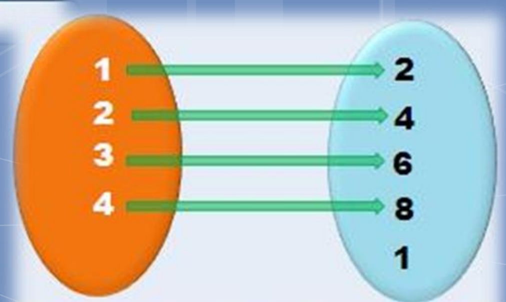
ONTO FUNCTION

BOTH ONE-ONE ONTO FUNCTION

NOT ONTO FUNCTION



NOT BOTH- ONE-ONE ONTO

KS091201
MATEMATIKA DISKRIT
(DISCRETE
MATHEMATICS)

Discrete Basic Structure: Functions

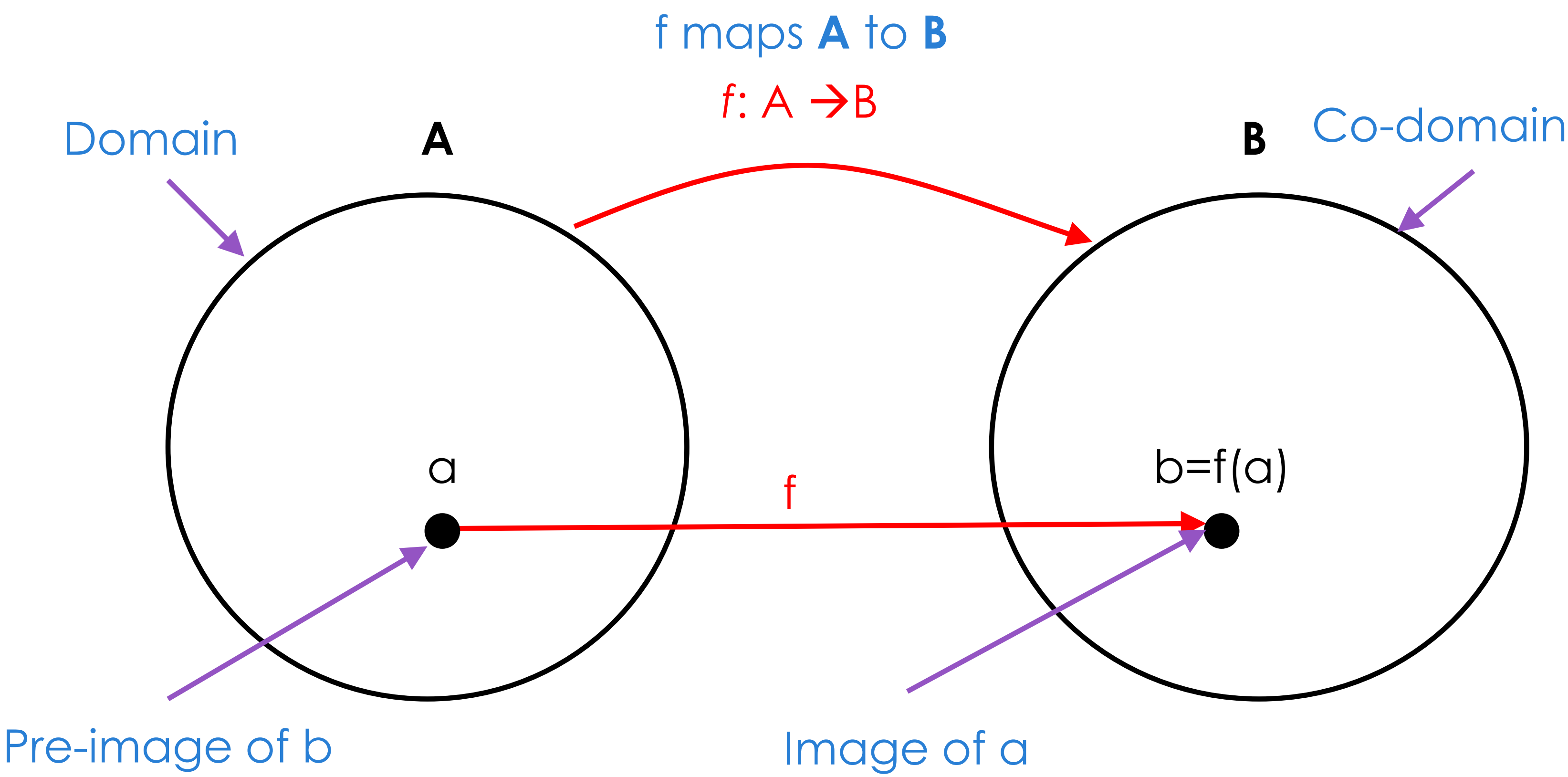
Discrete Math Team

Outline

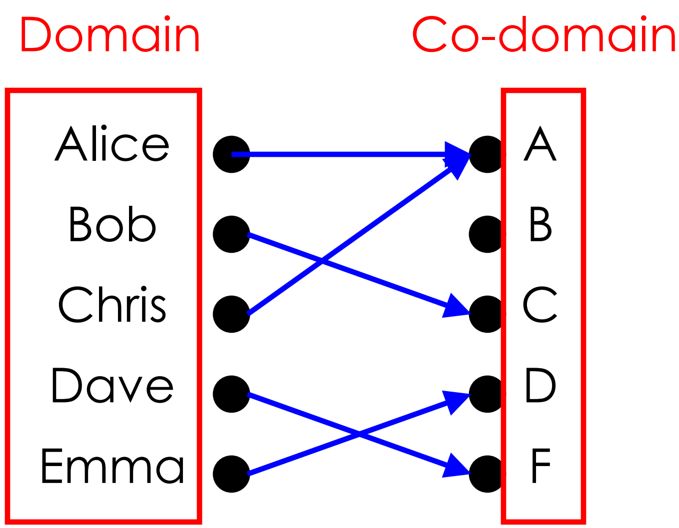
- Definition of function
- Function arithmetic
- One-to-one functions
- Onto functions
- Bijections
- Identity functions
