## Section 1.3: Predicate Logic

**Purpose of Section:** To introduce **predicate logic** (or **first-order logic**) which the language of mathematics. We see how predicate logic extends the language of sentential calculus studied in Sections 1.1 and 1.2 by the inclusion of universal and existential quantifiers, logical functions and variables.

### Introduction

The sentential calculus introduced in Lessons 1 and 2 is not sufficient to represent the types of assertions normally found in mathematics, and thus in the late 1800s logicians created a richer, more expressive language, called **predicate logic**[1]. Sentential logic as studied in Lessons 1 and 2 involve the truth or falsity of simple sentences, whereas predicate logic is richer and allows one to express concepts about *collections* of objects (maybe real numbers, natural numbers, or functions). For instance, when we say *"for any real number $x$ there exists a real number $y$ that satisfies $x < y$"* we are making a claim about the validity of $x < y$ over a collection of numbers $x$ and $y$. A truth table for such a sentence is infeasible for statements of this type since it would require an infinite number of rows to enumerate all possible values of $x$ and $y$. Nevertheless, sentences like this are the language of mathematics where information about the relevant variables are specified and propositions made about the variables. Whereas sentential calculus involves only simple sentences and their connectives, predicate logic includes *quantifiers, logical functions* and *variables,* allowing for a wider variety of ideas and concepts. Pick up your old calculus textbook and you will discover that theorems are stated in the language of predicate logic.

Predicate logic might be interpreted as a formal language representation for sentences in natural languages, such as English. The quantifiers of predicate logic act are the subject of a sentences, and logical functions, called predicates, act as the verbs of the sentence.

**Historical Note:** The development of predicate logic is generally attributed to the German logician Gottlob Frege (1848-1925), considered to be the most important logician of the $19^{th}$ century. Predicate logic with its quantifiers and predicates is the logical basis for today's mathematics. It was Frege's belief (misguided as it turned out) that all mathematics could be derived from logic.

### Quantifiers

Two phrases one hears again and again in any mathematical discussion are "*for all*," and "*there exists*." These expressions are called **quantifiers** in predicate logic and are necessary for the precise description of mathematical

[1] Predicate logic is also called *first-order logic* in contrast to sentential calculus, which is sometimes called *zero-order logic,* studied in Lessons 1 and 2.

concepts. For example, the statement $x < y$ is unclear, not having been explicitly told the meaning and extent of $x$ and $y$. Do we mean for *any* real number $x$ there *exists* a real number $y$ satisfying $x < y$? Or do we mean there *exists* an $x$ such that for *all* $y$ we have $x < y$. Clearly, the statements are *not* the same (one is true the other isn't). The point we are trying to make is that in precise mathematical discourse one does not simply write down expressions and equations without exactly specifying the meaning and extend of the variables involved.

This leads us to the two basic quantifiers of predicate logic. The phrase "**for all**," is called the **universal quantifier** and is denoted by $\forall$ (upside down capital A), and "**there exists**" is called the **existential quantifier** and denoted by $\exists$ (backwards capital E).

Inherent in the use of quantifies is the concept of a **universe**. It makes no sense to say "for all" unless we know "for all" *what*? Sometimes the universal set is pointed out and sometimes it is understood from the context of the statement. In mathematics there are several sets that serve as universes. We have listed some important sets which will act as universal sets.. The reader should remember the symbols used to denote for each set[2] such as increasing collection of set $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ since we will be referring to these sets in the remainder of the book..

$\mathbb{N} = \{1, 2, 3, ...\}$ (natural numbers)

$\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ (integers)

$\mathbb{Q} = \{ p/q : p \text{ and } q \text{ are integers with } q \neq 0 \}$ (rational numbers)

$\mathbb{R} = \{\text{real numbers}\}$

$\mathbb{C} = \{\text{complex numbers}\}$

[2]We have gotten ahead of ourselves and introduced a modicum of set notation, although we suspect most readers are well familiar with our mild use of sets and set notation.

Quantifiers of Predicate Logic Let $U$ be the universe under consideration.

