ITS Institut Teknologi Sepuluh Nopember

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KS091201
MATEMATIKA DISKRIT
(DISCRETE MATHEMATICS)

Discrete Basic Structure: Sequences and Summations

Discrete Math Team

Outline

- Definition of Sequence
- Sequence Examples
- Arithmetic Vs Geometric Sequences
- Fibonacci Sequence
- Determining Sequence Formula
- Useful Sequences
- Summations
- Evaluating Sequences
- Summation of A Geometric Series
- Double Summation
- Cardinality

Definitions of Sequence

- Sequence: an ordered list of elements
  - Like a set, but:
    - Elements can be duplicated
    - Elements are ordered
- A sequence is a function from a subset of **Z** to a set S
  - Usually from the positive or non-negative ints
  - a_n is the image of n
- a_n is a term in the sequence
- $\{a_n\}$ means the entire sequence
  - The same notation as sets!

Sequence examples

- $a_n = 3n$
  - The terms in the sequence are $a_1, a_2, a_3, \ldots$
  - The sequence $\{a_n\}$ is $\{ 3, 6, 9, 12, \ldots \}$
- $b_n = 2^n$
  - The terms in the sequence are $b_1, b_2, b_3, \ldots$
  - The sequence $\{b_n\}$ is $\{ 2, 4, 8, 16, 32, \ldots \}$
- Note that generally sequences are indexed from 1

