DOUBLE INTEGRALS

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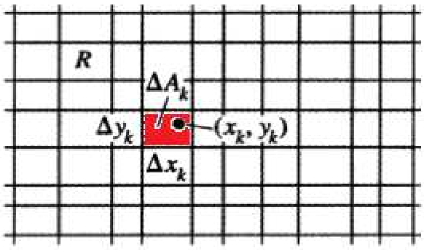
DEPT MATEMATIKA

Double Integrals

Suppose that $f(x, y)$ is defined on a rectangular region R given by

$$R: \quad a \le x \le b, \quad c \le y \le d.$$

We imagine R to be covered by a network of lines parallel to the x- and y-axes



If f is continuous throughout R, then, as we refine

$$\iint_R f(x, y)\, dx\, dy. = \iint_R f(x, y)\, dA = \lim_{\Delta A \to 0} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k.$$

Properties of Double Integrals

1. $$\iint_R kf(x,y)\,dA = k\iint_R f(x,y)\,dA \qquad \text{(any number } k\text{)}$$

2. $$\iint_R (f(x,y) \pm g(x,y))\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

3. $$\iint_R f(x,y)\,dA \ge 0 \quad \text{if} \quad f(x,y) \ge 0 \text{ on } R$$

4. $$\iint_R f(x,y)\,dA \ge \iint_R g(x,y)\,dA \quad \text{if} \quad f(x,y) \ge g(x,y) \text{ on } R$$

5. $$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA.$$

It holds when R is the union of two nonoverlapping rectangles R_1 and R_2

KOMPUTASI UNTUK DOUBLE INTEGRAL

Theorem 1

If $f(x, y)$ is continuous on the rectangular region $R : a \le x \le b,\ c \le y \le d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

Theorem 2

Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b,\ g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then
$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$
2. If R is defined by $c \le y \le d,\ h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then
$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

EXAMPLE $\boxed{\text{Calculate } \iint_R f(x, y)\, dA \text{ for}}$

$$f(x, y) = 1 - 6x^2 y \quad \text{and} \quad R: \; 0 \le x \le 2, \quad -1 \le y \le 1.$$

Solution

$$\iint_R f(x, y)\, dA = \int_{-1}^{1} \int_{0}^{2} (1 - 6x^2 y)\, dx\, dy = \int_{-1}^{1} \left[x - 2x^3 y \right]_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1} (2 - 16y)\, dy = \left[2y - 8y^2 \right]_{-1}^{1} = 4.$$

$$\int_{0}^{2} \int_{-1}^{1} (1 - 6x^2 y)\, dy\, dx = \int_{0}^{2} \left[y - 3x^2 y^2 \right]_{y=-1}^{y=1} dx$$

$$= \int_{0}^{2} \left[(1 - 3x^2) - (-1 - 3x^2) \right] dx = \int_{0}^{2} 2\, dx = 4.$$

EXAMPLE Calculate

$$\iint_R \frac{\sin x}{x}\, dA,$$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.

Solution If we integrate first with respect to y and then with respect to x, we find

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x}\, dy \right) dx = \int_0^1 \left(y \frac{\sin x}{x} \bigg]_{y=0}^{y=x} \right) dx = \int_0^1 \sin x\, dx$$

$$= -\cos(1) + 1 \approx 0.46.$$

If we reverse the order of integration and attempt to calculate

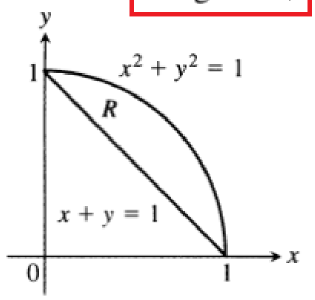
$$\boxed{\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy, \quad ???}$$

