Identification of a rank-dependent peer effect model

Eyo I. Herstad [1] Aristide Houndetoungan [2] Myungkou Shin [3]

[1] Erasmus University Rotterdam

[2] Université Laval

[3] University of Surrey

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Motivation

- Individuals interact in networks, creating spillovers/peer effects.

- The most common way of modeling this is through a linear-in-means (LiM) model

$$y_i = \beta_{\text{LiM}} \bar{y}_i + \mathbf{x}_i{}^{\mathsf{T}} \gamma + \varepsilon_i.$$

  $\bar{y}_i$ is the mean of peer outcomes for individual i: linear-in-means.
  β_{LiM} is the (homogeneous) peer effect coefficient.

- However, peer effects might be heterogeneous.
  1) Composition of your peer group matters beyond the first moment.
  2)

- How do we estimate peer effect while allowing for flexible pattern of heterogeneity?

What do we do?

- We introduce a model where the effect of each peer is endogenously determined;
  the distribution of peer outcomes matters.

- We show that the model is identified and construct an estimator.

- We show that the estimator performs fairly well even in small samples and
  relate it to other estimators in the literature.

- We apply the rank-dependent peer effect model to Add Health;
  the estimated pattern of heterogeneity in peer effect illustrate limitations of existing models.

Model

Identification

Implementation

Simulation

Empirical illustration

Conclusion

Rank-dependent peer effect model

- An econometrician observes $\{y_i, \mathbf{x}_i\}_{i=1}^{n}$.
  y_i is the outcome of interest for individual i; we suspect spillover in y_i.
  $\mathbf{x}_i$ is control covariates for individual i.

- In addition, the econometrician observes a $n \times n$ adjacency matrix A: a single network setup.
  Let $A_{i,j}$ denote the i-th row and j-th column element of A.
  $A_{i,j} \in \{0, 1\}$ indicates whether individual j is a 'friend' for individual i.

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- $d_i = \sum_{j=1}^n A_{i,j}$ is the number of friends for individual i.
  $\tilde{y}_{i,k}$ is the outcome of the k-th lowest performing friend of individual i.
  Namely, k-th ordered statistic from $\{y_j : A_{i,j} = 1\}$.

  Example: $A_{1,j} = \mathbf{1}\{j = 2, 3, 4\}$. Also, $y_2 = 3, y_3 = 1$ and $y_4 = 5$.
  Then, $d_1 = 3$ and $\left(\tilde{y}_{1,1}, \tilde{y}_{1,2}, \tilde{y}_{1,3}\right) = (1, 3, 5)$.
  "Individual 1 has three friends: individuals 2,3 and 4.
  The ordered peer outcomes for individual 1 are $(1, 3, 5)$."

Rank-dependent peer effect model

- Rank-dependent peer effect model: for $i = 1, \ldots, n$,

$$y_i = \sum_{k=1}^{d_i} \beta_{k,d_i} \tilde{y}_{i,k} + \mathbf{x}_i{}^\intercal \gamma + \varepsilon_i. \tag{1}$$

- $\beta_{k,d}$ is the peer effect coefficient on the k-th lowest peer outcome for someone with d friends.

  For example:

  - $\beta_{3,5}$ is the effect of the 3rd lowest performing friend on an individual with 5 friends.
  - $\beta_{3,3}$ is the effect of the highest performing friend on an individual with 3 friends.
  - $\beta_{1,1}$ is the effect of the sole friend on an individual with one friend.

Rank-dependent peer effect model: microfoundation

- When would the rank dependence in peer effect construction matter?
  Consider a simple quadratic utility framework: with $\tilde{\mathbf{y}}_i = \left(\mathbf{0}^\intercal, \tilde{y}_{i,1}, \ldots, \tilde{y}_{i,d_i}, \mathbf{0}^\intercal\right)^\intercal$,

$$U_i(y_i, \tilde{\mathbf{y}}_i) = \underbrace{\alpha_i y_i - \frac{1}{2}{y_i}^2}_{\text{private benefit}} + \underbrace{\sum_{k=1}^{d_i} \beta_{k,d_i} \tilde{y}_{i,k} \cdot y_i}_{\text{social benefit}} . \tag{2}$$

  The utility function consists of two parts:

  1) private benefit which is quadratic in y_i;

  2) social benefit which is linear in y_i, with coefficients depending on rank of the peer.

- The best response function for individual i leads to the rank-dependent peer effect model:

$$BR_i(\tilde{\mathbf{y}}_i) = \alpha_i + \sum_{k=1}^{d_i} \beta_{k,d_i} \tilde{y}_{i,k} \tag{3}$$

  with $\alpha_i = {\mathbf{x}_i}^\intercal \gamma + \varepsilon_i$.

Rank-dependent peer effect model: microfoundation

- The rank dependence would matter if the social benefit from having friends are in the form of

$$\sum_{k=1}^{d_i} \beta_{k,d_i} \tilde{y}_{i,k} \cdot y_i$$

  Note that β_{k,d_i} depends on k.
  "How my friend affects me depends on my other friends' outcomes."
  "Thus, how they affect me may differ from how they affect a third person."

- In particular, only their *relative position* matters.
  For example, when the social benefit is

$$\left(\beta_{\text{QiM},1}\bar{y}_i + \beta_{\text{QiM},2}\bar{y}_i^2\right) \cdot y_i$$

  a friend j's impact on individual i still depends on individual i's other friends outcomes.

Rank-dependent peer effect model: nested models

1) By letting $\beta_{k,d} = 0$ for every $k \neq 1$, we get the linear-in-minimum model:

$$y_i = \beta_{\min} \cdot \tilde{y}_{i,1} + \mathbf{x}_i^{\intercal} \gamma + \varepsilon_i.$$

2) By letting $\beta_{k,d} = 0$ for every $k \neq d$, we get the linear-in-maximum model:

$$y_i = \beta_{\max} \cdot \tilde{y}_{i,d_i} + \mathbf{x}_i^{\intercal} \gamma + \varepsilon_i.$$

3) By letting $\beta_{k,d} = \frac{1}{d}\beta_{\text{LiM}}$, we get the linear-in-means model:

$$y_i = \beta_{\text{LiM}} \cdot \bar{y}_i + \mathbf{x}_i^{\intercal} \gamma + \varepsilon_i.$$

4) By letting $\beta_{k,d} = \beta_{\text{LiMed}}\left(\mathbf{1}\left\{k = \frac{d+1}{2}, d \text{ is odd}\right\} + \frac{1}{2}\mathbf{1}\left\{k = \frac{d}{2}, \frac{d}{2} + 1, d \text{ is even}\right\}\right)$,

I get the linear-in-median model:

$$y_i = \beta_{\text{LiMed}} \cdot \text{med}\{y_j : A_{i,j} = 1\} + \mathbf{x}_i^{\intercal} \gamma + \varepsilon_i.$$

Rank-dependent peer effect model: nested models

5) More generally, we can model the peer effect to be linear in *quantiles* of peer outcomes. In this sense, the peer effect is a function of *empirical distribution* of peer outcomes.