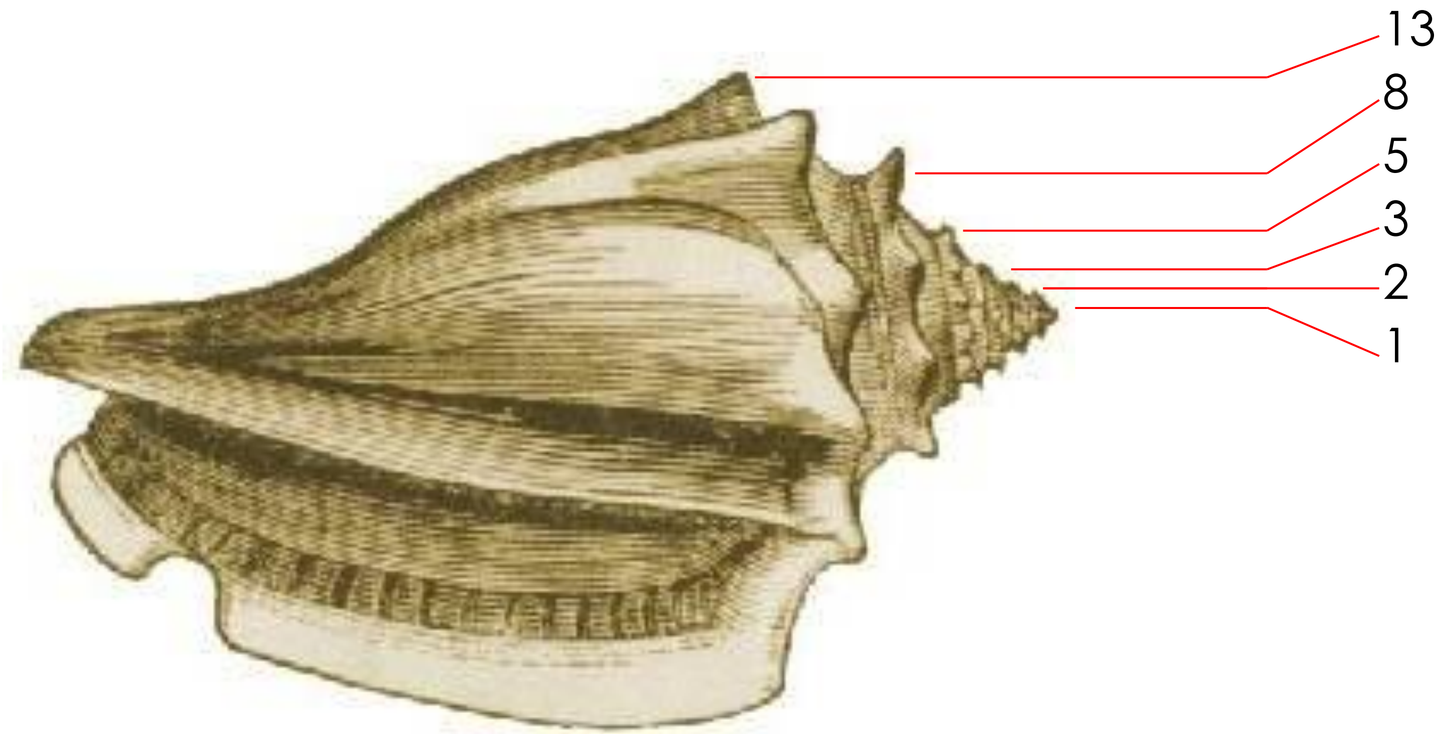
Arithmetic vs. Geometric Sequences

- Arithmetic sequences increase by a constant *amount*
  - $a_n = 3n$
  - The sequence $\{a_n\}$ is $\{ 3, 6, 9, 12, \ldots \}$
- Arithmetic Progression
  - $a, a+d, a+2d, \ldots, a+nd, \ldots$
  - $a_n = a + (n-1)\, d$
  - Discrete analogue of linear function $f(x) = dx + a$
- Geometric sequences increase by a constant *factor*
  - $b_n = 2^n$
  - The sequence $\{b_n\}$ is $\{ 2, 4, 8, 16, 32, \ldots \}$
- Geometric Progression
  - $a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots$
  - $a_n = ar^{n-1}$
  - Discrete analogue of exponential function $f(x) = ar^x$

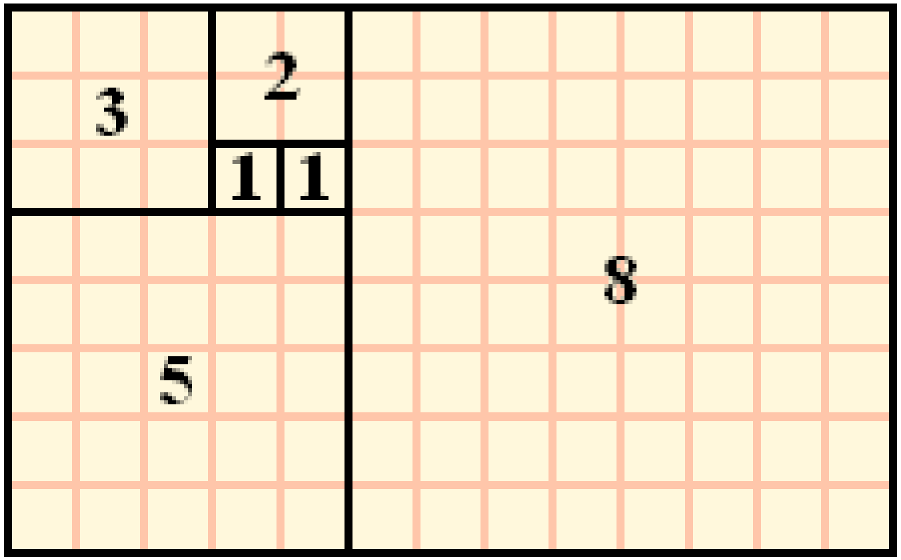
Fibonacci Sequence

- Sequences can be neither geometric nor arithmetic
  - $F_n = F_{n-1} + F_{n-2}$, where the first two terms are 1
    - Alternative, $F(n) = F(n-1) + F(n-2)$
  - Each term is the sum of the previous two terms
  - Sequence: { 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … }
  - This is the Fibonacci sequence
  - Full formula: $F(n) = \dfrac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5}\cdot 2^n}$

Fibonacci Sequence in Nature



Fibonacci Sequence Example



- Fibonacci references from http://en.wikipedia.org/wiki/Fibonacci_sequence

Determining the Sequence Formula

- Given values in a sequence, how do you determine the formula?

