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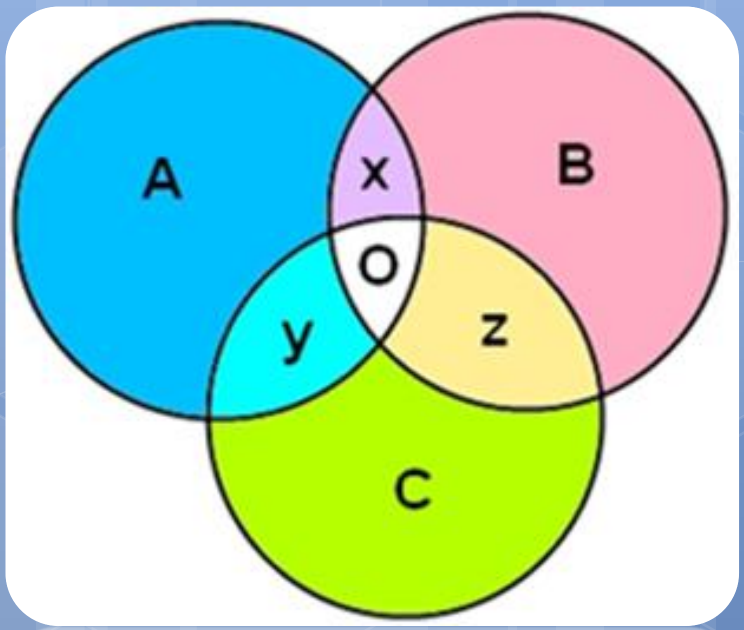
KS091201
MATEMATIKA DISKRIT (DISCRETE MATHEMATICS)



Discrete Basic Structure: Sets

Discrete Math Team

Outline

- What is a set?
- Set properties
- Specifying a set
- Often used sets
- The universal set
- Venn diagrams
- Sets of sets
- The empty set
- Set equality
- Subsets and Proper subsets
- Set cardinality
- Power sets
- Tuples
- Cartesian products
- Sets operation:
  - Union
  - Intersection
  - Disjoint
  - Difference
  - Symmtric difference
  - Complement
  - Set Identities
  - How to proof set identities

What is a set?

- A set is a group of "objects"
  - People in a class: { Alice, Bob, Chris }
  - Colors of a rainbow: { red, orange, yellow, green, blue, purple }
  - States of matter { solid, liquid, gas, plasma }
  - States in the US: { Alabama, Alaska, Virginia, … }
  - Sets can contain non-related elements: { 3, a, red, Virginia }
- Although a set can contain (almost) anything, we will most often use sets of numbers
  - All positive numbers less than or equal to 5: {1, 2, 3, 4, 5}
  - A few selected real numbers: { 2.1, π, 0, -6.32, e }

Set properties

- Order does not matter
  - We often write them in order because it is easier for humans to understand it that way
  - {1, 2, 3, 4, 5} is equivalent to {3, 5, 2, 4, 1}
- Sets are notated with curly brackets { }
- Sets do not have duplicate elements
  - Consider the set of vowels in the alphabet.
    - It makes no sense to list them as {a, a, a, e, i, o, o, o, o, o, u}
    - What we really want is just {a, e, i, o, u}
  - Consider the list of students in this class
    - Again, it does not make sense to list somebody twice
- Note that a list is like a set, but order does matter and duplicate elements are allowed
  - We won't be studying lists much in this class

Specifying a set

- Sets are usually represented by a capital letter (A, B, S, etc.)
- Elements are usually represented by an italic lower-case letter (*a*, x, *y*, etc.)
- Easiest way to specify a set is to list all the elements: A = {1, 2, 3, 4, 5}
  - Not always feasible for large or infinite sets
- Can use an ellipsis (…) when general pattern of the elements is obvious: B = {0, 1, 2, 3, …}
  - Can cause confusion.
    - Consider the set C = {3, 5, 7, …} What comes next?
    - If the set is all odd integers greater than 2, it is 9
    - If the set is all prime numbers greater than 2, it is 11

Specifying a set (cont.)

- Can use set-builder notation
  - D = {x | x is prime and x > 2}
  - E = {x | x is odd and x> 2}
  - The vertical bar means "such that"
  - Thus, set D is read (in English) as: "all elements x such that x is prime and x is greater than 2"
- A set is said to "contain" the various "members" or "elements" that make up the set
  - If an element x is a member of (or an element of) a set S, we use then notation $x \in S$
    - $4 \in \{1, 2, 3, 4\}$
  - If an element is not a member of (or an element of) a set S, we use the notation $x \notin S$
    - $7 \notin \{1, 2, 3, 4\}$
    - Virginia $\notin \{1, 2, 3, 4\}$

Often used sets

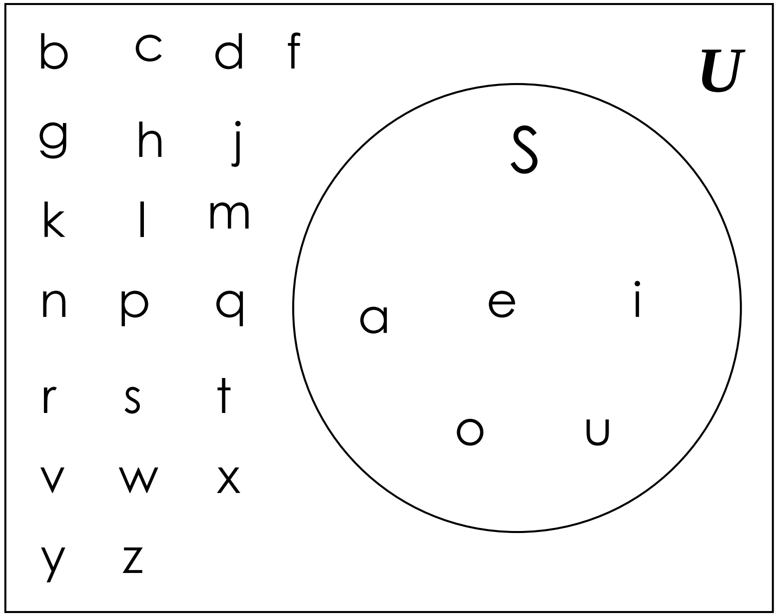
- **N** = {0, 1, 2, 3, …} is the set of natural numbers
- **Z** = {…, -2, -1, 0, 1, 2, …} is the set of integers
- $\mathbf{Z}^+$ = {1, 2, 3, …} is the set of positive integers (a.k.a whole numbers)
  - Note that people disagree on the exact definitions of whole numbers and natural numbers
- **Q** = $\{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0\}$ is the set of rational numbers
  - Any number that can be expressed as a fraction of two integers (where the bottom one is not zero)
- **R** is the set of real numbers

The universal set

- ***U*** is the universal set – the set of all of elements (or the "universe") from which given any set is drawn
  - For the set {-2, 0.4, 2}, ***U*** would be the real numbers
  - For the set {0, 1, 2}, ***U*** could be the natural numbers (zero and up), the integers, the rational numbers, or the real numbers, depending on the context
  - For the set of the students in this class, ***U*** would be all the students in the University (or perhaps all the people in the world)
  - For the set of the vowels of the alphabet, ***U*** would be all the letters of the alphabet
  - To differentiate ***U*** from $\cup$ (which is a set operation), the universal set is written in a different font (and in bold and italics)

Venn diagrams

- Represents sets graphically
  - The box represents the universal set
  - Circles represent the set(s)
- Consider set *S*, which is the set of all vowels in the alphabet
- The individual elements are usually not written in a Venn diagram



