P56 13.

$f(1) = 1^3 - 1 - 1 = -1 < 0$

$f(2) = 2^3 - 2 - 1 = 5 > 0$

$\therefore f(1) \cdot f(2) < 0$

$\therefore |p_n - p| < \frac{b-a}{2^n}$

不断代入可得 $p_{14} = 1.32477$

$|p_n - p| < \frac{b-a}{2^n} < 10^{-4}$

$\therefore 2^n > 1/10^{-4} = 10^4$

$\therefore n > \log_2 10^4 = 4\log_2 10 \approx 13.3$

$\therefore n_{min} = 14$

15. $p_n = \sum_{k=1}^{n} 1/k$

首先 $p_{n-1} = \sum_{k=1}^{n-1} 1/k$

$\therefore p_n - p_{n-1} = 1/n \Rightarrow \lim_{n\to\infty} (p_n - p_{n-1}) = \lim_{n\to\infty} 1/n = 0$

但是 p_n 依旧发散

$p_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + 1/2 + 1/3 + 1/4 \cdots + \frac{1}{2^n} \cdots + \frac{1}{2^{n+1}}$

$= 1 + 1/2 + (1/3 + 1/4) \cdots + \frac{1}{2^n} + \underbrace{\cdots + \frac{1}{2^{n+1}}}_{2^n}$

$> 1 + 1/2 + \frac{1}{4} \times 2 \cdots + \frac{1}{2^n} \cdot 2^{n-1} \cdots + \frac{1}{2^{n+1}} \cdot 2^n \cdots$

$> 1 + 1/2 \cdot n = n/2 + 1$

$\therefore$ 随 $n\uparrow$ $S(n)\uparrow$ $\therefore \sum_{k=1}^{n} 1/k$ 之和没有极限

$\therefore p_n = \sum_{k=1}^{n} \frac{1}{k}$ 发散

P64. 3.

a). $p_n = \frac{20}{21} p_{n-1} + \frac{1}{p_{n-1}^2}$ b). $p_n = \frac{2}{3} p_{n-1} + \frac{7}{p_{n-1}^2}$

c). $p_n = p_{n-1} - \frac{p_{n-1}(p_{n-1}^3 - 21)}{p_{n-1}^2 - 21}$

d). $p_n = \frac{\sqrt{21}}{\sqrt{p_{n-1}}}$

$1 \le P_n \le 21^{1/3}$ $\because P_0 = 1$

a) $\therefore P_1 = \frac{20}{21} + 1 = \frac{41}{21}$

$g_a' = \left|\frac{20}{21} - 2\frac{1}{P_{n-1}^3}\right|$ $g_a' \uparrow$ $\therefore g_a'max = \left|\frac{20}{21} - 2 \cdot \frac{1}{21}\right| = \left|\frac{18}{21}\right| = 6/7 \le 0.857 = k_a$

b). $g_b' = \frac{2}{3} - 14/P_{n-1}^3$ $g_b' \uparrow$ $g_b' = \left|\frac{2}{3} - 14/21\right| = 0$

$1 \le P_n \le 21^{1/3}$ $g_b'max = \left|\frac{2}{3} - 14/\left(\frac{23}{3}\right)^3\right| \le 0.636 = k_b$

$P_1 = \frac{23}{3}$

$g_b' = \left|\frac{2}{3} - 14/P_{n-1}^3\right|$

c). $P_n = P_{n-1} - \frac{P_{n-1}(P_{n-1}^3 - 21)}{P_{n-1}^2 - 21}$ $\because P_0 = 1$ $\therefore P_1 = 0$

$= P_{n-1}\left(1 - \frac{P_{n-1}^3 - 21}{P_{n-1}^2 - 21}\right)$ 又$\because P_1 = 0$ $\therefore P_n = 0$

$\therefore P_n$恒为0、$\therefore$ C方法不收敛于$21^{1/3}$

$= P_{n-1} \cdot \frac{P_{n-1}^2 - P_{n-1}^3}{P_{n-1}^2 - 21}$

$= \frac{P_{n-1}^3(1 - P_{n-1})}{P_{n-1}^2 - 21}$

d). $g_d = \left(\frac{21}{P_{n-1}}\right)^{1/2} = -\frac{1}{2} \cdot \frac{\sqrt{21}}{P_{n-1}^{3/2}}$ $\therefore g_d'max = \left|\frac{\sqrt{21}}{2} \cdot \frac{1}{(21^{1/3})^{3/2}}\right| \le 0.5 = k_d$

$|g_d'| = \left|\frac{\sqrt{21}}{2} \cdot \frac{1}{P_{n-1}^{3/2}}\right|$

$P_1 = \sqrt{21}$ $\therefore k_d > k_b > k_a$

$g_d'' = \frac{1}{2} \cdot \frac{21}{P_{n-1}^{3/2}} \cdot \frac{3}{2}$ $\therefore$ d快于b快于a

$\therefore g_d' \uparrow$ C不收敛于 $21^{1/3}$

P65. 19

a). $x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}$

$g(x) = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}$ $\quad g'(x_{n-1}) = \frac{1}{2} - \frac{1}{x_{n-1}^2}$

$g'(x_0) > \frac{1}{2} - \frac{1}{(\sqrt{2})^2} = 0$ $\quad \therefore g(x_0)\uparrow$

$x_1 > \frac{1}{2}x_0 + \frac{1}{x_0} = \frac{1}{2}\sqrt{2} + \frac{1}{\sqrt{2}} = \sqrt{2}$ $\quad \therefore x_1 > \sqrt{2}$

设 $x_k > \sqrt{2}$

则 $x_{k+1} > \frac{1}{2}\sqrt{2} + \frac{1}{\sqrt{2}}$ (同 x_0. x_1 时分析导数)

$\therefore x_n > \sqrt{2}$

$\because x_n > \sqrt{2}$ $\quad \therefore g'(x) = \frac{1}{2} - \frac{1}{x_{n-1}^2} \in (0, \frac{1}{2})$

$\therefore \exists k \in (0, \frac{1}{2})$ 使 $|g'(x)| < k$ 恒成立

$\therefore \exists$ 不动点 使 $p_n = g(p_{n-1})$ 成立

$x = \frac{1}{2}x + \frac{1}{x} \Rightarrow x = \sqrt{2}$

$\therefore$ 收敛于 $\sqrt{2}$

b). $0 < x_0 < \sqrt{2}$

$g'(x) = \frac{1}{2} - \frac{1}{x_{n-1}^2}$

$\therefore 0 < x < \sqrt{2}$ 时 $g'(x) < 0$ $\quad g(x)\downarrow$

$\because x_0 \in (0, \sqrt{2})$

$g(x_0) > g(\sqrt{2}) = \frac{1}{2}\sqrt{2} + \frac{1}{\sqrt{2}} = \sqrt{2}$

$\therefore x_1 = g(x_0) > \sqrt{2}$

c). 由 (a) 得若 $x_0 > \sqrt{2}$. 则数列收敛于 $\sqrt{2}$

由 (b) 得若 $\sqrt{2} > x_0 > 0$ 则 $x_1 > \sqrt{2}$

数列的前有限项不影响其收敛性与收敛值.

$\therefore$ 对 $\forall x_0 > 0$. 数列均收敛于 $\sqrt{2}$.