- Steps to consider:
  - Is it an arithmetic progression (each term a constant amount from the last)?
  - Is it a geometric progression (each term a factor of the previous term)?
  - Does the sequence is repeat (or cycle)?
  - Does the sequence combine previous terms?
  - Are there runs of the same value?

Determining the Sequence Formula

- 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, …
  - The sequence alternates 1's and 0's, increasing the number of 1's and 0's each time
- 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, …
  - This sequence increases by one, but repeats all even numbers once
- 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, …
  - Each term is twice the cube of *n*. The non-0 numbers are a geometric sequence (2^n) interspersed with zeros
- 3, 6, 12, 24, 48, 96, 192, …
  - Each term is twice the previous: geometric progression
  - $a_n = 3*2^{n-1}$

Determining the sequence formula

- 15, 8, 1, -6, -13, -20, -27, …
  - Each term is 7 less than the previous term
  - $a_n = 22 - 7n$
- 3, 5, 8, 12, 17, 23, 30, 38, 47, …
  - The difference between successive terms increases by one each time
  - $a_1 = 3$, $a_n = a_{n-1} + n$
  - $a_n = n(n+1)/2 + 2$
- 2, 16, 54, 128, 250, 432, 686, …
  - Each term is twice the cube of n
  - $a_n = 2*n^3$
- 2, 3, 7, 25, 121, 721, 5041, 40321
  - Each successive term is about n times the previous
  - $a_n = n! + 1$
  - Alternatively: $a_n = a_{n-1} * n - n + 1$

Useful Sequences

- $n^2 = 1, 4, 9, 16, 25, 36, \ldots$
- $n^3 = 1, 8, 27, 64, 125, 216, \ldots$
- $n^4 = 1, 16, 81, 256, 625, 1296, \ldots$
- $2^n = 2, 4, 8, 16, 32, 64, \ldots$
- $3^n = 3, 9, 27, 81, 243, 729, \ldots$
- $n! = 1, 2, 6, 24, 120, 720, \ldots$

Summations

- A summation:

upper limit

$$\sum_{j=m}^{n} a_j \quad \text{or} \quad \sum\nolimits_{j=m}^{n} a_j$$

lower limit

index of summation

- is like a for loop:

```
int sum = 0;
for ( int j = m; j <= n; j++ )
    sum += a(j);
```

Evaluating Sequences

- $\Sigma^{5}_{k=1}$ (k+1)
  $= 2 + 3 + 4 + 5 + 6 = 20$
- $\Sigma^{4}_{k=0}$ $(-2)^k$
  $= (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 = 11$
- $\Sigma^{10}_{k=1}$ 3
  $= 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 30$
- $\Sigma^{10}_{k=1}$ $(2^k - 2^{k-1})$
  $= (2^1-2^0) + (2^2-2^1) + (2^3-2^2) + \ldots (2^{10}-2^9) = 1023$
  - Note that each term (except the first and last) is cancelled by another term

Evaluating Sequences

- Let $S = \{ 1, 3, 5, 7 \}$
- What is $\Sigma_{j \in S}\, j$
  - $1 + 3 + 5 + 7 = 16$
- What is $\Sigma_{j \in S}\, j^2$
  - $1^2 + 3^2 + 5^2 + 7^2 = 84$
- What is $\Sigma_{j \in S}\, (1/j)$
  - $1/1 + 1/3 + 1/5 + 1/7 = 176/105$
- What is $\Sigma_{j \in S}\, 1$
  - $1 + 1 + 1 + 1 = 4$

Summation of A Geometric Series

- Sum of a geometric series:

$$\sum_{j=0}^{n} ar^{j} = \begin{cases} \dfrac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