Sets of sets

- Sets can contain other sets
  - $S = \{ \{1\}, \{2\}, \{3\} \}$
  - $T = \{ \{1\}, \{\{2\}\}, \{\{\{3\}\}\} \}$
  - $V = \{ \{\{1\}, \{\{2\}\}\}, \{\{\{3\}\}\}, \{\{1\}, \{\{2\}\}, \{\{\{3\}\}\}\} \}$
    - V has only 3 elements!
- Note that $1 \neq \{1\} \neq \{\{1\}\} \neq \{\{\{1\}\}\}$
  - They are all different

The empty set

- If a set has zero elements, it is called the empty (or null) set
  - Written using the symbol ∅
  - Thus, ∅ = { } ← **VERY IMPORTANT**
  - If you get confused about the empty set in a problem, try replacing ∅ by { }
- As the empty set is a set, it can be an element of other sets
  - { ∅, 1, 2, 3, x } is a valid set
- Note that ∅ ≠ { ∅ }
  - The first is a set of zero elements
  - The second is a set of 1 element (that one element being the empty set)
- Replace ∅ by { }, and you get: { } ≠ { { } }
  - It's easier to see that they are not equal that way

Set equality

- Two sets are equal if they have the same elements
  - $\{1, 2, 3, 4, 5\} = \{5, 4, 3, 2, 1\}$
    - Remember that order does not matter!
  - $\{1, 2, 3, 2, 4, 3, 2, 1\} = \{4, 3, 2, 1\}$
    - Since duplicate elements are not allowed!
- Two sets are not equal if they do not have the same elements
  - $\{1, 2, 3, 4, 5\} \neq \{1, 2, 3, 4\}$

Subsets

- If all the elements of a set S are also elements of a set T, then S is a subset of T
  - For example, if S = {2, 4, 6} and T = {1, 2, 3, 4, 5, 6, 7}, then S is a subset of T
  - This is specified by $S \subseteq T$
    - Or by $\{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6, 7\}$
- If S is not a subset of T, it is written as such: $S \not\subseteq T$
  - For example, $\{1, 2, 8\} \not\subseteq \{1, 2, 3, 4, 5, 6, 7\}$
- Note that any set is a subset of itself!
  - Given set S = {2, 4, 6}, since all the elements of S are elements of S, S is a subset of itself
  - This is kind of like saying 5 is less than or equal to 5
  - Thus, for any set S, $S \subseteq S$

Subsets (cont.)

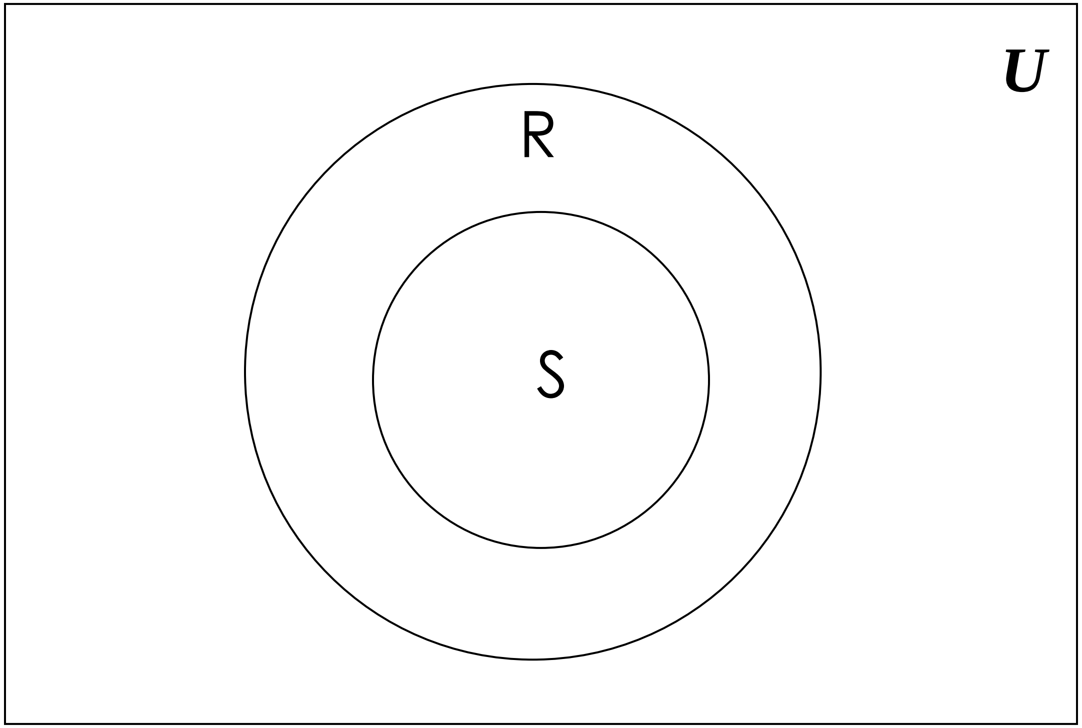
- The empty set is a subset of *all* sets (including itself!)
  - Recall that all sets are subsets of themselves
- *All* sets are subsets of the universal set
- A horrible way to define a subset:
  - $\forall x \, (x \in A \rightarrow x \in B)$
  - English translation: for all possible values of x, (meaning for all possible elements of a set), if x is an element of A, then x is an element of B
  - This type of notation will be gone over later

Proper Subsets

- If S is a subset of T, and S is not equal to T, then S is a proper subset of T
  - Let T = {0, 1, 2, 3, 4, 5}
  - If S = {1, 2, 3}, S is not equal to T, and S is a subset of T
  - A proper subset is written as $S \subset T$
  - Let R = {0, 1, 2, 3, 4, 5}. R is equal to T, and thus is a subset (but not a proper subset) of T
    - Can be written as: $R \subseteq T$ and $R \not\subset T$ (or just R = T)
  - Let Q = {4, 5, 6}. Q is neither a subset or T nor a proper subset of T
- The difference between "subset" and "proper subset" is like the difference between "less than or equal to" and "less than" for numbers
- The empty set is a proper subset of all sets other than the empty set (as it is equal to the empty set)

Proper subsets: Venn diagram

$$S \subset R$$



Set cardinality

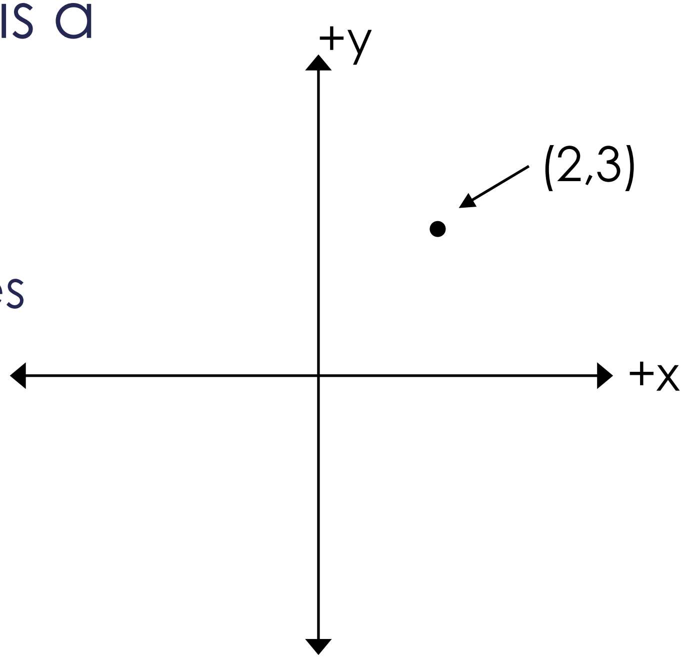
- The cardinality of a set is the number of elements in a set
  - Written as $|A|$
- Examples
  - Let $R = \{1, 2, 3, 4, 5\}$. Then $|R| = 5$
  - $|\varnothing| = 0$
  - Let $S = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. Then $|S| = 4$
- This is the same notation used for vector length in geometry
- A set with one element is sometimes called a singleton set

