卒論経過報告

尾山ゼミ/市村ゼミ

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- I uploaded this slide on my github page: https://github.com/myuuuuun/faculty_matching

Sections

Introduction

Subject

- テーマ: 「**School Choice Problem** において, **Strategy-Proof** な範囲内で **Assignment Maximization** を達成するメカニズムの開発」

- **School Choice Problem** は Abdulkadiroğlu and Sönmez (2003) によって定式化された問題で, 複数の学生と複数の学校を中央集権的なメカニズムを用いて上手くマッチさせる方法を考える
  - 各学生は 1 つの学校だけに入学できる一方, 各学校は定員の範囲内で複数の学生を入学させることができる
  - 学生だけを戦略的に行動する主体と考え, 学校側の戦略的行動は考慮に入れない. これにより Two-sided matching market のアルゴリズム (e.g. DA, Boston) と Object allocation(one-sided matching market) のアルゴリズム (e.g. SD, TTC) の両方が利用可能になる

- いずれかの学校に入学できる学生をできるだけ増やしたい (アンマッチになる学生の数をできるだけ減らしたい)

Motivation for Assignment Maximization

- DA は必ずしもマッチ数を大きくするメカニズムではない（後述）
- DA は stability がある点で優れている．しかし stability の制約のもとでは，全てのマッチングで (a) アンマッチとなる学生，(b) 学校の定員充足状況 は同じになる (Rural Hospital Theorem，後述)．したがって意味のある議論はできない．メカニズムによって決まった割り当てが強制できる (例えば進学選択のような) 場面では必ずしも stability は必要ではない
- アンマッチを出来るだけ出さないことが重要な場面は（進学選択以外にも）ある $\Rightarrow$ worth considering
    - 子どもと保育所のマッチング．(例えば皆が最寄りの保育園に入れることよりも) できるだけ多くの人がどこかしらの保育園に入れることが重要 (この文脈の先行研究: Sasaki and Ura (2016))
    - 研修医のマッチング (おそらく研修先がない状況はまずい)

Why under Strategy-proofness?

- Strategy-proofness が無い場合, 学生は「読み合い」をする必要がある (例: Boston メカニズムにおいて, 未内定のリスクが有るが優先度の高い進学先を志望するか, 確実に内定できる優先度の低い進学先を志望するか). これは学生にとってコストになる. Strategy-proofness があれば, そのような読み合いは不要 (一般的な Strategy-proofness のメリット)

- また Strategy-proofness が無い場合, 単純に提出された preference に基いてマッチ数を最大化しても, 真の preference の下での最大マッチ数が達成できるわけではない (学生の戦略的行動の結果). 特に, 制約のない Assignment Maximization を実現するメカニズムは学生の preference reduction に脆弱で, 極めて非効率なマッチングが実現する可能性がある (次スライド)

Why under Strategy-proofness?

- Strategy-proofness の制約の無い Assignment Maximization を考える
  - 常に最大マッチングを達成するメカニズムは Strategy-proof ではない．特に特定の状況下では極めて簡単な戦略的操作で自分のマッチ相手を改善できる．例えば，学生数=学校数=N，各学校の定員は 1 とし，全学生がアンマッチよりもいずれかの学校に入ることが望ましいと考えているとする．全員が正直に自分の選好を申告する時，最大マッチングのサイズは N である．いまプレイヤー 1 以外は正直申告をしているという仮定のもとで，プレイヤー 1 が自分の選好の第 1 位だけを残し，それ以外は unacceptable であると嘘の申告をしたとする．この時最大マッチングのサイズは依然 N であるが，それを達成するメカニズムは必ずプレイヤー 1 とプレイヤー 1 にとって順位第 1 位の学校をマッチさせるので，プレイヤー 1 は嘘の申告によって (弱い意味で) 自分の効用を改善することができる

Model and Properties

Model Settings of School Choice Problem

Definition (School Choice Problem)

School Choice Problem $(I, S, q, \succ, P)$ consists of: [1]