- Universal quantifier: $(\forall x \in U)\ P(x)$ means "For all (or any) $x$ in the set $U$, such that $P(x)$ is true"
- Existential quantifier: $(\exists x \in U)\ P(x)$ means "There exists an $x$ in the set $U$ such that $P(x)$ is true"
- Unique existential quantifier: $(\exists ! x \in U)\ P(x)$ means "There exists exactly one $x$ in the set $U$ such that $P(x)$ is true"

Example 1

- $(\forall x \in \mathbb{R})\left[(x<0) \vee (x \geq 0)\right]$

  For all real numbers $x$ either $x$ is less than zero or $x$ is greater than or equal to zero.

- $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x<y)$

  For all (or any) real numbers $x$ there exists a real number $y$ that satisfies $x<y$.

- $(\forall y \in \mathbb{R})(\exists ! x \in \mathbb{R})(2x+3=y)$ For any real number $y$ there exists one and only one $x$ that satisfies $2x+3=y$.

Margin Note: In books when one sees a statement like "If $x$ is an integer then $x$ is a rational number," one means $(\forall x)(x \in \mathbb{Z} \Rightarrow x \in \mathbb{Q})$ or $(\forall x \in \mathbb{Z})(x \in \mathbb{Q})$. In other words the universal quantifier is understood.

Predicates: As discussed predicate logic might be interpreted as English language translated into the logic of mathematics, complete with a subject and verb. The quantifiers are the subject, as when we say "for all real numbers $x$", and the predicate is proposition about the quantified variable. The predicate is a truth-valued function (of one or more variables) whose value is T or F. Typical predicates are

- $P(n)$: $n$ is a prime number
- $P(x)$: $x^2+1=0$
- $P(x,y)$: $x \leq y$
- $P(x,y,z)$: $x^2+y^2+z^2=1$

Predicates can depend on one or more variables. For the predicate $P(x,y): x \le y$ note that $P(1,2)$ is true, $P(5,3)$ is false.

In sentences like $(\forall x \in S)\ P(x)$ or $(\exists x \in S)\ P(x)$ the variable $x$ is called a **bound variable**[3], bound by the respective quantifier. A variable not bound is called a **free variable**, as in the phrase "$x$ is an even integer." In standard set notation $\{x \in \mathbb{R} : 0 \le x \le 1\}$ the variable $x$ is bound by the quantifier $x \in \mathbb{R}$, which we understand to mean "*for all* real numbers." In set notation the universal quantifier $\forall$ is generally omitted since it is understood.

Example 2 Translation of Predicate Logic

Below are English language interpretations of predicate logic sentences. Some sentences include more than one quantifier.

| Predicate Sentence | English Meaning | Truth Value |
|---|---|---|
| $(\forall x \in \mathbb{R})\ (x^2 \ge 0)$ | For any real number, its square is nonnegative. | true |
| $(\exists x \in \mathbb{N})$ $x$ is a prime number | There exists a prime number | true |
| $(\exists n \in \mathbb{N})\ (2 \nmid n)$ | There exists at least one odd natural number. | true |
| $(\exists n \in \mathbb{N})\ (2 \mid n)$ | There exists at least one even integer. | true |
| $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})\ (x = y+1)$ | For any natural number $x$ there exists a natural number $y$ satisfying $x = y+1$.. | false |
| $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\ (x < y)$ | For any real number $x$ there exist a real number $y$ greater than $x$. | true |
| $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})\ (x < y)$ | There exists a real number $x$ such that all real numbers $y$ are greater than $x$.. | false |
| $(\forall x > 0)(\exists y \in \mathbb{R})(y^2 = x)$ | For all positive real numbers $x$ there exists a real number $y$ such that the square of $y$ is $x$. | true |

Predicate Sentences
Table 1

[3] Students of integral calculus are well familiar with bound variables, sometimes called dummy variables.. The variable $x$ in the integral $\int_a^b f(x)\,dx$ is a bound (or dummy) variable.

