Sample Spaces, Events, Probability

Foundations of Data Analysis

January 17, 2019

Brain Teaser

You have cards numbered 1 through 10. If you pick five cards at random, what is the probability that you selected the cards 1, 2, 3, 4, 5 (not necessarily in that order)?

Sets

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$A = \{3, 8, 31\}$
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Not a valid set definition: $C = \{1, 2, 3, 4, 2\}$

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- The “empty” or “null” set has no elements:
  $$\emptyset = \{\ \}$$

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- Coin flip: $\Omega = \{H, T\}$
- Roll a 6-sided die: $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Pick a ball from a bucket of red/black balls: $\Omega = \{R, B\}$

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$$5 \in \mathbb{R}, \quad 17.42 \in \mathbb{R}, \quad \pi = 3.14159\ldots \in \mathbb{R}$$

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- Rationals:

$$\mathbb{Q} = \{\, p/q : p, q \in \mathbb{Z}, q \neq 0\}$$

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- $\emptyset \subseteq A$ for any set A

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  $\{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6\}$
- You flip a coin and it comes up "heads":
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- Your code takes longer than 5 seconds to run:
  $(5, \infty) \subseteq \mathbb{R}$

Set Operations: Union

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Note: If $A \cap B = \emptyset$, we say A and B are **disjoint**.

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$A^c = \{2, 4, 6\}$ "an even roll"

Set Operations: Difference

Definition

The **difference** of a set $A \subseteq \Omega$ and a set $B \subseteq \Omega$, denoted $A - B$, is the set of all elements in Ω that are in A and are not in B.

Example:

$A = \{3, 4, 5, 6\}$
$B = \{3, 5\}$
$A - B = \{4, 6\}$

Note: $A - B = A \cap B^c$

DeMorgan's Law

Complement of union or intersection:

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What is the English translation for both sides of the equations above?

Exercises

Check whether the following statements are true or false.
(Hint: you might use Venn diagrams.)

- $A - B \subseteq A$
- $(A - B)^c = A^c \cup B$
- $A \cup B \subseteq B$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

Probability

Definition

A **probability function** on a finite sample space Ω assigns every event $A \subseteq \Omega$ a number in $[0, 1]$, such that

1. $P(\Omega) = 1$
2. $P(A \cup B) = P(A) + P(B)$ when $A \cap B = \emptyset$

$P(A)$ is the **probability** that event A occurs.

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- $P(\{1, 2, 3\}) = 1/2$

Repeated Experiments

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Properties:
Order matters: $(1, 2) \neq (2, 1)$
Repeats are possible: $(1, 1) \in \mathbb{N} \times \mathbb{N}$

More Repeats

Repeating an experiment n times gives the sample space

$$\begin{aligned}\Omega^n &= \Omega \times \cdots \times \Omega \quad (n \text{ times}) \\ &= \{(x_1, x_2, \ldots, x_n) : x_i \in \Omega \text{ for all } i\}\end{aligned}$$

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If $|\Omega| = k$, then $|\Omega^n| = k^n$.

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Union of two overlapping events $A \cap B \neq \emptyset$:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Exercise

You are picking a number out of a hat, which contains the numbers 1 through 100. What are the following events and their probabilities?

- The number has a single digit
- The number has two digits
- The number is a multiple of 4
- The number is not a multiple of 4
- The sum of the number's digits is 5

Permutations

A **permutation** is an ordering of an n-tuple. For instance, the n-tuple $(1, 2, 3)$ has the following permutations:

$$(1, 2, 3), (1, 3, 2), (2, 1, 3)$$
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How many ways can you rearrange $(1, 2, 3, 4)$?

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Formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

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Example: You have cards numbered 1 through 10. If you pick five cards at random, what is the probability that you selected the cards 1, 2, 3, 4, 5 (not necessarily in that order)?

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We'll use the formula $P(A) = \frac{|A|}{|\Omega|}$.

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The total number of possible 5 card selections is

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So, finally the probability is

$$P(A) = \frac{|A|}{|\Omega|} = \frac{1}{252} \approx 0.00397 = 0.397\%$$