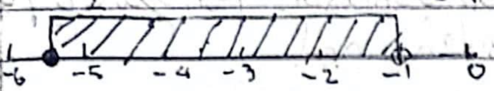
Pembahasan UTS Matematika I 2022/2023

1. a. $-5 \le 2x+6 < 4$

$-5-6 \le 2x < 4-6$

$-11 \le 2x < -2$

$-\frac{11}{2} \le x < -1$ $\{x \mid -\frac{11}{2} \le x < -1, x \in \mathbb{R}\}$



b. $\frac{2x-5}{x-5} \le 1$

$\frac{2x-5}{x-5} - 1 \le 0$

$\frac{2x-5-(x-5)}{x-5} \le 0$

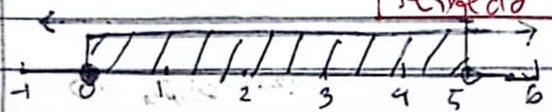
$\frac{2x-x-5+5}{x-5} \le 0$

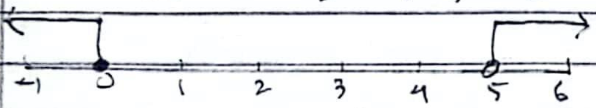
$\frac{x}{x-5} \le 0$

$(x \ge 0$ dan $x-5 < 0)$ atau

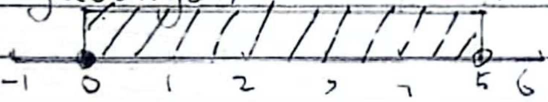
$(x \le 0$ dan $x-5 > 0)$

Note: agar pecahan bernilai ≤ 0, diperlukan pembilang dan penyebut yang berbeda tanda





Gabungan $\{x \mid 0 \le x < 5, x \in \mathbb{R}\}$



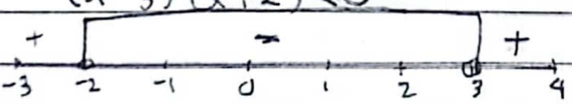
c. $x^2 - x < 6$

$x^2 - x - 6 < 0$

$\pm 1 \ \mp 6$

$\pm 2 \ \mp 3$

$(x-3)(x+2) < 0$



Note: Jika koefisien x^2 bernilai negatif, simbolnya menjadi $- + -$

$\{x \mid -2 < x < 3, x \in \mathbb{R}\}$

d. $x|x| \le |x-2|$

$x|x| - |x-2| \le 0$

$|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$ $\quad |x-2| = \begin{cases} x-2, & x \ge 2 \\ 2-x, & x < 2 \end{cases}$

I) $x < 0$

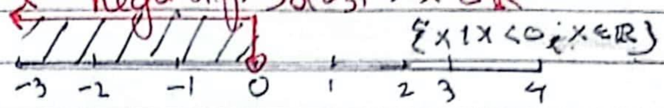
$x(-x) - (2-x) \le 0$

$-x^2 + x - 2 \le 0$

Nilai D : $b^2 - 4ac = 1^2 - 4\cdot(-1)(-2) = -3 <$

yang berarti seluruh nilai x akan menghasilkan f(x) negatif dikarenakan $D < 0$ dan koefisien x^2 negatif. Solusi : $x \in \mathbb{R}$

$\{x \mid x < 0, x \in \mathbb{R}\}$



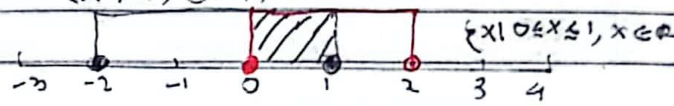
II $0 \le x < 2$

$x \cdot x - (2-x) \le 0$

$x^2 + x - 2 \le 0$

$(x+2)(x-1) \le 0$

$\{x \mid 0 \le x \le 1, x \in \mathbb{R}\}$



III $x \ge 2$

$x \cdot x - (x-2) \le 0$

$x^2 - x + 2 \le 0$

Nilai D : $1^2 - 4\cdot1\cdot2 = -7 < 0$, yang berarti grafik f(x) berada pada ~~sumb~~ positif, sehingga tidak ada solusi yang memenuhi.