For example,

$$y_i = \sum_{\tau=1}^{4} \beta_\tau \cdot \mathbb{F}_i^{-1}\big(\tau/5\big) + \mathbf{x}_i{}^\intercal \gamma + \varepsilon_i$$

where $\mathbb{F}_i(y) = \frac{1}{d_i} \sum_{j=1}^{n} A_{i,j} \mathbf{1}\{y_j \leq y\}$.

Let $\bar{d} = \max_{1 \leq i \leq n} d_i$. In the saturated model, there are $\frac{\bar{d}(\bar{d}+1)}{2}$ peer effect coefficients. Given *a priori* knowledge, we can develop a more parsimonious model.

Rank-depedent peer effect model: empirical practices

Table 5
Nonlinear specifications by gender[a]

Dep. var: UMD GPA

Panel A

	SAT-Q1	SAT-Q4	HSGPA-Q1	HSGPA-Q4	N	R^2
Men	.0891*	.0095	−.0593	.0395	2807	.3191
	(.0409)	(.0509)	(.0463)	(.0520)		
Women	.0336	.0622	.0049	−.0265	2748	.3181
	(.0409)	(.0427)	(.0404)	(.0355)		

Panel B

	Minsat	Maxsat	Minhsgpa	Maxhsgpa	N	R^2
Men	.0017	−.0131	.0558	−.0101	2807	.3178
	(.0196)	(.0263)	(.0677)	(.0712)		
Women	−.0050	.0278	−.0499	−.0917	2748	.3186
	(.0170)	(.0184)	(.0558)	(.0636)		

[a] Only observations in subsample (6) of Table 1 are included. Covariates included in addition to variables shown are identical to those used in the regressions to produce Table 4. “SAT-Q1” is a dummy for peer group’s average SAT score falling in the first (lowest) quartile of the same-gender distribution of mean peers’ SAT for the sample used to run the regression; “SAT-Q4” is a dummy for its falling in the fourth (highest) quartile. “HSGPA-Q1” and “HSGPA-Q4” are interpretable analogously. “Minsat” is the minimum SAT score among all peers; “Maxsat” is the maximum. “Minhsgpa” and “Maxhsgpa” are interpretable analogously. Standard errors are robust and clustered by peer group.

Figure 1: Foster [2006]

Bottom panel is nested in our model.

Rank-depedent peer effect model: empirical practices

Table 8
Quantile estimation results: fractions of weak and strong peers and quantiles of math score

Quantiles:	Average	0.10	0.25	0.50	0.75	0.90
Peer Characteristics						
Proportion below 25th pt	−0.854 (0.107)**	−1.677 (0.096)**	−1.347 (0.107)**	−0.900 (0.107)**	−0.363 (0.107)**	−0.050 (0.107)
Proportion above 75th pt	0.793 (0.094)**	0.182 (0.109)	0.402 (0.094)**	0.795 (0.094)**	1.333 (0.094)**	1.282 (0.094)**
Proportion of Male	−0.294 (0.083)**	−0.354 (0.177)*	−0.232 (0.083)**	−0.255 (0.083)**	−0.227 (0.083)**	−0.284 (0.083)**
Father's Education	−0.034 (0.016)*	−0.040 (0.034)	−0.047 (0.016)**	−0.049 (0.016)**	−0.026 (0.016)	0.032 (0.016)
Mother's Education	0.019 (0.016)	0.019 (0.040)	0.034 (0.016)*	0.012 (0.016)	0.007 (0.016)	−0.032 (0.016)*
Books over 200	−0.143 (0.094)	−0.130 (0.178)	−0.225 (0.094)*	−0.068 (0.094)	−0.012 (0.094)	−0.053 (0.094)
Computer at Home	−0.113 (0.074)	0.076 (0.203)	−0.059 (0.074)	−0.024 (0.074)	−0.243 (0.074)**	−0.294 (0.074)**
School Fixed Effects	No	No	No	No	No	No
Peer Characteristics						
Proportion below 25th pt	−0.261 (0.140)	−0.658 (0.236)**	−0.721 (0.182)**	−0.538 (0.125)**	−0.121 (0.200)	0.475 (0.233)*
Proportion above 75th pt	0.263 (0.128)*	0.117 (0.227)	−0.091 (0.183)	0.283 (0.135)*	0.435 (0.181)*	0.461 (0.218)*
Proportion of Male	−0.217 (0.108)*	−0.484 (0.198)*	−0.380 (0.120)**	−0.065 (0.113)	−0.278 (0.120)*	−0.298 (0.182)
Father's Education	−0.035 (0.024)	−0.064 (0.038)	−0.032 (0.038)	−0.057 (0.030)	−0.053 (0.039)	0.035 (0.045)
Mother's Education	0.018 (0.025)	0.008 (0.050)	0.042 (0.041)	−0.005 (0.028)	0.038 (0.041)	−0.043 (0.052)
Books over 200	0.019 (0.121)	−0.174 (0.204)	−0.309 (0.158)	0.004 (0.114)	0.253 (0.191)	0.303 (0.199)
Computer at Home	−0.113 (0.108)	−0.144 (0.216)	0.041 (0.168)	0.239 (0.141)	−0.114 (0.169)	−0.281 (0.201)
School Fixed Effects	Yes	Yes	Yes	Yes	Yes	Yes
Number of students	12,964	12,964	12,964	12,964	12,964	12,964

Note: The regressions are weighted by Within-Class Student Weight. Standard errors are in parentheses.
* The estimate is significant at the 0.05 level.
** Idem, 0.01.

Figure 2: Kang [2007]

Relative position in the overall distribution of outcome matters.

