Direct Methods for Solving Linear Systems

Numerical Analysis
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Kirchhoff’s laws

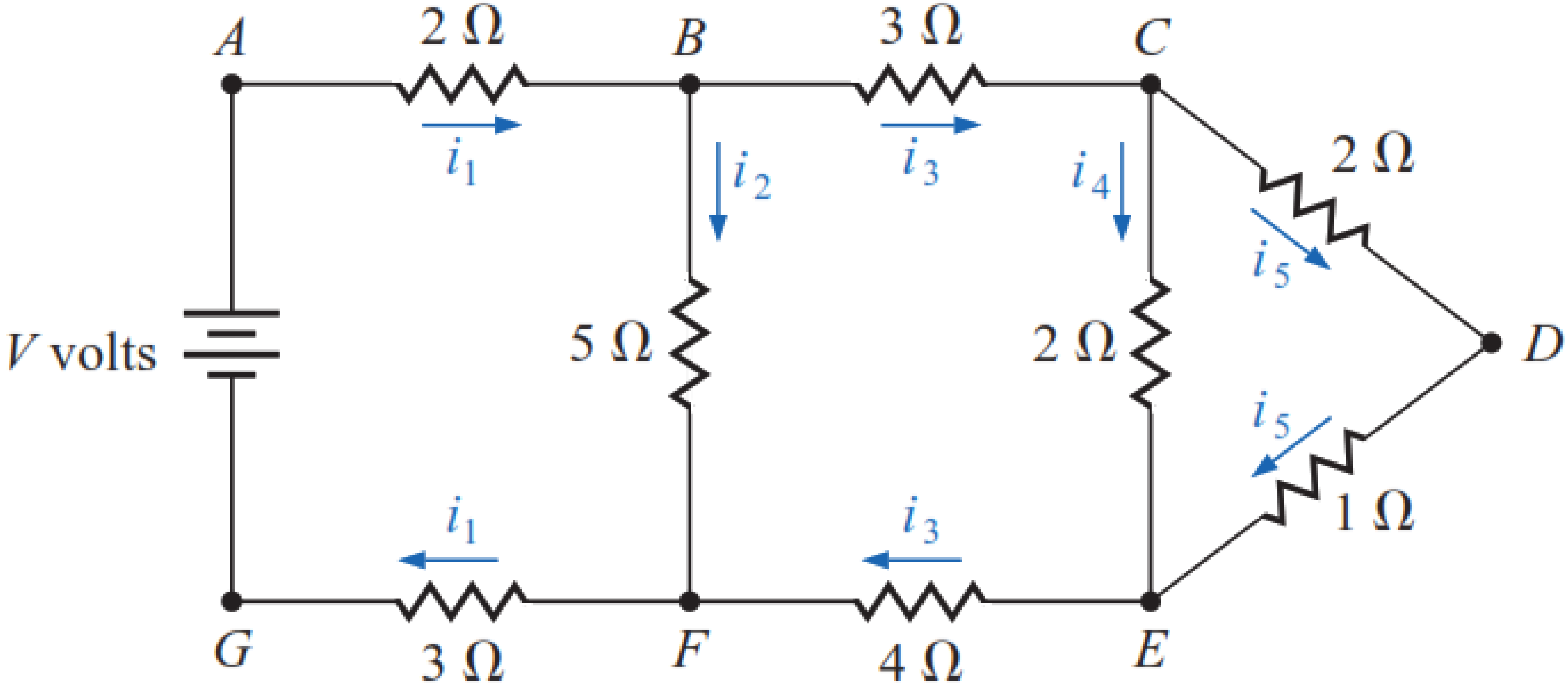
$$5i_1 + 5i_2 = V,$$

$$i_3 - i_4 - i_5 = 0,$$

$$2i_4 - 3i_5 = 0,$$

$$i_1 - i_2 - i_3 = 0,$$

$$5i_2 - 7i_3 - 2i_4 = 0.$$



Direct methods

$$
\begin{aligned}
E_1 &: \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\
E_2 &: \quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\
&\qquad\qquad\qquad\qquad \vdots \\
E_n &: \quad a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n.
\end{aligned}
\tag{6.1}
$$

Linear Systems of Equations

We use three operations to simplify the linear system given in (6.1):

1. Equation E_i can be multiplied by any nonzero constant λ with the resulting equation used in place of E_i. This operation is denoted $(\lambda E_i) \rightarrow (E_i)$.