- a finite set of students: $I = \{i_1, i_2, \ldots, i_n\}$
- a finite set of schools: $S = \{s_1, s_2, \ldots, s_m\}$
- a capacity vector: $q = (q_1, q_2, \ldots, q_m)$, where q_s denotes the capacity of the school $s \in S$
- a list of strict priorities of schools over students: $\succ = (\succ_s)_{s \in S}$, where $\succ_s$ is a linear order [2] over $I \cup \{s\}$ [3]. Here s denotes the outside option for $s \in S$
- a list of strict preferences of students over schools: $P = (P_i)_{i \in I}$, where P_i denotes student i's preference relation [4] over $S \cup \{i\}$. Here i denotes the outside option for $i \in I$

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[1] We mostly follow the notation in Abdulkadiroğlu et al. (2017)

[2] A binary relation R on a set X is a linear order(total order) if R satisfies (1) completeness: $\forall x, y \in X, xRy \vee yRx$, (2) transitivity: $\forall x, y, z \in X, (xRy \wedge yRz) \implies xRz$, and (3) antisymmetry: $\forall x, y \in X, (xRy \wedge yRx) \implies x = y$

[3] In general the school choice problem assumes all students are acceptable to each school

[4] Here we assume preferences are rational and strict, which implies they are linear order(s)

Model Settings of School Choice Problem

Definition (Matching)

A matching (or assignment or allocation) is a function $\mu : I \to S \cup I$ that satisfies following properties:

1. $\forall i \in I, \mu(i) \notin S \implies \mu(i) = i$
2. $\forall s \in S, \left|\mu^{-1}(s)\right| \leq q_s$, where $|I_1|$ denotes the size of $I_1 \subseteq I$

- $\mathcal{P}_i$ をプレイヤー $i \in I$ のあり得る P_i 全体の集合とし, $\mathcal{P}^I = \times_{i \in I} \mathcal{P}_i$ とおく
- $\mathcal{M}$ をあり得るマッチング μ 全体の集合とする
- $|\mu|$ を学生と学校のペアの総数, i.e., $|\mu| \equiv \sum_{s \in S} \left|\mu^{-1}(s)\right|$ と定義する
- 各学生 $i \in I$ に対し, R_i を, $s_1 P_i s_2 \iff [s_1 R_i s_2] \wedge \neg[s_2 R_i s_1]$ で定義する
- 同様に各学校 $s \in S$ に対し, $\succsim_s$ を, $i_1 \succ_s i_2 \iff [i_1 \succsim_i i_2] \wedge \neg[i_2 \succsim_i i_1]$ で定義する

Model Settings of School Choice Problem

Definition (Mechanism)

A (deterministic) mechanism $(\mathcal{P}^I, \phi)$ consists of:

1. a set of possible reported preference profiles $\mathcal{R}^I$
2. a function $\phi : \mathcal{P}^I \to \mathcal{M}$, which selects a matching $\mu \in \mathcal{M}$ based on a reported preference profile $P \in \mathcal{P}^I$ (and exogenously given $\succ$)

- Probabilistic mechanism(e.g. weak priority の仮定の下での probabilistic tie-breaking や probabilistic serial dictatorship) を考えることも出来るが, ここでは deterministic mechanism だけを考える
- 以降単に Choice function ϕ を用いて mechanism を表現する

Individual Rationality

Definition (Blocking by a student)

A student $i \in I$ **blocks a matching** $\mu \in \mathcal{M}$ if $iP_i\mu(i)$

Definition (Blocking by a school)

A school $s \in S$ **blocks a matching** $\mu \in \mathcal{M}$ if $\exists i \in \mu^{-1}(s)$ s.t. $s \succ_s i$

Definition (Individual Rationality)

A matching $\mu \in \mathcal{M}$ is **individually rational** if no students and no schools block μ. i.e.,

- $\forall i \in I, \mu(i) R_i i$, and
- $\forall s \in S, \forall i \in \mu^{-1}(s), i \succsim_s s$

- Assignment Maximization の文脈では, 断りのない限り, マッチングとは individually rational なマッチングを指すものとする

Non-Wastefulness, Fairness

Definition (Non-Wastefulness)

A matching $\mu \in \mathcal{M}$ is **non-wasteful** if

$$\forall s \in S, \forall i \in I,\ sP_i\mu(i) \implies \left[\left|\mu^{-1}(s)\right| = q_s \vee s \succ_s i\right]$$

Definition (Justified Envy)

A student $i \in I$ has **justified envy** for school $s \in S$ under a matching $\mu \in \mathcal{M}$ if

$$[sP_i\mu(i)] \ \wedge \ [\exists j \in \mu^{-1}(s) \text{ s.t. } i \succ_s j]$$