Rank-depedent peer effect model: empirical practices

Table 6—Peer Effects with Alternative Measures of Peer Ability

Measure of peer ability:	Average ability (1)	Maximum ability (2)	Minimum ability (3)	Tiger Woods is partner (4)	Average ability × (average ability − own ability) (5)
Own ability	0.672	0.673	0.672	0.674	0.672
	(0.039)	(0.039)	(0.039)	(0.039)	(0.039)
Peer ability	−0.035	−0.023	−0.031	−0.348	−0.016
	(0.040)	(0.032)	(0.040)	(0.462)	(0.014)
R^2	0.154	0.154	0.154	0.154	0.154
N	17,492	17,492	17,492	17,492	17,492

Measure of peer ability:	**1**{any partner in top 10%} (6)	**1**{any partner in top 25%} (7)	**1**{any partner in bottom 25%} (8)	**1**{any partner in bottom 10%} (9)
Own ability	0.673	0.672	0.673	0.673
	(0.039)	(0.039)	(0.039)	(0.039)
Peer ability	0.027	0.049	−0.041	−0.178
	(0.070)	(0.059)	(0.069)	(0.141)
R^2	0.154	0.154	0.154	0.154
N	17,492	17,492	17,492	17,492

Notes: Column 1 is reproduced from Table 3. Other columns present results from modifying baseline specifications as specified in equation (3) to support heterogeneous peer effects. The dependent variable is the golf score for the round. The ability variable is measured using the player's handicap. Standard errors are in parentheses and are clustered by playing group. All regressions weight each observation by the inverse of the sample variance of estimated ability of each player. All regressions include tournament-by-category fixed effects and round fixed effects. In column 9, average ability of playing partners is also included in regression. The estimated coefficient for this variable is −0.014 (0.048).

Figure 3: Guryan et al. [2009]

Again, some of the specifications focus on relative position in the overall distribution.

Rank-dependent peer effect model: comparison to existing models

- Tao and Lee [2014] provided a theoretical discussion for

$$y_i = \beta_{\mathsf{max}} \cdot \tilde{y}_{i,d_i} + \mathbf{x}_i{}^{\mathsf{T}}\gamma + \varepsilon_i.$$

- Boucher et al. [2024]'s model allows for nonlinear peer effects through a CES function:

$$\begin{aligned} y_i &= \lambda \left(\sum_{j=1}^{n} \frac{A_{i,j}}{d_i} y_j{}^{\rho} \right)^{\frac{1}{\rho}} + \mathbf{x}_i{}^{\mathsf{T}}\gamma + \varepsilon_i \\ &= \lambda \left(\sum_{k=1}^{d_i} \frac{1}{d_i} \tilde{y}_{i,k}^{\rho} \right)^{\frac{1}{\rho}} + \mathbf{x}_i{}^{\mathsf{T}}\gamma + \varepsilon_i. \end{aligned}$$

- Varying ρ gives (monotonically) higher/lower weights to peers with higher/lower outcomes.
  Tao and Lee [2014] is included as a special case of $\rho \to \infty$.
- But varying ρ cannot capture cases where only the "middle" matters, or only "extremes" matter, etc.

Rank-dependent peer effect model: comparison to existing models

- For comparison, think of the peer effect as 'product of peer outcomes.'
  Inputs are peer outcomes $\{y_j : A_{i,j} = 1\}$ and the output is the peer effect.

- Well-known properties of CES production functions are.

  1) Constant elasticity of substition $\frac{1}{1-\rho}$: $\rho \to \infty \Leftrightarrow$ only $\tilde{y}_{i,d_i}$ matters;
  $\rho = 1 \Leftrightarrow$ perfect subsitutes;
  $\rho \to 0 \Leftrightarrow$ Cobb-Douglas;
  $\rho \to -\infty \Leftrightarrow$ perfect complements; only $\tilde{y}_{i,1}$ matters.

  Also, $\rho > 1 \Leftrightarrow$ concave isoquant curve;
  $\rho < 1 \Leftrightarrow$ convex isoquant curve.

  2) Monotone marginal product: $\rho > 1 \Leftrightarrow$ diminishing marginal product;
  $\rho < 1 \Leftrightarrow$ increasing marginal product.

  3) Constant return to scale.

Rank-dependent peer effect model: comparison to existing models

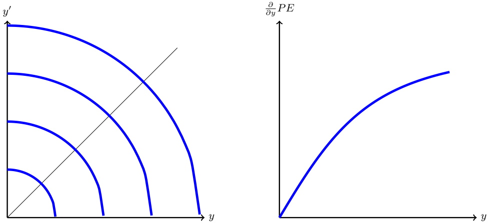

Figure 4: Isoquant curve and marginal product in CES peer effect, $\rho > 1$.

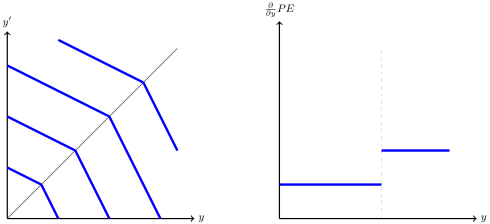

Figure 5: Isoquant curve and marginal product in rank-dependent peer effect, $(\beta_{12}, \beta_{22}) = (1, 2)$.

Rank-dependent peer effect model: comparison to existing theoretical results

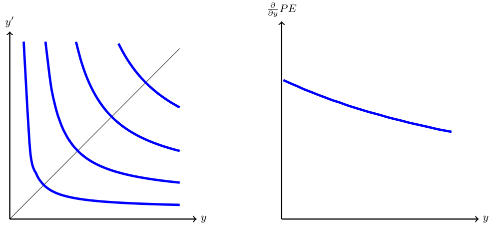

Figure 6: Isoquant curve and marginal product in CES peer effect, $\rho < 0$.

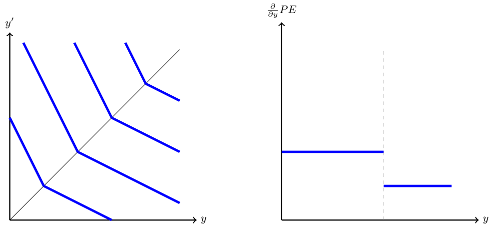

Figure 7: Isoquant curve and marginal product in rank-dependent peer effect, $(\beta_{12}, \beta_{22}) = (2, 1)$.

