Distributional Treatment Effect with Latent Rank Invariance

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Distributional treatment effect

Potential outcome setup: with $D \in \{0, 1\}$,

$$Y = D \cdot Y(1) + (1 - D) \cdot Y(0).$$

We do not observe $Y(1)$ and $Y(0)$ simultaneously; focus on ATE, LATE, etc.

Some questions can only be answered with **distribution** of treatment effect $Y(1) - Y(0)$.
"How many people are better off under the treatment?"
"How heterogeneous is the treatment effect at the individual level?"

Distributional treatment effect

Existing approaches

- Partial identification: put a bound on $\Pr\{Y(1) - Y(0) \leq y\}$

  Heckman et al. (1997); Fan and Park (2010); Fan et al. (2014); Firpo and Ridder (2019) Frandsen and Lefgren (2021); Kaji and Cao (2023) and more

- Independence: assume $Y(1) \perp\!\!\!\perp Y(0)$ or $Y(0) \perp\!\!\!\perp \big(Y(1) - Y(0)\big)$

  Heckman et al. (1997); Carneiro et al. (2003); Gautier and Hoderlein (2015); Noh (2023)

In this paper, we follow Carneiro et al. (2003) assuming a latent variable U such that

$$Y(1) \perp\!\!\!\perp Y(0) \,\big|\, U$$

and add **1)** nonparametric identification with flexible cond. dist. of $Y(d)$ given U
2) asymptotically normal estimator under a finite support assumption on U.

Distributional treatment effect: setup

An econometrican observes $\{Y_i, D_i, X_i, Z_i\}_{i=1}^{n}$:

$$Y_i = D_i \cdot Y_i(1) + (1 - D_i) \cdot Y_i(0).$$

$Y_i, X_i, Z_i \in \mathbb{R}$, $D_i \in \{0, 1\}$ and $\big(Y_i(1), Y_i(0), D_i, X_i, Z_i, U_i\big) \sim iid$.
X_i and Z_i are proxy variables for U_i. $U_i \in \mathbb{R}$.

Assumption 1. $\big(Y_i(1), Y_i(0), X_i\big) \perp\!\!\!\perp D_i \,\big|\, \big(Z_i, U_i\big)$.

- One of the proxy Z_i and the latent variable U_i are confounders.
- In proximal inference terminology,
  X_i is outcome-aligned proxy and Z_i is treatment-aligned proxy.
  Hu and Schennach (2008); Miao et al. (2018); Deaner (2023); Nagasawa (2022) and more

Assumption 2. $Y_i(1), Y_i(0), X_i, Z_i$ are mutually independent given U_i.

Distributional treatment effect: what is U?

When estimating quantile treatment effect with endogeneous treatment,
rank invariance/similarity is often used to extrapolate $Y_i(d)$ on $\{i : D_i = 1 - d\}$ and so.
Chernozhukov and Hansen (2005, 2006); Athey and Imbens (2006); Vuong and Xu (2017)
Callaway and Li (2019) and more

Rank invariance is strong;
the point identification of joint distribution of $\big(Y_i(1), Y_i(0)\big)$ is implied from

$$Y(1) \perp\!\!\!\perp Y(0) \mid \text{rank}.$$

In fact, $Y(d) \mid$ rank is nonrandom. Assumption 2 is a relaxed version of this.

With additional assumptions such as $\mathbf{E}[Y(d)|U = u]$ monotone in u,
"Conditional expectation of $Y_i(1)$ given U_i and that of $Y_i(0)$ given U_i have the same rank."
U_i can be thought of as a 'latent' or 'interim' rank.

Distributional treatment effect: conditional independence with proxy variable

When limited to $Y_i(1)$ and $Y_i(0)$, Assumption 2 is not binding; e.g. $U_i = Y_i(0)$.
Rather, Assumption 2 puts restriction on the *joint* distribution of $Y_i(1), Y_i(0), X_i$ and Z_i.

For example, for any $y \in \mathbb{R}$, there exists some w such that

$$f_{Y(1)|Y(0)}(y'|y) = \int_{\mathbb{R}} w(z) f_{Y|D=1,Z}(y|z) dz \qquad \forall y'$$

and likewise for X_i.
"Proxy variables creates sufficient variation in the dist. of $Y_i(1)$ to recover $F_{Y(1)|Y(0)}$ and vice versa."

The conditional independence assumption is a powerful assumption.

Distributional treatment effect: proxy variables *(past and future outcomes)*

Consider a short panel where $T = 3$ and $D_i = 1$ means being treated for $t = 2, 3$.

$$Y_{it}(d) = g_d\big(V_{it}, \varepsilon_{it}(d)\big).$$

There is a common shock V_{it} and treatment-status-specific shocks $\big(\varepsilon_{it}(0), \varepsilon_{it}(1)\big)$.

Assumption 2 holds when 1) $\{V_{it}\}_{t=1}^{3}$ is first-order Markov and
2) $\{V_{it}\}_{t=1}^{3}, \varepsilon_{i1}(0), \varepsilon_{i2}(1), \varepsilon_{i2}(0), \varepsilon_{i3}(1), \varepsilon_{i3}(0) \sim$ ind.

Y_{it} depends on Y_{it-1} only through V_{it} depending on V_{it-1}.

Distributional treatment effect: proxy variables *(repeated measurements)*

Suppose some error-ridden measurements of the latent variable U_i: X_i and Z_i.