2. Equation E_j can be multiplied by any constant λ and added to equation E_i with the resulting equation used in place of E_i. This operation is denoted $(E_i + \lambda E_j) \rightarrow (E_i)$.

3. Equations E_i and E_j can be transposed in order. This operation is denoted $(E_i) \leftrightarrow (E_j)$.

Linear Systems of Equations

The four equations

$$
\begin{aligned}
E_1: &\quad x_1 + x_2 \phantom{{}- x_3} + 3x_4 = 4, \\
E_2: &\quad 2x_1 + x_2 - x_3 + x_4 = 1, \\
E_3: &\quad 3x_1 - x_2 - x_3 + 2x_4 = -3, \\
E_4: &\quad -x_1 + 2x_2 + 3x_3 - x_4 = 4,
\end{aligned}
\tag{6.2}
$$

will be solved for x_1, x_2, x_3, and x_4. We first use equation E_1 to eliminate the unknown x_1 from equations E_2, E_3, and E_4 by performing $(E_2 - 2E_1) \rightarrow (E_2)$, $(E_3 - 3E_1) \rightarrow (E_3)$, and $(E_4 + E_1) \rightarrow (E_4)$. For example, in the second equation

$$(E_2 - 2E_1) \rightarrow (E_2)$$

Linear Systems of Equations

produces

$$(2x_1 + x_2 - x_3 + x_4) - 2(x_1 + x_2 + 3x_4) = 1 - 8(4).$$

which simplifies to the result shown as E_2 in

$$\begin{aligned}
E_1 &: \quad x_1 + x_2 \phantom{{}- x_3} + 3x_4 = 4, \\
E_2 &: \quad\quad - x_2 - x_3 - 5x_4 = -7, \\
E_3 &: \quad\quad - 4x_2 - x_3 - 7x_4 = -15, \\
E_4 &: \quad\quad\quad 3x_2 + 3x_3 + 2x_4 = 8.
\end{aligned}$$

Linear Systems of Equations

In the new system, E_2 is used to eliminate the unknown x_2 from E_3 and E_4 by performing $(E_3 - 4E_2) \rightarrow (E_3)$ and $(E_4 + 3E_2) \rightarrow (E_4)$. This results in

$$\begin{aligned}
E_1 :&\quad x_1 + x_2 \phantom{{}- x_3} + 3x_4 = 4, \\
E_2 :&\quad -x_2 - x_3 - 5x_4 = -7, \\
E_3 :&\quad 3x_3 + 13x_4 = 13, \\
E_4 :&\quad -13x_4 = -13.
\end{aligned} \tag{6.3}$$

The system of equations (6.3) is now in triangular (or reduced) form and can be solved for the unknowns by a backward-substitution process

Linear Systems of Equations

backward-substitution process

Since E_4 implies $x_4 = 1$, we can solve E_3 for x_3 to give

$$x_3 = \frac{1}{3}(13 - 13x_4) = \frac{1}{3}(13 - 13) = 0.$$

Continuing, E_2 gives

$$x_2 = -(-7 + 5x_4 + x_3) = -(-7 + 5 + 0) = 2,$$

and E_1 gives

$$x_1 = 4 - 3x_4 - x_2 = 4 - 3 - 2 = -1.$$

The solution to system (6.3), and consequently to system (6.2), is therefore, $x_1 = -1$, $x_2 = 2$, $x_3 = 0$, and $x_4 = 1$.

A linear system is often replaced by a matrix

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

A linear system is often replaced by a matrix

Determine the size and respective entries of the matrix

$$A = \begin{bmatrix} 2 & -1 & 7 \\ 3 & 1 & 0 \end{bmatrix}.$$

The matrix has two rows and three columns so it is of size 2×3. It entries are described by $a_{11} = 2$, $a_{12} = -1$, $a_{13} = 7$, $a_{21} = 3$, $a_{22} = 1$, and $a_{23} = 0$.

