Supplementary Appendix

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A Additional empirical results

A.1 Smoothness restriction on type-specific time fixed-effects

When the given dataset has relatively small number of units and/or time periods, carefully chosen smoothness restriction on the type-specific time fixed-effect can hugely improve the classification result, as shown in the simulations (see Section 6 of the main text). When there is no *a priori* knowledge on the smoothness restrictions, we suggest using cross-validated mean-square forecasting error, as a selection criterion on the smoothness restriction. For the main specification of the paper, we considered five smoothness restrictions:

$$\delta_t(k) = \delta(k), \quad \cdots \textit{constant}$$
$$\delta_t(k) = \Big(1, \mathbf{1}\{t \geq 1995\}\Big)\delta(k), \quad \cdots \textit{constant with a break}$$
$$\delta_t(k) = \Big(1, \mathbf{1}\{t \geq 1993\}, \mathbf{1}\{t \geq 1996\}\Big)\delta(k), \quad \cdots \textit{constant with two breaks}$$
$$\delta_t(k) = \Big(1, t - 1989\Big)\delta(k), \quad \cdots \textit{linear}$$
$$\delta_t(k) = \Big(1, t - 1989, (t - 1994)\mathbf{1}\{t \geq 1994\}\Big)\delta(k), \quad \cdots \textit{linear with a break}$$

Note that $\{\delta_t(k)\}_t$ are slopes for the type-specific time trend in outcome level; the above restrictions consider linear and quadratic time trends. To evaluate each of the restriction

specifications, we computed the mean-squared forecasting error, using the last three time periods: year 1997-1999. Firstly, we used year 1988-1996 ($T_0 = 8$) as training data and year 1997 as test data. Then, we used year 1988-1997 ($T_0 = 9$) as training data and year 1998 as test data. Lastly, we used year 1988-1998 ($T_0 = 10$) as training data and year 1999 as test data.

Table 1: Cross validation result with $K = 2$

MSFE	4.52	4.94	4.76	5.81	4.84
specification	Cons	Cons	Cons	Linear	Linear
# of breaks	0	1	2	0	1

Table 1 contains the mean-squared forecasting error of the $K = 2$ type classification using each smoothness restriction. Based on the cross validation result, we used the constant slope as our main empirical specification in the type classification step.

A.2 Type classification

In this subsection, we present the full classification result for $K = 2$. Below are the numbers of school districts in each states for Type 1 and Type 2. The number of treated school districts are denoted with red while the the number of never-treated school districts are denoted with black. Table 2 further summarizes the list and presents the number of school districts for each census region. The classification result suggests that the type classification captures heterogeneity across units that is not fully explained by the geographical location; it appears that the location is not a strong predictor of the school district's type.

Type 1 Alabama (3), Arkansas (1), Florida (2+4), Illinois (1), Kentucky (1+1), Mississippi (1), New York (1), North Carolina (3), Ohio (1), Pennsylvania (1), Texas (1), Wisconsin (1)

Type 2 Alabama (1), Arizona (1), Arkansas (1), California (2+1), Connecticut (2), Florida (5), Indiana (1), Maryland (1), Michigan (2+1), Mississippi (2), North Carolina (1), Pennsylvania (1), Texas (3+3)

	Northeast	Midwest	South	West
Type 1	2	2+1	10+7	-
Type 2	3	3+1	7+10	2+2

Table 2: Distribution of types across census regions

A.2.1 Sensitivity to the number of types

As a sensitivity analysis with regard to the number of types K, we considered the type classification under $K = 3, 4$ in addition to $K = 2$. To assess the sensitivity of the classification result to the number of types K, we firstly report the Bayesian information criterion for each value of K as suggested in Bonhomme and Manresa (2015); Janys and Siflinger (2024):

$$\frac{1}{nT_0}\sum_{i,t}\left(Y_{it} - Y_{it-1} - \hat{\delta}_t(\hat{k}_i) - X_{it}{}^{\intercal}\hat{\theta}\right)^2 + \hat{\sigma}^2\frac{K+n+p}{nT_0}\log nT_0$$

where $\hat{\sigma}^2$ is estimated with the largest $K = 4$.[1] Table 3 contains the information criterion for each number of types $K = 2, 3, 4$.

Table 3: BIC across $K = 2, 3, 4$

K	2	3	4
BIC	12.236	12.153	12.159

[1]The constant slope restriction $\delta_t(k) = \delta(k)$ is imposed for $K = 3$ and $K = 4$ as well and thus the number of parameter is set to be $K + n + p$: K constant slopes $\{\delta(k)\}_k$, n types $\{k_i\}_{i=1}^n$ and p control covariate coefficients θ. If the type-specific time fixed-effects were allowed to be fully heterogeneous across t, the number of parameters would be $KT_0 + n + p$.

Secondly, we compare the classification results across the different number of types. Table 4 finds seven groups of units depending on how their type estimate changes along with $K = 2, 3, 4$. For comparison across K, we reorder the types in the decreasing order of $\delta(k)$; the dissimilarity index rose the fastest for Type 1.

Table 4: Type classification comparison betweeon $K = 2$ and $K = 3$

Type seq.	$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(2,2,3)$	$(2,3,3)$	$(2,3,4)$
$K = 2$	1			2			
$K = 3$	1		2			3	
$K = 4$	1	2			3		4
# of units	9	10	3	10	11	3	4

Each column denotes a sequence of type estimates as K changes. For example, the first column finds number of units who were assigned to Type 1 in all of the three type classification results.

Table 3 suggests that the classification may suffer from overfitting when larger number of types is used: $K = 4$. In line with this observation, Table 4 shows us that increasing the number of types gives us types where only a few number of units are assigned: e.g., Type 4 when $K = 4$.

In the rest of the subsection, we report the descriptive statistics and the treatment effect estimates for $K = 3$ and $K = 4$. Table 5 and Table 6 contain within-type balancedness tests for $K = 3$ while Table 7 and Table 8 contain within-type balancedness tests for $K = 4$. Within each type, the control covariates are well-balanced across treatment status. Figure 1 and Figure 2 contain the treatment effect estimation results, respectively for $K = 3$ and $K = 4$. Similarly to $K = 2$ case, we find bigger treatment effect for school districts where the dissimilarity index in the pretreatment periods was rising faster. Lastly, Table 9 and Table 10 contain descriptive statistics for each type.

Table 5: Within-type Balancedness Test, $t = 1988$, $K = 3$

Type 1	treated	never-treated	Diff
1{central city}	0.29	0.58	-0.30
	(0.49)	(0.51)	(0.24)
% (white)	58.20	61.50	-3.30
	(19.05)	(21.29)	(9.47)
% (hispanic)	6.69	4.10	2.59
	(13.73)	(7.38)	(5.61)
% (free/reduced-price lunch)	39.71	37.80	1.91
	(10.82)	(17.95)	(6.60)
# (student)	47574	61230	-13656
	(30851)	(111488)	(34231)
N	7	12	-
p-value			0.519

Type 2	treated	never-treated	Diff
1{central city}	0.67	0.58	0.08
	(0.49)	(0.51)	(0.21)
% (white)	44.54	55.68	-11.13
	(21.83)	(20.05)	(8.56)
% (hispanic)	17.19	9.65	7.54
	(17.00)	(11.46)	(5.92)
% (free/reduced-price lunch)	37.87	34.80	3.07
	(15.83)	(17.67)	(6.85)
# (student)	84552	45499	39053
	(73316)	(50724)	(25736)
N	12	12	-
p-value			0.591

The table reports means of the school district characteristics and their differences across treatment status within each type. The p-value is for the null hypothesis that the means of differences between treated units and never-treated units are all zeros.

Table 6: Within-type Balancedness Test, $t = 1988$, $K = 3$

Type 3	treated	never-treated	Diff
1{central city}	0.50	0.80	-0.30
	(0.71)	(0.45)	(0.54)
% (white)	70.18	40.36	-2.98
	(3.68)	(25.51)	(11.70)
% (hispanic)	9.24	29.43	-20.19
	(7.02)	(26.92)	(13.02)
% (free/reduced-price lunch)	33.47	44.97	-11.51
	(0.68)	(17.54)	(7.86)
# (student)	39196	139532	-100336
	(34375)	(262980)	(120094)
N	2	5	-
p-value			0.930

The table reports means of the school district characteristics and their differences across treatment status within each type. The p-value is for the null hypothesis that the means of differences between treated units and never-treated units are all zeros.

Table 7: Within-type Balancedness Test, $t = 1988$, $K = 4$

Type 1	treated	never-treated	Diff
1{central city}	0.33	0.50	-0.17
	(0.58)	(0.55)	(0.40)
% (white)	49.63	52.51	-2.87
	(16.76)	(26.71)	(14.58)
% (hispanic)	12.58	6.51	6.07
	(21.40)	(10.12)	(13.03)
% (free/reduced-price lunch)	46.07	35.84	10.23
	(14.40)	(22.25)	(12.31)
# (student)	44764	101883	-57119
	(40819)	(152725)	(66655)
N	3	6	-
p-value			0.684

Type 2	treated	never-treated	Diff
1{central city}	0.46	0.50	-0.05
	(0.52)	(0.52)	(0.22)
% (white)	46.21	66.79	-20.58
	(23.69)	(13.75)	(8.17)
% (hispanic)	15.19	8.46	6.73
	(16.97)	(11.74)	(6.14)
% (free/reduced-price lunch)	37.53	34.17	3.36
	(11.50)	(15.81)	(5.73)
# (student)	92154	31723	60431
	(73361)	(22721)	(23071)
N	11	12	-
p-value			0.065

The table reports means of the school district characteristics and their differences across treatment status within each type. The p-value is for the null hypothesis that the means of differences between treated units and never-treated units are all zeros.

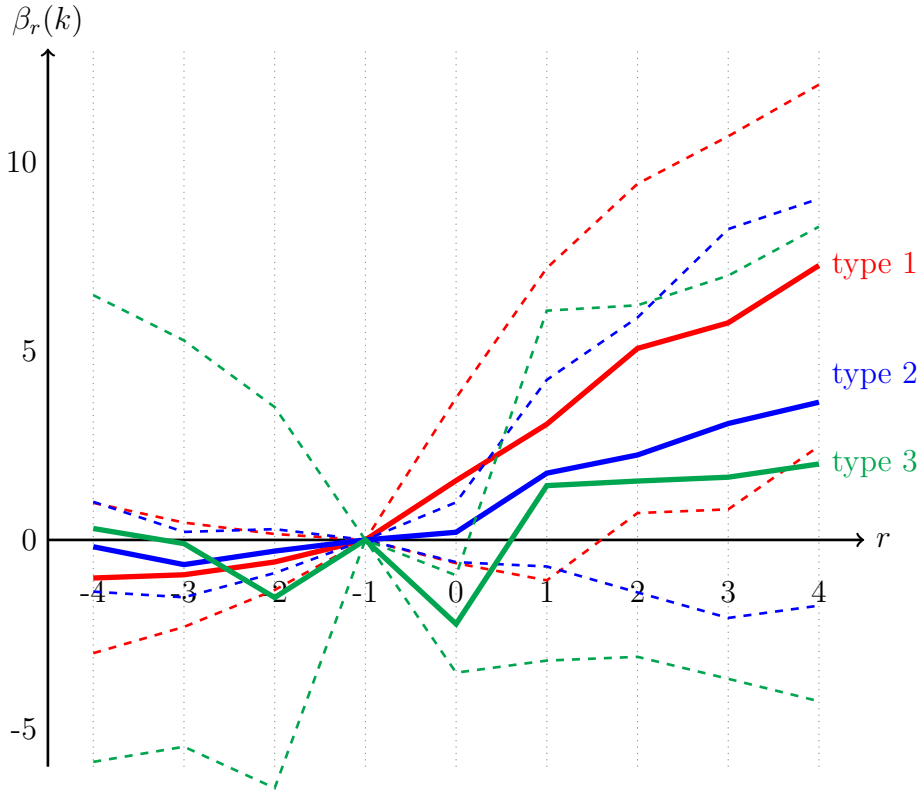
Table 8: Within-type Balancedness Test, $t = 1988$, $K = 4$

Type 3	treated	never-treated	Diff
1{central city}	0.80	0.89	-0.09
	(0.45)	(0.33)	(0.23)
% (white)	56.93	41.20	15.73
	(20.47)	(22.73)	(11.88)
% (hispanic)	9.67	18.77	-9.11
	(14.97)	(23.11)	(10.21)
% (free/reduced-price lunch)	36.29	44.66	-8.37
	(19.57)	(17.09)	(10.44)
# (student)	39932	104913	-64980
	(21910)	(197483)	(66553)
N	5	9	-
p-value			0.928

Type 4	treated	never-treated	Diff
1{central city}	0.50	0.50	0.00
	(0.71)	(0.71)	(0.71)
% (white)	70.18	60.28	9.90
	(3.68)	(18.37)	(13.25)
% (hispanic)	9.24	1.34	7.90
	(7.02)	(1.72)	(5.11)
% (free/reduced-price lunch)	33.47	34.51	-1.05
	(6.82)	(19.66)	(13.91)
# (student)	39196	21109	18087
	(34375)	(15180)	(26572)
N	2	2	-

The table reports means of the school district characteristics and their differences across treatment status within each type. The p-value is for the null hypothesis that the means of differences between treated units and never-treated units are all zeros; there are too few units in Type 4 for within-type balancedness test so there is no p-value reported for Type 4.