Carneiro et al. (2003) discusses a similar model, but with a factor structure:

$$
\begin{aligned}
Y_i(1) &= \lambda_i{}^\intercal f^1 + \varepsilon_i(1) \\
Y_i(0) &= \lambda_i{}^\intercal f^0 + \varepsilon_i(0) \\
X_i &= \lambda_i{}^\intercal f^x + \varepsilon_i^x \\
Z_i &= \lambda_i{}^\intercal f^z + \varepsilon_i^z
\end{aligned}
$$

$Y_i(1), Y_i(0)$ are potential earnings, depending on college attendance D_i.
λ_i is the latent ability of a student and (X_i, Z_i) are test scores.

Identification

Two distributional treatment effect parameters (DTE) are identified:

$$\Pr\{Y(1) \le y, Y(0) \le y'\} \quad \text{and} \quad \Pr\{Y(1) - Y(0) \le \delta\}.$$

Identification strategy:

1. Two proxy variables identify $Y(1) \mid U$ and $Y(0) \mid U$;
2. The conditional distributions and conditional independence

$$Y(1) \perp\!\!\!\perp Y(0) \,\big|\, U$$

   identify the conditional joint distribution of $\big(Y(1), Y(0)\big)$ given U;
3. Integrate out U to identify the unconditional joint distribution of $\big(Y(1), Y(0)\big)$.

Identification

We apply Hu and Schennach (2008) to treated subpopulation and to untreated subpopulation. more

Theorem 1.

Let Assumptions 1-3 or Assumptions 1-2, 4-5 hold. Then, the joint distribution of $\big(Y_i(1), Y_i(0)\big)$ and thus the distribution of the treatment effect $Y_i(1) - Y_i(0)$ are identified.

$$f_{Y(1),Y(0)}(y, y') = \int_{\mathbb{R}} f_{Y(1),Y(0)|U}(y, y'|u)du = \int_{\mathbb{R}} f_{Y(1)|U}(y|u) \cdot f_{Y(0)|U}(y'|u)du,$$

$$f_{Y(1)-Y(0)}(\delta) = \int_{\mathbb{R}} f_{Y(1)-Y(0)|U}(\delta|u)du = \int_{\mathbb{R}}\int_{\mathbb{R}} f_{Y(1)|U}(y+\delta|u) \cdot f_{Y(0)|U}(y|u)dydu.$$

We focus on two functions: $F_{Y(1),Y(0)}$ and $F_{Y(1)-Y(0)}$ (DTE).

Identification

Assumption 3-4 are **full rank/completeness** assumption on $f_{X|Z}$. A3 A4

For a continuous U_i, we additionally invoke

Assumption 5. $\mathbf{E}[Y_i(1) + Y_i(0)|U_i = u]$ is strictly increasing in u.
- 'Latent rank' interpretation.

The functional $\big(f_{Y(1)|U}(\cdot|u), f_{Y(0)|U}(\cdot|u)\big) \mapsto \int_{\mathbb{R}} y\big(f_{Y(1)|U}(y|u) + f_{Y(0)|U}(y|u)\big)dy$ finds a labeling on $\{f_{X|U}(\cdot|u)\}_u$.

Identification: falsification test

The conditional independence assumption is fundamentally untestable.

By extending the latent rank interpretation for both treatment regime, i.e.

Assumption 5'. $\mathbf{E}[Y_i(1)|U_i = u]$ and $\mathbf{E}[Y_i(0)|U_i = u]$ are strictly increasing in u.

we can have a sort of overidentification test on the null

$$f_{X|D=1,U}(\cdot|u) = f_{X|D=0,U}(\cdot|u) \qquad \forall u.$$

Identification: falsification test

Once $f_{X|D=1,U}(x|U_i)$ and $f_{X|D=0,U}(x|U_i)$ are identified from each subpopulation,

$$\min_{g \text{ monotone}} \mathbf{E}\left[\int_{\mathbb{R}} \left(f_{X|D=1,U}(x|g(U_i)) - f_{X|D=0,U}(x|U_i)\right)^2 du \middle| D_i = 0\right]$$

must be zero. g balances $U_i|D_i = 1$ and $U_i|D_i = 0$.

In the short panel context,

- cannot test the conditional independence *across treatment regime*.

- can somewhat test the *intertemporal* conditional independence, given random treatment.

"Can we construct a latent variable U that satisfies 1) intertemporal conditional independence and 2) no anticipation ($X_i \perp\!\!\!\perp D_i \mid U_i$)?"

Implementation

The estimation strategy is two-step:

Step 0. Assume $|\mathrm{supp}_U| < \infty$.

Step 1. Estimate $f_{U|D=d,Z}$ using *nonnegative matrix factorization*.

- Decompose $\mathbf{H} = \Big(f_{Y,X|D=d,Z}(y,x|z)\Big)_{(y,x),z}$ into $\Big(f_{Y(d),X|U}(y,x|u)\Big)_{(y,x),u}$ and $\Big(f_{U|D=d,Z}(u|z)\Big)_{u,z}$.

Step 2. *Plug-in GMM* to estimate DTE.

- Using $f_{U|D=d,Z}$, write $f_{Y(d)|U}$ as a linear combination of $\{f_{Y|D=d,Z}(\cdot|z)\}_z$.
- DTE parameters will be quadratic moments of (Y_i, D_i, Z_i).

Implementation: finite support

Part of Assumption 3 A3,

$U_i \in \{u^1, \cdots, u^K\}$ with known $K < \infty$.

