

Local Asymptotic Minimax Inference for Set-Identified Impulse Responses from Local Projection Instrumental Variable Models

Myung Hwan Seo (Seoul Nat. U.)
with Bonsoo Koo, Masaya Takano (Monash)
Seojeong (Jay) Lee (SNU),

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Background I

- A **Structural impulse response (IRs)**, the reaction of a dynamic system in response to some external (policy) change, is a key causal parameter in empirical macroeconomics.
 - 1 Typically, it is defined through the Structural Vector Autoregression (SVAR), e.g.

$$A_0 Y_t = A_1 Y_{t-1} + u_t,$$

under diverse identifying assumptions on the matrix A_0 or $Eu_t u_t'$.

- 2 The identification is also achieved by means of other external variables (or instruments).
- 3 Sign restrictions result in **set identification**.
- 4 It is often estimated by **Bayesian** approaches

Background II

- **Local Projection** and **Local Projection Instrumental Variable (LP-IV)** estimation are alternative popular approaches to SVAR.
 - 1 They are partial characterization of the system and intended to capture certain impulse responses only.
 - 2 They have not accommodated sign restrictions/set identification.
 - 3 In particular, it is not well understood what **Local Projection Instrumental Variable (LP-IV)** estimator identify under general conditions.

Background III

- Inference for set-identified structural IR can be nonstandard and challenging.
 - **Uniformity** over a class of data distributions (Imbens and Manski (2004), Stoye (2009))
 - Boundary points of the identified set are possibly **nondifferentiable functions**: (\Rightarrow no unbiased estimator, e.g. Hirano and Porter (2012))
 - The set may not be bounded.
- From the frequentist perspective (the set identified case):
 - Existing frequentist results: either conservative or only point-wise validity (Gafarov et al. (2018), Granziera et al. (2018))
 - Most Bayesian inference procedures can be invalid (Moon and Schorfheide (2012), Kitagawa et al. (2020)) and computationally demanding.
 - They are based on fully specified **SVAR**. No work exists based on Local Projection (LP).

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 - ➌ overcomes the issues related to the nondifferentiability of boundary.
 - ➍ computationally cheap.

Preview: LP-IV

Inference with external instruments

Stock and Watson (2018) consider identification of structural **IRs** to $\epsilon_{(1),t}$ in **LP-IV** (local projection with instrument variables) models with an instrument z_t for $\epsilon_{(1),t}$ where an LP-IV estimand is given by

$$\beta_{IV,(i,h)} = \mathbb{E}[y_{(i),t+h}z_t] / \mathbb{E}[y_{(1),t}z_t] \quad (1)$$

for $i = 1, \dots, n$ and $h \geq 0$. Then,

$$\theta_{h,(i,1)} = \beta_{IV,(i,h)} \quad i = 1, \dots, n \quad (2)$$

under the unit effect normalization and the following conditions:

Condition LP-IV

- ① $\mathbb{E}[\epsilon_{(1),t}z_t] \neq 0$. (**relevance**)
- ② $\mathbb{E}[\epsilon_{(2:n),t}z_t] = 0$. (**contemporaneous exogeneity**)
- ③ $\mathbb{E}[\epsilon_{t+j}z_t] = 0, \forall j \neq 0$. (**lead-lag exogeneity**)

Inference with external instruments: composite shock

- An instrument z_t may be correlated with multiple structural shocks (Jarociński and Karadi (2020), Braun and Brüggemann (2023), Koo et al. (2023)), in which case Condition LP-IV.2 (**contemporaneous exogeneity**) is violated.
- In this setup, Koo, Lee, Seo (2023), (**KLS23**), establish the LP-IV based set-identification of structural **IRs** under sign restrictions on a set of n_z instruments $\{z_t^{(k)}\}_{k=1}^{n_z}$, $n_z \in \mathbb{N}_+$.

Structural Vector Moving Average Model

$$\begin{array}{ccc}
 \begin{pmatrix} x_t \\ \vdots \\ y_t \end{pmatrix} & = \Theta(L) & \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \vdots \\ \varepsilon_{m,t} \end{pmatrix} \\
 n \times 1 & & m \times 1 \\
 \text{observed} & & \text{unobserved} \\
 \text{endogenous} & & \text{structural shocks}
 \end{array}$$

- $\varepsilon_{s,t}$'s are mutually uncorrelated, $E[\varepsilon_t] = 0$, $E[\varepsilon_t \varepsilon_t'] > 0$
- $\Theta(L) = \Theta_0 + \Theta_1 L + \Theta_2 L^2 + \dots$, L is the lag operator
- Θ_h : $n \times m$ matrix of impulse responses whose elements are

$$\theta_{h,ys} \equiv E[y_{t+h} | \varepsilon_{s,t} = 1] - E[y_{t+h} | \varepsilon_{s,t} = 0]$$

