Section 2. Predicate Logic

Discussion:

In Maths we use variables (usually ranging over numbers) in various ways.

How does x differ in what it represents in the following statements? x is real.

- $x^2 = 0$ — x represents one value, $x = 0$
- $x > 2$ — x represents some, but not all values
- $x + 0 = x$ — x represents all values
- $x^2 + 1 = 0$ — x represents no values

Definition: Predicate

A *predicate* is a sentence that contains one or more variables and becomes a statement when specific values are substituted for the variables.

Definition: Domain

The *domain* of a predicate variable consists of all values that may be substituted in place of the variable

Definition: Truth Set

If $P(x)$ is a predicate and x has domain D, the *truth set* of $P(x)$ is the set of all elements of D that make $P(x)$ true. The truth set is denoted $\{x \in D : P(x)\}$ and is read "the set of all x in D such that $P(x)$."

Examples:

- Let $P(x)$ be the predicate "$x^2 > x$" with $x \in \mathbb{R}$ i.e. domain the set of real numbers $\mathbb{R}$.

Write down $P(2), P(1), P(-2)$ and indicate which are true and which are false.

Determine the truth set of $P(x)$

$$\begin{array}{lllll} P(2): & 2^2 > 2 & \text{or} & 4 > 2 & \text{True} \\ P(1): & (1)^2 > 1 & \text{or} & 1 > 1 & \text{False} \\ P(-2): & (-2)^2 > (-2) & \text{or} & 4 > (-2) & \text{True} \end{array}$$

$$\{x \in \mathbb{R} : x^2 > x\} = \{x \in \mathbb{R} : x < 0 \vee x > 1\}$$

- Let $Q(n)$ be the predicate "n is factor of 8".

Determine the truth set of $Q(n)$ if $n \in \mathbb{Z}^+$

$$8 = \pm 1 \times \pm 8, \qquad 8 = \pm 2 \times \pm 4$$

$$\therefore \{n \in \mathbb{Z}^+ : \text{"}n \text{ is a factor of } 8\text{"}\} = \{1, 2, 4, 8\}$$

Exercises:

- Let $P(x)$ be the predicate "$x^3 > x$" with $x \in \mathbb{Z}$ i.e. domain the set of integers, $\mathbb{Z}$.

Write down $P(2), P(0), P(-2)$ and indicate which are true and which are false.

Determine the truth set of $P(x)$

- Let $Q(n)$ be the predicate "n is factor of 6".

Determine the truth set of $Q(n)$ if $n \in \mathbb{Z}$

2.1. Quantifiers

A way to obtain statements from predicates is to add *quantifiers*. Quantifiers are words that refer to quantities such as "all", "every", or "some" and tell for how many elements a given predicate is true.

2.1.1. Universal Quantifier

The symbol $\forall$ denotes "for all" and is called the *universal quantifier*.

Definition: Universal Statement

Let $P(x)$ be a predicate and D the domain of x. A *universal statement* is a statement of the form "$\forall x \in D, P(x)$". It is defined to be true if, and only if, $P(x)$ is true for every x in D. It is defined to be false if, and only if, $P(x)$ is false for at least one x in D. A value of x for which $P(x)$ is false is called a *counterexample* to the universal statement.

Examples:

- Write the sentence "All human beings are mortal" using the universal quantifier.

Let H be the set of human beings.
$\forall h \in H, h$ is mortal

- Consider $A = \{x_1, x_2, x_3\}$. With $\forall x \in A, P(x)$, the following must hold: $P(x_1) \wedge P(x_2) \wedge P(x_3)$

Thus there will be 3 predicates which must hold.

Exercises:

Write the following statements using the universal quantifier. Determine whether each statement is true or false.

- "All dogs are animals"

- The square of any real number is positive.

- Every integer is a rational number.

Exercises:

Write the following statements in words. Determine whether each statement is true or false.

- $\forall x \in \mathbb{N}, \sqrt{x} \in \mathbb{N}$

- $\forall x \in \mathbb{R},\ x^2 \neq -1.$

2.1.2. Existential Quantifier

The symbol $\exists$ denotes “there exists” and is called the *existential quantifier*.

