3	6	9	12	15
2	0.0349	0.0367	0.0384	0.0402	0.0419	0.0437	0.0454	0.0472	0.0489	0.0507	3	6	9	12	15
3	0.0524	0.0542	0.0559	0.0577	0.0594	0.0612	0.0629	0.0647	0.0664	0.0682	3	6	9	12	15
4	0.0699	0.0717	0.0734	0.0752	0.0769	0.0787	0.0805	0.0822	0.0840	0.0857	3	6	9	12	15
5	0.0875	0.0892	0.0910	0.0928	0.0945	0.0963	0.0981	0.0998	0.1016	0.1033	3	6	9	12	15
6	0.1051	0.1069	0.1086	0.1104	0.1122	0.1139	0.1157	0.1175	0.1192	0.1210	3	6	9	12	15
7	0.1228	0.1246	0.1263	0.1281	0.1299	0.1317	0.1334	0.1352	0.1370	0.1388	3	6	9	12	15
8	0.1405	0.1423	0.1441	0.1459	0.1477	0.1495	0.1512	0.1530	0.1548	0.1566	3	6	9	12	15
9	0.1584	0.1602	0.1620	0.1638	0.1655	0.1673	0.1691	0.1709	0.1727	0.1745	3	6	9	12	15
10	0.1763	0.1781	0.1799	0.1817	0.1835	0.1853	0.1871	0.1890	0.1908	0.1926	3	6	9	12	15
11	0.1944	0.1962	0.1980	0.1998	0.2016	0.2035	0.2053	0.2071	0.2089	0.2107	3	6	9	12	15
12	0.2126	0.2144	0.2162	0.2180	0.2199	0.2217	0.2235	0.2254	0.2272	0.2290	3	6	9	12	15
13	0.2309	0.2327	0.2345	0.2364	0.2382	0.2401	0.2419	0.2438	0.2456	0.2475	3	6	9	12	15
14	0.2493	0.2512	0.2530	0.2549	0.2568	0.2586	0.2605	0.2623	0.2642	0.2661	3	6	9	12	16
15	0.2679	0.2698	0.2717	0.2736	0.2754	0.2773	0.2792	0.2811	0.2830	0.2849	3	6	9	13	16
16	0.2867	0.2886	0.2905	0.2924	0.2943	0.2962	0.2981	0.3000	0.3019	0.3038	3	6	9	13	16
17	0.3057	0.3076	0.3096	0.3115	0.3134	0.3153	0.3172	0.3191	0.3211	0.3230	3	6	10	13	16
18	0.3249	0.3269	0.3288	0.3307	0.3327	0.3346	0.3365	0.3385	0.3404	0.3424	3	6	10	13	16
19	0.3443	0.3463	0.3482	0.3502	0.3522	0.3541	0.3561	0.3581	0.3600	0.3620	3	7	10	13	16
20	0.3640	0.3659	0.3679	0.3699	0.3719	0.3739	0.3759	0.3779	0.3799	0.3819	3	7	10	13	17
21	0.3839	0.3859	0.3879	0.3899	0.3919	0.3939	0.3959	0.3979	0.4000	0.4020	3	7	10	13	17
22	0.4040	0.4061	0.4081	0.4101	0.4122	0.4142	0.4163	0.4183	0.4204	0.4224	3	7	10	14	17
23	0.4245	0.4265	0.4286	0.4307	0.4327	0.4348	0.4369	0.4390	0.4411	0.4431	3	7	10	14	17
24	0.4452	0.4473	0.4494	0.4515	0.4536	0.4557	0.4578	0.4599	0.4621	0.4642	4	7	11	14	18
25	0.4663	0.4684	0.4706	0.4727	0.4748	0.4770	0.4791	0.4813	0.4834	0.4856	4	7	11	14	18
26	0.4877	0.4899	0.4921	0.4942	0.4964	0.4986	0.5008	0.5029	0.5051	0.5073	4	7	11	15	18
27	0.5095	0.5117	0.5139	0.5161	0.5184	0.5206	0.5228	0.5250	0.5272	0.5295	4	7	11	15	18
28	0.5317	0.5340	0.5362	0.5384	0.5407	0.5430	0.5452	0.5475	0.5498	0.5520	4	8	11	15	19
29	0.5543	0.5566	0.5589	0.5612	0.5635	0.5658	0.5681	0.5704	0.5727	0.5750	4	8	12	15	19
30	0.5774	0.5797	0.5820	0.5844	0.5867	0.5890	0.5914	0.5938	0.5961	0.5985	4	8	12	16	20
31	0.6009	0.6032	0.6056	0.6080	0.6104	0.6128	0.6152	0.6176	0.6200	0.6224	4	8	12	16	20
32	0.6249	0.6273	0.6297	0.6322	0.6346	0.6371	0.6395	0.6420	0.6445	0.6469	4	8	12	16	20
33	0.6494	0.6519	0.6544	0.6569	0.6594	0.6619	0.6644	0.6669	0.6694	0.6720	4	8	13	17	21
34	0.6745	0.6771	0.6796	0.6822	0.6847	0.6873	0.6899	0.6924	0.6950	0.6976	4	9	13	17	21
35	0.7002	0.7028	0.7054	0.7080	0.7107	0.7133	0.7159	0.7186	0.7212	0.7239	4	9	13	18	22
36	0.7265	0.7292	0.7319	0.7346	0.7373	0.7400	0.7427	0.7454	0.7481	0.7508	5	9	14	18	23
37	0.7536	0.7563	0.7590	0.7618	0.7646	0.7673	0.7701	0.7729	0.7757	0.7785	5	9	14	18	23
38	0.7813	0.7841	0.7869	0.7898	0.7926	0.7954	0.7983	0.8012	0.8040	0.8069	5	9	14	19	24
39	0.8098	0.8127	0.8156	0.8185	0.8214	0.8243	0.8273	0.8302	0.8332	0.8361	5	10	15	20	24
40	0.8391	0.8421	0.8451	0.8481	0.8511	0.8541	0.8571	0.8601	0.8632	0.8662	5	10	15	20	25
41	0.8693	0.8724	0.8754	0.8785	0.8816	0.8847	0.8878	0.8910	0.8941	0.8972	5	10	16	21	26
42	0.9004	0.9036	0.9067	0.9099	0.9131	0.9163	0.9195	0.9228	0.9260	0.9293	5	11	16	21	27
43	0.9325	0.9358	0.9391	0.9424	0.9457	0.9490	0.9523	0.9556	0.9590	0.9623	6	11	17	22	28
44	0.9657	0.9691	0.9725	0.9759	0.9793	0.9827	0.9861	0.9896	0.9930	0.9965	6	11	17	23	29

NATURAL TANGENTS

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
45	1.0000	1.0035	1.0070	1.0105	1.0141	1.0176	1.0212	1.0247	1.0283	1.0319	6	12	18	24	30
46	1.0355	1.0392	1.0428	1.0464	1.0501	1.0538	1.0575	1.0612	1.0649	1.0686	6	12	18	25	31
47	1.0724	1.0761	1.0799	1.0837	1.0875	1.0913	1.0951	1.0990	1.1028	1.1067	6	13	19	25	32
48	1.1106	1.1145	1.1184	1.1224	1.1263	1.1303	1.1343	1.1383	1.1423	1.1463	7	13	20	27	33
49	1.1504	1.1544	1.1585	1.1626	1.1667	1.1708	1.1750	1.1792	1.1833	1.1875	7	14	21	28	34
50	1.1918	1.1960	1.2002	1.2045	1.2088	1.2131	1.2174	1.2218	1.2261	1.2305	7	14	22	29	36
51	1.2349	1.2393	1.2437	1.2482	1.2527	1.2572	1.2617	1.2662	1.2708	1.2753	8	15	23	30	38
52	1.2799	1.2846	1.2892	1.2938	1.2985	1.3032	1.3079	1.3127	1.3175	1.3222	8	16	24	31	39
53	1.3270	1.3319	1.3367	1.3416	1.3465	1.3514	1.3564	1.3613	1.3663	1.3713	8	16	25	33	41
54	1.3764	1.3814	1.3865	1.3916	1.3968	1.4019	1.4071	1.4124	1.4176	1.4229	9	17	26	34	43
55	1.4281	1.4335	1.4388	1.4442	1.4496	1.4550	1.4605	1.4659	1.4715	1.4770	9	18	27	36	45
56	1.4826	1.4882	1.4938	1.4994	1.5051	1.5108	1.5166	1.5224	1.5282	1.5340	10	19	29	38	48
57	1.5399	1.5458	1.5517	1.5577	1.5637	1.5697	1.5757	1.5818	1.5880	1.5941	10	20	30	40	50
58	1.6003	1.6066	1.6128	1.6191	1.6255	1.6319	1.6383	1.6447	1.6512	1.6577	11	21	32	43	53
59	1.6643	1.6709	1.6775	1.6842	1.6909	1.6977	1.7045	1.7113	1.7182	1.7251	11	23	34	45	56
60	1.7321	1.7391	1.7461	1.7532	1.7603	1.7675	1.7747	1.7820	1.7893	1.7966	12	24	36	48	60
61	1.8040	1.8115	1.8190	1.8265	1.8341	1.8418	1.8495	1.8572	1.8650	1.8728	13	26	38	51	64
62	1.8807	1.8887	1.8967	1.9047	1.9128	1.9210	1.9292	1.9375	1.9458	1.9542	14	27	41	55	68
63	1.9626	1.9711	1.9797	1.9883	1.9970	2.0057	2.0145	2.0233	2.0323	2.0413	15	29	44	58	73
64	2.0503	2.0594	2.0686	2.0778	2.0872	2.0965	2.1060	2.1155	2.1251	2.1348	16	31	47	63	78
65	2.1445	2.1543	2.1642	2.1742	2.1842	2.1943	2.2045	2.2148	2.2251	2.2355	17	34	51	68	85
66	2.2460	2.2566	2.2673	2.2781	2.2889	2.2998	2.3109	2.3220	2.3332	2.3445	18	37	55	73	92
67	2.3559	2.3673	2.3789	2.3906	2.4023	2.4142	2.4262	2.4383	2.4504	2.4627	20	40	60	79	99
68	2.4751	2.4876	2.5002	2.5129	2.5257	2.5386	2.5517	2.5649	2.5782	2.5916	22	43	65	87	108
69	2.6051	2.6187	2.6325	2.6464	2.6605	2.6746	2.6889	2.7034	2.7179	2.7326	24	47	71	95	119
70	2.7475	2.7625	2.7776	2.7929	2.8083	2.8239	2.8397	2.8556	2.8716	2.8878	26	52	78	104	131
71	2.9042	2.9208	2.9375	2.9544	2.9714	2.9887	3.0061	3.0237	3.0415	3.0595	29	58	87	116	145
72	3.0777	3.0961	3.1146	3.1334	3.1524	3.1716	3.1910	3.2106	3.2305	3.2506	32	64	96	129	161
73	3.2709	3.2914	3.3122	3.3332	3.3544	3.3759	3.3977	3.4197	3.4420	3.4646	36	72	108	144	180
74	3.4874	3.5105	3.5339	3.5576	3.5816	3.6059	3.6305	3.6554	3.6806	3.7062	41	81	122	163	204
75	3.7321	3.7583	3.7848	3.8118	3.8391	3.8667	3.8947	3.9232	3.9520	3.9812	46	93	139	186	232
76	4.0108	4.0408	4.0713	4.1022	4.1335	4.1653	4.1976	4.2303	4.2635	4.2972	53	107	160	213	267
77	4.3315	4.3662	4.4015	4.4373	4.4737	4.5107	4.5483	4.5864	4.6252	4.6646					
78	4.7046	4.7453	4.7867	4.8288	4.8716	4.9152	4.9594	5.0045	5.0504	5.0970					
79	5.1446	5.1929	5.2422	5.2924	5.3435	5.3955	5.4486	5.5026	5.5578	5.6140					
80	5.6713	5.7297	5.7894	5.8502	5.9124	5.9758	6.0405	6.1066	6.1742	6.2432					
81	6.3138	6.3859	6.4596	6.5350	6.6122	6.6912	6.7720	6.8548	6.9395	7.0264					
82	7.1154	7.2066	7.3002	7.3962	7.4947	7.5958	7.6996	7.8062	7.9158	8.0285					
83	8.1443	8.2636	8.3863	8.5126	8.6427	8.7769	8.9152	9.0579	9.2052	9.3572					
84	9.5144	9.6768	9.8448	10.0187	10.1988	10.3854	10.5789	10.7797	10.9882	11.2048					
85	11.4301	11.6645	11.9087	12.1632	12.4288	12.7062	12.9962	13.2996	13.6174	13.9507					
86	14.3007	14.6685	15.0557	15.4638	15.8945	16.3499	16.8319	17.3432	17.8863	18.4645					
87	19.0811	19.7403	20.4465	21.2049	22.0217	22.9038	23.8593	24.8978	26.0307	27.2715					
88	28.6363	30.1446	31.8205	33.6935	35.8006	38.1885	40.9174	44.0661	47.7395	52.0807					
89	57.2900	63.6567	71.6151	81.8470	95.4895	114.5887	143.2371	190.9842	286.4777	572.9572					

