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Grey Duane model for reliability growth prediction under small sample uncertain failure data

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ABSTRACT

Traditional reliability growth models require failure data to adhere to a specific distribution, which greatly limits the applicability in data-scarce scenarios. Drawing from the characteristic in modeling small sample and poor information of grey system theory, this study introduces a novel grey Duane reliability growth prediction model, tailored for analyzing uncertainty in limited failure data. The characteristics and adaptability of the proposed grey Duane model (GDM) are analyzed and compared with two traditional reliability growth prediction models. Utilizing a first-order accumulation generation operation, a differential function is established to estimate the unknown parameters in GDM. The proposed GDM enables synchronized predictions of failure time, failure number, and instantaneous mean time between failures. To validate the applicability of GDM in reliability growth management, an industrial case in particular electronic equipment during an aircraft's flight-testing process is applied, whose results demonstrate its superior predictive accuracy compared to alternative models.

KEYWORDS

Aircraft equipment manufacturing; Duane model; grey system theory; reliability growth management; small sample

1. Introduction

Reliability growth is the process of gradually improving the reliability of a product over time by eliminating weak points in its design or manufacturing. The objective of reliability growth testing is to enhance product reliability by employing the "Test-Analyze-And-Fix" (TAAF) approach to address design flaws (Nadjafi and Gholami 2020). Reliability growth testing involves a repetitive cycle of testing and improvement within the TAAF framework, leading to continuous fluctuations and enhancements in the product's reliability level. Modeling the reliability growth process is a method of quantitatively describing this progression using mathematical models. Through growth models, it becomes possible to anticipate potential growth trends, devise growth plans, monitor the growth process, comprehend actual growth trends, and evaluate growth outcomes (Quigley and Walls 2003). Nevertheless, in complex reliability growth experiments, obtained data can be influenced by various factors such as extreme temperatures, communication delays, and equipment failures. Thus, the conventional assumptions of constant failure rates or reliability growth rates, along with their corresponding mathematical analytical methods, are no longer applicable. There is a need for more scientifically robust mathematical models for describing and analyzing reliability growth. This necessity is especially pronounced in the development and management of complex equipment, where expensive experimental costs and intricate operational environments result in sparse and irregular data. Based on the grey modeling approach, a reliability growth prediction method suitable for analyzing small-sample data is proposed.

The reliability growth model of a product reflects the patterns of its reliability growth during the development process. Many reliability growth models have been proposed to assess the changing reliability status of products. By analyzing test data from two hydraulic devices and three aircraft engines, electrical engineer Duane (1964) pointed out that as long as continuous improvements are made to the product, the cumulative failure rate and cumulative test time follow a straight line on a logarithmic coordinate paper. In 1966, Cox and Lewis (1966) hypothesized that system failure counts follow a nonhomogeneous Poisson process with an exponentiated intensity function, introducing the Cox–Lewis model. Subsequently, more models applied the nonhomogeneous Poisson process

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to reliability growth analysis (Pham and Zhang 2003; Li and Pham 2017; Okamura and Dohi 2021). In 1968, Virene utilized the Gompertz model from biology, which reflects the growth patterns of cell reproduction. The Gompertz curve's characteristic is that reliability starts growing slowly, then accelerates gradually, reaches a point of faster growth, and slows down again. Many systems in the development and early manufacturing stages exhibit this type of growth pattern (Fang and Yeh 2016). In 1974, Crow from the U.S. Army Materiel System Analysis Center provided a probabilistic interpretation of the Duane model and introduced the Army Materiel System Analysis Activity (AMSAA) model based on the nonhomogeneous Poisson process (Crow 1974). In the development phase, analyzing reliability growth is a preliminary exploration work. Predicting future observations and statistical measures based on current failure data, following specific probability distributions, holds promise for research and engineering. This approach enhances our understanding of potential failures, providing insights for optimizing development and improving overall system robustness. Haselgruber, Capser, and Vignati (2021) proposed using probability distributions to describe fault and maintenance data of flexible manufacturing systems. They processed data for mechanical, hydraulic, electronic, electrical, and software systems separately, fitting each set of data to known probability distribution types. The results indicated that the average failure interval time of the studied flexible manufacturing systems follows a Weibull distribution, and the maintenance interval time follows a log-normal distribution. Soltanali et al. (2021) addressed early-life reliability and proposed a novel time-related reliability estimation. Then, based on the concept of prospective multi-attribute decision-making, reliability allocation in the early design phase of a system is introduced to enhance product quality and system robustness. Considering the complexity growth, diversity, and uncertainty of system functions, Vineyard, Amoako-Gyampah, and Meredith (1999) provide a comparative framework for predicting operational reliability in the automotive manufacturing industry using soft computing and statistical techniques. Haber, Fargnoli, and Sakao (2020) utilized the fuzzy analytic hierarchy process to enhance the quality function deployment of product service systems. Based on component functional dependencies, Han et al. (2021) introduced a method for calculating average maintenance cost based on dynamic remaining useful life prediction after each maintenance action and applied functional importance to prioritize predictive maintenance component sets.

To handle the task of analyzing failure data with limited samples, Talafuse and Pohl (2017) employed an improved grey model to predict the reliability growth parameters of a system and studied how parameter estimation is influenced by systems whose failures follow multiple Weibull distributions. Due to its unique modeling mechanism and adaptability in analyzing small-sample data, grey models have been widely applied in various fields such as reliability growth prediction (Liu et al. 2020; Liu and Xie 2022), remaining life prediction (Zhao et al. 2020; Gu, Ge, and Li 2023), and equipment development (Pang et al. 2019; Du, Liu, and Liu 2021; Yang et al. 2021; Liu and Tang 2023). Many scholars have made improvements to the grey models from the perspectives of model structure and adaptability. Li and Xie (2023) deconstructed discrete structural components to analyze the relationship between grey predictive models and traditional models, providing a unified representation for the mixed relationship of increments and growth rates. Zeng, Ma, and Zhou (2020) proposed a new-structure grey Verhulst model. Xie, Liu, and Wu (2020) introduced a nonuniform interval reverse grey model. Considering the adaptability of the structure, Ding et al. (2021) proposed an adjustable time power item to enhance the data fitting capability of grey models. Wang and Jv (2021) proposed a nonlinear systematic grey model to predict the interdependent relationship of industrial structure. Proper data transformation techniques are important to ensure the accuracy of models. Building upon first-order accumulation, fractional-order accumulation (Wu et al. 2013), conformable fractional-order accumulation (Ma et al. 2020), and damping accumulation (Liu, Chen, and Wu 2021) have been proposed to extend the adaptability of grey models. To reduce system uncertainty, reasonable parameter optimization methods are also important modeling steps to ensure the precedence of information in grey models (Zhang and Liu 2010; Comert, Begashaw, and Huynh 2021; Liu et al. 2023; Wei 2023). Furthermore, seasonality (Wu et al. 2021; Zhou and Ding 2021), time-delay (Zhou et al. 2023), and self-adaptability (Xiao and Duan 2020; Tu and Chen 2021; Zhang, Yin, and Yang 2022) are also significant factors in grey system modeling.

In response to the insufficient information data during aircraft flight-test phases, a novel grey Duane model (GDM) is proposed for reliability assessment in this research. The main contributions are summarized as follows: (1) Improved assumptions for failure number and failure time allow GDM to remain effective for analyzing uncertainty in failure data that does not conform to a power-law process (PLP). (2) Utilizing grey generation operations, GDM can produce adaptive parameter estimates, enhancing the accuracy of model fitting. (3) The GDM enables accurate predictions of failure number, failure time, and instantaneous mean time between failures (MTBF) simultaneously.

