Use proof by induction to prove the following statements:

- The sum of the reciprocals of squares from 1 to n is less than or equal to 2 - (1/n).

Base Case: n=1

$\frac{1}{1^2} \overset{?}{\leq} 2-(\frac{1}{1})$

$1 \overset{?}{\leq} 2-1$

$1 \leq 1$ ✓

Induction Step: Assume true for n=k

$\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{k^2} \leq 2-(\frac{1}{k})$

Consider n=k+1

$\underbrace{\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{k^2}}+\frac{1}{(k+1)^2} \leq 2-(\frac{1}{k+1})$

$\leq 2-(\frac{1}{k})+\frac{1}{(k+1)^2}$

$= 2-(\frac{1}{k+1})[\frac{k+1}{k}-\frac{1}{k+1}]$

$= 2-\frac{1}{k+1}(\frac{k^2+k+1}{k(k+1)}) \leq 2-\frac{1}{k+1}$

$2-\frac{k^2+k+1}{k(k+1)^2} \leq 2-\frac{1}{k+1}$ $\quad \frac{k^2+k+1}{k(k+1)^2} \leq \frac{1}{k+1}$ $\quad k(k+1)^2 \leq (k+1)(k^2+k+1)$

$k^2+1 \leq k^2+k+1$ ✓

- There are n people in a room and each person want to shake hands once with each other person in the room. How many handshakes occur? Prove this by induction.

$\frac{n(n-1)}{2}$ handshakes occur with n people in a room

Base Case: n=1

$\frac{1(1-1)}{2}=0$

no handshakes with only one person!

Induction Step: Assume true for n=k

$\frac{k(k-1)}{2}$

Consider n=k+1

$\frac{k(k-1)}{2}+k$ more handshakes since the new person shakes hands with everyone but themselves

$\frac{k(k-1)}{2}+\frac{2k}{2}=\frac{k^2-k+2k}{2}=\frac{k^2+k}{2}=\frac{k(k+1)}{2}=\frac{(k+1)(k+1-1)}{2}$ ✓ → # of handshakes using formula

- Letting x = the square root of n, prove that the sum of the reciprocals of positive square roots from 1 to n lies between x and (2x-1)

Base Case: n=1

$\sqrt{1} \leq \frac{1}{\sqrt{1}} \leq 2\sqrt{1}-1$

$1 \leq 1 \leq 1$ ✓

Induction Step: Assume true for n=k

$\sqrt{k} \leq \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{k}} \leq 2\sqrt{k}-1$

Consider n=k+1

$\sqrt{k+1} \leq \underbrace{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{k}}}+\frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1}-1$

$\sqrt{k+1} \leq \sqrt{k}+\frac{1}{\sqrt{k+1}}$

$\sqrt{k+1} \leq \frac{\sqrt{k^2+k}+1}{\sqrt{k+1}}$

$k+1 \leq \sqrt{k^2+k}+1$

$k \leq \sqrt{k^2+k}$ ✓

$2\sqrt{k}\cancel{-1}+\frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1}\cancel{-1}$

$\frac{2\sqrt{k^2+1}+1}{\sqrt{k+1}} \leq 2\sqrt{k+1}$

$2\sqrt{k^2+1}+1 \leq 2k+2$

$2\sqrt{k^2+1} \leq 2k+1$

$\sqrt{k^2+1} \leq \frac{2k+1}{2}$

$k^2+1 \leq \frac{4k^2+4k+1}{4}$

$k^2+1 \leq k^2+k+\frac{1}{4}$ ✓

- Prove that every third Fibonacci number is even.

Base Case: $F_1=1$

$F_2=1$

$F_3=2$ → even

Induction Step: Assume true for n=k

F_{3k-2} → odd

F_{3k-1} → odd $\qquad F_{3k}=F_{3k-2}+F_{3k-1}$

F_{3k} → even $\qquad$ even = odd + odd

Consider n=k+1

$F_{3(k+1)-2}=F_{3k+3-2}=F_{3k+1}=F_{3k-1}+F_{3k}$ = odd + even → odd

$F_{3(k+1)-1}=F_{3k+3-1}=F_{3k+2}=F_{3k}+F_{3k+1}$ = even + odd → odd

$F_{3(k+1)}=F_{3k+3}=F_{3k+3}=F_{3k+1}+F_{3k+2}$ = odd + odd

= even ✓

from assumption above

$3k+3=3(k+1)$

↳ multiple of 3 (every third Fibonacci number)

- Prove that each positive integer can be written as a sum of distinct Fibonacci numbers. (This question is a little harder.)

Induction Step: Assume true for n ≤ k

$F_{i_1}+F_{i_2}+F_{i_3}+\dots=k$

Consider =k+1

$F_{i_1}+F_{i_2}+\dots+F_{i_n}=k+1$

if k+1 is a Fibonacci number → k+1 can be written as a sum of distinct Fibonacci numbers (k+1) ✓

if k+1 isn't a Fibonacci number

let f_j be the largest Fibonacci number less than k+1

$f_j < k+1 < f_{j+1}$

$0 < \underline{k+1-f_j} < f_{j+1}-f_j$

↳ by assumption can be written as sum of distinct Fibonacci numbers

if you add f_j which is a Fibonacci number you get k+1 which therefore can also be written as the sum of distinct Fibonacci numbers

Base Case: n=1

1=1

↳ first Fibonacci # or second ✓

(If you like, you could try showing that the actual sum in the first exercise equals one sixth of the square of pi.)

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{1}{6}\pi^2$$