Convolution

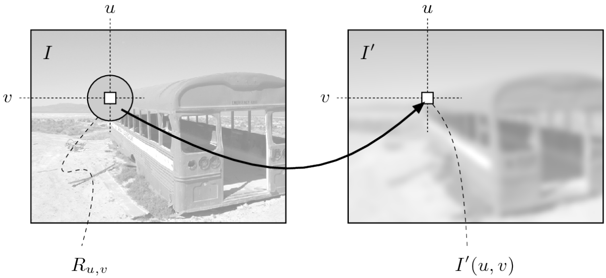
Foundations of Data Analysis

April 25, 2019

Spatial Filters

Definition

A **spatial filter** is an image operation where each pixel value $I(u, v)$ is changed by a function of the intensities of pixels in a neighborhood of (u, v).



What Spatial Filters Can Do

Blurring/Smoothing

$\rightarrow$

What Spatial Filters Can Do

Sharpening

$\rightarrow$

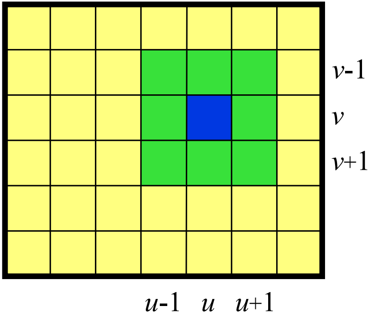
What Spatial Filters Can Do

Weird Stuff

$\rightarrow$

Example: The Mean of a Neighborhood

Consider taking the mean in a 3 $\times$ 3 neighborhood:



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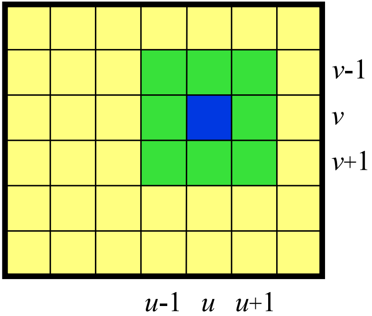
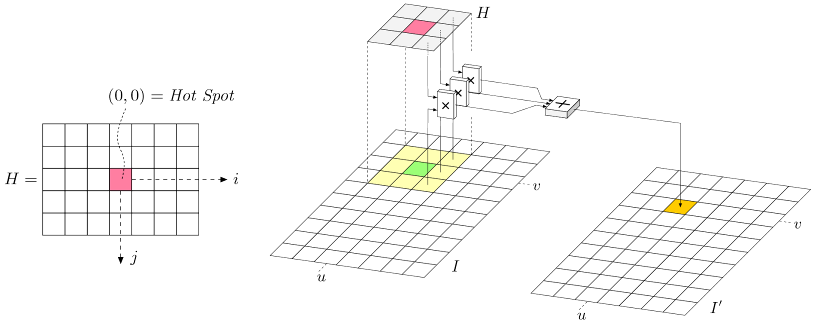


$$I'(u,v) = \frac{1}{9} \sum_{i=-1}^{1} \sum_{j=-1}^{1} I(u+i, v+j)$$

How a Linear Spatial Filter Works



H is the filter “kernel” or “matrix”

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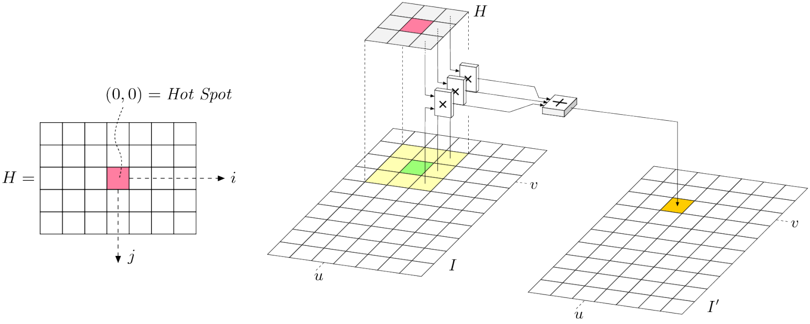

H is the filter “kernel” or “matrix”

For the neighborhood mean: $$H(i,j) = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

General Filter Equation

Notice that the kernel H is just a small image!