An LP-IV Estimand May Not Be Meaningful

- Suppose x_t is the policy variable and the shock to x_t is $\xi_t = \varepsilon_{1,t} + \varepsilon_{2,t}$
- Under Assumption 1 and the unit effect normalization

$$\beta_{IV} = w_1 \theta_{h,y1} + w_2 \theta_{h,y2}$$

where $w_s = \text{Cov}(z_t \varepsilon_{s,t}) / (\text{Cov}(z_t \varepsilon_{1,t}) + \text{Cov}(z_t \varepsilon_{2,t}))$

- However, a weight can be negative
- Why is this a problem?

$$\begin{array}{ccccccccc} \beta_h & = & w_1 & \theta_{h,y1} & + & w_2 & \theta_{h,y2} \\ \uparrow & & \uparrow & \uparrow & & \uparrow & \uparrow \\ 0 & & 2 & 1 & & -1 & 2 \end{array}$$

- "Negative weights" has been an important issue in econometrics in recent years, e.g.,
 - TWFE: de Chaisemartin and D'Haultfoeuill (2020, AER)
 - 2SLS: Mogstad, Torgovitsky, and Walters (2021, AER)

Two LP-IV estimands can yield an Identified Set

- Two instruments: z_t^A and z_t^B
- Two separate LP-IV estimands: β_h^A and β_h^B
- Intersection gives one of the following identified sets:

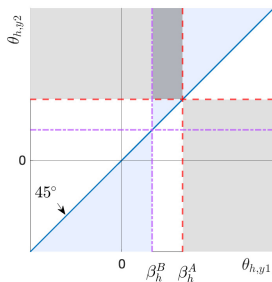
$$UL = \{\theta_{h,ys} : \theta_{h,ys} < \min(\beta_h^A, \beta_h^B) \text{ or } \max(\beta_h^A, \beta_h^B) < \theta_{h,ys}\} \quad (3)$$

$$U = \{\theta_{h,ys} : \theta_{h,ys} < \min(\beta_h^A, \beta_h^B)\} \quad (4)$$

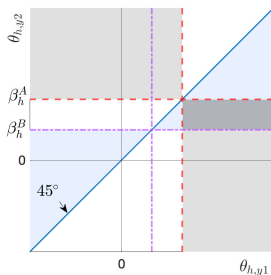
$$L = \{\theta_{h,ys} : \max(\beta_h^A, \beta_h^B) < \theta_{h,ys}\} \quad (5)$$

$$T = \{\theta_{h,ys} : \min(\beta_h^A, \beta_h^B) < \theta_{h,ys} < \max(\beta_h^A, \beta_h^B)\} \quad (6)$$

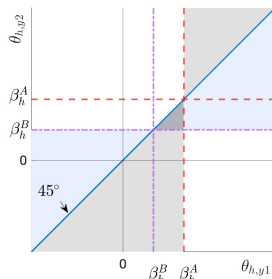
- T is the most informative, UL is the least



(d) $\Theta_{++}^A \cap \Theta_{+-}^B$, $\beta_h^A > \beta_h^B$



(e) $\Theta_{++}^A \cap \Theta_{-+}^B$, $\beta_h^A > \beta_h^B$



(f) $\Theta_{+-}^A \cap \Theta_{-+}^B$, $\beta_h^A > \beta_h^B$

- Two IVs with different signs ($+-$ vs $-+$) gives the informative bounds for both $\theta_{h,y1}$ and $\theta_{h,y2}$

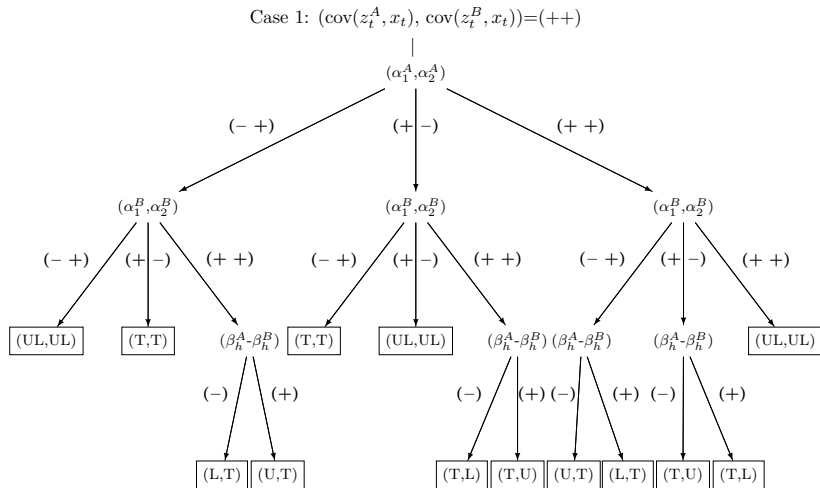
LP-IV as a Bound of the Identified Set

Proposition 1

Suppose that random variables z_t^A and z_t^B satisfy Assumption 1. We assume that $\text{cov}(z_t^A, x_t) > 0$, $\text{cov}(z_t^B, x_t) > 0$, and $\beta_h^A > \beta_h^B$ without loss of generality.

