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Sistem Persamaan Linear

LINEAR EQUATION

- A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \ ,$$

  where *b* and the coefficients $a_1, \ldots, a_n$ are real or complex numbers that are usually known in advance.

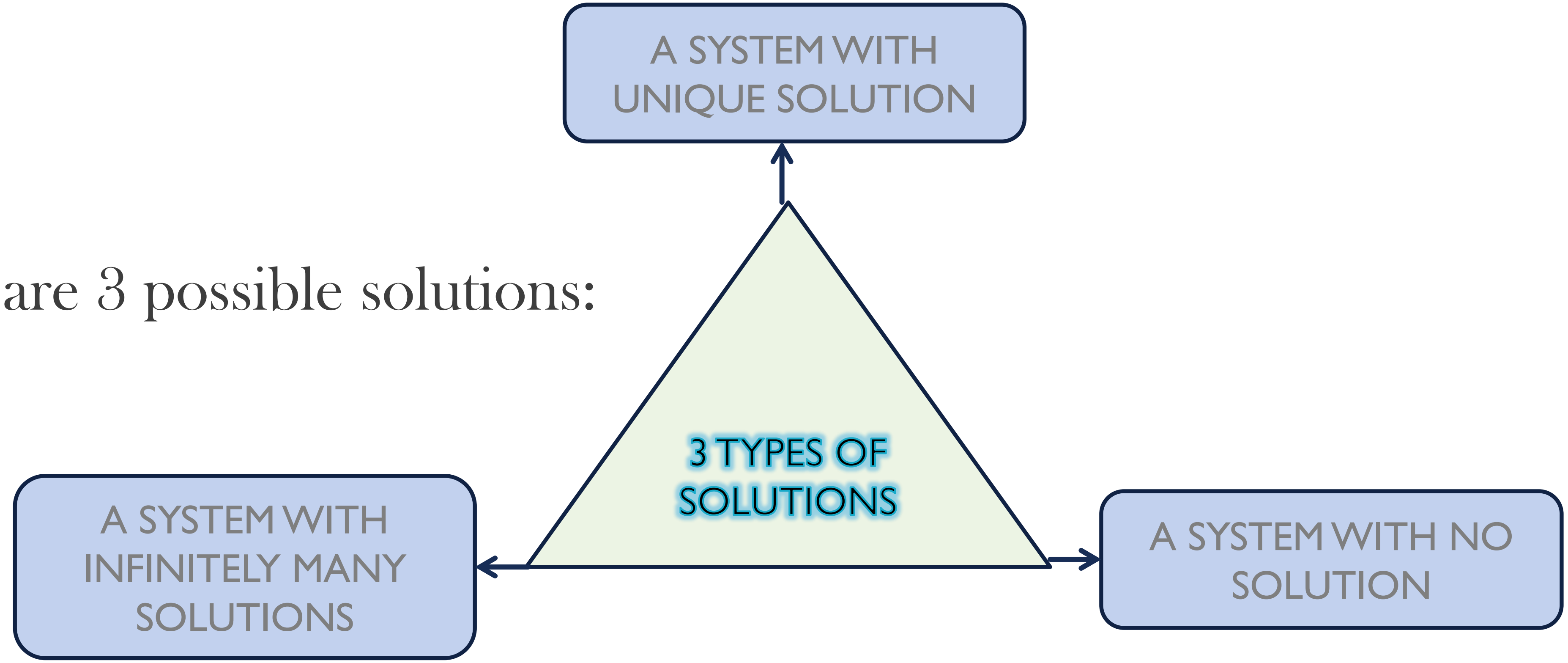
- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables — say, $x_1, \ldots, x_n$.

LINEAR EQUATION

- A **solution** of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.
- The set of all possible solutions is called the **solution set** of the linear system.
- Two linear systems are called **equivalent** if they have the same solution set.