A linear system is often replaced by a matrix

n-dimensional row vector

The $1 \times n$ matrix

$$A = [a_{11} \;\; a_{12} \;\; \cdots \;\; a_{1n}]$$

n-dimensional column vector

$n \times 1$ matrix

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$$

A linear system is often replaced by a matrix

An $n \times (n+1)$ matrix can be used to represent the linear system

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
\vdots \qquad\qquad\qquad & \quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n,
\end{aligned}$$

by first constructing

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

A linear system is often replaced by a matrix

Augmented Matrix

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$[A, \mathbf{b}] = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right],$$

A linear system is often replaced by a matrix

Augmented Matrix

$$\begin{aligned}
E_1: &\quad x_1 + x_2 \phantom{{}- x_3} + 3x_4 = 4,\\
E_2: &\quad 2x_1 + x_2 - x_3 + x_4 = 1,\\
E_3: &\quad 3x_1 - x_2 - x_3 + 2x_4 = -3,\\
E_4: &\quad -x_1 + 2x_2 + 3x_3 - x_4 = 4,
\end{aligned}$$

considering the augmented matrix:

$$\left[\begin{array}{rrrr:r}
1 & 1 & 0 & 3 & 4\\
2 & 1 & -1 & 1 & 1\\
3 & -1 & -1 & 2 & -3\\
-1 & 2 & 3 & -1 & 4
\end{array}\right]$$

A linear system is often replaced by a matrix

Augmented Matrix

considering the augmented matrix:

$$\left[\begin{array}{rrrr:r} 1 & 1 & 0 & 3 & 4 \\ 2 & 1 & -1 & 1 & 1 \\ 3 & -1 & -1 & 2 & -3 \\ -1 & 2 & 3 & -1 & 4 \end{array}\right]$$

Performing the operations as described in that example produces the augmented matrices

$$\left[\begin{array}{rrrr:r} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & -4 & -1 & -7 & -15 \\ 0 & 3 & 3 & 2 & 8 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{rrrr:r} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & 0 & 3 & 13 & 13 \\ 0 & 0 & 0 & -13 & -13 \end{array}\right].$$

Gaussian elimination with backward substitution

The general Gaussian elimination procedure applied to the linear system

$$
\begin{aligned}
E_1: &\quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\
E_2: &\quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\
&\qquad\qquad \vdots \qquad\qquad\qquad\quad \vdots \\
E_n: &\quad a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n,
\end{aligned}
\tag{6.4}
$$

is handled in a similar manner. First form the augmented matrix $\tilde{A}$:

$$
\tilde{A} = [A, \mathbf{b}] = \left[\begin{array}{cccc:c}
a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\
a_{21} & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\
\vdots & \vdots & & \vdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} & a_{n,n+1}
\end{array}\right],
\tag{6.5}
$$

Gaussian elimination with backward substitution

Provided $a_{11} \neq 0$, we perform the operations corresponding to

$$(E_j - (a_{j1}/a_{11})E_1) \rightarrow (E_j) \quad \text{for each } j = 2, 3, \ldots, n$$

provided $a_{ii} \neq 0$. This eliminates (changes the coefficient to zero) x_i in each row below the ith for all values of $i = 1, 2, \ldots, n-1$. The resulting matrix has the form:

$$\tilde{\tilde{A}} = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ 0 & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{nn} & a_{n,n+1} \end{array}\right],$$