Power sets

- Given the set S = {0, 1}. What are all the possible subsets of S?
  - They are: ∅ (as it is a subset of all sets), {0}, {1}, and {0, 1}
  - The power set of S (written as P(S)) is the set of all the subsets of S
  - P(S) = { ∅, {0}, {1}, {0,1} }
    - Note that |S| = 2 and |P(S)| = 4
- Let T = {0, 1, 2}. The P(T) = { ∅, {0}, {1}, {2}, {0,1}, {0,2}, {1,2}, {0,1,2} }
  - Note that |T| = 3 and |P(T)| = 8
- P(∅) = { ∅ }
  - Note that |∅| = 0 and |P(∅)| = 1
- If a set has *n* elements, then the power set will have 2^n elements

Tuples

- In 2-dimensional space, it is a (x, *y*) pair of numbers to specify a location
- In 3-dimensional (1,2,3) is not the same as (3,2,1) – space, it is a (x, *y*, z) triple of numbers
- In *n*-dimensional space, it is a *n*-tuple of numbers
  - Two-dimensional space uses pairs, or 2-tuples
  - Three-dimensional space uses triples, or 3-tuples
- Note that these tuples are **ordered**, unlike sets
  - the x value has to come first



Cartesian products

- A Cartesian product is a set of all ordered n-tuples where each "part" is from a given set
  - Denoted by A x B, and uses parenthesis (not curly brackets)
  - For example, 2-D Cartesian coordinates are the set of all ordered pairs **Z** x **Z**
    - Recall **Z** is the set of all integers
    - This is all the possible coordinates in 2-D space
  - Example: Given A = { a, b } and B = { 0, 1 }, what is their Cartiesian product?
    - C = A x B = { (a,0), (a,1), (b,0), (b,1) }
- Note that Cartesian products have only 2 parts in these examples (later examples have more parts)
- Formal definition of a Cartesian product:
  - $A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$

Cartesian products (cont.)

- All the possible grades in this class will be a Cartesian product of the set S of all the students in this class and the set G of all possible grades
  - Let S = { Alice, Bob, Chris } and G = { A, B, C }
  - D = { (Alice, A), (Alice, B), (Alice, C), (Bob, A), (Bob, B), (Bob, C), (Chris, A), (Chris, B), (Chris, C) }
  - The final grades will be a subset of this: { (Alice, C), (Bob, B), (Chris, A) }
    - Such a subset of a Cartesian product is called a relation (more on this later in the course)
- There can be Cartesian products on more than two sets
- A 3-D coordinate is an element from the Cartesian product of Z x Z x Z

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Sets Operation

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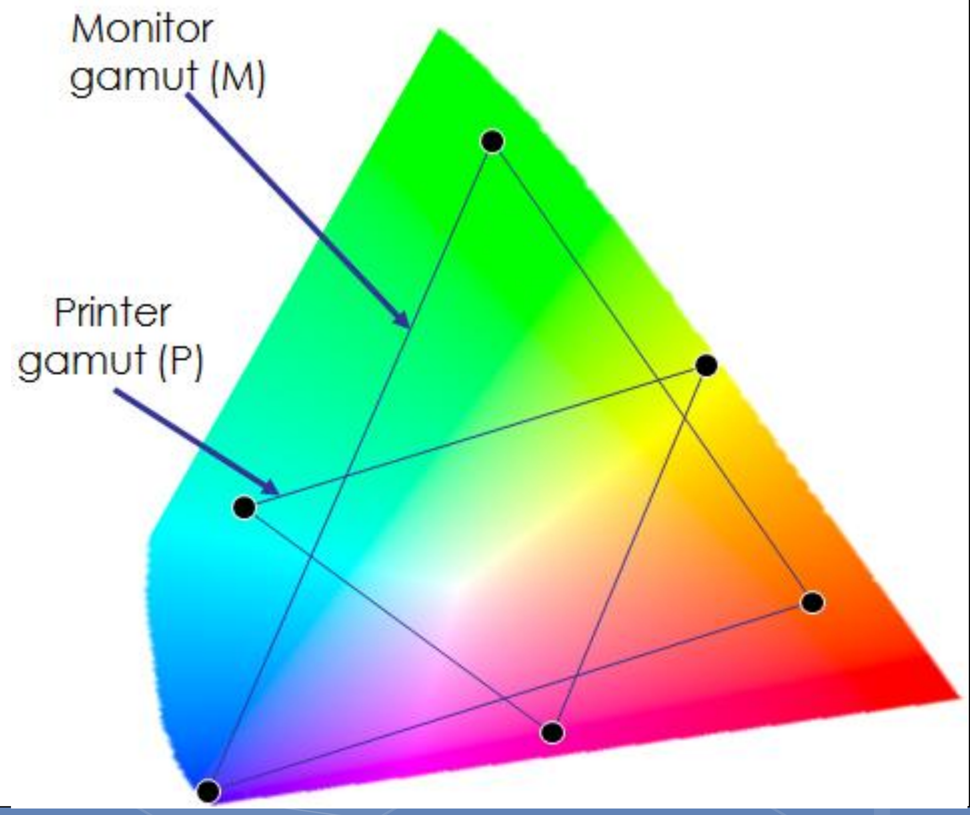
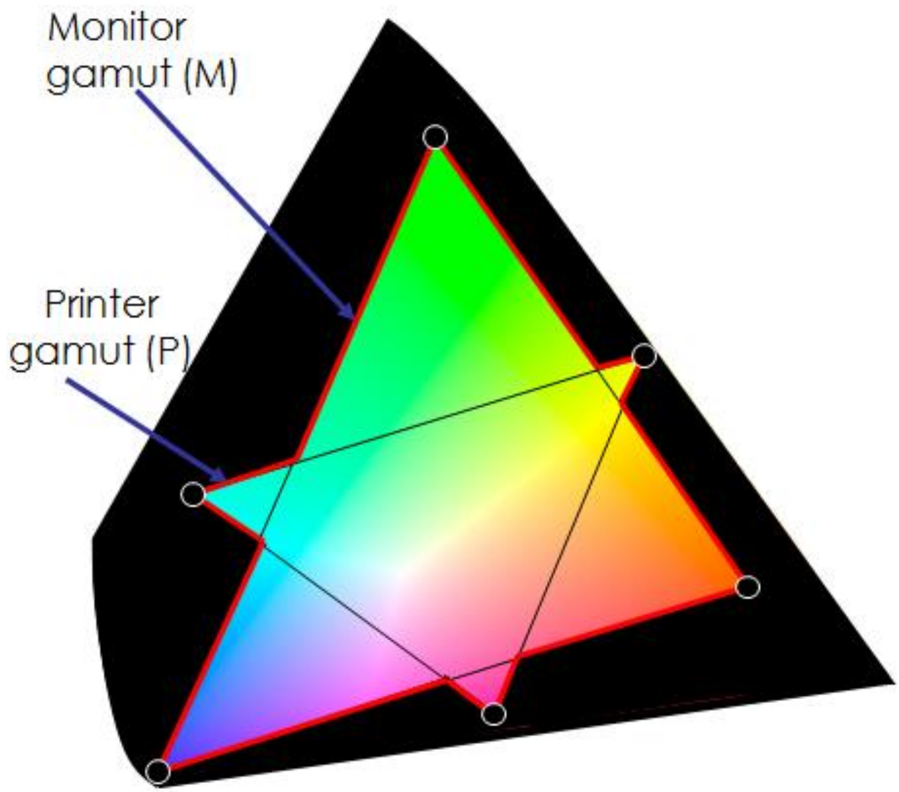
Sets of Colors