Reasoning behind the finite support assumption:

1. Finite mixture: Henry et al. (2014) and more.
   Discretization as approximation: Bonhomme et al. (2022) and more.
2. Low computational cost.
3. DTE parameters are identified with quadratic moments;
   a limiting distribution is derived from U stat. theory and Neyman orthogonality.
4. Identification is not tied to the finite support. sieve

Implementation: nonnegative matrix factorization

Recall the following matrix decomposition: given some partitions $\{\mathcal{Y}^m\}_m, \{\mathcal{X}_{m'}\}_{m'}, \{\mathcal{Z}^l\}_l$,

$$\mathbf{H}_d = \left(\Pr\left\{ Y_i \in \mathcal{Y}^m, X_i \in \mathcal{X}^{m'} \middle| D_i = d, Z_i \in \mathcal{Z}^l \right\} \right)_{(m,m'),l}$$
$$= \Gamma_d \cdot \Lambda_d$$

where $\Gamma_d = \left(\Pr\left\{ Y_i(d) \in \mathcal{Y}^m, X_i \in \mathcal{X}^{m'} \middle| U_i = u^k \right\} \right)_{(m,m'),k}$

$$\Lambda_d = \left(\Pr\left\{ U_i = u^k \middle| D_i = d, Z_i \in \mathcal{Z}^l \right\} \right)_{k,l}.$$

$\mathbf{H}_d$ is a discretization of $f_{Y,X|D=d,Z}$.

The full rank condition implies $|\text{supp}_Z| \geq K$; if $|\text{supp}_Z| > K$, use partition $\{\mathcal{Z}^l\}_{l=1}^K$.

Implementation: nonnegative matrix factorization

Solve the following nonnegative matrix factorization problem:

$$\left(\widehat{\Gamma}_0, \widehat{\Gamma}_1, \widehat{\Lambda}_0, \widehat{\Lambda}_1\right) = \arg\min \|\mathbb{H}_0 - \Gamma_0 \cdot \Lambda_0\|_F + \|\mathbb{H}_1 - \Gamma_1 \cdot \Lambda_1\|_F \tag{1}$$

subject to 1) $\Gamma_0, \Gamma_1, \Lambda_0, \Lambda_1$ are nonnegative.
Also, their columnwise sums are one. $\cdots$ *(linear constraints)*
2) Γ_0 and Γ_1 satisfy $Y_i(d) \perp\!\!\!\perp X_i \mid U_i$ $\cdots$ *(quadratic constraints)*
3) Γ_0 and Γ_1 imply the same marginal distribution of X_i $\cdots$ *(linear constraints)*

The objective becomes quadratic once we fix (Γ_0, Γ_1) or (Λ_0, Λ_1).
The quadratic constraint becomes linear once we fix Γ_X or $(\Gamma_{Y0}, \Gamma_{Y1})$.
(1) is solved iteratively. algorithm

Theorem 2. Under Assumptions 1-3,

$$\widehat{\Lambda}_0 \xrightarrow{p} \Lambda_0 \quad \text{and} \quad \widehat{\Lambda}_1 \xrightarrow{p} \Lambda_1$$

as $n \to \infty$, up to some permutation on $\{1, \cdots, K\}$.

The convergence rate is $n^{-\frac{1}{2}}$.
No additional assumptions needed; Assumptions 1-2 and full rank of $\mathbf{H}_X$.

Implementation: plug-in GMM

Once (Λ_0, Λ_1) are estimated, we can use

$$\begin{aligned}&\left(F_{Y(d)|U}(y|u^1) \quad \cdots \quad F_{Y(d)|U}(y|u^K)\right)\\&\qquad = \left(F_{Y|D=d,Z}(y|\mathcal{Z}^1) \quad \cdots \quad F_{Y|D=d,Z}(y|\mathcal{Z}^K)\right)(\Lambda_d)^{-1}.\end{aligned}$$

Distribution of $Y_i(d)$ given U_i are linear in (observed) distribution of Y_i given $D_i = d, Z_i$.

Let $\tilde{\Lambda}_d = \left(\tilde{\lambda}_{lk,d}\right)_{l,k} := (\Lambda_d)^{-1}$ for $d = 0, 1$.
We always get $\sum_{l=1}^K \tilde{\lambda}_{lk,d} = 1$ but $\tilde{\lambda}_{lk,d}$ may be negative.
"Extrapolation may need to happen unless $F_{Y|D=d,Z}(\cdot|\mathcal{Z}) = F_{Y(d)|U}(\cdot|u)$ for some $\mathcal{Z}$."

Implementation: plug-in GMM

Let $p_U(k) := \Pr\{U_i = u^k\} \qquad \forall k = 1, \cdots, K$

$p_{D,Z}(d,l) := \Pr\{D_i = d, Z \in \mathcal{Z}^l\} \qquad \forall d = 0, 1$ and $l = 1, \cdots, K$.

Then, quadratic moments identify DTE: with $w_{klm} = \dfrac{p_U(k)\tilde{\lambda}_{lk,0}\tilde{\lambda}_{mk,1}}{p_{D,Z}(0,l)p_{D,Z}(1,m)}$,

$$F_{Y(1),Y(0)}(y,y') = \sum_{k,l,m=1}^{K} w_{klm} \cdot \mathbf{E}\big[\mathbf{1}\{Y_i \le y, D_i = 1, Z_i \in \mathcal{Z}^m, Y_j \le y', D_j = 0, Z_j \in \mathcal{Z}^l\}\big]$$

$$F_{Y(1)-Y(0)}(\delta) = \sum_{k,l,m=1}^{K} w_{klm} \cdot \mathbf{E}\big[\mathbf{1}\{Y_i \le Y_j + \delta, D_i = 1, Z_i \in \mathcal{Z}^m, D_j = 0, Z_j \in \mathcal{Z}^l\}\big]$$

for all $(y,y') \in \mathbb{R}^2$ and $\delta \in \mathbb{R}$, with $(Y_i, D_i, Z_i) \perp\!\!\!\perp (Y_j, D_j, Z_j)$.