The structures of this study are outlined. Section 2 summarizes the preliminaries of the renowned Duane model and Donovan–Murphy (DM) model in reliability growth management. In Section 3, the definition of the proposed GDM is expounded, as well as its modeling procedures in reliability growth management and comparative characteristics to the Duane model and DM model. To validate the performance in predicting the cumulative failure numbers, accumulated failure time, and instantaneous MTBF of the proposed GDM, Section 4 compares the results with the aforementioned two reliability growth models by introducing two cases in literature. Section 5 applies the proposed GDM into a real industrial scenario for managing the reliability growth process in an aircraft's flight-testing process. In Section 6, we conclude the whole research and point out certain limitations of the GDM that require further refinement.

2. Preliminaries

2.1. Duane model

The Duane model (Gaudoin, Yang, Xie 2003) is the most commonly used reliability growth prediction model, characterized by its concise model representation, facilitating tracking, and evaluation of reliability growth processes. The Duane model states that during the development testing of a system, by continuously rectifying faults, the system's reliability level progressively improves with an increase in the number of tests conducted. On a logarithmic coordinate axis, the accumulated number of failures $N(t)$, forms a linear function relative to the testing time t, as follows:

$$\ln\frac{N(t)}{t} = \ln a - m\ln t \tag{1}$$

and

$$N(t) = at^{1-m}, t_1 < t < t_F, \tag{2}$$

where $m \in (0, 1]$ is reliability growth rate, a is a constant. t_1 is the first system failure time, and t_F is the target time for the test.

The cumulative failure rate of the system is

$$\lambda_c(t) = N(t)/t. \tag{3}$$

Then, the cumulative MTBF is

$$\theta_c(t) = 1/\lambda_c(t) = t/N(t) = t^m/a. \tag{4}$$

Assume the instantaneous failure rate of the system at time t is

$$\lambda(t) = \frac{\mathrm{d}N(t)}{\mathrm{d}t}, \tag{5}$$

thus, the instantaneous MTBF can be calculated as

$$\theta(t) = \frac{1}{\lambda(t)} = \frac{t^m}{a(1-m)}. \tag{6}$$

There are two ways to calculate the parameters of the Duane model, one is empirical method, and the other is data-driven method.

1. Empirical method refers to drawing an ideal growth curve by referring to the research experience and expert knowledge of similar projects before the start of a project, to determine the reliability growth coefficient m. Usually, $0.3 \le m \le 0.6$ (Melchers and Beck 2018). If $m < 0.3$, it indicates insufficient corrective measures. If $m > 0.6$, it indicates that strong failure analysis and corrective measures have been applied in reliability growth tests. If the first failure time point is known to be t_1, the initial MTBF of the system is $\theta(t_1)$. According to Eq. [4], we have

$$a = \frac{t_1^m}{\theta(t_1)}, \tag{7}$$

$$\theta(t) = \theta(t_1)\left(\frac{t}{t_1}\right)^m \frac{1}{1-m}, t_1 < t < t_F. \tag{8}$$

Following a predefined experimental schedule, the research team can continuously monitor whether the reliability growth level deviates from the predetermined target, allowing for the estimation of the expected time to achieve the reliability growth goal.

2. Data driven methods refer to the use of parameter estimation methods for numerical analysis of existing data. Converting the raw data in Eq. [4] into a logarithmic form, we have

$$\ln\theta_c(t) = -\ln a + m\ln t \tag{9}$$

By applying the ordinary least squares (OLS) method (Crowder 2017), the parameters in Eq. [9] are estimated as

$$\hat{m} = \frac{n\sum_{i=1}^n \ln\theta_c(t_i)\ln t_i - \left(\sum_{i=1}^n \ln\theta_c(t_i)\right)\left(\sum_{i=1}^n \ln \ln t_i\right)}{n\sum_{i=1}^n (\ln t_i)^2 - \left(\sum_{i=1}^n \ln t_i\right)^2}, \tag{10}$$

$$\hat{a} = \exp\left[\frac{1}{n}\left(\hat{m}\sum_{i=1}^{n}\ln\ln t_i - \sum_{i=1}^{n}\ln\theta_c(t_i)\right)\right]. \quad (11)$$

The advantage of the Duane model lies in its straightforward representation, making it conducive to devising reliability growth plans through graphical means. Furthermore, the model's parameters are easy to comprehend, allowing for reliability growth assessment in development management. However, the Duane model (1) is an empirical formula that fails to account for stochastic processes, and its point estimates lack high precision. In the Duane model, early failures strongly influence the resultant model. Taking the data from reference (Liu et al. 2020) as an example, the B-type (the mode that will be addressed by corrective actions in test) fault data during the experimental flight phase of aircraft was used to verify the fitting process of the model. As depicted in Figure 1, due to the natural logarithm function, failures occurring in the latter part of testing tend to cluster together, as there is almost no difference between the natural logarithms of accumulated values that are relatively large. This is especially crucial in the small-sample data analysis, where there isn't enough information to estimate its distribution or characteristics. Therefore, a rational method for estimating reliability coefficients is necessary.

2.2. Donovan–Murphy reliability growth model

Based on the variance stable transformation theory, Donovan and Murphy proposed a new reliability growth prediction model, abbreviated as the DM model (Donovan and Murphy 2000). The advantage of DM model is that it does not require logarithmic transformation of data and can intuitively draw images. The cumulative MTBF is estimated by

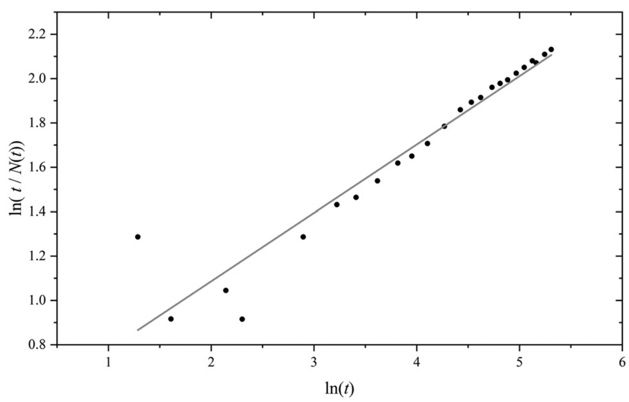

Figure 1. Disturbance of different observations on parameter estimation of Duane model (data sourced from literature (Liu et al. 2020)).

$$\theta_c(t) = \alpha + \beta\sqrt{t}. \quad (12)$$

The linear parameters α and β can be estimated by the OLS method. Due to $\theta_c(t) = t/N(t)$, this yields the estimated result of number of failures

$$\hat{N}(t) = \frac{t}{\alpha + \beta\sqrt{t}}, \quad (13)$$

and the instantaneous MTBF

$$\theta(t) = \frac{\mathrm{d}t}{\mathrm{d}\hat{N}(t)} = \frac{2\left(\alpha + \beta\sqrt{t}\right)^2}{2\alpha + \beta\sqrt{t}}. \quad (14)$$

Rearranging Eq. [13], one observes the effect of cumulative failure time t on the failure number $N(t)$ as

$$\hat{t} = \frac{1}{4}\left(\beta N(t) + \sqrt{\beta^2 N(t)^2 + 4\alpha N(t)}\right)^2. \quad (15)$$

Donovan and Murphy (2000) pointed out that the DM model and Duane model are mathematically equivalent when the slope is 0.5. And the new model fits the data more closely than the Duane model whenever the Duane slope is less than 0.5.