Definition (Fairness)

A matching $\mu \in \mathcal{M}$ is **fair** if no student has justfied envy for any school, i.e.,

$$\nexists (i, s) \in I \times S,\ [sP_i\mu(i)] \wedge [\exists j \in \mu^{-1}(s) \text{ s.t. } i \succ_s j]$$

Stability

Definition (Blocking by a pair)

A pair $(i,s) \in I \times S$ **blocks a matching** $\mu \in \mathcal{M}$ if

$$\left[sP_i\mu(i)\right] \;\wedge\; \left[\left[\left|\mu^{-1}(s)\right| < q_s \wedge i \succ_s s\right] \vee \left[\exists j \in \mu^{-1}(s) \text{ s.t. } i \succ_s j\right]\right]$$

Definition (Stability)

A matching $\mu \in \mathcal{M}$ is **pairwise stable** (or simply **stable**) if

- no student $i \in I$ and no school $s \in S$ block μ, and
- no pair $(i,s) \in I \times S$ blocks μ

Proposition (Stability as Fairness)

A matching $\mu \in \mathcal{M}$ is stable if and only if μ is individually rational, non-wasteful and fair

Pareto Efficiency

Definition (Pareto Domination)

A matching $\mu \in \mathcal{M}$ **Pareto dominates** another matching $\mu' \in \mathcal{M}$ if

- $\forall i \in I, \mu(i) R_i \mu'(i)$
- $\exists i \in I, \mu(i) P_i \mu'(i)$

Definition (Pareto Efficiency)

A matching $\mu \in \mathcal{M}$ is **Pareto efficient** if $\nexists \mu' \in \mathcal{M}$ that Pareto dominates μ

- Pareto efficiency implies individual rationality and non-wastefulness

Properties of Mechanism

Definition (Extension of Matching Properties to Mechanism 1)

A mechanism ϕ is individually rational / non-wasteful / fair / stable / Pareto efficient if $\forall P \in \mathcal{P}^I$, $\phi(P)$ is individually rational / non-wasteful / fair / stable / Pareto efficient

Definition (Extension of Matching Properties to Mechanism 2)

A mechanism ϕ Pareto dominates another mechanism ψ if

- $\exists P \in \mathcal{P}^I$, $\phi(P)$ Pareto dominates $\psi(P)$
- $\nexists P \in \mathcal{P}^I$, $\psi(P)$ Pareto dominates $\phi(P)$

- ある mechanism が [性質 X] だということは, 提出された preference list に基いて matching が X だというだけで, 真の preference で X だとは限らない. 実現した matching が本当に X であることを保証するためには, 次に述べる Strategy-proofness が重要

Strategy-Proofness

Definition (Strategy-Proofness)

A mechanism ϕ is (individually) **strategy-proof** if $\forall P \in \mathcal{P}^I, \forall i \in I, \forall P'_i \in \mathcal{P}_i$,

$$\phi(P)(i) \ R_i \ \phi(P'_i, P_{-i})(i)$$

- $\phi_i(P) \equiv \phi(P)(i)$ と定義する
- この時, $\phi(P)(i) \ R_i \ \phi(P'_i, P_{-i})(i) \iff \phi_i(P) \ R_i \ \phi_i(P'_i, P_{-i})$
- Preference revelation game において, 正直申告が弱支配戦略である, ということ
- Strategy-proofness があれば, 全ての学生が真の preference を報告することが期待できる

Size-wise Domination

- ここでは $\mathcal{M}$ を **individually rational** な **matching** の集合とする

Definition (Size-wise Domination of Matching)

A matching $\mu \in \mathcal{M}$ **size-wise dominates** another matching $\mu' \in \mathcal{M}$ if $|\mu| > |\mu'|$. A matching $\mu \in \mathcal{M}$ is **size-wise maximal** if $\nexists \mu' \in \mathcal{M}$ s.t. $|\mu'| > |\mu|$

Definition (Size-wise Domination of Mechanism)

A mechanism ϕ **size-wise dominates** another mechanism ψ if

- $\forall P \in \mathcal{P}^I, |\phi(P)| \geq |\psi(P)|$
- $\exists P \in \mathcal{P}^I, |\phi(P)| > |\psi(P)|$

