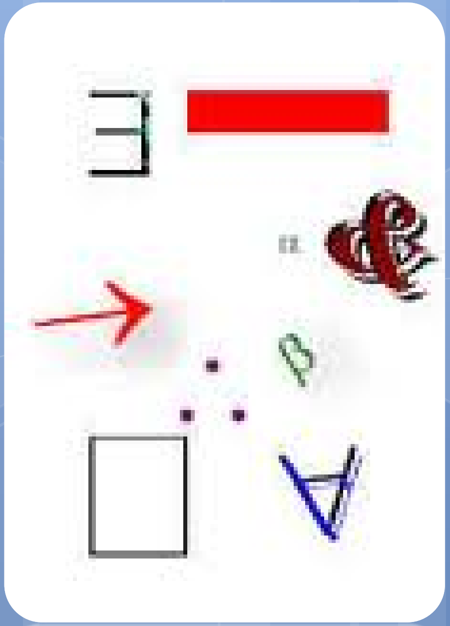
KS091201
MATEMATIKA DISKRIT
(DISCRETE MATHEMATICS)



PREDICATE & QUANTIFIER

Discrete Math Team

Outline

- Propositional function
- Function with multiple variables
- Quantifier
- Universal quantifier
- Existensial quantifier
- Binding variable
- Negating quantifier
- Translating from English
- Multiple quantifiers
- Order of quantifiers
- Negating multiple quantifiers

Propositional Functions

- Consider $P(x)$, as a symbolic notation of $x > 5$
  - $P(x)$: propositional function P at x (fungsi proposisi P untuk x)
  - x is subject
  - > 5 is predicate
  - $P(x)$ has no truth value when x is unknown
  - $P(x)$ become a proposition when we assigned certain value to x
  - The value given to x is taken from certain universe of discourse or domain (himpunan semesta)

Propositional Functions (cont.)

- Example:

  Consider $P(x) = x < 5$

  - $P(x)$ has no truth values (x is not given a value)
  - $P(1)$ is true: The proposition $1<5$ is true
  - $P(10)$ is false: The proposition $10<5$ is false
- Let $P(x) = x + 3 > x$
  - For what values of x is $P(x)$ true?

Function with Multiple Variables

- $P(x,y) = x + y = 0$
  - $P(1,2)$ is false, $P(1,-1)$ is true
- $P(x,y,z) = x + y = z$
  - $P(3,4,5)$ is false, $P(1,2,3)$ is true
- $P(x_1, x_2, x_3 \ldots x_n) = \ldots$

Quantifier

- A quantifier is "an operator that limits the variables of a proposition"
- In some cases, it's a more accurate way to describe things than Boolean propositions
- Process of bounding the variable x with a quantifier is called quantification
- Two types of quantifier will be discussed:
  - Universal quantifier
  - Existential quantifier

Universal Quantifier

- Represented by an upside-down *A*: $\forall$
  - It means "for all"
  - Let $P(x) = x+1 > x$

- We can state the following:
  - $\forall x\, P(x)$
  - English translation: "for all values of x, $P(x)$ is true"
  - English translation: "for all values of x, $x+1>x$ is true"

Universal Quantifier (cont.)

- But is that always true?
  - $\forall x\, P(x)$
- Let x = the character 'a'
  - Is 'a'+1 > 'a'?
- Let x = the state of East Java
  - Is East Java+1 > East Java?

- Don't forget to specify your universe!
  - What values x can represent
  - Called the "domain" or "universe of discourse"

Universal Quantifier (cont.)

- Let the universe be the real numbers.
- Let $P(x) = x/2 < x$
  - Not true for the negative numbers! (Called as **counterexample**)
  - Thus, $\forall x\ P(x)$ is false
    - When the domain is all the real numbers
- In order to prove that a universal quantification is true, it must be shown for **ALL** cases
- In order to prove that a universal quantification is false, it must be shown to be false for **only ONE** case

Universal Quantifier (cont.)

- Given some propositional function P(x)
- And values in the universe $x_1 \ldots x_n$
- The universal quantification $\forall x\, P(x)$ implies:

$$P(x_1) \land P(x_2) \land \ldots \land P(x_n)$$

Existensial Quantifier

- Represented by an backwards *E*: ∃
  - It means "there exists"
  - Let $P(x) = x^2 > 10$
- We can state the following:
  - $\exists x\ P(x)$
  - English translation: "there exists (a value of) *x* such that *P*(x) is true"
  - English translation: "for at least one value of *x*, $x^2 > 10$ is true"
- Note that you still have to specify your universe

Existensial Quantifier (cont.)

