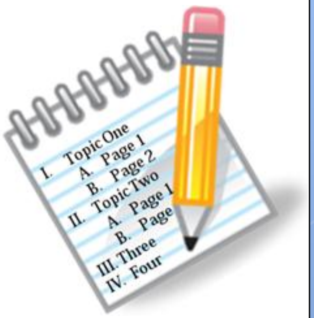
KS091201
MATEMATIKA DISKRIT
(DISCRETE
MATHEMATICS)

Number Theory: Integers, Division, Prime Number

Discrete Math Team

Outline



- Integer and Division
- Primes
- GCD (Great Common Divisor)
- LCM (least Common Multiple)

Division

- **Definition**: if a and b are integers ($a \neq 0$), a divides b if $\exists$ c such that $b = ac$.
- When a divides b, we say that a is a factor of b and that b is a multiple of a.
- **Notation**:
  - a | b : a divides b (a habis membagi b; b habis dibagi a)
  - a ∤ b : a does not divide b (a tidak habis membagi b; b tidak habis dibagi a)
- **Example**:
  - 3 | 7 ?   3 | 12?
  - 3 ∤ 7 since 7/3 is not an integer
  - 3 | 12 because 12/3 = 4

Theorem 1

- Let a, b, and c be integers, then:
  - If $a | b$ and $a | c$, then $a | (b + c)$
  - If $a | b$, then $a | bc$ for all integers c
  - If $a | b$ and $b | c$, then $a | c$

Proof:

- If $a | b$ and $a | c$, then $a | (b + c)$
  - $b = ma$ and $c = na$
  - $b + c = ma + na = (m + n)a$
  - $b + c = (m + n)a$
  - So, $a \mid (b + c)$

Proof

- If $a \mid b$, then $a \mid bc$ for all integers c
  - $b = ma, \; bc = (ma)c = (mc)a$
  - $bc = (mc)a$
  - So, $a \mid bc$

- If $a \mid b$ and $b \mid c$, then $a \mid c$
  - $b = ma, \; c = pb = p(ma) = (pm)a$
  - $c = (pm)a$
  - So, $a \mid c$

Corollary 1

- $a \mid b$ and $a \mid c \rightarrow a \mid mb + nc$

Proof:

- $b = pa$
- $c = qa$
- $mb = (mp)a$
- $nc = (nq)a$
- $mb + nc = (mp + nq)a$
- So, $a \mid mb + nc$

Division Algorithm

- **Theorem 2**: Let a be an integer and d a positive integer. Then there exist unique integers q and r, with $0 \leq r < d$, such that $a = dq + r$.

- **Definition**
  - $q = a$ **div** d; q = quotient, d = divisor, a = divident
  - $r = a$ **mod** d; r = remainder

Division Algorithm Examples

- What are the quotient and remainder when 101 is divided by 11?
  - $101 = 11 \cdot 9 + 2$
  - The quotient is: 9 = 101 **div** 11
  - The remainder is: 2 = 101 **mod** 11

- What are the quotient and remainder when -11 is divided by 3?
  - $-11 = 3(-4) + 1$
  - The quotient is: - 4 = - 11 **div** 3
  - The remainder is: 1 = - 11 **mod** 3
  - Note: the remainder can't be negative
  - $-11 = 3(-3) - 2 \rightarrow r = -2$ doesn't satisfy $0 \leq r < 3$

Modular Arithmetic

- **Definition**: If a and b are integers and m is positive integer, then a is congruent to b modulo m if m divides $a - b$.
- **Notation**:
  - $a \equiv b \pmod{m}$; a is congruent to b modulo m
  - $a \not\equiv b \pmod{m}$; a and b are not congruent to modulo m
- **Theorem 3**:
  - $a \equiv b \pmod{m}$ iff a **mod** $m = b$ **mod** m.
- **Example**:
  - $17 \equiv 12$ mod 5, 17 mod 5 = 2, 12 mod 5 = 2
  - $-3 \equiv 17$ mod 10, -3 mod 10 = 7, 17 mod 10 = 7

Modular Arithmetic

- **Theorem 4**:
  - Let m be a positive integer, $a \equiv b \pmod{m}$ iff $\exists\, k$ such that $a = b + km$.