- ① If $\Theta_{++}^A \cap \Theta_{++}^B$ or $\Theta_{+-}^A \cap \Theta_{+-}^B$ or $\Theta_{-+}^A \cap \Theta_{-+}^B$, then $\theta_{h,y1} \in UL$ and $\theta_{h,y2} \in UL$.
- ② If $\Theta_{++}^A \cap \Theta_{+-}^B$, then $\theta_{h,y1} \in T$ and $\theta_{h,y2} \in L$.
- ③ If $\Theta_{+-}^A \cap \Theta_{++}^B$, then $\theta_{h,y1} \in T$ and $\theta_{h,y2} \in U$.
- ④ If $\Theta_{++}^A \cap \Theta_{-+}^B$, then $\theta_{h,y1} \in L$ and $\theta_{h,y2} \in T$.
- ⑤ If $\Theta_{-+}^A \cap \Theta_{++}^B$, then $\theta_{h,y1} \in U$ and $\theta_{h,y2} \in T$.
- ⑥ If $\Theta_{+-}^A \cap \Theta_{-+}^B$ or $\Theta_{-+}^A \cap \Theta_{+-}^B$, then $\theta_{h,y1} \in T$ and $\theta_{h,y2} \in T$.

Reference for Practitioners



Example: Jarociński and Karadi (2020)

- The US Federal Open Market Committee (FOMC) announcements are usually considered as monetary policy shocks
- Observes that the interest rate and stock price surprises often move in the same direction upon the FOMC announcements
- FOMC announcement:

$$\tilde{\zeta}_t = \varepsilon_{mp,t} + \varepsilon_{cb,t}$$

- $\varepsilon_{mp,t}$: pure monetary policy shock
- $\varepsilon_{cb,t}$: the central bank information shock

IV and Sign Restrictions

- High-frequency financial market surprises (the change between 10 minutes before and 20 minutes after the announcements)
- Two IVs: z_t^{ff} and z_t^{sp}
- z_t^{ff} : high-frequency surprise in the fed funds futures
- z_t^{sp} : high-frequency surprise in the stock price
- Sign restrictions are

$$\begin{aligned}
 z_t^{ff} : \quad & \text{Cov}(z_t^{ff}, \varepsilon_{mp,t}) > 0, \quad \text{Cov}(z_t^{ff}, \varepsilon_{cb,t}) > 0 \quad \Rightarrow \Theta_{++}^{ff} \\
 z_t^{sp} : \quad & \text{Cov}(z_t^{sp}, \varepsilon_{mp,t}) < 0, \quad \text{Cov}(z_t^{sp}, \varepsilon_{cb,t}) > 0 \quad \Rightarrow \Theta_{-+}^{sp}
 \end{aligned}$$

Identified Set for Componentwise IR

- Recall

$$\beta_h^j = w_{mp}^j \theta_{h,mp} + w_{cb}^j \theta_{h,cb}$$

where

$$w_s^j = \frac{\text{Cov}(z_t^j, \varepsilon_{s,t})}{\text{Cov}(z_t^j, x_t)}$$

for $j = ff, sp$ and $s = mp, cb$

- $\theta_{h,mp}$: response of y_{t+h} to the unit change in the pure monetary shock (corresponding to 25BP change in x_t)
- $\theta_{h,cb}$: response of y_{t+h} to the unit change in the central bank information shock (corresponding to 25BP change in x_t)
- Obtain the identified sets L and T for each component IR by $\Theta_{++}^{ff} \cap \Theta_{-+}^{sp}$.

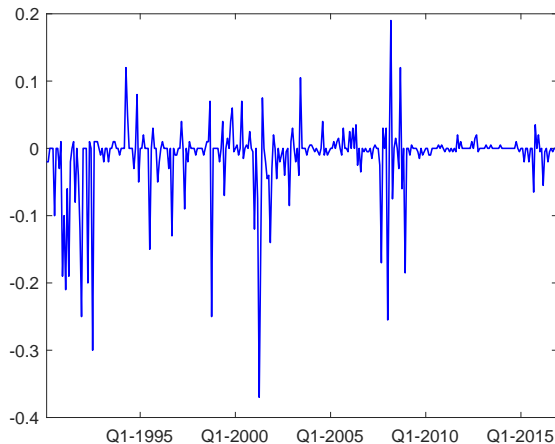
Data and Model

- Econometric model:

$$y_{t+h} = \mu_h + \beta_h x_t + \phi_h(L)' R_{t-1} + u_{t+h}$$

- y_t : macro variable of interest
- x_t : monthly average of the 1-year Treasury yield
- R_t : all the macro var including y_t , x_t , and z_t
- $\phi_h(L)$: a coeff vector of polynomial in the lag operator of order 12
- β_h : Responses to 25BP increase in 1-year government bond yield
- $\hat{\beta}_h^{ff}$, $\hat{\beta}_h^{sp}$: LP-IV est using z_t^{ff} and z_t^{sp} one at a time, respectively

high-frequency surprises in the fed funds futures



Source: Gertler and Karadi (2015)