- Inverse functions
- Composition of functions
- Some useful functions
- Proofing problems

Definition of a function

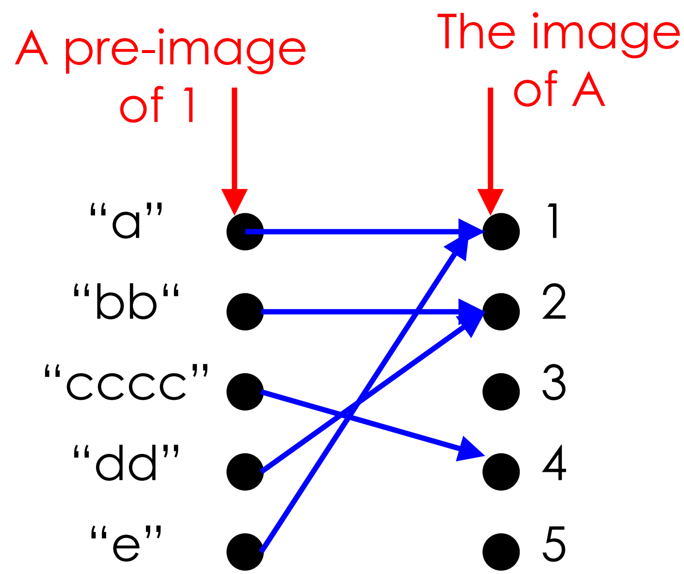
- A function takes an element from a set and maps it to a UNIQUE element in another set



More functions

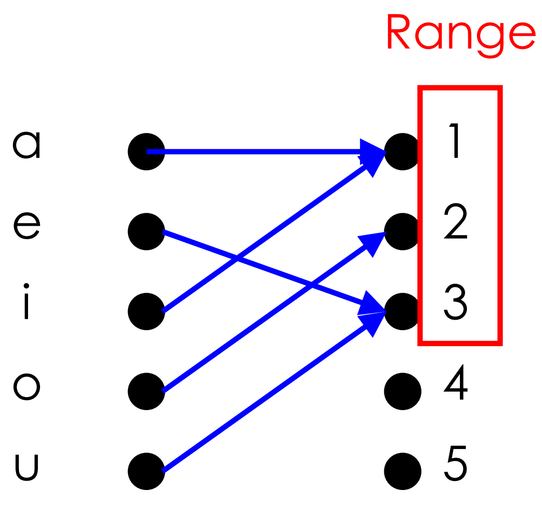


A class grade function

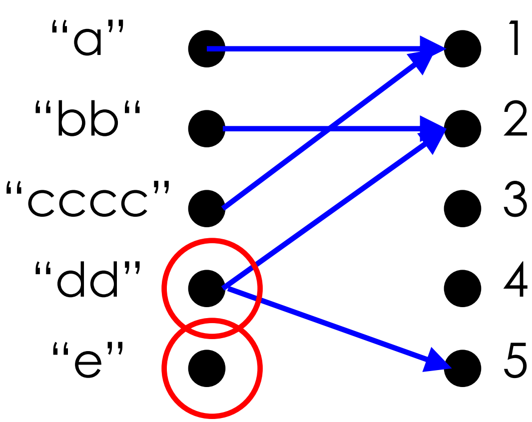


A string length function

Even More functions ☺



Some function…



Not a valid function!