3. Grey Duane model

3.1. Model definition and modeling steps

The traditional reliability model regards the initial state of the system as a sound state which assumes that $N(t_0) = a{t_0}^{1-m} = 0$. However, the initial system may have undergone multiple debugging or repairs, and the equipment may have potential existing faults before the first reliability test. Considering the impact of uncertain factors on the system, a constant b is introduced into the cumulative failure number equation, and Eq. [2] is modified into

$$N^{(0)}(t) = at^{1-m} + b, \quad (16)$$

which is abbreviated as grey Duane model (GDM), where, a and b are two parameters, m is the reliability growth rate. The modeling flowchart of GDM is shown in Figure 2. The specific modeling steps of the proposed GDM are as follows:

Step 1. Estimation of reliability growth coefficient m.

The identification of the hyperparameter m is a crucial process in reliability growth prediction models. Traditionally, Duane model employs a logarithmic transformation on the original data for computation, using a coordinate transformation method. However, this approach can introduce systematic errors. Due to the nature of the natural logarithm function, the

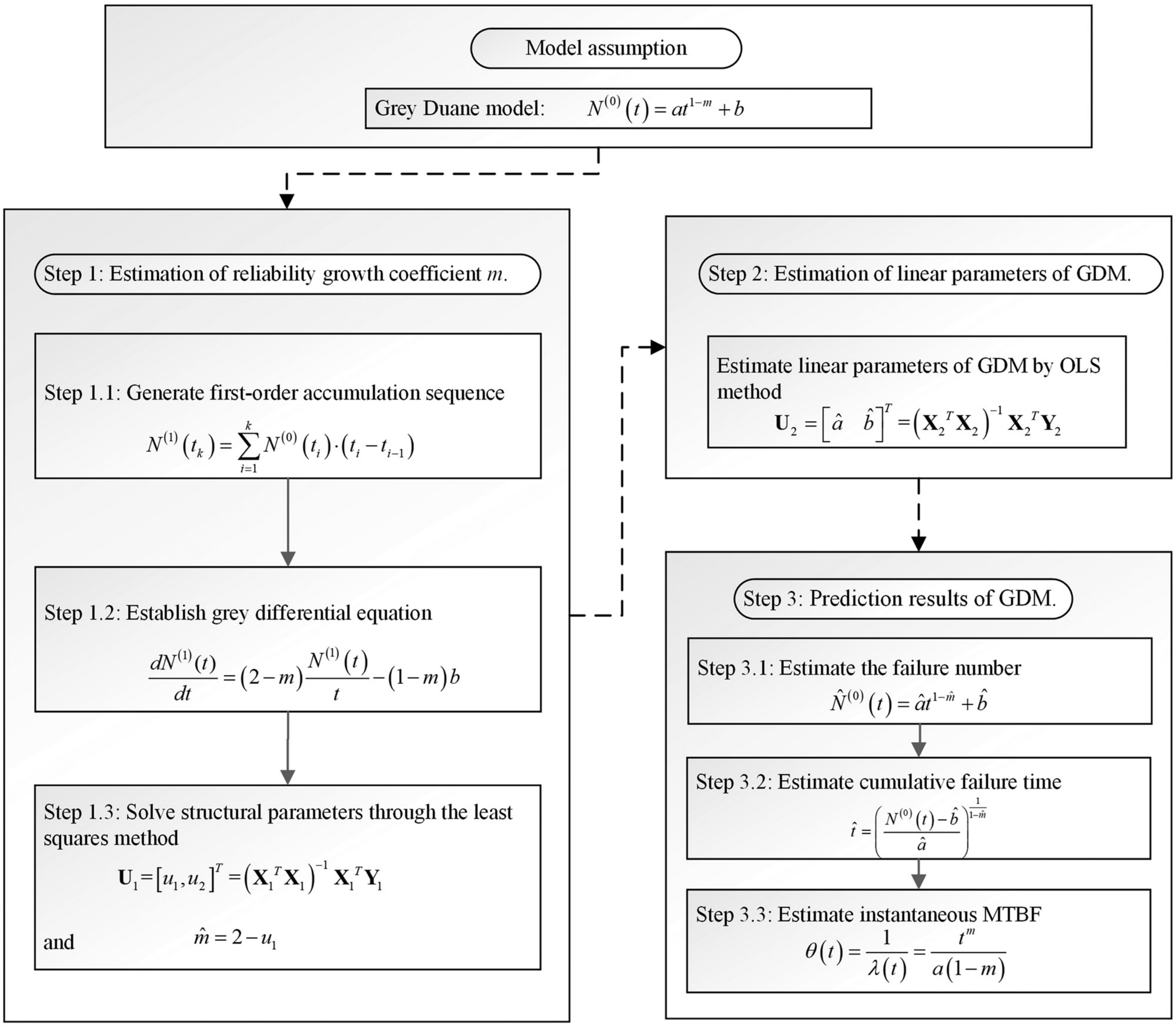

Figure 2. Flow chart of grey Duane model.

disparities between cumulative time and cumulative MTBF are diminished in later stages, making the Duane model susceptible to disturbances from early failure data. Conversely, without altering the data, the DM model sets a fixed hyperparameter $m = 0.5$, which avoids the data clustering issue. Nevertheless, when $m > 0.5$, the DM model's data fitting capability is weaker compared to the Duane model. In this study, the first-order accumulation operation of the grey modeling method is utilized to address the hyperparameter identification problem presented in Eq. [16].

Definition 1. Assume $\{N^{(0)}(t_i)\}_{i=1,2,\ldots,n}$ is a non-negative sequence, its first-order accumulation sequence is $\{N^{(1)}(t_i)\}_{i=1,2,\ldots,n}$, we have

$$N^{(1)}(t_k) = \sum_{i=1}^{k} N^{(0)}(t_i) \cdot (t_i - t_{i-1}), i = 1, 2, \ldots, n. \quad (17)$$

where $t_0 = 0$ is the start time of system.

The first-order accumulation of grey systems can be seen as an extension of the cumulative sum operator from discrete points to continuous time form. The discrete estimation of first-order accumulation sequence on a two-dimensional image is equivalent to the area of a continuous curve (see figure 1 of Wei, Xie, and Yang (2020)), thus, Eq. [17] can be integrated to replace the piecewise constant quadrature method

$$N^{(1)}(t_k) = \sum_{i=1}^{k} N^{(0)}(t_i) \cdot (t_i - t_{i-1}) = \int_0^{t_k} (as^{1-m} + b)\mathrm{d}s = \frac{a}{2-m} t_k^{2-m} + bt_k. \quad (18)$$

Bring Eq. [17] into [18], it yields

$$N^{(1)}(t_k) = \frac{1}{2-m} t_k N^{(0)}(t_k) + \frac{1-m}{2-m} b t_k. \tag{19}$$

