Have we misinterpreted CAPM for 40 years? A theoretical proof

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Notes:

1. During one of my GCAPM paper discussion with Dr. Harry Markowitz in 1999, I proposed to derive a special case of GCAPM equilibrium using the same assumption and mean-variance utilities as in the traditional CAPM. The result is this short paper. I am greatly in debt to Dr. Markowitz for his inspiration and encouragement.

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Abstract: The validity of CAPM has been contingent on its security market line hypothesis, which asserts that higher-beta-risk assets should carry higher expected returns. Owing to a lack of empirical support for that hypothesis, many have declared CAPM dead. However, by surrogating assets' following-period ex-post returns as asset expected returns, most empirical studies have misinterpreted CAPM. This paper shows that higher-beta-risk assets will not necessarily generate higher or lower ex-post returns and that CAPM is such a common sense theory that one can literally observe its ex-post return paradigms at work in the daily capital marketplace.

Sharpe, Linter, et al created CAPM, the first general equilibrium theory of the capital market, in 1964 and 1965. At the center of the theory is the security market line, which indicates that higher beta-risk assets should carry higher expected returns. As a theory, CAPM is very well received because of its parsimonious elegance and its common sense notion that any risk-averse investors would demand higher expected returns to compensate for taking higher risk.

In practice and in empirical tests, however, CAPM is not as well received. In the late 1960's, many practitioners, taking advantage of their superior computing power at that time, tried to put the CAPM theory to work by investing in higher-beta-risk assets. They were hoping that their higher expected returns would deliver higher ex post investment returns but were ultimately disappointed (Malkiel, 1990; Bernstein, 1992). At that time, academicians had just started to investigate the validity of CAPM. By and large, the CAPM related empirical studies available at that time (Fama, Fisher, Jensen and Roll, 1969; Blume, 1968) were promising in their support of CAPM's security market line (Fama, 1970). A few years later, in the early 1970's, more formal studies were conducted to test the validity of CAPM and the results were very positive (Black, Jensen and Scholes, 1972; Blume and Friend, 1973; Fama and MacBeth, 1973). However, those empirical results only brought a very brief period of euphoria for CAPM. Most of the later empirical studies (Rolls, 1977; Basu, 1977 and 1983; Banz, 1981; Bhandari, 1988; Chan, Hamao, Lakonishok, 1991; Fama and French, 1992) resoundingly rejected the existence of CAPM's security market line (Fama, 1991). Since then, many practitioners and academicians have declared CAPM dead.

This paper finds that both the aforementioned investment practice and empirical studies misinterpreted the relationship between CAPM's security market line and asset ex-post returns. In the aforementioned investment practice, practitioners have implicitly equated assets' ex-ante expected returns to assets' following-period ex-post returns. In the aforementioned empirical studies, economists have used (1) leading or lagging time-series regression coefficients to measure cross-sectional beta risk and (2) assets' following-period ex-post returns as a surrogate of assets' ex-ante expected returns. This paper proves that (1) higher-beta-risk assets do not necessarily generate higher or lower asset ex-post returns and (2) the levels of ex-ante beta risk should have little explanatory power to asset ex-post returns, especially when beta risk is measured by a time-series regression coefficient.

Adopting the highly criticized CAPM assumptions, this paper derives a special case of GCAPM equilibrium[1] (Fan, 2004). The equilibrium adds "asset price", whose variation defines asset ex-post returns, to the traditional CAPM equilibrium[2]. Through the price dimension, this paper derives two Asset Ex-Post Return Theorems for CAPM. They shows that, under CAPM's illustrative world, the main drivers of asset ex post returns are: (1) changes in asset expected return, (2) changes in asset beta risk, (3) changes in investors' taste for expected return, i.e. the capital market's unit expected-return premium, and (4) changes in investors' taste for beta risk, i.e. the capital market's unit beta-risk premium. According to the Theorem, beta risk is not the sole driver of asset ex-post returns and, as far as beta risk is concerned, it is the changes of beta risk – not the ex-ante level of beta risk – that determines asset ex-post returns.

Since asset ex-post returns are readily observable in the marketplace, the newly derived Asset Ex-Post Return Theorems provide a good hypothesis to test the validity of CAPM. Given that CAPM is such an intuitively appealing theory, one should not be surprised to find that one can literally observe CAPM's ex-post return paradigms at work in the daily marketplace – when an asset experiences an upward expected earnings revision, which will raise the asset's expected return, its price will rise and vice versa; when an asset's beta risk premium increases (either because the asset becomes more risky or investors become more risk averse), its price will fall and vice versa.

This paper consists of three sections. Section I derives the special case of GCAPM equilibrium. Section II derives the three-dimensional CAPM paradigms and discusses common CAPM misinterpretations. Section III concludes this paper.

I. A Special Case of GCAPM Equilibrium

A. CAPM assumptions

In Sharpe's illustrative world (1964), he assumes the following investment utility and market conditions.