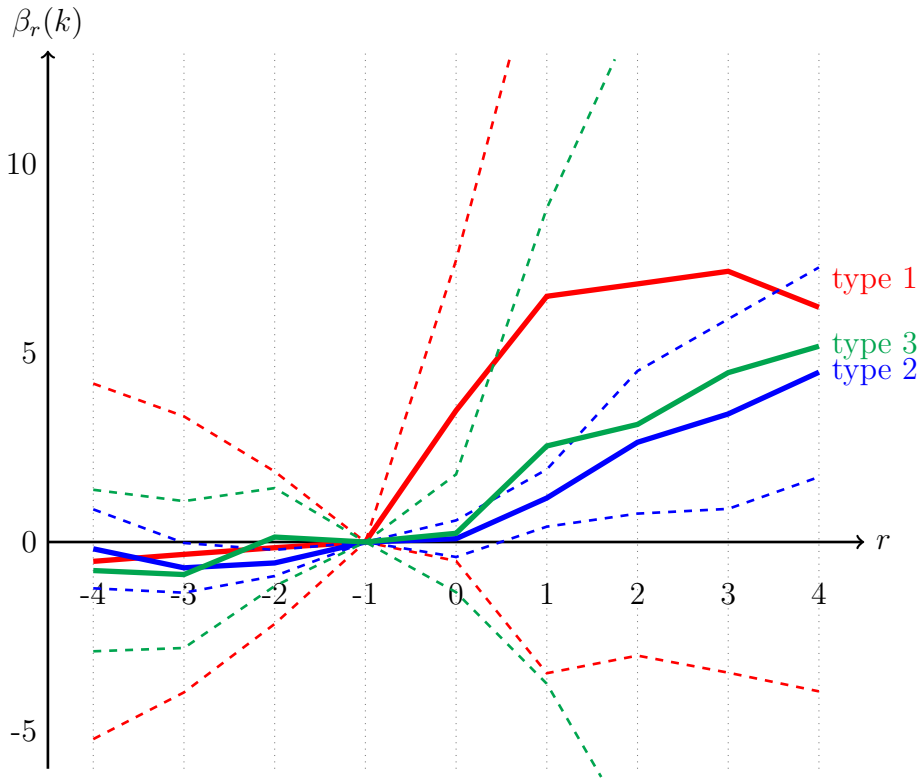
Figure 1: Type-specific ATT, $K = 3$



The graph reports the type-specific diff-in-diff estimates for the effect of dismissing court-mandated desegregation plan on the dissimilarity index of a school district. The dissimilarity index ranges from 0 to 100. In 1988, the average dissimilarity index was 34 and the standard deviation was 16.

Types are ordered in the decreasing order of $\delta(k)$; the dissimilarity index rose the fastest for Type 1 and the slowest for Type 3. The dashed lines denote the confidence intervals at 0.05 significance level, computed with asymptotic standard errors.

Figure 2: Type-specific ATT, $K = 4$



The graph reports the type-specific diff-in-diff estimates for the effect of dismissing court-mandated desegregation plan on the dissimilarity index of a school district. The dissimilarity index ranges from 0 to 100. In 1988, the average dissimilarity index was 34 and the standard deviation was 16.

Types are ordered in the decreasing order of $\delta(k)$; the dissimilarity index rose the fastest for Type 1 and the slowest for Type 3. The dashed lines denote the confidence intervals at 0.05 significance level, computed with asymptotic standard errors. The treatment effect estimates for Type 4 are omitted since there are too few units in Type 4 and therefore the esitmates are not as precisely estimated as for other types.

Table 9: Type-specific Descriptive Statistics, $t = 1988$, $K = 3$

	Type 1	Type 2	Type 3
dissimilarity index	28.21	37.41	39.79
	(13.32)	(18.51)	(16.04)
1{central city}	0.47	0.63	0.71
	(0.51)	(0.49)	(0.49)
% (white)	60.28	50.11	48.88
	(20.02)	(21.27)	(25.45)
% (hispanic)	5.05	13.42	23.66
	(9.89)	(14.69)	(24.25)
% (free/reduced-price lunch)	38.51	36.34	41.68
	(15.39)	(16.48)	(15.38)
# (student)	56199	65026	110865
	(89213)	(64801)	(220680)
N	19	24	7

The table reports the group means of the school district characteristics and their differences. The p-value for the null hypothesis that Type 1 and Type 2 share the same mean is 0.001. Similarly, the p-value for the null hypothesis of equal means between Type 1 and Type 3 is 0.218 and the p-value for the null hypothesis of equal means between Type 2 and Type 3 is 0.804.

Table 10: Type-specific Descriptive Statistics, $t = 1988$, $K = 4$

	Type 1	Type 2	Type 3	Type 4
dissimilarity index	27.11	34.00	39.17	34.45
	(16.50)	(13.57)	(21.92)	(12.83)
$\mathbf{1}\{$central city$\}$	0.44	0.48	0.86	0.50
	(0.52)	(0.51)	(0.36)	(0.58)
% (white)	51.55	56.95	46.82	65.23
	(22.76)	(21.45)	(22.54)	(12.24)
% (hispanic)	8.53	11.68	15.52	5.29
	(13.70)	(14.55)	(20.45)	(6.18)
% (free/reduced-price lunch)	39.25	35.78	41.67	33.99
	(19.68)	(13.71)	(17.74)	(11.37)
# (student)	86844	60625	81705	30153
	(125739)	(60474)	(158718)	(24078)
N	9	23	14	4

The table reports the group means of the school district characteristics and their differences. The null hypothesis that two types share the same mean is not rejected at the 0.05 significance level for any pair of two types except for Type 1 and Type 3, possibly due to small number of units per type.

A.2.2 Extrapolating type classification from never-treated units

As a robustness check on the $K = 2$ classification, we additionally conducted a type classification exercise only on the never-treated units. Out of the 50 school district, 29 school districts were never dismissed of the court-mandated desegregation plan until 2007, effectively giving us 19 untreated outcomes. The type classification was firstly done with the 29 never-treated units only, using $T = 19$, and then extrapolated to the 21 treated units, using all the available pretreatment outcomes for each unit. Since the number of time periods we use is longer, we considered different smoothness restrictions:

$$\delta_t(k) = \delta(k), \quad \cdots \textit{constant}$$
$$\delta_t(k) = \Big(1, \mathbf{1}\{t \geq 1999\}\Big)\delta(k), \quad \cdots \textit{constant with a break}$$
$$\delta_t(k) = \Big(1, \mathbf{1}\{t \geq 1996\}, \mathbf{1}\{t \geq 2002\}\Big)\delta(k), \quad \cdots \textit{constant with two breaks}$$
$$\delta_t(k) = \Big(1, t - 1989\Big)\delta(k), \quad \cdots \textit{linear}$$
$$\delta_t(k) = \Big(1, t - 1989, (t - 1998)\mathbf{1}\{t \geq 1998\}\Big)\delta(k), \quad \cdots \textit{linear with a break}$$

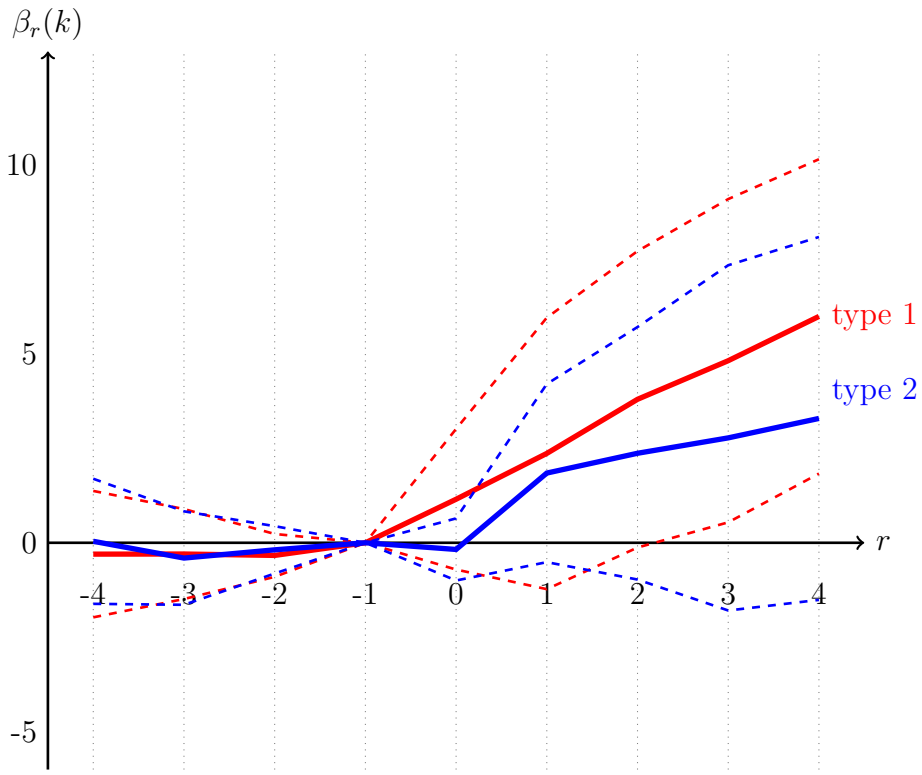
Table 11 conatins the cross validation results; the step function with two breaks is selected based on the mean squared forecasting error using the last three periods (year 2005-2007).

MSFE	2.674	2.590	2.569	2.646	2.716
specification	Cons	Cons	Cons	Linear	Linear
# of breaks	0	1	2	0	1

Table 11: Cross validation result with $K = 2$, never-treated units only

Table 12 compares the two classification results and Figure 3 contains the type-specific diff-in-diff estimates under the extrapolated classification result. The two classification results overlap for 90% of the units and the type-specific treatment effect estimates give us the same qualitative implication; the treatment effect estimates are bigger for Type 1 and the estimates are significant at 5% level only for Type 1.

Figure 3: Type-specific ATT, $K = 2$



The graph reports the type-specific diff-in-diff estimates for the effect of dismissing court-mandated desegregation plan on the dissimilarity index of a school district. The type classification was implemented with never-treated units and then extrapolated to treated units. The dissimilarity index ranges from 0 to 100. In 1988, the average dissimilarity index was 34 and the standard deviation was 16.

Type 1 is the type where the dissimilarity index was rising faster and Type 2 is the type where the dissimilarity index was rising slower:
$\hat{\delta}(1) = \big(2.35, -0.04, -0.72\big)^\intercal$ and $\hat{\delta}(2) = \big(0.71, 0.23, 0.06\big)^\intercal$.

	1	2	total
1	18	4	22
2	1	27	28
total	19	31	50

Table 12: Counts of school districts for each type

The rows are the types estimated with the population pretreatment outcomes and the columns are the types estimated with the never-treated units and extrapolated to the treated units.

A.2.3 Simulation results on misspecification

To show how the type-specific treatment effect estimator performs in a misspecified model, we conducted two additional simulation exercises. For the data generating process, we used the same DGP from the main text of the paper: for $t = -T_0 - 1, \cdots, 0$,

$$Y_{it} = \alpha_i + \delta(k)(t+1) + \beta(k_i)D_i\mathbf{1}\{t=0\} + U_{it},$$
$$U_{it} = \rho U_{it-1} + V_{it}.$$

$D_i, \alpha_i, U_{i,-T_0-1}, \{V_{it}\}_{t\leq 0}$ are mutually independent given k_i. $D_i\big|k_i \sim \text{Bernoulli}\big(\pi(k_i)\big)$ and

$$(\alpha_i, U_{i,-T_0-1})\,\big|k_i \sim \mathcal{N}\left(\begin{pmatrix}\alpha(k_i)\\ 0\end{pmatrix}, \begin{pmatrix}17 & 0\\ 0 & \sigma\end{pmatrix}\right),$$
$$V_{it}\big|k_i \overset{iid}{\sim} \mathcal{N}\Big(0, \sigma^2(1-\rho^2)\Big).$$

For the first case of misspecification, we consider a misspecified number of types; the true DGP satisfies the type-specific parallel trend assumption with the finite types, but the type classification step uses a smaller number of types than the truth. The values of the DGP

parameters are as follows: $K = 5$, $\sigma = 1.85$, $\rho = 0.60$,

$$\begin{pmatrix} \pi(1) & \pi(2) & \pi(3) & \pi(4) & \pi(5) \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$
$$\begin{pmatrix} \alpha(1) & \alpha(2) & \alpha(3) & \alpha(4) & \alpha(5) \end{pmatrix} = \begin{pmatrix} 37 & 39 & 35 & 36 & 38 \end{pmatrix}$$
$$\begin{pmatrix} \beta(1) & \beta(2) & \beta(3) & \beta(4) & \beta(5) \end{pmatrix} = \begin{pmatrix} 4 & 4 & 4 & 1 & 1 \end{pmatrix}$$
$$\begin{pmatrix} \delta(1) & \delta(2) & \delta(3) & \delta(4) & \delta(5) \end{pmatrix} = \begin{pmatrix} 2.06 & 1.66 & 1.26 & 0.4 & 0 \end{pmatrix}$$
$$\begin{pmatrix} \mu(1) & \mu(2) & \mu(3) & \mu(4) & \mu(5) \end{pmatrix} = \begin{pmatrix} \frac{1}{10} & \frac{1}{5} & \frac{1}{5} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

where $\mu(k) = \Pr\{k_i = k\}$. The DGP is 'fair' to the type-specific diff-in-diff estimator with $K = 2$, in the sense that the DGP can be reduced to a two-types DGP, when limited to $\pi(k)$ and $\beta(k)$. If the classification step with $K = 2$ successfully separate Type 1-3 and Type 4-5, the type-specific diff-in-diff estimator will be unbiased.

For the second case of misspecification, we consider weakly separated types. To model a latent type that is weakly separated, we let k_i be a continuous variable: $k_i \sim \text{unif}[0, 1]$. For the rest of the DGP parameters, we let

$$\begin{aligned} \pi(k) &= \frac{1}{3}\mathbf{1}\{k \leq 0.5\} + \frac{2}{3}\mathbf{1}\{k > 0.5\}, \\ \alpha(k) &= 37 + 2k, \\ \beta(k) &= 4\mathbf{1}\{k \leq 0.5\} + \mathbf{1}\{k > 0.5\}, \\ \delta(k) &= 1.66(1 - k). \end{aligned}$$

Again, the DGP is 'fair' to the type-specific diff-in-diff estimator with $K = 2$, in the sense that the type-specific diff-in-diff estimator is unbiased when the classification step with $K = 2$ successfully separate $\{i : k_i \leq 0.5\}$ and $\{i : k_i > 0.5\}$. Thus, for both of the misspecifications, the bias in the treatment effect estimation only comes from type misclassficiation.