Also not a valid function!

Function arithmetic

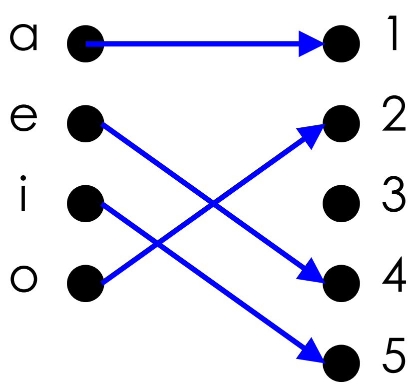
- Let $f_1(x) = 2x$
- Let $f_2(x) = x^2$
- Let f_1 and f_2 are function from A to **R**. Then f_1 and f_2 are also function from A to **R** defined by:

- $f_1 + f_2 = (f_1 + f_2)(x) = f_1(x) + f_2(x) = 2x + x^2$

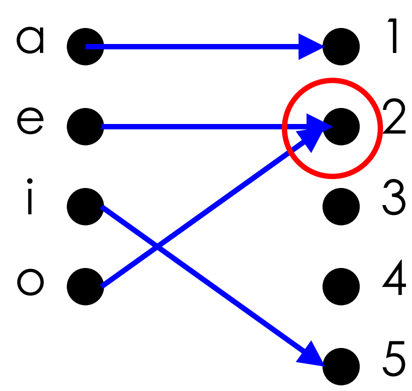
- $f_1 * f_2 = (f_1 * f_2)(x) = f_1(x) * f_2(x) = 2x * x^2 = 2x^3$

One-to-one functions

- A function *f*: A → B is one-to-one if each element in the co-domain has a unique pre-image
  - f is one to one ↔ $\forall a \forall b\ [\ f(a) = f(b) \rightarrow a = b]$
  - Or equivalently $\forall a\ \forall b\ [a \neq b \rightarrow f(a) \neq f(b)]$



A one-to-one function

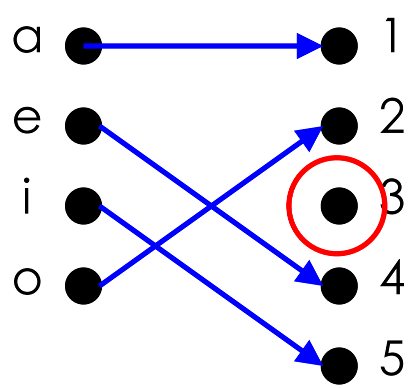


A function that is not one-to-one

More on one-to-one

- Injective is synonymous with one-to-one
- A function is an injection if it is one-to-one
- Meaning no 2 values map to the same result

- Note that there can be un-used elements in the co-domain



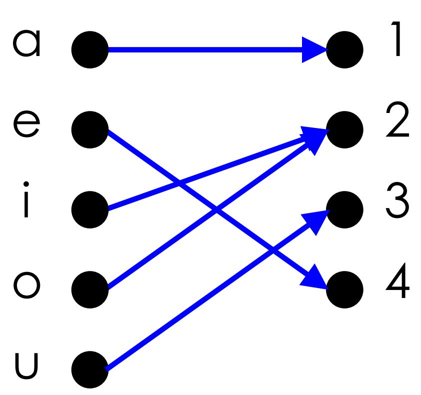
A one-to-one function

Example

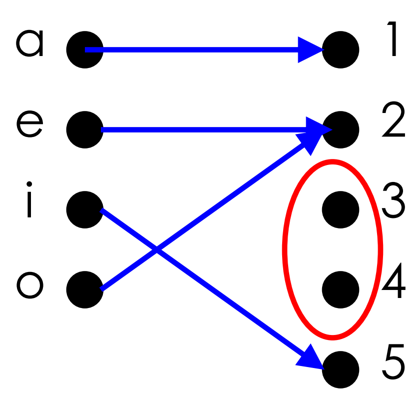
- Determine whether the function $f(x) = x^2$ from **Z** to **Z** is one to one.
- The function $f(x) = x^2$ is not one to one because for instance $f(1) = f(-1) = 1$, but $1 \neq -1$
- Is the function $f(x) = x + 1$ one to one?
- The function is one to one. **(Note:** $x + 1 \neq y + 1$ if $x \neq y$**)**