Then, we have

$$N^{(0)}(t_k) = (2-m)\frac{N^{(1)}(t_k)}{t_k} - (1-m)b. \tag{20}$$

Definition 2. Due to $\frac{\mathrm{d}N^{(1)}(t_k)}{\mathrm{d}t} \approx \frac{N^{(1)}(t_k)-N^{(1)}(t_{k-1})}{t_k-t_{k-1}} = N^{(0)}(t_k)$, the differential equation of GDM can describe the dynamic process of system failure frequency, that is

$$\frac{\mathrm{d}N^{(1)}(t)}{\mathrm{d}t} = (2-m)\frac{N^{(1)}(t)}{t} - (1-m)b. \tag{21}$$

Directly computing the parameters of Eq. [20] may lead to significant systemic errors due to first-order accumulation process. Utilizing the discretization of the differential Eq. [21] to solve for the parameters is a more reasonable approach, as it helps avoid errors stemming from integral transformations. Integrating both sides in Eq. [21] over the interval $[t_{k-1}, t_k]$, then, yields

$$\int_{t_{k-1}}^{t_k} \mathrm{d}N^{(1)}(t) = (2-m)\int_{t_{k-1}}^{t_k} \frac{N^{(1)}(t)}{t}\mathrm{d}t + (m-1)b\int_{t_{k-1}}^{t_k} \mathrm{d}t. \tag{22}$$

Using trapezoidal formula discretization rules to replace integral terms of Eq. [22], then, leads to differential discrete equation

$$N^{(1)}(t_k) - N^{(1)}(t_{k-1}) = \frac{2-m}{2}\left(\frac{N^{(1)}(t_k)}{t_k} + \frac{N^{(1)}(t_{k-1})}{t_{k-1}}\right)(t_k - t_{k-1}) + (m-1)b(t_k - t_{k-1}). \tag{23}$$

By the criterion for minimum sum of squared errors

$$\arg\min_{\hat{m}} \left\{ J_1 = \sum_{k=2}^{n} \begin{bmatrix} N^{(1)}(t_k) - N^{(1)}(t_{k-1}) \\ -\left(\frac{2-\hat{m}}{2}\left(\frac{N^{(1)}(t_k)}{t_k} + \frac{N^{(1)}(t_{k-1})}{t_{k-1}}\right) + (\hat{m}-1)b\right)(t_k - t_{k-1}) \end{bmatrix}^2 \right\}, \tag{24}$$

the OLS estimation of structural parameters is

$$\mathbf{U}_1 = [u_1, u_2]^T = \left(\mathbf{X}_1{}^T\mathbf{X}_1\right)^{-1}\mathbf{X}_1{}^T\mathbf{Y}_1, \tag{25}$$

where $\hat{m} = 2 - u_1$,

$$\mathbf{X}_1 = \begin{bmatrix} \left(\frac{N^{(1)}(t_2)}{t_2} + \frac{N^{(1)}(t_1)}{t_1}\right)\frac{(t_2-t_1)}{2} & t_2 - t_1 \\ \left(\frac{N^{(1)}(t_3)}{t_3} + \frac{N^{(1)}(t_2)}{t_2}\right)\frac{(t_3-t_2)}{2} & t_3 - t_2 \\ \vdots & \vdots \\ \left(\frac{N^{(1)}(t_n)}{t_n} + \frac{N^{(1)}(t_{n-1})}{t_{n-1}}\right)\frac{(t_n-t_{n-1})}{2} & t_n - t_{n-1} \end{bmatrix},$$

$$\mathbf{Y}_1 = \begin{bmatrix} N^{(1)}(t_2) - N^{(1)}(t_1) \\ N^{(1)}(t_3) - N^{(1)}(t_2) \\ \vdots \\ N^{(1)}(t_n) - N^{(1)}(t_{n-1}) \end{bmatrix}.$$

Step 2. Estimation of linear parameters in GDM.

To mitigate the discretization-induced system error, the linear parameters a and b can be estimated using Eq. [16]. Substituting the estimated hyperparameter $\hat{m}$ into Eq. [16], gives the OLS estimation criterion

$$\arg\min_{\hat{a},\hat{b}} \left\{ J_2 = \sum_{k=1}^{n} \left[N^{(0)}(t_k) - \left(\hat{a}t_k^{1-\hat{m}} + \hat{b}\right)\right]^2 \right\}. \tag{26}$$

This leads to

$$\mathbf{U}_2 = \begin{bmatrix} \hat{a} & \hat{b} \end{bmatrix}^T = \left(\mathbf{X}_2{}^T\mathbf{X}_2\right)^{-1}\mathbf{X}_2{}^T\mathbf{Y}_2, \tag{27}$$

where $\mathbf{X}_2 = \begin{bmatrix} t_1^{1-\hat{m}} & 1 \\ t_2^{1-\hat{m}} & 1 \\ \vdots & \vdots \\ t_n^{1-\hat{m}} & 1 \end{bmatrix}, \mathbf{Y}_2 = \begin{bmatrix} N^{(0)}(t_1) \\ N^{(0)}(t_2) \\ \vdots \\ N^{(0)}(t_n) \end{bmatrix}.$

Step 3. Prediction results of GDM.

Substituting $\hat{a}, \hat{b}$, and $\hat{m}$ into Eq. [16], then, leads to time response function of failure number

$$\hat{N}^{(0)}(t) = \hat{a}t^{1-\hat{m}} + \hat{b}. \tag{28}$$

Rearranging Eq. [16], one has the estimation of cumulative failure time

$$\hat{t} = \left(\frac{N^{(0)}(t) - \hat{b}}{\hat{a}}\right)^{\frac{1}{1-\hat{m}}}. \tag{29}$$

Similar to the Duane model, the system instantaneous failure rate of GDM is

$$\lambda(t) = \frac{\mathrm{d}N^{(0)}(t)}{\mathrm{d}t} = \frac{\mathrm{d}(at^{1-m} + b)}{\mathrm{d}t} = a(1-m)t^{-m}, \tag{30}$$

then leads to instantaneous MTBF $\theta(t) = \frac{1}{\lambda(t)} = \frac{t^m}{a(1-m)}$.

3.2. Model comparison and property analysis

The proposed GDM integrates the first-order accumulation and differential equation modeling from grey modeling methods into the traditional Duane model, thereby enhancing the model's applicability and the accuracy of parameter estimation. Grey methods are well-suited for analyzing reliability data due to the following reasons:

1. The observed data in Eq. [2] or [16] are non-negative and exhibit characteristics of a concave curve.
2. The first-order accumulation operation reduces the randomness of the original sequence, enhancing the accuracy of model fitting with insufficient data. This is particularly beneficial for reliability test data of complex equipment, which often consists of small samples.
3. The differential equation model can simulate the dynamic changes of a system, without requiring consideration of data distribution patterns and trends.

Table 1 lists the differences among the three models in terms of modeling assumptions and parameter estimation methods. The Duane model necessitates adherence to the power-law process for data and its parameter estimation involves a logarithmic transformation of the original data. The DM model does not require any data transformation, but it fixes the reliability growth rate at $m = 0.5$, limiting the model's flexibility. The proposed GDM introduces an uncertainty factor b to characterize the relationship between the failure number and cumulated failure time. By employing accumulation operations, the proposed GDM transforms the accumulated values $N^{(1)}(t_k)$ back to their original form $N^{(0)}(t)$ to solve for the reliability growth rate m. Then, differential Eq. [21] and its corresponding discrete form (23) are utilized to minimize the estimation error.