Rank-dependent peer effect model: comparison to existing models

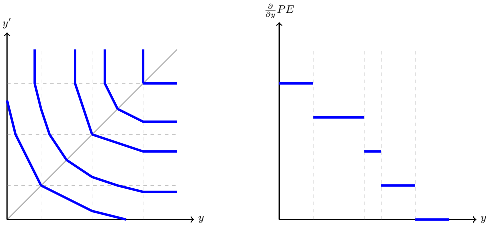

Figure 8: Isoquant curve and marginal product in rank-dependent peer effect, $(\beta_{15}, \cdots, \beta_{55}) = (4, 3, 2, 1, 0)$

When the number of friends is large,
our model can show a similar pattern of substitution and marginal product.

Rank-dependent peer effect model: comparison to existing models

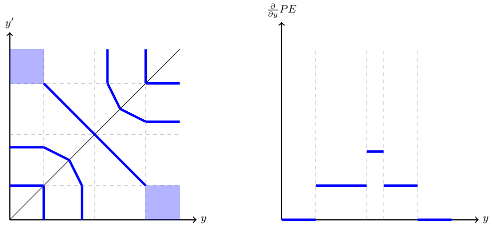

Figure 9: Isoquant curve and marginal product in rank-dependent peer effect, $(\beta_{15}, \cdots, \beta_{55}) = (0, 1, 2, 1, 0)$

1) There is an isoquant 'set.'

2) Marginal product is not monotone.

This pattern of peer effect construction is not allowed in the CES production function.

Rank-dependent peer effect model: comparison to existing models

In both CES peer effect production and rank-dependent peer effect production,
we have constant return to scale:

$$\lambda \left(\sum_{k=1}^{d_i} \frac{1}{d_i} \left(C \tilde{y}_{i,k} \right)^\rho \right)^{\frac{1}{\rho}} = C \lambda \left(\sum_{k=1}^{d_i} \frac{1}{d_i} \tilde{y}_{i,k}^\rho \right)^{\frac{1}{\rho}} \qquad \textit{(CES peer effect)}$$

$$\sum_{k=1}^{d_i} \beta_{k,d_i} \left(C \tilde{y}_{i,k} \right) = C \left(\sum_{k=1}^{d_i} \beta_{k,d_i} \tilde{y}_{i,k} \right) \qquad \textit{(rank-dependent peer effect)}$$

This would be relaxed if we include higher-order terms of $\tilde{y}_{i,k}$ or $\bar{y}_i$.

In addition, the rank-dependent peer effect model can allow for *return to the number of inputs*,
by letting $\sum_{k=1}^{d} \beta_{k,d}$ depend on d.
"Having three homogeneous friends may differ from having four homogeneous friends."
This is not allowed in the CES peer effect function.

Rank-dependent peer effect model: comparison to existing models

	linear-in-means	CES	rank-dependent
rate of substitution	constant	proportional to relative ratio	step
marginal product	constant	monotone	step
return to scale	constant	constant	constant
return to number of inputs	zero	zero	unrestricted

Table 1: Comparison to existing peer effect models.

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Identification: unique equilibrium

- For the rank-dependent peer effect model to be well-defined, the system of n nonlinear equations

$$y_1 = \sum_{k=1}^{d_1} \beta_{k,d_1} \tilde{y}_{1,k} + \mathbf{x}_1{}^{\mathsf{T}}\gamma + \varepsilon_1$$

$$\vdots$$

$$y_n = \sum_{k=1}^{d_n} \beta_{k,d_n} \tilde{y}_{n,k} + \mathbf{x}_n{}^{\mathsf{T}}\gamma + \varepsilon_n$$

needs to have a unique solution $\mathbf{y} = \big(y_1, \ldots, y_n\big)^{\mathsf{T}}$ given $\{\mathbf{x}_i, \varepsilon_i\}_{i=1}^n$.

- Then, the conditional distribution of $\{\varepsilon_i\}_{i=1}^n$ given $\{\mathbf{x}_i\}_{i=1}^n$ implies a conditional distribution of $\mathbf{y}$ given $\{\mathbf{x}_i\}_{i=1}^n$.

Identification: unique equilibrium

- A sufficient condition for the unique existence of outcome vector $\mathbf{y}$: the *total spillover for each individual* is less than one.

> **Assumption 1. (bounded peer effect)**
> For all $d \leq \bar{d} = \max_{i=1,\ldots,n} d_i$
> $$\sum_{k=1}^{d} |\beta_{k,d}| < 1$$

  - For individuals with one friend, this means $|\beta_{1,1}| < 1$.
  - For individuals with two friends, this means $|\beta_{1,2}| + |\beta_{2,2}| < 1$.

- This corresponds to the unique existence assumption of the linear-in-means model.
- Also, this corresponds to the stability condition in the autoregressive model.

Identification: unique equilibrium

Proposition 1.
Let Assumption 1 hold. Then, for any realization of $\{\mathbf{x}_i, A_{i,j}, \varepsilon_i\}_{1\leq i,j\leq n}$, the peer effect model (1) admits a unique equilibrium.

- Key part of the proof: we show

$$\mathbf{y} \mapsto \begin{pmatrix} \sum_{k=1}^{d_1} \beta_{k,d_1}\tilde{y}_{1,k} + \mathbf{x}_1{}^\intercal\gamma + \varepsilon_1 \\ \vdots \\ \sum_{k=1}^{d_n} \beta_{k,d_n}\tilde{y}_{n,k} + \mathbf{x}_n{}^\intercal\gamma + \varepsilon_n \end{pmatrix}$$

is a contraction mapping under Assumption 1.

Identification: reduced form

- Assume the network and covariates are exogenous to ε_i.

> **Assumption 2. (exogenous network and covariates)**
> For each $i = 1, \ldots, n$,
> $$\mathbf{E}\left[\varepsilon_i | \{\mathbf{x}_i, A_{i,j}\}_{1 \le i,j \le n}\right] = 0.$$

- Then under Assumption 1 and 2, we have

$$\mathbf{E}\left[y_i | \{\mathbf{x}_i, A_{i,j}\}_{1 \le i,j \le n}\right] = \sum_{j=1}^{n} \theta_{i,j} \mathbf{x}_j{}^{\mathsf{T}} \gamma + \eta_i$$

where $\theta_{i,j}, \eta_i$ are complicated functions of $A_{i,j}$ and β. more

Identification: reduced form

- In the linear-in-means model, the system of n linear equations is

$$\mathbf{y} = \beta G \mathbf{y} + \mathbb{X}\gamma + \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}.$$

where $G = \left(\frac{1}{d_i} A_{i,j}\right)_{i,j}$ and $\mathbb{X} = \begin{pmatrix} \mathbf{x}_1{}^\intercal \\ \vdots \\ \mathbf{x}_n{}^\intercal \end{pmatrix}$.