Let $H : R_H \rightarrow [0, K-1]$

$$I'(u,v) = \sum_{(i,j) \in R_H} I(u+i, v+j) \cdot H(i,j)$$

This is known as a **correlation** of I and H

What Does This Filter Do?

0	0	0
0	1	0
0	0	0

What Does This Filter Do?

0	0	0
0	1	0
0	0	0

Identity function (leaves image alone)

What Does This Filter Do?

$$\frac{1}{9}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

What Does This Filter Do?

$$\frac{1}{9}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

Mean (averages neighborhood)

What Does This Filter Do?

0	0	0
0	0	1
0	0	0

What Does This Filter Do?

0	0	0
0	0	1
0	0	0

Shift left by one pixel

What Does This Filter Do?

$$\frac{1}{9}\begin{array}{|c|c|c|}\hline -1 & -1 & -1 \\ \hline -1 & 17 & -1 \\ \hline -1 & -1 & -1 \\ \hline\end{array}$$

What Does This Filter Do?

$$\frac{1}{9}\begin{bmatrix} -1 & -1 & -1 \\ -1 & 17 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

Sharpen (identity minus mean filter)

Filter Normalization

- Notice that all of our filter examples sum up to one

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- Multiplying all entries in H by a constant will cause the image to be multiplied by that constant

$$
\begin{aligned}
I'(u,v) &= \sum_{i,j} I(u+i, v+j) \cdot (cH(i,j)) \\
&= c \sum_{i,j} I(u+i, v+j) \cdot H(i,j)
\end{aligned}
$$

Filter Normalization

- Notice that all of our filter examples sum up to one
- Multiplying all entries in H by a constant will cause the image to be multiplied by that constant
- To keep the overall brightness constant, we need H to sum to one

$$
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I'(u,v) &= \sum_{i,j} I(u+i, v+j) \cdot (cH(i,j)) \\
&= c \sum_{i,j} I(u+i, v+j) \cdot H(i,j)
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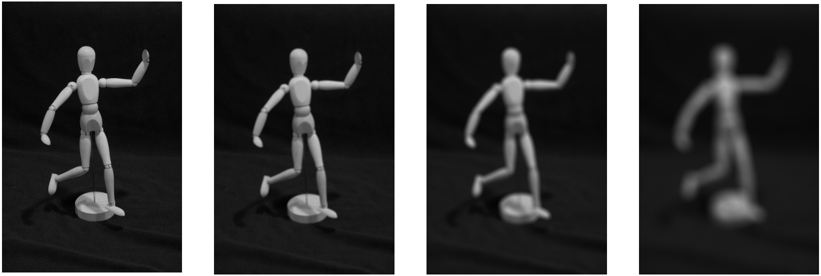
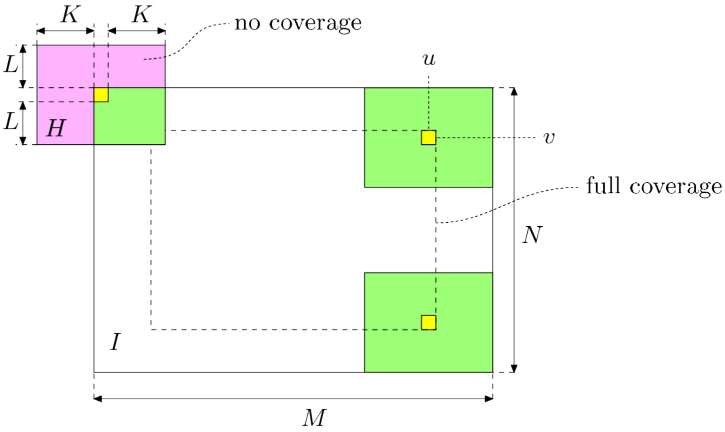
Effect of Filter Size

Mean Filters:



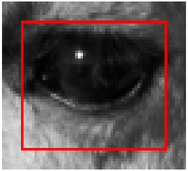
Original | 7×7 | 15×15 | 41×41

What To Do At The Boundary?



