Who's Who in Networks. Wanted: The Key Player

Ballester, Calvó-Armengol and Zenou (2006)

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- I uploaded this slide on my github page:

  https://github.com/myuuuuun/network2017

Sections

Summary

- The dependence of individual outcomes on group behavior is called **peer effects**
- Standard models assume peer effects are homogeneous across members. i.e., the marginal utility of a person taking an action can depend only on the average amount of actions group members take
- However generative models and empirical researches of peer effects suggest that they are heterogeneous
- This paper considers a game theoretic model which allows general pattern of bilateral influences and analyzes the resulting pattern of peer effects

Summary

- More precisely, this paper considers a finite n player game with linear-quadratic interdependent utility functions:

$$u_i(x_1, \ldots, x_n) = \alpha_i x_i + \frac{1}{2}\sigma_{ii} x_i^2 + \sum_{j \neq i} \sigma_{ij} x_i x_j$$

where $x_j \in \mathbb{R}_+$ is an action of the player j

- Linear-quadratic function form is restrictive, but we can consider this is a 2nd-order approximation (around 0) of general C^2 class utilities:

$$u_i(\boldsymbol{x}^{\mathrm{T}}) \simeq u_i(\mathbf{0}^{\mathrm{T}}) + \sum_j \frac{\partial}{\partial x_j} u_i(\mathbf{0}^{\mathrm{T}}) x_j + \frac{1}{2} \sum_j \frac{\partial^2}{\partial x_j^2} u_i(\mathbf{0}^{\mathrm{T}}) x_j^2 + \sum_{j \neq k} \frac{\partial^2}{\partial x_k \partial x_j} u_i(\mathbf{0}^{\mathrm{T}}) x_j x_k$$

$$= \{\text{const. for i}\} + \frac{\partial}{\partial x_i} u_i(\mathbf{0}^{\mathrm{T}}) x_i + \frac{1}{2} \frac{\partial^2}{\partial x_i^2} u_i(\mathbf{0}^{\mathrm{T}}) x_i^2 + \sum_{j \neq i} \frac{\partial^2}{\partial x_j \partial x_i} u_i(\mathbf{0}^{\mathrm{T}}) x_i x_j$$

Summary

- Note that the constant term does not matter for i's utility maximization given x_{-i}. The absolute level of utility matters only when we compare utilities between players
- Let $\Sigma = [\sigma_{ij}]$. This matrix summarizes the bilateral influences between players. We decompose this matrix additively into the 3 components:
  - **idiosyncratic effect**: reflects the concavity of the payoff function in own efforts
  - **global interaction effect**: is uniform across all players and reflects the strategic substitutability in efforts
  - **local interaction effect**: varies across pairs (i, j) and might be both strategic substitute and complement
- Using this decomposition, we first show there is a unique pure strategy Nash Equilibrium, which is an interior solution

Summary

- Next we see the main results. Theorem 1 states each player's equilibrium action is proportional to their **Bonacich network centrality**, which is an indicator measuring their importance for communication on the network
- By theorem 2, we show that the aggregate equilibrium outcome increases with the density and the size of the local interaction network
- Finally we consider an optimal network disruption policy: remove one player from the network that results in the maximum (or minimum) decrease in the aggregate equilibrium output. Theorem 3 claims such a player (**key player**) can be characterized by the **intercentrality**, which is another network centrality criterion

Model Settings

- Let's formally define our model. Consider a normal form game $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where
  - $N = \{1, 2, \dots, n\}$
  - $A_i = \mathbb{R}_+$ for all i
  - $u_i : \times_{i \in N} A_i \to \mathbb{R}$
    $u_i(x_1, x_2, \dots, x_n) = \alpha_i x_i + \frac{1}{2}\sigma_{ii} x_i^2 + \sum_{j \neq i} \sigma_{ij} x_i x_j \quad$ for all i
- For simplicity, we assume
  - $\alpha_i = \alpha > 0$ for all i
  - $\sigma_{ii} = \sigma < 0$ for all i

  But we consider general $\alpha_i > 0, \sigma_{ii} < 0$ case afterwards
- Observations:
  - $\sigma_{ii} < 0$ suggests that the payoff is strictly concave in own effort, that is, $\frac{\partial^2 u_i}{\partial x_i^2} = \sigma_{ii} < 0$

Model Settings

- Observations (Cont.):
  - For a pair $(i, j), i \neq j$, the bilateral effect from i's perspective is $\frac{\partial^2 u_i}{\partial x_i x_j} = \sigma_{ij}$. σ_{ij} is pair-dependent and can be of either sign. Also we allow the case $\sigma_{ij} \neq \sigma_{ji}$
  - If $\frac{\partial^2 u_i}{\partial x_i x_j} = \sigma_{ij} < 0$, it means x_i and x_j are **strategic substitutes** from i's perspective, i.e., an increase in j's effort triggers a downward shift in $i's$ response
  - On the other hand if $\frac{\partial^2 u_i}{\partial x_i x_j} = \sigma_{ij} > 0$, then x_i and x_j are **strategic complements** from i's perspective, i.e., an increase in j's effort triggers a upward shift in $i's$ response
  - Since $\sigma_{ij} \neq \sigma_{ji}$, it can happen that x_i and x_j are strategic substitutes from i's perspective though they are complements from j's perspective

Decomposition of Σ

- Let $\Sigma = [\sigma_{ij}]$. We also use this Σ to indicate the normal game
- Let $\underline{\sigma} = \min_{i,j \in N, i \neq j} \{\sigma_{ij}\}$ and $\bar{\sigma} = \max_{i,j \in N, i \neq j} \{\sigma_{ij}\}$
  i.e., $\underline{\sigma}, \bar{\sigma}$ are the minimum/maximum non-diagonal element of Σ
- We assume $\sigma < \min\{\underline{\sigma}, 0\}$
  - If $\underline{\sigma} \geq 0$, this assumption requires nothing additionally
  - If $\underline{\sigma} < 0$, this requires that own marginal returns decrease with the level of x_i, at least as much as cross-marginal returns $\frac{\partial^2 u_i}{\partial x_i x_j} = \sigma_{ij}$ (original HS condition)
- Define $\gamma = -\min\{\underline{\sigma}, 0\}$. By definition $\gamma \geq 0$
  - If actions are strategic substitutes for at least one pair (i, j), then $\underline{\sigma} < 0$ so $\gamma > 0$
  - Otherwise $\underline{\sigma} \geq 0$ and $\gamma = 0$

Decomposition of Σ

- Define $\lambda = \bar{\sigma} + \gamma = \bar{\sigma} - \min\{\underline{\sigma}, 0\}$. Then $\lambda \geq 0$
  - If $\underline{\sigma} < 0$, then $\lambda = \bar{\sigma} - \underline{\sigma} \geq 0$
  - Otherwise $\lambda = \bar{\sigma} \geq 0$
- We assume $\lambda > 0$. This assumption is not so problematic since $\lambda = 0 \Leftrightarrow \bar{\sigma} = \underline{\sigma}$. RHS means all off-diagonal elements are the same value, which is the homogeneous case
- In addition, if we add small ϵ to some off-diagonal elements of Σ, then the above equality does not hold. So this assumption is not so important practically

Decomposition of Σ

- Next define

$$g_{ij} = \begin{cases} \frac{\sigma_{ij}+\gamma}{\lambda} & i \neq j \\ 0 & \text{otherwise} \end{cases}$$

  - Since $\lambda = \bar{\sigma} + \gamma$, $g_{ij} \leq 1$
  - Also since

$$\sigma_{ij} + \gamma = \sigma_{ij} - \min\{\underline{\sigma}, 0\} = \begin{cases} \sigma_{ij} - \underline{\sigma} \geq 0 \\ \sigma_{ij} \geq 0 \quad (\because \sigma_{ij} \geq \underline{\sigma} \geq 0) \end{cases}$$

  so $g_{ij} \geq 0$

- g_{ij} measures the relative complementarity in efforts from i's perspective within (i, j) relative to the benchmark value $-\gamma \geq 0$
- Define $G = [g_{ij}]$. This is a 0-diagonal nonnegative square matrix so can be regarded as an adjacency matrix of a weighted directed graph g. In this model, the network structure comes from the utility parameter Σ