(1) Investors' investment utility is a function of their expected wealth e and the volatility of their expected wealth v, i.e.

$$u_i = f_i(e, v), \qquad \forall i \tag{1}$$

(2) Investors prefer higher expected wealth and lower wealth uncertainty, i.e.

$$\frac{\partial u_i}{\partial e} > 0 \quad and \quad \frac{\partial u_i}{\partial v} < 0, \qquad \forall i \tag{2}$$

(3) Investors can borrow and lend funds on equal terms, as a common pure rate of interest exists.

(4) Investors are assumed to agree on the prospects of all investments – the expected values, standard deviations, and correlation coefficients of asset future returns.

B. Utility Maximization Behavior

Let's denote any investment portfolio with n investment asset positions in a n-dimensional vector $\mathbf{x} = (x_1, \ldots, x_n)$, where x_i denote the amount of asset i holding in the portfolio. Correspondingly, let's denote the prices of those n assets in a n-dimensional price vector $\mathbf{p} = (p_1,\ldots,p_n)$, then the product $\mathbf{p.x}$ would denote the market value of any portfolio **x**. Similarly, we can denote an investor's capital endowment, i.e. his or her holdings in those n assets, in a n-dimensional vector $\mathbf{w} = (w_1,\ldots,w_n)$. Following the same rationale, the vector product $\mathbf{p.w} = y$ would denote the investor's capital endowment in the monetary term. Finally, let X denote the set of investment portfolios that can be constructed with the n investment assets, we can express any investor i's investment utility maximization mathematically as following:

$$\begin{aligned} \max \quad & u_i(\mathbf{x}(e, v)) \\ s.t. \quad & \mathbf{p.x} = y_i \\ & \mathbf{x} \in X \end{aligned} \tag{3}$$

In a competitive market place, i.e. no investors are large enough to manipulate capital asset prices, the following Theorem depicts how investors would behave under any given set of asset market prices.

The Competitive Utility Maximization Theorem[3]: In any competitive capital marketplace, investors will adjust their asset holdings such that their marginal utility of assets will be in proportion to asset market prices, i.e.

$$\frac{\dfrac{\partial u_i(\mathbf{x}(e, v))}{\partial x_j}}{\dfrac{\partial u_i(\mathbf{x}(e, v))}{\partial x_k}} = \frac{p_j}{p_k}, \qquad \forall\, x_j >> 0 \text{ and } x_k >> 0 \tag{4}$$

C. Cross-Sectional Asset Supply and Demand

For any given set of asset market prices, comparing an investor's desired asset holdings $\mathbf{x}_i(\mathbf{p}, y_i)$ with his or her asset endowment holdings $\mathbf{w}_i$, we can calculate the investor cross-sectional asset supply (sell) and demand (buy)[4]:

$$\mathbf{x}_i(\mathbf{p}, y_i) - \mathbf{w}_i = ((x_{i1}(\mathbf{p}, y_i) - w_{i1}), \ldots, (x_{in}(\mathbf{p}, y_i) - w_{in})) \tag{5}$$

In the vector expression on the right-hand side of equation (5), if the j-th vector element $(x_{ij}(\mathbf{p},y_i) - w_{ij})$ is greater than 0, this indicates that investor i would like to buy asset j in addition to his or her endowment holding, i.e. investor i is on the demand side of asset j, and vice versa.

With individual investors' cross-sectional asset supply and demand functions defined, a capital market's aggregate supply $\mathbf{S}(\mathbf{p})$ and demand $\mathbf{D}(\mathbf{p})$ functions in a m-investor world are simply the sum of the m individual investors' supply and demand functions, i.e.

$$\mathbf{S}(\mathbf{p}) = (\sum_{i=1}^{m} (x_{i1} - w_{i1}),\ \ldots,\ \sum_{i=1}^{m} (x_{in} - w_{in})); \qquad \forall\ [x_{ij} - w_{ij}] \leq 0$$
$$\mathbf{D}(\mathbf{p}) = (\sum_{i=1}^{m} (x_{i1} - w_{i1}),\ \ldots,\ \sum_{i=1}^{m} (x_{in} - w_{in})); \qquad \forall\ [x_{ij} - w_{ij}] \geq 0 \tag{6}$$

D. The Existence of Capital Market Equilibrium

The existence of capital market equilibrium depends on the existence of a set of market prices that can clear all investors' supply and demand for assets, i.e. there is no excess supply or demand of assets. Mathematically, a capital market equilibrium can be defined as the existence of an equilibrium price vector $\mathbf{p}^*$ that sets the following capital market's excess demand and supply for assets $\mathbf{z}(\mathbf{p}^*)$ to $\mathbf{0}$, i.e.

$$\mathbf{z}(\mathbf{p}^*) = \mathbf{D}(\mathbf{p}^*) + \mathbf{S}(\mathbf{p}^*) = \sum_{i=1}^{m} \mathbf{x}_i^*(\mathbf{p}^*, \mathbf{p}^* w_i) - \sum_{i=1}^{m} \mathbf{w}_i = \mathbf{0} \tag{7}$$

In CAPM's illustrative world, investors' mean-variance utilities are mathematically quadratic and concave. Arrow and Debreu (1954) have proved that a competitive capital market equilibrium not only exists, but is also unique under any set of concave utility functions. Hence, we have the following CAPM equilibrium Theorem.