A mechanism ϕ is **size-wise maximal** if ϕ is not dominated by any mechanism

Research Goals

- 以上の概念を用いて具体的な研究目標を定義する
  1. 他の strategy-proof メカニズムを size-wise dominate する strategy-proof メカニズムは存在するか? 特に Deferred Acceptance アルゴリズムを size-wise dominate する strategy-proof メカニズムは存在するか?
  2. 2 のようなメカニズムが存在しない場合, そのようなメカニズムが存在するような preference の最大部分集合 $\mathcal{P}_0 \subset \mathcal{P}$ はなにか?
  3. Preference revelation game に不完備情報を導入した場合, 既存の strategy-proof メカニズムと無制約に Assignment Maximization を達成するメカニズムとで, 均衡 (BNE) の下ではどちらのマッチ数が多くなるか?

Major Algorithms

Deferred Acceptance Algorithm

Definition (Deferred Acceptance Algorithm (Gale and Shapley, 1962))

Deferred Acceptance Algorithm(DA) $\phi^{\mathrm{DA}} : \mathcal{P}^I \to \mathcal{M}$ is defined as follows:

- N_k を step k で propose を行う学生の集合とする．$N_1 = I$ とする
- **Step** k　各 $i \in N_k$ に対し，まだ propose をしていない学校の中で，選好順が最も高い学校を s_i とする．
  - $s_i = i$ の場合，その学生をアンマッチとして確定する
  - それ以外の場合，i は s_i に propose する

  各学校 $s \in S$ は，step k で propose してきた acceptable な学生と step k-1 で仮マッチしていた学生を priority の高い順に並び替え，定員 q_s 人までと仮マッチする．それ以外の学生を reject し，全ての $s \in S$ について合わせて N_{k+1} とする．$|N_{k+1}| = 0$ の場合アルゴリズムを終了し，現在のマッチ (+アンマッチ) を $\phi^{\mathrm{DA}}(P)$ とする．それ以外の場合は step $k+1$ に進む

Deferred Acceptance Algorithm

procedure DEFERRED ACCEPTANCE(P)
  $S_i \leftarrow [P_i$ に従って $S \cup \{i\}$ を降順 (好ましい順) に並べたもの$], \forall i \in I$
  $M \leftarrow \emptyset, N \leftarrow I$; ▷ $M (\subseteq I \times (S \cup I))$ は仮マッチの集合, N は学生の queue
  while $N \neq \emptyset$ **do**
    $i \leftarrow N$.pop(); ▷ N から先頭の 1 人を取り出す
    $s \leftarrow S_i$.pop(); ▷ S_i から先頭の (最も選好順位の高い)1 校を取り出す
    if $s = i$ **then**
      M.push((i, i)); ▷ 選好表最上位が i なら, unmatch として確定
    else if $s \succ_s i$ **then**
      N.push(i); ▷ i が s にとって unacceptable なら, i を queue に戻す
    else if $|\{j | (j, s) \in M\}| < q_s$ **then**
      M.push((i, s)); ▷ s の枠が空いていれば, i と s をマッチ
    else
      $w \leftarrow \min\{j \mid (j, s) \in M\}$; ▷ s がマッチしている学生の内, 最も priority の低い人
      if $i \succ_s w$ **then**
        M.push((i, s)); ▷ s が自分より priority の低い人とマッチしていれば
        M.remove((w, s)); ▷ その人とのマッチを解消し, s と i をマッチ
        N.push(w); ▷ その後, w は学生の queue に戻す
      end if
    end if
  end while
  Define $\phi^{DA}(i) = \{s \mid (i, s) \in M\}$; **return** ϕ^{DA}; ▷ M を関数 ϕ^{DA} に変換して返す
end procedure

Deferred Acceptance Algorithm

- Properties of Deferred Acceptance Algorithm:
  - strategy-proof (Dubins and Freedman (1981), Roth (1982))
  - stable
  - Pareto dominates any other stable matching
  - not Pareto efficient

- Example: suppose true preferences and priorities are given by

$$P_{i_1} : s_2 \ \underline{s_1} \ i_1 \qquad\qquad \succ_{s_1} (1) : i_1 \ i_2 \ i_3 \ s_1$$

$$P_{i_2} : s_1 \ \underline{i_2} \qquad\qquad \succ_{s_2} (1) : i_3 \ i_1 \ s_2$$

$$P_{i_3} : s_1 \ \underline{s_2} \ i_3$$

The underlined schools are DA matchings. Here swapping i_1 and i_3's allocation Pareto improves both students. Thus DA is not Pareto efficient