Decomposition of Σ

- If $\sigma_{ij} = \sigma_{ji}$, then G is symmetric and g becomes undirected
- If $\sigma_{ij} \in \{\underline{\sigma}, \bar{\sigma}\}$ for all $i \neq j$ and $\underline{\sigma} \leq 0$, then

$$\text{For } i \neq j, \quad g_{ij} = \frac{\sigma_{ij} + \gamma}{\lambda} = \frac{\sigma_{ij} - \underline{\sigma}}{\bar{\sigma} - \underline{\sigma}} = \begin{cases} 1 & \text{if } \sigma_{ij} = \bar{\sigma} \\ 0 & \text{if } \sigma_{ij} = \underline{\sigma} \end{cases}$$

$$\text{For } i = j, \quad g_{ij} = 0$$

  So G is a 0-1 matrix and g becomes unweighted, undirected graph without self loops and multiple links

- Finally we define $\beta = -\sigma - \gamma$
  - Since $\sigma < \min\{\underline{\sigma}, 0\}$, $\beta = -\sigma + \min\{\underline{\sigma}, 0\} > 0$

Decomposition of Σ

- Now using γ, λ, β, we decompose Σ as follows:

$$\Sigma = -\beta I - \gamma U + \lambda G \tag{1}$$

where I denotes the identity matrix, U denotes the all 1 elements matrix

- This equality holds since
  - $\forall i \in N,\ \mathrm{RHS}_{ii} = -\beta - \gamma + 0 = (\sigma + \gamma) - \gamma = \sigma = \sigma_{ii}$
  - $\forall i, j \in N$ s.t. $i \neq j,\ \mathrm{RHS}_{ij} = 0 - \gamma + \lambda g_{ij} = -\gamma + \lambda \frac{\sigma_{ij} + \gamma}{\lambda} = \sigma_{ij}$
- Here
  - $-\beta I$ is idiosyncratic effect, which reflects the concavity of payoffs in own efforts
  - $-\gamma U$ is global interaction effect, which corresponds to a uniform substitutability in own efforts among all pairs
  - λG is local interaction effect, which is potentially heterogeneous across pairs

Decomposition of Σ

- By decomposition, the utility function can be rewritten as

$$
\begin{aligned}
u_i(x_1,\dots,x_n) &= \alpha x_i - \frac{1}{2}(\beta+\gamma)x_i^2 + \sum_{j\neq i}(-\gamma+\lambda g_{ij})x_i x_j \\
&= \alpha x_i - \frac{1}{2}(\beta-\gamma)x_i^2 - \gamma\sum_{j\in N} x_i x_j + \sum_{j\in N}\lambda g_{ij} x_i x_j \qquad (2)
\end{aligned}
$$

  for all $i \in N$ (Note $g_{ii}x_i x_i = 0$ since $g_{ii} = 0$)

- Let $\lambda^* = \frac{\lambda}{\beta}$ denote the strength of local interactions relative to own concavity

Preparation

- Next we introduce two network centrality measures
- As preparation, consider the following matrix. Given an induced graph g and a sufficiently small scalar $a \geq 0$, we define

$$M(g,a) = (I - aG)^{-1} = \sum_{k=0}^{\infty} a^k G^k \tag{3}$$

  where $G^k, k \geq 1$ is a k-th power of an adjacency matrix G of the network g and $G^0 = I$
- $g_{ij}^{[k]} \equiv [G^k]_{ij}$: (i,j) element of G^k indicates the (weighted) number of paths on g from the node i to j whose length is k
- By the following two facts, we see that for an enough small constant $a \geq 0$, the two equalities in (3) hold (the 2nd equality holds if $a < \frac{1}{\rho(G)}$, where $\rho(G)$ is the spectral radius of G)

Preparation

Fact (Fact 1, existence of $(\alpha I - A)^{-1}$)

Let A be a n-square matrix and $\alpha \in \mathbb{R}$. If α is an eigenvalue of A, then the matrix $(\alpha I - A)$ is not invertible. Otherwise $(\alpha I - A)$ is invertible

Proof.

$$\begin{aligned}(\alpha I - A) \text{ is not invertible } &\iff \det(\alpha I - A) = 0 \\ &\iff \alpha \text{ is an eigenvalue of } A\end{aligned}$$

$\square$

- Hence $(I - aG)^{-1}$ exists if $a \neq 0$ and $\frac{1}{a}$ is not an eigenvalue of G
- $(I - aG)^{-1}$ also exists if $a = 0$ (then $I - aG = I$ and this is invertible)

Preparation

Fact (Fact 2, existence of $\frac{1}{\alpha}\sum_{k=0}^{\infty}\frac{1}{\alpha^k}A^k$)

Let A be a n-square nonnegative matrix. If $\alpha \in \mathbb{R}$ satisfies

$$\alpha > \rho(A)$$

where $\rho(A) \equiv \max\{|\lambda| : \lambda \in \mathbb{C}, \det(\lambda I - A) = 0\}$ is a spectral radius of A, then $(\alpha I - A)$ is invertible and

$$(\alpha I - A)^{-1} = \frac{1}{\alpha}\sum_{k=0}^{\infty}\frac{1}{\alpha^k}A^k$$

holds

- This fact comes from the equivalence of Hawkins-Simon conditions, which is proved in the lecture 2

Preparation

Remark (Hawkins-Simon conditions (Extracted))

Let A be a n-square nonnegative matrix and $\alpha > 0$. The following conditions are equivalent:

1. For any $\boldsymbol{c} \geq \mathbf{0}$, there exists $\boldsymbol{x} \geq \mathbf{0}$ s.t. $(\alpha I - A)\boldsymbol{x} = \boldsymbol{c}$
2. $(\alpha I - A)$ is nonsingular and $(\alpha I - A)^{-1} \geq O$
3. $\alpha > \rho(A)$
4. $\frac{1}{\alpha} \sum_{k=0}^{\infty} \frac{1}{\alpha^k} A^k$ exists and equals to $(\alpha I - A)^{-1}$
5. $det(\alpha I - A)_k) > 0$ for all $k = 1, \ldots, n$, where B_k denotes the k-th principal minor matrix of B

- In the next slide we see that if $a < \frac{1}{\rho(A)}$, then the two equalities in (3) holds

Preparation

- Definition: for $G \geq O$ and $a \geq 0$

$$M(g, a) = (I - aG)^{-1} = \sum_{k=0}^{\infty} a^k G^k \qquad ((3)\ \text{revisited})$$

- Assume $a < \frac{1}{\rho(G)} \iff \frac{1}{a} > \rho(G)$
- If $a = 0$, the two equalities in (3) immediately holds
- Suppose $a > 0$. Since

$$(I - aG)^{-1} = \frac{1}{a}\left(\frac{1}{a}I - G\right)^{-1}$$

and $(\frac{1}{a}I - G)^{-1}$ exists by HS condition 2, $(I - aG)^{-1}$ also exists. By HS condition 4,

$$\frac{1}{a}\left(\frac{1}{a}I - G\right)^{-1} = \frac{1}{a}a\sum_{k=0}^{\infty} a^k G^k = \sum_{k=0}^{\infty} a^k G^k$$

thus the two equalities in (3) also holds also for $a > 0$

Preparation

- a is a discount factor that scales down the relative weight of longer paths
- $m_{ij}(g, a) \equiv [M(g, a)]_{ij} = \sum_{k=0}^{\infty} a^k g_{ij}^{[k]}$ counts the paths of all lengths from i to j, weighted by a^k
- Using $M(g, a)$, we introduce the **Bonacich centrality**

Bonacich centrality

Definition (Bonacich centrality)