Exercise 2.1

1. (i) $\sin\theta = \frac{6}{10}$, $\cos\theta = \frac{8}{10}$, $\tan\theta = \frac{6}{8}$, $\text{cosec}\,\theta = \frac{10}{6}$, $\sec\theta = \frac{10}{8}$, $\cot\theta = \frac{8}{6}$

(ii) $\sin\theta = \frac{7}{25}$, $\cos\theta = \frac{24}{25}$, $\tan\theta = \frac{7}{24}$, $\text{cosec}\,\theta = \frac{25}{7}$, $\sec\theta = \frac{25}{24}$, $\cot\theta = \frac{24}{7}$

(iii) $\sin\theta = \frac{35}{37}$, $\cos\theta = \frac{12}{37}$, $\tan\theta = \frac{35}{12}$, $\text{cosec}\,\theta = \frac{37}{35}$, $\sec\theta = \frac{37}{12}$, $\cot\theta = \frac{12}{35}$

(iv) $\sin\theta = \frac{9}{41}$, $\cos\theta = \frac{40}{41}$, $\tan\theta = \frac{9}{40}$, $\text{cosec}\,\theta = \frac{41}{9}$, $\sec\theta = \frac{41}{40}$, $\cot\theta = \frac{40}{9}$

2. (i) $\cos A = \frac{12}{15}$, $\tan A = \frac{9}{12}$, $\text{cosec}\,A = \frac{15}{9}$, $\sec A = \frac{15}{12}$, $\cot A = \frac{12}{9}$

(ii) $\sin A = \frac{8}{17}$, $\tan A = \frac{8}{15}$, $\text{cosec}\,A = \frac{17}{8}$, $\sec A = \frac{17}{15}$, $\cot A = \frac{15}{8}$

(iii) $\sin P = \frac{5}{13}$, $\cos P = \frac{12}{13}$, $\text{cosec}\,P = \frac{13}{5}$, $\sec P = \frac{13}{12}$, $\cot P = \frac{12}{5}$

(iv) $\sin\theta = \frac{15}{17}$, $\cos\theta = \frac{8}{17}$, $\tan\theta = \frac{15}{8}$, $\text{cosec}\,\theta = \frac{17}{15}$, $\cot\theta = \frac{8}{15}$

(v) $\sin\theta = \frac{60}{61}$, $\cos\theta = \frac{11}{61}$, $\tan\theta = \frac{60}{11}$, $\sec\theta = \frac{61}{11}$, $\cot\theta = \frac{11}{60}$

(vi) $\cos\theta = \frac{\sqrt{y^2 - x^2}}{y}$, $\tan\theta = \frac{x}{\sqrt{y^2 - x^2}}$, $\text{cosec}\,\theta = \frac{y}{x}$,

$\sec\theta = \frac{y}{\sqrt{y^2 - x^2}}$, $\cot\theta = \frac{\sqrt{y^2 - x^2}}{x}$

3. (i) 45° (ii) 0° (iii) 60° (iv) 30°

4. $\sin A = \frac{24}{26}$, $\cos A = \frac{10}{26}$, $\tan A = \frac{24}{10}$, $\text{cosec}\,A = \frac{26}{24}$, $\sec A = \frac{26}{10}$, $\cot A = \frac{10}{24}$

$\sin C = \frac{10}{26}$, $\cos C = \frac{24}{26}$, $\tan C = \frac{10}{24}$, $\text{cosec}\,C = \frac{26}{10}$, $\sec C = \frac{26}{24}$, $\cot C = \frac{24}{10}$

5. $\frac{17}{19}$ **6.** 1 **7.** $\frac{-63}{4}$ **8.** 1 **9.** $\frac{225}{64}$ **10.** (i) 1 (ii) 0

13. (i) $\sqrt{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{\sqrt{3}}{4}$ (iv) $\frac{25}{144}$ (v) 7 (vi) $\frac{4}{3}$ (vii) 9 (viii) 2

Exercise 2.2

1. (i) 1 (ii) 1 (iii) 1 (iv) 1 (v) 1 (vi) 1

2. (i) 0 (ii) 2 (iii) 0 (iv) 6 (v) 1 (vi) 9 (vii) 0 (viii) $\frac{3}{2}$ (ix) $\frac{1}{\sqrt{3}}$

3. (i) 60° (ii) 41° (iii) 55° (iv) 55° (v) 47° (vi) 60°

Exercise 2.3

1. (i) 0.4384 (ii) 0.3090 (iii) 0.7002 (iv) 0.9670 (v) 0.2113 (vi) 0.9760 (vii) 0.7623 (viii) 0.1841 (ix) 2.7475 (x) 1.1778 **2.** (i) $44^\circ 30'$ (ii) $14^\circ 54'$ (iii) $20^\circ 12'$ (iv) $76^\circ 30'$ (v) $89^\circ 6'$ **3.** (i) 1.2698 (ii) 1.3579 (iii) 1.0042 (iv) 4.4996 (v) 4.8098 **4.** 99.4134 cm^2 **5.** 14.6278 cm^2 **6.** 109.376 cm^2

7. 67.0389 cm^2 **8.** 13.8568 m **9.** 60° **10.** 8.09 cm **11.** 3.1056 cm **12.** 20.784 cm

Exercise 2.4

1. A 2. C 3. C 4. C 5. B 6. A 7. B 8. C 9. B 10. C 11. D 12. B 13. D 14. A 15. A

Mathematics

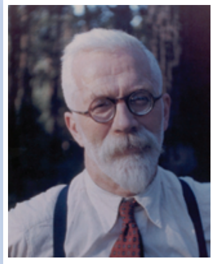
3 STATISTICS

"Statistical thinking today is as necessary for efficient citizenship as the ability to read and write"

Herbert. G. Wells

Main Targets

- To draw Histogram, Frequency Polygon
- To find Measures of Central Tendency : Mean, Median and Mode



SIR RONALD AYLMER FISHER FRS
1890 – 1962

Sir Ronald Aylmer Fisher FRS (17 February 1890 – 29 July 1962) was an English statistician, evolutionary biologist, geneticist, and eugenicist. Fisher is known as one of the chief architects of the neo-Darwinian synthesis, for his important contributions to statistics, including the analysis of variance (ANOVA), method of maximum likelihood, fiducial inference, and the derivation of various sampling distributions, and for being one of the three principal founders of population genetics. He was awarded the Linnean Society of London's prestigious Darwin–Wallace Medal in 1958.

3.1 Introduction

The subject statistics comprises the collection, organization, presentation, analysis and interpretation of data, assists in decision making. In the earlier classes you have studied about the collection of statistical data through primary sources and secondary sources. The data collected through these sources may contain a large number of numerical facts. These numerical facts must be arranged and presented in a tabular form in an orderly way before analysis and interpretation.

In case of some investigations, the classification and tabulation will give a clear picture of the significance of the data arranged so that no further analysis is required. However, these forms of presentation do not always prove to be interesting to the common man. One of the most convincing and appealing ways, in which statistical results may be presented is through diagrams and graphs.

3.2 Graphical Representation of Frequency Distribution

It is often said that "one picture is worth a thousand words." Indeed, statisticians have employed graphical techniques to more vividly describe the data. In particular, histograms and polygons are used to describe quantitative data that have been grouped into frequency, or percentage distributions.

A frequency distribution is organizing of raw data in tabular form, using classes and frequencies. A frequency distribution can be represented graphically by

(i) Histogram (ii) Frequency polygon
(iii) Smoothened frequency curve and (iv) Ogive or Cumulative frequency curve.

In this chapter we see the first two types of graphs, other two will be discussed in higher classes.

3.2.1 Histogram

Out of several methods of graphical representation of a frequency distribution, histogram is the most popular and widely used method. A histogram is a two dimensional graphical representation of continuous frequency distribution. In a histogram, rectangles are drawn such that the areas of the rectangles are proportional to the corresponding frequencies.

To draw a histogram with equal class intervals

1. Mark the intervals on the horizontal axis and the frequencies on the vertical axis.
2. The scales for both the axes need not be the same.
3. Class intervals must be exclusive. If the intervals are in inclusive form, convert them to the exclusive form.
4. Draw rectangles with class intervals as bases and the corresponding frequencies as lengths. The class limits are marked on the horizontal axis and the frequency is marked on the vertical axis. Thus a rectangle is constructed on each class interval.

Remark A histogram is similar to a bar graph. However, a histogram utilizes classes (intervals) and frequencies while a bar graph utilizes categories and frequencies . Histograms are used only for continuous data that is grouped.