Table 13 and Table 14 contain the simulation results. In the case of misspecified number of

types, the type-specific diff-in-diff estimator is significantly biased for the small pretreatment case ($T_0 = 10$), but the bias disappears for large pretreatment cases ($T_0 = 20, 30$). The overall performance of the type-specific diff-in-diff estimator is on par with that of the synthetic diff-in-diff estimator, which is a more flexible method. This result is intuitive since the true DGP admits a 'pseudo-true' DGP that makes the estimator unbiased with $K = 2$, by grouping Type 1-3 and Type 4-5, and the 'pseudo-true' types are well-separated at the margin. In the case of the continuous latent type, the type-specific diff-in-diff estimator does not perform well; there is no separation in terms of the type variable.[2] In contrast, the synthetic diff-in-diff estimator performs better for large pretreatment cases ($T_0 = 20, 30$) compared to the type-specific diff-in-diff estimator.

Thus, it is recommended that a user consider alternative estimators when they suspect that the finite type structure does not fully reflect the the unit-level heterogeneity. For example, by comparing the synthetic diff-in-diff estimate with the aggregate estimate from the type-specific diff-in-diff estimates, one can heuristically check if the finite type structure creates bias in estimation and continue to report the type-specific estimates to discuss the treatment effect heterogeneity, if the two estimates agree.

[2]The type-specific diff-in-diff estimator performs relatively well for large T_0 in the continuous type DGP when the correct smoothness restriction is imposed; this is because when the type k_i is known from within-unit information as $T_0 \to \infty$, the K-means objective function is minimized at an even split due to symmetry in the type distribution, which happens to lead to an unbiased estimator.

Table 13: $K = 2$ is used when $k_i \in \{1, \cdots, 5\}$

Bias					
(n, T_0)	DiD	SC	synthetic DiD	type-specific DiD	type-specific DiD
$(100, 10)$	-0.469	-0.177	-0.212	-0.165	-0.116
$(100, 20)$	-0.482	-0.069	-0.079	-0.033	-0.014
$(100, 30)$	-0.467	-0.027	-0.019	0.012	0.014
Constant slope	-	-	-	NO	YES

MSE					
(n, T_0)	DiD	SC	synthetic DiD	type-specific DiD	type-specific DiD
$(100, 10)$	0.437	0.478	0.254	0.241	0.230
$(100, 20)$	0.433	0.276	0.178	0.180	0.174
$(100, 30)$	0.442	0.298	0.195	0.200	0.200
Constant slope	-	-	-	NO	YES

Table 14: $K = 2$ is used when $k_i \in [0, 1]$

Bias					
(n, T_0)	DiD	SC	synthetic DiD	type-specific DiD	type-specific DiD
$(100, 10)$	-0.292	-0.203	-0.251	-0.289	-0.217
$(100, 20)$	-0.266	-0.060	-0.087	-0.189	-0.087
$(100, 30)$	-0.264	-0.042	-0.046	-0.113	-0.054
Constant slope	-	-	-	NO	YES

MSE					
(n, T_0)	DiD	SC	synthetic DiD	type-specific DiD	type-specific DiD
$(100, 10)$	0.258	0.376	0.256	0.260	0.242
$(100, 20)$	0.257	0.301	0.186	0.225	0.197
$(100, 30)$	0.230	0.262	0.166	0.177	0.168
Constant slope	-	-	-	NO	YES

B Proof for Lemma 1

The following lemma shows that whenever well-defined, the probability limit of a weighted, over-the-time mean of $\{Y_{it} - Y_{it-1}\}_t$ is a function of the latent type k_i.

Lemma B.1. *Assume that* $\Pr\{k_i = k\} > 0$ *and*

$$\lim_{T_0 \to \infty} \frac{1}{T_0} \sum_{t=-T_0}^{-1} \left(\mathbf{E}\left[Y_{it}(\infty) - Y_{it-1}(\infty) | k_i = k \right] \right)^2$$

exists for each k *and*

$$\frac{1}{T_0} \sum_{t=-T_0}^{-1} a_t \big(Y_{it}(E_i) - Y_{it-1}(E_i) - \mathbf{E}\left[Y_{it}(\infty) - Y_{it-1}(\infty) | k_i \right] \big) \xrightarrow{p} 0.$$

for any $\{a_t\}_t$ *such that* $\lim_{T_0 \to \infty} \frac{1}{T_0} \sum_t {a_t}^2$ *is finite. In addition, suppose that the following probabilistic limit exists:*

$$k_i^*(\{\tilde{a}_t\}_t) = \operatorname*{plim}_{T_0 \to \infty} \frac{1}{T_0} \sum_{t=-T_0}^{-1} \tilde{a}_t \left(Y_{it} - Y_{it-1} \right)$$

for some $\{\tilde{a}_t\}_t$ *such that* $\lim_{T_0 \to \infty} \frac{1}{T_0} \sum_t \tilde{a}_t^2$ *is finite. Then,*

$$k_i^*(\{\tilde{a}_t\}_t) = \lim_{T_0 \to \infty} \frac{1}{T_0} \sum_{t=-T_0}^{-1} \tilde{a}_t \delta_t(k_i). \tag{1}$$

Proof. The claim has two parts: the limit

$$\lim_{T_0 \to \infty} \frac{1}{T_0} \sum_{t=-T_0}^{-1} a_t \delta_t(k)$$

exists for each k whenever the probability limit $k_i^*(\{a_t\}_t)$ is well-defined and that

$$k_i^*(\{a_t\}_t) = \sum_{k=1}^{K} \lim_{T_0 \to \infty} \frac{1}{T_0} \sum_{t=-T_0}^{-1} a_t \delta_t(k) \mathbf{1}\{k_i = k\}.$$

To show the existence of the limits, assume to the contrary that for some k, $\frac{1}{T_0}\sum_t a_t\delta_t(k)$ does not converge. Let

$$\varepsilon^l = \lim_{T\to\infty}\inf_{T\geq T_0}\frac{1}{T_0}\sum_t a_t\delta_t(k),$$
$$\varepsilon^u = \lim_{T\to\infty}\sup_{T\geq T_0}\frac{1}{T_0}\sum_t a_t\delta_t(k).$$

Then, $\varepsilon^l < \varepsilon^u$. With $\varepsilon^* = \frac{\varepsilon^u-\varepsilon^l}{5}$, find that

$$\begin{aligned}
&\Pr\left\{k_i^*(\{a_t\}_t) \notin \left(\frac{1}{T_0}\sum_{t=-T_0}^{-1} a_t\delta_t(k_i) - \varepsilon^*, \frac{1}{T_0}\sum_{t=-T_0}^{-1} a_t\delta_t(k_i) + \varepsilon^*\right)\right\} \\
&= \Pr\left\{\left|\frac{1}{T_0}\sum_{t=-T_0}^{-1} a_t\delta_t(k_i) - k_i^*(\{a_t\}_t)\right| \geq \varepsilon^*\right\} \\
&\leq \Pr\left\{\left|\frac{1}{T_0}\sum_{t=-T_0}^{-1} a_t\left(Y_{it} - Y_{it-1}\right) - k_i^*(\{a_t\}_t)\right| \geq \frac{\varepsilon^*}{2}\right\} \\
&\qquad + \Pr\left\{\left|\frac{1}{T_0}\sum_{t=-T_0}^{-1} a_t\left(Y_{it} - Y_{it-1} - \delta_t(k_i)\right)\right| \geq \frac{\varepsilon^*}{2}\right\}.
\end{aligned}$$

The two probabilities on the RHS of the inequality both go to zero as $T_0 \to \infty$; the first probability goes to zero from the definition of $k_i^*(\{a_t\}_t)$ and the second probability goes to zero since $\frac{1}{T_0}\sum_{t=-T_0}^{-1} a_t\left(Y_{it} - Y_{it-1} - \delta_t(k_i)\right) = o_p(1)$.

Since $\Pr\{k_i = k\} > 0$, the convergence also implies that

$$p_t := \Pr\left\{k_i^*(\{a_s\}_s) \notin \left(\frac{1}{t}\sum_{s=-t}^{-1} a_s\delta_s(k) - \varepsilon^*, \frac{1}{t}\sum_{s=-t}^{-1} a_s\delta_s(k) + \varepsilon^*\right)\Big| k_i = k\right\}$$

converges to zero as $t \to \infty$. Find some $\tilde{T}$ such that $p_t \leq \frac{1}{3}$ for any $t \geq \tilde{T}$. From ε^l being a lim inf, there must exist some $T^* \geq \tilde{T}$ such that

$$\frac{1}{T^*}\sum_{t=-T^*}^{-1} a_t\delta_t(k) \leq \varepsilon^l + \varepsilon^*.$$

Then,

$$\begin{aligned}
&\Pr\left\{k_i^*(\{a_t\}_t) \leq \varepsilon^l + 2\varepsilon^* \middle| k_i = k\right\} \\
&\geq \Pr\left\{k_i^*(\{a_t\}_t) \in \left(\frac{1}{T^*}\sum_{t=-T^*}^{-1} a_t\delta_t(k_i) - \varepsilon^*, \frac{1}{T^*}\sum_{t=-T^*}^{-1} a_t\delta_t(k_i) + \varepsilon^*\right) \middle| k_i = k\right\} \\
&= 1 - p_{T^*} \geq \frac{2}{3}.
\end{aligned}$$

Likewise, from ε^u being a lim sup there must also exist some $T^{**} \geq \tilde{T}$ such that

$$\frac{1}{T^{**}}\sum_{t=-T^{**}}^{-1} a_t\delta_t(k) \geq \varepsilon^u - \varepsilon^*.$$

Then,

$$\begin{aligned}
p_{T^{**}} &= \Pr\left\{k_i^*(\{a_t\}_t) \notin \left(\frac{1}{T^{**}}\sum_{t=-T^{**}}^{-1} a_t\delta_t(k_i) - \varepsilon^*, \frac{1}{T^{**}}\sum_{t=-T^{**}}^{-1} a_t\delta_t(k_i) + \varepsilon^*\right)\right\} \\
&\geq \Pr\left\{k_i^*(\{a_t\}_t) \leq \varepsilon^u - 2\varepsilon^*\right\} \geq \Pr\left\{k_i^*(\{a_t\}_t) \leq \varepsilon^l + 2\varepsilon^*\right\} \geq \frac{2}{3}.
\end{aligned}$$

The second inequality holds since $\varepsilon^* = \frac{\varepsilon^u - \varepsilon^u}{5}$ and thus $\varepsilon^l + 2\varepsilon^* < \varepsilon^u - 2\varepsilon^*$. Since we set T such that $p_t \leq \frac{1}{3}$ for any $t \geq T$, we get a contradiction. Thus, the probabilistic limit implies K different limits. The second part of Equation (1) follows directly: $\frac{1}{T_0}\sum_{t=-T_0}^{-1} a_t\left(Y_{it} - Y_{it-1} - \delta_t(k_i)\right) = o_p(1)$. $k_i^*(\{a_t\}_t)$ has at most K points in its support whenever the probability limit is well-defined. □

Note that the probability limit of the weighted mean of $\{Y_{it} - Y_{it-1}\}_t$ discussed in Lemma B.1 is a function of the observable variables. Thus, we can extend the distribution of $\left(\{Y_{it}\}_{t=-\infty}^{T_1-1}, E_i\right)$ to the distribution of $\left(\{Y_{it}\}_{t=-\infty}^{T_1-1}, E_i, k_i^*(\{a_t\}_t)\right)$ whenever $k_i^*(\{a_t\}_t)$ is well-defined. From Lemma B.1, we know that whenever well-defined, $k_i^*(\{a_t\}_t)$ is indeed a function of k_i. Thus, it suffices to show that for any two different type $k \neq k$', we can find $\{a_t\}_t$ such that $k_i^*(\{a_t\}_t)$ separates $\{k_i = k\}$ and $\{k_i = k'\}$.

For notational brevity, let

$$\delta_t(k) = \mathbf{E}\left[Y_{it}(\infty) - Y_{it-1}(\infty) | k_i = k\right].$$

Then, by taking $\{a_t\}_t$ from $\{\delta_t(1)\}_t, \cdots, \{\delta_t(K)\}_t$, both claims follow trivially. Firstly, the condition in Lemma 1 and Assumption 4 guarantees the existence of

$$\lim_{T_0 \to \infty} \frac{1}{T_0} \sum_{t=-T_0}^{-1} \delta_t(k)\delta_t(k')$$

for any k, k'. Also, Assumption 4 guarantees that for any $k \neq k'$, either

$$\frac{1}{T_0} \sum_{t=-T_0}^{-1} \delta_t(k) \left(\delta_t(k') - \delta_t(k)\right) \quad \text{or} \quad \frac{1}{T_0} \sum_{t=-T_0}^{-1} \delta_t(k') \left(\delta_t(k) - \delta_t(k')\right)$$

is nonzero. Thus, either $\{a_t = \delta_t(k)\}_t$ or $\{a_t = \delta_t(k')\}_t$ successfully separates $\{k_i = k\}$ and $\{k_i = k'\}$; for example, when the first quantity is nonzero, $k_i^*(\{\delta_t(k)\}_t)$ has different values at $k_i = k$ and at $k_i = k'$. Thus, by conducting an exhaustive search on $\{a_t\}_t$ such that $k_i^*(\{a_t\}_t)$ is well-defined, we can construct a discrete random variable k_i^{**} that is bijective with k_i. Let $\{b^1, \cdots, b^K\} \subset \mathbb{R}^K$ denote the support of k_i^{**}. Then, the identification of the treatment effect parameters naturally follows from that $\left(\mathbf{E}\left[Y_{it}\mathbf{1}\{k_i = 1\}\right], \cdots, \mathbf{E}\left[Y_{it}\mathbf{1}\{k_i = K\}\right]\right)$ is a permutation of $\left(\mathbf{E}\left[Y_{it}\mathbf{1}\{k_i^{**} = b^1\}\right], \cdots, \mathbf{E}\left[Y_{it}\mathbf{1}\{k_i^{**} = b^K\}\right]\right)$ and so on.

