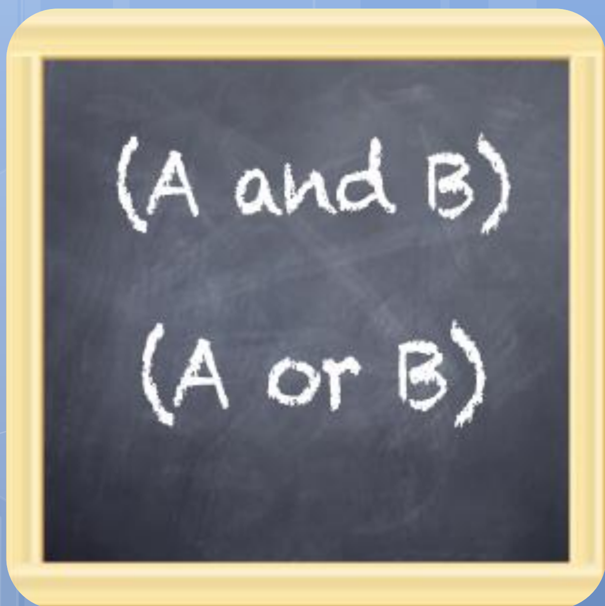
KS091201
MATEMATIKA DISKRIT
(DISCRETE MATHEMATICS )

# LOGIC & PROPOSITIONAL EQUIVALENCE




Discrete Math Team

# Outline

- Logic
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- Propositional Variables
- Logical Operators
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- Translating English Sentences
- Tautology, Contradiction, Equivalance
- Logical Equivalance
- Using Logical Equivalece for Proofing

# Logic

- Logic is the foundation of Mathematical reasoning (*penjabaran Matematika*).
- The rules of logic are used to distinguish between valid and invalid Mathematical arguments.
- Logic has numerous application in Computer Sciences such as in the design of computer circuits, the construction of computer programs, the verification of correctness of programs, etc.

# Proposition

- Proposition is a statement that can be either true or false, but not both.
  - “Washington DC is the capital of the USA.”
  - “Kuala Lumpur is a city in Indonesia.”
  - “3 = 2 + 1”
  - “3 = 2 + 2”

- Not propositions:
  - “is Cameron Diaz a Prime Minister?”
  - “x = 7”
  - “I am good student”

# Proposition (cont.)

- The following are not proposition
  - Instruction (*Kalimat perintah*)
  - Question (*Kalimat pertanyaan*)
  - Amazement (*Kalimat keheranan*)
  - Expectancy (*Kalimat harapan*)

# Propositional Variables

- We use propositional variables to refer to propositions
  - Usually are lower case letters starting with *p* (i.e. *p*, *q*, *r*, *s*, etc.)
  - A propositional variable can have one of two values: true (*T*) or false (*F*)

- A proposition can be…
  - A single variable: *p*
  - An operation of multiple variables: $p \wedge (q \vee \neg r)$

# Logical Operators

- Many mathematical statements are constructed by combining one or more propositions.
- New propositions, called compound propositions *(pernyataan gabungan/ pernyataan majemuk)*, are formed from existing propositions using logical operators.
- Consider the following examples:
  - $p$ = "Today is Sep, 11"
  - $q$ = "Today is my birthday"

# Logical Operator: Not

- A “not” operation switches (negates) the truth value
- Symbol: $\neg$ or ~
- $\neg p$ = “Today is not Sep, 11”

| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

# Logical Operator: And

- An "and" operation is true if both operands are true
- Symbol: $\wedge$
- $p \wedge q$ = "Today is Sep, 11 and today is my birthday"

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

# Logical Operator: Or

- An "or" operation is true if either operands or both are true
- Symbol: $\lor$
- $p \lor q$ = "Today is Sep, 11 or today is my birthday (or possibly both)"

| $p$ | $q$ | $p \lor q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

# Logical Operator: XOR

- An "xor" operation is true when exactly one of $p$ and $q$ is true, and false otherwise.
- Symbol: $p \oplus q$
- $p \oplus q$ = "Either today is Sep, 11 or today is my birthday (not both)"

| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

# Logical Operator: XOR (cont)

- When two simple propositions are combined using ‘or’, context will often provide the clue as to whether the inclusive or exclusive sense is intended.
- For instance, ‘*Tomorrow I will go swimming or play golf*’ seems to suggest that I will not do both in the same time and therefore points to an exclusive interpretation.

# Logical Operator: Conditional

- A conditional means "if $p$ then $q$"
- Symbol: $\rightarrow$
- $p \rightarrow q$ = "If today is Sep, 11, then today is my birthday"
- $p \rightarrow q = \neg p \vee q$

Antecedent/ hypothesis/ premise

Consequence/ conclusion

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

# Logical Operator: Conditional (cont)

- Let $p$ = "I pass my exams " and $q$ = "I will get drunk"
- I state: $p \rightarrow q$ = "If I pass my exams then I will get drunk"
- Note that if $p$ is false, then the conditional is true regardless of whether $q$ is true or false
- The above statement says nothing about what I will do if I *don't* pass my exams. I may get drunk or I may not, but in either case you could not accuse me of having made a false statement.
- The only circumstances in which I could be accused of uttering a falsehood is if I pass my exams and don't get drunk.