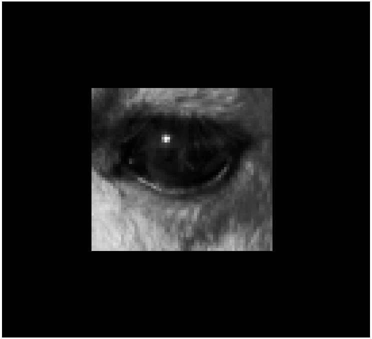
What To Do At The Boundary?

- Crop



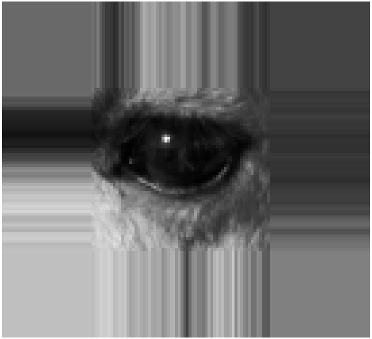
What To Do At The Boundary?

- Crop
- Pad



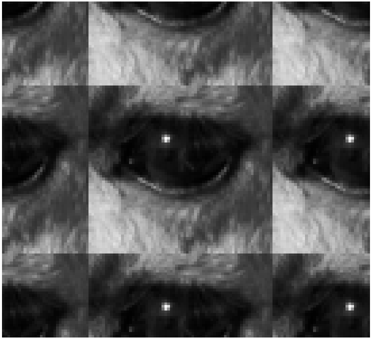
What To Do At The Boundary?

- Crop
- Pad
- Extend



What To Do At The Boundary?

- Crop
- Pad
- Extend
- Wrap



Convolution

Definition

Convolution of an image I by a kernel H is given by

$$I'(u,v) = \sum_{(i,j)\in R_H} I(u-i, v-j) \cdot H(i,j)$$

This is denoted: $I' = I * H$

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- Notice this is the same as correlation with H, but with negative signs on the I indices
- Equivalent to vertical and horizontal flipping of H:

$$I'(u,v) = \sum_{(-i,-j)\in R_H} I(u+i, v+j) \cdot H(-i,-j)$$

Linear Operators

Definition

A **linear operator** F on an image is a mapping from one image to another, $I' = F(I)$, that satisfies:

1. $F(cI) = cF(I)$,
2. $F(I_1 + I_2) = F(I_1) + F(I_2)$,

where I, I_1, I_2 are images, and c is a constant.

Both correlation and convolution are linear operators

Infinite Image Domains

Let's define our image and kernel domains to be infinite:

$$\Omega = \mathbb{Z} \times \mathbb{Z}$$

Remember $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Infinite Image Domains

Let's define our image and kernel domains to be infinite:

$$\Omega = \mathbb{Z} \times \mathbb{Z}$$

Remember $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Now convolution is an infinite sum:

$$I'(u, v) = \sum_{i=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} I(u-i, v-j) \cdot H(i, j)$$

This is denoted $I' = I * H$.

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We can still imagine that the image is defined on a bounded (finite) domain, $[0, w] \times [0, h]$, and is set to zero outside of this.

Properties of Convolution

Commutativity:

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In other words, we can leave the image fixed and slide the kernel or leave the kernel fixed and slide the image.

Properties of Convolution

Associativity:

$$\left(I * H_1\right) * H_2 = I * \left(H_1 * H_2\right)$$

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$$(I * H_1) * H_2 = I * (H_1 * H_2)$$

This means that we can apply H_1 to I followed by H_2, or we can convolve the kernels $H_2 * H_1$ and then apply the resulting kernel to I.

Properties of Convolution

Linearity:

$$(a \cdot I) * H = a \cdot (I * H)$$

$$(I_1 + I_2) * H = (I_1 * H) + (I_2 * H)$$

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This means that we can multiply an image by a constant before or after convolution, and we can add two images before or after convolution and get the same results.

Properties of Convolution

Shift-Invariance:

Let S be the operator that shifts an image I:

$$S(I)(u,v) = I(u+a, v+b)$$

Then

$$S(I * H) = S(I) * H$$

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Shift-Invariance:

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$$S(I)(u,v) = I(u+a, v+b)$$

Then

$$S(I * H) = S(I) * H$$

This means that we can convolve I and H and then shift the result, or we can shift I and then convolve it with H.