- Pick any 3 "primary" colors
- Triangle shows mixable color range (gamut) – the set of colors



Set operations: Union (Gabungan)

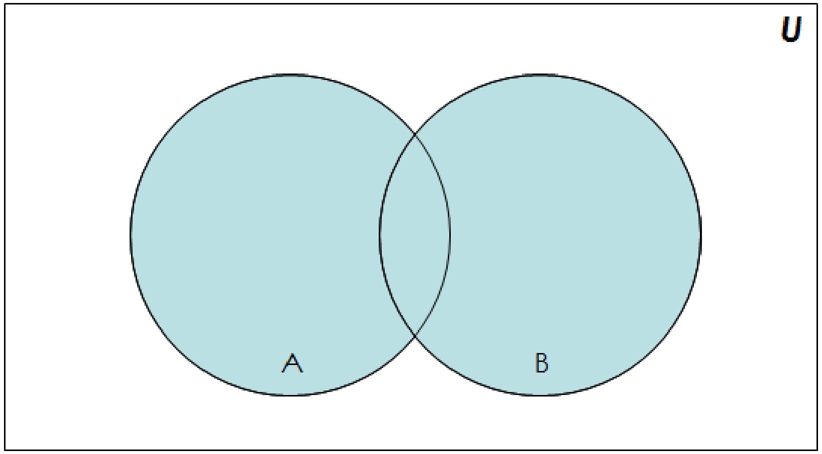
- A union of the sets contains all the elements in EITHER set
- Union symbol is usually a ∪
- Example:
  - C = M ∪ P



Set operations: Union (cont.)

- Formal definition for the union of two sets:
  $A \cup B = \{ x \mid x \in A \textbf{ or } x \in B \}$
- Further examples
  - $\{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$
  - {New York, Washington} ∪ {3, 4} = {New York, Washington, 3, 4}
  - $\{1, 2\} \cup \varnothing = \{1, 2\}$

A U B

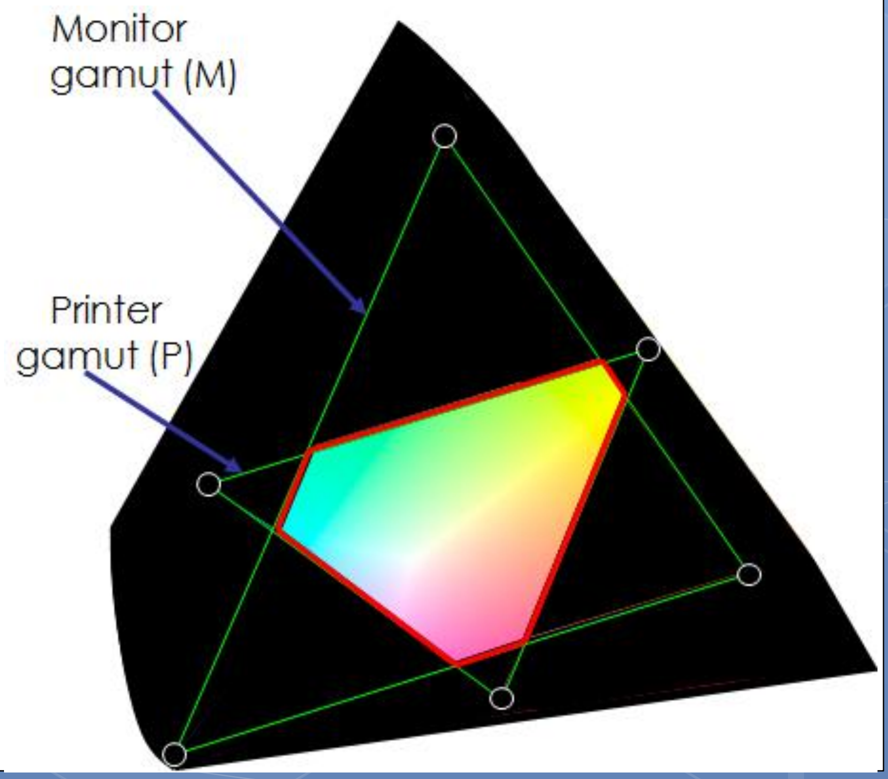


Properties of the union operation

- $A \cup \varnothing = A$ Identity law
- $A \cup \boldsymbol{U} = \boldsymbol{U}$ Domination law
- $A \cup A = A$ Idempotent law
- $A \cup B = B \cup A$ Commutative law
- $A \cup (B \cup C) = (A \cup B) \cup C$ Associative law

Set operations: Intersection (Irisan)

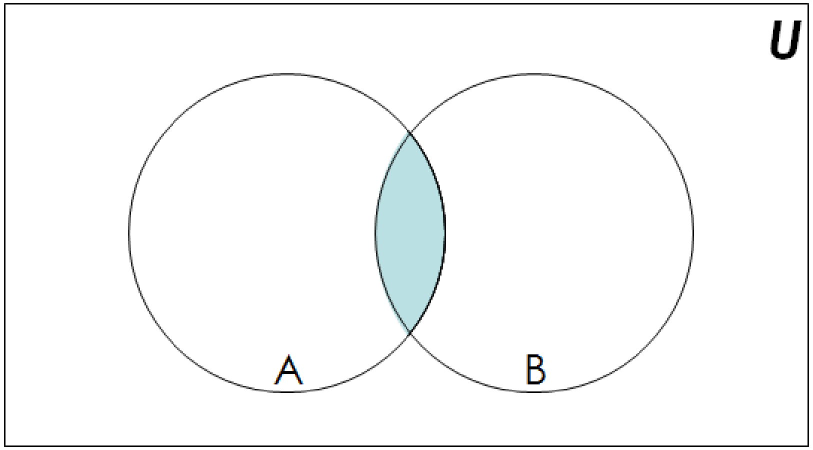
- An intersection of the sets contains all the elements in BOTH sets
- Intersection symbol is a $\cap$
- Example:
  $C = M \cap P$



Set operations: Intersection

- Formal definition for the intersection of two sets: $A \cap B = \{ x \mid x \in A$ **and** $x \in B \}$
- Further examples
  - $\{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$
  - $\{$New York, Washington$\} \cap \{3, 4\} = \varnothing$
    - No elements in common
  - $\{1, 2\} \cap \varnothing = \varnothing$
    - Any set intersection with the empty set yields the empty set

