PDF and firing rate of a LIF neuron

Reference: Brunel & Hakim (1999)

The membrane potential $V(t)$ of a LIF neuron is governed by

$$\tau_m \dot{V}(t) = -V(t) + RI(t), \tag{1}$$

where the input synaptic current

$$RI(t) = \tau_m J_E \sum_j \delta(t - t_j) - \tau_m J_I \sum_k \delta(t - t_k). \tag{2}$$

$\tau_m = RC$ is the membrane time constant. R and C are the membrane resistance and capacitance, respectively. J_E (J_I) is the amplitude of an excitatory (inhibitory) post-synaptic potential, whereas t_j (t_k) represents the time of the jth (kth) excitatory (inhibitory) input spike. When $V(t) = \theta$, $V(t)$ is reset to V_r and a pause for synaptic integration τ_r is imposed to mimic the refractory period. In the high input regime, the sum of synaptic inputs to a neuron can be approximated by a fluctuating input noise:

$$RI(t) \equiv \tau_m[\mu + \sigma\eta(t)], \tag{3}$$

where

$$\mu = J_E \nu_E - J_I \nu_I, \tag{4}$$
$$\sigma = \sqrt{J_E^2 \nu_E + J_I^2 \nu_I}. \tag{5}$$

$\eta(t)$ is a white noise random process such that $\langle \eta(t)\eta(t')\rangle = \delta(t - t')$. ν_E (ν_I) is the firing rate of the excitatory (inhibitory) input.

Normally μ and σ are constant but here we assume they are functions of V. After rescaling, we have

$$\dot{V}(t) = -\frac{V(t)}{\tau_m} + \mu(V) + \sigma(V)\eta(t) \tag{6}$$

and the associated Fokker-Planck equation is

$$\frac{\partial P(V,t)}{\partial t} = \frac{\partial}{\partial V}(\frac{V}{\tau_m} - \mu(V))P(V,t) + \frac{\sigma(V)^2}{2}\frac{\partial^2 P(V,t)}{\partial V^2}. \tag{7}$$

The continuity equation gives

$$\frac{\partial P(V,t)}{\partial t} = -\frac{\partial \Phi(V,t)}{\partial V}, \tag{8}$$

in which $\Phi(V,t)$ denotes the flux,

$$\Phi(V,t) = -(\frac{V}{\tau_m} - \mu(V))P(V,t) - \frac{\sigma(V)^2}{2}\frac{dP(V,t)}{dV}. \tag{9}$$

Consider the boundary conditions (BC). First, an absorbing BC at the threshold $V = \theta$,

$$P(\theta, t) = 0, \tag{10}$$

and thus using Eq. (9),

$$\frac{\partial P(\theta, t)}{\partial V} = -\frac{2\nu(t)}{\sigma(\theta)^2}. \tag{11}$$

$\nu(t) = \Phi(\theta, t)$ is the output rate of the neuron. At the reset V_r,

$$\frac{\partial P(V_r^+, t)}{\partial V} - \frac{\partial P(V_r^-, t)}{\partial V} = -\frac{2\nu(t)}{\sigma(\theta)^2}. \tag{12}$$

The normalization condition gives

$$\int_{-\infty}^{\theta} P(V, t) dV = 1. \tag{13}$$

In addition, $P(V, t)$ is continuous and satisfies

$$\lim_{V \to -\infty} P(V, t) = 0, \tag{14}$$

and

$$\lim_{V \to -\infty} V P(V, t) = 0. \tag{15}$$

For steady state,

$$\begin{aligned} \frac{\partial P(V, t)}{\partial t} &= 0 & (16)\\ \frac{d}{dV}\big[(\frac{V}{\tau_m} - \mu(V))P(V) + \frac{\sigma(V)^2}{2}\frac{dP(V)}{dV}\big] &= 0. & (17)\end{aligned}$$

There is no net flux for $V < V_r$ in steady state. Net flux flows from θ to V_r and to θ again.

$$\frac{d}{dV}\big[P(V)\text{exp}\big(2\int^V \frac{V - \mu(V)\tau_m}{\sigma(V)^2\tau_m} dV\big)\big] = -\frac{2\nu\Theta(V - V_r)}{\sigma(V)^2}\text{exp}\big(2\int^V \frac{V - \mu(V)\tau_m}{\sigma(V)^2\tau_m} dV\big), \tag{18}$$

which is defined for $\sigma(V) \neq 0$. Integrate both sides from V to θ:

$$P(V)\text{exp}\big(2\int^V \frac{V' - \mu(V')\tau_m}{\sigma(V')^2\tau_m} dV'\big) = \int_V^\theta \frac{2\nu\Theta(V' - V_r)}{\sigma(V')^2}\text{exp}\big(2\int^{V'} \frac{V'' - \mu(V'')\tau_m}{\sigma(V'')^2\tau_m} dV''\big) dV' \tag{19}$$

$$P(V) = \text{exp}\big(-2\int^V \frac{V' - \mu(V')\tau_m}{\sigma(V')^2\tau_m} dV'\big)\int_V^\theta \frac{2\nu\Theta(V' - V_r)}{\sigma(V')^2}\text{exp}\big(2\int^{V'} \frac{V'' - \mu(V'')\tau_m}{\sigma(V'')^2\tau_m} dV''\big) dV' \tag{20}$$

If μ and σ do not depend on V,

$$P(V) = \frac{2\nu}{\sigma^2}\text{exp}\big(-\frac{(V - \mu\tau_m)^2}{\sigma^2\tau_m}\big)\int_V^\theta \Theta(V' - V_r)\text{exp}\big(\frac{(V' - \mu\tau_m)^2}{\sigma^2\tau_m}\big) dV'. \tag{21}$$

To obtain the output rate, we make use of the normalization condition Eq. (13) and we have

$$\begin{aligned}
\frac{1}{\nu} &= \frac{2}{\sigma^2}\int_{-\infty}^{\theta} dV \mathsf{exp}\big(-\frac{(V-\mu\tau_m)^2}{\sigma^2\tau_m}\big)\int_{V}^{\theta}\Theta(V'-V_r)\mathsf{exp}\big(\frac{(V'-\mu\tau_m)^2}{\sigma^2\tau_m}\big)dV' \qquad (22)\\
&= 2\tau_m\int_{-\infty}^{\frac{\theta-\mu\tau_m}{\sigma\sqrt{\tau_m}}} dv\int_{v}^{\frac{\theta-\mu\tau_m}{\sigma\sqrt{\tau_m}}} du\Theta(u-\frac{V_r-\mu\tau_m}{\sigma\sqrt{\tau_m}})e^{u^2-v^2} \qquad (23)\\
&= 2\tau_m\int_{\frac{V_r-\mu\tau_m}{\sigma\sqrt{\tau_m}}}^{\frac{\theta-\mu\tau_m}{\sigma\sqrt{\tau_m}}} due^{u^2}\int_{-\infty}^{u} dve^{-v^2} \qquad (24)
\end{aligned}$$

Since the integrand is symmetric about $v=0$,

$$\begin{aligned}
\frac{1}{\nu} &= 2\tau_m\int_{y_r}^{y_\theta} due^{u^2}\int_{0}^{\infty} dve^{-(v-u)^2} \qquad (25)\\
&= \tau_m\int_{0}^{\infty} dve^{-v^2}[\frac{e^{2y_\theta v}-e^{2y_r v}}{v}], \qquad (26)
\end{aligned}$$

where $y_\theta=\frac{\theta-\mu\tau_m}{\sigma\sqrt{\tau_m}}$.