The CAPM Equilibrium Theorem[5]: Under CAPM's assumptions, a capital market equilibrium not only exists but is also unique.

II. The Three-Dimensional CAPM Paradigms

A. The Static GCAPM Theorem

Even though competitive capital market equilibriums may optimize many forms of social welfare function, the most interesting one to us is how capital markets weigh each individual investor's utility function in their equilibrium process. The answer to that question will provide us with the critical information connecting a capital market's expected-return and beta-risk premiums with individual investors' tastes of expected return and beta risk. The following Second Welfare Theorem sheds light on that very question.

The Second Welfare Theorem[6]: Competitive capital market equilibrium optimizes a social welfare function that weighs individual investors' investment utilities by the reciprocal of their marginal utilities of wealth at the cross-sectional equilibrium.

Mathematically, the aforementioned social welfare function W can be presented as:

$$W(u_1(\mathbf{x}_1(e,v)),\ldots,\ u_m(\mathbf{x}_m(e,v))) = \sum_{i=1}^{m} w_i u_i(\mathbf{x}_i(e,v)) \qquad (8)$$

where

$$w_i = \frac{1}{l_i^*} \quad and \quad l_i^* = \frac{\partial u_i(\mathbf{x}^*(e,v))}{\partial y_i} \qquad (9)$$

In equation (9), the asterisk denotes economic variables at cross-sectional capital market equilibrium. Assuming diminishing marginal utility of capital endowment, investors with larger capital endowments would generally have lower marginal utilities of capital endowment at capital market equilibrium, hence larger weights in the aforementioned social welfare function. In other words, larger investors' investment utilities, which reflect their financial needs, risk concerns, risk tolerances, and asset valuations[7], would have greater influence on competitive capital market equilibrium in general. This should not be a surprising result since larger investors have greater capital endowment, hence greater influence on the capital market's aggregate supply and demand for assets.

Labeling $w_i = 1/l_i^*$ as the "**wealth factor**" of any investor *i*, we can prove the following Static General Capital Asset Pricing Model (GCAPM) Theorem under CAPM's illustrative world.

The Static GCAPM Theorem[8]: In a competitive capital marketplace, cross-sectional capital asset prices reflect investors' wealth-factor-weighted marginal utilities of assets, i.e.

$$p_j = \sum_{i=1}^{m} w_i \frac{\partial u_i(\mathbf{x}_i(e,v))}{\partial x_j}; \quad \forall j \qquad (10)$$

Note that, in equation (10), p_j refers to any transient cross-sectional asset prices, not only the cross-sectional equilibrium asset prices.

B. The Static CAPM Paradigms

Applying partial differentiation on equation (10) with respect to "expected return" and "volatility risk", we have the following Corollary, which presents CAPM's security market line in a three-dimensional space.

Corollary: The Static CAPM Theorem[9]: Under CAPM's illustrative world, capital asset prices reflect investors' wealth-factor-weighted marginal utilities of assets' expected return and assets' market covariant volatility risk, which is defined as the expected price covariance between asset and market's risky portfolio.

Mathematically, the Corollary can be expressed as

$$p_j = a_m e_j - t_m C_{jm}$$
$$where \quad a_m = \sum_{i=1}^{m} w_i a_i \quad and \quad t_m = \sum_{i=1}^{m} 2 w_i t_i \tag{11}$$

In the first equation (11), $a_m > 0$ and $t_m > 0$ denote the capital market's unit price premium for asset expected return and market covariant volatility risk, which are denoted by e_j and C_{jm}, respectively. In the second equation (11), a_i and t_i denote individual investors' marginal utility of expected return and volatility risk, respectively. Equation (11) tells us that (1) the capital market's unit price premium for expected return equals the wealth-factor-weighted individual investor marginal utility of expected return and (2) the capital market's unit price premium for market covariant volatility risk equals twice that of the wealth-factor-weighted individual investor marginal utility of volatility risk.

Alternatively, equation (11) can be expressed in the following familiar CAPM form[10]:

$$p_j = a_m r_f + (a_m e_m - t_m V_m - a_m r_f) b_{jm}$$
$$or \quad p_j = p_{g_f} + (p_m - p_{r_f}) b_{jm}$$
$$where$$
$$r_f : \text{the riskfree interest rate}$$
$$b_{jm} : \text{ asset } j\text{'s beta risk; } \; b_{jm} = \frac{C_{jm}}{V_m} \tag{12}$$
$$e_m : \text{ the expected return of market's risky potfolio}$$
$$V_m : \text{ the volatilit y risk of the market's risky portfolio}$$
$$p_{g_f} : \text{ the price of the risk free security; } \; p_{g_f} = (a_m r_f)$$
$$p_m : \text{ the price of market's risky portfolio; } \; p_m = (a_m e_m - t_m V_m)$$

The following Exhibit 1 graphically presents equation (12) in a three-dimensional space. It shows that asset price has a linear relationship with beta risk similar to the relationship between asset expected return and beta risk. However, the 3-D security market line in Exhibit 1 could very well be downward sloping with respect to the price axis, when investors prefer the market's risk-free security to the market's risky portfolio – mathematically speaking, when $(a_m e_m - t_m V_m) < p_{g_f}$, the security market line will be downward sloping in the price-beta space. Hence, higher-beta-risk assets, which will carry higher expected returns, may not carry higher asset prices.