A ∩ B

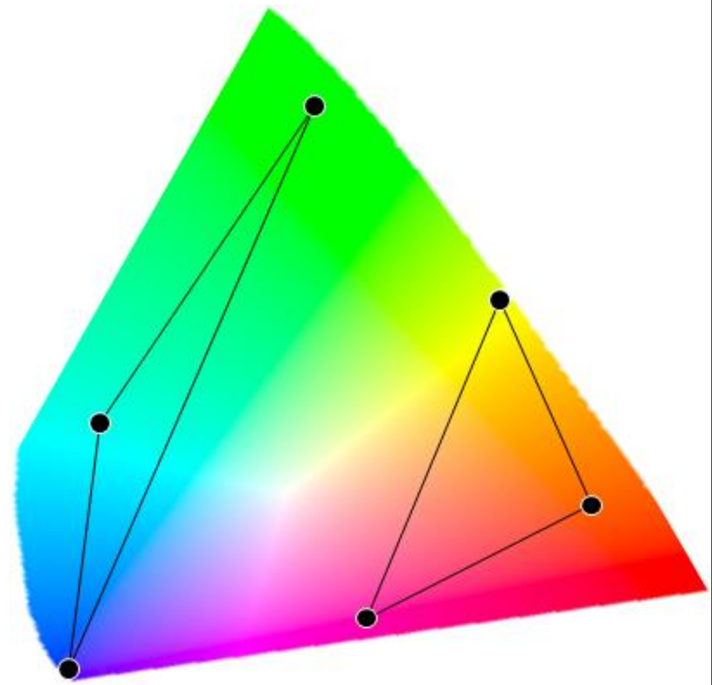


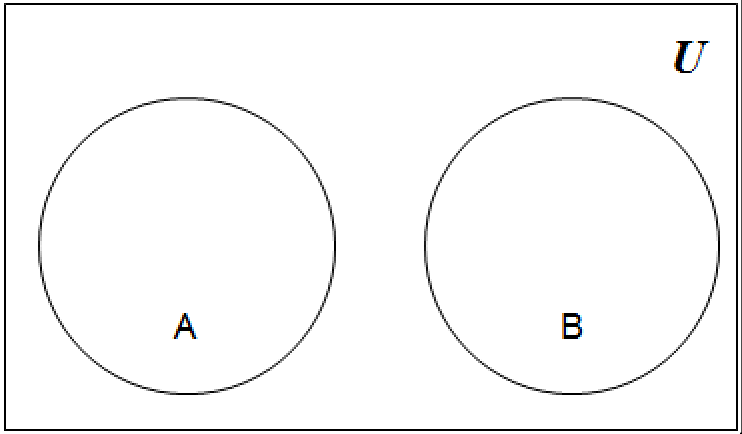
Properties of the intersection operation

- $A \cap \boldsymbol{U} = A$ — Identity law
- $A \cap \varnothing = \varnothing$ — Domination law
- $A \cap A = A$ — Idempotent law
- $A \cap B = B \cap A$ — Commutative law
- $A \cap (B \cap C) = (A \cap B) \cap C$ — Associative law

Disjoint sets

- Two sets are disjoint if they have NO elements in common
- Formally, two sets are disjoint if their intersection is the empty set
- Another example: the set of the even numbers and the set of the odd numbers



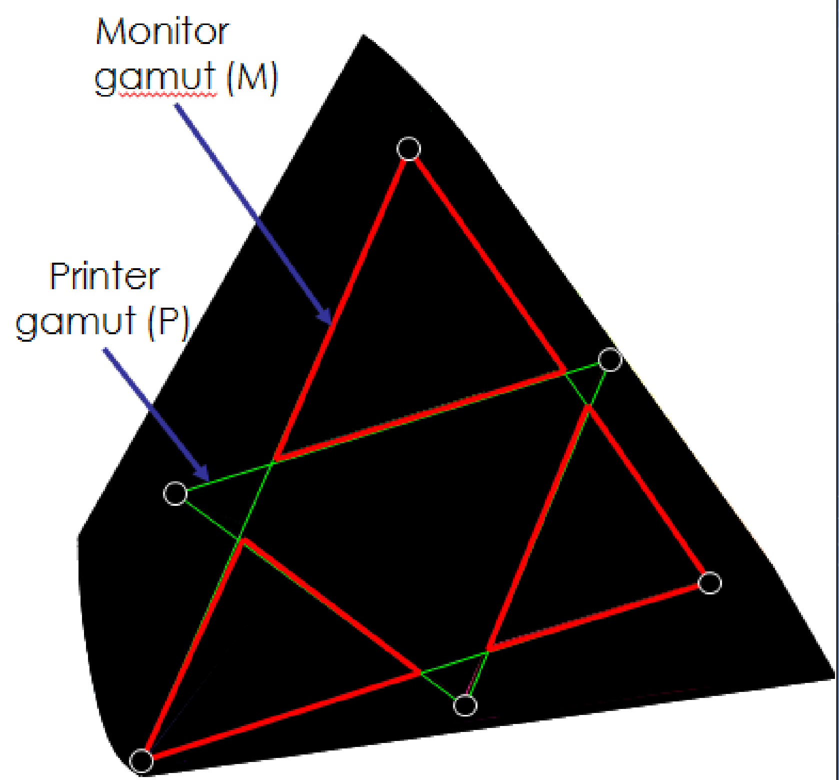


Disjoint sets (cont.)

- Formal definition for disjoint sets: two sets are disjoint if their intersection is the empty set
- Further examples
  - {1, 2, 3} and {3, 4, 5} are not disjoint
  - {New York, Washington} and {3, 4} are disjoint
  - {1, 2} and ∅ are disjoint
    - Their intersection is the empty set
  - ∅ and ∅ are disjoint!
    - Their intersection is the empty set

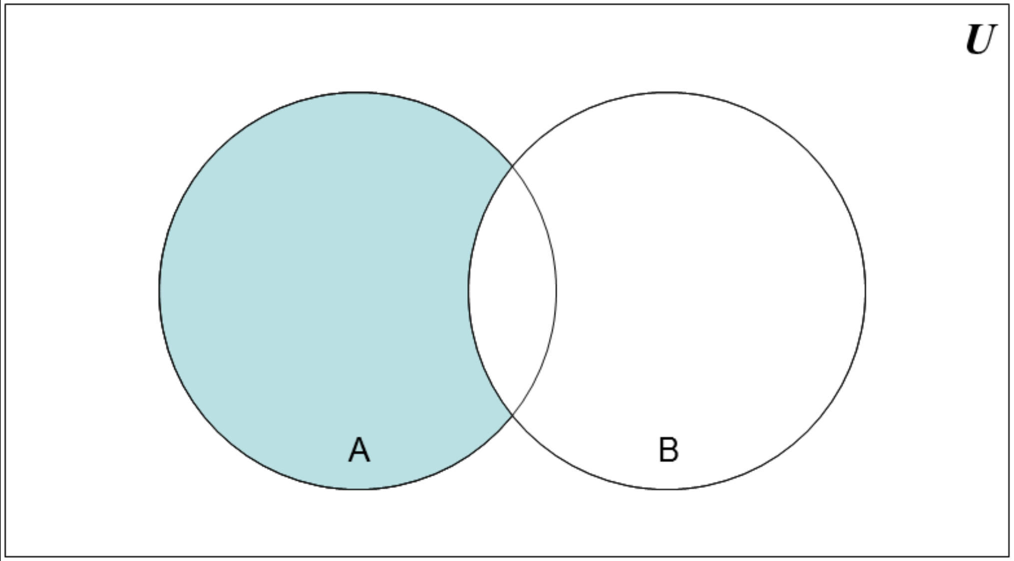
Set operations: Difference (Selisih)

- A difference of two sets is the elements in one set that are NOT in the other
- Difference symbol is a minus sign
- Example:
  - C = M - P
- Also visa-versa:
  - C = P - M