Implementation: plug-in GMM

Our (naive) estimator is a plug-in U statistics.

$$
\begin{aligned}
&\widehat{F}_{Y(1)-Y(0)}(\delta) \\
&= \sum_{k,l,m=1}^{K} \hat{w}_{klm} \cdot \binom{n}{2}^{-1} \sum_{i \neq j} \left(\frac{1}{2} \mathbf{1}\{Y_i \leq Y_j + \delta, D_i = 1, Z_i \in \mathcal{Z}^m, D_j = 0, Z_j \in \mathcal{Z}^l\} \right)
\end{aligned}
$$

and similarly for $\widehat{F}_{Y(1),Y(0)}$.

When the nuisance parameters $\left\{\tilde{\lambda}_{lk,0}, \tilde{\lambda}_{lk,1}\right\}_{l,k}$ and $\{p_U(k), p_{D,Z}(d,k)\}_{d,k}$ are known, the standard U statistics asymptotic theory applies.

In fact, (uniform) consistency is a direct corollary of Theorem 1.

Implementation: Neyman orthogonality

Λ is estimated with $n^{-\frac{1}{2}}$ rate.
To be robust to the first step estimation error, use an orthogonal score.

Three sets of nuisance parameters: $\{p_{D,Z}(d,k)\}_{d,k}$, $\{p_U(k)\}_k$ and $\left\{\tilde{\lambda}_{lk,d}\right\}_{l,k,d}$.

For $\{p_{D,Z}(d,k)\}_{d,k}$, we use $\mathbf{E}[\mathbf{1}\{D_i = d, Z_i \in \mathcal{Z}^k\}] - p_{D,Z}(d,k)$.
For $\{p_U(k)\}_k$ and $\left\{\tilde{\lambda}_{lk,d}\right\}_{l,k,d}$, no readily available moments since $\left(\widehat{\Lambda}_0, \widehat{\Lambda}_1\right)$ come from

$$\min \left\|\mathbb{H}_0 - \Gamma_0 \cdot \Lambda_0\right\|_F + \left\|\mathbb{H}_1 - \Gamma_1 \cdot \Lambda_1\right\|_F .$$

The FOCs are complex and introduce more nuisance paramters: (Γ_0, Γ_1).

Implementation: Neyman orthogonality

Instead, we use the quadratic constraints of conditional independence: more

$$\Pr\{Y_i \in \mathcal{Y}, X_i \in \mathcal{X} | U_i = u\} = \Pr\{Y_i \in \mathcal{Y} | U_i = u\} \cdot \Pr\{X_i \in \mathcal{X} | U_i = u\}$$

Let m be the score function for a DTE parameter
and ϕ be the score function for the nuisance parameters.

The **orthogonalized score** is

$$m(W_i, W_j) - \mu^\intercal \phi(W_i, W_j)$$

where $W_i = (Y_i, D_i, X_i, Z_i)$ and $\mu = \begin{pmatrix} \mathbf{E}\left[\frac{\partial}{\partial \tilde{\lambda}} \phi\right] \\ \mathbf{E}\left[\frac{\partial}{\partial p} \phi\right] \end{pmatrix}^+ \begin{pmatrix} \mathbf{E}\left[\frac{\partial}{\partial \tilde{\lambda}} m\right] \\ \mathbf{E}\left[\frac{\partial}{\partial p} m\right] \end{pmatrix}$.

μ exists from the full rank condition.

Theorem 3. Assumptions 1-3 hold. Then,

$$\sqrt{n}\left(\widehat{F}_{Y(1),Y(0)}(y,y') - F_{Y(1),Y(0)}(y,y')\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma(y,y')^2\right)$$
$$\sqrt{n}\left(\widehat{F}_{Y(1)-Y(0)}(\delta) - F_{Y(1)-Y(0)}(\delta)\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma(\delta)^2\right)$$

as $n \to \infty$.

Asymptotic variances are consistently estimated.

Implementation: choice of K

Choic of K is a nontrivial issue.

When using more partitions than needed,

$$\mathbf{H}_d = \left(\Pr\left\{ Y_i \in \mathcal{Y}^m, X_i \in \mathcal{X}^{m'} \middle| D_i = d, Z_i \in \mathcal{Z}^l \right\} \right)_{(m,m'),l}$$

is not full rank.

Cragg and Donald (1997); Bai and Ng (2002); Chen and Fang (2019) and more.

In the empirical illustration, I used smallest K such that

$$U_i \middle| D_i = 0 \overset{d}{\equiv} U_i \middle| D_i = 1$$

since the treatment D_i was randomly assigned.

Simulation

Monte Carlo simulations with a simple DGP with $K = 3$ and $Y_i, X_i, Z_i \in \{1, 2, 3\}$.
Nonnegative matrix factorization is applied to two 9×3 matrices.

Informativeness of the two proxy variables:

$$\Gamma_X = \left(\Pr\{X_i = x | U_i = u^k\} \right)_{x,k} = \begin{pmatrix} 0.800 & 0.100 & 0.067 \\ 0.133 & 0.800 & 0.133 \\ 0.067 & 0.100 & 0.800 \end{pmatrix}$$

$$\Lambda = \left(\Pr\{U_i = u^k | Z_i = z\} \right)_{z,k} = \begin{pmatrix} 0.840 & 0.091 & 0.040 \\ 0.077 & 0.772 & 0.055 \\ 0.083 & 0.137 & 0.905 \end{pmatrix}.$$

Their smallest singular values are 0.665 and 0.701.