C Proof for Theorem 2

In the rest of the proof, we will use the dot notation to denote the first difference: $\dot{Y}_{it} = Y_{it} - Y_{it-1}$, $\dot{X}_{it} = X_{it} - X_{it-1}$ and $\dot{U}_{it} = U_{it} - U_{it-1}$. Also, we will use the superscript naught to denote the true values of the parameters and the latent type variable: e.g. k_i^0 is the true type of unit i.

We prove Theorem 2 in the context of a linear model for outcome in level (see *Remark 5* of

the main text), since it involves some noteworthy technical complications compared to the case of a linear model for first-differenced outcomes. All of the arguments stays the same for the case of a linear model for first-differenced outcomes, by replacing $\dot{X}_{it}$ and $\dot{U}_{it}$ with X_{it} and U_{it}.

The proof for Theorem 1 is omitted since Theorem 2 nests Theorem 1 by connecting Assumption 5 to Assumption 7. Assumption 5-b is equivalent with parts of Assumption 7-b,d concerning U_{it} and $\delta_t(k)$. Assumption 5-e provides the weak dependence conditions as does Assumption 7-g. When investigating the proof in the context of Theorem 1, replace $\delta_t^0(k)$ with $\mathbf{E}[\dot{Y}_{it}(\infty)|k]$ and $\dot{U}_{it}$ with $\dot{Y}_{it}(E_i) - \mathbf{E}[\dot{Y}_{it}(\infty)|k_i^0]$.[3]

Step 1

The first step is to obtain an approximation of the objective function. Note that

$$\begin{aligned}
\widehat{Q}(\theta, \delta, \gamma) &= \frac{1}{nT_0} \sum_{i=1}^{n} \sum_{t=-T_0}^{-1} \left(\dot{Y}_{it} - \delta_t(k_i) - \dot{X}_{it}^{\intercal} \theta \right)^2 \\
&= \frac{1}{nT_0} \sum_{i,t} \left(\delta_t^0(k_i^0) - \delta_t(k_i) + \dot{X}_{it}^{\intercal}(\theta^0 - \theta) + \dot{U}_{it} \right)^2 \\
&= \frac{1}{nT_0} \sum_{i,t} \left\{ \left(\delta_t^0(k_i^0) - \delta_t(k_i) + \dot{X}_{it}^{\intercal}(\theta^0 - \theta) \right)^2 + \dot{U}_{it}^2 \right\} \\
&\quad + \frac{2}{nT_0} \sum_{i,t} \left(\delta_t^0(k_i^0) - \delta_t(k_i) + \dot{X}_{it}^{\intercal}(\theta^0 - \theta) \right) \dot{U}_{it}.
\end{aligned}$$

Let

$$\tilde{Q}(\theta, \delta, \gamma) = \frac{1}{nT_0} \sum_{i,t} \left\{ \left(\delta_t^0(k_i^0) - \delta_t(k_i) + \dot{X}_{it}^{\intercal}(\theta^0 - \theta) \right)^2 + \dot{U}_{it}^2 \right\}.$$

[3] In the case of a linear model for outcome in level, U_{it} corresponds to $Y_{it}(E_i) - \mathbf{E}[Y_{it}(\infty)|k_i]$. In the case of a linear model for first-differenced outcomes, $\dot{U}_{it}$ corresponds to $\dot{Y}_{it}(E_i) - \mathbf{E}\big[\dot{Y}_{it}(\infty)|k_i\big]$. Since the proof is written in the context of a linear model for outcome in level, $\dot{U}_{it}$ appears in the place of $\dot{Y}_{it}(E_i) - \mathbf{E}\big[\dot{Y}_{it}(\infty)|k_i\big]$.

Then,

$$\left|\widehat{Q}(\theta,\delta,\gamma)-\tilde{Q}(\theta,\delta,\gamma)\right| = \left|\frac{2}{nT_0}\sum_{i,t}\Big(\delta_t^0(k_i^0)-\delta_t(k_i)+\dot{X}_{it}^{\intercal}(\theta^0-\theta)\Big)\dot{U}_{it}\right|$$
$$\leq \left|\frac{2}{nT_0}\sum_{i,t}\Big(\delta_t^0(k_i^0)-\delta_t(k_i)\Big)\dot{U}_{it}\right| + \left|\frac{2}{nT_0}\sum_{i,t}\dot{X}_{it}^{\intercal}(\theta^0-\theta)\dot{U}_{it}\right|. \quad (2)$$

Firstly, find that

$$\left|\frac{1}{nT_0}\sum_{i,t}\delta_t^0(k_i^0)\dot{U}_{it}\right| \leq \sum_{k=1}^{K}\left|\frac{1}{nT_0}\sum_{i,t}\delta_t^0(k)\dot{U}_{it}\mathbf{1}\{k_i^0=k\}\right|$$
$$\leq \sum_{k=1}^{K}\left(\frac{1}{T_0}\sum_t \delta_t^0(k)^2\right)^{\frac{1}{2}}\left(\frac{1}{T_0}\sum_t\left(\frac{1}{n}\sum_i \dot{U}_{it}\mathbf{1}\{k_i^0=k\}\right)^2\right)^{\frac{1}{2}}$$
$$\leq M\sum_{k=1}^{K}\left(\frac{1}{n^2T_0}\sum_{i,j,t}\dot{U}_{it}\dot{U}_{jt}\mathbf{1}\{k_i^0=k_j^0=k\}\right)^{\frac{1}{2}} \xrightarrow{p} 0.$$

The first two inequalities are from separating the summation into types and applying Cauchy-Schwartz's inequality to over t. The third is from Assumption 7-b. It remains to prove the convergence in probability; for that we use Assumption 7-a,d. With some constant $C>0$ that only depends on $M>0$ from Assumption 7,

$$\mathbf{E}\left[\dot{U}_{it}\dot{U}_{jt}\mathbf{1}\{k_i^0=k_j^0=k\}\right] = \begin{cases}\mathbf{E}\big[\dot{U}_{it}^2\mathbf{1}\{k_i^0=k\}\big]\leq C & \text{if } i=j\\ \mathbf{E}\left[\dot{U}_{it}\mathbf{1}\{k_i^0=k\}\right]\mathbf{E}\left[\dot{U}_{jt}\mathbf{1}\{k_j^0=k\}\right]=0 & \text{if } i\neq j\end{cases}$$

since $\mathbf{E}[\dot{U}_{it}\mathbf{1}\{k_i^0=k\}]=\mathbf{E}[\dot{U}_{it}|k_i^0=k]\Pr\{k_i^0=k\}=0$.[4] Then,

$$\mathbf{E}\left[\frac{1}{nT_0}\sum_{i,j,t}\dot{U}_{it}\dot{U}_{jt}\mathbf{1}\{k_i^0=k_j^0=k\}\right]\leq C.$$

[4]In the case of Theorem 1,

$$\mathbf{E}[\dot{Y}_{it}(E_i)-\mathbf{E}[\dot{Y}_{it}(\infty)|k_i^0]|k_i^0=k]=\mathbf{E}[\mathbf{E}[\dot{Y}_{it}(E_i)-\dot{Y}_{it}(\infty)|k_i^0,E_i]|k_i^0=k]=0$$

from Assumption 2.

and $\frac{1}{nT_0}\sum_{i,j,t}\dot{U}_{it}\dot{U}_{jt}\mathbf{1}\{k_i^0=k_j^0=k\}=O_p(1)$ since $\frac{1}{nT_0}\sum_{i,j}\dot{U}_{it}\dot{U}_{jt}\mathbf{1}\{k_i^0=k_j^0=k\}$ is nonnegative and bounded in expectation. We can repeat this for the other quantity in the first term of (2).

Secondly, again from applying Cauchy-Schwartz's inequality,

$$\left|\frac{1}{nT_0}\sum_{i,t}\dot{X}_{it}^{\intercal}\left(\theta^0-\theta\right)\dot{U}_{it}\right| \leq \frac{1}{T_0}\sum_t\left\|\frac{1}{n}\sum_i\dot{U}_{it}\dot{X}_{it}\right\|_2\cdot\left\|\theta^0-\theta\right\|_2$$
$$\leq \frac{2M}{\sqrt{n}}\cdot\frac{1}{T_0}\sum_t\left(\frac{1}{n}\sum_{i,j}\dot{U}_{it}\dot{U}_{jt}\dot{X}_{it}^{\intercal}\dot{X}_{jt}\right)^{\frac{1}{2}}=\frac{2M}{\sqrt{n}}\cdot O_p(1)\xrightarrow{p}0.$$

The convergence in probability is from Assumption 7-a,d. Find that

$$\mathbf{E}\left[\dot{U}_{it}\dot{X}_{it}\right]=\mathbf{0},\qquad\qquad \mathbf{E}\left[\dot{U}_{it}^2\dot{X}_{it}^{\intercal}\dot{X}_{it}\right]\leq C$$

with some constant $C>0$ that only depends on $M>0$ from Assumption 7. Thus, from Jensen's inequality,

$$\frac{1}{T_0}\sum_t\mathbf{E}\left[\left(\frac{1}{n}\sum_{i,j}\dot{U}_{it}\dot{U}_{jt}\dot{X}_{it}^{\intercal}\dot{X}_{jt}\right)^{\frac{1}{2}}\right]\leq\frac{1}{T_0}\sum_t\left(\mathbf{E}\left[\frac{1}{n}\sum_{i,j}\dot{U}_{it}\dot{U}_{jt}\dot{X}_{it}^{\intercal}\dot{X}_{jt}\right]\right)^{\frac{1}{2}}\leq\sqrt{C}.$$

Then $\frac{1}{T_0}\sum_t\left(\frac{1}{n}\sum_{i,j}\dot{U}_{it}\dot{U}_{jt}\dot{X}_{it}^{\intercal}\dot{X}_{jt}\right)^{\frac{1}{2}}=O_p(1)$ since it is nonnegative and bounded in expectation. Thus, $\widehat{Q}(\theta,\delta,\gamma)-\tilde{Q}(\theta,\delta,\gamma)=o_p(1)$. Note that the bounding terms used in the consistency argument do not depend on (θ,δ,γ); the consistency is uniform.

Step 2

By plugging in the true parameters, $\tilde{Q}(\theta^0, \delta^0, \gamma^0) = \frac{1}{nT_0}\sum_{i,t}\dot{U}_{it}^2$ and

$$\begin{aligned}
\tilde{Q}(\theta, \delta, \gamma) - \tilde{Q}(\theta^0, \delta^0, \gamma^0) &= \frac{1}{nT_0}\sum_{i,t}\left(\delta_t^0(k_i^0) - \delta_t(k_i) + \dot{X}_{it}^{\intercal}(\theta^0 - \theta)\right)^2 \\
&\geq \frac{1}{nT_0}\sum_{i,t}\left(\dot{X}_{it}^{\intercal}(\theta^0 - \theta) - \bar{\dot{X}}_{k_i^0 \wedge k_i, t}^{\intercal}(\theta^0 - \theta)\right)^2 \\
&= \frac{1}{nT_0}\sum_{i,t}(\theta^0 - \theta)^{\intercal}\left(\dot{X}_{it} - \bar{\dot{X}}_{k_i^0 \wedge k_i, t}\right)\left(\dot{X}_{it} - \bar{\dot{X}}_{k_i^0 \wedge k_i, t}\right)^{\intercal}(\theta^0 - \theta) \\
&\geq \min_{\gamma \in \Gamma}\rho_n(\gamma) \cdot \|\theta^0 - \theta\|_2^2
\end{aligned}$$

with $\rho_n(\gamma)$ as defined in Assumption 7-h. Note that the unknowns in $\tilde{Q}(\theta, \delta, \gamma) - \tilde{Q}(\theta^0, \delta^0, \gamma^0)$ other than $(\theta^0 - \theta)$ are functions of (t, k_i^0, k_i). Thus, subtracting the group mean defined with (t, k_i^0, k_i) from $\dot{X}_{it}^{\intercal}(\theta^0 - \theta)$ is the lower bound for the sum of squares, giving us the first inequality.

From the result of Step 1 and the definition of the estimator being a minimizer of the objective function,

$$\begin{aligned}
\tilde{Q}(\hat{\theta}, \hat{\delta}, \hat{\gamma}) &= \widehat{Q}(\hat{\theta}, \hat{\delta}, \hat{\gamma}) + o_p(1) \\
&\leq \widehat{Q}(\theta^0, \delta^0, \gamma^0) + o_p(1) \\
&= \tilde{Q}(\theta^0, \delta^0, \gamma^0) + o_p(1)
\end{aligned}$$

and therefore

$$\min_{\gamma \in \Gamma}\rho_n(\gamma) \cdot \left\|\theta^0 - \hat{\theta}\right\|_2^2 \leq \tilde{Q}(\hat{\theta}, \hat{\delta}, \hat{\gamma}) - \tilde{Q}(\theta^0, \delta^0, \gamma^0) = o_p(1).$$

Therefore from Assumption 7-h,

$$\left\|\theta^0 - \hat{\theta}\right\|_2^2 = \frac{1}{\min_{\gamma\in\Gamma}\rho_n(\gamma)} \cdot \min_{\gamma\in\Gamma}\rho_n(\gamma)\left\|\theta^0 - \hat{\theta}\right\|_2^2 \xrightarrow{p} \frac{1}{\rho}\cdot 0 = 0.$$

We have consistency of $\hat{\theta}$.