### Order of Quantifiers Counts

Theorems in mathematics often begin with "*for all x there exists a y such that ⋯,*" or "*there exists an x such that for all y ⋯ .*" The question arises does the order of the quantifiers make a difference in the meaning of the sentence? The answer is yes and Figure 1 below illustrates the difference between the orders $\forall\forall, \exists\forall, \forall\exists$ and $\exists\exists$ of the quantifiers $\exists$ and $\forall$. To illustrate this difference, we present a simple scenario of a third grade class consisting of six students, 3 boys and 3 girls, where we denote the boys by the set $\boldsymbol{B}$ and the girls by the set $\boldsymbol{G}$, where

$$B = \{\text{Abe, Bob, Carl}\}$$
$$G = \{\text{Ann, Betty, Carol}\}$$

We now define the predicate

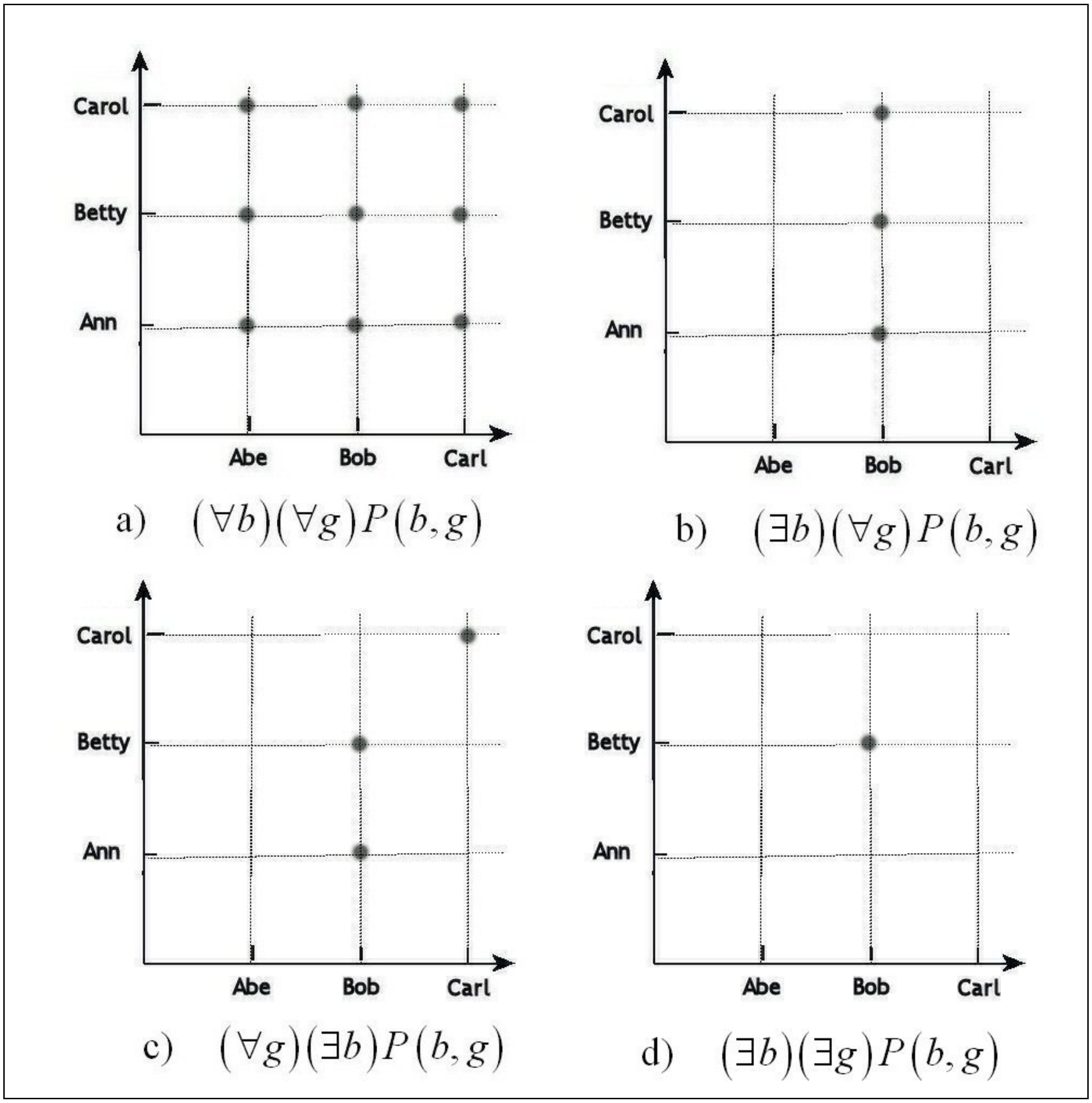
$$P(b,g) = \text{ boy } b \text{ likes girl } g$$