Onto functions

- A function *f*: A $\rightarrow$ B is onto if each element in the co-domain is an image of some pre-image
  - f is onto $\leftrightarrow$ $\forall y\ \exists x\ [\ f(x) = y\ ]$



An onto function

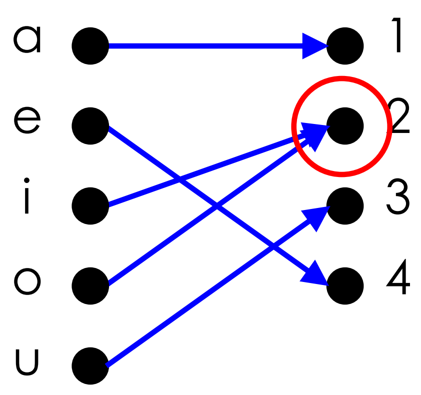


A function that is not onto

More on onto

- Surjective is synonymous with onto
- A function is an surjection if it is onto
- Meaning all elements in the right are mapped to

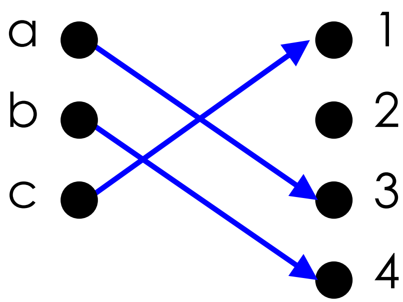
- Note that there can be multiply used elements in the co-domain



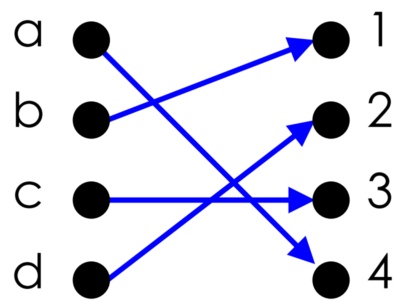
An onto function

Onto vs. one-to-one

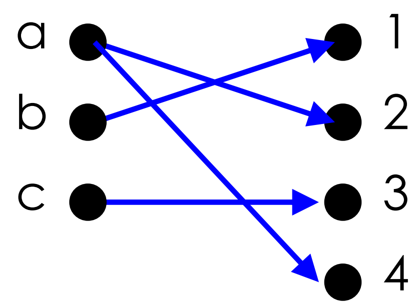
- Are the following functions onto, one-to-one, both, or neither?



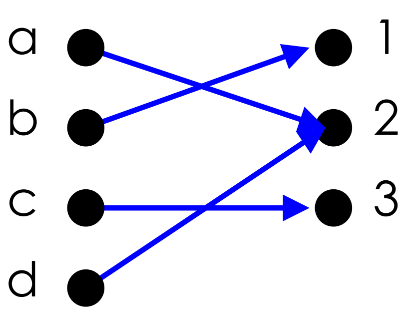
1-to-1, not onto



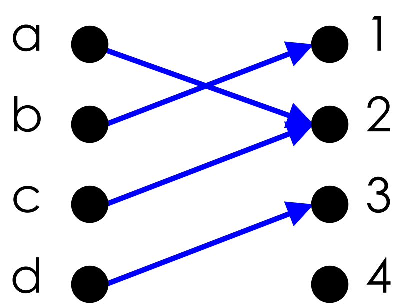
Both 1-to-1 and onto



Not a valid function



Onto, not 1-to-1



Neither 1-to-1 nor onto

Example

- Determine whether the function $f(x) = x^2$ from **Z** to **Z** is onto.

- The function $f(x) = x^2$ is not onto because for there is no integer x with $x^2 = -1$ for instance.

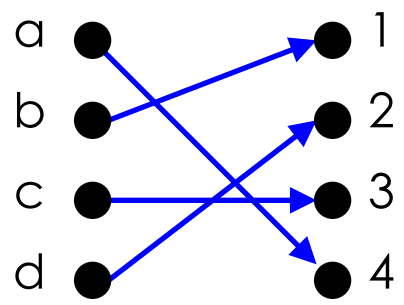
- Is the function $f(x) = x + 1$ onto?

