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Matriks
Matrix (Plural : Matrices)

MATRIX (PLURAL: MATRICES)

- **Matrix**
  - a rectangular array of elements written within brackets
  - Represented with a capital letter and classify by its dimension
- **Dimensions of a Matrix/Order of a Matrix**
  - determine by the number of horizontal rows and the number of vertical columns
- **Matrix Element**
  - each number in a matrix

WRITING THE DIMENSIONS OF A MATRIX.

$$\begin{array}{cc} & \begin{array}{cccc} & \text{4 columns} & & \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ A = & \begin{bmatrix} -5 & 0 & 1 & 2 \\ 3 & -4 & 3 & 2 \\ 3 & 1 & 4 & 2 \end{bmatrix} \begin{array}{l} \leftarrow \\ \leftarrow \quad \text{3 rows} \\ \leftarrow \end{array} \end{array}$$

Matrix A is a 3×4 matrix.

WRITE THE DIMENSIONS OR ORDER OF EACH MATRIX.

$$\begin{pmatrix} 4 & 6 & 5 \\ 2 & -3 & -7 \\ 1 & 0 & 9 \end{pmatrix} \quad \mathbf{3 \times 3}$$

$$\begin{pmatrix} -4 & 1/3 & -3 \end{pmatrix} \quad \mathbf{1 \times 3}$$

$$\begin{pmatrix} 4 & 5 & 0 \\ -2 & 0.5 & 17 \end{pmatrix} \quad \mathbf{2 \times 3}$$

$$\begin{pmatrix} 10 & 0 \\ 1 & -5 \\ -6.2 & 9 \end{pmatrix} \quad \mathbf{3 \times 2}$$

IDENTIFYING A MATRIX ELEMENT

$$\mathbf{A} = \begin{array}{cc} & \textit{column} \;\; j \rightarrow \\ \begin{array}{c} \textit{row} \;\; i \\ \downarrow \end{array} & \begin{bmatrix} -5 & 0 & 1 \\ 2 & 3 & -4 \\ -9 & 2 & 6 \\ 3 & 1 & 4 \end{bmatrix} \end{array}$$

a_{ij} denotes the element of the matrix A on the i^{th} row and j^{th} column.

Example:

Identify element a_{13} in Matrix A.

Answer: a_{13} means the element in row 1, column 3.

$$a_{13} = 1$$

IDENTIFY EACH MATRIX ELEMENT

$$A = \begin{bmatrix} -5 & 0 & 1 & 2 \\ 3 & -4 & -9 & 2 \\ 3 & 1 & 4 & 2 \end{bmatrix}$$

1. $a_{33} = \dots$
2. $a_{11} = \dots$
3. $a_{21} = \dots$
4. $a_{34} = \dots$
5. $a_{23} = \cdots$

IDENTIFY EACH MATRIX ELEMENT

$$A = \begin{bmatrix} -5 & 0 & 1 & 2 \\ 3 & -4 & -9 & 2 \\ 3 & 1 & 4 & 2 \end{bmatrix}$$

1. $a_{33} = 4$
2. $a_{11} = -5$
3. $a_{21} = 3$
4. $a_{34} = 2$
5. $a_{23} = -9$

ADDING AND SUBTRACTING MATRICES

- to add or subtract matrices A and B with the same dimensions, add or subtract the corresponding elements

***Note: you can only add or subtract matrices with the same dimensions.

EXAMPLE : MATRIX ADDITION AND SUBSTRACTION

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 & 7 \\ 3 & 4 & 5 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 6 & 7 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 & 3+7 \\ 7+3 & 8+4 & 9+5 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 10 \\ 10 & 12 & 14 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 6 & 7 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1-5 & 2-6 & 3-7 \\ 7-3 & 8-4 & 9-5 \end{bmatrix} = \begin{bmatrix} -4 & -4 & -4 \\ 4 & 4 & 4 \end{bmatrix}$$

FIND THE SUM OR DIFFERENCE OF EACH MATRIX.

1. $\begin{bmatrix} 1 & -2 & 0 \\ 3 & -5 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 9 & -3 \\ -9 & 6 & 12 \end{bmatrix}$

$$= \begin{bmatrix} 1+3 & -2+9 & 0-3 \\ 3-9 & -5+6 & 7+12 \end{bmatrix} = \begin{bmatrix} 4 & 7 & -3 \\ -6 & 1 & 19 \end{bmatrix}$$

FIND THE SUM OR DIFFERENCE OF EACH MATRIX.