where $b \in \boldsymbol{B}$ represents one of the 3 boys and $g \in \boldsymbol{G}$ one of the 3 girls. The four drawings in Figure 1 (a), (b), (c), and (d) illustrate which boys like which girls, a dot at the intersection of a boy and girl indicates a preference. For example, in Figure 1 a) every boy likes every girl, while in Figure 1 c) Bob likes Ann and Betty, and Carl likes Carol.

We will now argue the implications that

$$(\forall b)(\forall g)P(b,g) \Rightarrow (\exists b)(\forall g)P(b,g) \Rightarrow (\forall g)(\exists b)P(b,g) \Rightarrow (\exists b)(\exists g)P(b,g)$$

We analyze each of the drawings in Figure 1 one at a time:




Graphical Illustration $\forall\forall \Rightarrow \exists\forall \Rightarrow \forall\exists \Rightarrow \exists\exists$
Figure 1

Figure 1 (a) $(\forall b)(\forall g)P(b,g)$ In predicate logic this sentence reads "For every boy $b$ and every girl $g$, boy $b$ likes girl $g$. In more conversational English, this sentence reads every boy likes every girl, which we represent by placing a dot at the intersection of every boy and girl in Figure 1 (a).

Figure 1 (b) $(\exists b)(\forall g)P(b,g)$ This sentence says there exists a boy $b$ such that for any girl $g$, boy $b$ likes girl $g$. In other words there exists a boy, Bob in our example, that likes every girl. Clearly, this sentence holds if the sentence in (a) holds, which states *every* boy likes every girl, not just Bob.

Figure 1 (c) $(\forall g)(\exists b)P(b,g)$ This sentence says for every girl $g$ in the class, there exists a boy $b$ such that boy $b$ likes girl $g$. In other words, every girl in the class has at least one admirer, not necessarily the *same* boy as required in the more restrictive sentence in (b). In our example Ann and Betty are liked by Bob, and Carol is liked by Carl.

Figure 1 (d) $(\exists b)(\exists g)P(b,g)$ This sentence is necessary for all the previous ones. It reads there exists a boy and there exists a girl such that the boy likes the girl. Simply put, there is at least one romance taking place in the third grade class. Clearly, this sentence holds if the previous statement in (c) holds.

Our simple expedition into the third grade class demonstrates the general relationship

$$\boxed{(\forall x)(\forall y)P(x,y) \Rightarrow (\exists x)(\forall y)P(x,y) \Rightarrow (\forall y)(\exists x)P(x,y) \Rightarrow (\exists x)(\exists y)P(x,y)}$$

between the orders of the universal and existential quantifiers.

### Quantifiers in Calculus

As mentioned earlier predicate logic is the language of mathematics. The following definition of continuity of a function at a point is one such example.

Definition: *A real-valued function $f$ of a real variable $x$ is continuous at a point $x_0$ if and only if*[4]*:*

$$(\forall \varepsilon > 0)(\exists \delta > 0) \text{ such that } |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

We add the phrase "such that" in English only for clarification. Sometimes the phrase "such that" is denoted by ∍ (backwards epsilon), although this symbol is not a formal character in the predicate logic, only a shorthand symbol such as a stenographer would use. Using the shorthand ∍ the above definition of

[4] Note in the following predicate logic sentence we don't quantify the function $f$, the variable $x$, and the constant $x_0$ since they have been specified when we write "*A real-valued function f of a real variable $x$ is continuous at a point $x_0$.*"

continuity of $f$ at $x_0$ would be written

$$(\forall \varepsilon > 0)(\exists \delta > 0) \ni |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

In English this statement would read: For all epsilon greater than 0, there exists a delta greater than 0, such that if $x$ is closer to $x_0$ than delta, then $f(x)$ is closer to $f(x_0)$ than eplison. Don't you agree the symbolic languate of predicate logic is preferable?