3.2.2 Frequency Polygon

A frequency polygon uses the mid-point of a class interval to represent all the data in that interval. It is constructed by taking mid-points of class intervals on the horizontal axis and the frequencies on the vertical axis and joining these points. The two extremes are joined with the base in such a way that they touch the horizontal axis at half the distance of class interval outside the extreme points.

If we have to construct histogram and frequency polygon both, first draw the histogram and then join the mid-points of the tops of all the rectangles and finally the extreme points with the points outside the extreme rectangles.

Remark A histogram is often drawn as a guide, so that a frequency polygon can be drawn over the top.

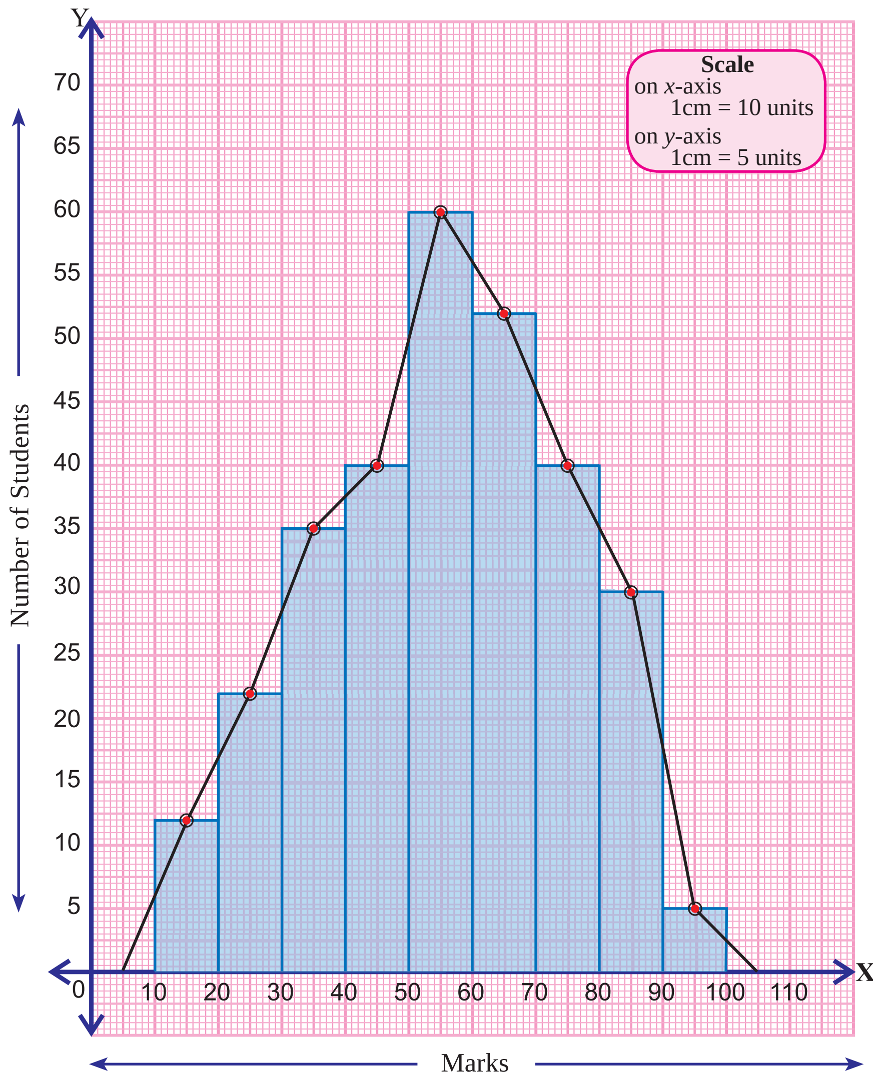
Example 3.1

Draw a histogram and frequency polygon to represent the following data.

Marks	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
No. of Students	12	22	35	40	60	52	40	30	5

Solution First we draw the histogram and then by joining the midpoints of the tops of the rectangles we draw the frequency polygon.

Histogram and Frequency Polygon



In the above example, the intervals are exclusive. Now, let us consider an example with inclusive intervals.

Example 3.2

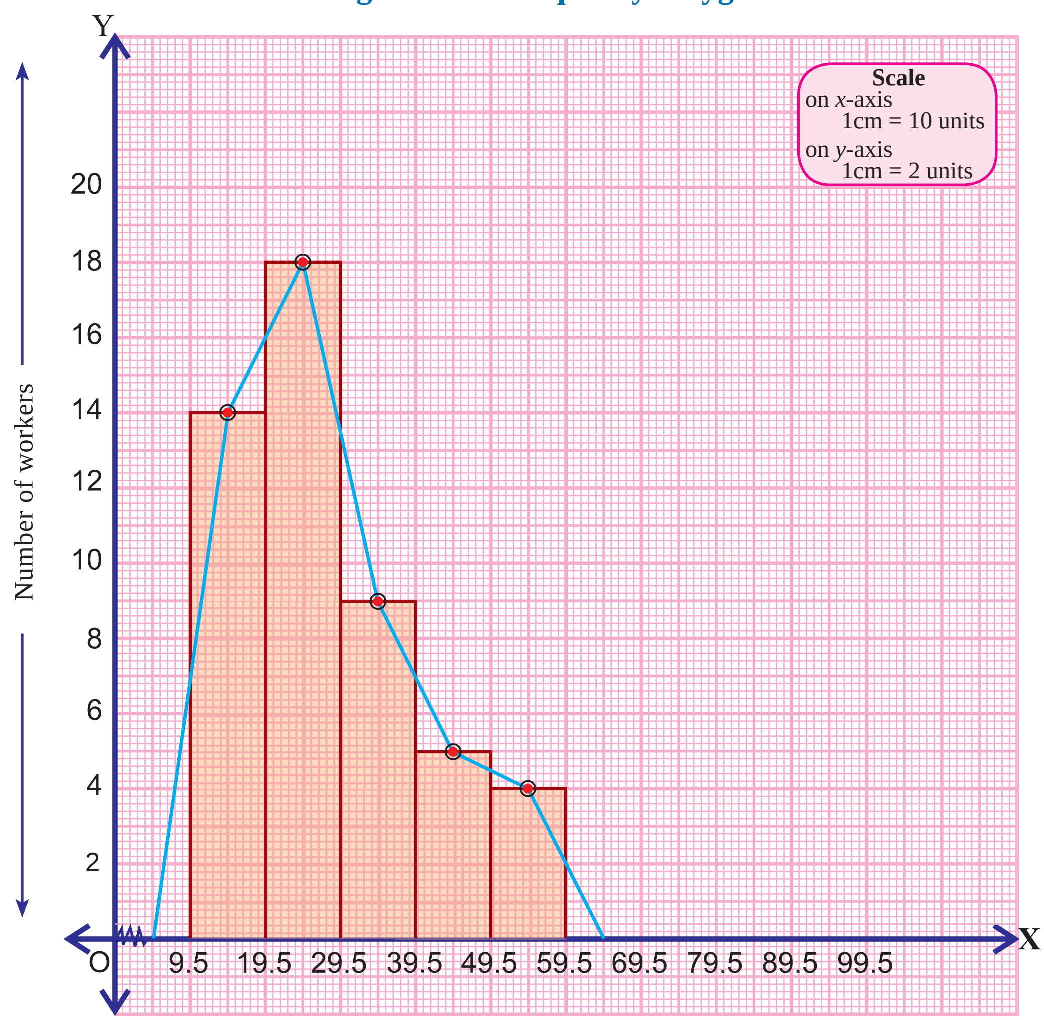
A survey was conducted in a small industrial plant having 50 workers to find the number of km each person commuted to work and the details are given below. Represent it by a histogram and frequency polygon.

Commuted Distance (km)	50-59	40-49	30-39	20-29	10-19
No. of Workers	4	5	9	18	14

Solution In the given table the class intervals are inclusive. So we convert them to the exclusive form and arrange the class intervals in ascending order.

Commuted Distance (km)	9.5-19.5	19.5-29.5	29.5-39.5	39.5-49.5	49.5-59.5
No. of Workers	14	18	9	5	4

Histogram and Frequency Polygon



Commuted distance (in k.m.)

Note

(i) The class intervals are made continuous and then the histogram is constructed.

(ii) If the scale along the horizontal axis does not start at the origin, a zig - zag curve is shown near the origin.

3.2.3 Histogram with Varying Base Width

Consider the following frequency distribution:

Time (seconds)	40-60	60-70	70-80	80-85	85-90	90-120
Frequency	100	60	90	70	60	90

The class interval 40-60 appears to be most popular, as it has the highest frequency. Note that this frequency 100 is spread across a time of 20 seconds. Although the class interval 80-85 has only a frequency of 70, this frequency is spread across a time of only 5 seconds.

So, we need to take into account the width of each class interval, before we draw the histogram, otherwise the histogram would not represent the data set correctly.

We do this by calculating the frequency density and modifying length of the rectangle.

Key Concept **Frequency Density**

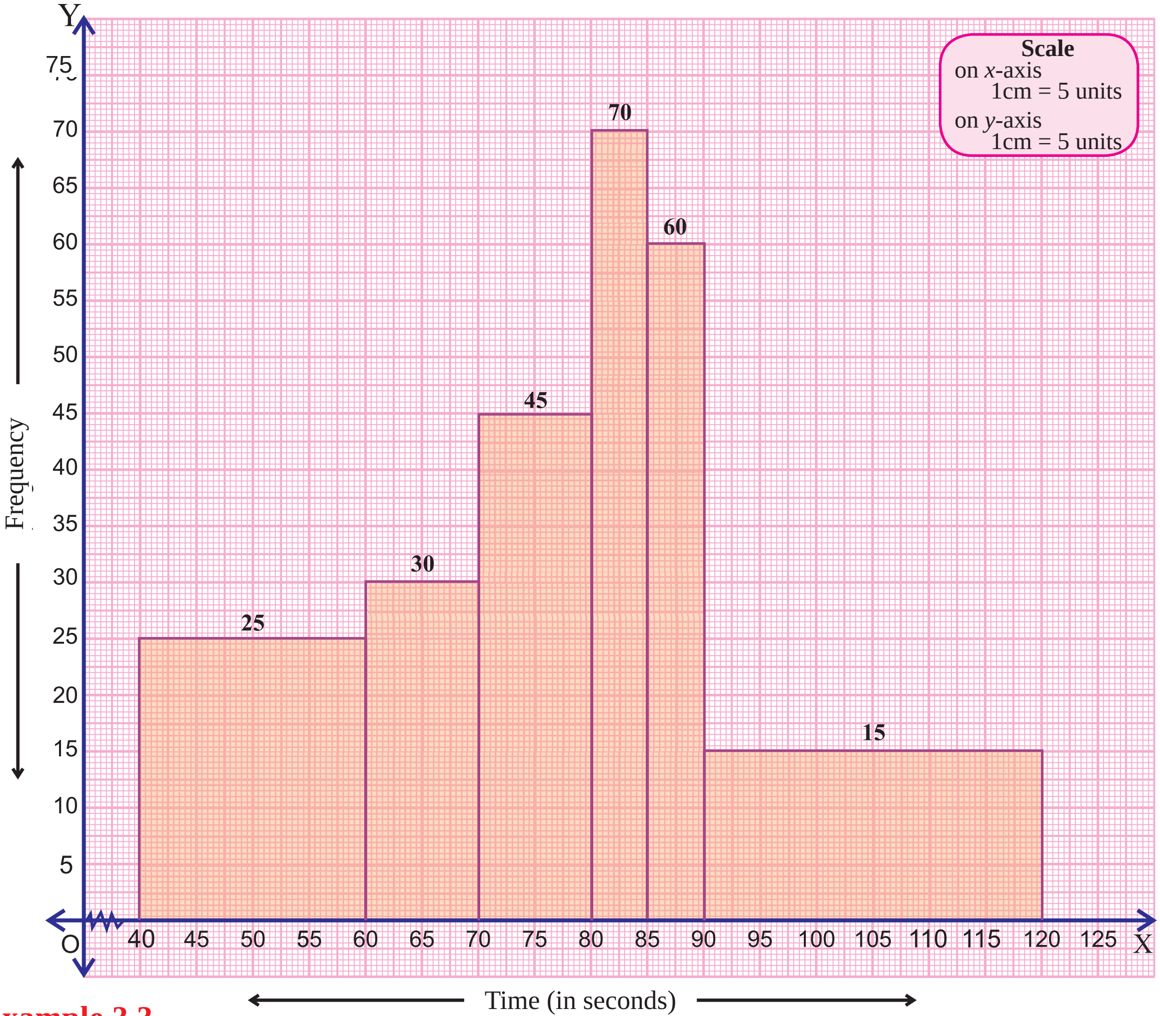
Frequency density = Frequency ÷ class width