2. $\begin{bmatrix} -12 & 24 \\ -3 & 5 \\ -1 & 10 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 2 & -4 \\ -1 & 5 \end{bmatrix}$

$$= \begin{bmatrix} -12-3 & 24+1 \\ -3+2 & 5-4 \\ -1-1 & 10+5 \end{bmatrix} = \begin{bmatrix} -15 & 25 \\ -1 & 1 \\ -2 & 15 \end{bmatrix}$$

FIND THE SUM OR DIFFERENCE OF EACH MATRIX.

3. $\begin{bmatrix} -3 & 5 \\ -1 & 10 \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix}$

$$= \begin{bmatrix} -3-(-3) & 5-1 \\ -1-2 & 10-(-4) \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -3 & 14 \end{bmatrix}$$

"SPECIAL" MATRICES

- Vector: a vector is an $m \times n$ matrix where either m OR $n = 1$ (but not both).

$$X = \begin{bmatrix} 12 \\ 9 \\ -4 \\ 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 7 & -22 & 14 \end{bmatrix}$$

"SPECIAL" MATRICES

- Square matrix: a square matrix is an $m \times n$ matrix in which $m = n$.

$$B = \begin{bmatrix} 7 & 3 & 2 \\ 8 & 4 & 1 \\ 6 & 5 & 9 \end{bmatrix}$$

- Entries m_{ii} are called the *diagonal* entries. The others are called *nondiagonal* entries

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

"SPECIAL" MATRICES

- Scalar: a scalar is an $m \times n$ matrix where BOTH m AND $n = 1$.

$$D = [17]$$

- Zero matrix: an $m \times n$ matrix of zeros.

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

“SPECIAL” MATRICES

- Diagonal Matrix: a square matrix whose nondiagonal elements are zero.

$$B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

- Identity Matrix: a square $(m \times m)$ matrix with 1s on the diagonal and zeros everywhere else.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

DIAGONAL AND TRACE

Let $A = [a_{ij}]$ be a square matrix. The diagonal or main diagonal of A consists of the elements with the same subscripts – that is

$$a_{11}, a_{22}, a_{33}, \dots, a_{nn}$$

The *trace* of A, written $tr(A)$, is the sum of diagonal elements, namely:

$$tr(A) = a_{11} + a_{22} + a_{33} + \dots + a_{nn}$$

Theorem:

- $tr(A + B) = tr(A) + tr(B)$
- $tr(kA) = k\ tr(A)$
- $tr(A^T) = tr(A)$
- $tr(AB) = tr(BA)$

EXAMPLE

Let A and B be the matrices with:

- Diagonal of $A = \{1, -4, 7\}$
- Diagonal of $B = \{2, 3, -4\}$

Find

1. $tr(A), tr(B)$
2. $tr(A + B)$
3. $tr(2A)$
4. $tr(A^T)$

EXAMPLE

Let A and B be the matrices with:

- Diagonal of $A = \{1, -4, 7\}$
- Diagonal of $B = \{2, 3, -4\}$

Find

1. $tr(A) = \mathbf{1 + (-4) + 7 = 4} \quad and \quad tr(B) = \mathbf{2 + 3 + (-4) = 1}$
2. $tr(A + B) = \mathbf{tr(A) + tr(B) = 4 + 1 = 5}$
3. $tr(2A) = \mathbf{2\ tr(A) = 2.4 = 8}$
4. $tr(A^T) = \mathbf{tr(A) = 4}$

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -4 & -4 \\ 5 & 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -5 & 1 \\ 0 & 3 & -2 \\ 1 & 2 & -4 \end{bmatrix}$, then find $tr(AB)$ and $tr(BA)$

$\boldsymbol{AB = \begin{bmatrix} 5 & 7 & -15 \\ -12 & 0 & 20 \\ 17 & 7 & -35 \end{bmatrix}}$**, then** $\boldsymbol{tr(AB) = 5 + 0 + (-35) = -30}$

$\boldsymbol{BA = \begin{bmatrix} 27 & 30 & 33 \\ -22 & -24 & -26 \\ -27 & -30 & -33 \end{bmatrix}}$**, then** $\boldsymbol{tr(BA) = 27 + (-24) + (-33) = -30}$

Although $AB \neq BA$, $tr(AB) = tr(BA)$ as mentioned in theorem

PROPERTIES: MATRIX ADDITION

If A, B, and C are *m x n* matrices, then Closure Property A + B is an *m x n* matrix

a. Commutative Property

   $A + B = B + A$

b. Associative Property for Addition

   $(A + B) + C = A + (B + C)$

c. "Additive Identity" Property → There exist a unique *m x n* matrix O such that

   $O + A = A + O = A$

d. "Additive Inverse" Property → For each A, there exists a unique opposite –A.