Definition: Existential Statement

Let $P(x)$ be a predicate and D the domain of x.
An *existential statement* is a statement of the form “$\exists x \in D, P(x)$”.
It is defined to be true if, and only if, $P(x)$ is true for at least one x in D.
It is defined to be false if, and only if, $P(x)$ is false for all x in D.

Examples:

- Write the sentence “Some people are vegetarians” using the existential quantifier.

Let H be the set of human beings.
$\exists h \in H, h$ is a vegetarian

- Consider $A = \{x_1, x_2, x_3\}$. With $\exists x \in A, P(x)$, the following must hold: $P(x_1) \vee P(x_2) \vee P(x_3)$

Thus there will be 1 predicate which must hold.

Exercises:

Write the following statements using the existential quantifier. Determine whether each statement is true or false.

- "Some cats are black"

- There is a real number whose square is negative.

- Some programs are structured.

Exercises:

Write the following statements in words. Determine whether each statement is true or false.

- $\exists m \in \mathbb{Z},\ m^2 = m$

- $\exists x \in \mathbb{R},\ x^2 = -1.$

- $\exists x \in \mathbb{Z}, \frac{1}{x} \notin \mathbb{Q}$

2.1.3. Negation of Universal Statements

Let $P(x)$ be a predicate and D the domain of x. The *negation* of a universal statement of the form:

$\forall x \in D, P(x)$ is logically equivalent to $\exists x \in D, \sim P(x)$

Symbolically $\sim(\forall x \in D, P(x)) \equiv \exists x \in D, \sim P(x)$

Example:

- Write down the negation of the following statement.

$\forall x \in \mathbb{R}, x^2 + 1 \geq 2x$

Negation:

$$\sim(\forall x \in \mathbb{R}, x^2 + 1 \geq 2x)$$
$$\equiv \exists x \in \mathbb{R}, \sim(x^2 + 1 \geq 2x)$$
$$\equiv \exists x \in \mathbb{R}, x^2 + 1 < 2x$$

False.

Exercises:

- Write down the negation of the following statement.

$$\forall x \in \mathbb{R}, x^2 \geq 0$$

- Write down the negation of the following statement.

$$\forall y \in \mathbb{R}, \left(y \neq 0 \Rightarrow \frac{y+1}{y} < 1 \right)$$

Example:

- Write the following statement using quantifiers. Find its negation and determine whether the statement or its negation is true, giving a brief reason..

"Every real number is either positive or negative."

Statement:

$$\forall x \in \mathbb{R},\ x < 0 \vee x > 0$$

Negation:

$$\begin{aligned}
&\sim (\forall x \in \mathbb{R},\ x < 0 \vee x > 0) \\
&\equiv \exists x \in \mathbb{R},\ \sim (x < 0 \vee x > 0) \\
&\equiv \exists x \in \mathbb{R},\ \sim (x < 0) \wedge \sim (x > 0) \\
&\equiv \exists x \in \mathbb{R},\ (x \geq 0) \wedge (x \leq 0) \\
&\equiv \exists x \in \mathbb{R}, x = 0
\end{aligned}$$

The true statement is the negation because $x = 0$ is neither positive nor negative.

Exercises:

- Write the following statement using quantifiers. Find the negation.

"The square of any integer is positive."

- Write the following statement using quantifiers. Find the negation.

"All computer programs are finite."

2.1.4. Negation of Existential Quantifiers

Let $P(x)$ be a predicate and D the domain of x. The *negation* of an existential statement of the form:

$\exists x \in D, P(x)$ is logically equivalent to $\forall x \in D, \sim P(x)$

Symbolically $\sim (\exists x \in D, P(x)) \equiv \forall x \in D, \sim P(x)$

Example:

- Write down the negation of the following statement.

$\exists x \in \mathbb{Q}, x^2 = 2$

Negation:

$$\begin{aligned} &\sim (\exists x \in \mathbb{Q}, x^2 = 2) \\ &\equiv \forall x \in \mathbb{Q}, \sim (x^2 = 2) \\ &\equiv \forall x \in \mathbb{Q}, x^2 \neq 2 \end{aligned}$$

The negation is true.