If C denotes the minimum class width of the data set, then the length of the rectangle is given by

$$\text{Length of the Rectangle} = \frac{\text{Frequency}}{\text{Class width}} \times C$$

Time (Seconds)	40-60	60-70	70-80	80-85	85-90	90-120
Frequency	100	60	90	70	60	90
Class Width	20	10	10	5	5	30
Length of the Rectangle	$\frac{100}{20} \times 5 = 25$	$\frac{60}{10} \times 5 = 30$	$\frac{90}{10} \times 5 = 45$	$\frac{70}{5} \times 5 = 70$	$\frac{60}{5} \times 5 = 60$	$\frac{90}{30} \times 5 = 15$

Histogram with Varying Base Length



Example 3.3

Draw a histogram to represent the following data set.

Marks	0-10	10-20	20-40	40-50	50-60	60-70	70-90	90-100
No. of students	4	6	14	16	14	8	16	5

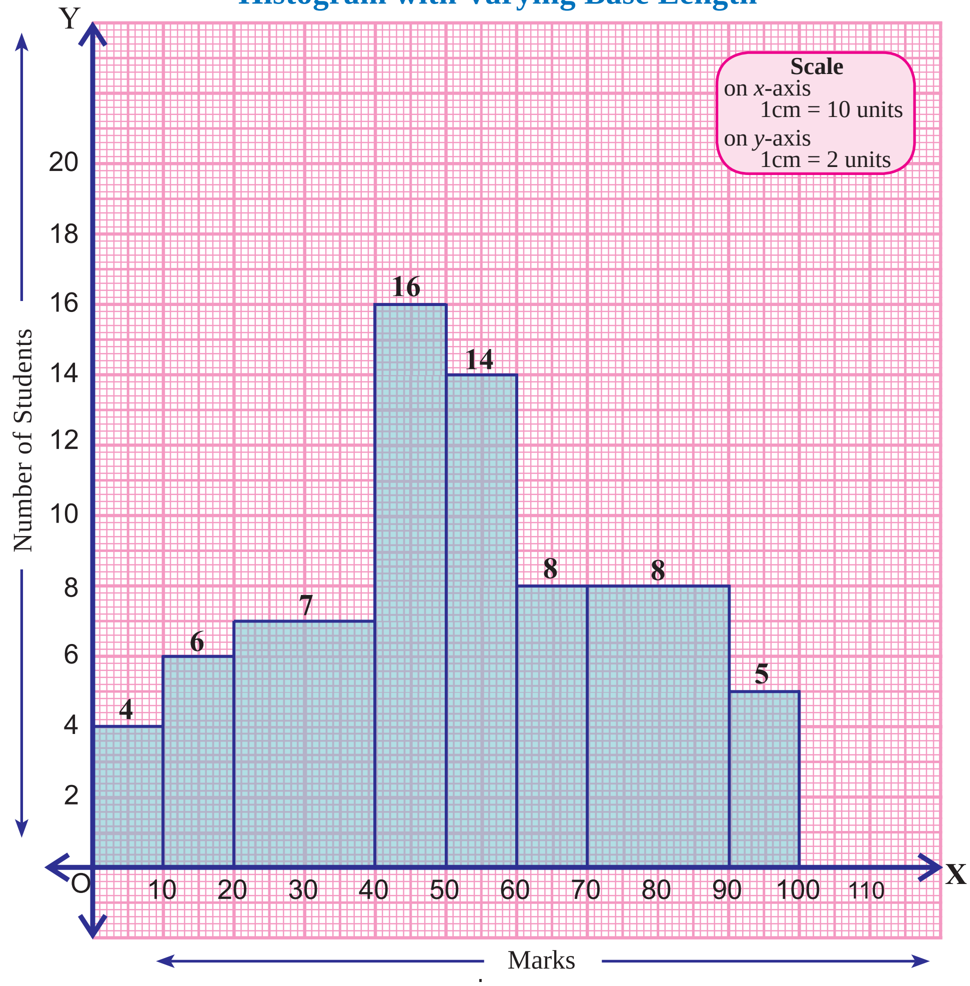
Solution The minimum of the class widths of the data set is 10. So, we draw rectangles with class intervals as bases and the lengths of the rectangles given by

$$\text{Length of Rectangle} = \frac{\text{Frequency}}{\text{Class width}} \times 10.$$

Thus, the histogram can be drawn as follows.

Marks	0-10	10-20	20-40	40-50	50-60	60-70	70-90	90-100
No. of students	4	6	14	16	14	8	16	5
Class width	10	10	20	10	10	10	20	10
Length of the Rectangle	$\frac{4}{10}\times 10$ $= 4$	$\frac{6}{10}\times 10$ $= 6$	$\frac{14}{20}\times 10$ $= 7$	$\frac{16}{10}\times 10$ $= 16$	$\frac{14}{10}\times 10$ $= 14$	$\frac{8}{10}\times 10$ $= 8$	$\frac{16}{20}\times 10$ $= 8$	$\frac{5}{10}\times 10$ $= 5$

Histogram with Varying Base Length



Exercise 3.1

1. Draw a histogram for the following distribution.

Class Interval	0-10	10-30	30-45	45-50	50-60
Frequency	8	28	18	6	10

2. Draw a histogram for the monthly wages of the workers in a factory as per data given below.

Monthly wages (₹)	2000-2200	2200-2400	2400-2800	2800-3000	3000-3200	3200-3600
No. of workers	25	30	50	60	15	10

3. The following distribution gives the mass of 48 objects measured to the nearest gram. Draw a histogram to illustrate the data.

Mass in (gms)	10-19	20-24	25-34	35-49	50-54
No. of objects	6	4	12	18	8

4. Draw a histogram to represent the following data.

Class interval	10-14	14-20	20-32	32-52	52-80
Frequency	5	6	9	25	21

5. The age (in years) of 360 patients treated in the hospital on a particular day are given below.

Age in years	10-20	20-30	30-50	50-60	60-70
No. of patients	80	50	80	120	30

Draw a histogram for the above data.

Measures of Central Tendency

One of the main objectives of statistical analysis is to get a single value that describes the characteristic of the entire data. Such a value is called the central value and the most commonly used measures of central tendencies are Arithmetic Mean, Median and Mode.

3.3 Mean

3.3.1 Arithmetic Mean - Raw Data

The arithmetic mean is the sum of a set of observations, positive, negative or zero, divided by the number of observations. If we have n real numbers $x_1, x_2, x_3 ...., x_n$, then their arithmetic mean, denoted by $\overline{x}$, is given by

$$\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} \text{ or } \overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ or } \overline{x} = \frac{\sum x}{n}$$

Remark $\overline{x} = \frac{\sum x}{n} \implies n\overline{x} = \sum x$. That is,

Total number of observations $\times$ Mean = Sum of all observations

Think and Answer !

Can a person of height 5 feet, who does not know swimming wade through a river which has an average depth of 4 feet to the other bank?

Example 3.4

Find the arithmetic mean of the marks 72, 73, 75, 82, 74 obtained by a student in 5 subjects in an annual examination.

Solution Here n = 5

$$\overline{x} = \frac{\sum x}{n} = \frac{72 + 73 + 75 + 82 + 74}{5} = \frac{376}{5} = 75.2$$

$\therefore$ Mean = 75.2

Example 3.5

The mean of the 5 numbers is 32. If one of the numbers is excluded, then the mean is reduced by 4. Find the excluded number.

Solution

Mean of 5 numbers = 32.

Sum of these numbers $= 32 \times 5 = 160$ $\qquad (\because n\overline{x} = \sum x)$

Mean of 4 numbers $= 32 - 4 = 28$

Sum of these 4 numbers $= 28 \times 4 = 112$

Excluded number = (Sum of the 5 given numbers) − (Sum of the 4 numbers)

$= 160 - 112 = 48$

3.3.2 Arithmetic Mean - Ungrouped Frequency Distribution

The mean (or average) of the observations $x_1, x_2, x_3 \ldots x_n$ with frequencies $f_1, f_2, f_3 \ldots f_n$ respectively is given by

$$\overline{x} = \frac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{f_1 + f_2 + \cdots + f_n} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

The above formula, more briefly, is written as $\overline{x} = \dfrac{\sum fx}{\sum f}$

Example 3.6

Obtain the mean of the following data.

x	5	10	15	20	25
f	3	10	25	7	5

Solution

x	f	fx
5	3	15
10	10	100
15	25	375
20	7	140
25	5	125
Total	$\sum f = 50$	$\sum fx = 755$

$$\text{Mean} = \frac{\sum fx}{\sum f} = \frac{755}{50} = 15.1$$

Mean = 15.1

3.3.3 Arithmetic Mean - Grouped Frequency Distribution

Consider the following frequency table.

Class interval (Marks)	0-10	10-20	20-30	30-40	40-50
Frequency (No. of students)	3	4	3	7	8

The first entry of the table says that 3 children got less than 10 marks but does not say anything about the marks got by the individuals. Now, for each class interval we require a point which could serve as the representative of the class interval. In the interval 0-10, we assume it as 5. That is, it is assumed that the frequency of each class interval is centered around its mid-point. Thus the mid-point or the class mark of each class can be chosen to represent the observations falling in that class.