Cara menentukan batas integral

1. Buat gambar region R
2. Tentukan bagian atas dan bawah kurvanya

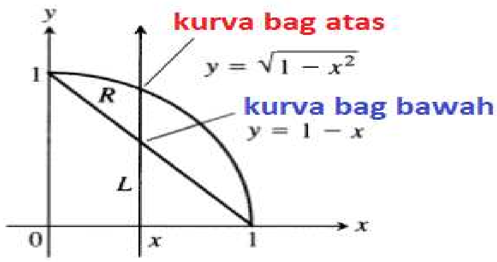
To evaluate $\iint_R f(x, y)\,dA$ ove

$\boxed{\text{a region } R,}$

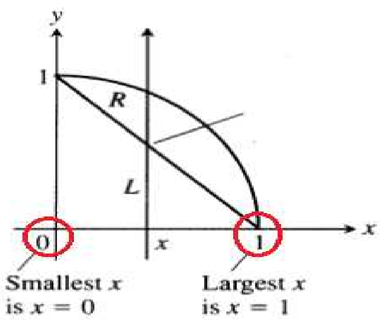


Cara menentukan batas pada integral anda (1)

Integral trhdp dy kemudian dx



1. Buat garis sejajar-Y
2. cari **kurva bag atas**
3. cari **kurva bag bawah**



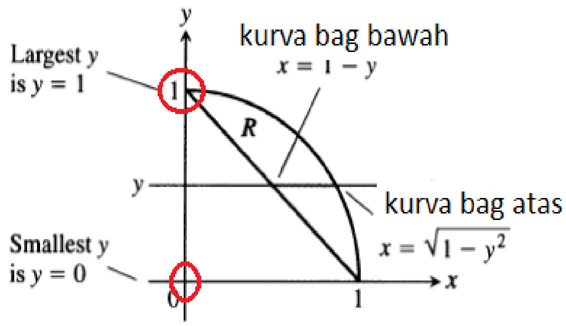
4. Cari titik potong kedua kurva tentukan nilai terkecil dan ter besar utk koordinat - X

$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=1}\int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx.$$

Cara menentukan batas pada integral anda (1)

INTEGRAL TRHDP DX KEMUDIAN DY

1. Buat garis sejajar - X
2. cari kurva bag atas (sbg fs y)
3. cari kurva bag bawah (sbg fs y)
4. cari titik potong kurva atas & kurva bawah. Tentukan nilai terbesar dan terkecil koordinta-Y

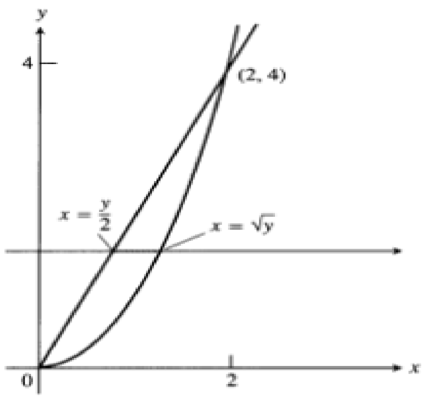


$$\iint_R f(x, y)\, dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\, dx\, dy.$$

EXAMPLE write an equivalent integral with the order of integration reversed.

$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$$

Solution The region of integration is given by the inequalities $x^2 \leq y \leq 2x$ and $0 \leq x \leq 2$. It is therefore the region bounded by the curves $y = x^2$ and $y = 2x$ between $x = 0$ and $x = 2$



The integral is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy.$$

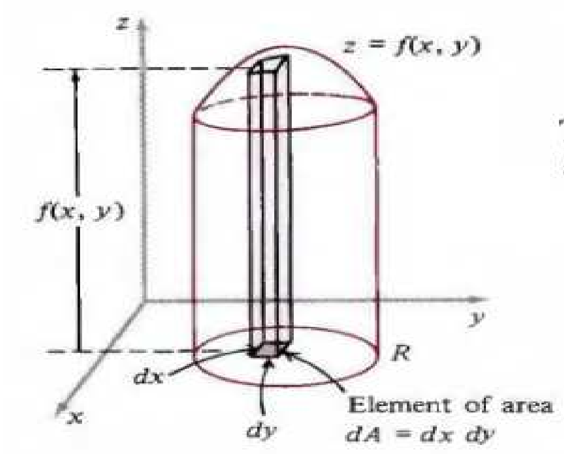
The common value of these integrals is 8.