   $A + (-A) = O$

IDENTIFY WHETHER THE TWO MATRICES ARE ADDITIVE INVERSE OR NOT.

1. $\begin{pmatrix} 14 & 5 \\ 0 & -2 \end{pmatrix}$, $\begin{pmatrix} -14 & -5 \\ 0 & 2 \end{pmatrix}$ Yes.

Find the "additive inverse" of the given matrix.

1. $\begin{pmatrix} -1 & 10 & -5 \\ 0 & 2 & -3 \end{pmatrix}$ => $\begin{pmatrix} 1 & -10 & 5 \\ 0 & -2 & 3 \end{pmatrix}$

SOLVING MATRIX EQUATIONS

- Matrix Equation
  - an equation in which the variable is a matrix
- Equal Matrices
  - matrices with the same dimensions and with equal corresponding elements

SOLVING A MATRIX EQUATION

- Solve for the matrix X.

$$X - \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 8 & 9 \end{pmatrix}$$

- Solution:

$$X - \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 8 & 9 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 8 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 2 \\ 11 & 11 \end{pmatrix}$$

EXERCISE

Solve for Matrix X.

1. $X + \begin{pmatrix} -1 & 0 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 10 & 7 \\ -4 & 4 \end{pmatrix}$

Answer: $X = \begin{pmatrix} 11 & 7 \\ -6 & -1 \end{pmatrix}$

2. $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} - X = \begin{pmatrix} 11 & 3 & -13 \\ 15 & -9 & 8 \end{pmatrix}$

Answer: $X = \begin{pmatrix} -9 & -2 & 12 \\ -15 & -11 & -7 \end{pmatrix}$

EXERCISE (CONT'D)

- Determine whether the two matrices in each pair are equal

1. $\begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$, $\begin{bmatrix} 8/2 & 18/3 & 16/2 \end{bmatrix}$

No, because they do not have the same dimensions.

2. $\begin{bmatrix} -2 & 3 \\ 5 & 0 \end{bmatrix}$, $\begin{bmatrix} -8/4 & 6 - 3 \\ 15/3 & 4 - 4 \end{bmatrix}$

Yes, because they have the same dimensions and the corresponding elements are equal.

FINDING UNKNOWN MATRIX ELEMENTS

- Solve the equation for x and y

$$\begin{pmatrix} x+8 & -5 \\ 3 & -y \end{pmatrix} = \begin{pmatrix} 38 & -5 \\ 3 & 4y-10 \end{pmatrix}$$

Solution:

$$x + 8 = 38$$
$$\mathbf{x = 30}$$

$$-y = 4y - 10$$
$$-5y = -10$$
$$\mathbf{y = 2}$$

EXERCISE

- Solve each unknown variable in each equation

1. $$\begin{pmatrix} 3x & 4 \end{pmatrix} = \begin{pmatrix} -9 & x + y \end{pmatrix}$$

$$x = -3, y = 7$$

2. $$\begin{pmatrix} 2 & 4 \\ 8 & 12 \end{pmatrix} = \begin{pmatrix} 4x - 6 & -10t + 5x \\ 4x & 15t + 1.5x \end{pmatrix}$$

$$\boldsymbol{x = 2; t = 3/5}$$

TRANSPOSE OF A MATRIX

- The transpose of an $r \times c$ matrix **M** is a $c \times r$ matrix called $\mathbf{M}^T$.
- Take every row and rewrite it as a column.
- Equivalently, flip about the diagonal

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

FACTS ABOUT TRANSPOSE

- Transpose is its own inverse: $(\mathbf{M}^T)^T = \mathbf{M}$ for all matrices $\mathbf{M}$.
- $\mathbf{D}^T = \mathbf{D}$ for all diagonal matrices $\mathbf{D}$ (including the identity matrix $\mathbf{I}$).