Class mark = $\frac{UCL + LCL}{2}$ (UCL= Upper Class Limit, LCL = Lower Class Limit)

Using the above formula the class marks for each of the class intervals are found out and are represented as x

$\bar{x} = \frac{\sum fx}{\sum f}$ can be used to find the mean of the grouped data.

In grouped frequency distribution, arithmetic mean may be computed by applying any one of the following methods:

(i) Direct Method

(ii) Assumed Mean Method

(iii) Step Deviation Method

Direct Method

When direct method is used, the formula for finding the arithmetic mean is

$$\bar{x} = \frac{\sum fx}{\sum f},$$

where x is the midpoint of the class interval and f is the frequency.

Steps :

(i) Obtain the midpoint of each class and denote it by x.

(ii) Multiply these midpoints by the respective frequency of each class and obtain the total of fx.

(iii) Divide $\sum fx$ by the sum $\sum f$ of the frequencies to obtain Mean.

Example 3.7

From the following table compute arithmetic mean by direct method.

Marks	0-10	10-20	20-30	30-40	40-50	50-60
No. of students	5	10	25	30	20	10

Solution

Marks	Midpoint (x)	No. of students (f)	fx
0-10	5	5	25
10-20	15	10	150
20-30	25	25	625
30-40	35	30	1050
40-50	45	20	900
50-60	55	10	550
		$\sum f = 100$	$\sum fx = 3300$

$$\overline{x} = \frac{\sum fx}{\sum f} = \frac{3300}{100} = 33$$

$\therefore$ Mean $= 33$

Assumed Mean Method

When assumed mean method is used, arithmetic mean is computed by applying the following formula.

$$\overline{x} = A + \frac{\sum fd}{\sum f},$$

where A is the assumed mean and $d = x - A$ is the deviation of midpoint x from assumed mean A.

Steps:

(i) Choose A as the assumed mean.

(ii) Find the deviation, $d = x - A$ for each class

(iii) Multiply the respective frequencies of each class by their deviations and obtain $\sum fd$.

(iv) Apply the formula $\overline{x} = A + \frac{\sum fd}{\sum f}$

Example 3.8

Calculate the arithmetic mean by assumed mean method for the data given in the above example.

Solution Let the assumed mean be A = 35

Marks	Mid-value (x)	No of students (f)	$d = x - 35$	fd
0-10	5	5	-30	-150
10-20	15	10	-20	-200
20-30	25	25	-10	-250
30-40	35	30	0	0
40-50	45	20	10	200
50-60	55	10	20	200
		$\sum f = 100$		$\sum fd = -200$

$$\overline{x} = A + \frac{\sum fd}{\sum f}$$

$$= 35 + \left(\frac{-200}{100}\right) = 35 - 2 = 33$$

Step Deviation Method

In order to simplify the calculation, we divide the deviation by the width of the class intervals. i.e calculate $\frac{x - A}{c}$ and then multiply by c in the formula for getting the mean of the data.

$$\overline{x} = A + \frac{\sum fd}{\sum f} \times c$$

Solution: Width of the class interval is c = 10

Marks	Mid-value	No of students f	$d = \frac{x - 35}{10}$	fd
0-10	5	5	-3	-15
10-20	15	10	-2	-20
20-30	25	25	-1	-25
30-40	35	30	0	0
40-50	45	20	1	20
50-60	55	10	2	20
		$\sum f = 100$		$\sum fd = -20$

$$\overline{x} = A + \frac{\sum fd}{\sum f} \times c \quad = 35 - \left(\frac{20}{100} \times 10\right) = 35 - 2 = 33$$

$\therefore$ Mean $= 33$

Mathematics

3.3.4 Properties of Mean

Property 1

Sum of the deviations taken from the arithmetic mean is zero.

If $x_1, x_2, x_3 ... x_n$ are n observations with mean $\overline{x}$ then $(x_1 - \overline{x}) + (x_2 - \overline{x}) + ... + (x_n - \overline{x}) = 0$

For example, the mean of 6, 8, 9, 14, 13 is 10. Consider the deviation of each observation from arithmetic mean.

Sum of the deviations from arithmetic mean is

$(6 - 10) + (8 - 10) + (9 - 10) + (14 - 10) + (13 - 10)$

$$= -4 + (-2) + (-1) + 4 + 3 = -7 + 7 = 0$$

Hence, from the above example, we observe that sum of the deviations from the arithmetic mean is zero.

Property 2

If each observation is increased by k then the mean of the new observations is the original mean increased by k.

i.e., suppose the mean of n observations $x_1, x_2, x_3 ... x_n$ is $\overline{x}$. Each observation is increased by k, then the mean of the new observation is $(\overline{x} + k)$.

For example, consider five numbers x_1, x_2, x_3, x_4 and x_5 whose mean is 20.

$$\text{i.e., } \overline{x} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} = 20$$

If each of the number is increased by 5, then the new numbers are

$x_1 + 5, x_2 + 5, x_3 + 5, x_4 + 5$ and $x_5 + 5$

Mean of these new numbers is

$$\frac{x_1 + 5 + x_2 + 5 + x_3 + 5 + x_4 + 5 + x_5 + 5}{5}$$

$$= \frac{x_1 + x_2 + x_3 + x_4 + x_5 + 25}{5}$$

$$= \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} + \frac{25}{5} = 20 + 5$$

= original mean + the increased value.

Hence, the original mean is increased by 5.

Property 3

If each observation is decreased by k, then the mean of the new observations is original mean decreased by k.

i.e., suppose the mean of n observations is $\overline{x}$. If each observation is decreased by k, then the mean of the new observation is $\overline{x} - k$.

For example, consider five numbers x_1, x_2, x_3, x_4 and x_5 whose mean is 20.

i.e., $\frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} = 20$

If each of the number is decreased by 5, then the new numbers are

$$x_1 - 5, x_2 - 5, x_3 - 5, x_4 - 5, x_5 - 5.$$

$$\text{New mean} = \frac{x_1 - 5 + x_2 - 5 + x_3 - 5 + x_4 - 5 + x_5 - 5}{5}$$

$$= \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} - \frac{25}{5}$$

$$= 20 - 5$$

= original mean - the decreased value.

Hence, the original mean is decreased by 5.

Property 4

If each observation is multiplied by k, $k \neq 0$, then the mean of the new observation is the original mean multiplied by k

i.e., suppose the mean of n observations $x_1, x_2, x_3 ... x_n$ is $\overline{x}$. If each observation is multiplied by k, $k \neq 0$, then the mean of the new observations is $k\overline{x}$.

For example, consider five numbers x_1, x_2, x_3, x_4 and x_5 whose mean is 20.

$$\text{Mean of these numbers} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} = 20$$

If each data is multiplied by 5, then the new observations are $5x_1, 5x_2, 5x_3, 5x_4, 5x_5$.

$$\text{New mean} = \frac{5x_1 + 5x_2 + 5x_3 + 5x_4 + 5x_5}{5}$$

$$= \frac{5(x_1 + x_2 + x_3 + x_4 + x_5)}{5} = 5\,(20)$$

= Five times the original mean.

Hence, the new mean is 5 times its original mean.

Property 5

If each observation is divided by k, $k \neq 0$, then the mean of new observations is the original mean divided by k.

i.e., suppose the mean of the n observations $x_1, x_2, x_3, x_4, x_5 \ldots x_n$ is $\overline{x}$. If each observation is divided by k, where $k \neq 0$, then the mean of the new observation is $\frac{\overline{x}}{k}$.

For example, consider five numbers x_1, x_2, x_3, x_4 and x_5 whose mean is 20. So,

$$\overline{x} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} = 20$$

Now we divide each number by 5. Let $y_1 = \frac{x_1}{5}$, $y_2 = \frac{x_2}{5}$, $y_3 = \frac{x_3}{5}$, $y_4 = \frac{x_4}{5}$ and $y_5 = \frac{x_5}{5}$. Then

$$\begin{aligned}
\overline{y} &= \frac{\frac{x_1}{5} + \frac{x_2}{5} + \frac{x_3}{5} + \frac{x_4}{5} + \frac{x_5}{5}}{5} \\
&= \frac{1}{5}\left(\frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}\right) = \frac{1}{5}(20) \\
&= \frac{1}{5}(\overline{x})
\end{aligned}$$

New mean is the original mean divided by 5.

Example 3.9

The mean mark of 100 students was found to be 40. Later on, it was found that a score of 53 was misread as 83. Find the correct mean corresponding to the correct score.

Solution Given that the total number of students n = 100, $\overline{x} = 40$. So,

Incorrect $\sum x = \overline{x} \times n = 40 \times 100 = 4000$

Correct $\sum x$ = Incorrect $\sum x$ − wrong item + correct item.

$= 4000 - 83 + 53 = 3970$

$$\text{Correct } \overline{x} = \frac{\text{correct} \sum x}{n} = \frac{3970}{100} = 39.7$$

Hence the correct mean is 39.7.

Exercise 3.2

1. Obtain the mean number of bags sold by a shopkeeper on 6 consecutive days from the following table

Days	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
No. of bags sold	55	32	30	25	10	20

2. The number of children in 10 families in a locality are 2, 4, 3, 4, 1, 6, 4, 5, x, 5. Find x if the mean number of children in a family is 4

3. The mean of 20 numbers is 59. If 3 is added to each number what will be the new mean?

4. The mean of 15 numbers is 44. If 7 is subtracted from each number what will be the new mean?

5. The mean of 12 numbers is 48. If each numbers is multiplied by 4 what will be the new mean?

6. The mean of 16 numbers is 54. If each number is divided by 9 what will be the new mean?

7. The mean weight of 6 boys in a group is 48 kg. The individual weights of 5 of them are 50kg, 45kg, 50kg, 42kg and 40kg. Find the weight of the sixth boy.

8. Using assumed mean method find the mean weight of 40 students using the data given below.

weights in kg.	50	52	53	55	57
No. of students	10	15	5	6	4

9. The arithmetic mean of a group of 75 observations was calculated as 27. It was later found that one observation was wrongly read as 43 instead of the correct value 53. Obtain the correct arithmetic mean of the data.

10. Mean of 100 observations is found to be 40. At the time of computation two items were wrongly taken as 30 and 27 instead of 3 and 72. Find the correct mean.

11. The data on number of patients attending a hospital in a month are given below. Find the average number of patients attending the hospital in a day.