Simulation

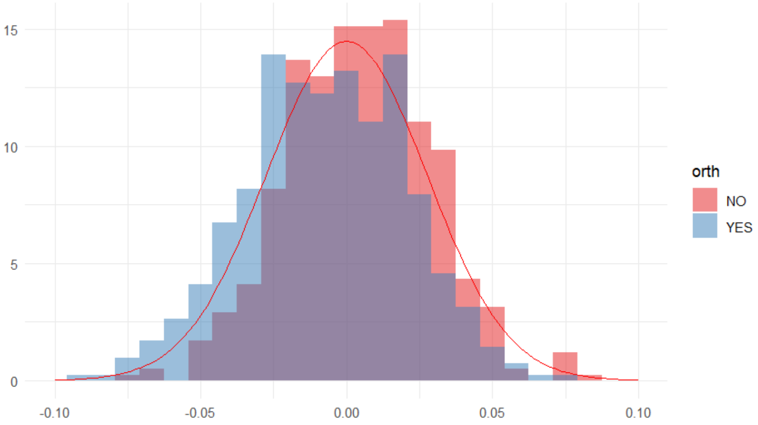

Figure 1: Histogram of $\hat{F}_{Y(1)-Y(0)}(0)^{(b)}$, $B = 500$.

Simulation

As we shift Λ, estimation worsens:

	$\widehat{F}_{Y(1)-Y(0)}$					
	$\sigma_{\min}(\Lambda) = 0.701$		$\sigma_{\min}(\Lambda) = 0.501$		$\sigma_{\min}(\Lambda) = 0.310$	
δ	bias	rMSE	bias	rMSE	bias	rMSE
-2	0.000	0.006	0.001	0.010	0.001	0.025
-1	-0.000	0.017	0.000	0.025	-0.002	0.052
0	-0.007	0.028	-0.012	0.040	-0.014	0.076
1	-0.009	0.025	-0.014	0.040	-0.015	0.084

Table 1: Bias and rMSE of DTE estimator, $B = 200$.

First step NMF worsens as Z_i gets less informative.

Simulation

	$\widehat{F}_{Y(1)-Y(0)}$		
	$\sigma_{\min}(\Lambda) = 0.701$	$\sigma_{\min}(\Lambda) = 0.501$	$\sigma_{\min}(\Lambda) = 0.310$
$\Pr\left\{F_{Y(1)-Y(0)}(-2) \in \widehat{CI}\right\}$	0.968	0.970	0.990
$\Pr\left\{F_{Y(1)-Y(0)}(-1) \in \widehat{CI}\right\}$	0.978	0.960	0.970
$\Pr\left\{F_{Y(1)-Y(0)}(0) \in \widehat{CI}\right\}$	0.960	0.975	0.990
$\Pr\left\{F_{Y(1)-Y(0)}(1) \in \widehat{CI}\right\}$	0.970	0.970	0.980
$\Pr\left\{\text{reject } F_{X\mid D=1,U} = F_{X\mid D=0,U}\right\}$	0.070	0.063	0.049

Table 2: Coverage of CI and type I error of falsification test, $B = 200$.

Empirical Illustration

I revisit Jones et al. (2019), which studies the effect of workplace wellness program.
The program *eligibility* was randomly assigned to employees at UIUC; intent-to-treat.
Using the University-provided health insurance data,
Jones et al. (2019) estimates its effect on medical spending.

The variables in the dataset are:

$$Y_i = \text{monthly medical spending over August 2016-July 2017}$$

$$D_i = \mathbf{1}\{\textit{eligible} \text{ for the wellness program starting in September 2016}\}$$

$$X_i = \text{monthly medical spending over July 2015-July 2016}$$

$$Z_i = \text{monthly medical spending over August 2017-January 2019}$$

"Underlying health status U_i depends on past health status, but not on medical spendings."

Empirical Illustration: setup

We used $K = 5$.
Partitions are constructed with $F_Y^{-1}(0), F_Y^{-1}(1/5), \cdots, F_Y^{-1}(1)$ and so on.

The test statistic on the null hypothesis $f_{X|D=1,U}(\cdot|u) = f_{X|D=0,U}(\cdot|u)$ for all u:
with $W_n = \left(\hat{f}_{X|D=1,U}(\mathcal{X}^m|u) - \hat{f}_{X|D=0,U}(\mathcal{X}^m|u)\right)_{m,u} \in \mathbb{R}^{25}$,

$$nW_n^{\intercal} Avar(W)^{-1} W_n = 16.435$$

The p-value is 0.901 .

Empirical Illustration: joint distribution of potential outcomes

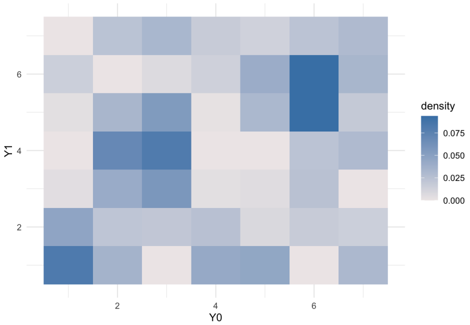

Figure 2: Joint density of $\big(Y(1), Y(0)\big)$.

y-axis is $Y(1)$ and x-axis $Y(0)$; each cell corresponds to $F_X^{-1}(0), F_X^{-1}(1/7), \cdots, F_X^{-1}(1)$.
No noticeable treatment effect; in Jones et al. (2019), p-values for ATE are 0.86-0.94.