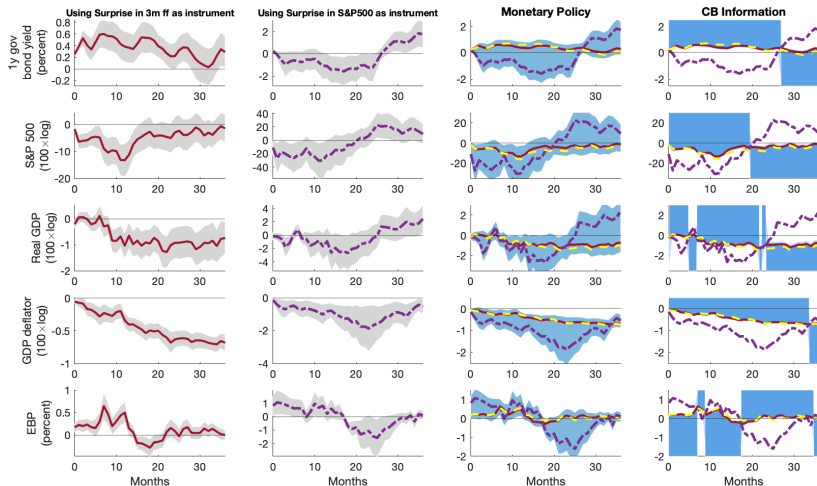


Figure: Responses to 25BP Increase in One-year Government Bond Yield

Summary

- IR analysis using LP-IV when the macro shock is composite
- LP-IV estimand is an affine combination of componentwise IR (non-negative weights under the same-sign condition)
- Main intuition is that the IV may be heterogeneously correlated with the macro shock elements, which generates heterogeneous treatment
- LP-IV estimand should be carefully interpreted as a structural IR
- Simple to implement identification strategies for componentwise IR are proposed

Besides, it is Not Okay to use 2SLS

- LP-2SLS or LP-GMM with multiple instruments, e.g., Stock and Watson (2018), Ramey and Zubairy (2018), **does not work**.
- Why? Andrews (2019) shows 2SLS is a linear combination of ind. IV estimand
- 2SLS can be non-causal even when the causal LP-IV estimands are combined
- An example is given in the paper
- Additionally, the overidentifying restrictions test is not recommended

Optimal Confidence Interval

Inference for an interval identified parameter

For a univariate parameter of interest $\theta \in \Omega_\theta \subset \mathbb{R}$, suppose the identified set Θ_0 for θ given the data distribution P is given by

$$\Theta_0(P) = [\theta_l, \theta_u] \quad (7)$$

where a pair of boundary points (θ_l, θ_u) is a functional of some reduced-form parameter $\beta(P) \in \Omega_\beta \subset \mathbb{R}^{n_\beta}$:

$$\Gamma(\beta(P)) = \begin{bmatrix} \theta_l \\ \theta_u \end{bmatrix} \quad (8)$$

for some mapping Γ from the parameter space Ω_β of β to the space of (θ_l, θ_u) :

$$\{(\theta_l, \theta_u)^\top \in \Omega_\theta \times \Omega_\theta : \theta_l \leq \theta_u\} \subset \mathbb{R}^2. \quad (9)$$

Optimal Confidence regions for the parameter

Our goal is the construction of **minimax optimal confidence regions (CRs)**.
Two notions of CRs under set-identification:

- 1 **CR** for the identified set Θ_0
- 2 **CR** for the "true" value θ_0 , cf. Imbens and Manski (2004).

Estimated identified set $\hat{\Theta} = [\hat{\theta}_l, \hat{\theta}_u]$

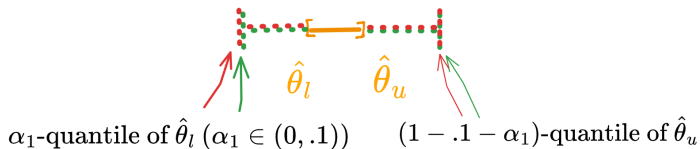


$\mathcal{CR}_{1-\alpha}$: **CR** for the point-identified case

with size .9

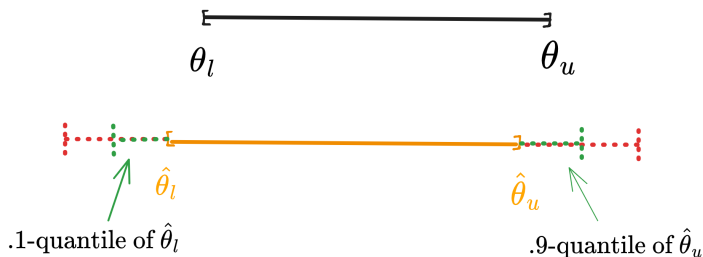
$$\bullet$$

$$\theta_l = \theta_u$$



$\mathcal{CR}_{1-\alpha}$: **CR** for the set-identified case

with size .9



Nonstandard inference: 1. Uniformity

In practice, it is typically unknown a priori whether θ is set- or point-identified.