Given a sufficiently small $a \geq 0$, the Bonacich centrality of the network g, $\boldsymbol{b}(g,a)$, is defined as

$$\boldsymbol{b}(g,a) = M\mathbf{1} = (I - aG)^{-1}\mathbf{1} = \left(\sum_{k=0}^{\infty} a^k G^k\right)\mathbf{1} \tag{4}$$

where $\mathbf{1} = (1,\ldots,1)^{\mathrm{T}}$. So the Bonacich centrality of node i, $b_i(g,a)$ is

$$b_i(g,a) = \sum_{j \in N} m_{ij}(g,a)$$

- $b_i(g,a)$ counts the total (weighted) number of paths starting from i
- We can decompose $b_i(g,a)$ into self loops and paths to other players: $b_i(g,a) = m_{ii}(g,a) + \sum_{j \neq i} m_{ij}(g,a)$

Bonacich centrality

- If $a > 0$, $a^0 G^0 = I$. Hence $m_{ii}(g,a) \geq 1$ and thus $b_i(g,a) \geq 1$
- If $a = 0$, then $\boldsymbol{b}(g,a) = M\mathbf{1} = (I)^{-1}\mathbf{1} = \mathbf{1}$. So $b_i(g,a) = 1$
- Using Bonacich centrality, we introduce another network centrality measure, called **intercentrality**

Intercentrality

Definition (Intercentrality)

Given a sufficiently small $a \geq 0$, the Intercentrality of the node i in the network g, $c_i(g, a)$, is defined as

$$c_i(g, a) = \frac{b_i(g, a)^2}{m_{ii}(g, a)}$$

- The Bonacich centrality $b_i(g, a)$ counts the total (weighted) number of paths that starts from i
- The intercentrality $c_i(g, a)$ is the sum of i's Bonacich centrality and i's contribution to every other player's Bonacich centrality(?)

$$\begin{aligned} c_i(g, a) &= \frac{(m_{ii}(g, a) + \sum_{j \neq i} m_{ij}(g, a))}{m_{ii}(g, a)} b_i(g, a) \\ &= b_i(g, a) + \frac{\sum_{j \neq i} m_{ij}(g, a)}{m_{ii}(g, a)} b_i(g, a) \end{aligned}$$

Intercentrality

- Holding $b_i(g,a)$ fixed, $c_i(g,a)$ decreases as $m_{ii}(g,a)$ increases:

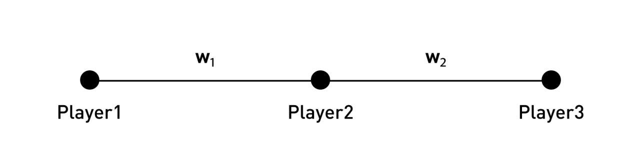
$$c_i(g,a) = \frac{b_i(g,a)^2}{m_{ii}(g,a)} = \frac{b_i(g,a)}{\frac{m_{ii}(g,a)}{b_i(g,a)}}$$

i.e., the intercentrality declines if the proportion of self loops in total paths from i increases

Example

Example

Consider the following undirected graph g



The adjacency matrix G of this graph is

$$G = \begin{pmatrix} 0 & w_1 & 0 \\ w_1 & 0 & w_2 \\ 0 & w_2 & 0 \end{pmatrix}$$

Example

Example (Cont.)

Therefore $M(g, a)$ given a is

$$
\begin{aligned}
M(g,a) = (I - aG)^{-1} &= \begin{pmatrix} 1 & -aw_1 & 0 \\ -aw_1 & 1 & -aw_2 \\ 0 & -aw_2 & 1 \end{pmatrix}^{-1} \\
&= \frac{1}{1 - a^2(w_1^2 + w_2^2)} \begin{pmatrix} 1 - a^2 w_2^2 & aw_1 & a^2 w_1 w_2 \\ aw_1 & 1 & aw_2 \\ a^2 w_1 w_2 & aw_2 & 1 - a^2 w_1^2 \end{pmatrix}
\end{aligned}
$$

For example, player 1, 2's Bonacich centralities and intercentralities are

$$
b_1(g,a) = \frac{1 + aw_1 + a^2 w_2(w_1 - w_2)}{1 - a^2(w_1^2 + w_2^2)}, \;\; b_2(g,a) = \frac{1 + a(w_1 + w_2)}{1 - a^2(w_1^2 + w_2^2)}
$$

$$
c_1(g,a) = \frac{1 + aw_1 + a^2 w_2(w_1 - w_2)}{1 - a^2 w_2^2}, \;\; c_2(g,a) = 1 + a(w_1 + w_2)
$$

Example

Example (Cont.)

Suppose $w_1 = 0$, i.e., the player 1 has no connection with others. Then two centralities are

$$b_1(g, a) = 1,\ b_2(g, a) = b_3(g, a) = \frac{1}{1 - aw_2}$$
$$c_1(g, a) = 1,\ c_2(g, a) = c_3(g, a) = 1 + aw_2$$

This seems to be an intuitive result because

- $b_1 = 1$, which is the minimum value of the Bonacich centrality, fits the fact that player 1 is isolated
- $b_2 = b_3$ and $c_2 = c_3$ reflect the symmetry of the role of player 2 and 3
- $b_1 < b_2 = b_3$ and $c_1 < c_2 = c_3$, which mean player 1 is less important in the network communication than 2 and 3

Example

Example

Next consider the following unweighted undirected graph g

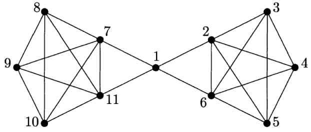


FIGURE 1.

The Bonacich centrality and intercentrality for $a = 0.1, 0.2$ are as follows:

TABLE I

Player Type	$a=0.1$ b_i	$a=0.1$ c_i	$a=0.2$ b_i	$a=0.2$ c_i
1	1.75	2.92	8.33	41.67*
2	1.88*	3.28*	9.17*	40.33
3	1.72	2.79	7.78	32.67

Main Results

- I will introduce the three main theorems of this paper
- In this section we focus on the case that Σ is symmetric, $\sigma_{ij} = \sigma_{ji}$. But some results still hold even if we remove this symmetric assumption
- Under the symmetric assumption, let $\mu_1(G)$ be the largest eigenvalue of G. $\mu_1(G)$ is well-defined since all eigenvalues of a real symmetric matrix are real values
- For any vector $\boldsymbol{y} \in \mathbb{R}^n$, define $y = y_1 + y_2 + \ldots + y_n$
- Before seeing the main theorems, we need to check some results using in the proofs

Preparation

Theorem (Perron-Frobenius Theorem I)

Let A be a square nonnegative matrix. A has an eigenvalue $\lambda_{PF}(A)$ that satisfies

1. $\lambda_{PF}(A) = \rho(A)$, *where $\rho(X)$ is the spectral radius of X, and*
2. *the corresponding eigenvector $\boldsymbol{x}$ satisfies $\boldsymbol{x} \geq \mathbf{0}, \boldsymbol{x} \neq \mathbf{0}$*

Theorem (Perron-Frobenius Theorem II)

Let A, B be n-square nonnegative matrices. If $A \geq B$, then $\lambda_{PF}(A) \geq \lambda_{PF}(B)$

- Perron-Frobenius theorem is proved in the lecture 2
- Proofs are also provided in 室田・杉原 (2013)[4, chapter 2.2]

Preparation

Lemma (Lemma 1)

Let A be a n-square nonnegative matrix. Given $S \subseteq N = \{1, 2, \dots, n\}, S \neq \emptyset$, define the submatrix of A, denoted by A_S, as follows

$$\text{Let } S = \{s_1, s_2, \dots, s_m\} \text{ s.t. } s_1 < s_2 < \dots < s_m. \text{ Then}$$

$$A_S = [A_{s_i, s_j}] = \begin{pmatrix} A_{s_1, s_1} & \dots & A_{s_1, s_m} \\ \vdots & \ddots & \vdots \\ A_{s_m, s_1} & \dots & A_{s_m, s_m} \end{pmatrix}$$

Then $\lambda_{PF}(A) \geq \lambda_{PF}(A_S)$

- This is an immediate result of the Perron-Frobenius theorem II and the next two facts

Preparation

Fact (Fact 3)

Let A be a n-square matrix. Consider a permutation of row and columns of A.