Empirical Illustration: treatment effect distribution

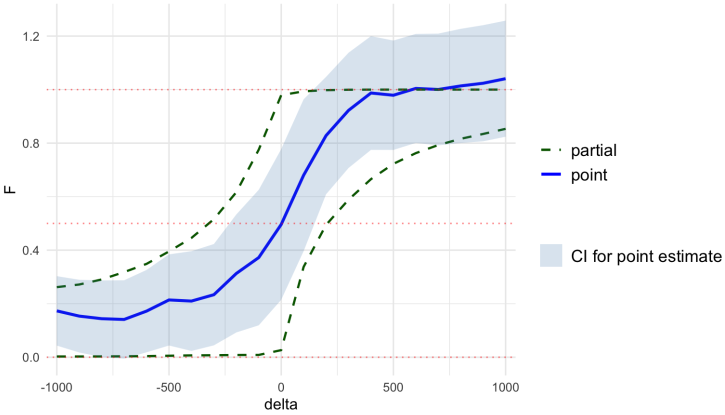

Figure 3: Marginal density of $Y(1) - Y(0)$.

Unclear whether the probability of getting benefited is bigger or smaller than 0.5. Thicker left tail.

Summary

- Assume a latent variable U such that

$$Y_i(1) \perp\!\!\!\perp Y_i(0) \;\Big|\; U_i.$$

  This assumption could be thought of as a 'latent rank invariance' condition when $\mathbf{E}[Y_i(d)|U_i = u]$ is monotone increasing in u.

- Use two proxy variables X_i and Z_i to identify the distribution of $Y_i(d)|U_i$.

- Nonnegative matrix factorization estimates distribution of U_i given (D_i, Z_i).

- An asymptotic distribution is derived for the plug-in GMM estimator.

Identification à la Hu and Schennach (2008)

An essential building block in the identification argument: $f_{Y,X|D,Z}$.
Fix y and d and discretize X_i and Z_i:

$$\mathbf{H} = \Big(f_{Y=y,X|D=d,Z}(x|z)\Big)_{x,z} = \Big(f_{X|U}(x|u)\Big)_{x,u} \cdot \text{diag}\Big(f_{Y|U}(y|u)\Big)_u \cdot \Big(f_{U|D=d,Z}(u|z)\Big)_{u,z}.$$

H is a $|\text{supp}_X| \times |\text{supp}_Z|$ matrix whose rows correspond to X_i and columns to Z_i.
Likewise, define $\mathbf{H}_X = \Big(f_{X|D=d,Z}(x|z)\Big)_{x,z}$.

Under Assumptions 1-2 and **full rank/completeness** of $\mathbf{H}_X$, A3 A4

$$\mathbf{H} \cdot (\mathbf{H}_X)^{-1} = \Big(f_{X|U}(x|u)\Big)_{x,u} \cdot \text{diag}\Big(\{f_{Y(d)=y|U}(u)\}_u\Big) \cdot \left(\Big(f_{X|U}(x|u)\Big)_{x,u}\right)^{-1}$$

Spectral decomposition identifies $f_{X|U}$.

Spectral Theorem of Hu and Schennach (2008)

Several deviations from Hu and Schennach (2008):

1. Two decomposition results; treated population and untreated population.
   Need to connect $\{f_{Y(1)|U}(\cdot|u)\}_u$ to $\{f_{Y(0)|U}(\cdot|u)\}_u$.

2. Mapping from $\{f_{X|U}(\cdot|u)\}_u$ to u to have distribution of U_i.

1. is easily solved.

Firstly, split the sample into two subsamples $\{i : D_i = 1\}$ and $\{i : D_i = 0\}$ and we get $\left\{f_{Y(1)|U}(\cdot|u), f_{X=1|U}(\cdot|u)\right\}_u$ and $\left\{f_{Y(0)|U}(\cdot|u), f_{X=0|U}(\cdot|u)\right\}_u$.

Under Assumption 1, $f_{X|D=1,U}(\cdot|u)$ and $f_{X|D=0,U}(\cdot|u)$ should be the same.

Spectral Theorem of Hu and Schennach (2008)

A linear operator $L_{Y=y,X|D=d,X}$ maps a density of Z_i to a density of $(Y_i(d) = y, X_i)$:

$$\left(L_{Y=y,X|D=d,Z}g\right)(x) = \int_{\mathbb{R}} f_{Y(d),X|D,Z}(y,x|d,z)g(z)dz.$$

From the decomposition based on Assumption 2, we get

$$L_{Y=y,X|D=d,Z} = L_{X|U} \cdot \Delta_{Y=y|U} \cdot L_{U|D=d,Z}$$

with similarly defined operators $L_{X|U}$, $L_{U|D=d,Z}$ and a diagonal operator $\Delta_{Y=y|U}$. Thus,

$$\begin{aligned} L_{Y=y,X|D=d,Z}\left(L_{X|D=d,Z}\right)^{-1} &= L_{X|U} \cdot \Delta_{Y=y|U} \cdot L_{U|D=d,Z} \cdot \left(L_{X|U} \cdot L_{U|D=d,Z}\right)^{-1} \\ &= \underbrace{L_{X|U} \cdot \Delta_{Y=y|U} \cdot \left(L_{X|U}\right)^{-1}}_{\text{spectral decomposition}}. \end{aligned}$$

back

Assumption 3

Assumption 3.

- **a.** *(finitely discrete U_i)* $U_i \in \{u^1, \cdots, u^K\}$.
- **b.** *(full rank)* $\left(f_{U|D=1,Z}(u|z)\right)_{u,z}$, $\left(f_{U|D=0,Z}(u|z)\right)_{u,z}$ and $\left(f_{X|U}(x|u)\right)_{x,z}$ have rank K.
- **c.** *(no repeated eigenvalue)* For any $k \neq k'$, there exist some $d \in \{0, 1\}$ and y such that

$$f_{Y(d)|U}(y \big| u^k) \neq f_{Y(d)|U}(y \big| u^{k'}).$$

"The latent heterogeneity U_i can be *at most* as rich/flexible as the proxy variables." back

Assumption 4

Assumption 4.

a. *(continuous U_i)* $U_i \in [0,1]$.