- Assumption 2 gives us the reduced form of

$$\mathbf{E}\left[\mathbf{y} | \{\mathbf{x}_i, A_{i,j}\}_{1 \le i,j \le n}\right] = (I_n - \beta G)^{-1} \mathbb{X}\gamma.$$

The conditional expectations are linear in $\{\mathbf{x}_i{}^\intercal \gamma\}_{i=1}^n$, without an intercept coefficient.

Identification: rewriting the rank-dependent peer effect model

Rewrite the rank-dependent peer effect to concatenate across different numbers of friends:

$$\sum_{k=1}^{d_i} \beta_{k,d_i} \tilde{y}_{i,k} = \tilde{\mathbf{y}}_i^{\mathsf{T}} \beta$$

where $\tilde{\mathbf{y}}_i$ is a $\frac{\bar{d}(\bar{d}+1)}{2}$-dimensional random vector such that

$$\tilde{\mathbf{y}}_i = \begin{pmatrix} \mathbf{0}^{\mathsf{T}} & \tilde{y}_{i,1} & \cdots & \tilde{y}_{i,d_i} & \mathbf{0}^{\mathsf{T}} \end{pmatrix}^{\mathsf{T}}$$

and β is a $\frac{\bar{d}(\bar{d}+1)}{2}$-dimensional parameter such that

$$\beta = \begin{pmatrix} \beta_{1,1} & \cdots & \beta_{1,\bar{d}} & \cdots & \beta_{\bar{d},\bar{d}} \end{pmatrix}^{\mathsf{T}}.$$

Identification: rewriting the rank-dependent peer effect model

Then,

$$y_i = \sum_{k=1}^{d_i} \beta_{k,d_i} \tilde{y}_{i,k} + \mathbf{x}_i{}^\intercal \gamma + \varepsilon_i = \begin{pmatrix} \tilde{\mathbf{y}}_i^\intercal & \mathbf{x}_i{}^\intercal \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} + \varepsilon_i.$$

By stacking across $i = 1, \ldots, n$,

$$\mathbf{y} = \mathbb{W} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}.$$

Suppose we have some instruments $\{\mathbf{z}_i\}_{i=1}^n$ that satisfy instrument exogeneity and relevance. more
$\mathbf{z}_i$ is also a $\frac{\bar{d}(\bar{d}+1)}{2}$-dimensional random vector.

Identification

Let

$$\mathbb{W} = \begin{pmatrix} \tilde{\mathbf{y}}_1^\intercal & & \mathbf{x}_1{}^\intercal \\ & \vdots & \\ \tilde{\mathbf{y}}_n^\intercal & & \mathbf{x}_n{}^\intercal \end{pmatrix} \quad \text{and} \quad \mathbb{Z} = \begin{pmatrix} \mathbf{z}_1{}^\intercal & & \mathbf{x}_1{}^\intercal \\ & \vdots & \\ \mathbf{z}_n{}^\intercal & & \mathbf{x}_n{}^\intercal \end{pmatrix}$$

Theorem 1.
Suppose that Assumptions 1-3 hold and $n \geq \frac{\bar{d}(\bar{d}+1)}{2} + l$. Then, β and γ are identified from the moment condition below:

$$\mathbf{E}\left[\mathbb{Z}^\intercal \mathbb{W} | \{\mathbf{x}_i, A_{i,j}\}_{1\leq i,j\leq n}\right] \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \mathbf{E}\left[\mathbb{Z}^\intercal \mathbf{y} | \{\mathbf{x}_i, A_{i,j}\}_{1\leq i,j\leq n}\right]. \tag{4}$$

Instruments

- We need instruments $\{\mathbf{z}_i\}_{i=1}^n$ such that
  i. $\mathbf{E}\left[\mathbf{z}_i{}^\intercal \varepsilon_i | \{\mathbf{x}_i, A_{i,j}\}_{1\leq i,j\leq n}\right]$ is zero for every i;
  ii. $\mathbf{E}\left[\mathbb{Z}^\intercal \mathbb{W} | \{\mathbf{x}_i, A_{i,j}\}_{1\leq i,j\leq n}\right]$ is full rank.
- We construct instruments that are functions of $\{\mathbf{x}_i, A_{i,j}\}_{1\leq i,j\leq n}$.
- In addition, we construct $\{\mathbf{z}_i\}_{i=1}^n$ so that $\mathbf{E}\left[\mathbb{Z}^\intercal \mathbb{W} | \{\mathbf{x}_i, A_{i,j}\}_{1\leq i,j\leq n}\right]$ is a diagonal block matrix.

Shape restriction

- In the fully saturated rank-dependent peer effect model, β is $\frac{\bar{d}(\bar{d}+1)}{2}$-dimensional.
- Imposing shape restriction on β may help in finite sample.

Implementation: instruments

Assumption 4. (Homoskedasticity)

$$\mathbf{E}\left[\varepsilon_i \varepsilon_j \middle| \{\mathbf{x}_i, A_{i,j}\}_{1\le i,j\le n}\right] = \sigma^2 \cdot \mathbf{1}\{i = j\}.$$

Given a set of just-identifying instruments $\{\mathbf{z}_i\}_{i=1}^n$ such that $\mathbf{z}_i \in \mathbb{R}^{\frac{\bar{d}(\bar{d}+1)}{2}+p}$, let

$$\begin{pmatrix} \hat{\beta}^{IV}(\{\mathbf{z}_i\}_{i=1}^n) \\ \hat{\gamma}^{IV}(\{\mathbf{z}_i\}_{i=1}^n) \end{pmatrix} = \left(\frac{1}{n} \sum_{i=1}^n \mathbf{z}_i \begin{pmatrix} \tilde{\mathbf{y}}_i \\ \mathbf{x}_i \end{pmatrix}^{\intercal} \right)^{-1} \frac{1}{n} \sum_{i=1}^n \mathbf{z}_i y_i$$

be the IV estimator for β and γ. Then,

$$\mathrm{Avar}\begin{pmatrix} \hat{\beta}^{IV}(\{\mathbf{z}_i\}_{i=1}^n) \\ \hat{\gamma}^{IV}(\{\mathbf{z}_i\}_{i=1}^n) \end{pmatrix} = \left(\frac{1}{n} \sum_{i=1}^n \mathbf{z}_i \begin{pmatrix} \mathbf{E}[\tilde{\mathbf{y}}_i | \{\mathbf{x}_i, A_{i,j}\}_{1\le i,j\le n}] \\ \mathbf{x}_i \end{pmatrix}^{\intercal} \right)^{-1} \mathrm{Var}\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbf{z}_i \varepsilon_i \right) \cdots .$$

be the asymptotic variance of the IV estimator under some weak dependence regularity conditions.