To compare the accuracy of the three models in terms of their state equations and parameter estimation, we utilized the data from Case 1 in Subsection 4.1 to test the fitting performance of the three models. The determination coefficient (R^2) was employed to evaluate the goodness of fit of the linear regression process.

$$R^2 = \frac{\sum_{i=1}^{n} \left(\hat{x}(t_i) - \bar{x}\right)^2}{\sum_{i=1}^{n} \left(x(t_i) - \bar{x}\right)^2}, \tag{31}$$

where $x(t_i)$ represents the actual observed value, $\hat{x}(t_i)$ represents the predicted value, n is the number of observed samples, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x(t_i)$ is the mean value.

The development process of complex equipment often involves small-scale testing, which results in insufficient data for accurate parameter estimation, and the evolution pattern of the data is not distinct. From the left three plots in Figure 3, it can be observed that the assumptions underlying the three models are reasonable, with determination coefficients $R^2 > 0.95$. Reliability growth data often approximates a power-law process, but the presence of uncertain factors tends to introduce disturbances in the data, thereby reducing the models' generalization ability. From the right three plots in Figure 3, it can be observed that the parameter estimation methods of the Duane and DM models only provide approximate estimates of the system's changing trend, resulting in relatively lower fitting accuracy. Conversely, the proposed GDM model exhibits the best fitting performance ($R^2 = 0.9812$). This demonstrates that the first-order accumulation operation in Eq. [17] can reduce the system's randomness and enhance the accuracy of the system parameter m.

Comparing the traditional Duane model with the proposed GDM, the new model possesses the following properties:

Property 1. The GDM can effectively approximate the fitting of power-law sequence $N^{(0)}(t) = at^{1-m} + b$.

Property 2. In the GDM, the relationship between the number of failures and accumulated time follows a logarithmic function $\ln \frac{N^{(0)}(t)-b}{t} = \ln a - m \ln t$. Consequently, reliability analysis does not require counting from the initial state of the development process, and the results of reliability tests from any time segment can be utilized for estimating instantaneous MTBF. The adaptive constant b does not contribute to the MTBF calculation, but it can enhance fitting and predictive accuracy of $\hat{N}^{(0)}(t_k)$.

Table 1. Comparison of three reliability growth prediction models.

Category	Duane model	DM model	GDM
Model assumption	$N(t) = at^{1-m}$	$N(t) = \frac{t}{\alpha+\beta\sqrt{t}}$	$N^{(0)}(t) = at^{1-m} + b$
Parameter estimation	$\ln \theta_c(t) = -\ln a + m \ln t$	$\theta_c(t) = \alpha + \beta\sqrt{t}$	See Eqs. [21] and [26]
Instantaneous MTBF	$\theta(t) = \frac{t^m}{a(1-m)}$	$\theta(t) = \frac{2(\alpha+\beta\sqrt{t})^2}{2\alpha+\beta\sqrt{t}}$	$\theta(t) = \frac{t^m}{a(1-m)}$

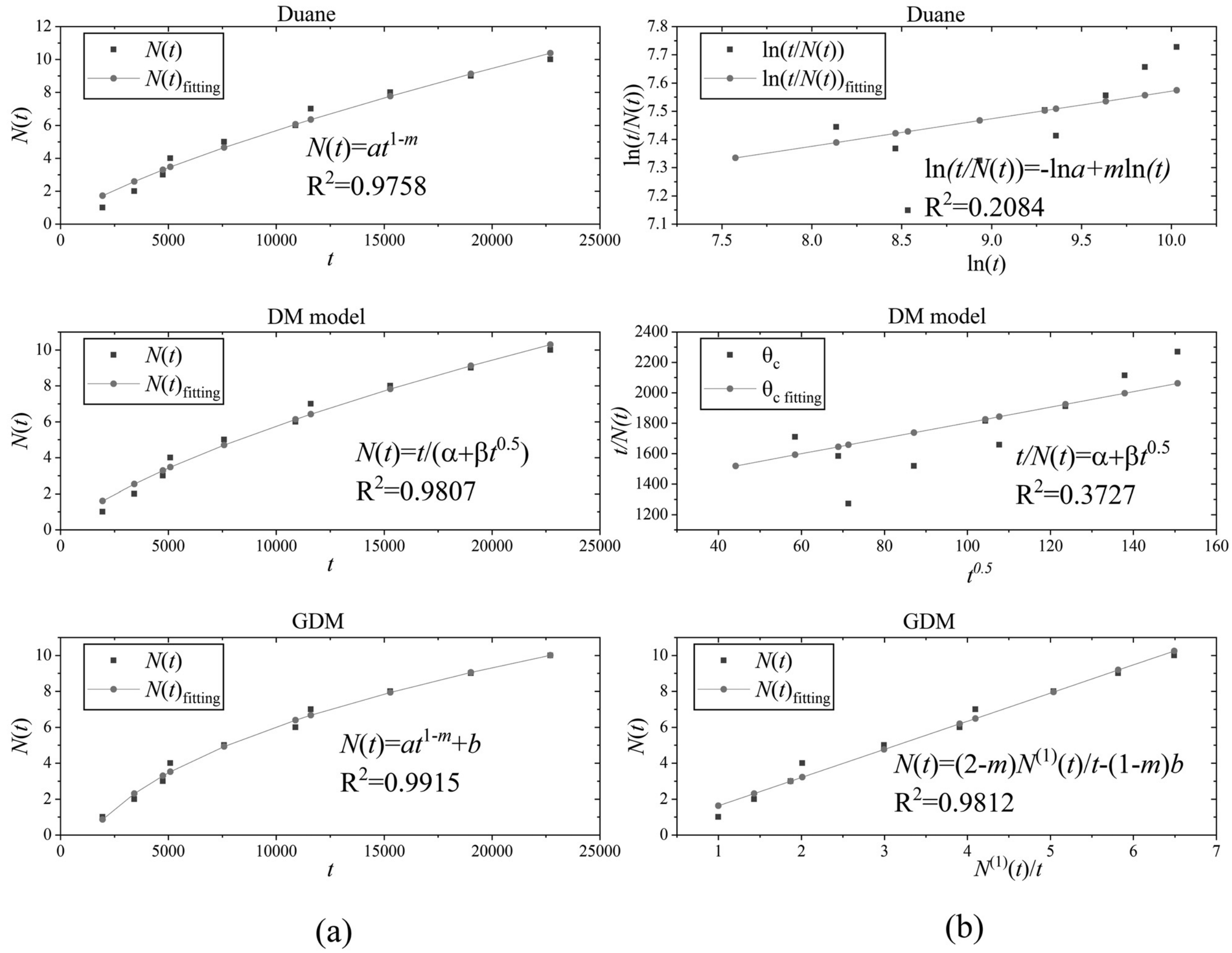

Figure 3. Comparison of fitting accuracy of reliability models: (a) model assumption; (b) parameter estimation.

Property 3. The traditional Duane model in Eq. [9] employs logarithmic data of cumulative MTBF to fit system parameters. Its cost function adheres to $\arg\min_{a,m}\left\{\sum_{k=1}^{n}\left[\ln\frac{t_k}{\hat{N}(t_k)}-\ln\frac{t_k}{N(t_k)}\right]^2\right\}$, but the calculation process is prone to error disturbances from early data. The proposed GDM model, however, obviates the need for coordinate transformation and directly models the observed data, leading to cost functions for parameter estimation as seen in Eqs. [24] and [26], thereby enhancing the accuracy of parameter estimation.

4. Verification

In this section, the performance of the proposed model was validated using case data from two references. A comparison was conducted through predictions in three aspects: failure number, accumulated failure time, and instantaneous MTBF.