Properties of Convolution

Theorem: The only shift-invariant, linear operators on images are convolutions.

Computational Complexity of Convolution

If my image I has size $M \times N$ and my kernel H has size $(2R+1) \times (2R+1)$, then what is the complexity of convolution?

$$I'(u,v) = \sum_{i=-R}^{R} \sum_{j=-R}^{R} I(u-i, v-j) \cdot H(i,j)$$

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Answer: $O(MN(2R+1)(2R+1)) = O(MNR^2)$.

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Answer: $O(MN(2R+1)(2R+1)) = O(MNR^2)$.
Or, if we consider the image size fixed, $O(R^2)$.

Which is More Expensive?

The following both shift the image 10 pixels to the left:

1. Convolve with a 21×21 shift operator (all zeros with a 1 on the right edge)
2. Repeatedly convolve with a 3×3 shift operator 10 times

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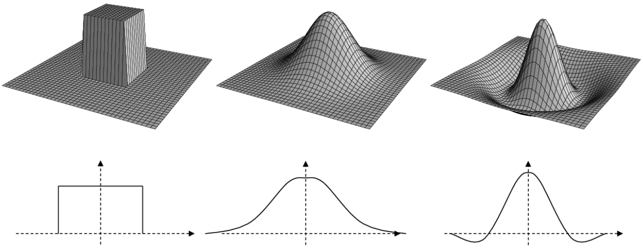
The first method requires $21^2 \cdot wh = 441 \cdot wh$.
The second method requires $(9 \cdot wh) \cdot 10 = 90 \cdot wh$.

Some More Filters

Box Gaussian Laplace



0	0	0	0	0
0	1	1	1	0
0	1	1	1	0
0	1	1	1	0
0	0	0	0	0

(a)

0	1	2	1	0
1	3	5	3	1
2	5	9	5	2
1	3	5	3	1
0	1	2	1	0

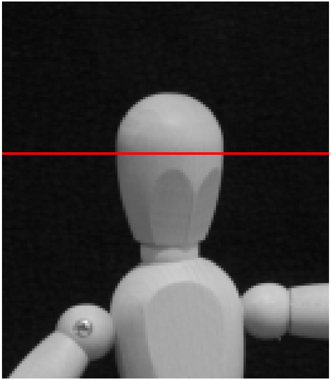
(b)

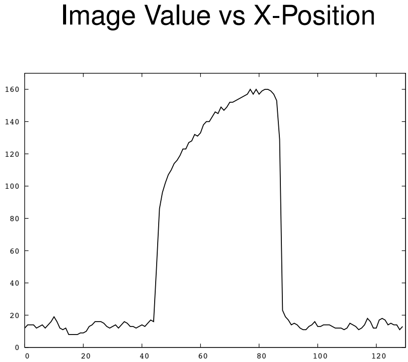
0	0	-1	0	0
0	-1	-2	-1	0
-1	-2	16	-2	-1
0	-1	-2	-1	0
0	0	-1	0	0

(c)

Edge Detection

What is an Edge?





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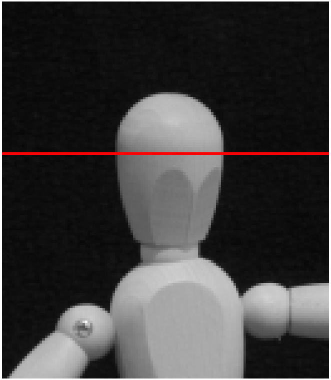
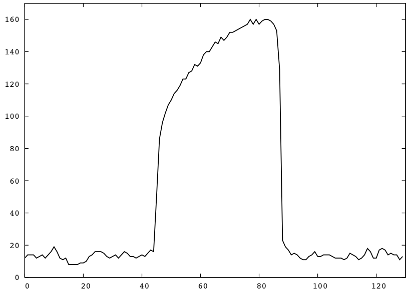


Image Value vs X-Position



An abrupt transition in intensity between two regions

What is an Edge?