At first glance, equation (12) does not seem to be realistically modeling real-world asset prices – according to equation (12) and Exhibit 1, assets that have the same beta risk will have the same price. In the real world, assets with the same beta risk could have very different prices. The truth is that CAPM does not observe the per share property of assets – both expected return and volatility risk are measured in the unit of percentage, which can apply to assets of any size. As a consequence, equation (12) prices assets in a hypothetical share unit where all assets have the same size of "per share" expected earnings (or cash flow) stream. To more realistically apply equation (12), one needs to prorate asset prices in proportion to the size of their per share expected earnings stream. For instance, a risk-free asset with $10,000 principal value should have a price twice that of a risk-free asset with $5,000 principal value. Similarly, if two assets have the same beta risk, which will also have the same expected return as mandated by CAPM's traditional 2-D security market line, an asset with $4 per share of earnings should have a price twice that of an asset with $2 per share of earnings.

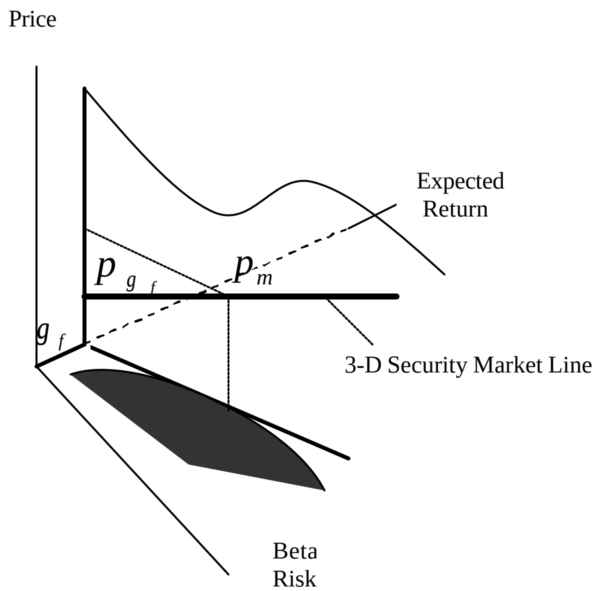

Exhibit 1: 3-D Security Market Line

C. The Dynamic CAPM Paradigm

Taking partial derivatives with respect to time on equation (11), we have the following Dynamic CAPM Theorem:

The Dynamic CAPM Theorem[11]: Under CAPM's illustrative world, asset price changes could reflect (1) changes in market's unit premium for expected return, (2) changes in individual asset's expected return, (3) changes in capital market's unit premium for market covariance risk, and (4) changes in individual asset's market covariant risk.

Mathematically, the Dynamic CAPM Theorem can be described in the following partial derivative equation,

$$\frac{\partial p_j^*}{\partial t} = \left(\frac{\partial a_m}{\partial t} e_j + a_m \frac{\partial e_j}{\partial t}\right) - \left(\frac{\partial t_m}{\partial t} C_{jm} + t_m \frac{\partial C_{jm}}{\partial t}\right) \qquad (13)$$

The Dynamic CAPM Theorem tells us that, even under CAPM's extremely simplified world, asset price changes are a very complex matter. First, if a market's unit price premium for expected return changes, holding all other asset-pricing attributes constant, all asset prices will change. Since a market's unit price premium, as defined in equation (11), equals a market's wealth-factor-weighted average of individual investors' marginal utility of expected return, its change may reflect many market events -- changes in investor wealth, investor wealth distribution, investor composition (e.g. foreign capital flow), investor financial needs, investor demographic, etc. Second, if a market's unit price premium for market covariant risk changes, holding all other asset-pricing attributes constant, all asset prices will change. Again, its change may reflect many market events -- changes in investor wealth, investor wealth distribution, investor composition, and investor risk tolerance, investor demographic, etc. Third, if a market's consensus on an individual asset's expected return changes, holding all other asset-pricing attributes constant, that individual asset price will change. Again, many economic factors may cause the consensus of individual asset's expected return to change. From the asset's macro environment point of view, events such as war, recession, inflation, regulatory environment changes, interest policy changes, etc. could affect a market's consensus on individual asset's expected return. From the asset-specific event point of view, events such as asset profitability, asset management team change, asset product-offering change, asset product competition change, financial stress, etc. could also affect a market's consensus on an asset's expected return. Fourth, if a market's consensus on an individual asset's market covariant risk changes, holding all other asset-pricing attributes constant, that individual asset's price will change. Again, many factors could drive such a consensus change. At the macro level, similar events such as war, recession, inflation, regulatory environment change, interest policy changes, etc. could affect a market's consensus on individual asset's (market covariant) risk. At the asset-specific level, events such as asset management team change, asset credit worthiness change, financial stress, etc. could cause a market's consensus on that assets' (market covariant) risk to change.