- The function is onto because for every integer y there is an integer x such that $f(x) = y$.
- **(Note:** $f(x) = y$ if and only if $x + 1 = y$, which holds if and only if $x = y - 1$**)**

Bijections

- Consider a function that is both one-to-one and onto:

- Such a function is a one-to-one correspondence, or a bijection



Identity functions

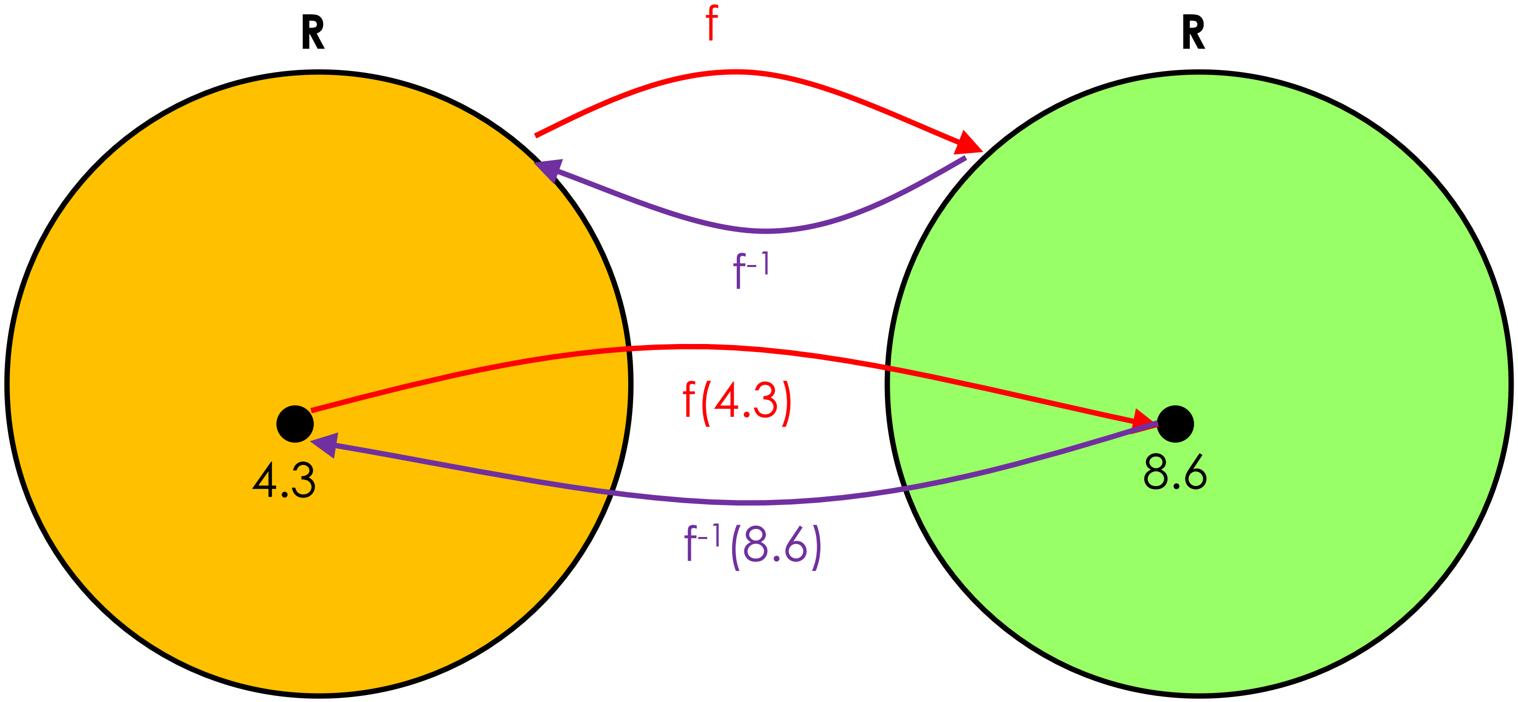
- A function such that the image and the pre-image are ALWAYS equal

- $f(x) = 1 * x$
- $f(x) = x + 0$

- The domain and the co-domain must be the same set

Inverse functions

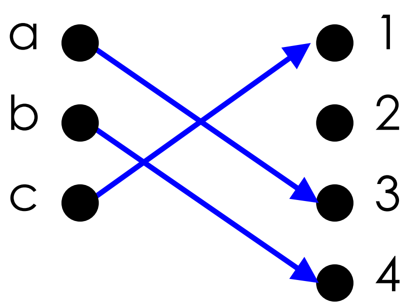
Let f(x) = 2*x



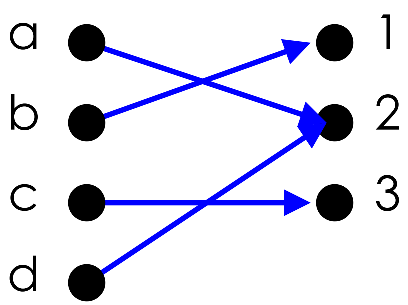
Then $f^{-1}(x) = x/2$

More on inverse functions

- Can we define the inverse of the following functions?



What is $f^{-1}(2)$?
Not onto!