- Example:

$$\sum_{j=0}^{10} 2^{n} = \frac{2^{10+1}-1}{2-1} = \frac{2048-1}{1} = 2047$$

Proof

- If r = 1, then the sum is:

$$\boxed{S=\sum_{j=0}^{n} a=(n+1)a}$$

$$S=\sum_{j=0}^{n} ar^{j}$$

$$rS=r\sum_{j=0}^{n} ar^{j}=\sum_{j=0}^{n} ar^{j+1}$$

$$=\sum_{k=1}^{n+1} ar^{k}$$

Shifting the index wih $k = j+1$

$$=\sum_{k=0}^{n} ar^{k}+\left(ar^{n+1}-a\right)$$

$$rS=S+\left(ar^{n+1}-a\right)$$

$$rS-S=\left(ar^{n+1}-a\right)$$

$$S(r-1)=\left(ar^{n+1}-a\right)$$

$$S=\frac{\left(ar^{n+1}-a\right)}{r-1}$$

Double Summations

- Like a nested for loop

$$\boxed{\sum_{i=1}^{4}\sum_{j=1}^{3} ij}$$

- Is equivalent to:

```
int sum = 0;
for ( int i = 1; i <= 4; i++ )
    for ( int j = 1; j <= 3; j++ )
        sum += i*j;
```

Cardinality

- For finite sets (only), cardinality is the number of elements in the set
- For finite and infinite sets, two sets *A* and *B* have the same cardinality if there is a one-to-one correspondence from *A* to *B*

Cardinality

- Example on finite sets:
  - Let $S = \{ 1, 2, 3, 4, 5 \}$
  - Let $T = \{ a, b, c, d, e \}$
  - There is a one-to-one correspondence between the sets
- Example on infinite sets:
  - Let $S = \mathbf{Z}+$
  - Let $T = \{ x \mid x = 2k \text{ and } k \in \mathbf{Z}+ \}$
  - One-to-one correspondence:

    $1 \leftrightarrow 2$ $2 \leftrightarrow 4$ $3 \leftrightarrow 6$ $4 \leftrightarrow 2$

    $5 \leftrightarrow 10$ $6 \leftrightarrow 12$ $7 \leftrightarrow 14$ $8 \leftrightarrow 16$

    Etc.
    - Note that here the '$\leftrightarrow$' symbol means that there is a correspondence between them, not the biconditional

More Definitions

- A set that is either finite or has the same cardinality as the set of $\mathbf{Z}^+$ is called countable.
- A set that is not countable is called uncountable.

- Countably infinite: elements can be listed
  - Anything that has the same cardinality as the positive integer
  - Example: rational numbers, odd integers, all integers
- Uncountably infinite: elements cannot be listed
  - Example: real numbers

- When an infinite set S is countable, we denote the cardinality of S by $\aleph_0$ (aleph null) -- $|S| = \aleph_0$