Since changes of asset prices define asset ex-post returns, we can easily deduce the following Asset Ex-Post Return Theorem from the Dynamic CAPM Theorem.

The CAPM Asset Ex-Post Return Theorem Holding other asset-pricing attributes constant, an asset price will increase, which will produce a positive ex-post return, only if (1) the asset's expected return increases, (2) the asset's market covariant risk decreases, (3) the capital market's unit expected return premium increases, or (4) the capital market's unit beta risk premium decreases; and vice versa.

The Ex-Post Asset Return Theorem is a natural by-product of the Dynamic CAPM Theorem. From equation (11), we know that an asset's price equals the asset's total expected return premium minus the asset's total market covariant risk premiums. Hence, anything that increases an asset's total expected-return premium will increase the asset's price and produces a positive ex-post asset return; and vice versa. In the same token, anything that increases an asset's total beta risk premium will decrease the asset's price and produce a negative ex-post asset return; and vice versa.

The Asset Ex-Post Return Theorem can also be presented in the following relative form:

The CAPM Asset Ex-Post Relative Return Theorem Holding other asset-pricing attributes constant, an asset price will increase more than the other asset prices only if (1) the asset's expected return increases more than the other assets', (2) the asset's market covariant risk decreases more than the other assets', (3) the capital market's unit expected return premium increases and the asset has higher expected return than other assets do, or (4) the capital market's unit beta risk premium decreases and the asset has higher market covariant risk than other assets do; and vice versa.

D. The CAPM Misinterpretation

The traditional CAPM is a cross-sectional asset-pricing model calibrated by two cross-sectional variables – ex-ante asset expected return assessment and ex-ante asset volatility risk assessment. Since neither of those two cross-sectional variables is readily observable, economists have to surrogate them with other more easily observable variables in their empirical tests. For the cross-sectional beta risk measurement, economists frequently use leading-period or lagging-period asset/market return regressions coefficient to measure it. This is a very popular practice because the mathematical definition of beta risk, as specified in the second notation item of equation (12), is very similar to that of a regression coefficient between asset return and market return. For expected return measurement, economists' most popular surrogate is an asset's following-period ex-post returns.

Categorically, most empirical tests of CAPM tried to prove the following three hypotheses (Fama, 1991): (1) cross-sectional expected returns are a positive linear function of cross-sectional beta risk, (2) beta is the only risk measure needed to explain the cross-sectional asset expected returns, and (3) which is a less-ambitious version of hypothesis (2) – whether cross-sectional beta risk can explain any asset ex-post returns at all, i.e. whether beta risk is relevant to asset pricing at all.

On hypothesis (1), by surrogating assets' expected returns with assets' following period ex-post returns, economists are actually testing "whether asset ex-post returns are a positive linear function of cross-sectional beta risk", which is a misinterpretation of CAPM. From the two Asset Ex-Post Return Theorems, we know that, as far as beta risk is concerned, it is the change of beta risk – not the level of ex-ante beta risk – that determines asset ex-post returns. Hence, higher-beta-risk assets will not necessarily generate higher or lower ex-post returns than lower-beta-risk assets. Not surprisingly, after surrogating assets' expected returns with assets' following period ex-post returns, most empirical tests on hypothesis (1) were negative (Fama and French, 1991; et al)[12].

On hypothesis (2), by surrogating asset following-period ex-post returns as asset expected returns in the hypothesis testing, economists are actually testing "whether beta is the only risk measure needed to explain asset ex-post returns", which again is a misinterpretation of CAPM. According to the Ex-Post Return Theorems, beta risk is only one of the four categorical factors that drive asset ex-post returns – not to mention that it is the change of beta risk -- not the level of beta risk -- that drives asset ex-post returns. Furthermore, as analyzed in the paragraph after the Dynamic CAPM Theorem, there is a whole school of economic factors behind each category of asset ex-post return drivers. Not surprisingly, economists (Basu, 1977 and 1983; Banz, 1981; Bhandari, 1988; Chan, Hamao, Lakonishok, 1991; Fama and French, 1992 & 1996) can find many non-beta-risk factors that can explain asset ex-post returns. As a matter of fact, few economists have already started to investigate how these so-called anomaly factors related to the four categories of CAPM's ex-post return drivers (Petkova and Zhang, 2003; Zhang, 2004).