Let $N = \{1, 2, \ldots, n\}$ and σ be a permutation of N, which means

$$\sigma : N \to N, \textit{bijective}$$

and define $A_\sigma = [A_{\sigma_i,\sigma_j}]$. Then for $\lambda \in \mathbb{C}$,

$$\lambda \in \{\textit{eigenvalues of } A\} \iff \lambda \in \{\textit{eigenvalues of } A_\sigma\}$$

- Let n=3. An example of σ and A_σ:

$$\sigma : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \; A_\sigma = \begin{pmatrix} A_{22} & A_{23} & A_{21} \\ A_{32} & A_{33} & A_{31} \\ A_{12} & A_{13} & A_{11} \end{pmatrix}$$

Preparation

Proof of Fact 3.

Given a permutation σ, define the row-permutation matrix P_σ as

$$P_\sigma = \begin{pmatrix} \boldsymbol{e}_{\boldsymbol{\sigma_1}}^{\mathrm{T}} \\ \boldsymbol{e}_{\boldsymbol{\sigma_2}}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{e}_{\boldsymbol{\sigma_n}}^{\mathrm{T}} \end{pmatrix}$$

where $\boldsymbol{e_i}$ is the $n \times 1$ vector that i-th element is 1 and others are 0
For any $n \times k$(k: arbitrary) matrix $X = (\boldsymbol{x}_{\boldsymbol{1}}^{\mathrm{T}}, \boldsymbol{x}_{\boldsymbol{2}}^{\mathrm{T}}, \ldots, \boldsymbol{x}_{\boldsymbol{n}}^{\mathrm{T}})^{\mathrm{T}}$,

$$P_\sigma X = \begin{pmatrix} \boldsymbol{e}_{\boldsymbol{\sigma_1}}^{\mathrm{T}} \\ \boldsymbol{e}_{\boldsymbol{\sigma_2}}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{e}_{\boldsymbol{\sigma_n}}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{\boldsymbol{1}}^{\mathrm{T}} \\ \boldsymbol{x}_{\boldsymbol{2}}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{x}_{\boldsymbol{n}}^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_{\boldsymbol{\sigma_1}}^{\mathrm{T}} \\ \boldsymbol{x}_{\boldsymbol{\sigma_2}}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{x}_{\boldsymbol{\sigma_n}}^{\mathrm{T}} \end{pmatrix}$$

holds. This is why we call P_σ row-permutation matrix.

Preparation

Proof of Fact 3 (Cont.)

Here P_σ^{T}, which is the transpose of the row-permutation matrix, becomes the column-permutation matrix, since for any $k \times n$(k: arbitrary) matrix $Y = (\boldsymbol{y_1}, \boldsymbol{y_2}, \ldots, \boldsymbol{y_n})$,

$$YP_\sigma^{\mathrm{T}} = (\boldsymbol{y_1}, \boldsymbol{y_2}, \ldots, \boldsymbol{y_n})(\boldsymbol{e_{\sigma_1}}, \boldsymbol{e_{\sigma_2}}, \ldots, \boldsymbol{e_{\sigma_n}}) = (\boldsymbol{y_{\sigma_1}}, \boldsymbol{y_{\sigma_2}}, \ldots, \boldsymbol{y_{\sigma_n}})$$

holds.

Example:

$$\text{Let } n = 3 \text{ and } \sigma : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}. \text{ For } X = \begin{pmatrix} \boldsymbol{x_1}^{\mathrm{T}} \\ \boldsymbol{x_2}^{\mathrm{T}} \\ \boldsymbol{x_3}^{\mathrm{T}} \end{pmatrix},$$

$$P_\sigma X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{x_1}^{\mathrm{T}} \\ \boldsymbol{x_2}^{\mathrm{T}} \\ \boldsymbol{x_3}^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x_2}^{\mathrm{T}} \\ \boldsymbol{x_3}^{\mathrm{T}} \\ \boldsymbol{x_1}^{\mathrm{T}} \end{pmatrix}$$

Preparation

Proof of Fact 3 (Cont.)

And for any $Y = (\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{y_3})$,

$$YP_\sigma^{\mathrm{T}} = (\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{y_3}) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = (\boldsymbol{y_2}, \boldsymbol{y_3}, \boldsymbol{y_1})$$

Here P_σ is an orthogonal matrix since

$$P_\sigma P_\sigma^{\mathrm{T}} = (\boldsymbol{e}_{\boldsymbol{\sigma_1}}^{\mathrm{T}}, \boldsymbol{e}_{\boldsymbol{\sigma_2}}^{\mathrm{T}}, \dots, \boldsymbol{e}_{\boldsymbol{\sigma_n}}^{\mathrm{T}})^{\mathrm{T}} (\boldsymbol{e}_{\boldsymbol{\sigma_1}}, \boldsymbol{e}_{\boldsymbol{\sigma_2}}, \dots, \boldsymbol{e}_{\boldsymbol{\sigma_n}}) = I$$
$$P_\sigma P_\sigma^{\mathrm{T}} = (\boldsymbol{e}_{\boldsymbol{\sigma_1}}, \boldsymbol{e}_{\boldsymbol{\sigma_2}}, \dots, \boldsymbol{e}_{\boldsymbol{\sigma_n}})(\boldsymbol{e}_{\boldsymbol{\sigma_1}}^{\mathrm{T}}, \boldsymbol{e}_{\boldsymbol{\sigma_2}}^{\mathrm{T}}, \dots, \boldsymbol{e}_{\boldsymbol{\sigma_n}}^{\mathrm{T}})^{\mathrm{T}} = I$$

hence $P_\sigma P_\sigma^{\mathrm{T}} = P_\sigma^{\mathrm{T}} P_\sigma = I$

Fix any n-square matrix A. Using the row/column permutation matrix,

$$A_\sigma = P_\sigma A P_\sigma^{\mathrm{T}}$$

Preparation

Proof of Fact 3 (Cont.)

Finally consider the characteristic equation of A_σ

$$
\begin{aligned}
det(A_\sigma - \lambda I) &= det(P_\sigma A P_\sigma^{\mathrm{T}} - \lambda I) \\
&= det(P_\sigma A P_\sigma^{\mathrm{T}} - \lambda P_\sigma I P_\sigma^{\mathrm{T}}) \qquad (\because P_\sigma P_\sigma^{\mathrm{T}} = I) \\
&= det(P_\sigma) det(A - \lambda I) det(P_\sigma^{\mathrm{T}}) \\
&= det(P_\sigma) det(A - \lambda I) det(P_\sigma^{-1}) \\
&= det(P_\sigma) det(A - \lambda I) \frac{1}{det(P_\sigma)} \\
&= det(A - \lambda I)
\end{aligned}
$$

This implies the characteristic equation of A_σ and A is equivalent. Therefore the eigenvalues of two matrices are also equivalent. $\square$

Preparation

Fact (Fact 4)

Let A be a n-square matrix. Consider an extended $(n+m)$-square matrix A_E such that

$$A_E = \begin{pmatrix} A & O_{n,m} \\ O_{m,n} & O_{n,n} \end{pmatrix}$$

Then for $\lambda \in \mathbb{C}$,

$$\lambda \in \{\textit{eigenvalues of } A\} \implies \lambda \in \{\textit{eigenvalues of } A_E\}$$

Proof of Fact 4.