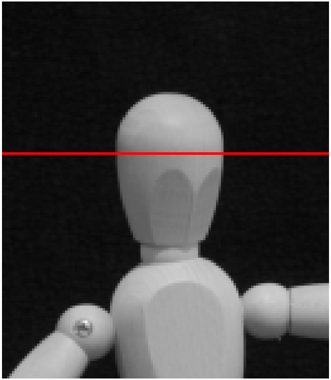


Image X-Derivative vs X-Position

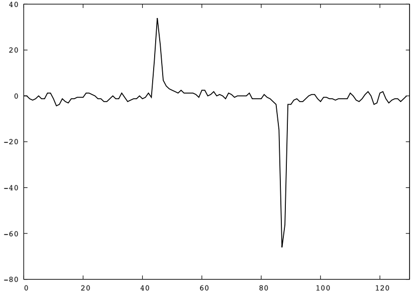


Image derivatives are high (or low) at edges

Review: Derivative of a Function

Given a function $f : \mathbb{R} \to \mathbb{R}$, its derivative is defined as

$$\frac{df}{dx}(x) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

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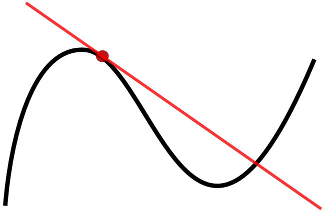
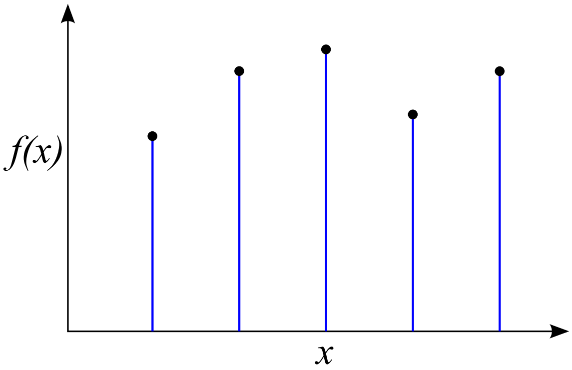


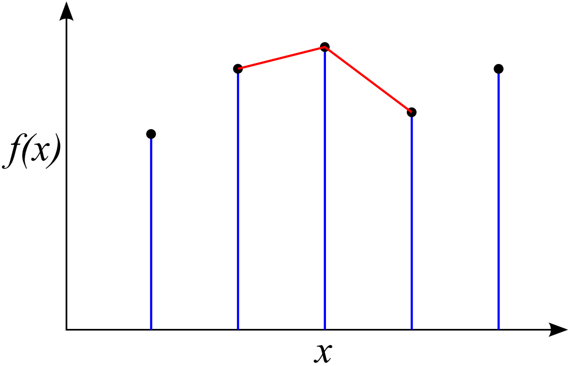
Derivative of f is the slope of the tangent to the graph of f

Derivatives of Discrete Functions



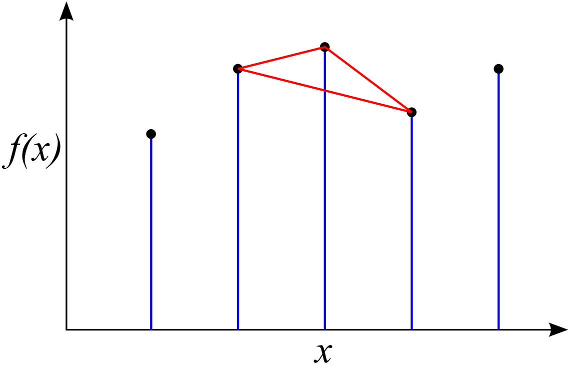
Discrete function defined on integer values of x

Derivatives of Discrete Functions



Slopes (derivatives) don’t match on left and right

Derivatives of Discrete Functions



Instead take the average of the two (or secant)