No. of patients	0-10	10-20	20-30	30-40	40-50	50-60
No. of days attending hospital	2	6	9	7	4	2

12. Calculate the arithmetic mean for the following data using step deviation method.

Marks	0-10	10-20	20-30	30-40	40-50	50-60
No. of students	8	15	22	20	10	5

13. In a study on patients, the following data were obtained. Find the arithmetic mean.

Age (in yrs)	10-19	20-29	30-39	40-49	50-59
No. of patients	1	0	1	10	13

14. The total marks obtained by 40 students in the Annual examination are given below

Marks	150 - 200	200 - 250	250 - 300	300 - 350	350 - 400	400 - 450	450 - 500
Students	2	3	12	10	4	6	3

Using step deviation method to find the mean of the above data.

Mathematics

15. Compute the arithmetic mean of the following distribution.

Class Interval	0 - 19	20 - 39	40 - 59	60 - 79	80 - 99
Frequency	3	4	15	14	4

3.4 Median

Median is defined as the middle item of the given observations arranged in order.

3.4.1 Median - Raw Data

Steps:

(i) Arrange the n given numbers in ascending or descending order of magnitude.

(ii) When n is odd, $\left(\frac{n+1}{2}\right)^{th}$ observation is the median.

(iii) When n is even the median is the arithmetic mean of the two middle values.

i.e., when n is even,

Median = Mean of $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observations.

Example 3.10

Find the median of the following numbers

(i) 24, 22, 23, 14, 15, 7, 21 (ii) 17, 15, 9, 13, 21, 32, 42, 7, 12, 10.

Solution

(i) Let us arrange the numbers in ascending order as below.

7, 14, 15, 21, 22, 23, 24

Number of items $n = 7$

$$\text{Median} = \left(\frac{n+1}{2}\right)^{th} \text{ observation} \quad (\because n \text{ is odd})$$

$$= \left(\frac{7+1}{2}\right)^{th} \text{ observation}$$

$$= 4^{th} \text{ observation} = 21$$

(ii) Let us arrange the numbers in ascending order

7, 9, 10, 12, 13, 15, 17, 21, 32, 42.

Number of items $n = 10$

Median is the mean of $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observations. $(\because n$ is even)

$$\left(\frac{n}{2}\right)^{th} \text{ observation} = \left(\frac{10}{2}\right)^{th} \text{ observation} = 5^{th} \text{ observation} = 13$$

$\left(\frac{n}{2}+1\right)^{th}$ observation= 6^{th} observation = 15.

$\therefore$ Median $= \frac{13+15}{2} = 14$

3.4.2 Median - Ungrouped Frequency Distribution

(i) Arrange the data in ascending or descending order of magnitude.

(ii) Construct the cumulative frequency distribution.

(iii) If n is odd, then Median = $\left(\frac{n}{2}+1\right)^{th}$ term.

(iv) If n is even, then Median = $\dfrac{\left\{\left(\frac{n}{2}\right)^{th} term + \left(\frac{n}{2}+1\right)^{th} term\right\}}{2}$

Example 3.11

Calculate the median for the following data.

Marks	20	9	25	50	40	80
No. of students	6	4	16	7	8	2

Solution Let us arrange marks in ascending order.

Marks	f	cf
9	4	4
20	6	10
25	16	26
40	8	34
50	7	41
80	2	43
	n = 43	

Here, n = 43, which is odd

Position of median $= \left(\frac{n+1}{2}\right)^{th}$ observation .

$= \left(\frac{43+1}{2}\right)^{th}$ observation.

$= 22^{nd}$ observation.

The above table shows that all items from 11 to 26 have their value 25. So, the value of 22^{nd} item is 25.

$\therefore$ Median = 25.

Example 3.12

Find the median for the following distribution.

Value	1	2	3	4	5	6
f	1	3	2	4	8	2

Solution

Value	f	cf
1	1	1
2	3	4
3	2	6
4	4	10
5	8	18
6	2	20
	n = 20	

$$n = 20 \text{ (even)}$$

$$\text{Position of the median} = \left(\frac{20+1}{2}\right)^{th} \text{th observation}$$

$$= \left(\frac{21}{2}\right)^{th} \text{th observation} = (10.5)^{th} \text{ observation}$$

The median then, is the average of the tenth and the eleventh items. The tenth item is 4, the eleventh item is 5.

$$\text{Hence median} = \frac{4+5}{2} = \frac{9}{2} = 4.5.$$

3.4.3 Median - Grouped Frequency Distribution

In a grouped frequency distribution, computation of median involves the following steps.

(i) Construct the cumulative frequency distribution.

(ii) Find $\frac{N}{2}^{th}$ term.

(iii) The class that contains the cumulative frequency $\frac{N}{2}$ is called the median class.

(iv) Find the median by using the formula:

$$\text{Median} = l + \frac{\frac{N}{2} - m}{f} \times c,$$

where l = Lower limit of the median class, f = Frequency of the median class

c = Width of the median class, N = The total frequency

m = cumulative frequency of the class preceeding the median class

Example 3.13

Find the median for the following distribution.

Wages (Rupees in hundreds)	0-10	10-20	20-30	30-40	40-50
No of workers	22	38	46	35	20

Solution

Wages	f	cf
0-10	22	22
10-20	38	60
20-30	46	106
30-40	35	141
40-50	20	161
	N = 161	

Here, $\frac{N}{2} = \frac{161}{2} = 80.5$. Median class is 20-30.

Lower limit of the median class $l = 20$

Frequency of the median class $f = 46$

Cumulative frequency of the class preceeding the median class $m = 60$

Width of the class $c = 10$

$$\text{Median}= l + \frac{\frac{N}{2} - m}{f} \times c$$

$$= 20 + \frac{80.5 - 60}{46} \times 10 = 20 + \frac{10}{46} \times 20.5$$

$$= 20 + \frac{205}{46} = 20 + 4.46 = 24.46$$

$$\therefore \text{ Median } = 24.46$$

Example 3.14

Find the median for the following data.

Marks	11-15	16-20	21-25	26-30	31-35	36-40
Frequency	7	10	13	26	9	5

Solution

Since the table is given in terms of inclusive type we convert it into exclusive type.

Marks	f	cf
10.5- 15.5	7	7
15.5-20.5	10	17
20.5-25.5	13	30
25.5-30.5	26	56
30.5-35.5	9	65
35.5-40.5	5	70
	N = 70	

$$N = 70, \quad \frac{N}{2} = \frac{70}{2} = 35$$

Median class is 25.5-30.5

Lower limit of the median class l = 25.5

Frequency of the median class f = 26

Cumulative frequency of the preceding median class m = 30

Width of the median class c = 30.5 − 25.5 = 5

$$\text{Median} = l + \frac{\frac{N}{2} - m}{f} \times c$$

$$= 25.5 + \frac{35 - 30}{26} \times 5 = 25.5 + \frac{25}{26} = 26.46$$

Exercise 3.3

1. Find the median of the following data.

   (i) 18,12,51,32,106,92,58

   (ii) 28,7,15,3,14,18,46,59,1,2,9,21

2. Find the median for the following frequency table.

Value	12	13	15	19	22	23
Frequency	4	2	4	4	1	5

3. Find the median for the following data.

Height (ft)	5-10	10-15	15-20	20-25	25-30
No of trees	4	3	10	8	5

4. Find the median for the following data.

Age group	0-9	10-19	20-29	30-39	40-49	50-59	60-69
No. of persons	4	6	10	11	12	6	1

5. Calculate the median for the following data

Class interval	1 - 5	6 - 10	11 - 15	16 - 20	21 - 25	26 - 30	31 - 35
Frequency	1	18	25	26	7	2	1

6. The following table gives the distribution of the average weekly wages of 800 workers in a factory. Calculate the median for the data given below.

Wages (₹ in hundres)	20 - 25	25 - 30	30 - 35	35 - 40	40 - 45	45 - 50	50 - 55	55 - 60
No. of persons	50	70	100	180	150	120	70	60

3.5 Mode

The Mode of a distribution is the value at the point around which the items tend to be most heavily concentrated.

3.5.1 Mode - Raw Data

In a raw data, mode can be easily obtained by arranging the observations in an array and then counting the number of times each observation occurs.

For example, consider a set of observations consisting of values 20,25,21,15,14,15.

Here, 15 occurs twice where as all other values occur only once. Hence mode of this data = 15.

Remark Mode can be used to measure quantitative as well as qualitative data. If a printing press turns out 5 impressions which we rate very sharp, sharp, sharp, sharp and Blurred, then the model value is sharp.

Example 3.15

The marks of ten students in a mathematics talent examination are 75,72,59,62, 72,75,71,70,70,70. Obtain the mode.

Solution Here the mode is 70, since this score was obtained by more students than any other.

Note A distribution having only one mode is called unimodal.

Example 3.16

Find the mode for the set of values 482,485,483,485,487,487,489.

Solution In this example both 485 and 487 occur twice. This list is said to have two modes or to be bimodal.

Note

(i) A distribution having two modes is called bimodal.

(ii) A distribution having three modes is called trimodal.

(iii) A distribution having more than three modes is called multimodal.

3.5.2 Mode - Ungrouped Frequency Distribution

In a ungrouped frequency distribution data the mode is the value of the variable having maximum frequency.

Example 3.17

A shoe shop in Chennai sold hundred pairs of shoes of a particular brand in a certain day with the following distribution.

Size of shoe	4	5	6	7	8	9	10
No of pairs sold	2	5	3	23	39	27	1

Find the mode of the following distribution.

Solution Since 8 has the maximum frequency with 39 pairs being sold the mode of the distribution is 8.

3.5.3 Mode - Grouped Frequency Distribution

In case of a grouped frequency distribution, the exact values of the variables are not known and as such it is very difficult to locate mode accurately. In such cases, if the class intervals are of equal width an appropriate value of the mode may be determined by using the formula

$$\text{Mode} = l + \left(\frac{f - f_1}{2f - f_1 - f_2}\right) \times c,$$

where l = lower limit of the modal class

f = frequency of modal class

c = class width of the modal class

f_1 = frequency of the class just preceeding the modal class.

f_2 = frequency of the class succeeding the modal class.

Mathematics

Example 3.18

Calculate the mode of the following data.

Size of item	10-15	15-20	20-25	25-30	30-35	35-40	40-45	45-50
No of items	4	8	18	30	20	10	5	2

Solution

Size of the item	f
10-15	4
15-20	8
20-25	18
25-30	30
30-35	20
35-40	10
40-45	5
45-50	2

Modal class is 25-30 since it has the maximum frequency.

Lower limit of the modal class $l = 25$

Frequency of the modal class $f = 30$

Frequency of the preceding the modal class $f_1 = 18$

Frequency of the class reducing the modal class $f_2 = 20$

Class width $c = 5$

$$\text{Mode} = l + \left(\frac{f - f_1}{2f - f_1 - f_2}\right) \times c$$

$$= 25 + \left(\frac{30 - 18}{60 - 18 - 20}\right) \times 5 = 25 + \frac{12 \times 5}{22}$$

$$= 25 + \frac{60}{22} = 25 + 2.73 = 27.73$$

$$\text{Mode} = 27.73$$

Exercise 3.4

1. The marks obtained by 15 students of a class are given below. Find the modal marks.
42,45,47,49,52,65,65,71,71,72,75,82,72,47,72

2. Calculate the mode of the following data.

Size of shoe	4	5	6	7	8	9	10
No. of Pairs sold	15	17	13	21	18	16	11

3. The age (in years) of 150 patients getting medical treatment in a hospital in a month are given below. Obtain its mode.