Deferred Acceptance Algorithm

- Major results:

Theorem (Rural Hospital Theorem (Roth, 1986))

For any preference $P \in \mathcal{P}^I$, the set of unmatched students and that of vacant seats of schools are exactly the same under all stable matchings

Theorem (Abdulkadiroğlu et al. (2009), Kesten and Kurino (2017))

No strategy-proof mechanism Pareto dominates DA

Top Trading Cycles

Definition (Top Trading Cycles (Shapley and Scarf (1974), Abdulkadiroğlu and Sönmez (2003)))

Top Trading Cycles(TTC) $\phi^{\mathrm{TTC}} : \mathcal{P}^I \to \mathcal{M}$ is defined as follows:

- N_k, L_k を step k で残っている学生, 学校の集合とする. $N_1 = I, L_1 = S$ とする
- **Step** k　次のようにして有向グラフを作る: すべての $i \in N_k$ と $s \in L_k$ をグラフのノードとみなす. 各 $i \in N_k$ は $L_k \cup \{i\}$ の中で最も選好順の高い学校に有向エッジを引く. 同様に各 $s \in L_k$ は $N_k \cup \{s\}$ の中で最も priority の高い学生に有向エッジを引く. こうして出来た有向グラフには少なくとも 1 つの cycle(閉路) が存在する. 各 cycle に対し, その中の学生と, その学生が指している学校をマッチさせる. N_k から cycle を構成した学生を取り除き, N_{k+1} とする. L_k から定員が満たされた学校を取り除き, L_{k+1} とする. $|N_{k+1}| = 0$ の場合アルゴリズムを終了し, 現在のマッチ (+アンマッチ) を $\phi^{\mathrm{TTC}}(P)$ とする. それ以外の場合は step $k+1$ に進む

Top Trading Cycles

- Properties of Top Trading Cycles:
  - strategy-proof
  - not stable(individually rational and non-wasteful, but not fair)
  - Pareto efficient

- Example: suppose true preferences and priorities are given by

$$
\begin{array}{ll}
P_{i_1} : \underline{s_1}\ \ i_1 & \succ_{s_1} (1) : i_3\ \ i_2\ \ i_1\ \ s_1 \\
P_{i_2} : s_1\ \ s_2\ \ \underline{s_3}\ \ i_2 & \succ_{s_2} (1) : i_1\ \ i_2\ \ i_3\ \ s_2 \\
P_{i_3} : \underline{s_2}\ \ s_1\ \ s_3\ \ i_3 & \succ_{s_3} (1) : i_2\ \ i_1\ \ i_3\ \ s_3
\end{array}
$$

The underlined schools are TTC matchings. Here (i_2, s_1) blocks $\phi^{\text{TTC}}(P)$ since $s_1 P_{i_2} s_3 \wedge i_2 \succ_{s_1} i_1$. Thus TTC is not stable

Results

Notes

- 以降, $\mathcal{M}$ は individually rational な matching 全体の集合を指すものとする
- 以降誤解のない場合は, ある mechanism ϕ と preference $P \in \mathcal{P}^I$ に対する各 $i \in I$ のマッチ相手 $\phi_{i_1}(P), \ldots, \phi_{i_n}(P)$ を並べて

$$\phi(P) = (\phi_{i_1}(P), \ldots, \phi_{i_n}(P))$$

とかく
- 以降 $|I| \leq \sum_{s \in S} q_s$ であるケースだけを扱う

Results from Afacany et al. (2017)

- 同じく Assignment maximization を扱っている Afacany et al. (2017) から, baseline になる結果をいくつか紹介する

Proposition (Proposition 1 in Afacany et al. (2017))

No strategy-proof mechanism is size-wise maximal(among all mechanisms)

Proposition (Proposition 2 in Afacany et al. (2017))

There is no size-wise domination between any pair of mechanisms among the DA, TTC, BM, and SD

Original Results

- 自分で出した (失敗を含めた) 結果を紹介する

Proposition (Proposition 1-1)

Consider the following mechanism ϕ: for a submitted $P \in \mathcal{P}^I$, conduct both DA and TTC. If $\left|\phi^{\mathrm{DA}}(P)\right| \geq \left|\phi^{\mathrm{TTC}}(P)\right|$, set $\phi(P) = \phi^{\mathrm{DA}}(P)$. Otherwise set $\phi(P) = \phi^{\mathrm{TTC}}(P)$. Then ϕ is not strategy-proof

Proof.