Exercises:

- Write down the negation of the following statement.

$\exists z \in \mathbb{Z}, (z \text{ is odd}) \vee (z \text{ is even})$

- Write down the negation of the following statement.

$\exists n \in \mathbb{N}, (n \text{ is even}) \wedge (\sqrt{n} \text{ is prime})$

Example:

- Write the following statement using quantifiers. Find its negation

"Some dogs are vegetarians."

Let D be the set of dogs.

Statement: $\exists d \in D,\ d \text{ is vegetarian}$

Negation:

$\sim (\exists d \in D,\ d \text{ is vegetarian})$

$\equiv \forall d \in D,\ \sim (d \text{ is vegetarian})$

$\equiv \forall d \in D,\ d \text{ is not vegetarian}$

All dogs are not vegetarian

Exercises:

- Write the following statement using quantifiers. Find the negation.

"There is a real number that is rational."

- Write the following statement using quantifiers. Find the negation.

"Some computer hackers are over 40."

- Write the following statement using quantifiers. Find the negation.

"Some animals are dogs."

2.1.5. Multiple Quantifiers

When a statement contains multiple quantifiers their order must be applied as written and will produce different results for the truth set.

Examples:

Write the following statements using quantifiers:

- "Everybody loves somebody."

Let H be the set of people.

Statement: $\forall x \in H, \exists y \in H, x$ loves y.

- "Somebody loves everyone."

Let H be the set of people.

Statement: $\exists x \in H, \forall y \in H, x$ loves y.

Exercises:

Write the following statements using quantifiers:

- “Everybody loves everybody.”

- The Commutative Law of Addition for $\mathbb{Z}$

- “Everyone had a mother.”

- “There is an oldest person.”

Examples:

Write the following statements without using quantifiers:

- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R},\ x + y = 0$

Statement: Given any real number, you can find a real number so that the sum of the two is *zero*. Alternatively: Every real number has an additive inverse.

- $\exists x \in \mathbb{R}, \forall y \in \mathbb{R},\ x + y = y$

Statement: There is a real number, which added to any other real number results in the other number.
Alternatively: Every real number has an additive identity.

Exercises:

Write the following statements without using quantifiers:

- $\forall c \in \text{colours}, \exists a \in \text{animals},\ a \text{ is coloured } c$

- $\exists b \in \text{books}, \forall p \in \text{people},\ p \text{ has read } b$

2.1.6. Interpreting Statements with Multiple Quantifiers

To establish the truth of a statement with more than one quantifier, take the action suggested by the quantifiers as being performed in the order in which the quantifiers occur.

Consider $A = \{x_1 , x_2 , x_3\}, B = \{y_1 , y_2\}$ and the predicate $P(x, y)$.

There will be 6 possible predicates:

$$P(x_1 , y_1) , P(x_1 , y_2),$$
$$P(x_2 , y_1), P(x_2, y_2),$$
$$P(x_3 , y_1) , P(x_3 , y_2).$$

- For $\forall x \in A, \forall y \in B, P(x, y)$ to be true the following must hold:

$$P(x_1 , y_1) \wedge P(x_1 , y_2)$$
$$\wedge P(x_2 , y_1) \wedge P(x_2, y_2)$$
$$\wedge P(x_3 , y_1) \wedge P(x_3 , y_2)$$

Thus there will be 6 predicates which must all be true. That is for all pairs (x, y), $P(x, y)$ must be true. It will be false if there is one pair (x, y), for which $P(x, y)$ is false.

- For $\forall x \in A, \exists y \in B, P(x, y)$ to be true, the following must hold:

$$P(x_1, y_1) \vee P(x_1, y_2)$$
$$\wedge P(x_2, y_1) \vee P(x_2, y_2)$$
$$\wedge P(x_3, y_1) \vee P(x_3, y_2)$$

Thus there will be 3 predicates which must be true. That is for every x there must be at least one y so that $P(x, y)$ is true. Given any element x in A you can find an element y in B, so that $P(x, y)$ is true. It will be false if there is one x in A for which $P(x, y)$ is false for every y in B.