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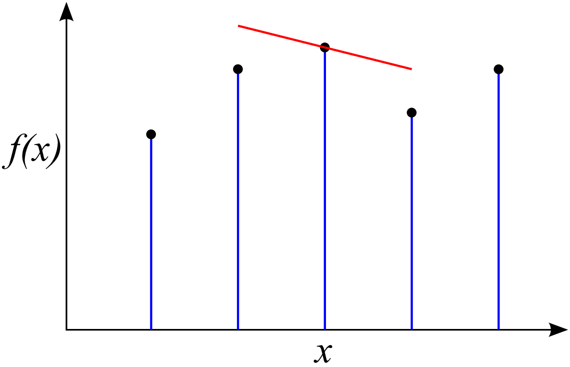

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Finite Differences

Forward Difference

$$\Delta_+ f(x) = f(x+1) - f(x) \qquad \text{right slope}$$

Finite Differences

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Backward Difference

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Finite Differences

Forward Difference

$$\Delta_+ f(x) = f(x+1) - f(x)$$

right slope

Backward Difference

$$\Delta_- f(x) = f(x) - f(x-1)$$

left slope

Central Difference

$$\Delta f(x) = \frac{1}{2}\left(f(x+1) - f(x-1)\right)$$

average slope

Finite Differences as Convolutions

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Take a convolution kernel: $H = [1 \ \ -1 \ \ 0]$

Finite Differences as Convolutions

Forward Difference

$$\Delta_+ f(x) = f(x+1) - f(x)$$

Take a convolution kernel: $H = [1 \; -1 \;\; 0]$

$$\Delta_+ f = f * H$$

(Remember that the kernel H is flipped in convolution)

Finite Differences as Convolutions

Central Difference

$$\Delta f(x) = \frac{1}{2}\left(f(x+1) - f(x-1)\right)$$

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Convolution kernel here is: $H = \left[\, \frac{1}{2} \;\; 0 \;\; -\frac{1}{2} \,\right]$

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Notice: Derivative kernels sum to zero!

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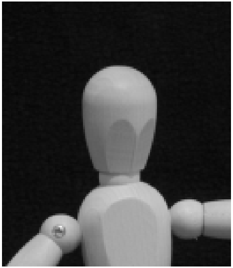
- Central differences:

$$\Delta_x I = I * H_x, \quad \text{and} \quad \Delta_y I = I * H_y,$$

$$\text{where} \;\; H_x = \begin{bmatrix} 0.5 & 0 & -0.5 \end{bmatrix} \;\; \text{and} \;\; H_y = \begin{bmatrix} -0.5 \\ 0 \\ 0.5 \end{bmatrix}$$

Derivatives of Images

x-derivative using central difference:



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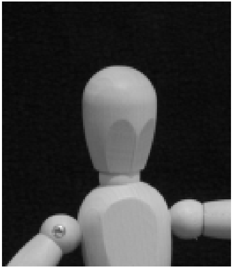

$$* \quad \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \quad =$$

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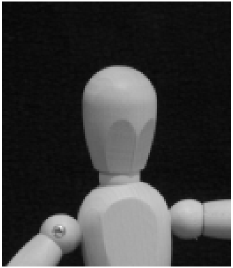

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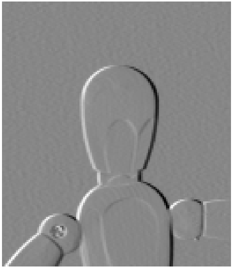
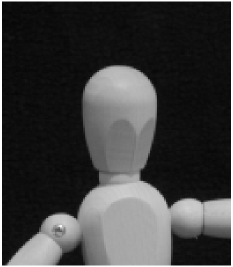


Derivatives of Images

y-derivative using central difference:



Derivatives of Images

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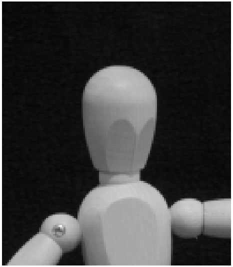

$$* \begin{bmatrix} 0.5 \\ 0 \\ -0.5 \end{bmatrix} =$$

Derivatives of Images

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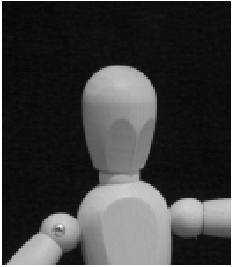