Consider the following preferences and priorities:

$$
\begin{array}{ll}
P_{i_1} : s_1 \; i_1 & \succ_{s_1} (1) : i_3 \; i_2 \; i_1 \; s_1 \\
P_{i_2} : s_1 \; s_2 \; s_3 \; i_2 & \succ_{s_2} (1) : i_1 \; i_2 \; i_3 \; s_2 \\
P_{i_3} : s_2 \; s_1 \; s_3 \; i_3 & \succ_{s_3} (1) : i_2 \; i_1 \; i_3 \; s_3
\end{array}
$$

Then $\phi^{\mathrm{DA}}(P) = (i_1, s_1, s_2)$ and $\phi^{\mathrm{TTC}}(P) = (s_1, s_3, s_2)$. So $\phi(P) = (s_1, s_3, s_2)$

Original Results

Proof (Cont.)

Consider i_2's manipulation $P'_{i_2} : s_1 \; i_2$. Then $\phi^{\text{DA}}(P_1, P'_{i_2}, P_3) = (i_1, s_1, s_2)$ and $\phi^{\text{TTC}}(P_1, P'_{i_2}, P_3) = (s_1, i_2, s_2)$. So $\phi(P_1, P'_{i_2}, P_3) = (i_1, s_1, s_2)$

Suppose i_2's true preference is P_2. Since i_2 can improve their matching $s_3 \to s_1$ by misreporting P'_{i_2}, ϕ is not strategy-proof. □

Proposition (Proposition 1-2)

Consider the following mechanism ϕ: for a submitted $P \in \mathcal{P}^I$, conduct both DA and TTC. If $\left|\phi^{\text{DA}}(P)\right| > \left|\phi^{\text{TTC}}(P)\right|$, set $\phi(P) = \phi^{\text{DA}}(P)$. Otherwise set $\phi(P) = \phi^{\text{TTC}}(P)$. Then ϕ is not strategy-proof

Original Results

Proof.

Consider the following preferences and priorities:

$$
\begin{array}{llll}
P_{i_1} : s_1 \ s_2 \ i_1 & & & \succ_{s_1} (1) : i_3 \ i_2 \ i_1 \ s_1 \\
P_{i_2} : s_1 \ i_2 & & & \succ_{s_2} (1) : i_1 \ i_2 \ i_3 \ s_2 \\
P_{i_3} : s_2 \ s_3 \ i_3 & & & \succ_{s_3} (1) : i_2 \ i_1 \ i_3 \ s_3
\end{array}
$$

Then $\phi^{\text{DA}}(P) = (s_2, s_1, s_3)$ while $\phi^{\text{TTC}}(P) = (s_1, i_2, s_2)$. So $\phi(P) = (s_2, s_1, s_3)$

Consider i_1's manipulation $P'_{i_1} : s_1 \ i_1$. Then $\phi^{\text{DA}}(P'_1, P_{i_2}, P_3) = (i_1, s_1, s_2)$ and $\phi^{\text{TTC}}(P'_1, P_{i_2}, P_3) = (s_1, i_2, s_2)$. So $\phi(P'_1, P_{i_2}, P_3) = (s_1, i_2, s_2)$

Suppose i_1's true preference is P_1. Since i_1 can improve their matching $s_2 \rightarrow s_1$ by misreporting P'_{i_1}, ϕ is not strategy-proof. $\square$

Original Results

Lemma (Lemma 1)

Fix $P \in \mathcal{P}^I$. Let $\mu^{\text{DA}} \equiv \phi^{\text{DA}}(P)$. If a matching $\mu \in \mathcal{M}$ satisfies $|\mu| > \left|\mu^{\text{DA}}\right|$, then $\exists i \in I$ such that $\mu^{\text{DA}}(i)\ P_i\ \mu(i)$

Proof.

Assume for a contradiction that $\forall i \in I, \mu(i)\ R_i\ \mu^{\text{DA}}(i)$.