As to hypothesis (3), it is a scaled-down version of the misinterpreted hypothesis (2). Given that economists have found so many non-beta-risk factors that can explain asset ex-post returns, they wonder if the beta risk is relevant at all in asset pricing. As stated before, it is the "change" of beta risk -- not the "level" of beta risk -- that drives asset ex-post returns; one should not be surprised to find that the level of beta risk does not explain asset ex-post returns at all (Fama and French, 1991; et al). Actually, by using multiple-period regression coefficients to measure beta risk, most economists have reinforced their negative empirical test results. A multiple-period regression coefficient captures the "average" or "invariant" portion of covariance between two variables. Since it is the "change" of beta risk that drives ex-post asset returns, regression coefficient, which captures the "invariant" portion of beta risk, will re-enforce the negative results of hypothesis (3) testing[13].

E. The validity of CAPM

Since the two ex-ante variables of the traditional CAPM are not readily observable, the best way to test the validity of CAPM is through the newly derived three-dimensional CAPM, especially through its two Asset Ex-Post Return Theorems.

While more rigorous empirical tests are required to claim the validity of CAPM, since CAPM is such an intuitively appealing theory, one shouldn't be surprised that one can literally observe CAPM's two ex-post return theorems at work daily in the capital marketplace. For instance, on an individual asset basis, we have frequently observed that when an asset experiences an upward expected earnings revision, which will increase the asset's expected return, its price will rise and vice versa. Equally frequently, we have observed that when an asset's beta risk premium increases (either because the asset becomes more risky or investors become more risk averse), its price will fall and vice versa. On the macro level, we also frequently observe that when a market experiences a major negative macro events such as war, recession, social turmoil, etc., which tend to increase the market's unit premium for beta risk, almost all risky (non-zero beta risk) asset prices will fall and lead to a major market downturn; and vice versa.

Furthermore, despite their misinterpretation of CAPM in the ex-post world, many academic empirical studies mentioned in this paper have provided non-contradicting evidences supporting the Asset Ex-Post Return Theorems: (1) higher-beta-risk assets will not necessarily generate higher ex-post returns, (2) the level of ex-ante beta risk does not explain asset ex-post returns[14], and (3) beta risk is not the only source for asset ex-post returns. In addition, some academic studies (Wang and Jagannathan, 1996; Lettau and Ludvigson, 2001; Wang, 2002; Bansal and Yaron, 2004) have shown evidence that time-varying economic risks, i.e. changes of beta risk, did explain asset ex-post returns.

III. Conclusion

Few economists have misinterpreted CAPM as a theory in its original ex-ante variables. In practices and in academic empirical studies, however, CAPM has been grossly misinterpreted in terms of its relationship with ex-post asset returns.

This paper derives a new CAPM equilibrium under the neoclassical framework and adds asset price, whose variation defines asset ex-post returns, as a new dimension to the traditional CAPM. The new equilibrium does not only explain how CAPM's security market line behaves in the ex-ante cross-sectional price space, but also provides a Dynamic CAPM Theorem and two Asset Ex-Post Return Theorems to illuminate CAPM paradigms in the ex-post world. The newly derived Theorems show that the empirical evidence that caused a premature pronouncement of CAPM's demise are actually consistent with CAPM's ex-post return paradigms: (1) higher-beta-risk assets will not necessarily generate higher ex-post returns, (2) except on some rare occasions, the level of ex-ante beta risk does not explain asset ex-post returns, and (3) beta risk is not the only source for asset ex-post returns.

CAPM is a parsimonious model established in an illustrative world. While its doctrine is rich and instructive, it provides few links to the real world and is very susceptible to misinterpretations. This paper has clarified a few of the most critical CAPM misinterpretations. Extending this paper's three-dimensional CAPM equilibrium, which is a special case of a more general equilibrium, into a heterogeneous world without CAPM's strenuous assumptions will shed even more light on how CAPM doctrines work in the real world[15].

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[1] For the general case of GCAPM, see "GCAPM I: A Microeconomic Theory of Investments (Fan, 1994-2004)," earlier versions of the paper are available at www.ssrn.com website. The 2004 version will be available soon at the same website.
[2] Note that, under totally different setup, Markowitz (1990) had also derived a version of CAPM equilibrium with a price dimension. Unfortunately, the model did not attract enough attention at that time.

[3] <proof> From the first order condition of equation (3), we have

$$\partial u_i(\mathbf{x}(e,v)) \,/\, \partial x_j = l_i \, p_j, \qquad \forall \; x_j >> 0 \tag{A1}$$

where the variable l_i is known as the Lagrange Multiplier, which reflects investor i's marginal utility of wealth. Dividing the first order condition of the j-th asset by that of the k-th asset, the Theorem is proved.

[4] Since cash is an asset holding itself, investors' budget constraints in equation (3) are an equality $\mathbf{p.x} = y_i$ instead of quasi-equality $\mathbf{p.x} \le y_i$. Through the equality constraint, we can derive each investor's demand function of assets in terms of investors' wealth endowment under a set of given market prices. First, let's express investors' investment utilities as functions of asset prices $\mathbf{p}$ and their wealth endowment y through the equality constraint:

$$\begin{aligned} v_i(\mathbf{p}, y_i) = \max \quad & u_i(\mathbf{x}(e, v)) \\ s.t. \quad & \mathbf{p.x} = y_i \\ & \mathbf{x} \in X \end{aligned} \tag{A2}$$

The new utility functions $v_i(\mathbf{p}, y_i)$ are investor i's **indirect utility function** because they are not in terms investors' direct preference attributes – expected wealth and wealth volatility. The indirect utility function allows us to derive each investor's optimal asset holdings for a given set of price through the following Proposition.