TRANSPOSE OF A VECTOR

If **v** is a row vector, $\mathbf{v}^T$ is a column vector and vice-versa

$$\begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T = \begin{bmatrix} x & y & z \end{bmatrix}$$

MULTIPLYING BY A SCALAR

- Can multiply a matrix by a scalar.
- Result is a matrix of the same dimension.
- To multiply a matrix by a scalar, multiply each component by the scalar.

$$k\mathbf{M} = k \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{bmatrix} = \begin{bmatrix} km_{11} & km_{12} & km_{13} \\ km_{21} & km_{22} & km_{23} \\ km_{31} & km_{32} & km_{33} \\ km_{41} & km_{42} & km_{43} \end{bmatrix}$$

MATRIX MULTIPLICATION

Multiplying an $r \times n$ matrix **A** by an $n \times c$ matrix **B** gives an $r \times c$ result **AB**.

$$\underset{\substack{r \times n \\ 4 \times 2}}{\overset{\mathbf{A}}{\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}}} \quad \underset{\substack{n \times c \\ 2 \times 5}}{\overset{\mathbf{B}}{\begin{bmatrix} ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix}}} = \underset{\substack{r \times c \\ 4 \times 5}}{\overset{\mathbf{AB}}{\begin{bmatrix} ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix}}}$$

MULTIPLICATION: RESULT

- Multiply an $r \times n$ matrix **A** by an $n \times c$ matrix **B** to give an $r \times c$ result **C = AB**.
- Then $\mathbf{C} = [c_{ij}]$, where c_{ij} is the dot product of the i-th row of **A** with the j-th column of **B**.
- That is:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \,.$$

EXAMPLE

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \end{bmatrix}$$

$$c_{24} = a_{21}b_{14} + a_{22}b_{24}$$

ANOTHER WAY OF LOOKING AT IT

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \quad \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \end{bmatrix}$$

$$c_{43} = a_{41}b_{13} + a_{42}b_{23}$$

2 X 2 CASE

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

2 X 2 EXAMPLE

$$\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 5 & 1/2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -7 & 2 \\ 4 & 6 \end{bmatrix}$$

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} -3 & 0 \\ 5 & 1/2 \end{bmatrix} \begin{bmatrix} -7 & 2 \\ 4 & 6 \end{bmatrix} \\ &= \begin{bmatrix} (-3)(-7) + (0)(4) & (-3)(2) + (0)(6) \\ (5)(-7) + (1/2)(4) & (5)(2) + (1/2)(6) \end{bmatrix} \\ &= \begin{bmatrix} 21 & -6 \\ -33 & 13 \end{bmatrix} \end{aligned}$$

3 X 3 CASE

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

3 X 3 EXAMPLE

$$\mathbf{A} = \begin{bmatrix} 1 & -5 & 3 \\ 0 & -2 & 6 \\ 7 & 2 & -4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -8 & 6 & 1 \\ 7 & 0 & -3 \\ 2 & 4 & 5 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 & -5 & 3 \\ 0 & -2 & 6 \\ 7 & 2 & -4 \end{bmatrix} \begin{bmatrix} -8 & 6 & 1 \\ 7 & 0 & -3 \\ 2 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot (-8) + (-5) \cdot 7 + 3 \cdot 2 & 1 \cdot 6 + (-5) \cdot 0 + 3 \cdot 4 & 1 \cdot 1 + (-5) \cdot (-3) + 3 \cdot 5 \\ 0 \cdot (-8) + (-2) \cdot 7 + 6 \cdot 2 & 0 \cdot 6 + (-2) \cdot 0 + 6 \cdot 4 & 0 \cdot 1 + (-2) \cdot (-3) + 6 \cdot 5 \\ 7 \cdot (-8) + 2 \cdot 7 + (-4) \cdot 2 & 7 \cdot 6 + 2 \cdot 0 + (-4) \cdot 4 & 7 \cdot 1 + 2 \cdot (-3) + (-4) \cdot 5 \end{bmatrix}$$

$$= \begin{bmatrix} -37 & 18 & 31 \\ -2 & 24 & 36 \\ -50 & 26 & -19 \end{bmatrix}$$