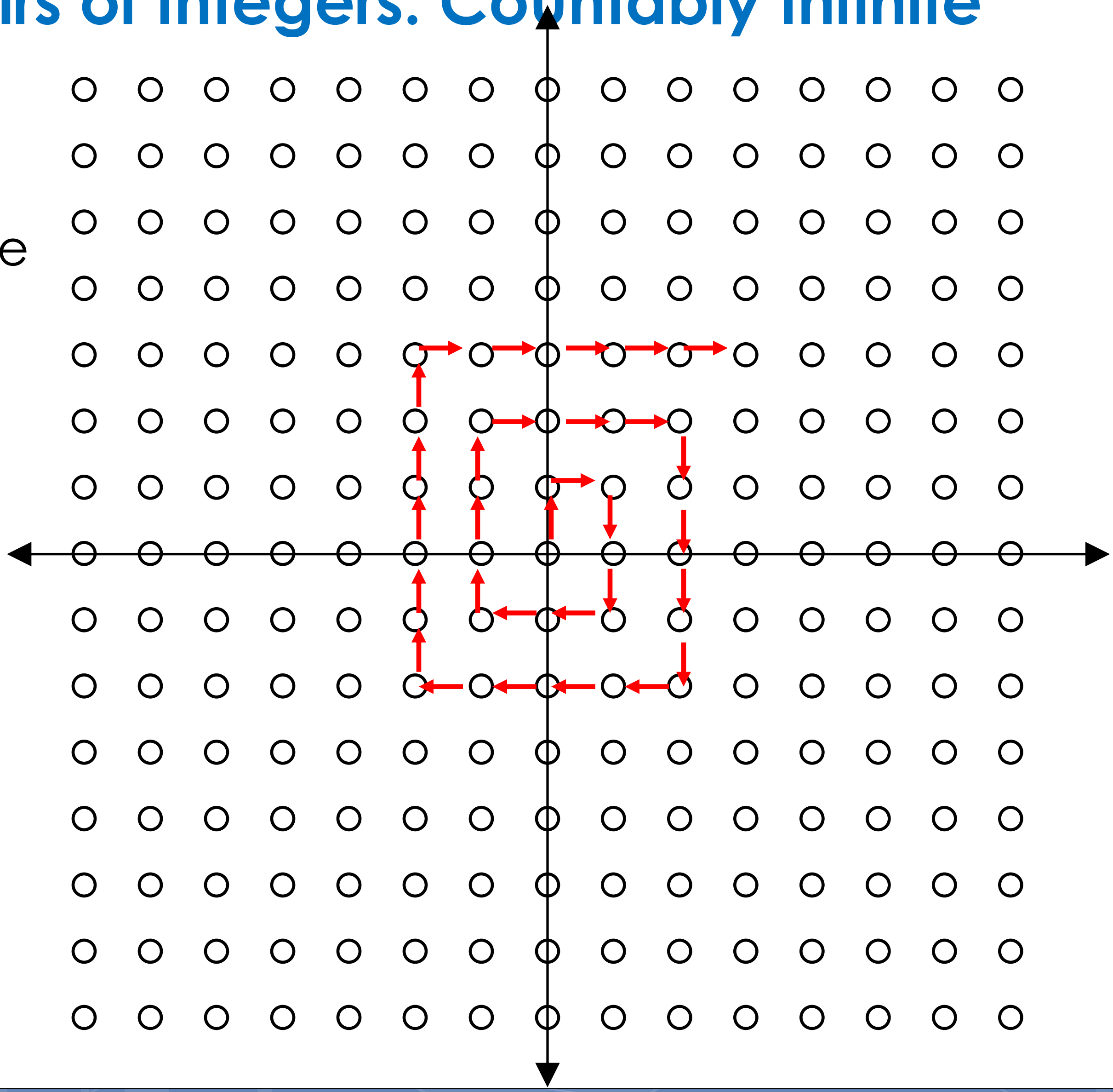
Showing a Set is Countably Infinite

- Done by showing there is a one-to-one correspondence between the set and the positive integers

- Examples
  - Even numbers
    - Shown two slides ago
  - Rational numbers
    - Shown next two slides
  - Ordered pairs of integers
    - Shown next slide

Ordered Pairs of Integers: Countably Infinite

A one-to-one correspondence



Show that the rational numbers are countably infinite

- First, let's show the positive rationals are countable
- See diagram:
- Can easily add 0 (add one column to the left)
- Can add negative rationals as well

$$
\begin{array}{cccccc}
1/1 & 2/1 & 3/1 & 4/1 & 5/1 & \cdots \\
1/2 & 2/2 & 3/2 & 4/2 & 5/2 & \cdots \\
1/3 & 2/3 & 3/3 & 4/3 & 5/3 & \cdots \\
1/4 & 2/4 & 3/4 & 4/4 & 5/4 & \cdots \\
1/5 & 2/5 & 3/5 & 4/5 & 5/5 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$