Age (yrs)	10-20	20-30	30-40	40-50	50-60	60-70
No of patients	12	14	36	50	20	18

4. For the following data obtain the mode.

Weight (in kg)	21-25	26-30	31-35	36-40	41-45	46-50	51-55	56-60
No of students	5	4	3	18	20	14	8	3

5. The ages of children in a scout camp are 13, 13, 14, 15, 13, 15, 14, 15, 13, 15 years. Find the mean, median and mode of the data.

6. The following table gives the numbers of branches and number plants in a garden of a school.

No. of branches	2	3	4	5	6
No. of plants	14	21	28	20	17

Calculate the mean, median and mode of the above data.

7. The following table shows the age distribution of cases of a certain disease reported during a year in a particular city.

Age in year	5 - 14	15 - 24	25 - 34	35 - 44	45 - 54	55 - 64
No. of cases	6	11	12	10	7	4

Obtain the mean, median and mode of the above data.

8. Find the mean, mode and median of marks obtained by 20 students in an examination. The marks are given below.

Marks	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
No. of students	1	4	5	8	2

Exercise 3.5

Choose the Correct Answer

1. The mean of the first 10 natural numbers is
   (A) 25 (B) 55 (C) 5.5 (D) 2.5
2. The Arithmetic mean of integers from –5 to 5 is
   (A) 3 (B) 0 (C) 25 (D) 10
3. If the mean of x, $x + 2$, $x + 4$, $x + 6$, $x + 8$ is 20 then x is
   (A) 32 (B) 16 (C) 8 (D) 4
4. The mode of the data 5, 5, 5, 5, 5, 1, 2, 2, 3, 3, 3, 4, 4, 4, 4 is
   (A) 2 (B) 3 (C) 4 (D) 5
5. The median of 14, 12, 10, 9, 11 is
   (A) 11 (B) 10 (C) 9.5 (D) 10.5
6. The median of 2, 7, 4, 8, 9, 1 is
   (A) 4 (B) 6 (C) 5.5 (D) 7
7. The mean of first 5 whole number is
   (A) 2 (B) 2.5 (C) 3 (D) 0
8. The Arithmetic mean of 10 number is –7. If 5 is added to every number, then the new Arithmetic mean is
   (A) –2 (B) 12 (C) –7 (D) 17
9. The Arithmetic mean of all the factors of 24 is
   (A) 8.5 (B) 5.67 (C) 7 (D) 7.5
10. The mean of 5 numbers is 20. If one number is excluded their mean is 15. Then the excluded number is
   (A) 5 (B) 40 (C) 20 (D) 10.

Points to Remember

The mean for grouped data

- The direct method : $\bar{x} = \dfrac{\sum fx}{\sum f}$
- The assumed mean method : $\bar{x} = A + \dfrac{\sum fd}{\sum f}$
- The step deviation method : $\bar{x} = A + \dfrac{\sum fd}{\sum f} \times C$
- The cumulative frequency of a class is the frequency obtained by adding the frequencies of all up to the classes preceeding the given class.
- The median for grouped date can be found by using the formula

  $$median = l + \frac{\frac{N}{2} - m}{f} \times c$$

- The mode for the grouped data can be found by using the formula

  $$\text{mode} = l + \left(\frac{f - f_1}{2f - f_1 - f_2}\right) \times c$$

Activity 1

Find the mean of 10, 20, 30, 40 and 50.

* Add 10 to each value and find the mean.
* Subtract 10 from each value and find the mean.
* Multiply each value by 10 and find the mean.
* Divide each value by 10 and find the mean.

Make a general statement about each situation by comparing with the properties of mean.

Activity 2

Give specific examples of your own in which,

(i) The median is preferred to arithmetic mean.

(ii) Mode is preferred to median.

(iii) Median is preferred to mode.

Activity 3

Record the shirt size of the students of your class. Form the frequency table and observe which is the most common size?

Activity 4

Divide the classroom into 5 goups. Each group record the marks of anyone of the subject in the summative assessment test conducted at the end of the first term and finds the average of the same. Compare your result with the other groups and find out the subject with the highest average.

Activity 5

Prepare 5 equal sized rectangular cards and name them as A, B, C, D, E.

A	**B**	**C**	**D**	**E**

Assign face value for each card as A = 5, B = 4, C = 3, D = 2, E = 1

Each student is given 10 chances to pickup the cards and record their face values. Each student obtains the mean of the recorded data.

The most lucky student is ....

Exercise 3.1

1. 4, 7, 6, 6, 5 **2.** 25, 30, 25, 60, 15, 5 **3.** 3, 4, 6, 6, 8 **4.** 5, 4, 3, 5, 3 **5.** 80, 50, 40, 120, 30

Exercise 3.2

1. 28.67 **2.** 6 **3.** 62 **4.** 37 **5.** 192 **6.** 6 **7.** 61kg **8.** 52.58 **9.** 27.13
10. 40.18 **11.** 28.67 **12.** 28 **13.** 48.1 **14.** 326.25 **15.** 55.5

Exercise 3.3

1. (i) 51 (ii) 14.5 **2.** 17 **3.** 19 **4.** 34.05 **5.** 14.7 **6.** 40

Exercise 3.4

1. 72 2. 7 3. 43.18 4. 41.75

Q. No.	Mean	Median	Mode
5.	14	14	13,15
6.	4.05	4	4
7.	32.1	31.2	27.8
8.	28	30	33.3

Exercise 3.5

1. C 2. B 3. B 4. D 5. A 6. C 7. A 8. A 9. D 10. B

4 PRACTICAL GEOMETRY

Main Targets

- To construct the Incentre
- To construct the Centroid

Johann Carl Friedrich Gauss

1777 – 1855

Johann Carl Friedrich Gauss (30 April 1777 – 23 February 1855) was a German mathematician and physical scientist who contributed significantly to many fields, including number theory, algebra, statistics, analysis, differential geometry, geodesy, geophysics, electrostatics, astronomy and optics. Gauss had a remarkable influence in many fields of mathematics and science and is ranked as one of history's most influential mathematicians. He referred to mathematics as "the queen of sciences".

He completed Disquisitiones Arithmeticae, his magnum opus, in 1798 at the age of 21, though it was not published until 1801. This work was fundamental in consolidating number theory as a discipline and has shaped the field to the present day.

4.1 Introduction

Geometry originated as a practical science concerned with surveying, measurements, areas, and volumes. Among the notable accomplishments one finds formulas for lengths, areas and volumes, such as the Pythagorean theorem, circumference and area of a circle, area of a triangle, volume of a cylinder, sphere, and a pyramid. A method of computing certain inaccessible distances or heights based on similarity of geometric figures is attributed to Thales. Development of astronomy led to emergence of trigonometry and spherical trigonometry, together with the attendant computational techniques.

In the first term we have learn to locate circumcentre and orthocentre of a triangle. Now let us learn to locate incentre and centroid.

4.2 Special line segments within Triangles

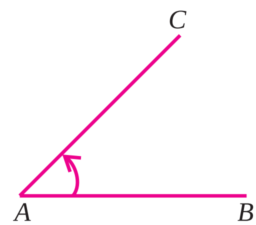
First let us learn to identify and to construct

(iii) Bisector of a given angle and

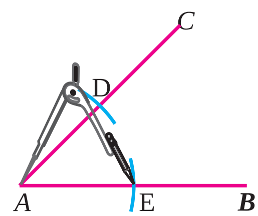
(iv) Line joining a given external point and the midpoint of a given line segment.

4.2.1 Construction of Angle Bisector

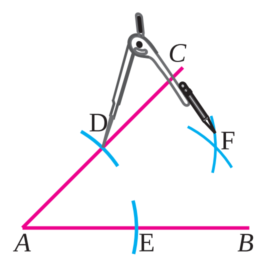
Step 1 : Draw the given angle $\angle CAB$ with the given measurement.



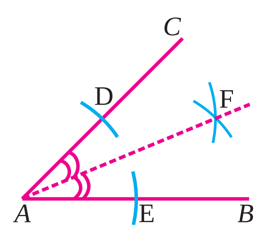
Step 2 : With A as centre and a convenient radius draw arcs to cut the two arms of the angle at D and E.



Step 3 : With D and E as centres and a suitable radius draw arcs to intersect each other at F.

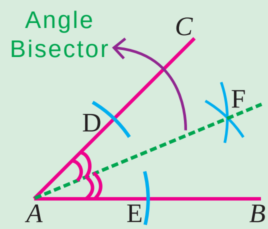


Step 4 : Join A and F to get the angle bisector AF of $\angle CAB$.



Key Concept | **Angle Bisector**

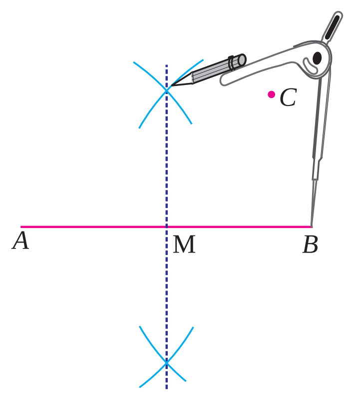
The line which divides a given angle into two equal angles is called the angle bisector of the given angle.



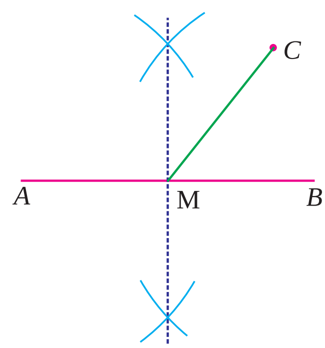
4.2.2 Construction of Line Joining a External Point and the Midpoint of a Line Segment

Step 1 : Draw a line segment AB with the given measurement and mark the given point C (external point).

Step 2 : Draw the perpendicular bisector of AB and mark the point of intersection M which is the mid point of line segment.

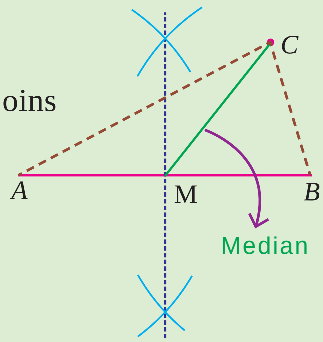


Step 3 : Join C and M to get the required line.



Key Concept **Median**

In a triangle, a median is the line segment that joins a vertex of the triangle and the midpoint of its opposite side.