Gabungan

$\{x \mid x \le 1, x \in \mathbb{R}\}$

2. a. Daerah Asal $f(x) = \frac{1}{x-3}$

Penyebut suatu pecahan tidak boleh 0

$x - 3 \ne 0$, $x \ne 3$

$D_f = \{x \mid x \ne 3, x \in \mathbb{R}\}$

b. Daerah Asal $g(t) = \sqrt{9-t^2}$

Didalam akar harus ≥ 0

$9 - t^2 \ge 0$

$t^2 - 9 \le 0$

$(t+3)(t-3) \le 0$

$D_f = \{t \mid -3 \le t \le 3, t \in \mathbb{R}\}$

3. Tentukan apakah fungsi genap/ganjil/bukan!

Fungsi Genap : $f(-x) = f(x)$

Fungsi Ganjil : $f(-x) = -f(x)$

a. $f(x) = \frac{x^3+3x}{x^4-3x^2+9}$

$f(-x) = \frac{(-x)^3+3(-x)}{(-x)^4-3(-x)^2+9}$

$= \frac{-x^3-3x}{x^4-3x^2+9}$

$= -\frac{x^3+3x}{x^4-3x^2+9} = -f(x)$

$\boxed{\text{Fungsi Ganjil}}$

b. $\phi(z) = \frac{2z+1}{z-1}$

$\phi(-z) = \frac{2(-z)+1}{(-z)+1}$

$= \frac{-2z+1}{-z+1}$

$\boxed{\text{Bukan fungsi genap dan ganjil}}$

4 a. Epsilon - Delta

Untuk setiap $\varepsilon > 0$, terdapat $\delta > 0$ yang memenuhi

$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$\left|\frac{2x^2-3x-2}{x-2} - 5\right| < \varepsilon$

$\left|\frac{2x^2-3x-2-5(x-2)}{x-2}\right| < \varepsilon$

$\left|\frac{2x^2-8x+8}{x-2}\right| < \varepsilon$

$\left|\frac{2(x-2)^2}{(x-2)}\right| < \varepsilon$

$2|x-2| < \varepsilon$

$|x-2| < \frac{\varepsilon}{2}$ set $\delta = \frac{\varepsilon}{2}$

$|x - x_0| < \delta$

$|x-2| < \frac{\varepsilon}{2}$

$2|x-2| < \varepsilon$

$\left|\frac{2(x-2)^2}{(x-2)}\right| < \varepsilon$

$|f(x) - L| < \varepsilon$ $\boxed{\text{Terbukti}}$

b. $\lim_{x \to 4} \frac{\sqrt{x^2+9}}{x} = \frac{\sqrt{4^2+9}}{4} = \frac{\sqrt{16+9}}{4} = \frac{\sqrt{25}}{4} = \frac{5}{4}$

5. a. $f(x)+g(x) = \sqrt{1+x} + \sqrt{1-x}$

$1+x \geq 0$ dan $1-x \geq 0$

$x \geq -1$ dan $x \leq 1$

$D_f = \{x \mid -1 \leq x \leq 1, x \in \mathbb{R}\}$

$f(x)-g(x) = \sqrt{1+x} - \sqrt{1-x}$

$D_f = \{x \mid -1 \leq x \leq 1, x \in \mathbb{R}\}$

$f(x)g(x) = \sqrt{(1+x)(1-x)}$

$= \sqrt{1-x^2}$

$D_f = \{x \mid -1 \leq x \leq 1, x \in \mathbb{R}\}$

$\frac{f(x)}{g(x)} = \frac{\sqrt{1+x}}{\sqrt{1-x}} = \sqrt{\frac{1+x}{1-x}}$

$D_f = \{x \mid -1 \leq x < 1, x \in \mathbb{R}\}$

Note: 1 tidak termasuk di dalam domain, dikarenakan penyebut pecahan tidak boleh nol

b. $f(x)+g(x) = x - \frac{1}{x} + x^2 + 1$

$D_f = \{x \mid x \neq 0, x \in \mathbb{R}\}$

$f(x)-g(x) = x - \frac{1}{x} - (x^2+1)$

$= x - \frac{1}{x} - x^2 - 1$

$D_f : \{x \mid x \neq 0, x \in \mathbb{R}\}$

$f(x)g(x) = \left(x - \frac{1}{x}\right)(x^2+1)$

$= x^3 + x - x - \frac{1}{x}$

$= x^3 - \frac{1}{x}$

$D_f : \{x \mid x \neq 0, x \in \mathbb{R}\}$

$\frac{f(x)}{g(x)} = \frac{x - \frac{1}{x}}{x^2+1}$

$= \frac{x^2-1}{x(x^2+1)}$ x^2+1 selalu > 0

$D_f : \{x \mid x \neq 0, x \in \mathbb{R}\}$

6. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

a. $f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - 2(x+h) - (x^2-2x)}{h}$