# Logical Operator: Conditional (cont)

- Alternate ways of stating a conditional:
  - $p$ implies $q$
  - If $p$, $q$
  - $p$ only if $q$
  - $p$ is sufficient for $q$
  - $q$ if $p$
  - $q$ whenever $p$
  - $q$ is necessary for $p$

# Logical Operator: Conditional (cont)

- Conditional statement:
  - If you use Ms. Word, then Windows is the operating system.
- Alternatively:
  - Ms. Word is sufficient for Windows
  - Windows is necessary for Ms. Word
- Ms. Word adalah syarat cukup bagi Windows, sedangkan Windows adalah syarat perlu bagi Ms. Word
- Ms. Word tidak dapat digunakan tanpa Windows tetapi Windows dapat digunakan tanpa Ms. Word

# Logical Operator: Conditional (cont)

| Propositions | | | | Conditional | Inverse | Converse | Contrapositive |
|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \to q$ | $\neg p \to \neg q$ | $q \to p$ | $\neg q \to \neg p$ |
| T | T | F | F | T | T | T | T |
| T | F | F | T | F | T | T | F |
| F | T | T | F | T | F | F | T |
| F | F | T | T | T | T | T | T |

- Two contrapositive propositions have the same truth values (equivalent)
- Inverse and converse are the opposite of the conditional, both of them have the same truth values
- These rules are useful in proofing

# Logical Operator: Bi-Conditional

- A bi-conditional means "*p* if and only if *q*"
- Symbol: $\leftrightarrow$
- Alternatively, it means "(if *p* then *q*) and (if *q* then *p*)"
- Note that a bi-conditional has the opposite truth values of the exclusive or

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

# Logical Operator: Bi-Conditional

- Let $p$ = "You can take the flight" and $q$ = "You buy a ticket"
- Then $p \leftrightarrow q$ means
  "You can take the flight if and only if you buy a ticket"
- Alternatively, it means "If you can take the flight, then you buy a ticket and if you buy a ticket then you take the flight"

# Boolean Operator Summary

| | | not | not | and | or | xor | conditional | Bi-conditional |
|---|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \wedge q$ | $p \vee q$ | $p \oplus q$ | $p \rightarrow q$ | $p \leftrightarrow q$ |
| T | T | F | F | T | T | F | T | T |
| T | F | F | T | F | T | T | F | F |
| F | T | T | F | F | T | T | T | F |
| F | F | T | T | F | F | F | T | T |

- Try to understand what they mean, don't just memorize the table ☺

# Presedence of Operators

- Just as in algebra, operators have precedence
  - 4+3*2 = 4+(3*2), not (4+3)*2
- Precedence order (from highest to lowest):

  $\neg \wedge \vee \rightarrow \leftrightarrow$
- This means that $p \vee q \wedge \neg r \rightarrow s \leftrightarrow t$ yields: $(p \vee (q \wedge (\neg r)) \rightarrow s) \leftrightarrow (t)$
- Not is *always* performed before any other operation

# Translating English Sentences

- Question 7 from Rosen, p. 17
  - $p$ = "It is below freezing"
  - $q$ = "It is snowing"

- It is below freezing and it is snowing
- It is below freezing but not snowing
- It is not below freezing and it is not snowing
- It is either snowing or below freezing (or both)
- If it is below freezing, it is also snowing
- It is either below freezing or it is snowing, but it is not snowing if it is below freezing
- That it is below freezing is necessary and sufficient for it to be snowing

- $p \wedge q$
- $p \wedge \neg q$
- $\neg p \wedge \neg q$
- $p \vee q$
- $p \rightarrow q$
- $((p \oplus q) \wedge (p \rightarrow \neg q)$
- $p \leftrightarrow q$

# Translating English Sentences

- "I have neither given nor received help on this exam"
- Let p = "I have given help on this exam"
- Let q = "I have received help on this exam"
- $\neg p \wedge \neg q$

# Translating English Sentences

- You can access the Internet from campus only if you are a computer science major or you are not a freshman.
- $a \rightarrow (c \vee \neg f)$

- You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old.
- $(f \wedge \neg s) \rightarrow \neg r$
- $r \rightarrow (\neg f \vee s)$

# Practice ☺

Some Students from Discrete Math (KS091201) went together to see cinema at Galaxy Mall last Sunday. From the clues below, can you determine the order in which they stood in the ticket queue?

- Yahya was in front of Andhino.
- Anggoro was behind Adityo and Dyah.
- Dyah was in front of Adi and Rani.
- Hendra was behind Anggoro, Dian and Rani.
- Rani was in front of Hendra, Dian and Harlen.
- Harlen was behind Dian, Adi and Hendra.
- Adityo was in front of Andhino.
- Dian was in front of Yahya, Anggoro and Adityo.
- Andhino was in front of Adi.
- Hendra was in front of Yahya and Harlen.
- Andhino was behind Adityo, Dyah and Hendra.
- Anggoro was in front of Andhino.
- Dian was behind Dyah.