IDENTITY MATRIX

- Recall that the identity matrix **I** (or $\mathbf{I}_n$) is a diagonal matrix whose diagonal entries are all 1.
- Now that we've seen the definition of matrix multiplication, we can say that **IM** = **MI** = **M** for all matrices **M** (dimensions appropriate)

$$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

MATRIX MULTIPLICATION FACTS

- Not commutative: in general $\mathbf{AB} \neq \mathbf{BA}$.
- Associative:

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

- Associates with scalar multiplication:

$$k(\mathbf{AB}) = (k\mathbf{A})\mathbf{B} = \mathbf{A}(k\mathbf{B})$$

- $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$
- $(\mathbf{M}_1\mathbf{M}_2\mathbf{M}_3\ldots\mathbf{M}_n)^T = \mathbf{M}_n^T\ldots\mathbf{M}_3^T\mathbf{M}_2^T\mathbf{M}_1^T$

ROW VECTOR TIMES MATRIX MULTIPLICATION

Can multiply a row vector times a matrix

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} =$$

$$\begin{bmatrix} xm_{11} + ym_{21} + zm_{31} & xm_{12} + ym_{22} + zm_{32} & xm_{13} + ym_{23} + zm_{33} \end{bmatrix}$$

MATRIX TIMES COLUMN VECTOR MULTIPLICATION

Can multiply a matrix times a column vector.

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xm_{11} + ym_{12} + zm_{13} \\ xm_{21} + ym_{22} + zm_{23} \\ xm_{31} + ym_{32} + zm_{33} \end{bmatrix}$$

ROW VS. COLUMN VECTORS

- Row vs. column vector matters now. Here's why: Let **v** be a row vector, **M** a matrix.
  - **vM** is legal, **Mv** is undefined
  - $\mathbf{M}\mathbf{v}^T$ is legal, $\mathbf{v}^T\mathbf{M}$ is undefined

COMMON MISTAKE

$\mathbf{Mv}^T \neq (\mathbf{vM})^T$, but $\mathbf{Mv}^T = (\mathbf{vM}^T)^T$ – compare the following two results:

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$= \begin{bmatrix} xm_{11} + ym_{21} + zm_{31} & xm_{12} + ym_{22} + zm_{32} & xm_{13} + ym_{23} + zm_{33} \end{bmatrix}$$

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xm_{11} + ym_{12} + zm_{13} \\ xm_{21} + ym_{22} + zm_{23} \\ xm_{31} + ym_{32} + zm_{33} \end{bmatrix}$$

VECTOR-MATRIX MULTIPLICATION FACTS I

Associates with vector multiplication.

- Let **v** be a row vector:

$$\mathbf{v}(\mathbf{AB}) = (\mathbf{vA})\mathbf{B}$$

- Let **v** be a column vector:

$$(\mathbf{AB})\mathbf{v} = \mathbf{A}(\mathbf{Bv})$$

VECTOR-MATRIX MULTIPLICATION FACTS 2

- Vector-matrix multiplication distributes over vector addition:

$$(\mathbf{v} + \mathbf{w})\mathbf{M} = \mathbf{vM} + \mathbf{wM}$$

- That was for row vectors **v**, **w**. Similarly for column vectors.

POWER OF MATRICES, POLYNOMIAL IN MATRICES

Let A be an n-square matrix over a field K. *Powers* of A are defined as follows:

$$A^2 = AA, \qquad A^3 = A^2A, \qquad \ldots, \qquad A^{n+1} = A^nA, \qquad \ldots, \qquad \text{and} \qquad A^0 = I$$

Polynomials in the matrix A are also defined. Specifically, for any polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where the a_i are scalars in K, $f(A)$ is defined to be the following matrix:

$$f(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n$$

[Note that $f(A)$ is obtained from $f(x)$ by substituting the matrix A for the variable x and substituting the scalar matrix a_0I for the scalar a_0.] If $f(A)$ is the zero matrix, then A is called a *zero* or *root* of $f(x)$.

EXAMPLE

Suppose $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$. Then

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} \quad \text{and} \quad A^3 = A^2A = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} -11 & 38 \\ 57 & -106 \end{bmatrix}$$

Suppose $f(x) = 2x^2 - 3x + 5$ and $g(x) = x^2 + 3x - 10$. Then

$$f(A) = 2\begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & -18 \\ -27 & 61 \end{bmatrix}$$

$$g(A) = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, A is a zero of the polynomial $g(x)$.