- Let $P(x) = x+1= x$
  - There is no numerical value x for which $x+1= x$
  - Thus, $\exists x$ P(x) is false
- Let $P(x) = x+1 = 0$
  - There is a numerical value for which $x+1= 0$
  - Thus, $\exists x\ P(x)$ is true

- In order to show an existential quantification is **true**, you only have to find **ONE** value
- In order to show an existential quantification is **false**, you have to show it's false for **ALL** values

Existensial Quantifier (cont.)

- Given some propositional function $P(x)$
- And values in the universe $x_1 \ldots x_n$
- The existential quantification $\exists x\, P(x)$ implies:

$$P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

Conclusion

Statement	When True	When False
$\forall x\ P(x)$	$P(x)$ is TRUE for every x	There is an x for which $P(x)$ is FALSE
$\exists\ x\ P(x)$	There is an x for which $P(x)$ is TRUE	$P(x)$ is FALSE for every x

Notes

- Recall that $P(x)$ is a propositional function
  - Let $P(x)$ be "$x > 0$"
- Recall that a proposition is a statement that is either true or false
  - $P(x)$ is not a proposition
- There are two ways to make a propositional function into a proposition:
  - Assign a certain value
    - For example, $P(-1)$ is false, $P(1)$ is true
  - Provide a quantification
    - For example, $\forall x\, P(x)$ is false and $\exists x\, P(x)$ is true
      - Let the universe of discourse be the real numbers

Binding Variable

- Let $P(x,y)$ be $x > y$
- Consider: $\forall x\ P(x,y)$
  - This is not a proposition!
  - What is y?
    - If it's 5, then $\forall x\ P(x,y)$ is false
    - If it's x-1, then $\forall x\ P(x,y)$ is true
- Note that y is not "bound" by a quantifier

Binding Variable (cont.)

- $(\exists x\, P(x)) \vee Q(x)$
  - The x in $Q(x)$ is not bound; thus not a proposition
- $(\exists x\, P(x)) \vee (\forall x\, Q(x))$
  - Both x values are bound; thus it is a proposition
- $(\exists x\, P(x) \wedge Q(x)) \vee (\forall y\, R(y))$
  - All variables are bound; thus it is a proposition
- $(\exists x\, P(x) \wedge Q(y)) \vee (\forall y\, R(y))$
  - The *y* in $Q(y)$ is not bound; this not a proposition

Negating Quantifiers

- Consider the statement:
  - All students in this class have Acer Laptop

- What is required to show the statement is false?
  - There exists a student in this class that does NOT has Acer Laptop

- To negate a universal quantification:
  - You negate the propositional function
  - AND you change to an existential quantification
  - $\neg \forall x\, P(x) = \exists x\, \neg P(x)$

Negating Quantifiers (cont.)

- Consider the statement:
  - There is a student in this class with Acer Laptop.
- What is required to show the statement is false?
  - All students in this class do not have Acer Laptop.
- Thus, to negate an existential quantification:
  - negate the propositional function
  - AND change to a universal quantification
  - $\neg \exists x\, P(x) = \forall x\, \neg P(x)$

Conclusion

Proposition	Negation	TRUE	FALSE
$\neg\exists x\, P(x)$	$\forall x\, \neg P(x)$	For all x, $P(x)$ is false	There is a value of x for which $P(x)$ is true
$\neg\forall x\, P(x)$	$\exists x\, \neg P(x)$	There is a value of x for which $P(x)$ is false	For all x, $P(x)$ is true

Translating from English

- What about if the universe of discourse is all people?
  - S(x) be "x is a student in this class"
  - C(x) be "x has studied Calculus"

  - Every student in this class has studied Calculus.
  - $\forall x\ (S(x) \wedge C(x))$
    - This is wrong! Why?
    - It means that "All people are students in this class and have studied Calculus"
  - $\forall x\ (S(x) \rightarrow C(x))$
    - It means that "For every person x, if x is student in this class, then x has studied Calculus"

Translating from English

- Consider:
  - "Every student in this class has visited Manado or Cianjur"

- Let:
  - S(x) be "x is a student in this class"
  - M(x) be "x has visited Manado"
  - C(x) be "x has visited Cianjur"

Translating from English

- Consider: “Some students have visited Manado”
  - Rephrasing: “There exists a student who has visited Manado”
- $\exists x\ M(x)$
  - True if the universe of discourse is all students
- What about if the universe of discourse is all people?
  - $\exists x\ (S(x) \rightarrow M(x))$
    - This is wrong! Why?
    - The statement is true although there is someone not in the class
  - $\exists x\ (S(x) \wedge M(x))$
    - There is a person x who is a student in this class and who has visited Manado

Translating from English

- Consider: "Every student in this class has visited Cianjur or Manado"

- $\forall x\ (M(x) \lor C(x))$
  - When the universe of discourse is all students in this class

- $\forall x\ (S(x) \rightarrow (M(x) \lor C(x)))$
  - When the universe of discourse is all people

Multiple Quantifiers

- You can have multiple quantifiers on a statement

- $\forall x \exists y\ P(x, y)$
  - "For all x, there exists a y such that $P(x,y)$"
  - Example: $\forall x \exists y\ (x+y = 0)$

- $\exists x \forall y\ P(x,y)$
  - There exists an x such that for all y $P(x,y)$ is true"
  - Example: $\exists x \forall y\ (x*y = 0)$

Order of quantifiers

- $\exists x \forall y$ and $\forall x \exists y$ are not equivalent!