Step 3

In this step, we show that $\{\hat{\delta}_t(\hat{k}_i)\}_{i,t}$ is close to $\{\delta_t^0(k_i^0)\}_{i,t}$ in terms of the l_2 norm.

$$\begin{aligned}
&\left|\tilde{Q}(\hat{\theta},\hat{\delta},\hat{\gamma}) - \tilde{Q}(\theta^0,\hat{\delta},\hat{\gamma})\right| \\
&= \left|\frac{1}{nT_0}\sum_{i,t}\left(\delta_t^0(k_i^0) - \hat{\delta}_t(\hat{k}_i) + \dot{X}_{it}^{\intercal}(\theta^0-\hat{\theta})\right)^2 - \frac{1}{nT_0}\sum_{i,t}\left(\delta_t^0(k_i^0) - \hat{\delta}_t(\hat{k}_i)\right)^2\right| \\
&\leq \left|\frac{2}{nT_0}\sum_{i,t}\left(\delta_t^0(k_i^0) - \hat{\delta}_t(\hat{k}_i)\right)\dot{X}_{it}^{\intercal}(\theta^0-\hat{\theta}) + \frac{1}{nT_0}\sum_{i,t}\left(\dot{X}_{it}^{\intercal}(\theta^0-\hat{\theta})\right)^2\right| \\
&\leq \frac{4M}{nT_0}\sum_{i,t}\|\dot{X}_{it}\|_2 \cdot \left\|\theta^0-\hat{\theta}\right\|_2 + \frac{1}{nT_0}\sum_{i,t}\|\dot{X}_{it}\|_2^2 \cdot \left\|\theta^0-\hat{\theta}\right\|_2^2 = o_p(1).
\end{aligned}$$

The second inequality is from Assumption 7-b and Cauchy-Schwartz inequality on $|\dot{X}_{it}^{\intercal}(\theta^0 - \hat{\theta})|$. Note that for any n, $\frac{1}{nT_0}\sum_{i,t}\|\dot{X}_{it}\|_2^2$ is bounded in expectation by $4M$ from Assumption 7.d and thus $O_p(1)$. Likewise, $\frac{1}{nT_0}\sum_{i,t}\|\dot{X}_{it}\|_2$ is bounded in expectation by $2\sqrt{M}$ from Jensen's inequality. Since we have shown $\hat{\theta} \xrightarrow{p} \theta^0$, we have the last equality. Then,

$$\begin{aligned}
&\frac{1}{nT_0}\sum_{i,t}\left(\delta_t^0(k_i^0) - \hat{\delta}_t(\hat{k}_i)\right)^2 + \frac{1}{nT_0}\sum_{i,t}\dot{U}_{it}^2 \\
&= \tilde{Q}(\theta^0,\hat{\delta},\hat{\gamma}) = \tilde{Q}(\hat{\theta},\hat{\delta},\hat{\gamma}) + o_p(1) = \widehat{Q}(\hat{\theta},\hat{\delta},\hat{\gamma}) + o_p(1) \\
&\leq \widehat{Q}(\theta^0,\delta^0,\gamma^0) + o_p(1) = \frac{1}{nT_0}\sum_{i,t}\dot{U}_{it}^2 + o_p(1).
\end{aligned}$$

$\frac{1}{nT_0}\sum_{i,t}\left(\delta_t^0(k_i^0) - \hat{\delta}_t(\hat{k}_i)\right)^2 = o_p(1)$. For Theorem 1, the result holds directly from Step 1.

Step 4

In this step, we find some permutation on $\left\{\hat{\delta}_t(k)\right\}_{t,k}$ so that $\frac{1}{nT_0}\sum_{i,t}\left(\delta_t^0(k_i^0)-\hat{\delta}_t(k_i^0)\right)^2$ is close to zero. Note that $\widehat{Q}(\theta,\delta,\gamma)$ does not vary for any $(\theta,\tilde{\delta},\tilde{\gamma})$ defined with a permutation on $(1,\cdots,K)$: with σ, a permutation on $\{1,\cdots,K\}$, letting $\tilde{k}_i=\sigma(k_i)$ and $\tilde{\delta}_t(\sigma(k))=\delta_t(k)$ gives us $\widehat{Q}(\theta,\delta,\gamma)=\widehat{Q}(\theta,\tilde{\delta},\tilde{\gamma})$. Thus, we want to define a bijection on $\{1,\cdots,K\}$ to match $\hat{k}$ with true k^0, to have classification result. Define a function σ by letting

$$\sigma(k)=\arg\min_{\tilde{k}}\frac{1}{T_0}\sum_{t=-T_0}^{-1}\left(\delta_t^0(k)-\hat{\delta}_t(\sigma(k))\right)^2$$

for each k. First, let us show that σ actually lets the objective go to zero for each k: fix k,

$$\begin{aligned}
&\min_{\tilde{k}}\frac{1}{T_0}\sum_{t=-T_0}^{-1}\left(\delta_t^0(k)-\hat{\delta}_t(\sigma(k))\right)^2\\
&\leq\frac{n}{\sum_i\mathbf{1}\{k_i^0=k\}}\cdot\min_{\tilde{k}}\frac{1}{nT_0}\sum_{i,t}\left(\delta_t^0(k)-\hat{\delta}_t(\sigma(k))\right)^2\mathbf{1}\{k_i^0=k\}\\
&\leq\frac{n}{\sum_i\mathbf{1}\{k_i^0=k\}}\cdot\frac{1}{nT_0}\sum_{i,t}\left(\delta_t^0(k_i^0)-\hat{\delta}_t(\hat{k}_i)\right)^2\xrightarrow{p}0
\end{aligned}$$

as $n\to\infty$. From Assumption 7-f, we have the convergence.

For some $k,\tilde{k}$ such that $k\neq\tilde{k}$,

$$\begin{aligned}
&\left(\frac{1}{T_0}\sum_t\left(\hat{\delta}_t(\sigma(k))-\hat{\delta}_t(\sigma(\tilde{k}))\right)^2\right)^{\frac{1}{2}}\\
&\geq\left(\frac{1}{T_0}\sum_t\left(\delta_t^0(k)-\delta_t^0(\tilde{k})\right)^2\right)^{\frac{1}{2}}\\
&\qquad-\left(\frac{1}{T_0}\sum_t\left(\delta_t^0(k)-\hat{\delta}_t(\sigma(k))\right)^2\right)^{\frac{1}{2}}-\left(\frac{1}{T_0}\sum_t\left(\delta_t^0(\tilde{k})-\hat{\delta}_t(\sigma(\tilde{k}))\right)^2\right)^{\frac{1}{2}}\\
&\xrightarrow{p}c(k,\tilde{k})>0
\end{aligned}$$

from Assumption 7.c. Thus, $\Pr\{\sigma\text{ is not bijective}\}\leq\sum_{k\neq\tilde{k}}\Pr\left\{\sigma(k)=\sigma(\tilde{k})\right\}\to 0$ as $n\to$

∞.

Before proceeding to the next step, let us drop the σ notation. Based on σ, we can construct a bijection $\tilde{\sigma} : \{1, \cdots, K\} \to \{1, \cdots, K\}$ such that

$$\frac{1}{T}\sum_t \left(\delta_t^0(k) - \hat{\delta}_t(\tilde{\sigma}(k))\right)^2 \xrightarrow{p} 0 \tag{3}$$

as $n \to \infty$ for all k, by letting $\tilde{\sigma} = \sigma$ whenever σ is bijective. From now on, I will drop $\tilde{\sigma}$ by always rearranging $(\hat{\theta}, \hat{\delta}, \hat{\gamma})$ so that $\tilde{\sigma}(k) = k$.

Step 5

Here, we study the probability of the K-means algorithm assigning a wrong type to an arbitrary unit i.

$$\begin{aligned}
\Pr\left\{\hat{k}_i \neq k_i^0\right\} &\leq \sum_{\tilde{k} \neq k_i^0} \Pr\left\{\frac{1}{T_0}\sum_t \left(\dot{Y}_{it} - \hat{\delta}_t(\tilde{k}) - \dot{X}_{it}^\intercal \hat{\theta}\right)^2 \leq \frac{1}{T_0}\sum_t \left(\dot{Y}_{it} - \hat{\delta}_t(k_i^0) - \dot{X}_{it}^\intercal \hat{\theta}\right)^2\right\} \\
&= \sum_{\tilde{k} \neq k_i^0} \Pr\left\{\frac{2}{T_0}\sum_t \left(\hat{\delta}_t(k_i^0) - \hat{\delta}_t(\tilde{k})\right) \cdot \left(\dot{Y}_{it} - \frac{\hat{\delta}_t(k_i^0) + \hat{\delta}_t(\tilde{k})}{2} - \dot{X}_{it}^\intercal \hat{\theta}\right) \leq 0\right\}.
\end{aligned}$$

The inequality is from the second stage of the K-means algorithm. Then,

$$\begin{aligned}
&\Pr\left\{\hat{k}_i \neq k_i^0\right\} \\
&\leq \sum_{\tilde{k} \neq k_i^0} \Pr\left\{\frac{2}{T_0}\sum_t \left(\hat{\delta}_t(k_i^0) - \hat{\delta}_t(\tilde{k})\right) \cdot \left(\delta_t^0(k_i^0) - \frac{\hat{\delta}_t(k_i^0) + \hat{\delta}_t(\tilde{k})}{2} + \dot{X}_{it}^\intercal(\theta^0 - \hat{\theta}) + \dot{U}_{it}\right) \leq 0\right\} \\
&\leq \sum_k \sum_{\tilde{k} \neq k} \Pr\left\{\frac{2}{T}\sum_t \left(\hat{\delta}_t(k) - \hat{\delta}_t(\tilde{k})\right) \cdot \left(\delta_t^0(k) - \frac{\hat{\delta}_t(k) + \hat{\delta}_t(\tilde{k})}{2} + \dot{X}_{it}^\intercal(\theta^0 - \hat{\theta}) + \dot{U}_{it}\right) \leq 0\right\}.
\end{aligned}$$

Let

$$
\begin{aligned}
A_{ik\tilde{k}} &= \frac{1}{T_0}\sum_t \left(\hat{\delta}_t(k) - \hat{\delta}_t(\tilde{k})\right)\dot{U}_{it} + \frac{1}{T_0}\sum_t \left(\hat{\delta}_t(k) - \hat{\delta}_t(\tilde{k})\right)\dot{X}_{it}^{\intercal}(\theta^0 - \hat{\theta}) \\
&\quad + \frac{1}{T_0}\sum_t \left(\hat{\delta}_t(k) - \hat{\delta}_t(\tilde{k})\right)\cdot\left(\delta_t^0(k) - \frac{\hat{\delta}_t(k) + \hat{\delta}_t(\tilde{k})}{2}\right) \\
B_{ik\tilde{k}} &= \frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)\dot{U}_{it} + \frac{1}{2T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)^2.
\end{aligned}
$$

Note that $A_{ik\tilde{k}}$ depends on the estimator $(\hat{\theta}, \hat{\delta}, \hat{\gamma})$ while $B_{ik\tilde{k}}$ does not. Then,

$$
\Pr\left\{\hat{k}_i \neq k_i^0\right\} \leq \sum_k \sum_{\tilde{k}\neq k} \Pr\left\{A_{ik\tilde{k}} \leq 0\right\} \leq \sum_k \sum_{\tilde{k}\neq k} \Pr\left\{B_{ik\tilde{k}} \leq |B_{ik\tilde{k}} - A_{ik\tilde{k}}|\right\} \tag{4}
$$

We will show that $A_{ik\tilde{k}}$ and $B_{ik\tilde{k}}$ are sufficiently close to each other and that $\Pr\{B_{ik\tilde{k}} \leq 0\}$ converges to zero sufficiently fast.

$$
\begin{aligned}
|B_{ik\tilde{k}} - A_{ik\tilde{k}}| \leq{}& \left|\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \hat{\delta}_t(k)\right)\dot{U}_{it}\right| + \left|\frac{1}{T_0}\sum_t \left(\delta_t^0(\tilde{k}) - \hat{\delta}_t(\tilde{k})\right)\dot{U}_{it}\right| \\
&+ \left|\frac{1}{T_0}\sum_t \left(\hat{\delta}_t(k) - \hat{\delta}_t(\tilde{k})\right)\dot{X}_{it}^{\intercal}(\theta^0 - \hat{\theta})\right| \\
&+ \left|\frac{1}{2T_0}\sum_t \left(\delta_t^0(k) - \hat{\delta}_t(k)\right)\cdot\left(\delta_t^0(k) - \hat{\delta}_t(k) - \delta_t^0(\tilde{k}) + \hat{\delta}_t(\tilde{k})\right)\right| \\
&+ \left|\frac{1}{2T_0}\sum_t \left(\delta_t^0(\tilde{k}) - \hat{\delta}_t(\tilde{k})\right)\cdot\left(\delta_t^0(\tilde{k}) - \hat{\delta}_t(\tilde{k}) - \delta_t^0(k) - \hat{\delta}_t(k)\right)\right|.
\end{aligned}
$$

We apply Cauchy-Schwartz's inequality to each of the five terms so that we can use the consistency result in (3). For the first term,

$$
\left|\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \hat{\delta}_t(k)\right)\dot{U}_{it}\right| \leq \left(\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \hat{\delta}_t(k)\right)^2\right)^{\frac{1}{2}}\left(\frac{1}{T_0}\sum_t \dot{U}_{it}^2\right)^{\frac{1}{2}}
$$

and similarly for the second term. As for the third term, from Assumption 7-b,

$$\begin{aligned}\left|\frac{1}{T_0}\sum_t\left(\hat{\delta}_t(k)-\hat{\delta}_t(\tilde{k})\right)\dot{X}_{it}^{\intercal}(\theta^0-\hat{\theta})\right| &\leq \frac{1}{T_0}\sum_t\left|\hat{\delta}_t(k)-\hat{\delta}_t(\tilde{k})\right|\cdot\|\dot{X}_{it}\|_2\cdot\left\|\theta^0-\hat{\theta}\right\|_2 \\ &\leq 2M\left(\frac{1}{T_0}\sum_t\|\dot{X}_{it}\|_2\right)\cdot\left\|\theta^0-\hat{\theta}\right\|_2\end{aligned}$$