- Ruling out point-identification leads to under-coverage when θ is (or is close to be) indeed point-identified.

As pointed out by Imbens and Manski (2004), a **CR** for the parameter θ needs to be uniformly valid over a given class \mathcal{P} of data distributions P :

$$\inf_{P \in \mathcal{P}} \inf_{\theta \in \Theta_0(P)} \Pr_P(\theta \in \mathcal{CR}_{1-\alpha}) \geq 1 - \alpha. \quad (10)$$

to ensure finite-sample validity under (near) point-identification.

Nonstandard inference: 2. Nondifferentiability of Γ

Recall

$$\begin{bmatrix} \theta_l \\ \theta_u \end{bmatrix} = \Gamma(\beta(P)) \quad (11)$$

for some functional Γ of a finite-dimensional reduced form parameter β .

Typically, a \sqrt{T} -consistent and asymptotically normal estimator $\hat{\beta}$ of $\beta(P)$ is available:

$$\sqrt{T}(\hat{\beta} - \beta(P)) \xrightarrow{d} \mathcal{N}[\mathbf{0}, \Sigma] \quad (12)$$

where Σ is positive definite.

Let $(\hat{\theta}_l, \hat{\theta}_u)$ be a (naive) plug-in estimator for (θ_l, θ_u) defined as

$$\begin{bmatrix} \hat{\theta}_l \\ \hat{\theta}_u \end{bmatrix} = \Gamma(\hat{\beta}) \quad (13)$$

Nonstandard inference: 2. Nondifferentiability of Γ

If Γ is differentiable, $(\hat{\theta}_I, \hat{\theta})_u$ is \sqrt{T} -consistent and asymptotically normal by the continuous mapping theorem.

However, for set-identified IRs, Γ may be only continuous but not differentiable at $\beta(P)$, in which case Hirano and Porter (2012) show

- There exists no unbiased estimator for (θ_I, θ_u) .
- There exists no regular estimator for (θ_I, θ_u) .
- Asymptotic properties of estimators are not invariant to \sqrt{T} -local perturbation of P .
- e.g. $\max(\beta), \min(\beta)$ are non-differentiable on the line that $\beta_1 = \beta_2$.

The asymptotic distribution of an estimator for (θ_I, θ_u) is not normal in general.

Nonstandard inference: 2. Remarks on nondifferentiability issues

If Γ is not differentiable, a "good" estimator $\check{\beta}$ (e.g., semiparametric efficient) for β does not necessarily lead to a "good" plug-in estimator $\Gamma(\check{\beta})$.

Then, what are desirable properties of $\mathcal{CR}_{1-\alpha}$ for θ we wish to achieve?

- $\mathcal{CR}_{1-\alpha}$ has uniform level of $1 - \alpha$ over some family of \sqrt{T} -local perturbations of P .
- The average length of $\mathcal{CR}_{1-\alpha}$ is as small as possible while maintaining the uniform level.

We formulate this problem as a minimization problem of a **local asymptotic minimax risk**, Hájek (1972), for $\mathcal{CR}_{1-\alpha}$.

Local asymptotic minimax inference

- Song (2014) and Fang (2014) study local asymptotic minimax optimality in point estimation problems of a univariate parameter when Γ is possibly nondifferentiable.
- We consider local asymptotic minimax problems for confidence regions, which are random sets which depend on a multivariate parameter vector and nonlinear constraints.

Element-wise additive corrections

- Given a \sqrt{T} -consistent and asymptotically normal estimator $\hat{\beta}$, we consider a plug-in type estimator $\Gamma(\tilde{\beta})$ with element-wise corrections where

$$\tilde{\beta} = \hat{\beta} + \tilde{c}/\sqrt{T} \quad (14)$$

where \tilde{c} is a sequence of $\mathbb{R}^{n_\beta} \times 1$ random vectors which converges in probability to some nonrandom limit $c \in \mathbb{R}^{n_\beta}$.

- Optimality of element-wise corrections of the form (14) is known for some class of loss functions (Van der Vaart (1992), Song (2014), Fang (2014)).

on Bayesian inference for set-identified models

- For point-identified models, a Bayesian credible set is an asymptotically valid confidence set from the frequentist view by the Bernstein–von Mises theorem.
- For **set-identified** models, it is not (Moon and Schorfheide (2012)).
 - The prior belief is not asymptotically negligible.
 - "asymptotically Bayesian highest-posterior- density sets exclude parts of the estimated identified set, whereas it is well known that frequentist confidence sets extend beyond the boundaries of the estimated identified set."
 - Many of existing empirical works belong to this case.
- Giacomini and Kitagawa (2021) propose a multiple-prior Bayesian inference approach which provides a robust Bayesian credible set for the identified set with the correct frequentist coverage probability.