Let λ be an eigenvector of A and $\boldsymbol{x} \neq \mathbf{0}$ an eigenvector that belongs to λ. Let $\boldsymbol{x_E} = \begin{pmatrix} \boldsymbol{x} \\ \mathbf{0}_{\boldsymbol{m}} \end{pmatrix}$. Then using the block matrix multiplication formula, $A_E \boldsymbol{x_E} = \lambda \boldsymbol{x_E}$ holds and thus λ is also an eigenvalue of A_E □

Preparation

- Then let's prove the lemma 1

Proof of Lemma 1.

Fix any n-square nonnegative matrix A and $S \subseteq N, S \neq \emptyset$

Let $S = \{s_1, \ldots, s_n\}$ s.t. $s_1 < \ldots < s_m$ and A_S be a m-square submatrix of A given by S

Define a n-square nonnegative matrix $\underline{A} = [\underline{A}_{ij}]$ such that

$$\underline{A}_{ij} = \begin{cases} A_{ij} & i \in S \wedge j \in S \\ 0 & \text{otherwise} \end{cases}$$

And define another n-square nonnegative matrix A_{SE} such that

$$A_{SE} = \begin{pmatrix} A_S & O_{m,n-m} \\ O_{n-m,m} & O_{n-m,n-m} \end{pmatrix}$$

Preparation

Proof of Lemma 1 (Cont.)

1. Since $A \geq \underline{A} \geq O$, by the Perron-Frobenius theorem II, $\lambda_{PF}(A) \geq \lambda_{PF}(\underline{A})$
2. A_E is a permutated matrix of $\underline{A}$ (permutating rows and columns in the same permutation σ). Hence by the fact 3, $\lambda_{PF}(\underline{A}) = \lambda_{PF}(A_{SE})$
3. By the fact 4, $\{\text{eigenvalues of } A_{SE}\} \supseteq \{\text{eigenvalues of } A_S\}$. Thus $\rho(A_{SE}) \geq \rho(A_S)$.
   And by the Perron-Frobenius theorem I, since A_{SE}, A_S are nonnegative, $\lambda_{PF}(A_{SE}) = \rho(A_{SE})$ and $\rho(A_S) = \lambda_{PF}(A_S)$.
   Therefore $\lambda_{PF}(A_{SE}) \geq \lambda_{PF}(A_S)$

Using 1-3,

$$\lambda_{PF}(A) \geq \lambda_{PF}(\underline{A}) = \lambda_{PF}(A_{SE}) \geq \lambda_{PF}(A_S)$$

$\square$

- Using this lemma 1, we prove the theorem 1

Theorem 1

Theorem (Ballester et al.(2006) Theorem 1)

1. *The matrix* $(\beta I - \lambda G)^{-1}$ *is well defined and nonnegative if and only if* $\beta > \lambda\mu_1(G)$
2. *If* $\beta > \lambda\mu_1(G)$*, then the game* Σ *has a unique pure strategy Nash equilibrium* $\boldsymbol{x}^*(\Sigma)$*, which is an interior solution and given by*

$$\boldsymbol{x}^*(\Sigma) = \frac{\alpha}{\beta + \gamma b(g, \lambda^*)} \boldsymbol{b}(g, \lambda^*) \tag{5}$$

- The equilibrium actions are proportional to each player's Bonacich centrality
- Note we defined $\lambda^* = \frac{\lambda}{\beta}$ $(\beta > 0)$ and assumed $\lambda > 0$. Hence $\lambda^* > 0$
- Also note $b(g, \lambda^*) = \sum_{i \in N} b_i(g, \lambda^*)$
- If $(\beta I - \lambda G)$ is nonsingular, then by the symmetry of Σ, Σ is nonsingular. Note $\Sigma = -\beta I - \gamma U + \lambda G$

Theorem 1

Proof.

1. Since G is nonnegative, by the Perron-Frobenius theorem I, $\rho(G)$ is an eigenvalue of G. Therefore $\mu_1(G) = \rho(G) \geq 0$. Then

$$\beta > \lambda\mu_1(G) \iff \frac{1}{\lambda^*} > \rho(G)$$

This is equivalent to HS condition 3. Therefore HS condition 2 also holds so $(\frac{1}{\lambda^*}I - G)^{-1}$ exists and $(\frac{1}{\lambda^*}I - G)^{-1} \geq O$. Here

$$(\beta I - \lambda G)^{-1} = \frac{1}{\lambda}\left(\frac{1}{\lambda^*}I - G\right)^{-1}$$

$$\begin{aligned} \therefore \beta > \lambda\mu_1(G) &\iff \frac{1}{\lambda^*} > \rho(G) \\ &\iff (\frac{1}{\lambda^*}I - G)^{-1} \text{ exists and nonnegative} \\ &\iff (\beta I - \lambda G)^{-1} \text{ exists and nonnegative} \end{aligned}$$

Theorem 1

Proof (Cont.)

2. First we show that there exists an interior Nash equilibrium. By FOC of the utility functions for each i,

$$\frac{\partial u_i}{\partial x_i} = \alpha_i + \sum_{j \in N} \sigma_{ij} x_j = 0$$

Using a matrix notation, these FOCs are summarized as

$$-\Sigma \boldsymbol{x} = [\beta I + \gamma U - \lambda G]\boldsymbol{x} = \alpha \mathbf{1} \tag{6}$$

Since Σ is nonsingular, (6) has the unique solution, denoted by $\boldsymbol{x}^*$ Using the fact $U\boldsymbol{x}^* = x^*\mathbf{1}$,

$$(6) \iff \beta(I - \lambda^* G)\boldsymbol{x}^* = (\alpha - \gamma x^*)\mathbf{1}$$

Theorem 1

Proof (Cont.)

Since $(\beta I - \lambda G)$ is invertible, $(I - \lambda^* G)$ is also invertible. So

$$\beta \boldsymbol{x}^* = (\alpha - \gamma x^*)(I - \lambda^* G)^{-1}\mathbf{1} = (\alpha - \gamma x^*)\boldsymbol{b}(g, \lambda^*)$$

By summing up the elements of the vector,

$$\beta x^* = (\alpha - \gamma x^*) b(g, \lambda^*) \ \therefore x^* = \frac{\alpha b(g, \lambda^*)}{\beta + \gamma b(g, \lambda^*)}$$

So

$$\begin{aligned} \boldsymbol{x}^* &= \frac{(\alpha - \gamma x^*)}{\beta} \boldsymbol{b}(g, \lambda^*) \\ &= \frac{\alpha}{\beta}\left(1 - \frac{\gamma b(g, \lambda^*)}{\beta + \gamma b(g, \lambda^*)}\right) \boldsymbol{b}(g, \lambda^*) \\ &= \frac{\alpha}{\beta + \gamma b(g, \lambda^*)} \boldsymbol{b}(g, \lambda^*) \end{aligned}$$

Theorem 1

Proof (Cont.)

Next we show that corner solutions are not optimal.

For any n-square matrix Y and $S \subseteq N, S \neq \emptyset$, define Y_S as in the Lemma 1. Similarly define the subvector of $\boldsymbol{y}$, $\boldsymbol{y}_S$, that consists of s-th element for $s \in S$.

Consider the decomposition of Σ_S:

$$\Sigma_S = -\beta I_S - \gamma U_S + \lambda G_S$$

G_S is the adjacency matrix of the subnetwork induced by the players S

By the lemma 1, $\mu_1(G) = \rho(G) \geq \rho(G_S) = \mu_1(G_S)$ holds and thus

$$\beta > \lambda\mu_1(G) \geq \lambda\mu_1(G_S)$$

holds

Theorem 1

Proof (Cont.)

Assume there exists a non-interior Nash Equilibrium $\boldsymbol{y}^*$.
Define $S \subseteq N$ as

$$s \in S \iff y_i^* > 0$$

Note that $S \neq \emptyset$ since

$$\frac{\partial}{\partial x_i} u_i(\mathbf{0}^{\mathrm{T}}) = \alpha > 0$$

so $\boldsymbol{y}* = \mathbf{0}$ cannot be a Nash Equilibrium

Consider the FOCs of players in S. By the same argument as in the interior solution case, FOCs are summarized as

$$-\Sigma_S \boldsymbol{y}_S = [\beta I_S + \gamma U_S - \lambda G_S]\boldsymbol{y}_S = \alpha \mathbf{1}_S$$
$$\iff \beta(I_S - \lambda^* G_S)\boldsymbol{y}_S^* = (\alpha - \gamma y_S^*)\mathbf{1}_S$$

Theorem 1

Proof (Cont.)

