**2.1.1** **Order of Concepts in Schoolbooks**

Learning math takes deliberate, repetitive practice at one’s own level (Clements and Sarama [2014]; Hofman et al. [2018]). It turns out that a lot of steps are needed to be able to do multiplication with numbers. At the start, numbers must be taught, and the first learning steps for counting begin with words of number and how they need to be associated with quantities. This is usually taught with verbal chants and frequently repeated reciter of the verbal numbers is required. Next is the connection that must be made with objects and actions in combination with counting of small numbers up to 10 and 20. A bit more difficult is the explicit understanding of cardinality and keeping track of objects that have not yet been counted, next to understanding errors in other’s counting. These steps are followed by counting backwards and counting at random starting points. More difficult skills like being able to skip counter with 10, 20, 30 till 100, and converging to counter with patterns using strategies. Continuing in counter of imagined items and keeping track of those and all this towards counter of quantitative units/place values as described by Clements and Sarama [2014]. Only after mastery of counting, operation on numbers can be though.

Schools usually have standard mathematical books with exercises. As can be read in Clements and Sarama [2014], for multiplication, elementary school books begin in grade 2 with multiplication in the form of repeated addition (2x3 = 2+2+2 or 3+3). It is assumed that children have mastered addition and subtraction problems with grouping numbers into groups of ten (11 is a group of 10 and a group of 1) and the development of place value concepts (4 in 400 means something else that the 4 in 40: 435 is 400+30+5 a.k.a. expanded notation). Building on the understanding of numbers greater than ten, of grouping and of place value. In the third grade, children practice more difficult exercises on tables, graphs

and plots and how to solve problems. At the age of 9, children development should include understanding 3D array structure and how multiplication is needed to measure the volume. It isn’t always clear what the specific concepts are that children learn, but they learn by practicing it within other skills. Clements and Sarama [2014] summarize knowledge on how children learn and explain "learning trajectories" that help diagnose the skills of students in mathematics. Learning trajectories are the natural developmental progressions in learning. It consists of a metamathematical goal, developmental path and instructional activities. Additionally, curricula in early childhood education is discussed.

For the Dutch school system, the fundamental level of control (1F) at the end of primary school is ordered by law by the Dutch government (Ministry of Education, Culture and Science). In 2006 core objectives were ordered and in 2009 reference levels were added. Core objectives describe what primary school needs to offer students, and the reference levels describe what students need to master at the end of primary school. Because these laws are compact and abstractly formulated, SLO published papers with explanations and examples. SLO, the national expertise center for curriculum development, describes the subgoals of the curriculum of students from class 2- 8 for mathematics as can be read in Noteboom et al. [2011] and Noteboom et al. [2017]. They work with a target level, called 1S, that is made consulting experts in mathematics (methods developers, researchers, curriculum developers), educational practice (teachers, guides) and organizations working for the government like the Inspection of Education. S1 can be interpreted as deepening and expansion of F1.

**2.1.2** **Mathematical Concepts**

The concepts used for this proposal are derived from the reference windows and domains described in Noteboom et al. [2011] and Noteboom et al. [2017], consisting of Numbers, Proportions, Measuring and Geometry and Connections. In each of these domains, reference windows are described with i) Notation, language and meaning: here the numbers, symbols and relationships in the math notation, meaning and pronunciation are noted. ii) Connections to each other: connections between concepts, notation, numbers and pronunciation in daily situations are given. And iii) Usage: using the skills to solve math problems. For each domain, characteristics of knowledge and skills are described: have facts and concepts ready to use, being able to reproduce examples, get routines, use techniques, and steps to approach a problem. Other examples are the knowledge why concepts and methods are used, being able to formalize, abstract over and generalize in addition to having insight.

