KS091201
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(DISCRETE
MATHEMATICS)

RULES OF INFERENCE

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Outline

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- Modus Tollens
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Valid Arguments

- An Argument in propositional logic is a sequence of propositions.
- All but the final proposition are called premises.
- The final proposition is called conclusion.
- An argument is valid if the truth of all premises implies that the conclusion is true.
  - i.e. $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology.

Modus Ponens

- Consider $(p \wedge (p \rightarrow q)) \rightarrow q$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q))$	$(p \wedge (p \rightarrow q)) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore\ q \end{array}$$

Modus Ponens Example

- Assume you are given the following two statements:
  - "you are in this class"
  - "if you are in this class, you will get a grade"
- Let p = "you are in this class"
- Let q = "you will get a grade"
- By Modus Ponens, you can conclude that you will get a grade

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

Modus Tollens

- Assume that we know: $\neg q$ and $p \rightarrow q$
  - Recall that $p \rightarrow q \equiv \neg q \rightarrow \neg p$ (contrapositive)

- Thus, we know $\neg q$ and $\neg q \rightarrow \neg p$
- We can conclude $\neg p$

$$\begin{array}{rl} & \neg\, q \\ & \underline{p \rightarrow q} \\ \therefore & \neg\, p \end{array}$$

Modus Tollens Example

- Assume you are given the following two statements:
  - "you will not get a grade"
  - "if you are in this class, you will get a grade"

- Let *p* = "you are in this class"
- Let *q* = "you will get a grade"

- By Modus Tollens, you can conclude that you are not in this class

$$\begin{array}{l} \neg q \\ \underline{p \rightarrow q} \\ \therefore \neg p \end{array}$$

Addition & Simplification

- Addition: If you know that p is true, then $p \vee q$ will ALWAYS be true

$$\frac{p}{\therefore\; p \vee q}$$

- Simplification: If $p \wedge q$ is true, then p will ALWAYS be true

$$\frac{p \wedge q}{\therefore\; q}$$

Example

- We have the hypotheses:
  - "It is not sunny this afternoon and it is colder than yesterday"
  - "We will go swimming only if it is sunny"
  - "If we do not go swimming, then we will take a canoe trip"
  - "If we take a canoe trip, then we will be home by sunset"
- Does this imply that "we will be home by sunset"?

- $((\neg p \wedge q) \wedge (r \rightarrow p) \wedge (\neg r \rightarrow s) \wedge (s \rightarrow t)) \rightarrow t$???
  - When
    - p = "It is sunny this afternoon"
    - q = "it is colder than yesterday"
    - r = "We will go swimming"
    - s = "we will take a canoe trip"
    - t = "we will be home by sunset"

Example

1.	$\neg p \wedge q$	1st hypothesis
2.	$\neg p$	Simplification using step 1
3.	$r \rightarrow p$	2nd hypothesis
4.	$\neg r$	Modus tollens using steps 2 & 3
5.	$\neg r \rightarrow s$	3rd hypothesis
6.	s	Modus ponens using steps 4 & 5
7.	$s \rightarrow t$	4th hypothesis
8.	t	Modus ponens using steps 6 & 7

- We showed that:
  - $[(\neg p \wedge q) \wedge (r \rightarrow p) \wedge (\neg r \rightarrow s) \wedge (s \rightarrow t)] \rightarrow t$
  - That when the 4th hypothesis is true, then the implication is true
  - In other words, we showed the above is a tautology!

More Rules of Inference

- Conjunction: if p and q are true separately, then $p \wedge q$ is true

$$\begin{array}{l} p \\ \underline{q \qquad} \\ \therefore p \wedge q \end{array}$$

- Disjunctive syllogism: If $p \vee q$ is true, and p is false, then q must be true

$$\begin{array}{l} p \vee q \\ \underline{\neg p \qquad} \\ \therefore q \end{array}$$

- Resolution: If $p \vee q$ is true, and $\neg p \vee r$ is true, then $q \vee r$ must be true

$$\begin{array}{l} p \vee q \\ \underline{\neg p \vee r} \\ \therefore q \vee r \end{array}$$

- Hypothetical syllogism: If $p \rightarrow q$ is true, and $q \rightarrow r$ is true, then $p \rightarrow r$ must be true

$$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array}$$

Summary: Rules of Inference

Modus ponens	p $p \rightarrow q$ $\therefore q$	Modus tollens	$\neg q$ $p \rightarrow q$ $\therefore \neg p$
Hypothetical syllogism	$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	Disjunctive syllogism	$p \vee q$ $\neg p$ $\therefore q$
Addition	p $\therefore p \vee q$	Simplification	$p \wedge q$ $\therefore p$
Conjunction	p q $\therefore p \wedge q$	Resolution	$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$

