Notes: Conditional Probability

Foundations of Data Analysis

January 21, 2020

Brain Teaser:

1. If I have two children, one of whom is a boy, what is the probability that the other child is a boy?
2. If I have two children, and the *oldest* is a boy, what is the probability that the other child is a boy?
3. If I have two children, one of whom is a girl, what is the probability that the other child is a boy?
4. If I have two children, and one is a boy born on a Tuesday, what is the probability that the other child is a boy?

Review of "English translation" for events:

- $A \cap B =$ "both events A and B happen"
- $A \cup B =$ "either event A or B (or both) happens"
- $A^c =$ "event A does not happen"

Set Theory Rules: (try drawing Venn diagrams of these)

- Definition of set difference: $A - B = A \cap B^c$ "event A happens, but B does not"
- Associative Law:
$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$
- Commutative Law:
$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$
- Distributive Law:
$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
- DeMorgan's Law:
$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

Counting:

- **Number of permutations of n items: (a.k.a. number of unique orderings)** $n! = n \times (n-1) \times (n-2) \times \cdots \times 2$

- **Number of ways to select k items out of n choices:** $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
  (here order does not matter, just which k items you select)

Probability Rules:

- **Equally likely outcomes:** $P(A) = \frac{|A|}{|\Omega|}$
- **Inclusion-Exclusion Rule:** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- **Complement Rule:** $P(A^c) = 1 - P(A)$
- **Difference Rule:** $P(A - B) = P(A) - P(A \cap B)$

Exercise: Try deriving these rules from the definition of a probability function. Draw a Venn diagram to convince yourself they work.

Conditional Probability:

$$P(A|B) = \text{“the probability of event } A \text{ given that we know } B \text{ happened”}$$
$$\text{Formula: } P(A|B) = P(A \cap B)/P(B)$$

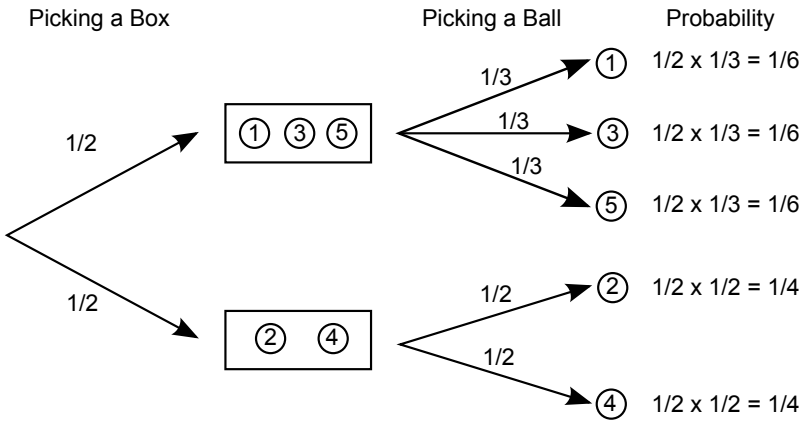
Multiplication Rule:

$$P(A \cap B) = P(A|B)P(B)$$

Tree diagrams to compute “two stage” probabilities (B = first stage, A = second stage):

1. First branch computes probability of first stage: $P(B)$
2. Second branch computes probability of second stage, given the first: $P(A|B)$
3. Multiply probabilities along a path to get final probabilities $P(A \cap B)$

Example: You are given two boxes with balls numbered 1 - 5. One box contains balls 1, 3, 5, and the other contains balls 2 and 4. You first pick a box at random, then pick a ball from that box at random. What is the probability that you pick a 2?



Exercise: You are analyzing the effectiveness of online advertising for a company that sells widgets. The company finds that 50% of traffic to their website comes from clicks of online ads. In addition, 20% of visitors to their website both had clicked an online ad and purchased a widget. If a person clicks on the company’s ad, what is the probability that they will purchase a widget?

Exercise: In Charlottesville the sky is overcast on about 40% of days. If it is overcast, there is a 25% chance that it will also be windy. What is the probability that it is both overcast and windy?