FUNGSI GAMMA

IMPROPER INTEGRAL KHUSUS

$$\boxed{\text{Prove that } \int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}\,/\,2.}$$

Let $I_M = \int_0^M e^{-x^2}\,dx = \int_0^M e^{-y^2}\,dy$ and let $\lim_{M\to\infty} I_M = I$. Then

$$\begin{aligned}
I_M^2 &= \left(\int_0^M e^{-x^2}\,dx\right)\left(\int_0^M e^{-x^2}\,dy\right) \\
&= \int_0^M \int_0^M e^{-(x^2+y^2)}\,dx\,dy \\
&= \iint_M e^{-(x^2+y^2)}\,dx\,dy
\end{aligned}$$

Since the integrand is positive, we have

$$\iint_M e^{-(x^2+y^2)}\,dx\,dy \leqq I_M^2 \leqq \iint_M e^{-(x^2+y^2)}\,dx\,dy$$

IMPROPER INTEGRAL KHUSUS

Prove that $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}\,/2.$

Using polar coordinates, we have

$$\int_{\phi=0}^{\pi/2}\int_{p=0}^{M} e^{-\rho^2}\rho\,d\rho\,d\phi \leqq I_M^2 \leqq \int_{\phi=0}^{\pi/2}\int_{\pi=0}^{M\sqrt{2}} e^{-\rho^2}\rho\,d\rho\,d\phi$$

or

$$\frac{\pi}{4}(1-e^{M^2}) \leqq I_M^2 \leqq \frac{\pi}{4}(1-e^{-2M^2})$$

Then, taking the limit as $M \to \infty$, we find

$$\lim_{M\to\infty} I^2{}_M = I^2 = \pi/4 \quad \text{and} \quad \boxed{I = \sqrt{\pi}\,/2}$$

DEFINISI

Fungsi Gamma dinotasikan dengan $\Gamma(n)$ dan didefinisikan dengan

$$\boxed{\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \ \ldots} \qquad (4.1)$$

Teorema IV.1. Misalkan $\lim x^p f(x) = c$, dengan c suatu konstanta, maka

$\int_a^\infty f(x)dx$ konvergen untuk $p > 1$, dan c suatu konstanta berhingga.

Teorema IV.2. Misalkan $\lim_{x \to 0} x^p f(x) = c$, dengan c suatu konstanta, maka

$\int_a^b f(x)dx$ konvergen untuk $p < 1$, dan c suatu konstanta berhingga.

SIFAT-SIFAT

Dari (4.1)

$$\boxed{\Gamma(n+1)} = \int_0^{\infty} x^n e^{-x} dx$$

$$= -\left[x^n e^{-x} \Big|_0^{\infty} - n \int_0^{\infty} x^{n-1} e^{-x} dx \right]$$

$$= n \int_0^{\infty} x^{n-1} e^{-x} dx \qquad \boxed{= n\,\Gamma(n)}$$

Rumus lain yang melibatkan fungsi gamma

$$\boxed{\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}}, \quad 0 < x < 1 \ldots$$

FS GAMMA : CONTOH

Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-1/2} e^{-x}\, dx.$$

Letting $x = u^2$ this integral becomes

$$2\int_0^\infty e^{-u^2}\, du = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}$$

FS GAMMA : CONTOH

Evaluate each integral. $\boxed{\int_0^\infty \sqrt{y}\, e^{-y^2}\, dy.}$

Letting $y^3 = x$, the intergral becomes

$$\int_0^\infty \sqrt{x^{1/3}}\; e^{-x} . \frac{1}{3} x^{-2/3} dx = \frac{1}{3}\int_0^\infty x^{-1/2} e^{-x} dx$$

$$= \frac{1}{3}\Gamma\left(\frac{1}{2}\right)$$

$$\boxed{= \frac{\sqrt{\pi}}{3}}$$

FS GAMMA : CONTOH

Akan ditentukan hasil dari $\Gamma(-\frac{5}{2})$

Dari rumus rekursif (4.3), dapat ditulis $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$, sehingga

$$\Gamma(-\tfrac{5}{2}) = \frac{\Gamma(-\frac{5}{2}+1)}{-\frac{5}{2}} = \frac{\Gamma(-\frac{3}{2})}{-\frac{5}{2}}$$

$$= \frac{\Gamma(-\frac{1}{2})}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)} = \frac{\Gamma(\frac{1}{2})}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}$$

$$\boxed{= -\frac{8}{15}\sqrt{\pi}}$$

FS GAMMA : LATIHAN

1. Tentukan $\int_0^\infty e^{-x^2}\,dx$ dan $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\,dx$!

2. Tentukan hasil dari $\int_0^1 (\ln x)^4\,dx$!

3. Tunjukkan $\int_0^\infty \frac{e^{-st}}{\sqrt{t}}\,dt = \sqrt{\frac{\pi}{s}},\ s > 0$!

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Dengan fungsi Gamma, hitung :

1 $\int_0^\infty x^6 e^{-3x}\,dx,$

2 $\int_0^\infty x^2 e^{-2x^2}\,dx.$

3 $\int_0^\infty y^3 e^{-2y^5}\,dy.$

4 $\int_0^1 (x \ln x)^3\,dx,$

5 $\int_0^1 \sqrt[3]{\ln(1/x)}\,dx.$