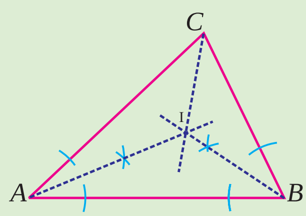
4.3 The Points of Concurrency of a Triangle

As we have already learnt how to draw the Angle Bisector and Median, now let us learn to locate the Incentre and Centroid of a given triangle.

4.3.1 Construction of the Incentre of a Triangle

Key Concept **Incentre**

The point of concurrency of the internal angle bisectors of a triangle is called the incentre of the triangle and is denoted by *I*.

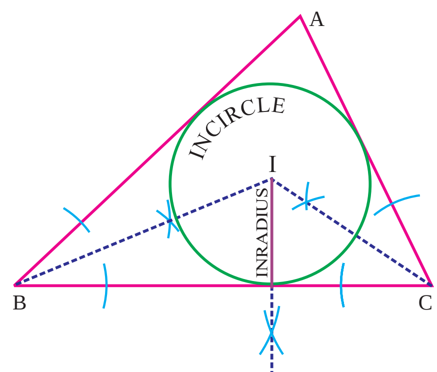


Incircle The circle drawn with the incentre (*I*) as centre and touching all the three sides of a triangle is the incircle of the given triangle.

Inradius The radius of the incircle is called the inradius of the triangle.

(or)

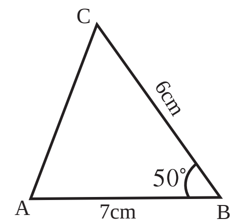
It is the shortest distance of any side of the triangle from the incentre *I*.



Example 4.1

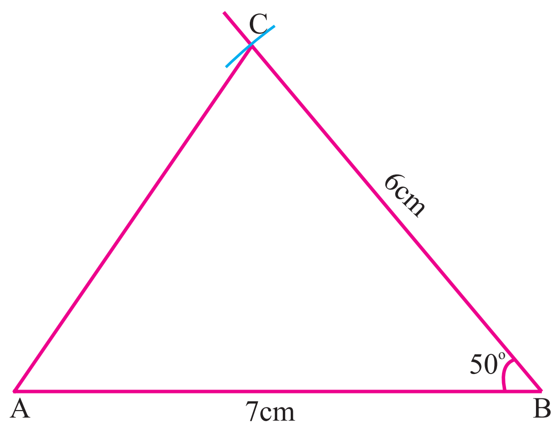
Construct the incentre of ΔABC with $AB = 7cm$, $\angle B = 50^o$ and $BC = 6cm$. Also draw the incircle and measure its inradius.

Solution

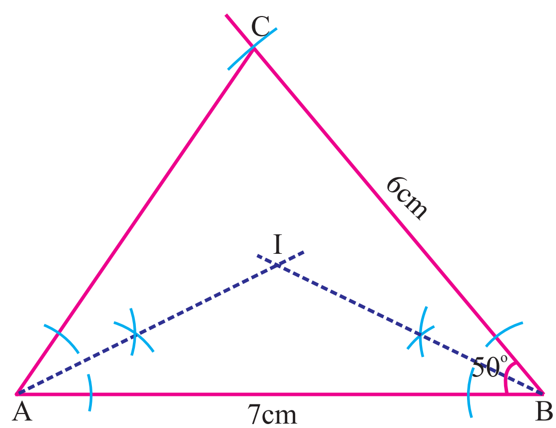


Rough Diagram

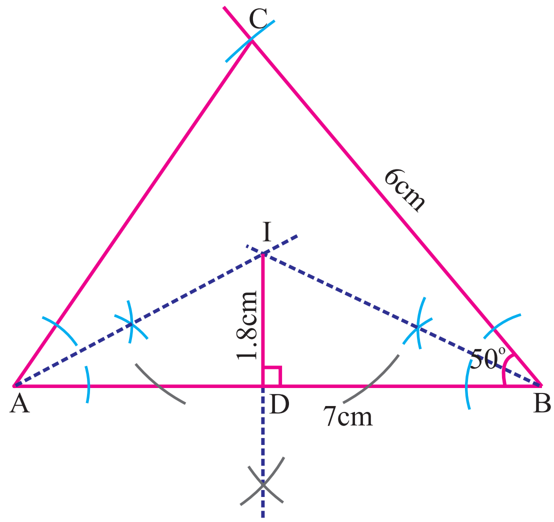
Step 1 : Draw the ΔABC with the given measurments.



Step 2 : Construct the angle bisectors of any two angles (A and B) and let them meet at I. Then I is the incentre of ΔABC

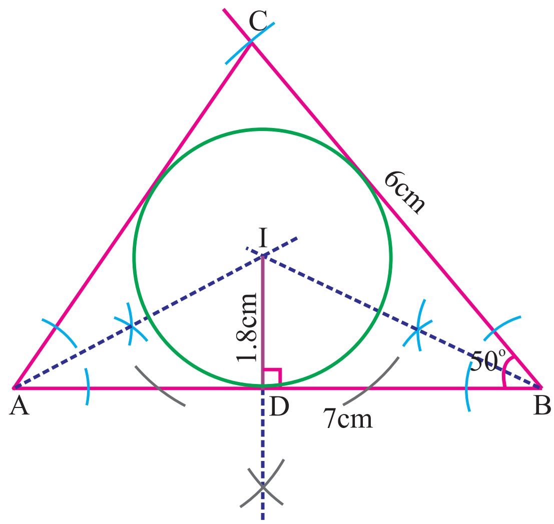


Step 3 : With I as an external point drop a perpendicular to any one of the sides to meet at D.



Step 4 : With I as centre and ID as radius draw the circle. This circle touches all the sides of the triangle.

Inradius = 1.8 cm



Remark The incentre of any triangle always lies inside the triangle.

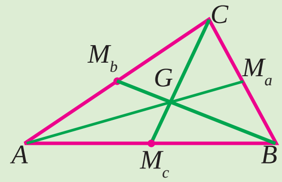
Exercise 4.1

1. Draw the incircle of ΔABC, where AB = 9 cm, BC = 7cm, and AC = 6cm.
2. Draw the incircle of ΔABC in which AB = 6 cm, AC = 7 cm and $\angle A = 40^\circ$. Also find its inradius.
3. Construct an equilateral triangle of side 6cm and draw its incircle.
4. Construct ΔABC in which AB = 6 cm, AC = 5 cm and $\angle A = 110^\circ$. Locate its incentre and draw the incircle.

4.3.2 Construction of the Centroid of a Triangle.

Key Concept | **Centroid**

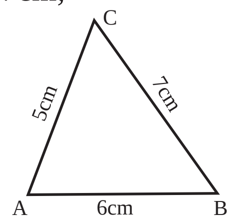
The point of concurrency of the medians of a triangle is called the Centroid of the triangle and is usually denoted by *G*.



Example 4.2

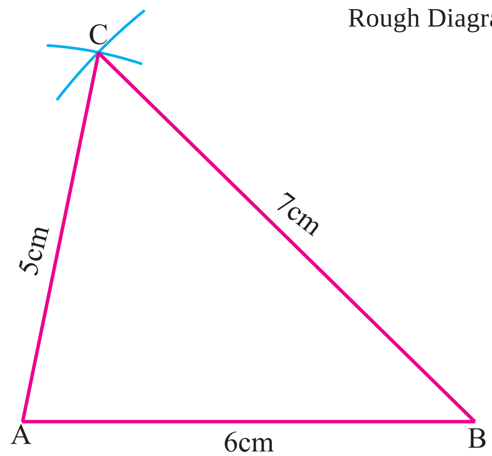
Construct the centroid of ΔABC whose sides are AB = 6cm, BC = 7cm, and AC = 5cm.

Solution

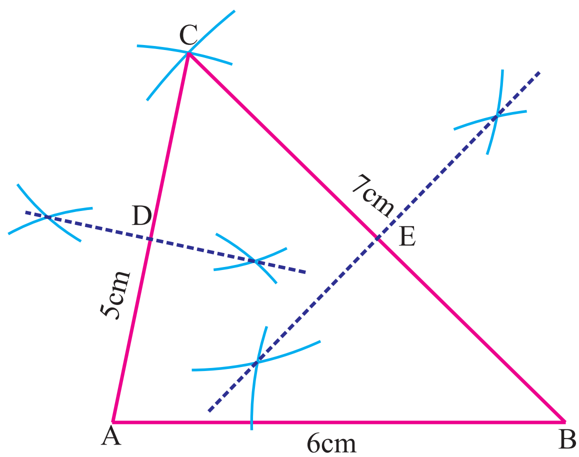


Rough Diagram

Step 1 : Draw ΔABC using the given measurements.

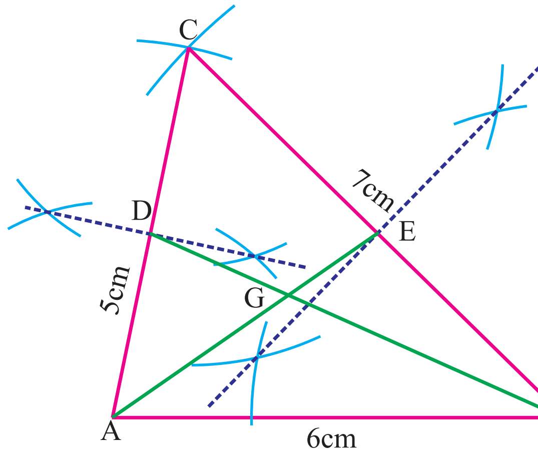


Step 2 : Construct the perpendicular bisectors of any two sides (AC and BC) to find the mid points D and E of AC and BC respectively .



Step 3 : Draw the medians AE and BD and let them meet at G.

The point G is the centroid of the given ΔABC



Remark
(i) Three medians can be drawn in a triangle.
(ii) The centroid divides a median in the ratio 2:1 from the vertex.
(iii) The centroid of any triangle always lie inside the triangle.

Exercise 4.2

1. Construct the ΔABC such that AB = 6cm, BC = 5cm and AC = 4cm and locate its centroid.
2. Draw and locate the centroid of triangle LMN with LM = 5.5cm, $\angle M = 100^\circ$, MN = 6.5cm.
3. Draw a equilateral triangle of side 7.5cm and locate the centriod.
4. Draw the right triangle whose sides are 3cm, 4cm and 5cm and construct its centroid.
5. Draw the ΔPQR, where $PQ = 6cm$, $\angle P = 110^o$ and QR = 8cm and construct its centroid.

Activity 1

Obective : To find the incentre of a given triangle using paper folding.

Procedure : Construct the angle bisectors of the given triangle (by making a pair of sides to coincide). The meeting point of these angle bisectors is the incentre of the given triangle.

Activity 2

Obective : To find the centroid of a triangle using paper folding.

Procedure : Using the activities learnt in the first term construct the medians of the given triangular sheet of paper. The point of intersection of the medians is the centroid of the given triangle.

'I can, I did'

Student's Activity Record

Subject :

Sl. No.	Date	Lesson No.	Topic of the Lesson	Activities	Remarks