Define $I_0 \equiv \{i \in I \mid \mu(i)\ P_i\ \mu^{\text{DA}}(i)\}$. Since $|\mu| > \left|\mu^{\text{DA}}\right|$ implies $\exists i \in I$, $\mu^{\text{DA}}(i) = i$ and $\mu(i) \in S$, $|I_0| > 0$.

Consider the following Pareto improvement procedure:

- **Step** 0 Pick up any $i_1 \in I$. Define $I_1 \equiv I_0 \setminus \{i\}$
- **Step** $k(\geq 1)$ Let $s \equiv \mu(i_k)$. By the non-wastefulness of DA, $sP_{i_k}\mu^{\text{DA}}(i_k)$ implies $s \succ_s i_k$ or $\left|(\mu^{\text{DA}})^{-1}(s)\right| = q_s$. Since μ is individually rational, $s = \mu(i_k)$ implies the latter is the case.

Original Results

Proof (Cont.)

- **Step** $k(\geq 1)$(Cont.) Hence $\exists j \in I$ such that $\mu^{\text{DA}}(j) = s$ and $\mu(j) \neq s$. This implies $sP_j\mu(j)$ or $\mu(j)P_js$, since P_i is strict.
  - If $sP_j\mu(j)$, then that contradicts the assumption
  - If $\mu(j)P_js$, then $j \in I_k$. When $I_k = \emptyset$, this is a contradiction. Otherwise let $i_{k+1} \equiv j$ and $I_{k+1} \equiv I_k \setminus \{i_{k+1}\}$

  This procedure finishes in finite steps since $|I| < \infty$. So at some step k, a contradction occurs. This completes the proof

$\square$

Corollary (Corollary 1)

Fix $P \in \mathcal{P}^I$. Let $\mu^{\text{DA}} \equiv \phi^{\text{DA}}(P)$. If a matching $\mu \in \mathcal{M}$ satisfies $|\mu| > |\mu^{\text{DA}}|$, then $\exists i \in I$ such that $\mu^{\text{DA}}(i)\ P_i\ \mu(i)\ P_i\ i$

Original Results

Proposition (Proposition 2)

Consider the following mechanism ϕ:

$$\phi(P) = \begin{cases} \mu \quad \text{s.t. } |\mu| > \left|\phi^{\text{DA}}(P)\right| & \text{if } P = \bar{P} \\ \phi^{\text{DA}}(P) & \text{otherwise} \end{cases}$$

Then ϕ is not strategy-proof

Proof.

By the lemma 1, $\exists i \in I$ s.t. $\phi_i^{\text{DA}}(\bar{P})\bar{P}_i\phi_i(\bar{P})$. Fix such i. Let $s \equiv \phi_i^{\text{DA}}(\bar{P})$

Consider the following preference manipulation

$$P_i' : [\text{same schools and orders above } s] \; s \; i$$

Then $\phi_i(P_i', \bar{P}_{-i}) = \phi_i^{\text{DA}}(P_i', \bar{P}_{-i}) = s$. So if i's true preference is $\bar{P}_i$, they can be better of by reporting P_i'. Therefore ϕ is not strategy-proof $\square$

Original Results

Claim (Claim 1)

No strategy-proof mechanism is size-wise maximal among all strategy-proof mechanisms

- Since we already know that DA does not size-wise dominate TTC, it is enough to show that **no strategy-proof mechanism size-wise dominates DA**

- Motivating example: assume a mechanism ϕ size-wise dominates DA. Consider the following preferences and priorities

$$
\begin{aligned}
P_{i_1} &: s_1 \ i_1 & \qquad \succ_{s_1} (1) &: i_3 \ i_2 \ i_1 \ s_1 \\
P_{i_2} &: s_2 \ s_1 \ s_3 \ i_2 & \qquad \succ_{s_2} (1) &: i_1 \ i_2 \ i_3 \ s_2 \\
P_{i_3} &: s_2 \ s_1 \ s_3 \ i_3 & \qquad \succ_{s_3} (1) &: i_2 \ i_1 \ i_3 \ s_3
\end{aligned}
$$

Here $\phi^{DA}(P) = (i_1, s_2, s_1)$. Assume at this P, $|\phi(P)| > \left|\phi^{DA}(P)\right|$ holds. Therefore $\phi(P) = (s_1, s_2, s_3)$ or (s_1, s_3, s_2)