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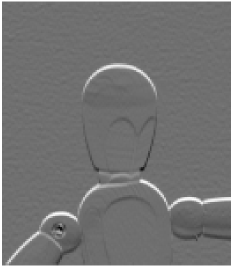

Combining x and y Derivatives

The **discrete gradient** of $I(x, y)$ is the 2D vector:

$$\nabla I(x, y) = \begin{bmatrix} \Delta_x I(x, y) \\ \Delta_y I(x, y) \end{bmatrix}$$

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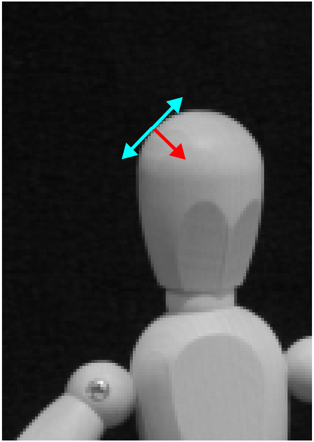
$$\nabla I(x, y) = \begin{bmatrix} \Delta_x I(x, y) \\ \Delta_y I(x, y) \end{bmatrix}$$

The **gradient magnitude** is

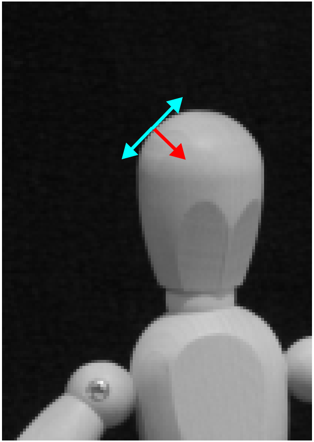
$$\|\nabla I(x, y)\| = \sqrt{(\Delta_x I(x, y))^2 + (\Delta_y I(x, y))^2}$$

Image Gradient



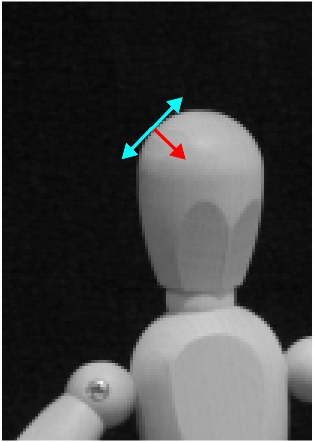
- Gradient points in direction of maximal increasing intensity

Image Gradient



- Gradient points in direction of maximal increasing intensity
- Length (magnitude) of gradient equals amount of change in that direction

Image Gradient



- Gradient points in direction of maximal increasing intensity
- Length (magnitude) of gradient equals amount of change in that direction
- Gradient is perpendicular (90 degrees) to edge contour

Convolutional Neural Networks (CNNs)

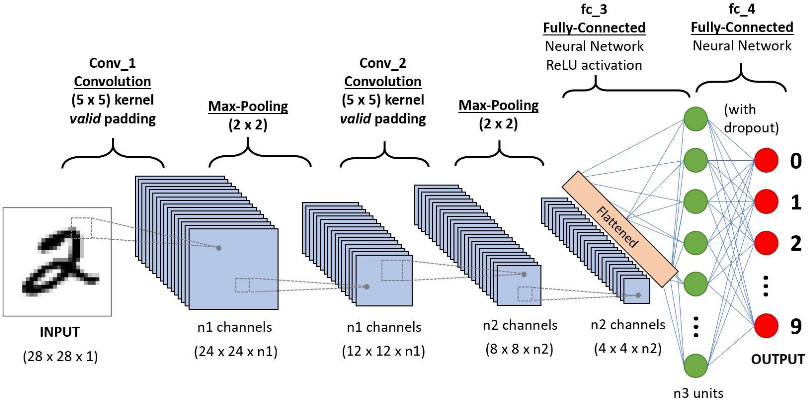
Learning a Filter

w_1	w_2	w_3
w_4	w_5	w_6
w_7	w_8	w_9

?

Filter consists of weights that need to be learned.

Convolutional Neural Networks



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