- For $\exists x \in A, \forall y \in B, P(x, y)$ to be true, the following must hold:

$$P(x_1, y_1) \wedge P(x_1, y_2)$$
$$\vee P(x_2, y_1) \wedge P(x_2, y_2)$$
$$\vee P(x_3, y_1) \wedge P(x_3, y_2)$$

Thus there will be 2 predicates which must be true. That is there is one x that when paired with any y, $P(x, y)$ is true. You can find one element x in A that with all elements y in B, $P(x, y)$ is true. It will be false if for every x in A, there is a y in B for which $P(x, y)$ is false.

- For $\exists x \in A, \exists y \in B, P(x, y)$ to be true, the following must hold:

$$P(x_1, y_1) \vee P(x_1, y_2)$$
$$\vee P(x_2, y_1) \vee P(x_2, y_2)$$
$$\vee P(x_3, y_1) \vee P(x_3, y_2)$$

Thus there will be 1 predicate which must be true. That is there is one x that when paired with one y, $P(x, y)$ is true. You can find one element x in A and one element y in B, $P(x, y)$ is true. It will be false if for all pairs (x, y), $P(x, y)$ is false.

Summary:

Statement	**When true?**	**When false?**
$\forall x, \forall y, P(x, y)$	$P(x, y)$ is true for all pairs (x, y)	There is a pair (x, y) for which $P(x, y)$ is false
$\forall x, \exists y, P(x, y)$	For every x, there is a y for which $P(x, y)$ is true	There is an x such that $P(x, y)$ is false for every y
$\exists x, \forall y, P(x, y)$	There is an x such that $P(x, y)$ is true for every y	For every x, there is a y for which $P(x, y)$ is false
$\exists x, \exists y, P(x, y)$	There is a pair (x, y) for which $P(x, y)$ is true	$P(x, y)$ is false for all pairs (x, y)

2.1.7. Negation of Statements with Multiple Quantifiers.

To negate statements with multiple quantifiers, each quantifier is negated and the predicate must be negated.

- To negate $\forall x \in A, \forall y \in B, P(x, y)$

$$\sim \left(\forall x \in A, \forall y \in B, P(x, y)\right) \equiv \exists x \in A, \exists y \in B, \sim P(x, y)$$

- To negate $\forall x \in A, \exists y \in B, P(x, y)$

$$\sim \left(\forall x \in A, \exists y \in B, P(x, y)\right) \equiv \exists x \in A, \forall y \in B, \sim P(x, y)$$

- To negate $\exists x \in A, \forall y \in B, P(x, y)$

$$\sim \left(\exists x \in A, \forall y \in B, P(x, y)\right) \equiv \forall x \in A, \exists y \in B, \sim P(x, y)$$

- To negate $\exists x \in A, \exists y \in B, P(x, y)$

$$\sim \left(\exists x \in A, \exists y \in B, P(x, y)\right) \equiv \forall x \in A, \forall y \in B, \sim P(x, y)$$

Examples:

Write the negation of the following:

- Statement: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R},\ x + y = 0$

Negation:

$$\sim \left(\forall x \in \mathbb{R}, \exists y \in \mathbb{R},\ x + y = 0\right)$$
$$\equiv \exists x \in \mathbb{R}, \forall y \in \mathbb{R},\ x + y \neq 0$$

False: Take $y = -x$, then $x + y = x - x = 0$

- Statement: $\exists x \in \mathbb{R}, \forall y \in \mathbb{R},\ xy = 1$

Negation:

$$\sim \left(\exists x \in \mathbb{R}, \forall y \in \mathbb{R},\ xy = 1\right)$$
$$\equiv \forall x \in \mathbb{R}, \exists y \in \mathbb{R},\ xy \neq 1$$

True: Take $y = -x$, then $xy = -x^2 \neq 1$

Exercises:

Write the negation of the following:

- Statement: $\forall c \in \text{colours}, \exists a \in \text{animals},\ a \text{ is coloured } c$

- Statement: $\exists b \in \text{books}, \forall p \in \text{people},\ p \text{ has read } b$