LATIHAN : TULIS DALAM BENTUK dxdy

$$\boxed{\int_0^1 \int_2^{4-2x} dy\, dx}$$

$$\boxed{\int_0^1 \int_{1-x}^{1-x^2} dy\, dx}$$

VOL. DGN DOUBLE INTEGRAL



The height of this column is $f(x, y)$, so its volume is

$$\boxed{dV = f(x, y)\, dA.}$$

$$\boxed{V = \iint dV = \iint_R f(x, y)\, dA.}$$

EXAMPLE Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

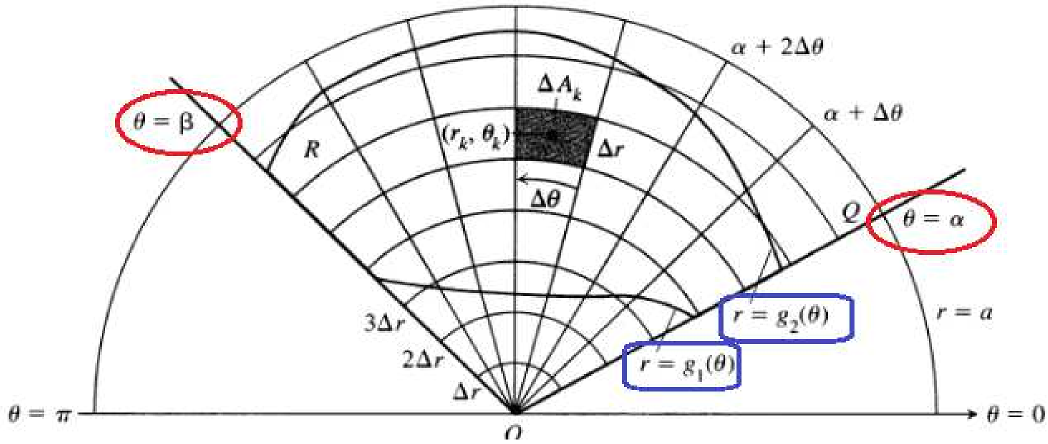
$$z = f(x, y) = 3 - x - y.$$

Solution

For any x between 0 and 1, y may vary from $y = 0$ to $y = x$ Hence,

$$\boxed{\begin{aligned} V &= \int_0^1 \int_0^x (3 - x - y)\,dy\,dx = \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx \\ &= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_{x=0}^{x=1} = 1. \end{aligned}}$$

DOUBLE INT. PADA KOOR POLAR



$$\iint_R f(r,\theta)\,dA = \boxed{\int_{\theta=\alpha}^{\theta=\beta}} \boxed{\int_{r=g_1(\theta)}^{r=g_2(\theta)}} f(r,\theta)\,r\,dr\,d\theta.$$

HUBUNGAN POLAR & KARTESIAN

Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad x^2 + y^2 = r^2, \qquad \frac{y}{x} = \tan\theta \qquad (2)$$

EXAMPLE 4

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$

Changing Cartesian Integrals into Polar Integrals

Step 1: Substitute $x = r\cos\theta$ and $y = r\sin\theta$, and replace $dx\,dy$ by $r\,dr\,d\theta$ in the Cartesian integral.

Step 2: Supply polar limits of integration for the boundary of R.

The Cartesian integral then becomes

$$\iint_R f(x, y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta,$$

EXAMPLE Evaluate

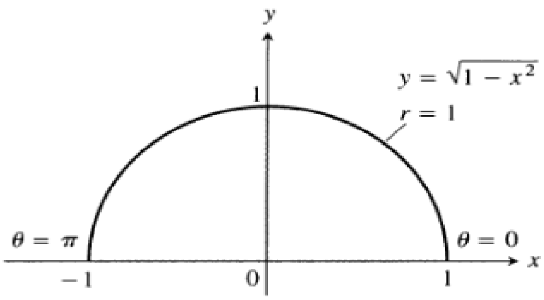
$$\iint_R e^{x^2+y^2}\,dy\,dx,$$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$ (Fig. 13.28).

Solution

$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta$$

$$= \int_0^{\pi}\frac{1}{2}(e-1)\,d\theta = \frac{\pi}{2}(e-1).$$



13.28 $0 \le r \le 1, \quad 0 \le \theta \le \pi.$

Latihan

If R is the region bounded by the lines $y = x$, $y = 0$, $x = 1$, evaluate the double integral

$$\iint_R \frac{dx\,dy}{(1 + x^2 + y^2)^{3/2}}$$

by changing to polar coordinates.

Evaluate the integral

$$\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} (x^2 + y^2)\,dy\,dx$$

by changing to polar coordinates.