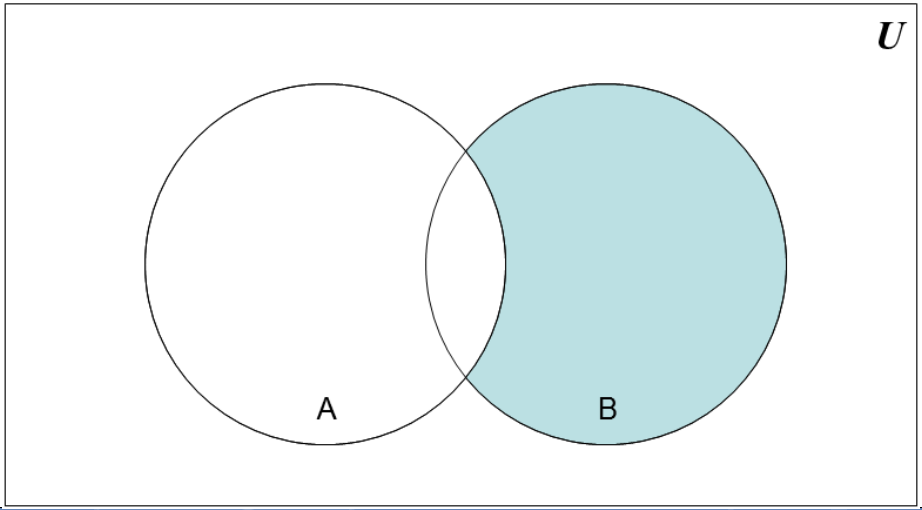


Set operations: Difference (cont.)

A - B



B - A

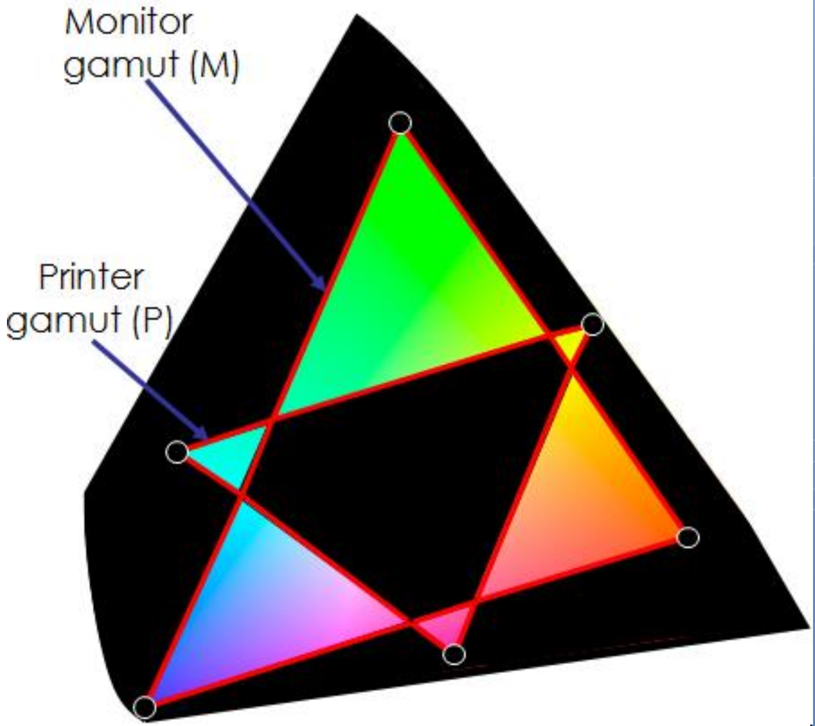


Set operations: Difference (cont.)

- Formal definition for the difference of two sets:
  $A - B = \{ x \mid x \in A \textbf{ and } x \notin B \}$
  $A - B = A \cap \overline{B}$ ← Important!
- Further examples
  - $\{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\}$
  - {New York, Washington} - {3, 4} = {New York, Washington}
  - $\{1, 2\} - \varnothing = \{1, 2\}$
    - The difference of any set S with the empty set will be the set S

Set operations: Symmetric Difference

- A symmetric difference of the sets contains all the elements in either set but NOT both
- Symetric diff. symbol is a $\oplus$
- Example: $C = M \oplus P$



Set operations: Symmetric Difference

- Formal definition for the symmetric difference of two sets:

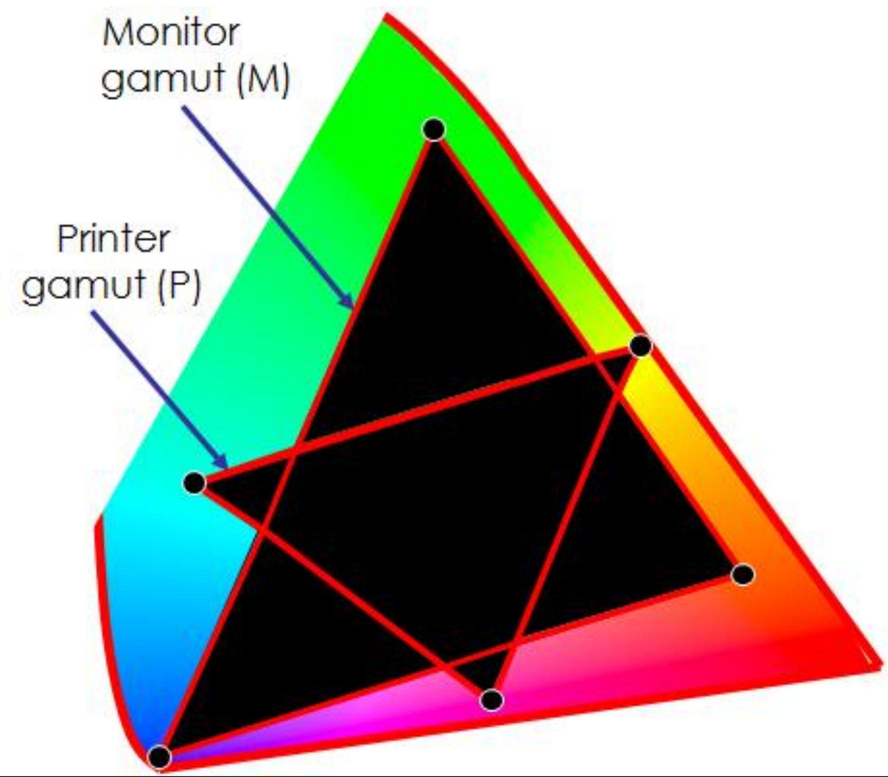
  $A \oplus B = \{ x \mid (x \in A \text{ or } x \in B) \text{ and } x \notin A \cap B\}$

  $A \oplus B = (A \cup B) - (A \cap B)$ ← Important!