Proofing Example

- "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on"
  - $(\neg r \vee \neg f) \rightarrow (s \wedge d)$
- "If the sailing race is held, then the trophy will be awarded"
  - $s \rightarrow t$
- "The trophy was not awarded"
  - $\neg t$
- Can you conclude: "It rained"?
  - r

Proofing Example

1. $\neg t$ — 3rd hypothesis
2. $s \rightarrow t$ — 2nd hypothesis
3. $\neg s$ — Modus tollens using steps 2 & 3
4. $(\neg r \vee \neg f) \rightarrow (s \wedge d)$ — 1st hypothesis
5. $\neg(s \wedge d) \rightarrow \neg(\neg r \vee \neg f)$ — Contrapositive of step 4
6. $(\neg s \vee \neg d) \rightarrow (r \wedge f)$ — DeMorgan's law and double negation law
7. $\neg s \vee \neg d$ — Addition from step 3
8. $r \wedge f$ — Modus ponens using steps 6 & 7
9. r — Simplification using step 8

Fallacy of Affirming the Conclusion

- Consider the following:

$$\begin{array}{l} q \\ \underline{p \rightarrow q} \\ \therefore p \end{array} \qquad \begin{array}{l} q \\ \underline{\neg q \rightarrow \neg p} \\ \therefore p \end{array}$$

- Is this true?

p	q	$p \rightarrow q$	$q \wedge (p \rightarrow q))$	$(q \wedge (p \rightarrow q)) \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

- Not a valid rule!

Fallacy Example 1

- Assume you are given the following two statements:
  - "you will get a grade"
  - "if you are in this class, you will get a grade"
- Let p = "you are in this class"
- Let q = "you will get a grade"

$$\begin{array}{l} q \\ \underline{p \to q} \\ \therefore p \end{array}$$

- You CANNOT conclude that you are in this class
  - You could be getting a grade for another class

Fallacy of denying the hypothesis

- Consider the following:

$$\begin{array}{l} \neg p \\ \underline{p \rightarrow q} \\ \therefore \neg q \end{array}$$

- Is this true?

p	q	$p \rightarrow q$	$\neg p \wedge (p \rightarrow q))$	$(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$
T	T	T	F	T
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

- Not a valid rule!

Fallacy Example 2

- Assume you are given the following two statements:
  - "you are not in this class"
  - "if you are in this class, you will get a grade"

- Let p = "you are in this class"
- Let q = "you will get a grade"

$$\begin{array}{l} \neg p \\ p \rightarrow q \\ \hline \therefore \neg q \end{array}$$

- You CANNOT conclude that you will not get a grade
  - You could be getting a grade for another class

Rules of Inference for Universal Quantifier

- Assume that we know that $\forall x\, P(x)$ is true
  - Then we can conclude that $P(c)$ is true
    - Here c stands for some specific constant
  - This is called "universal instantiation"

- Assume that we know that $P(c)$ is true for any value of c
  - Then we can conclude that $\forall x\, P(x)$ is true
  - This is called "universal generalization"

Rules of Inference for Existential Quantifier

- Assume that we know that $\exists x\, P(x)$ is true
  - Then we can conclude that $P(c)$ is true for some value of c
  - This is called "existential instantiation"

- Assume that we know that $P(c)$ is true for some value of c
  - Then we can conclude that $\exists x\, P(x)$ is true
  - This is called "existential generalization"

Proofing Example 1

- Given the hypotheses:
  - "Linda, a student in this class, owns a red convertible." $C(\text{Linda})$ $R(\text{Linda})$
  - "Everybody who owns a red convertible has gotten at least one speeding ticket" $\forall x\ (R(x) \rightarrow T(x))$
- Can you conclude: "Somebody in this class has gotten a speeding ticket"? $\exists x\ (C(x) \wedge T(x))$

Proofing Example 1

1.	$\forall x\ (R(x) \rightarrow T(x))$	3^{rd} hypothesis
2.	$R(\text{Linda}) \rightarrow T(\text{Linda})$	Universal instantiation using step 1
3.	$R(\text{Linda})$	2^{nd} hypothesis
4.	$T(\text{Linda})$	Modes ponens using steps 2 & 3
5.	$C(\text{Linda})$	1^{st} hypothesis
6.	$C(\text{Linda}) \wedge T(\text{Linda})$	Conjunction using steps 4 & 5
7.	$\exists x\ (C(x) \wedge T(x))$	Existential generalization using step 6

- Thus, we have shown that "Somebody in this class has gotten a speeding ticket"