4.1. Case 1. Failures versus time in an aerial flying system

The case data in Case 1 is sourced from reference (Nadjafi and Gholami 2022). The failure number and accumulated failure time of an aircraft's power system under 50°C conditions were recorded. The GDM and the traditional Duane model were separately employed for calculations using the same data sample. The first 9 data points were input into the models, with the final one data point serving as a test. To contrast the predictive performance of the models, the mean absolute percentage error (MAPE) and root mean square error (RMSE) were utilized as measures of error assessment. Furthermore, by inputting model parameters into Eqs. [2] and [16], the failure time can be concurrently estimated as follow:

$$\mathrm{MAPE}=\sum_{i=1}^{n}\left|\frac{\hat{x}(t_i)-x(t_i)}{x(t_i)}\right|\times 100\%,\tag{32}$$

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\hat{x}(t_i) - x(t_i)\right)^2}, \tag{33}$$

where $x(t_i)$ is the actual value of failure time or failures number, $\hat{x}(t_i)$ represents the corresponding predicted value.

$$\hat{t}_{\text{Duane}} = \left(\frac{N(t)}{\hat{a}_{\text{Duane}}}\right)^{\frac{1}{1-\hat{m}_{\text{Duane}}}}, \tag{34}$$

$$\hat{t}_{\text{GDM}} = \left(\frac{N(t) - \hat{b}_{\text{GDM}}}{\hat{a}_{\text{GDM}}}\right)^{\frac{1}{1-\hat{m}_{\text{GDM}}}}. \tag{35}$$

Tables 2 and 3 present a comparison of the predictive outcomes from the three models. The results from Table 2 reveal that, for estimates of accumulated failure time and failure number, the proposed GDM displays the most favorable predictive performance. This functionality is crucial for reliability forecasting models to anticipate the next occurrence of a fault. Due to restricted sample data and the perturbing impact of uncertain factors, the conventional Duane model struggles to provide accurate failure data estimates. The estimated reliability growth coefficient of the Duane model, at $m = 0.057$, significantly underestimates the true level of reliability improvement. Data from Table 2 demonstrates that the actual instantaneous MTBF for the 9th and 10th data points already exceed 3500 h, with the true value for out-of-sample data's instantaneous MTBF being 3676 h. The existing Duane and DM models are unable to accurately estimate such small-sample uncertainty issues, and their predictive results are unacceptable. In contrast, the estimated instantaneous MTBF from the proposed GDM closely aligns with the actual value. The reliability growth coefficient of the GDM is $m = 0.446$. The estimated instantaneous MTBF for the 9th and 10th trials is 3761.5 and 4069.96 h, respectively, conforming to the trend of reliability growth during the aircraft equipment development process.

4.2. Case 2. Reliability testing for a hypothetical system

The data originates from reference (Cahoon, Sanborn, and Wilson 2021). The 12 failure times of a hypothetical repairable system are counted to describe the nonstationary process of system failures. The initial eleven

Table 2. Prediction results of failure time and failures number in Case 1.

$N(t)$	t	Duane model (a, m) (0.001, 0.057)		DM model (α, β) (1415.686, 3.470)		GDM (a, m, b) (0.048, 0.446,–2.050)	
		$\hat{N}(t)$	$\hat{t}$	$\hat{N}(t)$	$\hat{t}$	$\hat{N}(t)$	$\hat{t}$
1	1950	1.23	1566.40	1.24	1552.39	1.16	1777.18
2	3418	2.09	3267.90	2.11	3225.48	2.33	2965.66
3	4750	2.85	5024.41	2.87	4981.74	3.21	4417.46
4	5090	3.04	6817.62	3.06	6807.87	3.41	6121.43
5	7588	4.42	8638.64	4.42	8696.22	4.76	8068.79
6	10,890	6.22	10,482.12	6.13	10,641.66	6.27	10,252.38
7	11,601	6.60	12,344.50	6.48	12,640.45	6.57	12,666.17
8	15,288	8.56	14,223.20	8.29	14,689.69	7.99	15,304.94
9	19,024	10.52	16,116.30	10.04	16,787.08	9.29	18,164.16
Fitting MAPE (%)		11.26	12.23	10.45	11.80	**8.10**	**8.37**
Fitting RMSE		0.68	1275.46	0.55	1102.12	**0.32**	**660.03**
10	22,700	12.43	18,022.29	11.71	18,930.72	10.45	21,239.79
Forecasting MAPE (%)		24.30	20.61	17.10	16.60	**4.52**	**6.43**
Forecasting RMSE		2.43	4677.71	1.71	3769.28	**0.45**	**1460.21**

The numbers of bold characters are the results of the proposed model.

Table 3. Instantaneous MTBF estimation of Case 1 by three reliability prediction models.

$N(t)$	t	Cumulative MTBF $t/N(t)$	Instantaneous MTBF $\frac{dt}{dN(t)} = \frac{t_k - t_{k-1}}{N(t_k) - N(t_{k-1})}$	Instantaneous MTBF estimation		
				Duane model	DM Model	GDM
			In-sample data			
1	1950	1950	1950	1682.84	1649.44	1361.47
2	3418	1709	1468	1737.94	1726.74	1748.84
3	4750	1583.333	1332	1771.09	1783.69	2025.41
4	5090	1272.5	340	1778.13	1796.95	2088.85
5	7588	1517.6	2498	1819.36	1883.62	2496.16
6	10,890	1815	3302	1857.49	1979.33	2932.73
7	11,601	1657.286	711	1864.25	1998.04	3016.66
8	15,288	1911	**3687**	1894.02	2087.42	**3411.92**
9	19,024	2113.778	**3736**	1917.94	2168.12	**3761.50**
			Out-sample data			
10	22,700	2270	**3676**	1937.49	2240.56	**4069.96**

The numbers of bold characters are the results of the proposed model.

data points were used for model training, while the final data point was reserved for validation. As Table 4 reveals, the proposed GDM exhibits the most impressive predictive performance, with prediction errors of 0.68% for failure number and 1.46% for failure time. The DM model's predictive capability slightly outperforms that of the Duane model. The reliability growth coefficients for the three models are $m_{\mathrm{Duane}} = 0.414$, $m_{\mathrm{DM}} = 0.5$, and $m_{\mathrm{GDM}} = 0.598$, respectively. As shown in Table 5, the three models yield nearly identical instantaneous MTBF values.

5. Applications

In this section, the presented GDM is utilized for forecasting the reliability growth of a particular electronic equipment during an aircraft's flight-testing phase. The case background and comparative models are introduced, followed by a comprehensive discussion and analysis of the experimental outcomes.

5.1. Research background

Equipment in the flight-testing phase possesses its own distinct characteristics owing to the intricate nature of external flight environments and various imposed constraints. During the flight-testing phase, equipment is subjected to real environmental stresses, encompassing both natural and on-board conditions, which expose representative fault patterns. By harnessing innovative technologies, processes, and materials, improvements and corrections are applied to these fault patterns, resulting in an upward trajectory of equipment reliability. Given the substantial research and development costs and time required for each experiment, the availability of failure data for some equipment is often constrained. Uncertainty factors such as communication disruptions, extreme weather, or signal interference during the experimental process introduce irregularities into the data, posing challenges for reliability growth prediction and assessment.

Table 4. Prediction results of failure time and failures number in Case 2.