Last, for the fourth term, from Assumption 7-b,

$$\begin{aligned}&\left|\frac{1}{2T_0}\sum_t\left(\delta_t^0(k)-\hat{\delta}_t(k)\right)\cdot\left(\delta_t^0(k)-\hat{\delta}_t(k)-\delta_t^0(\tilde{k})+\hat{\delta}_t(\tilde{k})\right)\right| \\ &\leq 2M\left(\frac{1}{T_0}\sum_t\left(\delta_t^0(k)-\hat{\delta}_t(k)\right)^2\right)^{\frac{1}{2}}\end{aligned}$$

and similarly for the fifth term. From Assumption 7-d, both $\frac{1}{T_0}\sum_t\dot{U}_{it}^2$ and $\frac{1}{T_0}\sum_t\|\dot{X}_{it}\|_2$ are uniformly bounded in expectation and thus $O_p(1)$. To use (3), choose an arbitrary $\eta>0$ and focus only on the event of

$$\left\|\theta^0-\hat{\theta}\right\|_2,\left(\frac{1}{T_0}\sum_t\left(\delta_t^0(k)-\hat{\delta}_t(k)\right)\right)^{\frac{1}{2}}<\eta \tag{5}$$

for all k. When (5) is true, with some constant $C>0$,

$$|B_{ik\tilde{k}}-A_{ik\tilde{k}}|\leq\eta C\left(\left(\frac{1}{T_0}\sum_t\dot{U}_{it}^2\right)^{\frac{1}{2}}+\frac{1}{T_0}\sum_t\|\dot{X}_{it}\|_2+1\right).$$

Note that C only depend on M from Assumption 7 and does not depend on η. Let $D(\eta)$ be

a binary random variable which equals one if (5) holds true for all k. Then,

$$\begin{aligned}
&\Pr\left\{B_{ik\tilde{k}} \leq \left|B_{ik\tilde{k}} - A_{ik\tilde{k}}\right|, D(\eta) = 1\right\} \\
&\leq \Pr\left\{B_{ik\tilde{k}} \leq \eta C\left(\left(\frac{1}{T_0}\sum_t \dot{U}_{it}^2\right)^{\frac{1}{2}} + \frac{1}{T_0}\sum_t \|\dot{X}_{it}\|_2 + 1\right)\right\} \\
&\leq \Pr\left\{\frac{1}{T_0}\sum_t \dot{U}_{it}^2 \geq M^*\right\} + \Pr\left\{\frac{1}{T_0}\sum_t \|\dot{X}_{it}\|_2 \geq M^*\right\} \\
&\qquad + \Pr\left\{B_{ik\tilde{k}} \leq \eta C(\sqrt{M^*} + M^* + 1)\right\} \qquad (6)
\end{aligned}$$

for any arbitrary $M^* > 0$. Let $M^* = \max\{4\sqrt{M} + 1, 4\tilde{M}\}$ since $\mathbf{E}[\dot{U}_{it}^2]$ is uniformly bounded by $4\sqrt{M}$ from Assumption 7-d.[5]

Now, we show that all three probabilities in (6) go to zero. For that, we use Lemma B5 of Bonhomme and Manresa (2015). For the first quantity, find that

$$\begin{aligned}
\Pr\left\{\frac{1}{T_0}\sum_t \dot{U}_{it}^2 \geq M^*\right\} &\leq \Pr\left\{\frac{1}{T_0}\sum_t \dot{U}_{it}^2 \geq 4\sqrt{M} + 1\right\} \\
&\leq \Pr\left\{\frac{1}{T_0}\sum_t \left(\dot{U}_{it}^2 - \mathbf{E}\big[\dot{U}_{it}^2\big]\right) \geq 1\right\}.
\end{aligned}$$

[5]In cases of the linear model for first-differenced outcomes and Theorem 1, a similar uniform bound on $\mathbf{E}[U_{it}^2]$ and $\mathbf{E}[(\dot{Y}_{it}(E_i) - \mathbf{E}[\dot{Y}_{it}(\infty)|k_i^0])^2]$ can be found.

Let $Z_t = \dot{U}_{it}^2 - \mathbf{E}[\dot{U}_{it}^2]$. WTS $\{Z_t\}_{t=1}^{T_0}$ satisfies the condition given in Assumption 7-g.

$$
\begin{aligned}
&\Pr\{|Z_t| \geq z\} \\
&= \Pr\left\{|U_{it} - U_{it-1}| \geq \sqrt{\mathbf{E}[\dot{U}_{it}^2] + z}\right\} + \Pr\left\{|U_{it} - U_{it-1}| \leq \sqrt{\mathbf{E}[\dot{U}_{it}^2] - z}\right\} \\
&\leq \Pr\left\{|U_{it}| \geq \frac{\sqrt{\mathbf{E}[\dot{U}_{it}^2] + z}}{2}\right\} + \Pr\left\{|U_{it-1}| \geq \frac{\sqrt{\mathbf{E}[\dot{U}_{it}^2] + z}}{2}\right\} + \mathbf{1}\{z \leq \mathbf{E}[\dot{U}_{it}^2]\} \\
&\leq 2\exp\left(1 - \left(\frac{\sqrt{\mathbf{E}[\dot{U}_{it}^2] + z}}{2b}\right)^{d_2}\right) + \mathbf{1}\{z \leq \mathbf{E}[\dot{U}_{it}^2]\} \\
&\leq 2\exp\left(1 - \left(\frac{z}{4b^2}\right)^{\frac{d_2}{2}}\right) + \mathbf{1}\{z \leq \mathbf{E}[\dot{U}_{it}^2]\}.
\end{aligned}
$$

We want to find some $\tilde{b}$ and $\tilde{d}_2$ such that

$$\Pr\{|Z_t| \geq z\} \leq \exp\left(1 - \left(\frac{z}{\tilde{b}}\right)^{\tilde{d}_2}\right).$$

Note that the RHS crosses one when $z = \tilde{b}$. It suffices to show

$$2\exp\left(1 - \left(\frac{z}{4b^2}\right)^{\frac{d_2}{2}}\right) + \mathbf{1}\{z \leq \mathbf{E}[\dot{U}_{it}^2]\} \leq \exp\left(1 - \left(\frac{z}{\tilde{b}}\right)^{\tilde{d}_2}\right) \tag{7}$$

for $z \geq \tilde{b}$. Fix some $\tilde{d}_2 \in \left(0, \frac{d_2}{2}\right)$ and let

$$\tilde{b} = \max\left\{4\sqrt{M} + 1, 4b^2\left(\log 2 + 1\right)^{\frac{2}{d_2}}, 4b^2\left(\frac{2\tilde{d}_2}{d_2}\right)^{\frac{2}{d_2}}\right\}.$$

Since $\tilde{b} \geq 4\sqrt{M} \geq \mathbf{E}[\dot{U}_{it}^2]$, (7) for $z \geq \tilde{b}$ is equivalent with

$$\exp\left(\left(\frac{z}{4b^2}\right)^{\frac{d_2}{2}} - \left(\frac{z}{\tilde{b}}\right)^{\tilde{d}_2}\right) \geq 2 \qquad \Leftrightarrow \qquad \left(\frac{z}{4b^2}\right)^{\frac{d_2}{2}} - \left(\frac{z}{\tilde{b}}\right)^{\tilde{d}_2} \geq \log 2.$$

The inequality holds at $z = \tilde{b}$ and the LHS in the last inequality strictly increases in z since

$$\frac{d_2}{2} \cdot \frac{z^{\frac{d_2}{2}-1}}{(2b)^{d_2}} - \frac{\tilde{d}_2 z^{\tilde{d}_2-1}}{\tilde{b}^{\tilde{d}_2}} = z^{\tilde{d}_2-1}\left(\frac{d_2}{2(2b)^{d_2}} z^{\frac{d_2}{2}-\tilde{d}_2} - \frac{\tilde{d}_2}{\tilde{b}^{\tilde{d}_2}}\right) \geq 0$$

for all $z \geq \tilde{b}$. Z_t is strongly mixing since $\dot{U}_{it}^2$ is a (Borel-)measurable function of (U_{it}, U_{it-1}) and we can always find $\tilde{a}$ and $\tilde{d}_1$ such that $\exp(-\tilde{a}t^{\tilde{d}_1}) \leq \exp(-a(t-1)^{d_1})$ for all t. Thus, conditions in Assumption 7-g holds for Z_t. Thus, from Lemma B5 of Bonhomme and Manresa (2015), for any $\nu > 0$,

$$T_0{}^{\nu} \Pr\left\{\frac{1}{T_0}\sum_t \dot{U}_{it}^2 \geq M^*\right\} = o(1).$$

For Theorem 1, find that a similar result holds with $\Pr\left\{\frac{1}{T_0}\sum_t (\dot{Y}_{it}(e) - \mathbf{E}[\dot{Y}_{it}(\infty)|k_i^0])^2 \geq M^*\right\}$. Since E_i has finite support, $T_0{}^{\nu} \Pr\left\{\frac{1}{T_0}\sum_t (\dot{Y}_{it}(E_i) - \mathbf{E}[\dot{Y}_{it}(\infty)|k_i^0])^2 \geq M^*\right\} = o(1)$.

For the second quantity, find that

$$\begin{aligned}\Pr\left\{\frac{1}{T_0}\sum_t \|\dot{X}_{it}\|_2 \geq M^*\right\} &\leq \Pr\left\{\frac{2}{T_0}\sum_{t=-T_0-1}^{-1} \|X_{it}\|_2 \geq 4\tilde{M}\right\} \\ &\leq \Pr\left\{\frac{1}{T_0+1}\sum_{t=-T_0-1}^{-1} \|X_{it}\|_2 \geq \tilde{M}\right\}\end{aligned}$$

From Assumption 7-d, for any $\nu > 0$,

$$T_0{}^{\nu} \Pr\left\{\frac{1}{T_0}\sum_t \|\dot{X}_{it}\|_2 \geq M^*\right\} \leq (T_0+1)^{\nu} \Pr\left\{\frac{1}{T_0+1}\sum_{t=-T_0-1}^{-1} \|X_{it}\|_2 \geq \tilde{M}\right\} = o(1).$$

For the last quantity, let $\eta^* = \frac{c^*}{4C(M^*+\sqrt{M^*}+1)}$ with $c^* = \frac{\min_{k,k'} c(k,k')}{2} > 0$. Then,

$$\begin{aligned}
&\Pr\left\{B_{ik\tilde{k}} \leq \eta^* C(M^* + \sqrt{M^*} + 1)\right\} \\
&\leq \Pr\left\{\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)\dot{U}_{it} \leq \eta^* C(M^* + \sqrt{M^*} + 1) - \frac{c^*}{2}\right\} \\
&\qquad + \mathbf{1}\left\{\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)^2 \leq c^*\right\} \\
&\leq \Pr\left\{\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)\dot{U}_{it} \leq -\frac{c^*}{4}\right\} + \mathbf{1}\left\{\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)^2 \leq c^*\right\}.
\end{aligned}$$

For the first term, use Lemma B5 of Bonhomme and Manresa (2015) again. From Assumption 7-b, we have

$$\Pr\left\{\left|\left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)\dot{U}_{it}\right| \geq z\right\} \leq \Pr\left\{|\dot{U}_{it}| \geq \frac{z}{2M}\right\}.$$

By applying similar argument from before, we can prove the tail property given in Assumption 7-g for $\left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)\dot{U}_{it}$ with any k and $\tilde{k}$. Also, the first part of Assumption 7-g is satisfied since $\left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)\dot{U}_{it}$ is a (Borel-)measurable function of $\left(U_{it}, U_{it-1}\right)$.[6] For any $\nu > 0$,

$$T_0{}^{\nu} \Pr\left\{\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)\dot{U}_{it} \leq -\frac{c^*}{4}\right\} = o(1).$$

Again, in the case of Theorem 1, note that E_i has finite support and repeat

$$T_0{}^{\nu} \Pr\left\{\frac{1}{T_0}\sum_t \left(\mathbf{E}[\dot{Y}_{it}(\infty)|k_i^0 = k] - \mathbf{E}[\dot{Y}_{it}(\infty)|k_i^0 = \tilde{k}]\right)\left(\dot{Y}_{it}(e) - \mathbf{E}[\dot{Y}_{it}(\infty)|k_i^0]\right)\right\} = o(1)$$

for every e. For the second term, Assumption 7-c assumes that $\mathbf{1}\{\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)^2 \leq c^*\} = 0$ when n is large and therefore $o(T^{-\nu})$ for any $\nu > 0$.

[6] Here, I am treating $\left\{\delta_t^0(k)\right\}_{t,k}$ as if uniformly fixed across n. This can be relaxed by assuming $\left\{\delta_t^0(k)\right\}_{t,k}$ is also a strongly mixing random process as in Bonhomme and Manresa (2015).

Finally, going back to (4) and (6), thanks to K being finite,

$$\Pr\left\{\hat{k}_i \neq k_i^0, D(\eta^*) = 1\right\} = o(T^{-\nu}). \tag{8}$$

Step 6

In this step let us discuss the probability of assigning a wrong type at least to one unit. As $n \to \infty$, for any $\nu > 0$

$$\begin{aligned}
&\Pr\left\{\sup_i \mathbf{1}\{\hat{k}_i \neq k_i^0\} > 0\right\} \\
&\leq \Pr\left\{\sum_i \mathbf{1}\{\hat{k}_i \neq k_i^0\} > 0, D(\eta^*) = 1\right\} + \Pr\{D(\eta^*) = 0\} \\
&\leq n \cdot \Pr\left\{\hat{k}_i \neq k_i^0, D(\eta^*) = 1\right\} + \Pr\{D(\eta^*) = 0\} \\
&= o(n{T_0}^{-\nu}) + o(1).
\end{aligned}$$

The last equality holds from (8).