Original Results

- Suppose $\phi(P) = (s_1, s_2, s_3)$ is the case. Consider $P'_{i_3} : s_2\ s_1\ i_3$. Then DA matching does not change: $\phi^{DA}(P_{i_1}, P_{i_2}, P'_{i_3}) = (i_1, s_2, s_1)$. By SP and maximality, $\phi(P_{i_1}, P_{i_2}, P'_{i_3}) = (s_1, s_2, i_3)$

  Next consider $P'_{i_1} : s_1\ s_3\ i_1$. Then DA matching will be: $\phi^{DA}(P'_{i_1}, P_{i_2}, P'_{i_3}) = (s_3, s_2, s_1)$. By SP and maximality $\phi(P_{i_1}, P'_{i_2}, P'_{i_3}) = (s_1, s_3, s_2)$.

  Finally consider $P'_{i_2} : s_2\ i_2$. Then $\phi^{DA}(P') = (s_3, s_2, s_1)$. However by SP, $\phi_{i_2}(P') = i_2$, which contradicts the maximality of ϕ

Original Results

- Suppose $\phi(P) = (s_1, s_3, s_2)$ is the case. Consider P'_{i_2} : s_2 i_2. Then the DA matching does not change: $\phi^{\mathrm{DA}}(P_{i_1}, P'_{i_2}, P_{i_3}) = (i_1, s_2, s_1)$. By SP and maximality, $\phi(P_{i_1}, P_{i_2}, P'_{i_3})$ is (a): (s_1, i_2, s_2) or (b): (s_1, i_2, s_3)

  If (a) is the case, consider P'_{i_1} : s_1 s_3 i_1. Then DA matching will be: $\phi^{\mathrm{DA}}(P'_{i_1}, P'_{i_2}, P_{i_3}) = (s_3, s_2, s_1)$. By SP and maximality $\phi(P'_{i_1}, P'_{i_2}, P_{i_3}) = (s_1, s_2, s_3)$. Next consider P'_{i_3} : s_2 $s_1 i_3$. Then the DA matching does not change, while the maximum size of ϕ is 2 by SP, which contradicts the maximality

  If (b) is the case, consider P'_{i_3} : s_2 s_1 i_3. Then $\phi^{\mathrm{DA}}(P') = (i_1, s_2, s_1)$. By SP and maximality, $\phi(P_{i_1}, P'_{i_2}, P'_{i_3}) = (s_1, s_2, i_3)$. Next consider P'_{i_1} : s_1 s_3 i_1. While $\left|\phi^{\mathrm{DA}}(P')\right| = 3$, $|\phi(P')|$ is at most 2, which contradicts the maximality of ϕ

$\square$

Original Results

- In general...
  - Fix $P \in \mathcal{P}^I$ s.t. $|\phi(P)| > |\phi^{\mathrm{DA}}(P)|$
  - $\phi^{\mathrm{DA}}(P)$ でマッチしている人の preference を順に，DA のマッチングの直後が outside option になるように削る．これによって DA のマッチングは変わらない
  - 上の操作で，最終的に $|\phi(P')|$ は $|\phi^{\mathrm{DA}}(P)|$ と等しくなる
  - DA でマッチしていなかった人 1 人に対して，$\phi(P')$ のもとで定員の空いている学校が outside option の直前に来るような preference を考えて，矛盾を出す

From now on......

- claim 1 を一般に示す
- (claim 1 が示されたとして) そのようなメカニズムが存在するような preference の最大部分集合 $\mathcal{P}_0 \subset \mathcal{P}$ はなにかを調べる
- Preference revelation game に不完備情報を導入した場合, 既存の strategy-proof メカニズムと無制約に Assignment Maximization を達成するメカニズムとで, 均衡 (BNE) の下ではどちらのマッチ数が多くなるかを調べる. 特に次の EAM メカニズムについて検証する

EAM mechanisim from Afacany et al. (2017)

Theorem (Theorem 1 in Afacany et al. (2017))

There is a (non strategy-proof) size-wise maximal and Pareto efficient mechanism, called Efficient Assignment Maximizing mechanisim (EAM)

Proposition (Proposition 5 in Afacany et al. (2017))

EAM mechanism has a unique Nash equilibrium (in the preference revelation game) outcome that is equivalent to the outcome of SD given some common priority

Theorem (Theorem 2 in Afacany et al. (2017))

EAM mechanism is not size-wise dominated in equilibrium

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