$N(t)$	t	Duane model (a, m) (0.554, 0.414)		DM model (α, β) (0.872, 1.143)		GDM (a, m, b) (1.558, 0.598, −1.721)	
		$\hat{N}(t)$	$\hat{t}$	$\hat{N}(t)$	$\hat{t}$	$\hat{N}(t)$	$\hat{t}$
1	3	1.05	2.74	1.05	2.78	0.70	4.00
2	9	2.01	8.94	2.09	8.35	2.05	8.72
3	20	3.21	17.86	3.34	16.58	3.48	15.76
4	25	3.65	29.18	3.79	27.45	3.96	25.41
5	41	4.88	42.71	5.00	40.94	5.21	37.92
6	50	5.48	58.29	5.58	57.04	5.79	53.54
7	69	6.62	75.83	6.65	75.77	6.83	72.48
8	91	7.79	95.24	7.73	97.11	7.84	94.94
9	128	9.51	116.44	9.27	121.07	9.24	121.12
10	151	10.48	139.38	10.12	147.64	9.99	151.19
11	182	11.69	164.00	11.17	176.83	10.91	185.33
Fitting MAPE (%)		5.19	8.97	**4.31**	**7.52**	5.89	8.25
Fitting RMSE		0.38	8.26	0.24	4.64	**0.22**	**3.40**
12	227	13.31	190.25	12.54	208.63	12.08	223.70
Forecasting MAPE (%)		10.91	16.19	4.53	8.09	**0.68**	**1.46**
Forecasting RMSE		1.31	36.75	0.54	18.37	**0.08**	**3.30**

The numbers of bold characters are the results of the proposed model.

Table 5. Instantaneous MTBF estimation of Case 2 by three reliability prediction models.

$N(t)$	t	Cumulative MTBF	Instantaneous MTBF	Instantaneous MTBF estimation		
		$t/N(t)$	$\frac{dt}{dN(t)}$	Duane model	DM Model	GDM
			In-sample data			
1	3	3.00	3	4.85	4.37	2.07
2	9	4.50	6	7.65	7.15	3.99
3	20	6.67	11	10.65	10.45	6.44
4	25	6.25	5	11.68	11.64	7.36
5	41	8.20	16	14.33	14.81	9.89
6	50	8.33	9	15.56	16.32	11.14
7	69	9.86	19	17.78	19.13	13.50
8	91	11.38	22	19.94	21.93	15.93
9	128	14.22	37	22.96	25.97	19.53
10	151	15.10	23	24.59	28.19	21.56
11	182	16.55	31	26.56	30.94	24.11
			Out-sample data			
12	227	18.92	45	29.11	34.53	27.51

The failure data for specific electronic equipment were recorded over its initial twelve TAAF process, enabling the estimation of reliability growth trends during the flight-testing phase. Additionally, it is assumed that the objective of the trials is to achieve an MTBF of 15 h. The failure number and cumulative failure time need to be forecasted to evaluate the equipment's developmental progress.

5.2. Comparison of model results

To compare the predictive performance of the DGM model, the Duane model and the GDM model were selected as comparative models. The first ten data samples were utilized for model calculation, while the last two data points were reserved for testing. Table 6 reveals that in terms of failure number and cumulative failure time prediction, the proposed GDM model exhibits the most accurate predictive performance with a MAPE of less than 10%. Furthermore, the predictive performance of the DM model surpasses that of the traditional Duane model. This is primarily attributed to the DM model's fixed reliability growth rate set at $m = 0.5$. The calculated reliability growth rate for the proposed GDM model is $m = 0.487$, similar to the DM model. We observed that the DM model sustains stable predictive performance when the actual reliability approaches 0.5. The parameter estimation of the traditional Duane model, $m = 0.240$, underestimates the growth rate of system reliability. Consequently, the Duane model's performance in small-sample reliability prediction is weak, and its accuracy in predicting failure time and number is unsatisfactory.

All three models yield distinct estimates for instantaneous MTBF, attributed to the different reliability coefficients m. For estimating the out-of-sample data's MTBF, the Duane model underestimates the system's reliability level, primarily due to early data disturbances resulting in smaller parameter m for the Duane model. Results from Table 7 demonstrate that the instantaneous MTBF estimations for the 12th data point are 6.92, 8.82, and 11.02 for the Duane model, DM model, and GDM, respectively. The estimate

Table 6. Prediction results of failure time and failures number of aircraft electronic equipment.

$N(t)$	t	Duane model (a, m) (0.528, 0.240)		DM model (α, β) (1.710, 0.475)		GDM (a, m, b) (1.491, 0.487, −1.496)	
		$\hat{N}(t)$	$\hat{t}$	$\hat{N}(t)$	$\hat{t}$	$\hat{N}(t)$	$\hat{t}$
1	3.62	1.40	2.32	1.38	2.45	1.39	2.73
2	5	1.80	5.76	1.80	5.69	1.91	5.27
3	8.53	2.70	9.82	2.75	9.53	2.98	8.60
4	9.99	3.04	14.33	3.11	13.93	3.36	12.73
5	18.1	4.78	19.22	4.85	18.87	5.09	17.64
6	25.11	6.13	24.42	6.14	24.32	6.29	23.31
7	30.26	7.06	29.91	7.00	30.27	7.07	29.76
8	37.26	8.27	35.65	8.08	36.71	8.04	36.98
9	45.42	9.62	41.63	9.25	43.63	9.06	44.95
10	52.07	10.67	47.81	10.13	51.04	9.82	53.67
Fitting MAPE (%)		10.92	14.07	8.92	11.19	**7.06**	**7.45**
Fitting RMSE		0.47	2.44	0.34	1.55	**0.27**	**1.22**
11	60.62	11.98	54.20	11.21	58.93	10.74	63.15
12	71.49	13.58	60.77	12.48	67.28	11.82	73.38
Forecasting MAPE (%)		11.02	12.80	2.95	4.34	**1.93**	**3.41**
Forecasting RMSE		1.31	8.84	0.37	3.21	**0.22**	**2.24**

The numbers of bold characters are the results of the proposed model.

Table 7. Comparison of instantaneous MTBF prediction by three models.

$N(t)$	t	Cumulative MTBF	Instantaneous MTBF	Instantaneous MTBF estimation		
		$t/N(t)$	$\frac{dt}{dN(t)}$	Duane model	DM Model	GDM
			In-sample data			
1	3.62	3.62	3.62	3.39	3.16	2.58
2	5	2.50	1.38	3.66	3.43	3.02
3	8.53	2.84	3.53	4.16	3.99	3.91
4	9.99	2.50	1.46	4.32	4.19	4.23
5	18.1	3.62	8.11	4.98	5.12	5.64
6	25.11	4.19	7.01	5.39	5.77	6.62
7	30.26	4.32	5.15	5.63	6.20	7.25
8	37.26	4.66	7.00	5.92	6.73	8.02
9	45.42	5.05	8.16	6.21	7.29	8.84
10	52.07	5.21	6.65	6.42	7.71	9.44
			Out-sample data			
11	60.62	5.51	8.55	6.65	8.22	**10.17**
12	71.49	5.96	10.87	6.92	8.82	**11.02**

The numbers of bold characters are the results of the proposed model.

Table 8. Reliability growth prediction of aircraft electronic equipment by three models.