- **Proof**:
  - If $a \equiv b \pmod{m}$, then $m \mid (a - b)$.
  - This means that $\exists\, k$ such that $a - b = km$, so that $a = b + km$.
  - Conversely, if $\exists\, k$ such that $a = b + km$, then $km = a - b$.
  - Hence, m divides $a - b$, so that $a \equiv b \pmod{m}$

Modular Arithmetic

- **Theorem 5**:
  - Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then
    - $a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$

- **Proof:**
  - Because $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, there are integers s and t with $b = a + sm$ and $d = c + tm$. Hence:
    - $b + d = (a + sm) + (c + tm) = (a + c) + m(s + t)$
    - $bd = (a + sm)(c + tm) = ac + m(at + cs + stm)$
  - Hence, $a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$

Corollary 2

- Let m be a positive integer, a and b be integers. Then:
  - $(a + b)$ **mod** $m \equiv ((a$ **mod** $m) + (b$ **mod** $m))$ **mod** m
  - $ab \equiv (a$ **mod** $m)(b$ **mod** $m)$ (**mod** m)
- **Proof:**
  - By the definitions of mod m and congruence modulo m, we know that $a \equiv (a$ **mod** $m)$(**mod** m) and $b \equiv (b$ **mod** $m)$(**mod** m)
  - Hence theorem 5 tells us that:
    - $(a + b)$ **mod** $m \equiv ((a$ **mod** $m) + (b$ **mod** $m))$ **mod** m
    - $ab \equiv (a$ **mod** $m)(b$ **mod** $m)$ (**mod** m)

Caesar Cipher

- Alphabet to number: a~0, b~1, … , z~25.
- Encryption: $f(p) = (p + k)$ **mod** 26.
- Decryption: $f^{-1}(p) = (p - k)$ **mod** 26.
  - Caesar used $k = 3$.
- This is called a substitution cipher
  - You are substituting one letter with another

Caesar Cipher Example

- Encrypt "go cavaliers"
  - Translate to numbers: g = 6, o = 14, etc.
    - Full sequence: 6, 14, 2, 0, 21, 0, 11, 8, 4, 17, 18
  - Apply the cipher to each number: $f(6) = 9$, $f(14) = 17$, etc.
    - Full sequence: 9, 17, 5, 3, 24, 3, 14, 11, 7, 20, 21
  - Convert the numbers back to letters 9 = j, 17 = r, etc.
    - Full sequence: jr wfdydolhuv

- Decrypt "jr fdydolhuv"
  - Translate to numbers: j = 9, r = 17, etc.
    - Full sequence: 9, 17, 5, 3, 24, 3, 14, 11, 7, 20, 21
  - Apply the cipher to each number: $f^{-1}(9) = 6$, $f^{-1}(17) = 14$, etc.
    - Full sequence: 6, 14, 2, 0, 21, 0, 11, 8, 4, 17, 18
  - Convert the numbers back to letters 6 = g, 14 = 0, etc.
    - Full sequence: go cavaliers

Caesar Cipher Example

- Encrypt "MEET YOU IN THE PARK"
  - Translate to numbers:
    - Full sequence: 12, 4, 4, 19, 24, 14, 20, 8, 13, 19, 7, 4, 15, 0, 17, 10
  - Apply the cipher to each number:
    - Full sequence: 15, 7, 7, 22, 1, 7, 23, 11, 16, 22, 10, 7, 18, 3, 20, 13
  - Convert the numbers back to letters:
    - Full sequence: PHHW BRX LQ WKH SDUN

Caesar Cipher Example

- What letter replaces the letter K when the function $f(p) = (7p + 3)$ **mod** 26 is used for encryption?

Solution:

- First, note that 10 represents K, then using the encryption function specified, it follows that $f(10) = (7 \,.\, 10 + 3)$ **mod** $26 = 21$
- Because 21 represents V, K is replaced by V in the encrypted message.

Prime Numbers

- **Definition**: A positive integer p is prime if the only positive factors of p are 1 and p
  - If there are other factors, it is composite
  - Note that 1 is not prime!
  - It's not composite either – it's in its own class
- **Definition**: An integer n is composite if and only if there exists an integer a such that $a \mid n$ and $1 < a < n$

Fundamental theorem of arithmetic

- Every positive integer greater than 1 can be uniquely written as a prime or as the product of two or more primes where the prime factors are written in order of non-decreasing size

- Examples
  - 100 = 2 * 2 * 5 * 5
  - 182 = 2 * 7 * 13
  - 29820 = 2 * 2 * 3 * 5 * 7 * 71

Composite Factors

- If n is a composite integer, then n has a prime divisor less than or equal to the square root of n

Showing a number is prime

- Show that 113 is prime
- Solution
  - The only prime factors less than $\sqrt{113}$ = 10.63 are 2, 3, 5, and 7
  - Neither of these divide 113 evenly
  - Thus, by the fundamental theorem of arithmetic, 113 must be prime