Proofing Example 2

- Given the hypotheses:
  - "There is someone in this class who has been to France" $\exists x\,(C(x) \wedge F(x))$
  - "Everyone who goes to France visits the Louvre" $\forall x\,(F(x) \rightarrow L(x))$
- Can you conclude: "Someone in this class has visited the Louvre"? $\exists x\,(C(x) \wedge L(x))$

Proofing Example 2

1.	$\exists x\,(C(x) \wedge F(x))$	1st hypothesis
2.	$C(y) \wedge F(y)$	Existential instantiation using step 1
3.	$F(y)$	Simplification using step 2
4.	$C(y)$	Simplification using step 2
5.	$\forall x\,(F(x) \rightarrow L(x))$	2nd hypothesis
6.	$F(y) \rightarrow L(y)$	Universal instantiation using step 5
7.	$L(y)$	Modus ponens using steps 3 & 6
8.	$C(y) \wedge L(y)$	Conjunction using steps 4 & 7
9.	$\exists x\,(C(x) \wedge L(x))$	Existential generalization using step 8

- Thus, we have shown that "Someone in this class has visited the Louvre"

Proofing Example 3

- Show that these premises: "A student in this class has not read the book" and "Everyone in this class passed the first exam" have the conclusion: "Someone who passed the first exam has not read the book"
- Let:
  - C(x): "x is in the class"
  - B(x) : "x has read the book"
  - P(x) : "x passed the first exam"
- Premises:
  - $\exists x\ (C(x) \land \neg B(x))$
  - $\forall x\ (C(x) \rightarrow P(x))$
- Conclusion: $\exists x\ (P(x) \land \neg B(x))$

Proofing Example 3

1	$\exists x\,(C(x) \land \neg B(x))$	Premise 1
2	$C(a) \land \neg B(a)$	Existential instantiation from (1)
3	$C(a)$	Simplification from (2)
4	$\forall x\,(C(x) \rightarrow P(x))$	Premise 2
5	$C(a) \rightarrow P(a)$	Universal instantiation from (4)
6	$P(a)$	Modus ponens from (3) and (5)
7	$\neg B(a)$	Simplification from (2)
8	$P(a) \land \neg B(a)$	Conjunction from (6) and (7)
9	$\exists x\,(P(x) \land \neg B(x))$	Existential generalization from (8)

Proofing Example 4

Explain which rules of inference are used for each step

- "David, a student in this class, knows how to write programs in JAVA. Everyone who knows how to write programs in JAVA can get high-paying job. Therefore, someone in this class can get a high-paying job."
- Let:
  - C(x): "x is in the class"
  - J(x): "x knows how to write programs in JAVA"
  - H(x): "x can get high-paying job"
- Premises:
  - $C(David); J(David); \forall x\,(J(x) \rightarrow H(x))$
- Conclusion: $\exists x\,(C(x) \wedge H(x))$

Proofing Example 4

1	$\forall x\ (J(x) \rightarrow H(x))$	Premise 3
2	$\forall x\ (J(David) \rightarrow H(David))$	Universal instantiation from (1)
3	$J(David)$	Premise 2
4	$H(David)$	Modus ponens from (2) and (3)
5	$C(David)$	Premise 2
6	$C(David) \wedge H(David)$	Conjunction from (4) and (5)
7	$\exists x\ (C(x) \wedge H(x))$	Simplification from (2)

Proofing Example 5

- "Somebody in this class enjoy whale watching. Every person who enjoys whale watching cares about ocean pollution. Therefore, there is a person in this class who cares about ocean pollution."
- Let:
  - $C(x)$: "x is in the class"
  - $W(x)$: "x enjoys whale watching"
  - $P(x)$: "x cares about ocean pollution"
- Premises:
  - $\exists x\ (C(x) \wedge W(x))$
  - $\forall x\ (W(x) \rightarrow P(x))$
- Conclusion: $\exists x\ (C(x) \wedge P(x))$

Proofing Example 5

1	$\exists x\,(C(x) \land W(x))$	Premise 1
2	$(C(a) \land W(a))$	Existensial instantiation from (1)
3	$W(a)$	Simplification from (2)
4	$\forall x\,(W(x) \rightarrow P(x))$	Premise 2
5	$W(a) \rightarrow P(a)$	Universal instantiation from (4)
6	$P(a)$	Modus Ponens from (3) and (5)
7	$C(a)$	Simplification from (2)
8	$(C(a) \land P(a))$	Conjunction fro (6) and (7)
9	$\exists x\,(C(x) \land P(x))$	Existensial generalization from (8)

How do you know which one to use?

- **Experience**!
- In general, use quantifiers with statements like "for all" or "there exists"