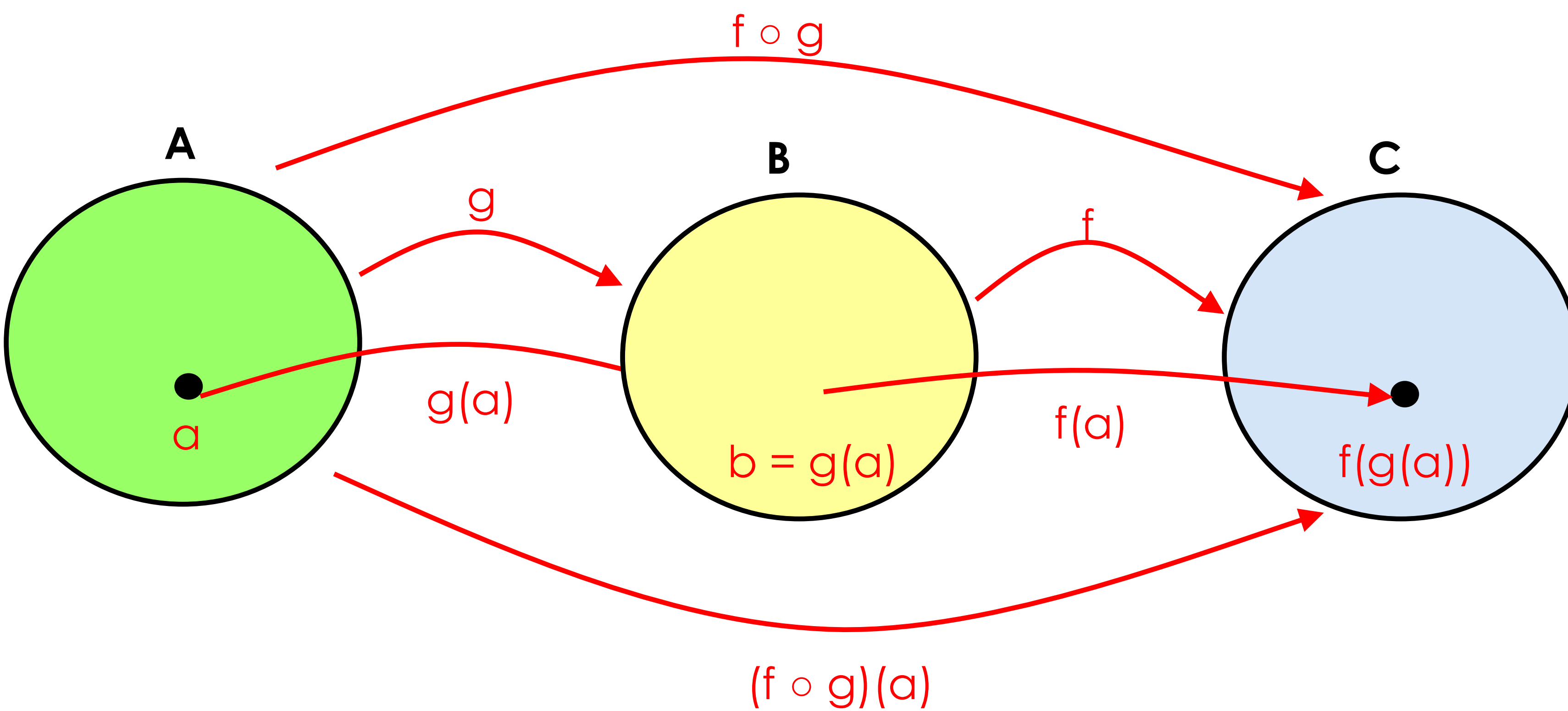
What is $f^{-1}(2)$?
Not 1-to-1!

- An inverse function can ONLY be done defined on a bijection

Compositions of functions

- Let $(f \circ g)(x) = f(g(x))$
- Let $f(x) = 2x + 3$ Let $g(x) = 3x + 2$
- $g(1) = 5$, $f(5) = 13$
- Thus, $(f \circ g)(1) = f(g(1)) = 13$