□

D Proof for Corollary 3

The first part of the proof is the same with Corollary 2. The second part follows the proof of Theorem 2 of Callaway and Sant'Anna (2021). Fix some t, k and e such that $0 \leq e \leq t \leq T_1 - 1$ and $\mu(k, e) > 0$. Then, it satisfies that $t - e \leq \bar{r}_k$ from Assumption 6.

Step 1

Firstly, let us show that $\widehat{ATT}_t(k,e)$ is close to the infeasible estimator using the true types $\{k_i^0\}_{i=1}^n$:

$$\widetilde{ATT}_t(k,e) = \frac{\sum_{i=1}^n (Y_{it} - Y_{i,e-1})\,\mathbf{1}\{k_i^0 = k, E_i = e\}}{\sum_{i=1}^n \mathbf{1}\{k_i^0 = k, E_i = e\}} - \frac{\sum_{i=1}^n (Y_{it} - Y_{i,e-1})\,\mathbf{1}\{k_i^0 = k, E_i > t\}\pi_e(X_i,k,\hat{\xi})/\pi_{t+}(X_i,k,\hat{\xi})}{\sum_{i=1}^n \mathbf{1}\{k_i^0 = k, E_i > t\}\pi_e(X_i,k,\hat{\xi})/\pi_{t+}(X_i,k,\hat{\xi})}.$$

Find that

$$\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n \left(Y_{it} - Y_{i,e-1}\right)\left(\mathbf{1}\{\hat{k}_i = k, E_i > t\} - \mathbf{1}\{k_i^0 = k, E_i > t\}\right)\frac{\pi_e(X_i,k,\hat{\xi})}{\pi_{t+}(X_i,k,\hat{\xi})}\right|$$
$$\leq \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n \left(Y_{i,e+r} - Y_{i,e-1}\right)^2\right)^{\frac{1}{2}} \cdot \left(\frac{1}{n}\sum_{i=1}^n \mathbf{1}\{\hat{k}_i \neq k_i^0\}\right)^{\frac{1}{2}} \sup_i \left|\frac{\pi_e(X_i,k,\hat{\xi})}{\pi_{t+}(X_i,k,\hat{\xi})}\right|.$$

$\sup_i \pi_e/\pi_{t+}$ is bounded by $1/\varepsilon^\pi$ from Assumption 9-c. $\frac{1}{n}\sum_{i=1}^n (Y_{i,e+r} - Y_{i,e-1})^2$ is bounded in expectation uniformly over e and r from Assumption 9-a and therefore $O_p(1)$. From Theorem 2,

$$\Pr\left\{\sum_{i=1}^n \mathbf{1}\{\hat{k}_i \neq k_i\} > \varepsilon^2\right\} \leq \Pr\left\{\sup_i \mathbf{1}\{\hat{k}_i \neq k_i\} > 0\right\} = o(n{T_0}^{-\nu}) + o(1)$$

for any $\nu, \epsilon > 0$. Since $n{T_0}^{-\nu^*} \to 0$ as $n \to \infty$, $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n \mathbf{1}\{\hat{k}_i \neq k_i^0\}\right)^{\frac{1}{2}} = o_p(1)$.

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n (Y_{it} - Y_{i,e-1})\mathbf{1}\{\hat{k}_i = k, E_i > t\}\frac{\pi_e(X_i,k,\hat{\xi})}{\pi_{t+}(X_i,k,\hat{\xi})}$$
$$= \frac{1}{\sqrt{n}}\sum_{i=1}^n (Y_{it} - Y_{i,e-1})\mathbf{1}\{k_i^0 = k, E_i > t\}\frac{\pi_e(X_i,k,\hat{\xi})}{\pi_{t+}(X_i,k,\hat{\xi})} + o_p(1)$$

By the same argument,

$$\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\{\hat{k}_i = k, E_i > t\}\frac{\pi_e(X_i,k,\hat{\xi})}{\pi_{t+}(X_i,k,\hat{\xi})} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\{k_i^0 = k, E_i > t\}\frac{\pi_e(X_i,k,\hat{\xi})}{\pi_{t+}(X_i,k,\hat{\xi})} + o_p(1).$$

The same applies to the other term without π_e/π_{t+}. Note that $\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\{k_i^0 = k, E_i = e\}$ and $\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\{k_i^0 = k, E_i > t\}\frac{\pi_e}{\pi_{t+}}$ both have nonzero probabilistic limits; for the latter, apply Assumption 9-c and find that it is bounded from below by $\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\{k_i^0 = k, E_i > t\}\varepsilon^{\pi}$. Thus,

$$\sqrt{n}\left(\widehat{ATT}_t(k,e) - \widetilde{ATT}_t(k,e)\right) = o_p(1).$$

Step 2

In this step, we rewrite $ATT_t(k,e)$ in a way that it connects to $\widetilde{ATT}_t(k,e)$:

$$ATT_t(k,e) = \mathbf{E}\left[Y_{it}(e) - Y_{i,e-1}(e)|k_i^0 = k, E_i = e\right] - \mathbf{E}\left[Y_{it}(\infty) - Y_{i,e-1}(\infty)|k_i^0 = k, E_i = e\right].$$

Find that from Assumption 9-b,

$$\begin{aligned}
&\mathbf{E}\left[Y_{it}(\infty) - Y_{i,e-1}(\infty)|k_i^0 = k, E_i = e\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[Y_{it}(\infty) - Y_{i,e-1}(\infty)|X_i, k_i^0 = k\right]|k_i^0 = k, E_i = e\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[Y_{it} - Y_{i,e-1}|X_i, k_i^0 = k, E_i > t\right]|k_i^0 = k, E_i = e\right] \\
&= \frac{\mathbf{E}\left[\mathbf{E}\left[Y_{it} - Y_{i,e-1}|X_i, k_i^0 = k, E_i > t\right]\mathbf{1}\{k_i^0 = k, E_i = e\}\right]}{\Pr\{k_i^0 = k, E_i = e\}}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{E}\left[Y_{it}-Y_{i,e-1}|X_i,k_i^0=k,E_i>t\right]\mathbf{1}\{k_i^0=k,E_i=e\}\right]\\
&=\mathbf{E}\left[\frac{\mathbf{E}\left[(Y_{it}-Y_{i,e-1})\,\mathbf{1}\{k_i^0=k,E_i>t\}|X_i\right]\Pr\left\{k_i^0=k,E_i=e|X_i\right\}}{\Pr\left\{k_i^0=k,E_i>t|X_i\right\}}\right]\\
&=\mathbf{E}\left[(Y_{it}-Y_{i,e-1})\,\mathbf{1}\{k_i^0=k,E_i>t\}\cdot\frac{\Pr\left\{k_i^0=k,E_i=e|X_i\right\}}{\Pr\left\{k_i^0=k,E_i>t|X_i\right\}}\right]
\end{aligned}$$

and

$$\begin{aligned}
\Pr\left\{k_i^0=k,E_i=e\right\}&=\mathbf{E}\left[\mathbf{1}\{k_i^0=k,E_i=e\}\cdot\frac{\Pr\left\{k_i^0=k,E_i>t|X_i\right\}}{\Pr\left\{k_i^0=k,E_i>t|X_i\right\}}\right]\\
&=\mathbf{E}\left[\mathbf{1}\{k_i^0=k,E_i>t\}\cdot\frac{\Pr\left\{k_i^0=k,E_i=e|X_i\right\}}{\Pr\left\{k_i^0=k,E_i>t|X_i\right\}}\right].
\end{aligned}$$

The second to the last equality holds since $\Pr\left\{E_i>t|k_i^0=k,X_i\right\}\geq\varepsilon^{\pi}>0$ from Assumption 9-c and $\mu(k,e')>0$ for some $e'>t$ from Assumption 6.

For notational brevity, let

$$\begin{aligned}
W_i&=\mathbf{1}\{k_i^0=k,E_i>t\}\pi_e(X_i,k,\xi^0)/\pi_{t+}(X_i,k,\xi^0),\\
\widehat{W}_i&=\mathbf{1}\{k_i^0=k,E_i>t\}\pi_e(X_i,k,\hat{\xi})/\pi_{t+}(X_i,k,\hat{\xi}).
\end{aligned}$$

Then,

$$\begin{aligned}
ATT_t(k,e)&=\frac{\mathbf{E}\left[(Y_{it}-Y_{i,e-1})\mathbf{1}\{k_i^0=k,E_i=e\}\right]}{\mathbf{E}\left[\mathbf{1}\{k_i^0=k,E_i=e\}\right]}-\frac{\mathbf{E}\left[(Y_{it}-Y_{i,e-1})W_i\right]}{\mathbf{E}\left[W_i\right]}\\
\widetilde{ATT}_t(k,e)&=\frac{\frac{1}{n}\sum_i(Y_{it}-Y_{i,e-1})\mathbf{1}\{k_i^0=k,E_i=e\}}{\frac{1}{n}\sum_i\mathbf{1}\{k_i^0=k,E_i=e\}}-\frac{\frac{1}{n}\sum_i(Y_{it}-Y_{i,e-1})\widehat{W}_i}{\frac{1}{n}\sum_i\widehat{W}_i}
\end{aligned}$$

Step 3

Now, let us derive an asymptotic linear approximation of $\widetilde{ATT}_t(k,e)$. Find that

$$\sqrt{n}\left(\widetilde{ATT}_t(k,e) - ATT_t(k,e)\right) = A_n - B_n$$

where

$$A_n = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^n (Y_{it} - Y_{i,e-1})\,\mathbf{1}\{k_i^0 = k, E_i = e\}}{\tilde{\mu}(k,e)} - \sqrt{n}\frac{\mathbf{E}\left[(Y_{it} - Y_{i,e-1})\mathbf{1}\{k_i^0 = k, E_i = e\}\right]}{\mu(k,e)}$$
$$B_n = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^n (Y_{it} - Y_{i,e-1})\,\widehat{W}_i}{\overline{\widehat{W}}_n} - \sqrt{n}\frac{\mathbf{E}\left[(Y_{it} - Y_{i,e-1})W_i\right]}{\mathbf{E}\left[W_i\right]}$$

where $\tilde{\mu}(k,e) = \frac{1}{n}\sum_{i=1}^n \mathbf{1}\{k_i^0 = k, E_i = e\}$ and $\overline{\widehat{W}}_n = \frac{1}{n}\sum_{i=1}^n \widehat{W}_i$.

Before deriving the asymptotic approximation, let us provide some useful expansions and probabilistic convergences. Firstly, apply the first-order Taylor's expansion to $\widehat{W}_i$ with regard to $\hat{\xi}$ around ξ^0:

$$\widehat{W}_i = W_i + \mathbf{1}\{k_i^0 = k, E_i > t\}\frac{\partial}{\partial \xi}\left(\frac{\pi_e(X_i,k,\xi)}{\pi_{t+}(X_i,k,\xi)}\right)^{\intercal}\Bigg|_{\xi\in\left(\xi^0,\hat{\xi}\right)}\left(\hat{\xi} - \xi^0\right).^7 \tag{9}$$

The first-order remainder term is $O_p(1/\sqrt{n})$ since $\|\hat{\xi} - \xi^0\|_2 = O_p(1/\sqrt{n})$ from asymptotic normality of $\hat{\xi}$ and $\frac{\partial}{\partial \xi}\frac{\pi_e}{\pi_{t+}} = O_p(1)$ from Assumption 9-d and the convergence of $\hat{\xi}$ to ξ^0:

$$\begin{aligned}\left|\frac{\partial}{\partial \xi}\left(\frac{\pi_e(X_i,k,\xi)}{\pi_{t+}(X_i,k,\xi)}\right)^{\intercal}\Bigg|_{\xi\in\left(\xi^0,\hat{\xi}\right)}\left(\hat{\xi} - \xi^0\right)\right| &\leq \left\|\frac{\partial}{\partial \xi}\left(\frac{\pi_e(X_i,k,\xi)}{\pi_{t+}(X_i,k,\xi)}\right)\Bigg|_{\xi\in\left(\xi^0,\hat{\xi}\right)}\right\|_2 \left\|\hat{\xi} - \xi^0\right\|_2 \\ &= O_p(1)O_p\left(\frac{1}{\sqrt{n}}\right).\end{aligned}$$

[7] A slight abuse of notation is used here; formally, the remainder term in the expansion is an integral of the gradient $\frac{\partial}{\partial \xi}\frac{\pi_e}{\pi_{t+}}$ and there may not be a convex combination of ξ^0 and $\hat{\xi}$ that gives us the integral. However, since we assume uniform bound on the gradient from Assumption 9-d-ii when $\hat{\xi}$ is close to ξ^0, the integral will also be bounded by the same constant.