b. *(bounded density)* All marginal and conditional densities of $\big(Y_i(1), Y_i(0), X_i, Z_i, U_i\big)$ are bounded.

c. *(completeness)* Let $f_{X|Z,d}$ denote the conditional density of X_i given $(D_i = d, Z_i)$.

$$\int_{\mathbb{R}} |g(x)|dx \quad \text{and} \quad \int_{\mathbb{R}} g(x) f_{X|Z,d}(x|z)d(x) = 0 \quad \forall d, z$$

implies $g(x) = 0$. Assume similarly for $f_{X|U}$.

d. *(no repeated eigenvalue)* $\forall u \neq u'$, there exists $d \in \{0,1\}$ such that

$$\Pr\left\{ f_{Y(d)|U}(Y_i(d)|u) \neq f_{Y(d)|U}(Y_i(d)|u')|D_i = d \right\} > 0.$$

back

Nonnegative matrix factorization

The objective function in (1) is quadratic with linear constraints, once we fix two out of the three matrices $\Gamma_X, \Gamma_Y, \Lambda$.

Thus, find the (local) minima by iterating among three objects:

1. Given $\left(\Gamma_0^{(s)}, \Gamma_1^{(s)}\right)$, update (Λ_0, Λ_1).
2. Given $\left(\Lambda_0^{(s+1)}, \Lambda_1^{(s+1)}, {\Gamma_{Y0}}^{(s)}, {\Gamma_{Y1}}^{(s)}\right)$, update Γ_X.
3. Given $\left(\Lambda_0^{(s+1)}, \Lambda_1^{(s+1)}, {\Gamma_X}^{(s+1)}\right)$, update $(\Gamma_{Y0}, \Gamma_{Y1})$.
4. Iterate **1**-**3** until convergence.

In practice, use may initial values to find the global minimum. back

Sieve MLE

To allow for a continuous U_i, we can directly construct a likelihood using sieves:

$$f_{Y,X|D=d,Z,n}(y,x|z;\theta) = \int_{\mathbb{R}} f_{Y(d)|U,n}(y|u;\theta) \cdot f_{X|U,n}(x|u;\theta) \cdot f_{U|D=d,Z,n}(u|z;\theta)du.$$

The latent rank interpretation is simple to impose with Bernstein polynomials:
a Bernstein polynomial of degree m is

$$g_m(u) = \sum_{k=0}^{m} \theta_k u^k (1-u)^{m-k}.$$

Then, monotonicity of $\int_0^1 u g_m(u)du$ is a set of linear constraints on $\{\theta_k\}_{k=0}^{m}$.

back

Sieve MLE

Theorem 4. Let Assumptions 1-2,4-6 hold. Then,

$$\left\|\hat{f}_{Y(1),Y(0)} - f_{Y(1),Y(0)}\right\|_{\infty} \xrightarrow{p} 0$$

as $n \to \infty$ and for any $(y, y') \in \mathbb{R}^2$ and $\delta \in \mathbb{R}$,

$$\sqrt{n}\left(\hat{f}_{Y(1),Y(0)}(y, y') - f_{Y(1),Y(0)}(y, y')\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma(y, y')^2\right)$$
$$\sqrt{n}\left(\widehat{\Pr\left\{Y_i(1) - Y_i(0) \leq \delta\right\}} - \Pr\left\{Y_i(1) - Y_i(0) \leq \delta\right\}\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma(\delta)^2\right)$$

as $n \to \infty$.

Assumption 6

a. Functions in $\{\Theta_n\}_{n=1}^{\infty} \cup \Theta$ is uniformly bounded. Θ is convex.

b. $f_{Y(1)|U}, f_{Y(0)|U}, f_{X|U}, f_{U|D=1,Z}, f_{U|D=0,Z}$ are in the interior of $\Lambda_c^{\gamma_1}([0,1]^2)$ with $\gamma_1 > 1$. Also, for any $\theta \in \Theta_n$ for some n,

$$f_{Y(1)|U,n}(\cdot;\theta), f_{Y(0)|U,n}(\cdot;\theta), f_{X|U,n}(\cdot;\theta), f_{U|D=1,Z,n}(\cdot;\theta), f_{U|D=0,Z,n}(\cdot;\theta) \in \Lambda_c^{\gamma_1}([0,1]^2)$$

and $\log f_{Y,X|D,Z}(\cdot;\theta) \in \Lambda_c^{\gamma}([0,1]^4)$ with $\gamma > 2$.

c. $\mathbf{E}\left[\left(\log f_{Y,X|D,Z}(Y_i, X_i|D_i, Z_i)\right)^2\right] < \infty$. There exists measurable functions h_1, h_2 such that

$$\begin{aligned} & h_1(y,d,x,z) \\ & \leq \frac{1}{f_{Y,X|D,Z}(y,x|d,z;\theta)} \left(\int_0^1 \frac{f_{Y(d)|U}(y|u;\theta) f_{X|U}(x|u;\theta) f_{U|D=d,Z}(u|z;\theta)}{f_{Y(d)|U}(y|u;\theta) + f_{X|U}(x|u;\theta) + f_{U|D=d,Z}(u|z;\theta)} du \right) \\ & \leq h_2(y,d,x,z) \end{aligned}$$