Since $\beta > \lambda\mu_1(G_S)$ and so $(I_S - \lambda^* G_S)$ is invertible,

$$
\begin{aligned}
\boldsymbol{y}_S^* &= \frac{\alpha - \gamma y_S^*}{\beta}(I - \lambda^* G_S)^{-1}\mathbf{1}_S = \frac{\alpha - \gamma y_S^*}{\beta}\boldsymbol{b}(g_S, \lambda^*) \\
\therefore\ y_i^* &= \begin{cases} \frac{\alpha - \gamma y_S^*}{\beta} b_i(g_S, \lambda^*) & i \in S \\ 0 & i \in N \setminus S \end{cases}
\end{aligned} \tag{7}
$$

Now for $i \in N \setminus S$, in order to keep $y_i^* = 0$ as a best response,

$$
\frac{\partial}{\partial x_i} u_i(\boldsymbol{y}^{\mathrm{T}}) = \alpha - \gamma y_S^* + \lambda \sum_{j \in S} g_{ij} y_j^* \leq 0,\ \forall i \in N \setminus S
$$

has to hold (the 1st equality comes from (2))

Theorem 1

Proof (Cont.)

Using (7) , this can be rewritten as

$$(\alpha - \gamma y_S^*)\left(1 + \lambda^* \sum_{j \in S} g_{ij} b_j(g_S, \lambda^*)\right) \leq 0$$

Since $1 + \lambda^* \sum_{j \in S} g_{ij} b_j(g_S, \lambda^*) \geq 0$, this means $\alpha - \gamma y_S^* \leq 0$. But then again by (7) , $y_i^* \leq 0, \forall i \in S$. Contradiction.

□

- (5) also implies that each player contributes to the aggregate outputs in proportion to their Bonacich centrality since

$$x_i^*(\Sigma) = \frac{b_i(g, \lambda^*)}{b(g, \lambda^*)} x^*(\Sigma)$$

Remarks of Theorem 1

Remark (Remark 1)

Consider the general $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i > 0$ differs across players. Then using the weighted Bonacich centrality $\boldsymbol{b}_{\boldsymbol{\alpha}}(g, \lambda^*) = (I - \lambda^* G)^{-1}\boldsymbol{\alpha}$, the variation of (5) holds

Remark (Remark 2)

Consider the general σ_{ii} in addition to the general $\boldsymbol{\alpha}$. Define $\tilde{\boldsymbol{\alpha}}$ and $\tilde{\Sigma}$ as $\tilde{\alpha}_i = \frac{\alpha_i}{|\sigma_{ii}|}$ and $\tilde{\sigma_{ij}} = \frac{\sigma_{ij}}{|\sigma_{ii}|}$. Using the weighted Bonacich centrality, the variation of (5) holds

Remark (Remark 3)

In the proof of the theorem 1, the symmetry of Σ is used only for the existence of Σ^{-1}. So if we assume explicitly that Σ is invertible, then without symmetry we can show the theorem 1

- If G is a 0-1 symmetric, the theorem 1 holds under a (stronger) sufficient condition that is easy to check (w/o $\rho(G)$)

Corollary 1

Corollary (Ballester et al.(2006) Corollary 1)

Suppose

1. $\sigma_{ij} \in \{\underline{\sigma}, \bar{\sigma}\}$ *for all* $i \neq j$, *and* $\underline{\sigma} \leq 0$
2. *the glaph* g *induced by the decomposition* (1) *is connected*

Let m *be the number of all nodes in* g. *If* $\beta > \lambda\sqrt{2m + n - 1}$, *then the unique Nash equlibrium of the game* Σ *is given by* (5)

- Note that $2m = \sum_{i,j} g_{ij}$
- If the condition 1 is satisfied, then the induced graph g becomes undirected and unweighted, as we see in the section of decomposition of Σ
- If g is connected, by the following theorem, $\sqrt{2m - n + 1} \geq \rho(G)$ holds and thus $\beta > \lambda\sqrt{2m - n + 1} \implies \beta > \lambda\rho(G) = \lambda\mu_1(G)$ holds (note $\lambda > 0$). Therefore by the statement 2 of the theorem 1, the corollary 1 immediately holds

Corollary 1

Theorem (Yuan(1988)[2] Theorem 1)

Suppose A is the adjacency matrix of an unweighted undirected connected graph g. Then $\rho(A)$ satisfies

$$\rho(A) \leq \sqrt{2m - n + 1} \tag{8}$$

The equality holds if and only if the graph g is one of the following graphs

1. *the star graph*
2. *the complete graph*

Proof.

Let $A = \begin{pmatrix} \boldsymbol{A}_1^{\mathrm{T}} \\ \boldsymbol{A}_2^{\mathrm{T}} \\ \vdots \\ \boldsymbol{A}_n^{\mathrm{T}} \end{pmatrix}$. Let λ be any eigenvalue of A and $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ be the corresponding eigenvector with $\|\boldsymbol{x}\|_2 = 1$

Corollary 1

Proof (Cont.)

Define a vector $\boldsymbol{x}(i)$ whose j-th element $x_j(i)$ is

$$x_j(i) = \begin{cases} 0 & \text{if } j \text{ nadj } i \\ x_j & \text{otherwise} \end{cases}$$

where " j nadj i " means j in the non-adjacency nodes of i

Since $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $A_{ij} = 0$ if j nadj i,

$$\lambda x_i = \boldsymbol{A_i}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{A_i}^{\mathrm{T}}\boldsymbol{x}(i)$$

Let $d_i = \sum_j A_{ij}$ (row sum of A_i^{T} i.e., degree of the node i)

Corollary 1

Proof (Cont.)

By the Cauchy-Schwartz inequality,

$$\begin{aligned}
\lambda^2 x_i^2 &= |\langle \boldsymbol{A_i},\ \boldsymbol{x}(i)\rangle|^2 \\
&\leq (\|\boldsymbol{A_i}^{\mathrm{T}}\|_2)^2(\|\boldsymbol{x}(i)\|_2)^2 \\
&= (\sum_j A_{ij}^2)(\sum_{j\neq i} x_j^2) \\
&= (\sum_j A_{ij})(\sum_{j\neq i} x_j^2) \qquad (\because A_{ij} = 0 \text{ or } 1\ ) \\
&= d_i(1 - \sum_{j \text{ nadj } i} x_j^2) \qquad (\because \|\boldsymbol{x}\|_2 = 1)
\end{aligned} \tag{9}$$

By summing up for i,

$$\lambda^2 \leq 2m - \sum_i d_i \left(\sum_{j \text{ nadj } i} x_j^2 \right) \tag{10}$$

Corollary 1

Proof (Cont.)

Now

$$\begin{aligned}
\sum_{i=1} d_i \left(\sum_{j \text{ nadj } i} x_j^2 \right) &= \sum_{i=1}^{n} d_i x_i^2 + \sum_{i=1}^{n} d_i \left(\sum_{\substack{j \text{ nadj } i \\ j \neq i}} x_j^2 \right) \\
&\geq \sum_{i=1}^{n} d_i x_i^2 + \sum_{i=1}^{n} \left(\sum_{\substack{j \text{ nadj } i \\ j \neq i}} x_j^2 \right) \qquad (11) \\
&\qquad (\because d_i \geq 1, \ \forall i, \text{ since } g \text{ is connected}) \\
&= \sum_{i=1}^{n} d_i x_i^2 + \sum_{k=1}^{n} (n - d_k - 1) x_k^2 \\
&= n - 1
\end{aligned}$$

Corollary 1

Proof (Cont.)