Implementation: instruments

Proposition 2.

Let Assumptions 1-2 and 4 hold and let

$$\mathbf{z}_i{}^* = \mathbf{E}[\tilde{\mathbf{y}}_i | \{\mathbf{x}_i, A_{i,j}\}_{1 \le i,j \le n}] \quad \forall i = 1, \cdots, n.$$

Then, for any set of instruments $\{\mathbf{z}_i\}_{i=1}^n$ satisfying Assumption 3,

$$\text{Avar}\begin{pmatrix} \hat{\beta}^{IV}(\{\mathbf{z}_i{}^*\}_{i=1}^n) \\ \hat{\gamma}^{IV}(\{\mathbf{z}_i{}^*\}_{i=1}^n) \end{pmatrix}^{-1} - \text{Avar}\begin{pmatrix} \hat{\beta}^{IV}(\{\mathbf{z}_i\}_{i=1}^n) \\ \hat{\gamma}^{IV}(\{\mathbf{z}_i\}_{i=1}^n) \end{pmatrix}^{-1}$$

is positive semi-definite.

This is the same result as Chamberlain [1987], Newey [1990].

Implementation: instruments

- Recall the reduced form representation:

$$\mathbf{E}\left[y_i | \{\mathbf{x}_i, A_{i,j}\}_{1\le i,j\le n}\right] = \sum_{j=1}^{n} \theta_{i,j} \mathbf{x}_j{}^\intercal \gamma + \eta_i.$$

We can similarly derive reduced form representation for cond. exp. of $\tilde{y}_{i,k}$:

$$\mathbf{E}[\tilde{\mathbf{y}}_i | \{\mathbf{x}_i, A_{i,j}\}_{1\le i,j\le n}] = \sum_{j=1}^{n} \tilde{\Theta}_{i,j} \mathbf{x}_j{}^\intercal \gamma + \tilde{\eta}_i.$$

- The classic instrument for peer effects is based on the same observation: Bramoullé et al. [2009]
  The optimal instruments are

$$\begin{aligned}\mathbf{E}\left[G\mathbf{y} | \{\mathbf{x}_i, A_{i,j}\}_{1\le i,j\le n}\right] &= G\left(I_n - \beta G\right)^{-1} \mathbb{X}\gamma \\ &= \left(G + \beta G^2 + \cdots\right) \mathbb{X}\gamma\end{aligned}$$

Using average of friends' characteristics is to use first-order approximation of optimal instruments.

Implementation: instruments

- In the linear-in-means model, we use average of peer characteristics to instrument for average of peer outcome.

- Similarly, we suggest using (component-wise) ordered statistics of peer characteristics to instrument ordered statistics of peer outcomes

$$\mathbf{z}_i = \begin{pmatrix} \mathbf{0}^\intercal & \tilde{\mathbf{x}}_{i,1} & \cdots & \tilde{\mathbf{x}}_{i,d_i} & \mathbf{0}^\intercal \end{pmatrix}^\intercal$$

  where $\tilde{x}_{i,k}$ collects the component-wise k-th ordered statistics of $\{x_j : A_{i,j} = 1\}$.

- In the reduced form $\mathbf{E}[\tilde{\mathbf{y}}_i | \{\mathbf{x}_i, A_{i,j}\}_{1 \le i,j \le n}] = \sum_{j=1}^{n} \tilde{\Theta}_{i,j} \mathbf{x}_j{}^\intercal \gamma + \tilde{\eta}_i$

  $\Theta_{i,j}$ is unknown even when β, γ are known.

  This is because $\tilde{\Theta}_{i,j}$ involves probability of a specific ordering on $\mathbf{y}$.

  We may not have the nice 'first-order approximation' interpretation.

  Relevance can be directly checked from dataset.

Implementation: possible resampling improvement

- We are considering a bootstrap-based improvement on the instruments.

- Given first-step estimators based on naive instruments $\hat{\beta}^{FS}$ and $\hat{\gamma}^{FS}$:

  1. Calculate $\hat{\varepsilon}_i = y_i - \tilde{\mathbf{y}}_i^\intercal \hat{\beta}^{FS} - \mathbf{x}_i{}^\intercal \hat{\gamma}^{FS}$.

  2. For $b = 1, ..., B$:

    i. draw $\varepsilon_i^{(b)} \overset{\text{iid}}{\sim} \text{unif}\{\hat{\varepsilon}_1, \cdots, \hat{\varepsilon}_n\}$ for $i = 1, \cdots, n$.

    ii. Solve for $\mathbf{y}^{(b)}$ using the rank-dependent peer effect model.

  3. Use $\left\{ \frac{1}{B} \sum_{b=1}^{B} \tilde{\mathbf{y}}_i^{(b)} \right\}_{i=1}^{n}$ as instruments.

- In simulation, this procedure improves estimation performance.
  Currently working on some theoretical justification.

Implementation: shape restriction

- The dimension of β is $\frac{\bar{d}(\bar{d}+1)}{2}$. In Add Health, at most 10 friends are reported; $\beta \in \mathbb{R}^{45}$.

- In practice, it may be beneficial to impose some shape restriction on β.

  1) Sparsity

  Only a finite number of quantiles matter and β is zero elsewhere.

