Normal Forms & Propositional Resolution

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Resolution Rule

$$\frac{A \lor C,\ B \lor \neg C}{A \lor B}$$

- dengan A, B, C merupakan formula proposisi
- Formula $A \lor B$ disebut **resolvent dari** $A \lor C$ dan $B \lor \neg C$,
- Dan ditulis $A \lor B = \text{Res}(A \lor C,\ B \lor \neg C)$
- Berikan contohnya!

TERMINOLOGI (Definisi istilah)

1. *A **literal** is a propositional variable or its negation.*

2. *An **elementary disjunction (respectively elementary conjunction**) is any literal or a disjunction (respectively conjunction) of two or more literals.*

3. *A **disjunctive normal form** (DNF) is any elementary conjunction or a disjunction of two or more elementary conjunctions.*

4. *A **conjunctive normal form** (CNF) is any elementary disjunction or a conjunction of two or more elementary disjunctions.*

Contoh:

- p, $\neg q$, $p \vee \neg q$, $p \vee \neg p \vee q \vee \neg r$ are elementary disjunctions;
- p, $\neg q$, $\neg p \wedge q$, $\neg p \wedge q \wedge \neg r \wedge \neg p$ are elementary conjunctions;
- p, $\neg q$, $p \wedge \neg q$, $p \vee \neg q$, $(p \wedge \neg p) \vee \neg q$, $(r \wedge q \wedge \neg p) \vee (\neg q \wedge p) \vee (\neg r \wedge p)$ are disjunctive normal forms;
- p, $\neg q$, $p \wedge \neg q$, $p \vee \neg q$, $p \wedge (\neg p \vee \neg q)$, $(r \vee q \vee \neg r) \wedge \neg q \wedge (\neg p \vee r)$ are conjunctive normal forms

Cara Membentuk Equivalent Normal Form

- **Method 1** *This method is based on the following algorithm, which transforms any formula into a DNF, respectively CNF*

- **Method 2** *This second method constructs the normal forms directly from the truth table of the given formula. We will outline it for DNF.*

Cara membentuk equivalent normal form

- Proses dalam metode diatas, formula proposisi dapat disederhakan menggunakan hukum-hukum logika yang ada.

$$p \vee \neg p \equiv \top, p \wedge \neg p \equiv \bot, , ; p \wedge \top \equiv p, p \wedge \bot \equiv \bot; p \vee \top \equiv \top, p \vee \bot \equiv p.$$

Metode 1

1. Eliminate all occurrences of ↔ and → using the logical equivalences

- $A \rightarrow B \equiv \neg A \vee B$,
- $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$.

2. Import all negations to stand directly in front of the propositional variables, using the logical equivalences listed above.

3. For DNF: distribute all conjunctions over disjunctions using the distributive law: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

4. Respectively, for CNF: distribute all disjunctions over conjunctions using the other distributive law: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

Contoh :

$(p \wedge \neg r) \rightarrow (p \leftrightarrow \neg q)$

$$\equiv \neg(p \wedge \neg r) \vee ((p \rightarrow \neg q) \wedge (\neg q \rightarrow p))$$
$$\equiv (\neg p \vee \neg\neg r) \vee ((\neg p \vee \neg q) \wedge (\neg\neg q \vee p))$$
$$\equiv \neg p \vee r \vee ((\neg p \vee \neg q) \wedge (q \vee p))$$

For a DNF we further distribute $\wedge$ over $\vee$ and simplify:

$$\equiv \neg p \vee r \vee (((\neg p \vee \neg q) \wedge q) \vee ((\neg p \vee \neg q) \wedge p))$$
$$\equiv \neg p \vee r \vee ((\neg p \wedge q) \vee (\neg q \wedge q)) \vee ((\neg p \wedge p) \vee (\neg q \wedge p))$$
$$\equiv \neg p \vee r \vee ((\neg p \wedge q) \vee \bot) \vee (\bot \vee (\neg q \wedge p))$$
$$\equiv \neg p \vee r \vee (\neg p \wedge q) \vee (\neg q \wedge p).$$

For a CNF we distribute $\vee$ over $\wedge$ and simplify:

$$\equiv (\neg p \vee r \vee \neg p \vee \neg q) \wedge (\neg p \vee r \vee q \vee p)$$
$$\equiv (\neg p \vee r \vee \neg q) \wedge (\top \vee r \vee q)$$
$$\equiv (\neg p \vee r \vee \neg q) \wedge \top$$
$$\equiv \neg p \vee r \vee \neg q.$$

Contoh Metode 2

The formula $p \leftrightarrow \neg q$ has a truth table

p	q	$p \leftrightarrow \neg q$
T	T	F
T	F	T
F	T	T
F	F	F

The corresponding DNF is $(p \land \neg q) \lor (\neg p \land q)$. Check this!

Latihan 1

Bentuk ke DNF atau CNF

(a) $\neg(p \leftrightarrow q)$

(b) $((p \rightarrow q) \wedge \neg q) \rightarrow p$

(c) $(p \leftrightarrow \neg q) \leftrightarrow r$

(d) $p \rightarrow (\neg q \leftrightarrow r)$

(e) $(\neg p \wedge (\neg q \leftrightarrow p)) \rightarrow ((q \wedge \neg p) \vee p)$

Clausal Form

1. *A clause is essentially an elementary disjunction* $l1 \lor \cdots \lor ln$ *but written as a set of literals* $\{l1, \ldots, ln\}$.