Gaussian elimination with backward substitution

The new linear system is triangular,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= a_{1,n+1}, \\
a_{22}x_2 + \cdots + a_{2n}x_n &= a_{2,n+1}, \\
\ddots \qquad \vdots \qquad & \qquad \vdots \\
a_{nn}x_n &= a_{n,n+1},
\end{aligned}$$

so *backward substitution* can be performed. Solving the nth equation for x_n gives

$$x_n = \frac{a_{n,n+1}}{a_{nn}}.$$

Gaussian elimination with backward substitution

Solving the $(n-1)$st equation for x_{n-1} and using the known value for x_n yields

$$x_{n-1} = \frac{a_{n-1,n+1} - a_{n-1,n}x_n}{a_{n-1,n-1}}.$$

Continuing this process, we obtain

$$x_i = \frac{a_{i,n+1} - a_{i,n}x_n - a_{i,n-1}x_{n-1} - \cdots - a_{i,i+1}x_{i+1}}{a_{ii}} = \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}},$$

for each $i = n-1, n-2, \cdots, 2, 1$.

Gaussian elimination with backward substitution

Gaussian elimination procedure is described more precisely, although more intricately, by forming a sequence of augmented matrices $\tilde{A}^{(1)}, \tilde{A}^{(2)}, \ldots, \tilde{A}^{(n)}$, where $\tilde{A}^{(1)}$ is the matrix $\tilde{A}$ given in (6.5) and $\tilde{A}^{(k)}$, for each $k = 2, 3, \ldots, n$, has entries $a_{ij}^{(k)}$, where:

$$a_{ij}^{(k)} = \begin{cases} a_{ij}^{(k-1)}, & \text{when } i = 1, 2, \ldots, k-1 \text{ and } j = 1, 2, \ldots, n+1, \\ 0, & \text{when } i = k, k+1, \ldots, n \text{ and } j = 1, 2, \cdots, k-1, \\ a_{ij}^{(k-1)} - \dfrac{a_{i,k-1}^{(k-1)}}{a_{k-1,k-1}^{(k-1)}} a_{k-1,j}^{(k-1)}, & \text{when } i = k, k+1, \ldots, n \text{ and } j = k, k+1, \ldots, n+1. \end{cases}$$

Gaussian elimination with backward substitution

Thus

$$\tilde{A}^{(k)} = \left[\begin{array}{cccccccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1,k-1}^{(1)} & a_{1k}^{(1)} & \cdots & a_{1n}^{(1)} & a_{1,n+1}^{(1)} \\ 0 & a_{22}{}^{(2)} & a_{23}^{(2)} & \cdots & a_{2,k-1}^{(2)} & a_{2k}^{(2)} & \cdots & a_{2n}^{(2)} & a_{2,n+1}^{(2)} \\ \vdots & \ddots & \ddots & & \vdots & \vdots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & a_{k-1,k-1}^{(k-1)} & a_{k-1,k}^{(k-1)} & \cdots & a_{k-1,n}^{(k-1)} & a_{k-1,n+1}^{(k-1)} \\ \vdots & & & \ddots & 0 & a_{kk}^{(k)} & \cdots & a_{kn}^{(k)} & a_{k,n+1}^{(k)} \\ \vdots & & & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & a_{nk}^{(k)} & \cdots & a_{nn}^{(k)} & a_{n,n+1}^{(k)} \end{array}\right] \tag{6.6}$$

represents the equivalent linear system for which the variable x_{k-1} has just been eliminated from equations $E_k, E_{k+1}, \ldots, E_n$.

Gaussian elimination with backward substitution

The procedure will fail if one of the elements $a_{11}^{(1)}, a_{22}^{(2)}, a_{33}^{(3)}, \ldots, a_{n-1,n-1}^{(n-1)}, a_{nn}^{(n)}$ is zero because the step

$$\left(E_i - \frac{a_{i,k}^{(k)}}{a_{kk}^{(k)}}(E_k)\right) \to E_i$$

either cannot be performed (this occurs if one of $a_{11}^{(1)}, \ldots, a_{n-1,n-1}^{(n-1)}$ is zero), or the backward substitution cannot be accomplished (in the case $a_{nn}^{(n)} = 0$). The system may still have a solution, but the technique for finding the solution must be altered. An illustration is given in the following example.