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 - Many of existing empirical works belong to this case.
- Giacomini and Kitagawa (2021) propose a multiple-prior Bayesian inference approach which provides a robust Bayesian credible set for the identified set with the correct frequentist coverage probability.
 - However, GK relies crucially on the assumption that Γ is differentiable and the set is bounded.

Local Asymptotically Normal (LAN) limit experiment

We consider the local drifting sequence $\{\beta_T(\mathfrak{h})\}$ of the reduced form regular parameter defined as

$$\beta_T(\mathfrak{h}) = \beta + \mathfrak{h}/\sqrt{T}, \quad \mathfrak{h} \in \mathcal{H} \quad (15)$$

where $\mathcal{H} \subset \mathbb{R}^2$.

Assumption 1 (Regular estimator $\hat{\beta}$ of β)

① $\hat{\beta}$ is a regular asymptotically linear estimator of $\beta(P_0)$:

$$\sqrt{T}(\hat{\beta} - \beta_T(\mathfrak{h})) \xrightarrow{P_{\mathfrak{h},T}} \mathcal{N}[\mathbf{0}, \Sigma] \quad \forall \mathfrak{h} \in \mathcal{H} \quad (16)$$

where Σ is positive definite.

② The estimator $\hat{\Sigma}$ converges in probability to Σ uniformly over $\mathfrak{h} \in \mathcal{H}$.

Local Asymptotic Minimax Optimal **CR**

We are interested in the local asymptotic minimax optimal confidence region for θ at a level $1 - \alpha$ in terms of the minimum average length. More specifically, we construct a random set $\mathcal{CR}_{1-\alpha}$ such that

$$\mathcal{R}(\mathcal{CR}_{1-\alpha}) = \sup_{b \in [0, \infty)} \liminf_{T \rightarrow \infty} \sup_{\mathfrak{h} \in \mathcal{H}_b} \mathbb{E}_{T, \mathfrak{h}} \left[\sqrt{T} \text{vol}(\mathcal{CR}_{1-\alpha} \setminus \Theta_0(P_{T, \mathfrak{h}})) \right] \quad (17)$$

subject to

$$\sup_{b \in [0, \infty)} \limsup_{T \rightarrow \infty} \sup_{\mathfrak{h} \in \mathcal{H}_b} \sup_{\theta \in \Theta_0(P_{T, \mathfrak{h}})} \Pr_{T, \mathfrak{h}}(\theta \notin \mathcal{CR}_{1-\alpha}) \leq \alpha. \quad (18)$$

where $\mathcal{H}_b = \{\mathfrak{h} \in \mathcal{H} : \|\mathfrak{h}\| \leq b\}$.

Leading case

Here, we focus

① on the class of

$$\begin{aligned}\Theta_0(P) &= [\min(\beta_{(1)}(P), \beta_{(2)}(P)), \max(\beta_{(1)}(P), \beta_{(2)}(P))] \\ &= [\beta_{(1)}(P), \beta_{(2)}(P)] \cup [\beta_{(2)}(P), \beta_{(1)}(P)]\end{aligned}\quad (19)$$

② on the near-identified case where both $\beta_{(1)}$ and $\beta_{(2)}$ is in a \sqrt{T} -local neighborhood of θ , i.e. $\beta = (\theta, \theta)^\top$ in (15).

Then,

$$\Theta_0(P_{T,h}) = \begin{cases} [\beta_{T,(1)}(h_{(1)}), \beta_{T,(2)}(h_{(2)})] & \text{if } h_{(1)} \leq h_{(2)} \\ [\beta_{T,(2)}(h_{(2)}), \beta_{T,(1)}(h_{(1)})] & \text{if } h_{(1)} > h_{(2)} \end{cases}$$

Class of CR's

- we restrict our attention to a plug-in type estimator with element-wise additive correction terms:

$$\begin{aligned} \mathcal{CR}_{1-\alpha} = & \left[\min(\hat{\beta}_{(1)} + \tilde{c}_{l,(1)} / \sqrt{T}, \hat{\beta}_{(2)} + \tilde{c}_{l,(2)} / \sqrt{T}), \right. \\ & \left. \max(\hat{\beta}_{(1)} + \tilde{c}_{u,(1)} / \sqrt{T}, \hat{\beta}_{(2)} + \tilde{c}_{u,(2)} / \sqrt{T}) \right] \end{aligned} \quad (20)$$

- in line with Chernozhukov, Lee, Rosen (2013) in the sense that we do not construct the critical values based on the distributions of the boundary estimators, like $[\min(\hat{\beta}) + c_1 / \sqrt{T}, \max(\hat{\beta}) + c_2 / \sqrt{T}]$.
- optimality of element-wise corrections of this form is known for some class of loss functions (Van der Vaart (1992), Song (2014), Fang (2014)) in case of the point estimation.