Now, apply the second-order Taylor's expansion to $\widehat{W}_i$:

$$\widehat{W}_i = W_i + \mathbf{1}\{k_i^0 = k, E_i > t\} \frac{\partial}{\partial \xi} \left(\frac{\pi_e(X_i, k, \xi)}{\pi_{t+}(X_i, k, \xi)} \right)^\intercal \Bigg|_{\xi=\xi^0} \left(\hat{\xi} - \xi^0 \right)$$
$$+ \mathbf{1}\{k_i^0 = k, E_i > t\} \left(\hat{\xi} - \xi^0 \right)^\intercal \frac{\partial^2}{\partial \xi \partial \xi^\intercal} \left(\frac{\pi_e(X_i, k, \xi)}{\pi_{t+}(X_i, k, \xi)} \right) \Bigg|_{\xi \in \left(\xi^0, \hat{\xi}\right)} \left(\hat{\xi} - \xi^0 \right). \quad (10)$$

Note that the second-order remainder term is $o_p(1/\sqrt{n})$ from Assumption 9-d and the asymptotic normality of $\hat{\xi}$. Lastly, find that from (9) and $\frac{1}{n}\sum_i (Y_{it} - Y_{i,e-1})^2$ being bounded in expectation,

$$\left| \frac{1}{n} \sum_{i=1}^n (Y_{it} - Y_{i,e-1}) \left(\widehat{W}_i - W_i \right) \right| = O_p\left(\frac{1}{\sqrt{n}} \right),$$
$$\frac{1}{n} \sum_{i=1}^n (Y_{it} - Y_{i,e-1}) \widehat{W}_i = \mathbf{E}\left[(Y_{it} - Y_{i,e-1}) W_i \right] + O_p\left(\frac{1}{\sqrt{n}} \right). \quad (11)$$

The $O_p(1/\sqrt{n})$ term in the second equality comes from applying the CLT to $(Y_{it} - Y_{i,e-1}) W_i$ and the $O_p(1/\sqrt{n})$ term from the first equality. Likewise, we have

$$\overline{\widehat{W}}_n = \mathbf{E}[W_i] + O_p(1/\sqrt{n}). \quad (12)$$

To drive the asymptotic approximation of B_n, apply the second-order Taylor's expansion to B_n with regard to $1/\overline{\widehat{W}}_n$ around $1/\mathbf{E}[W_i]$. As argued in the Step 1, $\mathbf{E}[W_i] > 0$ from Assumption 9-c; the derivatives are defined:

$$\frac{\frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_{it} - Y_{i,e-1}) \widehat{W}_i}{\overline{\widehat{W}}_n}$$
$$= \frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_{it} - Y_{i,e-1}) \widehat{W}_i \left(\frac{1}{\mathbf{E}[W_i]} - \frac{1}{\mathbf{E}[W_i]^2} \left(\overline{\widehat{W}}_n - \mathbf{E}[W_i] \right) + \frac{2}{\widetilde{W}_n^3} \left(\overline{\widehat{W}}_n - \mathbf{E}[W_i] \right)^2 \right)$$
$$= \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{(Y_{it} - Y_{i,e-1}) \widehat{W}_i}{\mathbf{E}[W_i]} - \frac{\mathbf{E}[(Y_{it} - Y_{i,e-1}) W_i]}{\mathbf{E}[W_i]^2} \sqrt{n} \left(\overline{\widehat{W}}_n - \mathbf{E}[W_i] \right) + o_p(1).$$

with some $\widetilde{W}_n$ between $\widehat{\overline{W}}_n$ and $\mathbf{E}[W_i]$. The second equality holds from (12) and (11). Then, from (10) and $\frac{1}{n}\sum_i (Y_{it} - Y_{i,e-1})^2$ being bounded in expectation,

$$
\begin{aligned}
&\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^n (Y_{it} - Y_{i,e-1})\widehat{W}_i}{\widehat{\overline{W}}_n} \\
&= \frac{1}{\mathbf{E}[W_i]} \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_{it} - Y_{i,e-1}) W_i + o_p(1) \\
&\quad + \frac{1}{n}\sum_{i=1}^n \frac{(Y_{it} - Y_{i,e-1})\mathbf{1}\{k_i^0 = k, E_i > t\}}{\mathbf{E}[W_i]} \frac{\partial}{\partial \xi} \left(\frac{\pi_e(X_i, k, \xi)}{\pi_{t+}(X_i, k, \xi)} \right)^{\intercal} \Bigg|_{\xi = \xi^0} \cdot \sqrt{n}\left(\hat{\xi} - \xi^0\right) \\
&\quad - \frac{\mathbf{E}[(Y_{it} - Y_{i,e-1})W_i]}{\mathbf{E}[W_i]^2} \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^n (W_i - \mathbf{E}[W_i]) \\
&\quad - \frac{\mathbf{E}[(Y_{it} - Y_{i,e-1})W_i]}{\mathbf{E}[W_i]} \cdot \frac{1}{n}\sum_{i=1}^n \frac{\mathbf{1}\{k_i^0 = k, E_i > t\}}{\mathbf{E}[W_i]} \frac{\partial}{\partial \xi} \left(\frac{\pi_e(X_i, k, \xi)}{\pi_{t+}(X_i, k, \xi)} \right)^{\intercal} \Bigg|_{\xi = \xi^0} \cdot \sqrt{n}\left(\hat{\xi} - \xi^0\right).
\end{aligned}
$$

To find the probability limits of the quantities in the second and the fourth terms, let

$$
\begin{aligned}
\bar{B}_1 &= \frac{1}{\mathbf{E}[W_i]} \cdot \mathbf{E}\left[(Y_{it} - Y_{i,e-1})\, \mathbf{1}\{k_i^0 = k, E_i > t\} \frac{\partial}{\partial \xi} \left(\frac{\pi_e(X_i, k, \xi)}{\pi_{t+}(X_i, k, \xi)} \right) \Bigg|_{\xi = \xi^0} \right] \\
\bar{B}_2 &= \frac{1}{\mathbf{E}[W_i]} \cdot \mathbf{E}\left[\mathbf{1}\{k_i^0 = k, E_i > t\} \frac{\partial}{\partial \xi} \left(\frac{\pi_e(X_i, k, \xi)}{\pi_{t+}(X_i, k, \xi)} \right) \Bigg|_{\xi = \xi^0} \right].
\end{aligned}
$$

Note that the sample analogues for $\bar{B}_1$ and $\bar{B}_2$ with ξ^0 replaced with $\hat{\xi}$ are consistent for $\bar{B}_1$ and $\bar{B}_2$ from Assumption 9-d; $\bar{B}_1$ and $\bar{B}_2$ are consistently estimable. Consequently,

$$
\begin{aligned}
B_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{W_i}{\mathbf{E}[W_i]} \left(Y_{it} - Y_{i,e-1} - \frac{\mathbf{E}[(Y_{it} - Y_{i,e-1})W_i]}{\mathbf{E}[W_i]} \right) \\
+ \left(\bar{B}_1 - \frac{\mathbf{E}[(Y_{it} - Y_{i,e-1})W_i]}{\mathbf{E}[W_i]} \bar{B}_2 \right)^{\intercal} \cdot \sqrt{n}\left(\hat{\xi} - \xi^0\right) + o_p(1).
\end{aligned}
$$

By repeating the same argument for A_n,

$$
A_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\mathbf{1}\{k_i^0 = k, E_i = e\}}{\mu(k, e)} \left(Y_{it} - Y_{i,e-1} - \frac{\mathbf{E}[(Y_{it} - Y_{i,e-1})\mathbf{1}\{k_i^0 = k, E_i = e\}]}{\mu(k, e)} \right) + o_p(1).
$$

Note the asymptotic linear approximation given in Corollary 3 holds for $\hat{\xi}$ as well from the proof for Corollary 2. We can construct score functions l^1 and l^0 as follows:

$$A_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l^1_{tke}\left(\{Y_{it}\}_{t\geq -1}, k_i^0, E_i\right) + o_p(1),$$
$$B_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l^0_{tke}\left(\{Y_{it}\}_{t\geq -1}, X_i, k_i^0, E_i\right) + o_p(1).$$

Note that l^π, the score function from the asymptotic linear approximation for $\hat{\xi}$, appears in l^0. Now we have

$$\begin{aligned}
&\sqrt{n}\left(\widehat{ATT}_t(k,e) - ATT_t(k,e)\right) \\
&= \left(1, -1\right) \begin{pmatrix} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l^1_{tke}\left(\{Y_{it}\}_{t\geq -1}, k_i^0, E_i\right) \\ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l^0_{tke}\left(\{Y_{it}\}_{t\geq -1}, X_i, k_i^0, E_i\right) \end{pmatrix} + o_p(1).
\end{aligned}$$

The asymptotic linear approximation is derived for $\widehat{ATT}_t(k,e)$.

Step 4

To derive asymptotic distribution of $\hat{\beta}_r(k)$, consider

$$\begin{aligned}
&\frac{\hat{\mu}(k,e)}{\sum_{e'\leq T_1-1-r} \hat{\mu}(k,e')} \cdot \sqrt{n}\widehat{ATT}_t(k,e) - \frac{\mu(k,e)}{\sum_{e'\leq T_1-1-r} \mu(k,e')} \cdot \sqrt{n} ATT_t(k,e) \\
&= \frac{\hat{\mu}(k,e)}{\sum_{e'\leq T_1-1-r} \hat{\mu}(k,e')} \cdot \sqrt{n}\left(\widehat{ATT}_t(k,e) - ATT_t(k,e)\right) \\
&\quad + \sqrt{n}\left(\frac{\hat{\mu}(k,e)}{\sum_{e'\leq T_1-1-r} \hat{\mu}(k,e')} - \frac{\mu(k,e)}{\sum_{e'\leq T_1-1-r} \mu(k,e')}\right) \cdot ATT_t(k,e).
\end{aligned}$$

Note that $\hat{\mu}$ is constructed with the estimated type $\hat{k}_i$, instead of the true type k_i^0. By taking the second-order Taylor's expansion of $\sum_{e'} \hat{\mu}(k,e')$ around $\sum_{e'} \mu(k,e')$,

$$\begin{aligned}\sqrt{n}\left(\frac{\hat{\mu}(k,e)}{\sum_{e'}\hat{\mu}(k,e')} - \frac{\mu(k,e)}{\sum_{e'}\mu(k,e')}\right) &= \sqrt{n}\left(\frac{\hat{\mu}(k,e)}{\sum_{e'}\mu(k,e')} - \frac{\mu(k,e)}{\sum_{e'}\mu(k,e')}\right) \\ &\quad - \frac{\hat{\mu}(k,e)}{\left(\sum_{e'}\mu(k,e')\right)^2}\sqrt{n}\left(\sum_{e'}\Big(\hat{\mu}(k,e') - \mu(k,e')\Big)\right) \\ &\quad + \frac{2\hat{\mu}(k,e)}{\tilde{\mu}^3}\sqrt{n}\left(\sum_{e'}\Big(\hat{\mu}(k,e') - \mu(k,e')\Big)\right)^2\end{aligned}$$

with some $\tilde{\mu}$ between $\sum_{e'} \mu(k,e')$ and $\sum_{e'} \hat{\mu}(k,e')$. The second-order remainder term is $o_p(1)$ since $\sqrt{n}\left(\sum_{e'}\left(\hat{\mu}(k,e') - \mu(k,e')\right)\right) = O_p(1)$ from Step 1 and $\sum_{e'} \mu(k,e')$ is nonzero by taking $r \leq \bar{r}_k$ from Assumption 6. Thus,

$$\begin{aligned}&\sqrt{n}\left(\frac{\hat{\mu}(k,e)}{\sum_{e'}\hat{\mu}(k,e')} - \frac{\mu(k,e)}{\sum_{e'}\mu(k,e')}\right) \\ &= \sqrt{n}\left(\frac{\hat{\mu}(k,e) - \mu(k,e)}{\sum_{e'}\mu(k,e')}\right) - \frac{\mu(k,e)}{\left(\sum_{e'}\mu(k,e')\right)^2}\sqrt{n}\left(\sum_{e'}\Big(\hat{\mu}(k,e') - \mu(k,e')\Big)\right) + o_p(1) \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{1}\{\hat{k}_i = k, E_i = e\} - \mu(k,e)}{\sum_{e'}\mu(k,e')} \\ &\quad - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mu(k,e)\big(\mathbf{1}\{\hat{k}_i = k, E_i \leq T_1 - 1 - r\} - \sum_{e'}\mu(k,e')\big)}{\big(\sum_{e'}\mu(k,e')\big)^2} + o_p(1).\end{aligned}$$

Again, repeating the same argument from Step 1, we can find that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{1}\{\hat{k}_i = k, E_i = e\} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{1}\{k_i^0 = k, E_i = e\} + o_p(1).$$

Thus,

$$\begin{aligned}
&\sqrt{n}\left(\frac{\hat{\mu}(k,e)}{\sum_{e'}\hat{\mu}(k,e')}-\frac{\mu(k,e)}{\sum_{e'}\mu(k,e')}\right)\\
&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{1}\{k_i^0=k,E_i=e\}-\mu(k,e)}{\sum_{e'}\mu(k,e')}\\
&\quad-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mu(k,e)\big(\mathbf{1}\{k_i^0=k,E_i\le T_1-1-r\}-\sum_{e'}\mu(k,e')\big)}{\big(\sum_{e'}\mu(k,e')\big)^2}+o_p(1).
\end{aligned}$$

Let l^{μ} denote the score function in the asymptotic linear approximation:

$$\sqrt{n}\left(\frac{\hat{\mu}(k,e)}{\sum_{e'}\hat{\mu}(k,e')}-\frac{\mu(k,e)}{\sum_{e'}\mu(k,e')}\right)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}l_{ke}^{\mu}(k_i^0,E_i)+o_p(1).$$

Combining all of the results so far, we get

$$\begin{aligned}
&\sqrt{n}\left(\hat{\beta}_r(k)-\beta_r(k)\right)\\
&=\sum_{e\le T_1-1-r}\left(\frac{\hat{\mu}(k,e)}{\sum_{e'}\hat{\mu}(k,e')}\cdot\sqrt{n}\widehat{ATT}_t(k,e)-\frac{\mu(k,e)}{\sum_{e'}\mu(k,e')}\cdot\sqrt{n}ATT_t(k,e)\right)\\
&=\sum_{e\le T_1-1-r}\frac{\mu(k,e)}{\sum_{e'}\mu(k,e')}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(l_{e+r,k,e}^{1}\left(\{Y_{it}\}_{t\ge 0},k_i^0,E_i\right)-l_{e+r,k,e}^{0}\left(\{Y_{it}\}_{t\ge 0},X_i,k_i^0,E_i\right)\right)\\
&\qquad+\sum_{e\le T_1-1-r}ATT_t(k,e)\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}l_{ke}^{\mu}(k_i^0,E_i)+o_p(1).
\end{aligned}$$

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