Therefore we get

$$\lambda^2 \leq 2m - \sum_i d_i \left(\sum_{j \text{ nadj } i} x_j^2 \right) \leq 2m - n + 1$$

for any eigenvector λ

By the Perron-Frobenius theorem I, A has an eigenvector $\lambda = \rho(A)$. So

$$\rho(A) \leq \sqrt{2m - n + 1}$$

holds.

Next we check when the two sides are equal.

The equality holds $\implies$ g is star or complete: To show RHS, only holding (11) with equality is needed

Corollary 1

Proof (Cont.)

By (11) ,

$$\sum_{i=1} d_i(\sum_{j \text{ nadj } i} x_j^2) = \sum_{i=1} \sum_{j \text{ nadj } i} x_j^2$$

Hence for all i, $d_i = 1$ or $d_i = n-1$ should hold. This implies g is

1. the complete graph, which means $d_i = n-1$ for all n players, or
2. the star graph, which means $d_i = n-1$ for one player and $d_i = 1$ for other $n-1$ players

($\because$) suppose $d_i = n-1$ for $2 \leq m \leq n-1$ players. Then every player is connected to at least 2 players, that contradicts $d_i = 1$ for some i. Suppose $d_i = n-1$ for 0 players. The graph is no longer connected

g **is star or complete** $\implies$ **the equality holds**: **(skip)**. (Suppose g is complete or star. Since it is obvious that (11) holds with equality, we focus on (9) ...)

Corollary 1

Proof (Cont.)

Note: it is known that the eigenvalues of g are

1. for the complete graph, $\lambda = n - 1, 1$
2. for the star graph, $\lambda = \pm\sqrt{n-1}, 0$

And

1. for the complete graph, $\sqrt{2m - n + 1} = \sqrt{n(n-1) - (n-1)} = n - 1$
2. for the star graph, $\sqrt{2m - n + 1} = \sqrt{2(n-1) - (n-1)} = \sqrt{n-1}$

Therefore we can also show the equality condition by proving the above facts

□

Theorem 2

- This theorem 2 states that the equlibrium aggregate outcome is related to the partial order of Σ
- Increasing the local complementary leads to increasing the aggregate equilibrium output
- For n-square matrix Σ, Σ', define $\Sigma' \gneq \Sigma \iff \forall i, j,\ \sigma'_{ij} \geq \sigma_{ij}$ and $\exists i, j,\ \sigma'_{ij} > \sigma_{ij}$

Theorem (Ballester et al.(2006) Theorem 2)

Let Σ, Σ' be n-square symmetric matrices such that $\Sigma' \gneq \Sigma \geq O$.

If $\beta > \lambda\mu_1(G)$ and $\beta' > \lambda'\mu_1(G')$ for the parameters given by the decomposition (1) , then $x^*(\Sigma') > x^*(\Sigma)$

Theorem 2

Proof.

Define D such that $\Sigma' = \Sigma + \lambda D$

By $\Sigma' \gneq \Sigma \geq O$ and $\lambda > 0$, $\forall i, j, d_{ij} \geq 0$ and $\exists i, j, d_{ij} > 0$

By $\beta > \lambda\mu_1(G)$ and $\beta' > \lambda'\mu_1(G')$, the first statement of the theorem 1 holds. Then the FOCs in the proof of the theorem 1, (6) , hold so

$$-\Sigma \boldsymbol{x}^*(\Sigma) = -\Sigma' \boldsymbol{x}^*(\Sigma') = \alpha \mathbf{1}$$

with $\boldsymbol{x}^*(\Sigma), \boldsymbol{x}^*(\Sigma') > \mathbf{0}$

Theorem 2

Proof (Cont.)

By the symmetry of Σ,

$$\begin{aligned}
\alpha x^*(\Sigma') &= \alpha(\boldsymbol{x}^*(\Sigma'))^{\mathrm{T}}\mathbf{1} \\
&= -(\boldsymbol{x}^*(\Sigma'))^{\mathrm{T}}\Sigma\boldsymbol{x}^*(\Sigma) \\
&= (\boldsymbol{x}^*(\Sigma'))^{\mathrm{T}}(\lambda D - \Sigma')\boldsymbol{x}^*(\Sigma) \\
&= (\boldsymbol{x}^*(\Sigma'))^{\mathrm{T}}(\lambda D - (\Sigma')^{\mathrm{T}})\boldsymbol{x}^*(\Sigma) \\
&= \lambda(\boldsymbol{x}^*(\Sigma'))^{\mathrm{T}}D\boldsymbol{x}^*(\Sigma) + (-\Sigma' x^*(\Sigma'))^{\mathrm{T}}\boldsymbol{x}^*(\Sigma) \\
&= \alpha x^*(\Sigma) + \lambda(\boldsymbol{x}^*(\Sigma'))^{\mathrm{T}}D\boldsymbol{x}^*(\Sigma)
\end{aligned}$$

Since $\boldsymbol{x}^*(\Sigma), \boldsymbol{x}^*(\Sigma') > \mathbf{0}$ and $D \gneq O$, (the second term) > 0

Then by $\alpha > 0$, $x^*(\Sigma') > x^*(\Sigma)$ holds $\square$

Theorem 2

$$\boldsymbol{b}(g, \lambda^*) = (I - \lambda^* G)^{-1} \mathbf{1} \qquad \text{((4) , (5) revisited)}$$

$$\boldsymbol{x}^*(\Sigma) = \frac{\alpha}{\beta + \gamma b(g, \lambda^*)} \boldsymbol{b}(g, \lambda^*)$$

- Note: $\gamma = -\min\{\underline{\sigma}, 0\}$, $\lambda = \bar{\sigma} - \gamma$, $\beta = -\gamma - \sigma$, $\lambda^* = \lambda/\beta$
- An increase in the cross effects from Σ to $\Sigma' \gneq \Sigma$ can have opposite effects
  1. $\lambda^* G$ can increase if λ or G increase, if β decreases, or both. Then local complementarity increases and so does the equilibrium outcome for each player
  2. $\lambda^* G$ and γ can decrease. This is the case, for example, if Σ' is obtained from Σ by only increasing $\underline{\sigma}(\Sigma)$. Then the total number of weighted paths $b(g, \lambda^*)$ decreases, but the impact of γ dominates, so the aggregate output increases

Theorem 3

- Finally let's consider an optimal network disruption policy: remove one player from the network that results in the maximum (or minimum) decrease in the aggregate equilibrium output
- Here we assume Σ is symmetric, $\sigma_{ij} \in \{\underline{\sigma}, \bar{\sigma}\}$ and $\underline{\sigma} \geq 0$ so that g becomes unweighted undirected graph
- Additionally assume that $\forall v \in \{\underline{\sigma}, \bar{\sigma}\}, \sigma_{ij} = v$ for at least 2 different pairs. This condition guarantees that any player removal does not change β, γ, λ
- We denote by G^{-i} the adjacency matrix that obtained from G by setting to zero the elements of i-th row and column and by g the graph of it
- Define Σ^{-i} in the same way

Theorem 3

- Then we formally define the optimal network disruption problem

Definition (Optimal network disruption problem)

Assume all the above assumptions are satisfied. The optimal network disruption problem is $\max \{x^*(\Sigma) - x^*(\Sigma^{-i}) \mid i = 1, \ldots, n\}$, which is equivalent to

$$\min_{i \in N} \; x^*(\Sigma^{-i}) \tag{12}$$

- This is a finite optimization problem so there exists at least one solution
- Let i^* gives the solution to (12) . We call i^* **key player**
- Theorem 3 states the key player is geometrically characterized by the intercentrality

Theorem 3

Theorem (Ballester et al.(2006) Theorem 3)

Assume all the above mentioned assumptions are satisfied. If $\beta > \lambda\mu_1(G)$, the key player i^* that solves (12) has the highest intercentrality of parameter λ^*, that is,

$$c_i^*(g, \lambda^*) \geq c_i(g, \lambda^*), \quad \forall i \in N$$

- The proof is based on the following technical lemma

Lemma (Ballester et al.(2006) Lemma 1)

Assume $M(g, a) = (I - aG)^{-1}$ is well defined and nonnegative. Then

$$m_{ij}(g, a)m_{ik}(g, a) = m_{ii}(g, a)[m_{jk}(g, a) - m_{jk}(g^{-i}, a)]$$

for all $j \neq i, k \neq i$

Theorem 3

Proof of Ballester et al.(2006) Lemma 1.