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- in line with Chernozhukov, Lee, Rosen (2013) in the sense that we do not construct the critical values based on the distributions of the boundary estimators, like $[\min(\hat{\beta}) + c_1 / \sqrt{T}, \max(\hat{\beta}) + c_2 / \sqrt{T}]$.
- optimality of element-wise corrections of this form is known for some class of loss functions (Van der Vaart (1992), Song (2014), Fang (2014)) in case of the point estimation.
- However, the minimax optimization problem (20) can be **computationally prohibitive**.

Under this limit experiment

The risk $\ddot{\mathcal{R}}_\alpha(c; \hat{\beta})$ simplifies to

$$\begin{aligned} \ddot{\mathcal{R}}_{1-\alpha}(c; \hat{\beta}) &= \mathbb{E} \left[\max(Z_{(1)}(\Sigma) + c_{u,(1)}, Z_{(2)}(\Sigma) + c_{u,(2)}) \right] \\ &\quad - \mathbb{E} \left[\min(Z_{(1)}(\Sigma) + c_{l,(1)}, Z_{(2)}(\Sigma) + c_{l,(2)}) \right] \end{aligned} \quad (21)$$

subject to

$$\begin{aligned} \inf_{v \in \{\tilde{v} \in \mathbb{R}^2; \tilde{v}_{(1)} \tilde{v}_{(2)} \leq 0\}} \Pr \Big(\max(Z_{(1)}(\Sigma) + c_{u,(1)} + v_{(1)}, Z_{(2)}(\Sigma) + c_{u,(2)} + v_{(2)}) \geq 0 \\ \geq \min(Z_{(1)}(\Sigma) + c_{l,(1)} + v_{(1)}, Z_{(2)}(\Sigma) + c_{l,(2)} + v_{(2)}) \Big) \geq 1 - \alpha. \end{aligned} \quad (22)$$

where

$$\begin{bmatrix} Z_{(1)}(\Sigma) \\ Z_{(2)}(\Sigma) \end{bmatrix} \sim \mathcal{N} \left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma \right] \quad (23)$$

Theorem: Optimal CI for θ_0

- Let $c_{l,(1)} = -c_{u,(1)}$, $c_{u,(2)} = -c_{l,(2)}$ and

$$(c_{(1)}^*, c_{(2)}^*) = \arg \min c_{u,(1)} - c_{l,(2)} \quad \text{s.t. (22) holds.}$$

- Then, the unique minimizer $c' = ((c'_u)^\top, (c'_l)^\top)^\top$ of the local asymptotic minimax risk $\mathcal{R}(\mathcal{CR}_{1-\alpha})$ over $c \in \mathbb{R}^4$ given $\hat{\beta}$ is characterized as

- 1 If $\sigma_1 = \sigma_2$,

$$c' = (c_{(1)}^*, c_{(2)}^*, -c_{(1)}^*, -c_{(2)}^*). \quad (24)$$

- 2 If $\sigma_1 < \sigma_2$, there exist some $\eta^* > 0$ and $\lambda_{\eta^*} \in (0, 1)$ such that

$$c' = (c_{(1)}^* + \eta^*, c_{(2)}^* - \lambda_{\eta^*} \eta^*, -c_{(1)}^* - \eta^*, -c_{(2)}^* + \lambda_{\eta^*} \eta^*)^\top. \quad (25)$$

- 3 If $\sigma_1 > \sigma_2$, there exist some $\eta^{**} \geq 0$ and $\lambda_{\eta^{**}} \in (0, 1)$ such that

$$c' = (c_{(1)}^* - \lambda_{\eta^{**}} \eta^{**}, c_{(2)}^* + \eta^{**}, -c_{(1)}^* + \lambda_{\eta^{**}} \eta^{**}, -c_{(2)}^* - \eta^{**})^\top. \quad (26)$$

Monte Carlo Simulation

Monte Carlo experiments: Design (Instruments)

- 2 instruments $z_t^{(1)}$ and $z_t^{(2)}$ correlated with $\epsilon_{j,t}$, $j = 1, 2$:

$$\begin{bmatrix} z_t^{(1)} \\ z_t^{(2)} \end{bmatrix} = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{bmatrix} \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} \quad (27)$$

where $u_t \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}[\mathbf{0}, I_2]$ and $u_t \perp \epsilon_t$.

- $\alpha_{1,1} > 0$, $\alpha_{1,2} > 0$ and $\alpha_{2,1} > 0$, $\alpha_{2,2} < 0$, $\alpha_{2,1} + \alpha_{2,2} > 0$ as in the example
- The LP-IV estimators

$$\hat{\beta}_{(1),h} = \frac{\mathbb{E}_T[z_t^{(1)} y_{t+h}]}{\mathbb{E}_T[z_t^{(1)} x_t]}, \quad \hat{\beta}_{(2),h} = \frac{\mathbb{E}_T[z_t^{(2)} y_{t+h}]}{\mathbb{E}_T[z_t^{(2)} x_t]}. \quad (28)$$

Monte Carlo experiments: Design

- The lag-order of SVMA: $\bar{L} = 16$
- The sample size $T = 500$
- Coefficient for $(z_t^{(1)}, z_t^{(2)})$:

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{bmatrix} = \begin{bmatrix} 2 & .4 \\ 1.2 & -.5 \end{bmatrix}$$

- For each replication, we construct CRs for each $\theta_{1,h}$ given $\{y_t, x_t, z_t^{(1)}, z_t^{(2)}\}$ with level .9 and compute empirical coverage probabilities and the average values of the upper and lower bounds of the CR (# of replications = 300).