Proposition[4]: (Roy's Identity) Let $\mathbf{x}_{ij}(\mathbf{p}, y_i)$ denote investor i's Marshallian asset demand function for asset j, then

$$x_{ij}(\mathbf{p}, y_i) = \frac{\dfrac{\partial v_i(\mathbf{p}, y_i)}{\partial p_j}}{\dfrac{\partial v_i(\mathbf{p}, y_i)}{\partial y_i}} \qquad \forall i \;\; and \;\; j \tag{A3}$$

provided that the right-hand side of equation (7) is well defined, i.e. $|p_j| >> 0$ and $y_i >> 0$.

[5] <proof> see Arrow and Debreu, 1954.

[6] <Proof> From the following social welfare function maximization

$$\begin{aligned} \max \quad & \sum_{i=1}^{m} w_i u_i(\mathbf{x}_i) \\ s.t. \quad & \sum_{i=1}^{m} \mathbf{x}_i - \mathbf{w}_i = \mathbf{0} \\ & \mathbf{x}_i \in X \quad \forall i \end{aligned} \tag{A4}$$

, we have the following first order condition

$$w_i \mathbf{D} u_i(x_i^*) = \mathbf{q} \tag{A5}$$

One natural solution to equation (A5) is to set the vector $\mathbf{q}$ equal to the equilibrium price vector $\mathbf{p}$ and set w_i equal the reciprocal of individual investor's marginal utility of capital endowment at the capital market equilibrium λ^*. From equation (A1), we know that equation (A5) is satisfied. (Q.E.D.)

[7] Since CAPM assumes homogeneous assessments of asset volatility risk and expected return, this will not be an issue here (See the general case of GCAPM for a discussion in the heterogeneous world).

[8] <Proof> Taking the first-order, partial derivative with respect to asset j on equation (8), we have

$$\frac{\partial W(u_1(\mathbf{x}_1),...,u_m(\mathbf{x}_m))}{\partial x_j} = \sum_{i=1}^{m} w_i \frac{\partial u_i(\mathbf{x}_i)}{\partial x_j} \tag{A6}$$

Substitute equation (A1) into the right-hand side of the above equation, we have

$$\sum_{i=1}^{m} w_i \frac{\partial u_i(\mathbf{x})}{\partial x_j} = \sum_{i=1}^{m} w_i l_i p_j = p_j \sum_{i=1}^{m} w_i l_i \tag{A7}$$

From the Second Welfare Theorem, we know that $w_i = 1/\lambda_i^*$, where λ_i^* is investor i's marginal utility of wealth at the capital market equilibrium:

$$\sum_{i=1}^{m} w_i \frac{\partial u_i(\mathbf{x})}{\partial x_j} = p_j \sum_{i=1}^{m} w_i l_i = p_j \sum_{i=1}^{m} \frac{l_i}{l_i^*} \tag{A8}$$

Re-arranging equation (A8) and let

$$k = \frac{1}{\sum_{i=1}^{m} \frac{l_i}{l_i^*}} \tag{A9}$$

we have the following Equation:

$$p_j = k \sum_{i=1}^{m} w_i \frac{\partial u_i(\mathbf{x})}{\partial x_j}; \quad \forall j \tag{A10}$$

Since, during the capital market equilibrium process, investors are at least as well off as they were before the capital market equilibrium, we know that k is a bounded positive variable, i.e.

$$l_i \geq l_i^* > 0 \quad ; \quad \forall i \quad or \quad \frac{1}{m} \geq k > 0 \tag{A11}$$

where m is the number of investors in the capital market. Since utility functions are scalable by positive constants, scaling investors' utility function by the positive variable k, let

$$p_j = k \sum_{i=1}^{m} w_i \frac{\partial u_i(\mathbf{x})}{\partial x_j} = \sum_{i=1}^{m} w_i \left(\frac{\partial (k u_i(\mathbf{x}))}{\partial x_j}\right) = \sum_{i=1}^{m} w_i \frac{\partial u_i'(\mathbf{x})}{\partial x_j} \quad ; \quad \forall j \tag{A12}$$

(Q.E.D.)

Note that during the equilibrium process

$$l_i \to l_i^* \quad \forall\, i\,; \quad k = \frac{1}{\sum_{i=1}^{m} \frac{l_i}{l_i^*}} \to \frac{1}{m} > 0 \; and \; p_j \to p_j^* \quad \forall\, j \tag{A13}$$

Throughout the equilibrium process, the same scaling factor k applies across all investor utilities, i.e. there are no biases in the utility scaling. One can interpret k as a currency conversion factor between asset prices and investor's raw utility units. As asset prices converge to the equilibrium prices, k is converging to $1/m$. Since k is converging to a lower value $1/m$ during a market equilibrium process, each dollar is worth more and more in terms of investors' raw utility units during a capital market equilibrium process. However, the wealth factor w_i will also change over the equilibrium process. At the beginning of equilibrium process, larger investors have larger wealth factor to begin with due to their lower marginal utility of wealth. At the end of equilibrium, those investors who become richer during the equilibrium process will increase their wealth factor more than the others.