**Historical Note:** Frege's 1879 seminal work *Begriffsschrift* ("Conceptial Notation") marked the beginning of a new era in logic, which allowed for the quantification of mathematical variables, just in time for the more precise **arithmetization of analysis** of calculus, being carried out in the late 1800s by mathematicians like the German Karl Weierstrass (1815-1897).

## Negation in Predicate Logic

In the next few sections when we introduce strategies for proving mathematical theorems, it is necessary to know how to negate quantified statements. The following table shows how a few statements in predicate logic are negated. We don't always state the universe if it is assumed given.

| Sentence | Negation |
|---|---|
| $(\forall x \in S)\ P(x)$ | $(\exists x \in S)\ (\sim P(x))$ |
| $(\exists x \in S)\ P(x)$ | $(\forall x \in S)\ (\sim P(x))$ |
| $(\forall x)(\exists y)\ P(x, y)$ | $(\exists x)(\forall y)\ (\sim P(x, y))$ |
| $(\exists x)(\forall y)\ P(x, y)$ | $(\forall x)(\exists y)\ (\sim P(x, y))$ |
| $(\forall x)(\forall y) P(x, y)$ | $(\exists x)(\exists y)(\sim P(x, y))$ |
| $(\exists x)(\exists y) P(x, y)$ | $(\forall x)(\forall y)(\sim P(x, y))$ |

Negation in Predicate Logic
Table 2

Table 3 gives some negations in common English.

| Sentence | Negation |
|---|---|
| Every day it rains. | There exists a day when it doesn't rain. |
| There exists a number that is transcendental. | Every number is not transcendental. |
| All prime numbers are odd. | There exists a prime number that is not odd. |
| At least one day I will go to class | I will never go to class. |

Common Predicate Sentences and Their Negatives
Table 3

Example 3 (Negations)

State the negation of the following quantified propositions.

a) $(\forall x \in S)(x \geq 0)$

b) $(\exists n \in S)(n \text{ is a prime number})$

c) $(\forall x)(\exists y)(xy = 10)$ (which we read "for all $x$, there exists a $y$ such that $xy = 10$")

d) $(\exists x)(\forall y)(xy \neq 10)$ (which we read "there exists an $x$ such that for all $y$ we have $xy \neq 10$)

Solution

a) $\sim(\forall x \in S)(x \geq 0) \equiv (\exists x \in S)(x < 0)$
b) $\sim(\exists n \in S)(n \text{ is a prime number}) \equiv (\forall n \in S)(n \text{ is not a prime number})$
c) $\sim(\forall x)(\exists y)(xy = 10) \equiv (\exists x)(\forall y)(xy \neq 10)$
d) $\sim(\exists x)(\forall y)(xy \neq 10) \equiv (\forall x)(\exists y)(xy = 10)$

Historical Note: Although Frege is credited with the development of predicate logic, Aristotle anticipated quantifiers 2000 years earlier in his development of syllogisms, the most famous being "All people are mortal. Socrates is a person. Therefore Socrates is mortal." Also earlier In the $19^{th}$ century the English logician Augustus DeMorgan developed the notion of quantifiers. However, it was Frege who developed the complete theory of predicate logic as we know it today.

Quantifiers and Conjunctions

The universal quantifier $(\forall)$ is related to the logical conjunction $(\wedge)$ by

$$(\forall x)\left[P(x) \wedge Q(x)\right] \;\Leftrightarrow\; (\forall x)P(x) \wedge (\forall x)Q(x)$$

For example, suppose we have a universe of students $x$ at a university where the predicates are taken to be

$$P(x) = \text{ student } x \text{ likes peanuts}$$
$$Q(x) = \text{ student } x \text{ likes qumquats}$$

Here $(\forall x)(P(x) \wedge Q(x))$ means every student $x$ likes both peanuts and qumquats, which is true if and only if every student likes peanuts and every student likes qumquats, which is represented by $(\forall x)P(x) \wedge (\forall x)Q(x)$