Showing a number is composite

- Show that 899 is prime

- Solution
  - Divide 899 by successively larger primes (up to $\sqrt{899} = 29.98$), starting with 2
  - We find that 29 and 31 divide 899

Primes are infinite

- Theorem (by Euclid): There are infinitely many prime numbers

- Proof by contradiction
- Assume there are a finite number of primes
- List them as follows: $p_1, p_2 \ldots, p_n$.
- Consider the number $q = p_1 p_2 \ldots p_n + 1$
  - This number is not divisible by any of the listed primes
    - If we divided p_i into q, there would result a remainder of 1
  - We must conclude that q is a prime number, not among the primes listed above
    - This contradicts our assumption that all primes are in the list $p_1, p_2 \ldots, p_n$.

The prime number theorem

- The ratio of the number of primes not exceeding x and x/ln(x) approaches 1 as x grows without bound
  - Rephrased: the number of prime numbers less than x is approximately x/ln(x)
  - Rephrased: the chance of an number x being a prime number is
    1 / ln(x)

- Consider 200 digit prime numbers
  - ln $(10^{200}) \approx 460$
  - The chance of a 200 digit number being prime is 1/460
  - If we only choose odd numbers, the chance is 2/460 = 1/230

Greatest common divisor

- The greatest common divisor of two integers a and b is the largest integer d such that $d \mid a$ and $d \mid b$
  - Denoted by gcd (a, b)
- Examples
  - gcd (24, 36) = 12
  - gcd (17, 22) = 1
  - gcd (100, 17) = 1

Relative primes

- Two numbers are relatively prime if they don't have any common factors (other than 1)
  - Rephrased: a and b are relatively prime if $\gcd(a, b) = 1$
- $\gcd(25, 39) = 1$, so 25 and 39 are relatively prime

Pairwise relative prime

- A set of integers a_1, a_2, … a_n are pairwise relatively prime if, for all pairs of numbers, they are relatively prime
  - Formally: The integers a_1, a_2, … a_n are pairwise relatively prime if gcd $(a_i, a_j) = 1$ whenever $1 \leq i < j \leq n$.

- Example: are 10, 17, and 21 pairwise relatively prime?
  - gcd (10,17) = 1, gcd (17, 21) = 1, and gcd (21, 10) = 1
  - Thus, they are pairwise relatively prime
- Example: are 10, 19, and 24 pairwise relatively prime?
  - Since gcd (10,24) ≠ 1, they are not

More on gcd's

- Given two numbers *a* and *b*, rewrite them as:

  $$a = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}, b = p_1^{b_1} p_2^{b_2} \ldots p_n^{b_n}$$

- Example: gcd (120, 500)
  - $120 = 2^3 * 3 * 5 = 2^3 * 3^1 * 5^1$
  - $500 = 2^2 * 5^3 = 2^2 * 3^0 * 5^3$
- Then compute the gcd by the following formula:

  $$\gcd(a, b) = p_1^{\min(a_1, b_1)} p_2^{\min(a_2, b_2)} \ldots p_n^{\min(a_n, b_n)}$$

  - Example: gcd $(120,500) = 2^{\min(3,2)} 3^{\min(1,0)} 5^{\min(1,3)} = 2^2 3^0 5^1 = 20$

Least common multiple

- The least common multiple of the positive integers *a* and *b* is the smallest positive integer that is divisible by both *a* and *b*.
  - Denoted by lcm (*a*, *b*)
  - $\mathrm{lcm}(a,b) = p_1^{\max(a_1,b_1)} p_2^{\max(a_2,b_2)} \ldots p_n^{\max(a_n,b_n)}$
- Example: lcm(10, 25) = 50
- What is lcm (95256, 432)?
  - $95256 = 2^3 3^5 7^2$; $432 = 2^4 3^3$
  - $\mathrm{lcm}(2^3 3^5 7^2, 2^4 3^3) = 2^{\max(3,4)} 3^{\max(5,3)} 7^{\max(2,0)} = 2^4 3^5 7^2$ $= 190512$

lcm and gcd theorem

- Let *a* and *b* be positive integers. Then $a*b = \gcd(a, b) * \operatorname{lcm}(a, b)$

- Example: gcd (10, 25) = 5, lcm (10, 25) = 50
  - 10*25 = 5*50

- Example: gcd (95256, 432) = 216, lcm (95256, 432) = 190512
  - 95256*432 = 216*190512