[9] <Proof> From equation (10), we can apply the following partial differentiation

$$p_j = \sum_i^m w_i \frac{\partial u_i(\mathbf{x}_i(e,v))}{\partial e} \frac{\partial e_{\mathbf{x}_i}}{\partial x_j} + \sum_i^m w_i \frac{\partial u_i(\mathbf{x}_i(e,v))}{\partial v} \frac{\partial v_{\mathbf{x}_i}}{\partial x_j} \tag{A14}$$

Let $a_i = \frac{\partial u_i(\mathbf{x}_i(e,v))}{\partial e} > 0 \; and \; t_i = -\left(\frac{\partial u_i(\mathbf{x}_i(e,v))}{\partial v}\right) > 0$ denote investor i's marginal utility of expected return and volotility risk, respectively; equation (A14) can be simplified as

$$p_j = \sum_i^m w_i a_i \frac{\partial e_{\mathbf{x}_i}}{\partial x_j} - \sum_i^m w_i t_i \frac{\partial v_{\mathbf{x}_i}}{\partial x_j} \tag{A15}$$

In equation (A14) and (A15), $\frac{\partial e_{\mathbf{x}_i}}{\partial x_j} \; and \; \frac{\partial v_{B\mathbf{x}_i}}{\partial x_j}$ denote asset j's contribution to portfolio $\mathbf{x}_i$'s expected return and volatitlity risk. Under CAPM's illustrative world, we can express the expected return and volatility of portfolio $\mathbf{x}_i$ as

$$e_{\mathbf{x}_i} = \sum_{j=1}^{m} x_j e_j \qquad and \qquad v_{B\mathbf{x}_i} = \sum_{j=1}^{m} \sum_{k=1}^{m} x_j x_k \, r_{jk} \, s_j \, s_k \tag{A16}$$

where $e_j \; and \; s_j$ are asset j's expected return and standard deviation risk, respectively and r_{jk} denote the correlation between asset j and asset k. Carry out the partial derivatives, we have

$$\frac{\partial e_{\mathbf{x}_i}}{\partial x_j} = e_j \quad and \quad \frac{\partial v_{\mathbf{x}_i}}{\partial x_j} = 2C_{j\mathbf{x}_i} \tag{A17}$$

where $C_{j\mathbf{x}_i}$ denotes asset j's covariance with portfolio $\mathbf{x}_i$. Applying CAPM's assumptions, Sharpe, Linter, et al, proved that all investors would hold the same optimal risky portfolio – the market's risky portfolio $\mathbf{x}_m$ (Sharpe, 1964), i.e.

$$\mathbf{x}_i = \mathbf{x}_m \quad and \quad C_{j\mathbf{x}_i} = C_{jm} \quad \forall i \tag{A18}$$

Hence, equation (A15) can be expressed as

$$p_j = a_m e_j - t_m C_{jm} \tag{A19}$$

and $$a_m = \sum_{i=1}^{m} w_i a_i \ and \quad t_m = 2\sum_{i=1}^{m} w_i t_i \tag{A20}$$

(Q.E.D.)

[10] In CAPM (Sharpe, 1964), investors are assumed to have the following standard mean-variance form of utility:

$$u_i(\mathbf{x}_i(e,v)) = e_{\mathbf{x}_i} - t_i v_{\mathbf{x}_i} \tag{A21}$$

With the utility functional form, Sharpe proved the following famous security market line:

$$e_j = r_f + (e_m - g_f) b_{jm}, where\ b_{jm} = \frac{C_{jm}}{V_m} \tag{A22}$$

Plugging equation (A22) into equation (A19), we have

$$\begin{aligned} p_j &= a_m r_f + (a_m e_m - t_m V_m - a_m r_f) b_{jm} \\ &= p_{g_f} + (p_m - p_{r_f}) b_{jm} \end{aligned} \tag{A23}$$

Note that $p_{g_f} = a_m r_f$ and $p_m = (a_m e_m - t_m V_m)$.

[11] The Theorem is a straightforward description of the partial differentiation of equation (11) with respect to time, as shown in equation (13).

[12] The level of ex-ante beta risk will drive asset ex-post returns proportionally, only when the market's unit beta-risk premium changes and all other asset-pricing attributes stay constant during the empirical test period. According to the second Asset Ex-Post Return Theorem, holding other things constant, when the market's unit beta risk premium increases, higher-beta risk assets will generate "lower" ex-post returns and vice versa – still, higher-beta-risk assets will not always generate "higher" ex-post returns.

[13] See footnote 12.

[14] Except for the rare occasion mentioned in footnote 12.

[15] See footnote 1 for reference.