On the other hand, for the existential quantifier $(\exists)$, we have

$$(\exists x)\left[P(x) \wedge Q(x)\right] \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$$

which in the context of students at our university, states

> *if* there exists a student who likes both peanuts and qumquats, then there exists a student who likes peanuts and there exists a student who likes qumquats

which is clearly true since we simply take a student who likes both peanuts and qumquats. On the other hand, the converse

$$(\exists x)P(x) \wedge (\exists x)Q(x) \not\Rightarrow (\exists x)\left[P(x) \wedge Q(x)\right]$$

does *not* always hold since there might be a student who likes peanuts and a student who likes qumquats, but that doesn't mean there is a student who likes both peanuts and qumquats (every student likes *exactly* one of the selection of peanuts or qumquats).

### Quantifiers and Disjunctions

For the existential quantifier $(\exists)$ and the logical disjunction $(\vee)$, we have

$$(\exists x)\left[P(x) \vee Q(x)\right] \Leftrightarrow (\exists x)P(x) \vee (\exists x)Q(x)$$

which states the logical equivalence of

"*there is at least one* $x$ *such that* $P(x)$ or $Q(x)$ *is true*"

with the sentence

"t*here exists an* $x$ *such that* $P(x)$ *is true or there exists an* $x$ *such that* $Q(x)$ *is true*."

However, the universal quantifier $(\forall)$ acting on the logical disjunction $(\vee)$ obeys the laws

$$(\forall x)P(x)\vee(\forall x)Q(x)\Rightarrow(\forall x)\left[P(x)\vee Q(x)\right]$$
$$(\forall x)\left[P(x)\vee Q(x)\right]\not\Rightarrow(\forall x)P(x)\vee(\forall x)Q(x)$$

Here $(\forall x)P(x)\vee(\forall x)Q(x)$ means $P(x)$ is true for all $x$ or $Q(x)$ is true for all $x$, which implies the weaker statement that for all $x$, either $P(x)$ or $Q(x)$, denoted by $(\forall x)\left[P(x)\vee Q(x)\right]$. [Clearly, if every student likes peanuts or if every student likes qumquats, then every student likes peanuts or qumquats.]

On the other hand

$$(\forall x)\left[P(x)\vee Q(x)\right] \text{ does } \textit{not} \text{ imply } (\forall x)P(x)\vee(\forall x)Q(x)$$

For example $(\forall x)\left(P(x)\vee Q(x)\right)$ means every student likes peanuts or qumquats, but that does not imply $(\forall x)P(x)\vee(\forall x)Q(x)$, which says there are three situations: 100% of the student body likes peanuts, 100% of the student body likes qumquats, or 100% of the student body likes both peanuts and qumquats. (In other words, we don't have 75% peanut lovers and 25% qumquat lovers.)

We summarize the above relationships between quantifiers and connectives by Table 2.

| | $\wedge$ | $\vee$ |
|---|---|---|
| $\forall$ | $(\forall)(P\wedge Q)\Leftrightarrow\forall P\wedge\forall Q$ | $\forall P\vee\forall Q\Rightarrow(\forall)(P\vee Q)$ |
| $\exists$ | $(\exists)(P\wedge Q)\Rightarrow\exists P\wedge\exists Q$ | $(\exists)(P\vee Q)\Leftrightarrow\exists P\vee\exists Q$ |

Quantifiers and Conjunctions and Disjunctions
Table 2

**Historical Note:** The American logician Charles Saunders Pierce (1839-1941) introduced **second-order logic**, which in addition to quantifying variables like $x, y, \ldots$ also quantifies functions and entire sets of variables. However, for most mathematics, first-order logic is adequate. Pierce also developed first-order logic on his own, but Frege carried out his research earlier and is generally given credit its development. It was Pierce who coined the word "first-order" logic.