Broadly, the time-line for this proposal contains the grades 1 to 8. Through grade 1 to 4, numbers and their operations are learned. In Grade 4 addition and subtraction are learned by mental arithmetic. In the 5th grade they continue with numbers up to 1000. Additionally, they get multiplication and division by mental arithmetic and the beginning of decimal numbers up to two decimals and continue this in grade 6. In grade 7 sums up to three decimals are practiced together with addition and subtraction of unequal decimal numbers. More over, multiplication with 3 or 4 numbers (4x253) is thought together with the inverse relationship between multiplication and division. In grade 8 they learn the relation between division and decimal numbers as can be derived from the publication from Noteboom et al. [2017]

Given the information on the domains and the actual question provided in the data, the following concepts were pinpointed for multiplication and most of the examples come from Noteboom et al. [2011]. It should be noted that some of these are more of an effect than a concept. Take for example recursive items where a number is multiplied by itself, it is not really a concept that makes it easier to give the correct answer, but nonetheless these will most likely be picked up by data analysis because they are mostly rated easier than closely related items (6x6 v.s. 6x7). The tables are the basic multiplication problems for mental arithmetic (2x1,2x2,2x3,2x4 ect.) Tables #1-#2 and up represent the tables but with more numbers where strategies will need to be used.

1. Tables one: x1
2. Tables 2-4: x2, x3 and x4
3. Tables five: x5
4. Tables ten: x10
5. Tables 6-9: x6, x7, x8 and x9
6. Recursive items: 2x2; 8x8
7. Mirror items: 7x6 = 6x7
8. Times eleven, twelve and fifteen: x11, x12 and x15
9. Multiples of ten: 8x10; 20x40; 50x500
10. Times ten, hundred or thousands: x100, x1000; 65x1000
11. Times decimals tenths, hundredths: x0,1, x0,01
12. Times decimals thousands: x0,001
13. Tens times decimals: 1000x2,50; 10x0,45
14. Times decimal harder: 14,3x5
15. Tables #1-#2: 6x28
16. Tables #1-#3: 7x165
17. Tables #2-#2: 37x83
18. Tables #2-#3: 52x853
19. Tables #3-#3: 234x523
20. Times zero: x0; 0x10, 10x0
21. Zero times decimal: 0x0,4

It should be noted that there is a differences in the mastery levels of the concepts. The same kind of concepts can be solved with different attributes. Solving a question with pen and paper is easier than solving it mentally. Or being able to explain why strategies work, for example, being able to explain why the terms 3x5 = 5x3 is reversible, but that this strategy will not work for other operators like 3-5 != 5-3. Therefore, some associated skills involving multiplication are mentioned here.

* Understanding why a ten-point position system is used. Why it is more efficient to count in 10s than in 9s.
* Expanded notation: How much is the ’n’ worth in n, n0, 3n, 0,n5. To be able to explain that the ’4’ in 40, 34 or 0,45 have a different value.
* Place value concepts: Splitting of numbers in thousands, hundreds, tens and ones or from decimals to integers. How much is 4351? 4000+300+51+1, How many times does 0,01 fits in 10? And likewise 3,25 = 3 + 0,2 + 0,05.
* Understanding notation (..x..=..) same as (..=..x..)
* Comparing numbers: What is the value of a number compared to each other and their representation of the number line.
* Applying numbers in length, area, volume, time, money, dates, weight and temperature and understanding these scales.
* Understanding volume under graph or legend on a map and the ratio. Take for example the ratio of inhabitants for a city and a village.
* Distributive Strategy: Splitting a question into subquestions, for example 7x18 = 7x10 + 7x8.
* Strategy to double: Seeing that 6x24 is the double of 3x24 and using this to solve questions.
* Strategy to half: Seeing that 10x12 = 120, so 5x120 is the half.
* Analogies: 7x5=35 and 7x50=350.
* Strategy to group: Similar to splitting, 2x5x8 = (2x5)x8 = 10x8. Or understanding that 4x18+2x18 = 6x18.
* Associative Strategy 12x7x8=(12x7)x8).
* Strategy to compensate: For example 4x99 = 4x(100-1) = 4x100 - 4x1. Or 3x2,98 = 3x3,00
  + 3x.02.
* Understanding inversive relation between multiplication and division. ?x25=100 for 100:25. Or how many times does 0.5 fit in 10?
* Multiplication and division (3x (1/2)).
* Interpret a story and formulating the right approach. For example, a class of 22 students go to the Zoo with three supervisors, a ticket costs 3 euro for children and 5 euro for

adults. Other example, we want paint the wall, the width of the room is 12 and the height is 3, how much *m*2 paint do we need? We can paint 1 *m*2 in a minute, how much time do we need to paint all?

* Estimating totals: Being able to given an estimation of the correct answer.
* Calculation on calculator (and control by estimation).
* Understanding ’residue (average childbirth 1,75).
* Understanding wrong answers of others.