Monte Carlo experiments: Comparison with a naive estimator

We compare a local asymptotic minimax CR with the following naive CR:

$$\left[\hat{\beta}_l - T^{-1/2} \hat{\sigma}_l q_{1-\alpha_1}, \hat{\beta}_u + T^{-1/2} \hat{\sigma}_u q_{1-\alpha_2} \right]$$

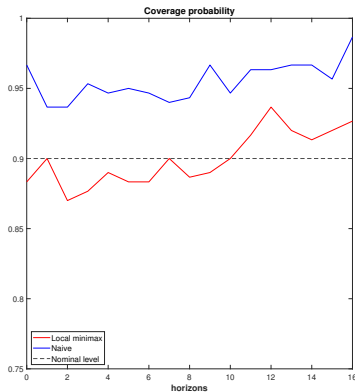
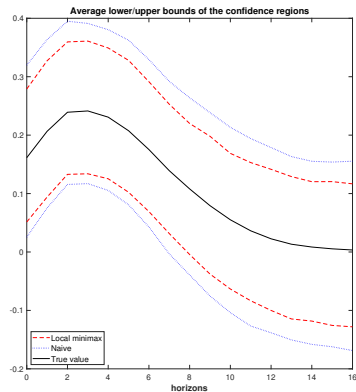
where $l \in \{1, 2\}$ is the index of the smaller of $\{\hat{\beta}_{(1)}, \hat{\beta}_{(2)}\}$:

$$\hat{\beta}_l = \min \left(\hat{\beta}_{(1)}, \hat{\beta}_{(2)} \right), \quad \hat{\beta}_u = \max \left(\hat{\beta}_{(1)}, \hat{\beta}_{(2)} \right),$$

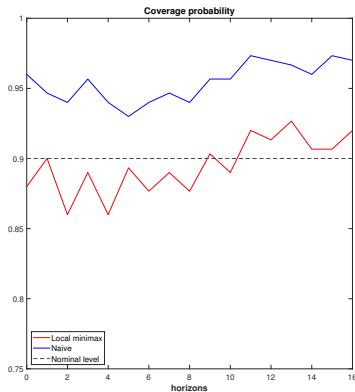
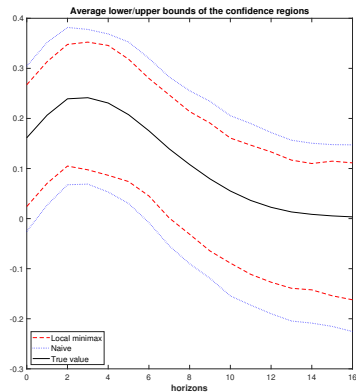
$(\hat{\sigma}_l, \hat{\sigma}_u)$ are corresponding estimated asymptotic standard errors.

A pair $(\alpha_1, \alpha_2) \in (0, 1)^2$ satisfies $(1 - \alpha_1)(1 - \alpha_2) = 1 - \alpha$ and is chosen to minimize the length of the CR: $\hat{\sigma}_u q_{1-\alpha_1} + \hat{\sigma}_l q_{1-\alpha_1}$.

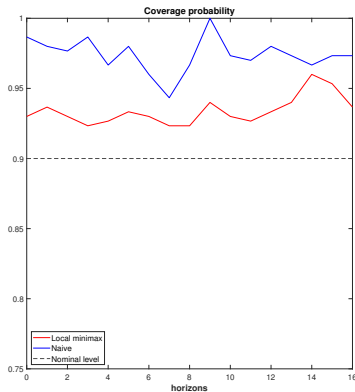
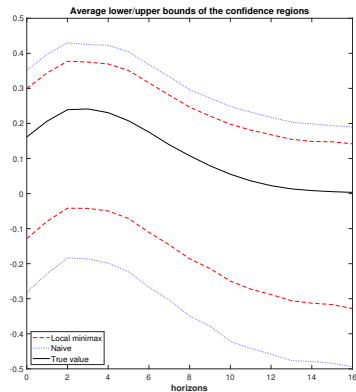
Coverage and Length of CI's

 $\delta = 0$ 

Coverage and Length of CI's

 $\delta = 1$ 

Coverage and Length of CI's

 $\delta = 10$ 

Conclusion

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Conclusion

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 - ② **shortest average length confidence interval** for IR maintaining uniform validity.
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Thank you for your attention!

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