Problems

1. Write the following sentences in the symbolic language of predicate logic. The universe of each variable is given in paranthesis.

a) If $a|b$ and $b|c$, then $a|c$, where $a,b,c$ are integers. (Integers)

b) 4 does not divide $n^2+2$ for any integer (Integers)

c) $x^3+x+1=0$ for some real $x$ (Real numbers)

d) Only fools fall in love. (All people)

e) Everybody loves mathematics. (All people)

f) For every positive real number $a$ there exists a real number $x$ that satisfies $e^x=a$. (Real numbers)

g) For every positive real number $\varepsilon>0$ there exists a real number $\delta>0$ such that $|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon$. (Real numbers)

h) Everyone always attends class. (All students)

i) The equation $x^2+1=0$ has no solution (Real numbers)

j) The equation $x^2-2=0$ has no solution (Rational numbers)

k) In every mathematics class there is always one genius. (Students)

2. **(True or False?)** Which of the following are true in the given universe? The universe is given in parenthesis.

a) $(\forall x)(x \le x)$ (Real numbers)

b) $(\exists x)(x^2=2)$ (Real numbers)

c) $(\exists x)(x^2=2)$ (Rational numbers)

d) $(\exists x)(x^2+x+1=0)$ (Real numbers)

e) $(\forall x)[x \equiv 1 (\mathrm{mod}\, 5)]$ (Integers)

f) $(\exists ! x)(e^x = 1)$ (Real numbers)

g) $(\forall x)(x \le x)$

h) $(\forall x)(x \le x)$

i) $(\forall x)(x \le x)$

j) $(\forall x)(x \le x)$

4. **(Expanding Universes)** In which universe $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are the following sentences true?

a) For every $x$ in the universe there exists a $y = 1 - x$ in the universe.

b) For every $x \neq 0$ in the universe there exists a $1/x$ in the universe.

c) For every $x$ in the universe there exists a solution of $x^2 - 2 = 0$ in the universe.

d) For every $x$ in the universe there exists a solution of $x^2 + 1 = 0$ in the universe.

5. **(Not as Easy as It Looks)** Tell if $(\exists x)[x \text{ is even } \Rightarrow 5 \le x \le 10]$ is true or false for the given universe $U$.

a) $U = \{4\}$
b) $U = \{3\}$
c) $U = \{6,8,10\}$
d) $U = \{6,8,10,12\}$
e) $U = \{6,7,8,10,12\}$

6. **(Small Universe)** Which of the following statements are true for the

7. universe $U = \{1, 2, 3\}$.

a) $1 < 0 \Rightarrow (\exists x)(x < 0)$

b) $(\exists x)(\forall y)(x \le y)$

c) $(\forall x)(\exists y)(x \le y)$

d) $(\exists x)(\exists y)(y = x + 1)$

e) $(\forall x)(\forall y)(xy = yx)$

f) $(\forall x)(\exists y)(y \le x + 1)$

8. **(Well-Known Universe)** Letting

$R(x)$: $x$ is a rational number
$I(x)$: $x$ is an irrational number

which of the following statements are true for the universe of real numbers.

a) $(\forall x)[I(x) \vee R(x)]$

b) $(\forall x)[I(x) \wedge R(x)]$

c) $(\forall x)R(x) \vee (\forall x)I(x)$

d) $(\forall x)[R(x) \vee I(x)] \Rightarrow [(\forall x)R(x) \vee (\forall x)I(x)]$

9. State the following famous theorems in mathematics in the symbolic language of predicate calculus.

a) **(Bolzano's Intermediate Value Theorem)** If $f$ is a continuous function on an interval $[a,b]$ and if $f$ changes sign from negative to positive (or vice versa), then there exists a $c$ between $a$ and $b$ such that $f(c) = 0$.

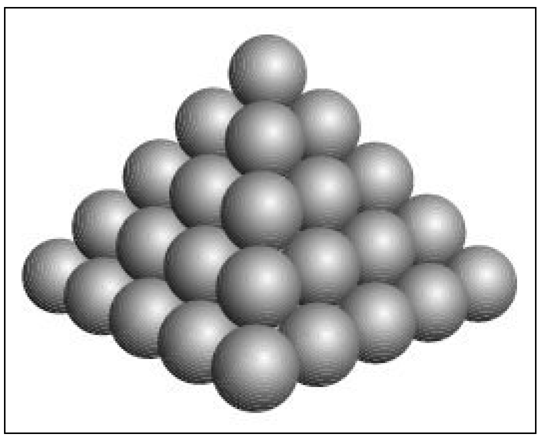
b) **(Fermat's Last Theorem)** If $n$ is an integer greater than 2, then there are no nonzero integer values of $a, b, c$ that satisfy $a^n + b^n = c^n$.