Assumption 6 II

for all $\theta \in \Theta$ and $\mathbf{E}\left[(h_1(Y_i, D_i, X_i, Z_i,))^2\right], \mathbf{E}\left[(h_2(Y_i, D_i, X_i, Z_i))^2\right] < \infty$. ALso, There exist a measurable function h_3 such that

$$\begin{aligned}&\frac{1}{2f_{Y,X|D,Z}(y,x|d,z;\theta)^2}\left(\int_0^1 \frac{f_{Y(d)|U}(y|u;\theta)f_{X|U}(x|u;\theta)f_{U|D=d,Z}(u|z;\theta)}{f_{Y(d)|U}(y|u;\theta)+f_{X|U}(x|u;\theta)+f_{U|D=d,Z}(u|z;\theta)}du\right)^2\\ &\quad+\frac{1}{f_{Y,X|D,Z}(y,x|d,z;\theta)}\int_0^1\left(f_{Y(d)|U}(y|u;\theta)+f_{X|U}(x|u;\theta)+f_{U|D=d,Z}(u|z;\theta)\right)du\\ &\leq h_3(y,d,x,z)\end{aligned}$$

for all $\theta \in \Theta$ and $\mathbf{E}\left[(h_3(Y_i, D_i, X_i, Z_i,))^2\right] < \infty$.

d. $\|\Pi_n\theta^0 - \theta^0\|_\infty = o\left(n^{-\frac{1}{4}}\right)$ as $n \to \infty$ where

$$\Pi_n\theta^0 = \arg\max_{\theta\in\Theta_n} \mathbf{E}\left[\log f_{Y,X|D,Z}(Y_i, X_i|D_i, Z_i;\theta)\right]$$

Also, $p_n \to \infty$, $\frac{p_n \log n}{\sqrt{n}} \to 0$ as $n \to \infty$.

Assumption 6 III

e. With some $c_1, c_2 > 0$,

$$c_1 \mathbf{E}\left[\log \frac{f_{Y,X|D,Z}(Y_i, X_i|D_i, Z_i; \theta^0)}{f_{Y,X|D,Z}(Y_i, X_i|D_i, Z_i; \theta)}\right] \leq \left\|\theta - \theta^0\right\|^2 \leq c_2 \mathbf{E}\left[\log \frac{f_{Y,X|D,Z}(Y_i, X_i|D_i, Z_i; \theta^0)}{f_{Y,X|D,Z}(Y_i, X_i|D_i, Z_i; \theta)}\right]$$

holds for any $\theta \in \Theta_n$ such that $\|\theta - \theta^0\|_\infty = o(1)$.

f. Let p_1 be the degree of a tensor product Bernstein polynomial used in approximating $f_{Y(1)|U}$ to Θ_n and similarly define p_0, p_X, p_{1Z} and p_{0Z}; for example, $p_1 = (p^y + 1) \cdot (p^u + 1)$. With some abuse of notation, let $\{\theta_{j,1}\}_{j=1}^{p_1}$ denote the basis functions used in approximating $f_{Y(1)|U}$ and similarly define $\{p_{j,0}\}_{j=1}^{p_0}, \cdots, \{p_{j,0Z}\}_{j=1}^{p_{0Z}}$.

Assumption 6 IV

Let

$$\frac{d}{d\theta_1}\log f_{Y,X|D,Z}(Y_i,X_i|D_i,Z_i;\theta^0)\left[\{\theta_{j,1}\}_{j=1}^{p_1}\right]=\begin{pmatrix}\frac{d}{d\theta_1}\log f_{Y,X|D,Z}(Y_i,X_i|D_i,Z_i;\theta^0)\left[\theta_{1,1}\right]\\ \vdots \\ \frac{d}{d\theta_1}\log f_{Y,X|D,Z}(Y_i,X_i|D_i,Z_i;\theta^0)\left[\theta_{p_1,1}\right]\end{pmatrix}$$

$$W_n(Y_i,D_i,X_i,Z_i)=\begin{pmatrix}\frac{d}{d\theta_1}\log f_{Y,X|D,Z}(Y_i,X_i|D_i,Z_i;\theta^0)\left[\{\theta_{j,1}\}_{j=1}^{p_1}\right]\\ \vdots \\ \frac{d}{d\theta_{0Z}}\log f_{Y,X|D,Z}(Y_i,X_i|D_i,Z_i;\theta^0)\left[\{\theta_{j,0Z}\}_{j=1}^{p_{0Z}}\right]\end{pmatrix}$$

and

$$\Omega_n=\mathbf{E}\left[W_n(Y_i,D_i,X_i,Z_i)\left(W_n(Y_i,D_i,X_i,Z_i)\right)^{\intercal}\right].$$

Then, the smallest eigenvalue of Ω_n is bounded away from zero uniformly across n.

Additional moments

The quadratic moment is

$$\sum_{l=1}^{K} \frac{\tilde{\lambda}_{lk,d}}{p_{D,Z}(d,l)} \cdot \mathbf{E}\left[\frac{1}{2}\mathbf{1}\{Y_i \in \mathcal{Y}, D_i = d, X_i \in \mathcal{X}, Z_i \in \mathcal{Z}^l\}\right]$$
$$+ \sum_{m=1}^{K} \frac{\tilde{\lambda}_{mk,d}}{p_{D,Z}(d,m)} \cdot \mathbf{E}\left[\frac{1}{2}\mathbf{1}\{Y_j \in \mathcal{Y}, D_j = d, X_j \in \mathcal{X}, Z_i \in \mathcal{Z}^m\}\right]$$
$$- \sum_{l=1}^{K}\sum_{m=1}^{K} \frac{\tilde{\lambda}_{lk,d}\tilde{\lambda}_{mk,d}}{p_{D,Z}(d,l) \cdot p_{D,Z}(d,m)} \mathbf{E}[\mathbf{1}\{Y_i \in \mathcal{Y}, D_i = d, Z_i \in \mathcal{Z}^l, X_j \in \mathcal{X}, D_j = d, Z_j \in \mathcal{Z}^m\}] = 0$$

with $(Y_i, D_i, Z_i) \perp\!\!\!\perp (Y_j, D_j, Z_j)$. back

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