# Answer

- Dyah-Rani-Dian-Adityo-Anggoro-Hendra-Yahya-Andhino-Adi-Harlen

# Practice ☺

- If $p$, $q$, $r$ are $T$, $F$, $F$ respectively. What is the value ($T$ or $F$) of:

  $(p \leftrightarrow \neg r \vee q \wedge p \rightarrow r) \oplus (p \wedge r)$
- ***FALSE***

# Tautology, Contradiction, Equivalence

- **Tautology**: a statement (compound props.) that's always true no matter what the truth values of the propositions
  - $p \vee \neg p$ will always be true
- **Contradiction**: a statement (compound props.) that's always false
  - $p \wedge \neg p$ will always be false
- **Contingency**: a statement (compound props.) that's neither a tautology nor contradiction.
- A **logical equivalence** means that the two sides always have the same truth values. Or *in other word* $p \rightarrow q$ is tautology
  - Symbol is $\equiv$ or $\Leftrightarrow$ (we'll use $\equiv$)

# Logical Equivalence

- Identity law $p \wedge T \equiv p$

| $p$ | T | $p \wedge$T |
|---|---|---|
| T | T | T |
| F | T | F |

- Commutative law $p \wedge q \equiv q \wedge p$

| p | q | p∧q | q∧p |
|---|---|---|---|
| T | T | T | T |
| T | F | F | F |
| F | T | F | F |
| F | F | F | F |

# Logical Equivalence (cont.)

- Associative law $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

| p | q | r | $p \wedge q$ | $(p \wedge q) \wedge r$ | $q \wedge r$ | $p \wedge (q \wedge r)$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | T | F | T | F | F | F |
| T | F | T | F | F | F | F |
| T | F | F | F | F | F | F |
| F | T | T | F | F | T | F |
| F | T | F | F | F | F | F |
| F | F | T | F | F | F | F |
| F | F | F | F | F | F | F |

# Summary of Logical Equivalence

| | | | |
|---|---|---|---|
| $p \wedge T \equiv p$<br>$p \vee F \equiv p$ | Identity Laws | $(p \vee q) \vee r \equiv p \vee (q \vee r)$<br>$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | Associative laws |
| $p \vee T \equiv T$<br>$p \wedge F \equiv F$ | Domination Law | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$<br>$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | Distributive laws |
| $p \vee p \equiv p$<br>$p \wedge p \equiv p$ | Idempotent Laws | $\neg (p \wedge q) \equiv \neg p \vee \neg q$<br>$\neg (p \vee q) \equiv \neg p \wedge \neg q$ | De Morgan's laws |
| $\neg(\neg p) \equiv p$ | Double negation law | $p \vee (p \wedge q) \equiv p$<br>$p \wedge (p \vee q) \equiv p$ | Absorption laws |
| $p \vee q \equiv q \vee p$<br>$p \wedge q \equiv q \wedge p$ | Commutative Laws | $p \vee \neg p \equiv T$<br>$p \wedge \neg p \equiv F$ | Negation laws |
| $p \rightarrow q \equiv \neg p \vee q$ | Definition of Implication | $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ | Definition of Biconditional |

# The Liar Paradox

- This statement is false. (A)
- If (A) is true, then "This statement is false" is true. Therefore (A) must be false.
- The hypothesis that (A) is true leads to the conclusion that (A) is false, a contradiction.
- If (A) is false, then "This statement is false" is false. Therefore (A) must be true.
- The hypothesis that (A) is false leads to the conclusion that (A) is true, another contradiction.
- Either way, (A) is both true and false, which is a paradox.
- http://en.wikipedia.org/wiki/Liar_paradox

# Proof using Logical Equivalence

| | |
|---|---|
| $(p \rightarrow r) \vee (q \rightarrow r)$ | |
| $\equiv (\neg p \vee r) \vee (\neg q \vee r)$ | Definition of implication |
| $\equiv \neg p \vee r \vee \neg q \vee r$ | Associative |
| $\equiv \neg p \vee \neg q \vee r \vee r$ | Commutative |
| $\equiv (\neg p \vee \neg q) \vee (r \vee r)$ | Associative |
| $\equiv \neg (p \wedge q) \vee r$ | De Morgan, Idempotent |
| $\equiv (p \wedge q) \rightarrow r$ | Definition of implication |

# Proof using Logical Equivalence

- Show that $(p \wedge q) \rightarrow (p \vee q)$ is a Tautology.

(Proof)

| | |
|---|---|
| $(p \wedge q) \rightarrow (p \vee q)$ | |
| $\equiv \neg (p \wedge q) \vee (p \vee q)$ | Def. of Implication |
| $\equiv (\neg p \vee \neg q) \vee (p \vee q)$ | De Morgan |
| $\equiv (\neg p \vee p) \vee (\neg q \vee q)$ | Commutative, Associative |
| $\equiv T \vee T$ | Negation |
| $\equiv T$ | Identity |