c) **(Euler's Theorem)** If $P$ is any regular polyhedra, and $v, e, f$ represent the number of vertices, edges, and faces, respectively of the polyhedra, then $v - e + f = 2$.

d) **Binomial Theorem)** If $a, b$ are real numbers and $n$ is a nonnegative integer, then

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k}$$

e) **(Kepler Theorem[5])** No packing of congruent spheres in three-dimensional space has a density greater than $\pi / \sqrt{18} \approx 0.74$.



10. **(Negation)** Negate the following sentences both in words and symbolically in the language of predicate logic.

a) All women are moral.

b) Every player on the basketball team was over six feet tall.

c) For any real number $y$ there exists a real number $x$ that satisfies $y = \tan x$.

d) The exists a real number $x$ such that $0 < x < 5$ and $x^3 - 8 = 0$.

[5]This theorem was called the Kepler Conjecture until 1998 when the conjecture was proven in the affirmative by Thomas Hales of the University of Pittsburg. The proof, however, is 111 pages long and requires the use of a computer to verify 5000 specific cases of configurations of balls to prove the theorem.

e) The equation $a^n + b^n = c^n$ does not have nonzero integer solutions $a,b,c$ for $n$ a natural number $n > 2$.

f) $(\forall x)[P(x) \Rightarrow Q(x)]$

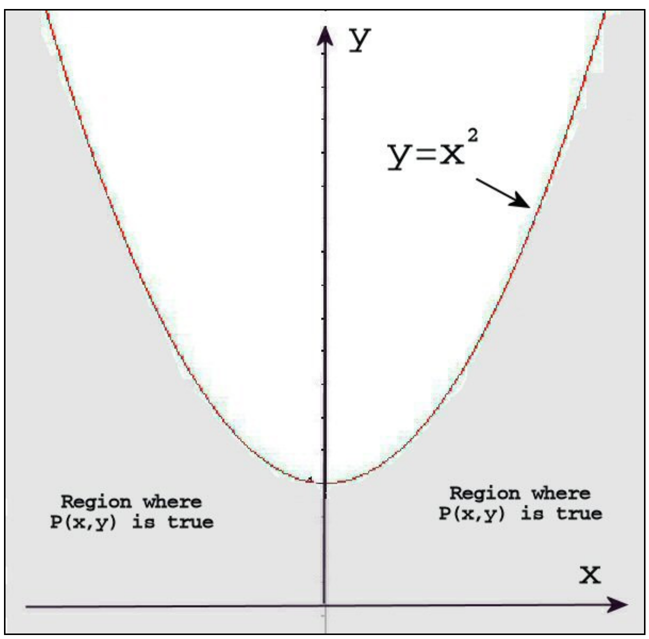
g) $(\forall x)\left[x > 0 \Rightarrow (\exists y)(x^2 = y)\right]$

h) $P \Rightarrow (Q \wedge R)$

i) $(P \wedge Q) \Rightarrow R$

10. **(An Old Friend)** Write a statement in predate logic that defines the points in the plane (i.e. $\Re^2$) whose distance from the origin (0,0) is 1.

11. **(Graph to the Rescue)** Let the universe be the Cartesian plane with proposition $P(x,y): y \le x^2 + 1$. The points $(x,y)$ in the plane where the proposition is true is drawn in Figure 3.




Truth Values of $P(x,y)$
Figure 3

Determine which of the following sentences are true?

a) $(\forall x)(\forall y)P(x,y)$

b) $(\forall x)(\exists y)P(x, y)$

c) $(\exists x)(\forall y)P(x, y)$

d) $(\exists x)(\exists y)P(x, y)$

e) $(\exists y)(\forall x)P(x, y)$

f) $(\forall y)(\exists x)P(x, y)$

12. State the denial of the words of wisdom attributed to Abraham Lincoln: “You can fool some of the people, all the time and all the people some of the time, but you can’t fool all the people all of the time.”