- Further examples
  - $\{1, 2, 3\} \oplus \{3, 4, 5\} = \{1, 2, 4, 5\}$
  - {New York, Washington} $\oplus$ {3, 4} = {New York, Washington, 3, 4}
  - $\{1, 2\} \oplus \varnothing = \{1, 2\}$
    - The symmetric difference of any set S with the empty set will be the set S

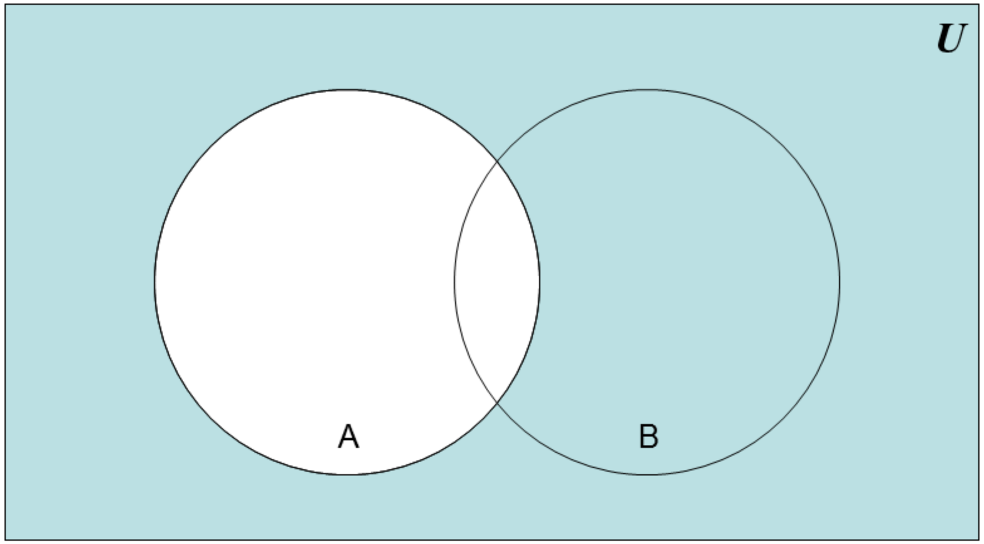
Complement sets

- A complement of a set is all the elements that are NOT in the set
- Complement symbol is a bar above the set name: $\overline{P}$ or $\overline{M}$
- Alternative symbol:
  - P^C or M^C

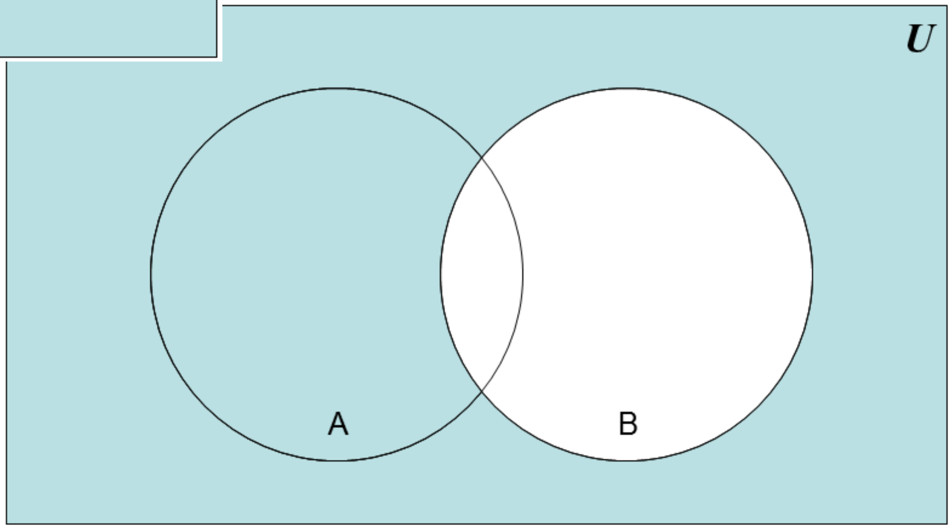


Complement sets (cont.)

$\overline{A}$



$\overline{B}$



Complement sets (cont.)

- Formal definition for the complement of a set: $\overline{A} = \{ x \mid x \notin A \} = A^c$
  - Or $\boldsymbol{U} - A$, where $\boldsymbol{U}$ is the universal set
- Further examples (assuming $\boldsymbol{U} = \mathbf{Z}$)
  - $\overline{\{1, 2, 3\}} = \{ \ldots, -2, -1, 0, 4, 5, 6, \ldots \}$
- Properties of complement sets
  - $\overline{\overline{A}} = A$ Complementation law
  - $A \cup \overline{A} = \boldsymbol{U}$ Complement law
  - $A \cap \overline{A} = \varnothing$ Complement law

Set identities

- Set identities are basic laws on how set operations work
  - Many have already been introduced on previous slides
- Just like logical equivalences!
  - Replace $\cup$ with $\vee$
  - Replace $\cap$ with $\wedge$
  - Replace $\emptyset$ with F
  - Replace $\boldsymbol{U}$ with T

Recap of set identities

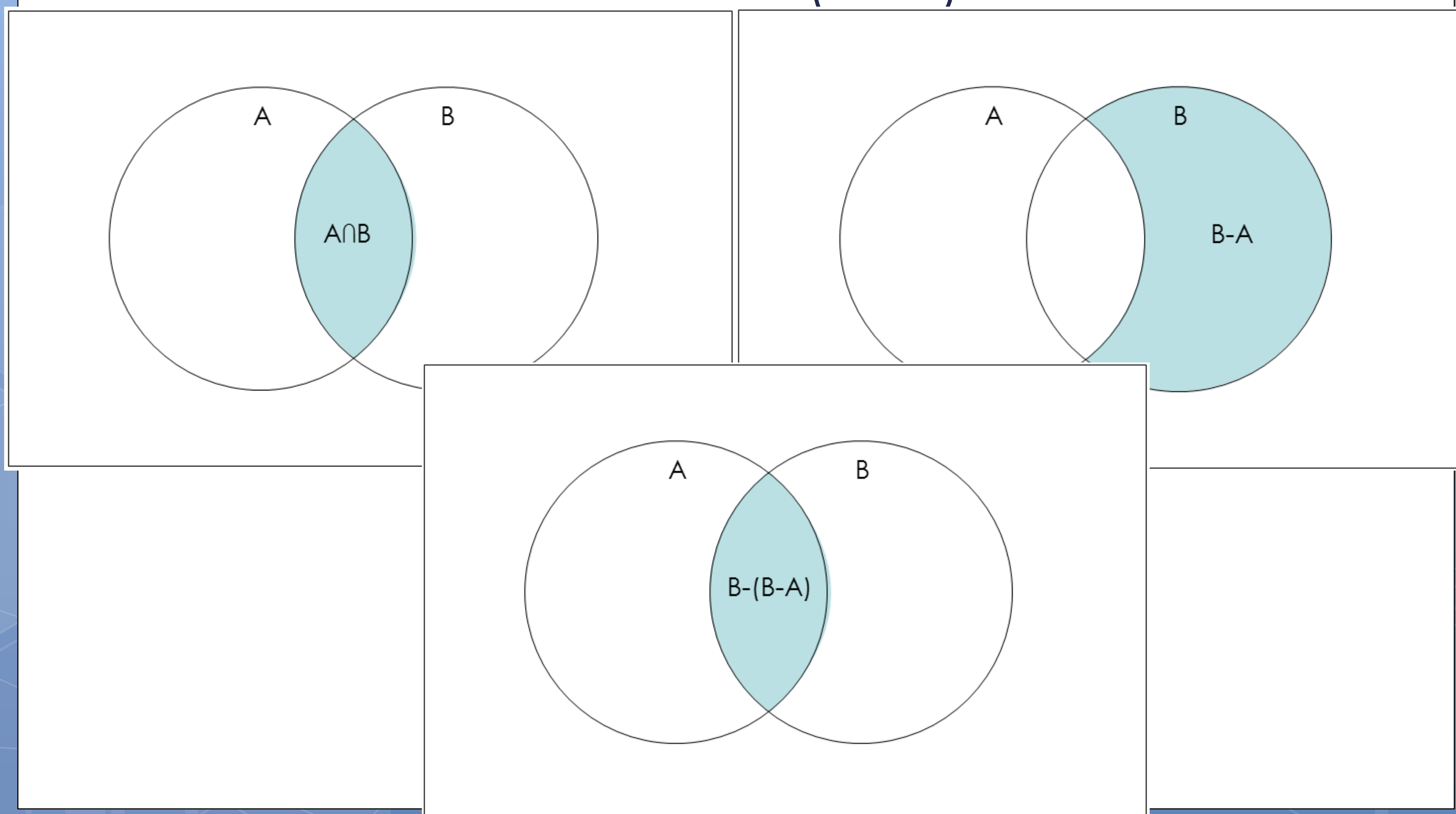
$A \cup \varnothing = A$ $A \cap U = A$	Identity Law	$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination law
$A \cup A = A$ $A \cap A = A$	Idempotent Law	$(A^c)^c = A$	Complement Law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Law	$(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$	De Morgan's Law
$A \cup (B \cup C)$ $= (A \cup B) \cup C$ $A \cap (B \cap C)$ $= (A \cap B) \cap C$	Associative Law	$A \cap (B \cup C) =$ $(A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) =$ $(A \cup B) \cap (A \cup C)$	Distributive Law
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Law	$A \cup A^c = U$ $A \cap A^c = \varnothing$	Complement Law

How to prove a set identity?