  For example, if only the 0, 1/3, 2/3, and 1 quantiles matter,

$$\beta_{1,1} = \beta^0 + \beta^{1/3} + \beta^{2/3} + \beta^1,$$

$$\begin{pmatrix} \beta_{1,2} & \beta_{2,2} \end{pmatrix} = \begin{pmatrix} \beta^0 + \beta^{1/3} & \beta^{2/3} + \beta^1 \end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix} \beta_{1,10} & \cdots & \beta_{10,10} \end{pmatrix} = \begin{pmatrix} \beta^0 & 0 & 0 & \beta^{1/3} & 0 & 0 & \beta^{2/3} & 0 & 0 & \beta^1 \end{pmatrix}$$

This generalizes linear-in-minimum, linear-in-maximum, linear-in-median, etc.

Implementation: shape restriction

- In practice, it may be beneficial to impose some shape restriction on β.

2) Smoothness

For example, construct a linear spline function on $[0,1]$ with two knots at $(1/3, 2/3)$:

$$\beta(\tau) = \beta^0 + \beta^1\tau + \beta^2\,(\tau - 1/3)\,\mathbf{1}\{\tau \geq 1/3\} + \beta^3\,(\tau - 2/3)\,\mathbf{1}\{\tau \geq 2/3\}.$$

Then,

$$\beta_{1,1} = \int_0^1 \beta(\tau)d\tau = \beta^0 + \frac{\beta^1}{2} + \frac{2\beta^2}{9} + \frac{\beta^3}{18}$$

$$\begin{pmatrix}\beta_{1,2} & \beta_{2,2}\end{pmatrix} = \begin{pmatrix}\int_0^{\frac{1}{2}} \beta(\tau)d\tau & \int_{\frac{1}{2}}^1 \beta(\tau)d\tau\end{pmatrix} = \begin{pmatrix}\frac{\beta^0}{2} + \frac{\beta^1}{8} + \frac{\beta^2}{72} & \frac{\beta^0}{2} + \frac{3\beta^1}{8} + \frac{13\beta^2}{72} + \frac{\beta^3}{18}\end{pmatrix}$$

$$\vdots$$

This generalizes linear-in-means, linear-in-sums, etc.

Simulation: DGP

- The DGP is the rank-dependent peer effect model with sparsity:
  four quantiles at 0, 1/3, 2/3, 1 matter.
- The sample is 50 networks of 50 people.
  Network adjacency matrix A is calibrated to match the observed networks in Add Health dataset.
- There are two iid-distributed covariates and two contextual effects.
  With $x_{i,1} \sim \mathcal{N}(0,1)$ and $x_{i,2} \sim$ Poisson(2), $\mathbf{x}_i = \left(1, x_{i,1}, x_{i,2}, \bar{x}_{i,1}, \bar{x}_{i,2}\right)^\intercal$.

  With iid-distributed error terms $\varepsilon_i \sim \mathcal{N}(0, 0.7)$,

$$y_i = \tilde{\mathbf{y}}_i^\intercal \beta + {\mathbf{x}_i}^\intercal \gamma + \varepsilon_i$$

- $\gamma = \left(1, -0.5, 1, -0.2, 0.6\right)^\intercal$ and we considered six different βs.
- As comparison, we also estimated the linear-in-means and the CES peer effect models.

Simulation: results

Quantiles				LiM	CES	
β^0	$\beta^{1/3}$	$\beta^{2/3}$	β^1	β_{LiM}	ρ	λ
DGP A: $(\beta^0, \beta^{1/3}, \beta^{2/3}, \beta^1) = (0, 0.05, 0.2, 0.3)$						
-0.000	0.050	0.199	0.301	0.401	8.767	0.587
(0.002)	(0.004)	(0.008)	(0.007)	(0.015)	(0.437)	(0.004)
DGP B: $(\beta^0, \beta^{1/3}, \beta^{2/3}, \beta^1) = (0.3, 0.2, 0.05, 0)$						
0.300	0.200	0.049	0.001	0.656	-5.411	0.525
(0.004)	(0.009)	(0.017)	(0.011)	(0.015)	(0.248)	(0.006)
DGP C: $(\beta^0, \beta^{1/3}, \beta^{2/3}, \beta^1) = (0, 0.275, 0.275, 0)$						
-0.000	0.275	0.274	0.001	0.564	1.898	0.597
(0.003)	(0.006)	(0.011)	(0.009)	(0.007)	(0.092)	(0.006)

Table 2: Comparison across estimators

Simulation: results

Quantiles				LiM	CES	
β^0	$\beta^{1/3}$	$\beta^{2/3}$	β^1	β_{LiM}	ρ	λ
DGP D: $(\beta^0, \beta^{1/3}, \beta^{2/3}, \beta^1) = (0.275, 0, 0, 0.275)$						
0.275	0.000	-0.001	0.276	0.527	-2.325	0.441
(0.003)	(0.007)	(0.013)	(0.010)	(0.008)	(0.353)	(0.012)
DGP E: $(\beta^0, \beta^{1/3}, \beta^{2/3}, \beta^1) = (-0.05, 0.35, 0.15, 0.1)$						
-0.050	0.350	0.149	0.101	0.549	2.513	0.609
(0.003)	(0.005)	(0.011)	(0.008)	(0.008)	(0.111)	(0.006)
DGP F (LiM model): $\beta_{\text{LiM}} = 0.55$						
0.114	0.166	0.139	0.132	0.550	1.003	0.550
(0.003)	(0.007)	(0.013)	(0.010)	(0.004)	(0.067)	(0.005)

Table 3: Comparison across estimators

Model

Identification

Implementation

Simulation

Empirical illustration

Conclusion

Empirical illustration: setup

- We apply our methods to Wave I of Add-health.
- Around 90,000 students from 144 schools.
  Student list up to 5 male and 5 female best friends.
  - Final sample contains approx. 75,000 students in 141 schools.
  - Average number of friends is 3.47, with 22% having no friends.
- To show how patterns can vary, we show results for:
  academic performance, extracurricular activities, smoking, and self-esteem.
- Covariates are age, sex, ethnicity, race, mothers education and mothers employment,
  as well as contextual variables of all individual covariates.

Empirical illustration: choice of shape restriction

specification test p-value 3 vs. 4	specification test p-value 4 vs. 5	KP test statistic	Sargan test p-value
Academic achievements (GPA)			
0.006	0.368	3528.619	0.146
Extracurricular activities			
0.002	0.593	2932.638	0.047
Smoking			
0.000	0.123	1704.396	1.000
Self-esteem			
0.723	0.997	1583.295	0.214

Table 4: Specification test and instrument relevance/validity tests

As a specification test, we used Smith [1992]'s test,
This compares the performance of an alternative model applied to $\mathbf{y}$
against the performance of the alternative model applied to fitted value $\hat{\mathbf{y}}$ from the null model.