2. *The empty clause is a clause containing no literals; a unit clause is a clause containing only one literal.*

3. *A clausal form is a (possibly empty) set of clauses, written as a list:* $C1 \cdots Ck$. *It represents the conjunction of these clauses.*

Clausal Form

- Every CNF can be rewritten in a clausal form, and therefore every propositional formula is equivalent to one in a clausal form.

- For instance, the clausal form of the CNF formula $(p \lor \neg q \lor \neg r) \land \neg p \land (\neg q \lor r)$ *is* $\{p, \neg q, \neg r\}\{\neg p\}\{\neg q, r\}$.

- The Resolution Rule can be rewritten for clauses as follows:

$$\frac{\{A_1, \ldots, C, \ldots, A_m\}\{B_1, \ldots, \neg C, \ldots, B_n\}}{\{A_1, \ldots, A_m, B_1, \ldots, B_n\}}.$$

- The clause {A1,…, Am, B1,…, Bn} is a **resolvent of** the clauses {A1,…, C,…, Am} and {B1,…, ¬C,…, Bn}.

Contoh

Example 3.6.7

Some examples of applications of clausal resolution:

$$\frac{\{p, q, \neg r\}\{\neg q, \neg r\}}{\{p, \neg r\}},$$

$$\frac{\{\neg p, q, \neg r\}\{r\}}{\{\neg p, q\}},$$

$$\frac{\{\neg p\}\{p\}}{\{\,\}}.$$

Contoh

- Terdapat dua clauses yang dapat mempunyai lebih dari satu resolvents, misal:

$$\frac{\{p, \neg q\}\{\neg p, q\}}{\{p, \neg p\}}, \quad \frac{\{p, \neg q\}\{\neg p, q\}}{\{\neg q, q\}}.$$

- Namun tidak berarti :

$$\frac{\{p, \neg q\}\{\neg p, q\}}{\{\}}.$$

Resolution – based Derivations

- The idea behind the method of resolution is similar to that of semantic tableaux.

- In order to determine whether a logical consequence $A1, \ldots, An \models B$ holds using the method of resolution, we negate the conclusion B and transform each of the formulae A1,…, An, $\neg$B into clausal form.

- Then we test whether the resulting set of clauses is unsatisfiable, by looking for a resolution-based derivation of the empty clause from that set of clauses.

Definisi

- *A **resolution-based derivation** of a formula B from a list of formulae A1, …, An is a derivation of the empty clause {} from the set of clauses obtained from the formulae A1, …, An, ¬B, by successive applications of the Rule of Propositional Resolution.*

Contoh

Example 3.6.9

Using Propositional Resolution, check the logical consequence $p \to q, q \to r \models p \to r$.

Solution

First, we transform each of $p \to q, q \to r, \neg(p \to r)$ to clausal form. The resulting set of clauses is:

$$C_1 = \{\neg p, q\}, C_2 = \{\neg q, r\}, C_3 = \{p\}, C_4 = \{\neg r\}.$$

Now we apply the Resolution Rule successively:

$$C_5 = \{q\} = Res(C_1, C_3);$$
$$C_6 = \{r\} = Res(C_2, C_5);$$
$$C_6 = \{\} = Res(C_4, C_6).$$

The derivation of the empty clause completes the proof that $p \to q, q \to r \models p \to r$.

- Catatan :
  - $p \to q \equiv \neg p \vee q$
  - $q \to r \equiv \neg q \vee r$
  - $\neg (p \to r) \equiv \neg (\neg p \vee r) \equiv \neg \neg p \wedge \neg r \equiv p \wedge \neg r$

Contoh

Example 3.6.10

Check whether $(\neg p \to q), \neg r \models p \vee (\neg q \wedge \neg r)$ holds.

Solution
First transform $(\neg p \to q), \neg r, \neg(p \vee (\neg q \wedge \neg r))$ to clausal form:

$$C_1 = \{p, q\}, C_2 = \{\neg r\}, C_3 = \{\neg p\}, C_4 = \{q, r\}.$$

Now, applying resolution successively:

$$C_5 = Res(C_1, C_3) = \{q\};$$
$$C_6 = Res(C_2, C_4) = \{q\}.$$

At this stage, no new applications of the Propositional Resolution Rule are possible hence the empty clause is not derivable. Therefore, $(\neg p \to q), \neg r \not\models p \vee (\neg q \wedge \neg r)$.

- Catatan :
  - $(\neg p \to q) \equiv \neg\neg p \vee q \equiv p \vee q$
  - $\neg (p \vee (\neg q \wedge \neg r) \equiv \neg p \wedge \neg (\neg q \wedge \neg r) \equiv \neg p \wedge (\neg\neg q \vee \neg\neg r)\ ) \equiv \neg p \wedge (q \vee r)$

Latihan 2

- Using the method of resolution check which of the following formulae are tautologies.

(a) $((p \rightarrow q) \rightarrow q) \rightarrow q$

(b) $((p \rightarrow q) \wedge (p \rightarrow \neg q)) \rightarrow \neg p$

(c) $((p \vee q) \rightarrow \neg r) \rightarrow \neg(\neg q \wedge r)$

(d) $((p \rightarrow q) \wedge (p \rightarrow r)) \rightarrow (p \rightarrow (q \wedge r))$

Latihan 3

- Using the method of resolution check the following logical consequences.

(a) $\neg p \to q,\ \neg p \to \neg q \vDash p$

(b) $p \to r,\ q \to r \vDash (p \vee q) \to r$

(c) $(p \wedge q) \to r \vDash (q \to r) \vee (p \to r)$