$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{2x} - 2h - \cancel{x^2} + \cancel{2x}}{h}$

$= \lim_{h \to 0} \frac{2xh + h^2 - 2h}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(2x+h-2)}{\cancel{h}}$

$= \lim_{h \to 0} 2x + h - 2$

$= 2x - 2$

$f'(2) = 2 \cdot 2 - 2$

$= 4 - 2$

$= 2$

b. $f'(t) = \lim_{h \to 0} \frac{\frac{1}{2}(t+h)^2 + 1 - (\frac{1}{2}t^2 + 1)}{h}$

$= \lim_{h \to 0} \frac{\frac{1}{2}(t^2 + 2th + h^2) + \cancel{1} - \frac{1}{2}t^2 - \cancel{1}}{h}$

$= \lim_{h \to 0} \frac{\cancel{\frac{1}{2}t^2} + th + \frac{1}{2}h^2 - \cancel{\frac{1}{2}t^2}}{h}$

$= \lim_{h \to 0} \frac{th + \frac{1}{2}h^2}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(t + \frac{1}{2}h)}{\cancel{h}}$

$= \lim_{h \to 0} t + \frac{1}{2}h$

$= t$

$f'(2) = 2.$

7 a. $\frac{d}{dx}(y^3+7y-x^3) = \frac{d}{dx}(0)$

$3y^2 \cdot \frac{dy}{dx} + 7 \cdot \frac{dy}{dx} - 3x^2 = 0$

$(3y^2+7)\frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{3x^2}{3y^2+7}$

$\frac{d^2}{dx^2}(y^3+7y-x^3) = \frac{d}{dx}(0)$

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3x^2}{3y^2+7}\right) = 0$ $\qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v-uv'}{v^2}$

$\frac{d^2y}{dx^2} \frac{6x(3y^2+7)-3x^2\left(6y\frac{dy}{dx}\right)}{(3y^2+7)^2}$

$\frac{d^2y}{dx^2} = \frac{6x(3y^2+7)-18x^2\left(\frac{3x^2}{3y^2+7}\right)}{(3y^2+7)^2}$

$\frac{d^2y}{dx^2} = \frac{6x(3y^2+7)^2-54x^4}{(3y^2+7)^3}$

b. $\frac{d}{dx}(x^3y^4-1) = \frac{d}{dx}(0)$

$3x^2y^4 + x^3 \cdot 4y^3 \cdot \frac{dy}{dx} = 0$

$4x^3y^3\frac{dy}{dx} = -3x^2y^4$

$\frac{dy}{dx} = -\frac{3x^2y^4}{4x^3y^3}$

$\frac{dy}{dx} = -\frac{3y}{4x}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{3y}{4x}\right)$

$= -\frac{3}{4}\left(\frac{\frac{dy}{dx}\cdot x - y\cdot(1)}{x^2}\right)$

$= -\frac{3}{4}\left(\frac{-\frac{3y}{4x}x - y}{x^2}\right)$

$= -\frac{3}{4}\left(\frac{-3y-4y}{4x^2}\right)$

$= -\frac{3}{16}\left(\frac{-7y}{x^2}\right)$

c. $\frac{d}{dx}(y) = \frac{d}{dx}\left(\sqrt{\sin(xy^2)}\right)$

$\frac{dy}{dx} = \frac{1}{2\sqrt{\sin(xy^2)}}\cos(xy^2)\left(1\cdot y^2 + x\cdot 2y\cdot\frac{dy}{dx}\right)$

$\frac{2\sqrt{\sin(xy^2)}}{\cos(xy^2)}(1-2xy)\frac{dy}{dx} = y^2$

$\frac{dy}{dx} = \frac{\cos(xy^2)\cdot y^2}{2\sqrt{\sin(xy^2)}(1-2xy)}$

$\frac{d^2y}{dx} = \frac{d}{dx}\left(\frac{\cos(xy^2)\cdot y^2}{2\sqrt{\sin(xy^2)}(1-2xy)}\right)$

$= \frac{\left(-\sin(xy^2)\left(y^2+2xy\frac{dy}{dx}\right)\cdot y^2 + \cos(xy^2)\cdot 2y\cdot\frac{dy}{dx}\right)\left(2\sqrt{\sin(xy^2)}(1-2xy)\right) - \cos(xy^2)\cdot y^2\left(\frac{\cos(xy^2)y^2}{2\sqrt{\sin(xy^2)}(1-2xy)}(1-2xy) + 2\sqrt{\sin(xy^2)}\left(1-2y-2x\frac{dy}{dx}\right)\right)}{\left(2\sqrt{\sin(xy^2)}(1-2xy)\right)^2}$