TYPES OF SOLUTIONS TO SYSTEMS OF LINEAR EQUATIONS

There are 3 possible solutions:



A SYSTEMS WITH UNIQUE SOLUTION

Consider the system:

$$x_1 + 2x_2 = 3$$
$$0x_1 + x_2 = 2$$

Augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

The system has unique solution: $x_2 = 2, \quad x_1 = -1$

A SYSTEMS WITH INFINITELY MANY SOLUTION

Consider the system:

$$x_1 + 2x_2 = 3$$
$$0x_1 + 0x_2 = 0$$

Augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

The system has many solutions: let $x_2 = s$, where s is called a free variable.
Then, $x_1 = 3 - 2s$,

A SYSTEMS WITH NO SOLUTION

- Consider the system: $$\begin{aligned} x_1 + 2x_2 &= 3 \\ 0x_1 + 0x_2 &= 2 \end{aligned}$$
- Augmented matrix: $$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 2 \end{array}\right]$$
- The system has no solution, since coefficient of x_2 is '0'.

$$(0x_2 \neq 2)$$

SOLVING SYSTEMS OF EQUATIONS

Systems of linear equations :

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
&\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
\end{aligned}
$$

SOLVING SYSTEMS OF EQUATIONS

4 methods used to solve systems of equations.

1) The Inverse of the Coefficient Matrix
2) Gauss Elimination
3) Gauss-Jordan Elimination
4) Cramer's Rule

THE INVERSE OF THE COEFFICIENT MATRIX

SOLVING SYSTEMS OF EQUATIONS

Matrix Form: $\boldsymbol{AX = B}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

To find $\boldsymbol{X}$: $\boldsymbol{X = A^{-1} B}$

THE INVERSE OF THE COEFFICIENT MATRIX

Method : $X = A^{-1} B$

Example:

Solve the system by using A^{-1}, the inverse of the coefficient matrix:

$$x + y + z = 0$$
$$2x - y + z = -1$$
$$-x + 3y - z = -8$$

THE INVERSE OF THE COEFFICIENT MATRIX

Solution:

$$A=\begin{bmatrix}1 & 1 & 1\\ 2 & -1 & 1\\ -1 & 3 & -1\end{bmatrix} \quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \quad B=\begin{bmatrix}0\\ -1\\ -8\end{bmatrix}$$

$AX=B:$

$$\begin{bmatrix}1 & 1 & 1\\ 2 & -1 & 1\\ -1 & 3 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ -1\\ -8\end{bmatrix}$$

THE INVERSE OF THE COEFFICIENT MATRIX

Find A^{-1} : $\boxed{A^{-1} = \frac{1}{|A|} adj[A]}$

Cofactor of A : $\begin{bmatrix} -2 & 1 & 5 \\ 4 & 0 & -4 \\ 2 & 1 & -3 \end{bmatrix}$

Therefore:

$$adj[A] = \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 1 \\ 5 & -4 & -3 \end{bmatrix}, \quad |A| = 4$$

THE INVERSE OF THE COEFFICIENT MATRIX

Find X: $\boxed{X = A^{-1}B}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 1 \\ 5 & -4 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ -8 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ -2 \\ 7 \end{bmatrix} \qquad \therefore x = -5, y = -2, z = 7$$

THE INVERSE OF THE COEFFICIENT MATRIX

Example 2:

Solve the system by using A^{-1}, the inverse of the coefficient matrix:

$$\begin{aligned} x - 3y + z &= 1 \\ 2x - y + 2z &= 2 \\ x + 2y - 3z &= -1 \end{aligned}$$

Answer : $x = \frac{1}{2},\ y = 0,\ z = \frac{1}{2}$

GAUSS ELIMINATION

GAUSS ELIMINATION

- Consider the systems of linear eq:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
&\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
\end{aligned}
$$

GAUSS ELIMINATION

Write in augmented form : $[A|B]$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right]$$

Using ERO, such that A may be reduce in REF/Upper Triangular

GAUSS ELIMINATION

Example:

Solve the system by using Gauss Elimination method:

$$x + y + z = 0$$
$$2x - y + z = -1$$
$$-x + 3y - z = -8$$

GAUSS ELIMINATION

Solution:

Write in augmented form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 1 & -1 \\ -1 & 3 & -1 & -8 \end{array}\right]$$

Reduce to REF : (Diagonal = 1)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 1 & -1 \\ -1 & 3 & -1 & -8 \end{array}\right] \underset{\mathrm{H}_{31(1)}}{\overset{\mathrm{H}_{21(-2)}}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & -1 \\ 0 & 4 & 0 & -8 \end{array}\right] \overset{H_{32(\frac{4}{3})}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & -1 \\ 0 & 0 & -\frac{4}{3} & -\frac{28}{3} \end{array}\right] \underset{H_{3\left(-\frac{3}{4}\right)}}{\overset{H_{2(-\frac{1}{3})}}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$$\therefore \quad z = 7$$

$$y + \frac{1}{3} z = \frac{1}{3}$$

$$y + \frac{1}{3}.7 = \frac{1}{3}$$

$$y = \frac{1}{3} - \frac{7}{3}$$

$$y = -2$$

$$x + y + z = 0$$

$$x - 2 + 7 = 0$$

$$x + 5 = 0$$

$$x = -5$$

GAUSS JORDAN ELIMINATION

GAUSS JORDAN ELIMINATION

Written in augmented form : $[A|B]$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right]$$