Failure number $\hat{N}(t)$	Duane model		DM model		GDM	
	(a, m) (0.555, 0.261)		(α, β) (1.639, 0.494)		(a, m, b) (1.180, 0.438, −0.981)	
	$\hat{t}$	$MTBF_{inst}$	$\hat{t}$	$MTBF_{inst}$	$\hat{t}$	$MTBF_{inst}$
13	71.51	7.45	78.10	9.44	81.13	13.26
14	79.06	7.64	87.80	9.94	91.73	13.99
15	86.80	7.83	97.99	10.44	102.91	14.71
16	94.72	8.01	108.68	10.94	114.63	15.42

provided by the proposed GDM model is closer to the true value of 11.02. Benefiting from its unique first-order accumulation mechanism and differential equation model representation, the GDM can more accurately estimate the developmental status of uncertain systems, yielding precise estimates.

5.3. Further prediction

For aircraft equipment, characterized by strict requirements, repeated usage, and significant costs associated with complex systems, accurately assessing reliability and achieving reliability growth throughout the entire lifecycle presents a crucial challenge. The reliability growth objectives for complex equipment should be determined based on practical engineering needs and feasibility, following comprehensive evaluation by technical teams. Moreover, the reliability and development experience of similar equipment serve as vital references when formulating ideal reliability growth curves for new aircraft equipment.

Without the test set, all twelve data samples are brought into GDM. The subsequent five failure time and instantaneous MTBF are listed in Table 8. The traditional Duane model's MTBF estimates deviate from reality due to underestimation of reliability growth speed. In fact, by the 11th test, the system's reliability MTBF had already reached 8.55 h, significantly exceeding the Duane model's forecast for the 16th test. The DM model outperforms the Duane model in small-sample scenarios, as it does not require coordinate system transformation in its calculations, effectively reducing systematic errors. The proposed GDM demonstrates superior predictive performance for both failure times and instantaneous MTBF. According to the GDM's calculations, the MTBF for the 16th test reaches 15.42 h, satisfying the system's reliability development requirement. The failure time for the 16th test is 114.65 h. The aircraft equipment development process necessitates tracking and controlling reliability growth. Based on available test data, it is reasonable to construct reliability growth plans, as depicted in Figure 4. If subsequent test results deviate from targets, implementing effective measures to

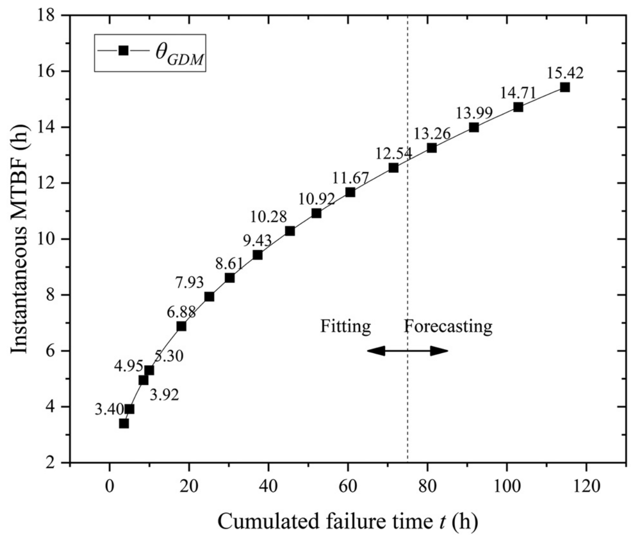

Figure 4. The estimation of instantaneous MTBF by GDM.

enhance the development process and promote reliability growth becomes essential.

5.4. Discussion of model effectiveness and limitations

The results in Tables 7 and 8 demonstrate that the proposed GDM method performs better in evaluating and estimating the expected MTBF of electronic devices. Consequently, we can assert that this gray modeling method enhances the effectiveness of the traditional Duane model in scenarios involving small sample predictions. From the perspective of uncertain data analysis, this study provides an effective modeling approach for fault data that does not conform to the power-law process.

The case data experiments in Section 4 reveal that small sample data analysis is susceptible to the influence of disturbance factors, and conventional large sample statistical methods cannot accurately estimate the developmental status of uncertain systems. Noisy observational data can reduce the accuracy of model parameter estimates. Indeed, gray modeling methods can make reliability growth prediction models more effective in small sample data analysis. The employed gray accumulation operation serves as an excellent denoising operator, with reversibility that helps mitigate the impact of random observational data on prediction results. Table 1 compares the modeling methods and underlying assumptions of GDM, the Duane model, and the DM model. The GDM model exhibits high accuracy ($R^2 > 0.9$) in estimating structural parameters. The flexible growth coefficient, m, can adapt to different scenarios of reliability growth experiments, facilitating the development of rational reliability growth plans based on actual engineering conditions.

From the perspective of engineering practice in predicting the reliability of aircraft electronic equipment, it is imperative to enhance the flexibility of models for making accurate decisions based on data. Firstly, GDM focuses on the initial state of the system, recognizing that electronic devices may undergo multiple debugging and maintenance processes before reliability testing. Consequently, GDM is a three-parameter nonlinear system, allowing for adaptive correction of missing data on uncertain system initial points. Additionally, GDM can simultaneously estimate fault time and instantaneous MTBF, which is beneficial for decision support processes in maintenance prediction and reliability forecasting.

Despite these positive improvements, the proposed method still has certain limitations. First, constrained by cost and testing requirements, the research results stem from a single case study, lacking external validity. The randomness of faults in actual engineering may result in unstable deviations in predicted outcomes. Second, the parameter-solving method of GDM is a two-stage regression calculation process. The solution of nonlinear and linear parameters is interrelated in two sequential parts, and the complexity of the algorithm may limit the model's usage and generalization. From a mathematical computation perspective, heuristic algorithms, or nonlinear parameter optimization methods serve as viable alternatives. Our approach aims to provide a short-term prediction and control tool for reliability growth experiments in small sample scenarios. While GDM has its advantages, it is crucial to recognize that, due to the inherent constraints of gray modeling methods, the performance of GDM may decline with an increase in data volume. Therefore, the error propagation process of the model should be further explored.

6. Conclusions

This study introduces a novel reliability growth prediction model. Built upon the grey modeling approach, the proposed GDM is well-suited for analyzing uncertainty in small-sample data. Numerical cases have demonstrated several advantages of the proposed method over traditional models: (1) The GDM does not require data transformation, thereby avoiding the aggregation of logarithmic function transformed data and interference from early-stage failure data. (2) The GDM can simultaneously provide accurate estimates for failure number, failure time, and instantaneous MTBF. (3) In scenarios with limited available data, the GDM accurately estimates model parameters, benefiting from the first-order cumulative generation operation.

The reliability analysis methods for engineering products, stemming from diverse backgrounds and manufacturing conditions, exhibit notable variations. The application of the GDM model to real-world reliability prediction cases has convincingly demonstrated its efficacy in handling uncertain data. However, it's essential to note that this application is currently confined to the reliability test data of aircraft electronic equipment. The proposed novel method requires validation for its effectiveness, preferably in a more extensive data sample or across diverse industries. Despite its strengths, it is crucial to acknowledge that the performance of GDM may experience degradation as the volume of data increases, owing to the inherent constraints of the grey modeling approach. Looking ahead, it is necessary is to explore more streamlined approaches that simplify this accumulation operation for enhanced efficiency.

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Disclosure statement

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Data availability statement

All data, models, and code generated or used during the study appear in the submitted article. Code is available at https://github.com/mytruth126/grey-Duane-model.

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