Gaussian elimination with backward substitution

Represent the linear system

$$
\begin{aligned}
E_1: &\quad x_1 - x_2 + 2x_3 - x_4 = -8,\\
E_2: &\quad 2x_1 - 2x_2 + 3x_3 - 3x_4 = -20,\\
E_3: &\quad x_1 + x_2 + x_3 = -2,\\
E_4: &\quad x_1 - x_2 + 4x_3 + 3x_4 = 4,
\end{aligned}
$$

as an augmented matrix and use Gaussian Elimination to find its solution.

The augmented matrix is

$$
\tilde{A} = \tilde{A}^{(1)} = \left[\begin{array}{rrrr:r} 1 & -1 & 2 & -1 & -8 \\ 2 & -2 & 3 & -3 & -20 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & -1 & 4 & 3 & 4 \end{array}\right].
$$

Gaussian elimination with backward substitution

$$\tilde{A} = \tilde{A}^{(1)} = \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -8 \\ 2 & -2 & 3 & -3 & -20 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & -1 & 4 & 3 & 4 \end{array}\right].$$

Performing the operations

$$(E_2 - 2E_1) \to (E_2),\ (E_3 - E_1) \to (E_3), \quad \text{and} \quad (E_4 - E_1) \to (E_4),$$

gives

$$\tilde{A}^{(2)} = \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -8 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right].$$

Gaussian elimination with backward substitution

$$\tilde{A}^{(2)} = \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -8 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right].$$

The diagonal entry $a_{22}^{(2)}$, called the **pivot element**, is 0, so the procedure cannot continue in its present form. But operations $(E_i) \leftrightarrow (E_j)$ are permitted, so a search is made of the elements $a_{32}^{(2)}$ and $a_{42}^{(2)}$ for the first nonzero element. Since $a_{32}^{(2)} \neq 0$, the operation $(E_2) \leftrightarrow (E_3)$ is performed to obtain a new matrix,

$$\tilde{A}^{(2)'} = \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right].$$

Gaussian elimination with backward substitution

$$\tilde{A}^{(2)'} = \left[\begin{array}{rrrr:r} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right].$$

Since x_2 is already eliminated from E_3 and E_4, $\tilde{A}^{(3)}$ will be $\tilde{A}^{(2)'}$, and the computations continue with the operation $(E_4 + 2E_3) \rightarrow (E_4)$, giving

$$\tilde{A}^{(4)} = \left[\begin{array}{rrrr:r} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right].$$

Gaussian elimination with backward substitution

Finally, the matrix is converted back into a linear system that has a solution equivalent to the solution of the original system and the backward substitution is applied:

$$x_4 = \frac{4}{2} = 2,$$

$$x_3 = \frac{[-4 - (-1)x_4]}{-1} = 2,$$

$$x_2 = \frac{[6 - x_4 - (-1)x_3]}{2} = 3,$$

$$x_1 = \frac{[-8 - (-1)x_4 - 2x_3 - (-1)x_2]}{1} = -7.$$

Gaussian elimination with backward substitution

Example 2 illustrates what is done if $a_{kk}^{(k)} = 0$ for some $k = 1, 2, \ldots, n-1$. The kth column of $\tilde{A}^{(k-1)}$ from the kth row to the nth row is searched for the first nonzero entry. If $a_{pk}^{(k)} \neq 0$ for some p,with $k + 1 \leq p \leq n$, then the operation $(E_k) \leftrightarrow (E_p)$ is performed to obtain $\tilde{A}^{(k-1)'}$. The procedure can then be continued to form $\tilde{A}^{(k)}$, and so on. If $a_{pk}^{(k)} = 0$ for each p, it can be shown (see Theorem 6.17 on page 398) that the linear system does not have a unique solution and the procedure stops. Finally, if $a_{nn}^{(n)} = 0$, the linear system does not have a unique solution, and again the procedure stops.

Algorithm 6.1 summarizes Gaussian elimination with backward substitution. The algorithm incorporates pivoting when one of the pivots $a_{kk}^{(k)}$ is 0 by interchanging the kth row with the pth row, where p is the smallest integer greater than k for which $a_{pk}^{(k)} \neq 0$.

Exercise

Please check "kulon"