- Using ERO, such that A may be reduce in RREF/IDENTITY (***DIAGONAL = 1, OTHER ENTRIES = 0***)

Reduce to RREF : (Diagonal = 1, Other entries = 0)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 1 & -1 \\ -1 & 3 & -1 & -8 \end{array}\right] \underset{\mathrm{H}_{31(1)}}{\overset{\mathrm{H}_{21(-2)}}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & -1 \\ 0 & 4 & 0 & -8 \end{array}\right] \overset{H_{32(\frac{4}{3})}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & -1 \\ 0 & 0 & -\frac{4}{3} & -\frac{28}{3} \end{array}\right] \underset{H_{3\left(-\frac{3}{4}\right)}}{\overset{H_{2(-\frac{1}{3})}}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$$\underset{\mathrm{H}_{13(-1)}}{\overset{\mathrm{H}_{23(-\frac{1}{3})}}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \end{array}\right] \overset{H_{12(-1)}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

Example 2:

Solve the system by Gauss elimination.

$$x - 3y + z = 1$$
$$2x - y + 2z = 2$$
$$x + 2y - 3z = -1$$

Answer : $x = \frac{1}{2},\ y = 0,\ z = \frac{1}{2}$

GAUSS JORDAN ELIMINATION

Example 4:

Solve the system by Gauss Jordan elimination.

$$\begin{aligned} x+y+z&=4 \\ x-y-z&=0 \\ x-y+z&=0 \end{aligned}$$

$$x=2,\ y=2,\ z=0$$

Answer :

CRAMMER RULES

CRAMER'S RULE

Theorem 5

Cramer's Rule for 3x3 system

Given the system:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_1 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_2 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

with :

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

CRAMER'S RULE

If :

$$D_{x_1} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \quad D_{x_2} = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \quad D_{x_3} = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Then :

$$x = \frac{D_{x_1}}{D} \quad y = \frac{D_{x_2}}{D} \quad z = \frac{D_{x_3}}{D}$$

CRAMER'S RULE

$$D_{x_1} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \quad D_{x_2} = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \quad D_{x_3} = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Step to solve n linear inhomogeneous equations in n variables proceed as follows:

i. Compute $|D|$, the determinant of the coefficient matrix, and, if $|D| \neq 0$, proceed to the next step.

ii. Compute the modified coefficient determinants $|D_i|, i = 1, 2, \ldots, n$ where D_i is derived from D by replacing the i-th column of D by the inhomogeneous vector B ;

iii. The solutions $x_1, x_2, \ldots, x_n$ are given by $x_i = \dfrac{|D_i|}{|D|}$ for $i = 1, 2, \ldots, n$.

iv. If $|D| = 0$ the Cramer's rule cannot be applied. In such a case, either a unique solution to the system does not exist or there is no solution.

CRAMER'S RULE

Example 5:

Solve the system by using the Cramer's Rule.

$$x + y + z = 4$$
$$x - y - z = 0$$
$$x - y + z = 0$$

CRAMER'S RULE

<u>Solution</u>

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Determinant of A :

$$|A| = D = 1\begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = -4$$

CRAMER'S RULE

$$D_x = \begin{vmatrix} 4 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{vmatrix} = 4\begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix} = -8 \quad \rightarrow \quad x = \frac{-8}{-4} = 2$$

$$D_y = \begin{vmatrix} 1 & 4 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = -4\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -8 \quad \rightarrow \quad y = \frac{-8}{-4} = 2$$

$$D_x = \begin{vmatrix} 1 & 1 & 4 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = 4\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = 0 \quad \rightarrow \quad z = \frac{0}{-4} = 0$$

CRAMER'S RULE

<u>Example 6</u>:

Solve the system by using Cramer's Rule.

$$x + 2y = z - 1$$

$$x = 4 + y - z$$

$$x + y - 3z = -2$$

Answer :

$$x = 2,\ y = -1,\ z = 1$$