Compositions of functions

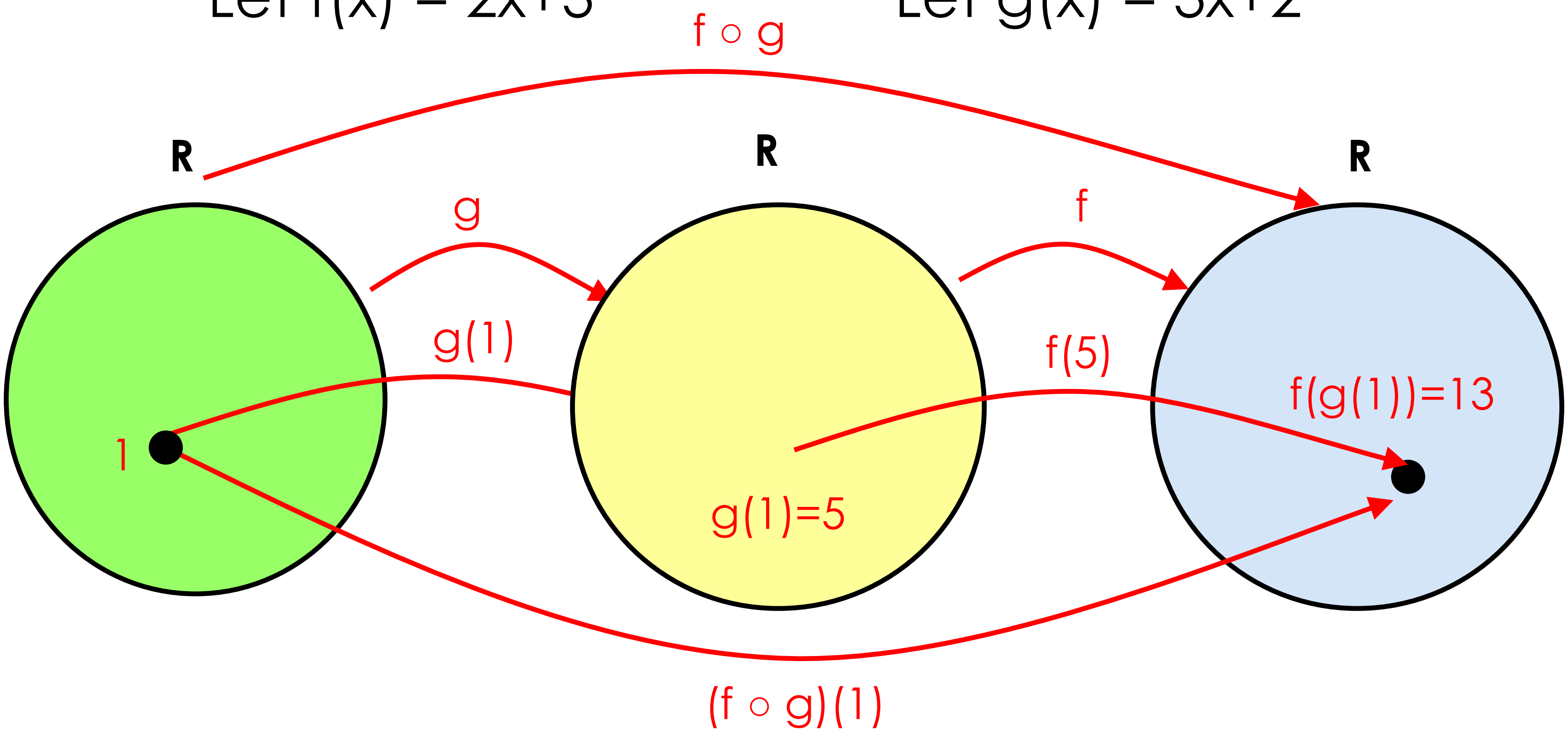


$$(f \circ g)(x) = f(g(x))$$

Compositions of functions

Let $f(x) = 2x+3$

Let $g(x) = 3x+2$



$f(g(x)) = 2(3x+2)+3 = 6x+7$

Compositions of functions

Does $f(g(x)) = g(f(x))$?

Let $f(x) = 2x+3$

Let $g(x) = 3x+2$

$f(g(x)) = 2(3x+2)+3 = 6x+7$

$g(f(x)) = 3(2x+3)+2 = 6x+11$

Not equal!

Function composition is not commutative!

- Note: fungsi yang paling kanan dioperasikan paling awal, selanjutnya fungsi di samping kirinya, and so forth.