Empirical illustration: result

Quantile				LIM	CES	
β^0	$\beta^{1/3}$	$\beta^{2/3}$	β^1	β_{LiM}	ρ	λ
Academic achievements (GPA)						
0.068	0.149	0.641	-0.120	0.815	0.611	0.807
(0.054)	(0.108)	(0.115)	(0.062)	(0.041)	(0.692)	(0.041)
Extracurricular activities						
-0.079	0.538	0.244	-0.006	0.795	-0.116	0.695
(0.091)	(0.140)	(0.088)	(0.023)	(0.051)	(0.448)	(0.034)
Smoking						
-0.124	0.386	0.359	0.115	0.811	1.466	0.711
(0.086)	(0.115)	(0.065)	(0.024)	(0.045)	(0.801)	(0.122)
Self-esteem						
0.112	0.148	0.229	-0.023	0.530	-9.940	0.383
(0.059)	(0.119)	(0.087)	(0.028)	(0.156)	(13.357)	(0.101)

Table 5: Comparison across estimators and outcomes— continued

Conclusion

- We propose a rank-dependent peer effect model.

- The model nests various existing peer effect model, and can capture new patterns of peer effect.

  1) heterogenous pattern of substitution, including isoquant set;

  2) nonmonotonic marginal product;

  3) return to scale in terms of number of friends.

- In implementation, we suggest

  1) ordered statistics of friends' characteristics as instruments;

  2) shape restriction on β, using sparsity or smoothness.

- The differences in estimated spillovers are empirically relevant and impact policy relevant objects: Sacerdote [2001], Zimmerman [2003], Booij et al. [2017] and more.

References I

Adam S Booij, Edwin Leuven, and Hessel Oosterbeek. Ability peer effects in university: Evidence from a randomized experiment. *The review of economic studies*, 84(2):547–578, 2017.

Vincent Boucher, Michelle Rendall, Philip Ushchev, and Yves Zenou. Toward a general theory of peer effects. *Econometrica*, 92(2):543–565, 2024.

Yann Bramoullé, Habiba Djebbari, and Bernard Fortin. Identification of peer effects through social networks. *Journal of Econometrics*, 150(1):41–55, 2009.

Gary Chamberlain. Asymptotic efficiency in estimation with conditional moment restrictions. *Journal of econometrics*, 34(3):305–334, 1987.

Gigi Foster. It's not your peers, and it's not your friends: Some progress toward understanding the educational peer effect mechanism. *Journal of Public Economics*, 90(8-9):1455–1475, 2006.

Jonathan Guryan, Kory Kroft, and Matthew J Notowidigdo. Peer effects in the workplace: Evidence from random groupings in professional golf tournaments. *American Economic Journal: Applied Economics*, 1(4):34–68, 2009.

References II

Changhui Kang. Classroom peer effects and academic achievement: Quasi-randomization evidence from south korea. *Journal of Urban Economics*, 61(3):458–495, 2007.

Whitney K Newey. Efficient instrumental variables estimation of nonlinear models. *Econometrica: Journal of the Econometric Society*, pages 809–837, 1990.

Bruce Sacerdote. Peer effects with random assignment: Results for dartmouth roommates. *The Quarterly Journal of Economics*, 116(2):681–704, 2001.

Richard J Smith. Non-nested tests for competing models estimated by generalized method of moments. *Econometrica: Journal of the Econometric Society*, 60(4):973–980, 1992.

Ji Tao and Lung-fei Lee. A social interaction model with an extreme order statistic. *The Econometrics Journal*, 17(3):197–240, 2014.

David J Zimmerman. Peer effects in academic outcomes: Evidence from a natural experiment. *Review of Economics and statistics*, 85(1):9–23, 2003.

Definitions of θ and η

Let π be an ordering on $\{1,\ldots,n\}$ in terms of $\{y_i\}_{i=1}^n$;
let $\mathbb{B}(\pi)$ denote a $n \times n$ peer effect coefficient matrix such that

$$\begin{pmatrix} \sum_{k=1}^{d_1} \beta_{k,d_1} \tilde{y}_{1,k} \\ \vdots \\ \sum_{k=1}^{d_n} \beta_{k,d_n} \tilde{y}_{n,k} \end{pmatrix} = \mathbb{B}(\pi)\mathbf{y}$$

given the ordering π. Then,

$$\theta_{i,j} = \sum_{\pi \in \Pi} \theta_{i,j}(\pi) \mathrm{Pr}_n \{\pi\}$$

$$\eta_i = \sum_{\pi \in \Pi} \sum_{j=1}^{n} \theta_{i,j}(\pi) \mathbf{E}[\varepsilon_j | \pi] \cdot \mathrm{Pr}_n \{\pi\}$$

and $\big(\theta_{i,1}(\pi), \ldots, \theta_{i,n}(\pi)\big)$ is the i-th row of the $n \times n$ matrix $(I_n - \mathbb{B}(\pi))^{-1}$. Back

Assumption 3

- (exogeneity) $\{z_{i,1}, \ldots, z_{i,\bar{d}}\}_{i=1}^{n}$ are known, predetermined functions of $\{\mathbf{x}_i, A_{i,j}\}_{1 \leq i,j \leq n}$.
- (relevance) The construction of the instrument z_i and the reduced-form representation of

$$\mathbf{E}\left[\tilde{y}_{i,k}\right] = \sum_{j=1}^{n} \tilde{\theta}_{i,k,j} x_j{}^{\intercal} \gamma + \tilde{\eta}_{i,k}$$

from Corollary 1 satisfy that

$$\sum_{i=1}^{n} \begin{pmatrix} z_{i,d} \\ \mathbf{x}_i \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{n} \tilde{\theta}_{i,1,j} x_j{}^{\intercal} \gamma + \tilde{\eta}_{i,1} \\ \vdots \\ \sum_{j=1}^{n} \tilde{\theta}_{i,d_i,j} x_j{}^{\intercal} \gamma + \tilde{\eta}_{i,d_i} \\ \mathbf{x}_i \end{pmatrix}^{\intercal} \mathbf{1}\{d_i = d\}$$

has full rank, for each $d = 1, \ldots, \bar{d}$.

T1 P2