Let $g^s_{j(i)k}$ (resp. $g^s_{j(i^0)k}$) be the weight of s-length paths from j to k that contain i (resp. do not contain i)

By the symmetry of Σ, $m_{jk}(g, a) = m_{kj}(g, a), \forall j, k, \forall g$. Then

$$
\begin{aligned}
& m_{ii}(g, a)[m_{jk}(g, a) - m_{jk}(g^{-i}, a)] \\
&\qquad = \sum_{p=0}^{\infty} a^p g^{[p]}_{ii} \left(\sum_{q=0}^{\infty} a^q (g^{[q]}_{jk} - g^{[q]}_{j(i^0)k})\right) \\
&\qquad = \sum_{p=1}^{\infty} a^p \sum_{\substack{r+s=p \\ r\geq 0, s\geq 1}} g^{[r]}_{ii} (g^{[s]}_{jk} - g^{[s]}_{j(i^0)k}) \quad (\because g^{[0]}_{jk} - g^{[0]}_{j(i^0)k} = 0) \\
&\qquad = \sum_{p=1}^{\infty} a^p \sum_{\substack{r+s=p \\ r\geq 0, s\geq 2}} g^{[r]}_{ii} g^{[s]}_{j(i)k} \quad (\because j\text{-}i\text{-}k \text{ path is at least of length } 2)
\end{aligned}
$$

Theorem 3

Proof of Ballester et al.(2006) Lemma 1 (Cont.)

$$
\begin{aligned}
&= \sum_{p=1}^{\infty} a^p \sum_{\substack{r+s+t=p \\ r\geq 1, s\geq 0, t\geq 1}} g_{ji}^{[r]} g_{ii}^{[s]} g_{ik}^{[t]} \\
&= \sum_{p=1}^{\infty} a^p \sum_{\substack{r+s=p \\ r\geq 1, s\geq 1}} g_{ji}^{[r]} g_{ik}^{[s]} \\
&= m_{ij}(g,a) m_{ik}(g,a)
\end{aligned}
$$

□

Proof of Theorem 3.

By $\beta > \lambda\mu_1(G)$, $M(g, \lambda^*)$ is well-defined and nonnegative

Since $\mu_1(G) \geq \mu_1(G^{-i})$ by the Perron-Frobenius Theorem, $\beta > \lambda\mu_1(G^{-i})$ holds and thus $M(g^{-i}, \lambda^*)$ also exists and is nonnegative

Theorem 3

Proof of Theorem 3 (Cont.)

Since $x^*(\Sigma^{-i}) = \frac{\alpha b(g^{-i},\lambda^*)}{\beta+\gamma b(g^{-i},\lambda^*)}$,

$$\frac{\partial x^*(\Sigma^{-i})}{\partial b} = \frac{\alpha(\beta+\gamma b) - \alpha b \gamma}{\text{positive}} = \frac{\alpha\beta}{\text{positive}} > 0$$

Therefore $x^*(\Sigma^{-i})$ is increasing in $b(g^{-i}, \lambda^*)$

So the problem (12) is equivalent to

$$\min_{i \in N} b(g^{-i}, \lambda^*)$$

Define $b_{ji}(g, \lambda^*) = b_j(g, \lambda^*) - b_j(g^{-i}, \lambda^*)$ for all $j \neq i$. This is the contribution of i to j's Bonacich centrality

Summing over all $j \neq i$ and adding $b_i(g, \lambda^*)$ both sides gives

Theorem 3

Proof of Theorem 3 (Cont.)

$$d_i(g,\lambda^*) \equiv b_i(g,\lambda^*) + \sum_{j\neq i} b_{ji}(g,\lambda^*) = b(g,\lambda^*) - b(g^{-i},\lambda^*)$$

Here the solution to **(??)** becomes $i^* \in \text{argmin}_{i\in N} d_i(g,\lambda^*)$

We have

$$\begin{aligned} d_i(g,\lambda^*) &= b_i(g,\lambda^*) + \sum_{j\neq i}(b_j(g,\lambda^*) - b_j(g^{-i})) \\ &= b_i(g,\lambda^*) + \sum_{j\neq i}\sum_{k=1}^{n}(m_{jk}(g,\lambda^*) - m_{jk}(g^{-i})) \end{aligned}$$

Theorem 3

Proof of Theorem 3 (Cont.)

By the result of Ballester et al.(2006) lemma 1,

$$
\begin{aligned}
d_i(g,\lambda^*) &= b_i(g,\lambda^*) + \sum_{j\neq i}\sum_{k=1}^{n} \frac{m_{ij}(g,\lambda^*) - m_{ik}(g^{-i},\lambda^*)}{m_{ii}(g,\lambda^*)} \\
&= b_i(g,\lambda^*)\left(1 + \sum_{j\neq i}\frac{m_{ij}(g,\lambda^*)}{m_{ii}(g,\lambda^*)}\right) \\
&= \frac{b_i(g,\lambda^*)^2}{m_{ii}(g,\lambda^*)}
\end{aligned}
$$

□

Theorem 3

Example

Again consider the following unweighted undirected graph g

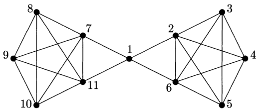


FIGURE 1.

The Bonacich centrality and intercentrality for $a = 0.1, 0.2$ are as follows:

TABLE I

	$a = 0.1$		$a = 0.2$	
Player Type	b_i	c_i	b_i	c_i
1	1.75	2.92	8.33	41.67*
2	1.88*	3.28*	9.17*	40.33
3	1.72	2.79	7.78	32.67

Therefore if $a = 0.1(0.2)$, removal of the player 2(1) results in the maximum decrease in the aggregate output

Theorem 3

Corollary (Corollary 2)

Assume all the above mentioned assumptions are satisfied. If $\beta > \lambda\mu_1(G)$, the key player i^* that solves $\max_{i \in N} \; x^*(\Sigma^{-i})$ has the lowest intercentrality of parameter λ^*, that is,

$$c_i^*(g, \lambda^*) \leq c_i(g, \lambda^*), \quad \forall i \in N$$

Remark (Remark 4)

When Σ is not symmetric, the theorem 3 and corollary 2 still hold, where the modified intercentrality is given by $\tilde{c}_i(g, a) = b_i(g, a)\frac{\sum_{j=1}^{n} m_{ji}(g, a)}{m_{ii}(g, a)}$

Discussion and Extensions

- Theorem 3 characterize the key player when the planner's objective function is the total output. Suppose instead the objective function is total welfare, $W^*(\Sigma) = \sum u_i(x^*(\Sigma)) = \frac{\beta+\gamma}{2} \sum x_i^*(\Sigma)^2$. If $\gamma = 0$, then $W^*(\Sigma) = \frac{\alpha^2}{2\beta} \sum b_i(g, \lambda^*)^2$. Also in this case, the key player can be characterized geometrically(?)
- The intercentrality measure can be generalized for the group index. Then we can consider the optimal group removal problem. Note that the solution to the problem can be different from the sequential removal of players
- Beyond the optimal player removal problem, we can also analyze the optimal tax/subsidy policies to hold the aggregate output to some planner's objective value
- We can also deal with the dynamic network. That is, we define the two stage game that at first stage players choose to stay/drop their edges and at second stage they solve the Σ. For this literature, see Calvó-Armengol and Zenou(2004) [3]

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