- $\exists x \forall y \; P(x,y)$
  - $P(x,y) = (x+y = 0)$ is false

- $\forall x \exists y \; P(x,y)$
  - $P(x,y) = (x+y = 0)$ is true

Negating multiple quantifiers

- Recall negation rules for single quantifiers:
  - $\neg\forall x\, P(x) = \exists x\, \neg P(x)$
  - $\neg\exists x\, P(x) = \forall x\, \neg P(x)$
  - Essentially, you change the quantifier(s), and negate what it's quantifying
- Examples:
  - $\neg(\forall x \exists y\, P(x,y)) = \exists x \forall y\, \neg P(x,y)$
  - $\neg(\forall x \exists y \forall z\, P(x,y,z)) = \exists x \forall y \exists z\, \neg P(x,y,z)$

Negating multiple quantifiers (cont.)

- Consider $\neg(\forall x \exists y\ P(x,y)) = \exists x \forall y\ \neg P(x,y)$
  - The left side is saying "for all x, there exists a *y* such that *P* is true"
  - To disprove it (negate it), you need to show that "there exists an x such that for all *y*, *P* is false"

- Consider $\neg(\exists x \forall y\ P(x,y)) = \forall x \exists y\ \neg P(x,y)$
  - The left side is saying "there exists an x such that for all *y*, *P* is true"
  - To disprove it (negate it), you need to show that "for all x, there exists a *y* such that *P* is false"

Translating Quantifiers

- Let N(x) be the statement "x has visited North Dakota", where he domain consist of the students in your school. Express each of these quantifications in English.
  a) $\exists x\, N(x)$

     Some students in the school have visited North Dakota.

     There exists a student in the school who has visited N.D.
  b) $\forall x\, N(x)$

     Every student in the school has visited North Dakota.

     All students in the school have visited North Dakota.
  c) $\neg \exists x\, N(x)$: negation of part a)

     No student in the school has visited North Dakota.

     There does not exist a student in the school who has visited N.D.

Translating Quantifiers

- Let N(x) be the statement "x has visited North Dakota", where he domain consist of the students in your school. Express each of these quantifications in English.

d) $\exists x \neg N(x)$

Some students in the school have not visited North Dakota.

There exists a student in the school who has not visited N.D.

e) $\neg \forall x\, N(x)$: negation of part b)

It is not true that every student in the school has visited N.D.

Not all students in the school have visited N.D.

f) $\forall x \neg N(x)$

All students in the school have not visited North Dakota.

(common English sentence takes this sentence, incorrectly, the answer of part e)

Note: c) and f) are equivalent; d) and e) are also equivalent. But both pairs are not equivalent to each other.

Translating Quantifiers

Note: The domain is all integers

- The product of two negative integers is positive
  - $\forall x \forall y\ ((x<0) \land (y<0) \rightarrow (xy > 0))$
  - Why conditional instead of and?
- The average of two positive integers is positive
  - $\forall x \forall y\ ((x>0) \land (y>0) \rightarrow ((x+y)/2 > 0))$
- The difference of two negative integers is not necessarily negative
  - $\exists x \exists y\ ((x<0) \land (y<0) \land (x-y \geq 0))$
  - Why and instead of conditional?
- The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers
  - $\forall x \forall y\ (|x+y| \leq |x| + |y|)$

Translating Quantifiers

Note:The domain is all real numbers

- $\exists x \forall y\ (x+y = y)$
  - There exists an additive identity for all real numbers
- $\forall x \forall y\ (((x \geq 0) \wedge (y < 0)) \rightarrow (x-y > 0))$
  - A non-negative number minus a negative number is greater than zero
- $\exists x \exists y\ (((x \leq 0) \wedge (y \leq 0)) \wedge (x-y > 0))$
  - The difference between two non-positive numbers is not necessarily non-positive (i.e. can be positive)
- $\forall x \forall y\ (((x \neq 0) \wedge (y \neq 0)) \leftrightarrow (xy \neq 0))$
  - The product of two non-zero numbers is non-zero if and only if both factors are non-zero