- For example: $A \cap B = B - (B - A)$
- There are four methods to prove:
  - Use the basic set identities
  - Use membership tables
  - Prove each set is a subset of each other
    - This is like proving that two numbers are equal by showing that each is less than or equal to the other
  - Use set builder notation and logical equivalences

What we are going to prove?

$$A \cap B = B - (B - A)$$



Proof by Set Identities

Prove that A ∩ B = B - (B - A)

$A \cap B = B - (B \cap \overline{A})$	Definition of difference
$= B \cap \overline{(B \cap \overline{A})}$	Definition of difference
$= B \cap (\overline{B} \cup \overline{\overline{A}})$	DeMorgan's law
$= B \cap (\overline{B} \cup A)$	Complementation law
$= (B \cap \overline{B}) \cup (B \cap A)$	Distributive law
$= \varnothing \cup (B \cap A)$	Complement law
$= (B \cap A)$	Identity law
$= A \cap B$	Commutative law

What is a membership table?

- Membership tables show all the combinations of sets an element can belong to
  - 1 means the element belongs, 0 means it does not
- Consider the following membership table:

A	B	A ∪ B	A ∩ B	A - B
1	1	1	1	0
1	0	1	0	1
0	1	1	0	0
0	0	0	0	0

- The top row is all elements that belong to both sets A and B
  - Thus, these elements are in the union and intersection, but not the difference

What is a membership table?

- Membership tables show all the combinations of sets an element can belong to
  - 1 means the element belongs, 0 means it does not
- Consider the following membership table:

A	B	A U B	A ∩ B	A - B
1	1	1	1	0
1	0	1	0	1
0	1	1	0	0
0	0	0	0	0

- The second row is all elements that belong to set A but not set B
  - Thus, these elements are in the union and difference, but not the intersection

What is a membership table?

- Membership tables show all the combinations of sets an element can belong to
  - 1 means the element belongs, 0 means it does not
- Consider the following membership table:

A	B	A U B	A ∩ B	A - B
1	1	1	1	0
1	0	1	0	1
0	1	1	0	0
0	0	0	0	0

- The third row is all elements that belong to set B but not set A
  - Thus, these elements are in the union, but not the intersection or difference

What is a membership table?

- Membership tables show all the combinations of sets an element can belong to
  - 1 means the element belongs, 0 means it does not
- Consider the following membership table:

A	B	A U B	A ∩ B	A - B
1	1	1	1	0
1	0	1	0	1
0	1	1	0	0
0	0	0	0	0

- The bottom row is all elements that belong to neither set A or set B
  - Thus, these elements are neither the union, the intersection, nor difference

Proof by membership tables

- The following membership table shows that $A \cap B = B - (B - A)$

A	B	$A \cap B$	B-A	B-(B-A)
1	1	1	0	1
1	0	0	0	0
0	1	0	1	0
0	0	0	0	0

- Because the two indicated columns have the same values, the two expressions are identical
- This is similar to Propositional logic!

Proof by showing each set is a subset of the other

- Assume that an element is a member of one of the identities
  - Then show it is a member of the other
- Repeat for the other identity

- We are trying to show:
  - $(x \in A \cap B \rightarrow x \in B-(B-A)) \wedge (x \in B-(B-A) \rightarrow x \in A \cap B)$
  - This is the biconditional:
  - $x \in A \cap B \leftrightarrow x \in B-(B-A)$

- Not good for long proofs

Proof by showing each set is a subset of the other

- Assume that $x \in B-(B-A)$
  - By definition of difference, we know that $x \in B$ and $x \notin B-A$
- Consider $x \notin B-A$
  - If $x \in B-A$, then (by definition of difference) $x \in B$ and $x \notin A$
  - Since $x \notin B-A$, then only one of the inverses has to be true (DeMorgan's law): $x \notin B$ or $x \in A$
- So we have that $x \in B$ and ($x \notin B$ or $x \in A$)
  - It cannot be the case where $x \in B$ and $x \notin B$
  - Thus, $x \in B$ and $x \in A$
  - This is the definition of intersection
- Thus, if $x \in B-(B-A)$ then $x \in A \cap B$

Proof by showing each set is a subset of the other

- Assume that $x \in A \cap B$
  - By definition of intersection, $x \in A$ and $x \in B$
- Thus, we know that $x \notin B-A$
  - B-A includes all the elements in B that are also not in A not include any of the elements of A (by definition of difference)
- Consider B-(B-A)
  - We know that $x \notin B-A$
  - We also know that if $x \in A \cap B$ then $x \in B$ (by definition of intersection)
  - Thus, if $x \in B$ and $x \notin B-A$, we can restate that (using the definition of difference) as $x \in B-(B-A)$
- Thus, if $x \in A \cap B$ then $x \in B-(B-A)$

Proof by set builder notation and logical equivalences

- First, translate both sides of the set identity into set builder notation
- Then modify one side to make it identical to the other
  - Do this using logical equivalences

Proof by set builder notation and logical equivalences

$B-(B-A)$	Original statement
$=\{x \mid x\in B \wedge x\notin (B-A)\}$	Definition of difference
$=\{x \mid x\in B \wedge \neg(x\in (B-A))\}$	Negating "element of"
$=\{x \mid x\in B \wedge \neg(x\in B \wedge x\notin A)\}$	Definition of difference
$=\{x \mid x\in B \wedge (x\notin B \vee x\in A)\}$	DeMorgan's Law
$=\{x \mid (x\in B \wedge x\notin B) \vee (x\in B \wedge x\in A)\}$	Distributive Law
$=\{x \mid (x\in B \wedge \neg(x\in B)) \vee (x\in B \wedge x\in A)\}$	Negating "element of"
$=\{x \mid F \vee (x\in B \wedge x\in A)\}$	Negation Law
$=\{x \mid x\in B \wedge x\in A\}$	Identity Law
$=A\cap B$	Definition of intersection

Computer representation of sets

- Assume that U is finite (and reasonable!)
  - Let U be the alphabet
- Each bit represents whether the element in U is in the set
- The vowels in the alphabet:
  `abcdefghijklmnopqrstuvwxyz`
  `10001000100000100000100000`
- The consonants in the alphabet:
  `abcdefghijklmnopqrstuvwxyz`
  `01110111011111011111011111`

Computer representation of sets

- Consider the union of these two sets:

  ```
   10001000100000100000100000
  ∨01110111011111011111011111
   11111111111111111111111111
  ```

- Consider the intersection of these two sets:

  ```
   10001000100000100000100000
  ∧01110111011111011111011111
   00000000000000000000000000
  ```