8. a. $f(x) = x^3 + x$

$f(-x) = (-x)^3 + (-x)$

$= -x^3 - x$

$= -(x^3+x) = -f(x)$ $\boxed{\text{Ganjil}}$

b. $f(x) = |x|$

$f(-x) = |-x|$

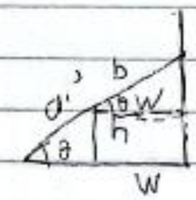
$= |x| = f(x)$ $\boxed{\text{Genap}}$

c. $f(x) = x + \cos(x)$

$f(-x) = (-x) + \cos(-x)$

$= -x + \cos(x)$ $\boxed{\text{Bukan Keduanya}}$

9.



Note:
$\sin = \frac{de}{mi}$ $\qquad \cos = \frac{sa}{mi}$

$\sin\theta = \frac{h}{a}$ $\qquad \cos\theta = \frac{w}{b}$

$a = \frac{h}{\sin\theta}$ $\qquad b = \frac{w}{\cos\theta}$

$P(\theta) = \frac{h}{\sin\theta} + \frac{w}{\cos\theta} = h(\sin\theta)^{-1} + w(\cos\theta)^{-1}$

Note:
Untuk mendapatkan minimum/maksimum, turunan pertama fungsi sama dengan nol.

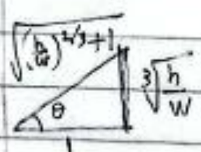
$P'(\theta) = h\left(-\frac{\cos\theta}{\sin^2\theta}\right) + w\left(-\frac{-\sin\theta}{\cos^2\theta}\right) = 0$

$w\frac{\sin\theta}{\cos^2\theta} = h\frac{\cos\theta}{\sin^2\theta}$

$\frac{h}{w} = \frac{\sin^3\theta}{\cos^3\theta}$

$\frac{h}{w} = \tan^3\theta$

$\tan\theta = \sqrt[3]{\frac{h}{w}} \rightarrow \theta = \tan^{-1}\left(\sqrt[3]{\frac{h}{w}}\right)$



$\sin\theta = \frac{\sqrt[3]{\frac{h}{w}}}{\sqrt{\left(\frac{h}{w}\right)^{2/3}+1}}$

$\cos\theta = \frac{1}{\sqrt{\left(\frac{h}{w}\right)^{2/3}+1}}$

$P\left(\tan^{-1}\left(\sqrt[3]{\frac{h}{w}}\right)\right) = \boxed{\left(\sqrt{\left(\frac{h}{w}\right)^{2/3}+1}\right)\left(\frac{h}{\sqrt[3]{\frac{h}{w}}} + w\right)}$

10 Note:

Fungsi Naik : $f'(x) > 0$

Fungsi Turun : $f'(x) < 0$

Fungsi Cekung ke atas: $f''(x) > 0$

Fungsi Cekung ke Bawah: $f''(x) < 0$

$f(x) = x^3 - 9x^2 - 15x - 5$

$f'(x) = 3x^2 - 18x - 15$

$f''(x) = 6x - 18$

$3x^2 - 18x - 15 = 0$

$x^2 - 6x - 5 = 0$

$$x_{1,2} = \frac{6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot (-5)}}{2}$$

$$= \frac{6 \pm \sqrt{36 + 20}}{2}$$

$$= \frac{6 \pm \sqrt{56}}{2}$$

$$= 3 \pm \sqrt{14}$$

$+ \quad 3-\sqrt{14} \quad - \quad 3+\sqrt{14} \quad +$

Fungsi Naik : $\{x \mid x < 3-\sqrt{14}$ dan $x > 3+\sqrt{14}, x \in \mathbb{R}\}$

Fungsi Turun : $\{x \mid 3-\sqrt{14} < x < 3+\sqrt{14}, x \in \mathbb{R}\}$

$6x - 18 = 0$

$6x = 18$

$x = 3$

$- \quad 3 \quad +$

Fungsi Cekung ke atas : $\{x \mid x > 3, x \in \mathbb{R}\}$

Fungsi Cekung ke Bawah: $\{x \mid x < 3, x \in \mathbb{R}\}$