Useful functions

- Floor: $\lfloor x \rfloor$ means take the greatest integer less than or equal to the number
- Ceiling: $\lceil x \rceil$ means take the lowest integer greater than or equal to the number
- round(x) = floor(x+0.5) = $\lfloor x+0.5 \rfloor$

Sample floor/ceiling questions

- Find these values

$\lfloor 1.1 \rfloor$	1
$\lceil 1.1 \rceil$	2
$\lfloor -0.1 \rfloor$	-1
$\lceil -0.1 \rceil$	0
$\lceil 2.99 \rceil$	3
$\lceil -2.99 \rceil$	-2
$\lfloor ½+\lceil ½ \rceil \rfloor$	$\lfloor ½+1 \rfloor = \lfloor 3/2 \rfloor = 1$
$\lceil \lfloor ½ \rfloor + \lceil ½ \rceil + ½ \rceil$	$\lceil 0 + 1 + ½ \rceil = \lceil 3/2 \rceil = 2$

Ceiling and floor properties

Let n be an integer

(1a) $\lfloor x \rfloor = n$ if and only if $n \le x < n+1$

(1b) $\lceil x \rceil = n$ if and only if $n-1 < x \le n$

(1c) $\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$

(1d) $\lceil x \rceil = n$ if and only if $x \le n < x+1$

(2) $x-1 < \lfloor x \rfloor \le x \le = \lceil x \rceil < x+1$

(3a) $\lfloor -x \rfloor = - \lceil x \rceil$

(3b) $\lceil -x \rceil = - \lfloor x \rfloor$

(4a) $\lfloor x+n \rfloor = \lfloor x \rfloor + n$

(4b) $\lceil x+n \rceil = \lceil x \rceil + n$

Ceiling property proof

- Prove rule 4a: $\lfloor x+n \rfloor = \lfloor x \rfloor + n$
  - Where n is an integer
  - Will use rule 1a: $\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$
- Direct proof!
  - Let $m = \lfloor x \rfloor$
  - Thus, $m \leq x < m+1$ (by rule 1a)
  - Add n to both sides: $m+n \leq x+n < m+n+1$
  - By rule 4a, $m+n = \lfloor x+n \rfloor$
  - Since $m = \lfloor x \rfloor$, $m+n$ also equals $\lfloor x \rfloor + n$
  - Thus, $\lfloor x \rfloor + n = m+n = \lfloor x+n \rfloor$

Factorial

- Factorial is denoted by $n!$
- $n! = n * (n-1) * (n-2) * \ldots * 2 * 1$
- Thus, $6! = 6 * 5 * 4 * 3 * 2 * 1 = 720$
- Note that $0!$ is defined to equal 1

Proving function problems

- Let *f* be a function from A to B, and let S and T be subsets of A. Show that

$$a)\ f(S \cup T) = f(S) \cup f(T)$$
$$b)\ f(S \cap T) \subseteq f(S) \cap f(T)$$

Proving function problems

- $f(S \cup T) = f(S) \cup f(T)$
- Will show that each side is a subset of the other
- Two cases!

- Show that $f(S \cup T) \subseteq f(S) \cup f(T)$
  - Let $b \in f(S \cup T)$. Thus, $b=f(a)$ for some $a \in S \cup T$
  - Either $a \in S$, in which case $b \in f(S)$
  - Or $a \in T$, in which case $b \in f(T)$
  - Thus, $b \in f(S) \cup f(T)$
- Show that $f(S) \cup f(T) \subseteq f(S \cup T)$
  - Let $b \in f(S) \cup f(T)$
  - Either $b \in f(S)$ or $b \in f(T)$ (or both!)
  - Thus, $b = f(a)$ for some $a \in S$ or some $a \in T$
  - In either case, $b = f(a)$ for some $a \in S \cup T$

Proving function problems

- $f(S \cap T) \subseteq f(S) \cap f(T)$

- Let $b \in f(S \cap T)$. Then $b = f(a)$ for some $a \in S \cap T$
- This implies that $a \in S$ and $a \in T$
- Thus, $b \in f(S)$ and $b \in f(T)$
- Therefore, $b \in f(S) \cap f(T)$

Proving function problems

- Let *f* be an invertible function from Y to Z
- Let *g* be an invertible function from X to Y
- Show that the inverse of $f \circ g$ is:
  - $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

Proving function problems

- Thus, we want to show, for all z ∈ Z and x ∈ X

$$\left((f \circ g) \circ \left(g^{-1} \circ f^{-1}\right)\right)(z) = z$$
$$\left(\left(g^{-1} \circ f^{-1}\right) \circ (f \circ g)\right)(x) = x$$

$$\begin{aligned}
\left((f \circ g) \circ \left(g^{-1} \circ f^{-1}\right)\right)(z) &= (f \circ g)\left(\left(g^{-1} \circ f^{-1}\right)(z)\right) \\
&= (f \circ g)\left(g^{-1}\left(f^{-1}(z)\right)\right) \\
&= f\left(g\left(g^{-1}\left(f^{-1}(z)\right)\right)\right) \\
&= f\left(f^{-1}(z)\right) \\
&= z
\end{aligned}$$

- The second equality is similar