Canonical Correlation Analysis (CCA)

Foundations of Data Analysis

March 29, 2023

Example: Psych Measures vs. Test Scores

	locus_of_control	self_concept	motivation	read	write	math	science	female
0	-0.84	-0.24	1.00	54.8	64.5	44.5	52.6	1
1	-0.38	-0.47	0.67	62.7	43.7	44.7	52.6	1
2	0.89	0.59	0.67	60.6	56.7	70.5	58.0	0
3	0.71	0.28	0.67	62.7	56.7	54.7	58.0	0
4	-0.64	0.03	1.00	41.6	46.3	38.4	36.3	1

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Question: Are these psychological measures related to scores on standardized tests?

Correlation

Covariance between two random samples: $x_i, y_i \in \mathbb{R}$

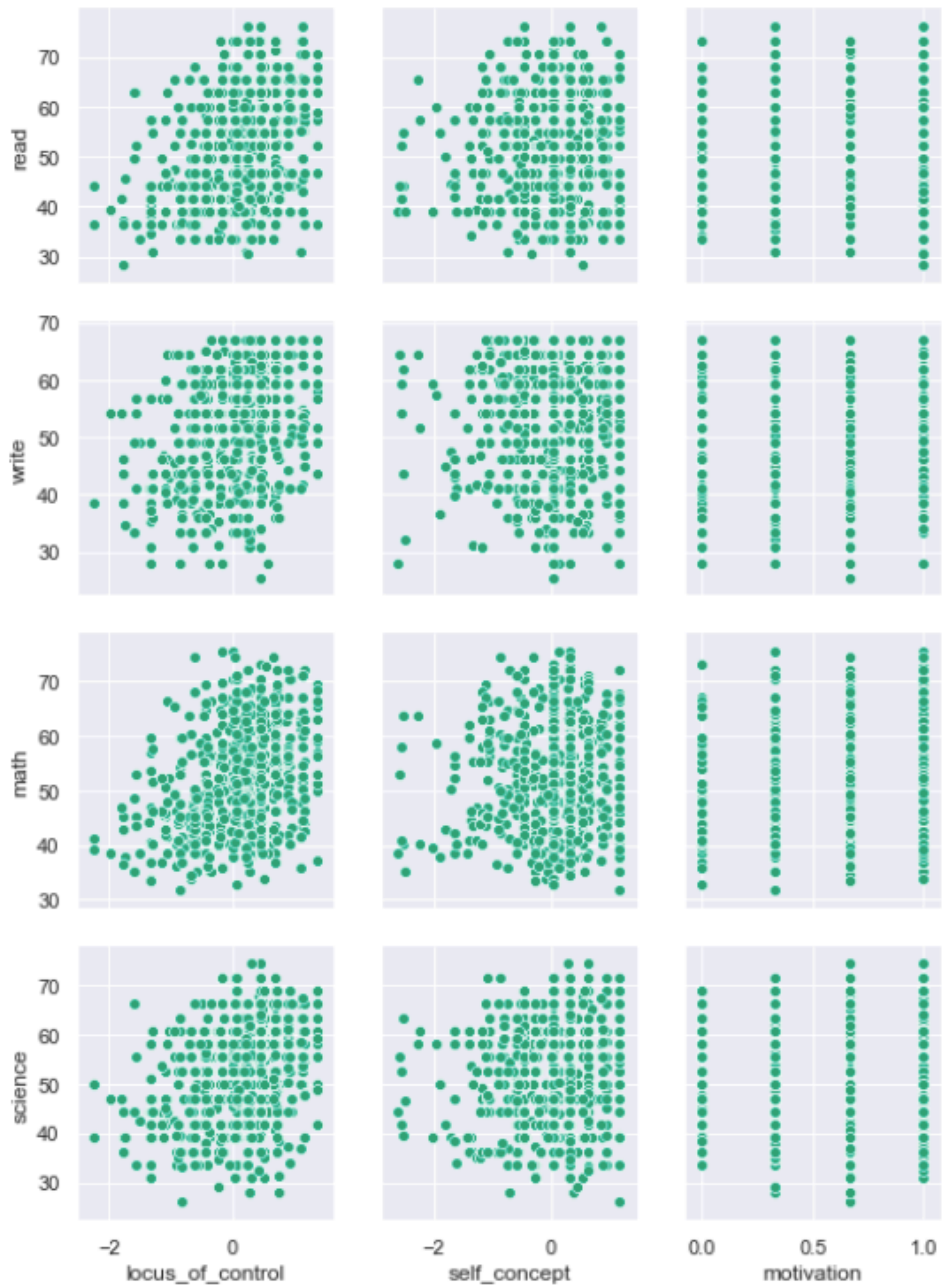
$$\text{cov}(x, y) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$

Measures how x “covaries” with y

Proportional to correlation:

$$\text{corr}(x, y) = \frac{\text{cov}(x, y)}{\text{sd}(x)\text{sd}(y)}$$

Pairwise Relationships



Pairwise Correlations:

0.3735	0.2106	0.0606
0.3588	0.2542	0.0194
0.3372	0.1950	0.0535
0.3246	0.1156	0.0698

Canonical Correlation Analysis (CCA)

Group your data table into two sets of variables/datasets:

$$X : n \times d_X \qquad\qquad Y : n \times d_Y$$

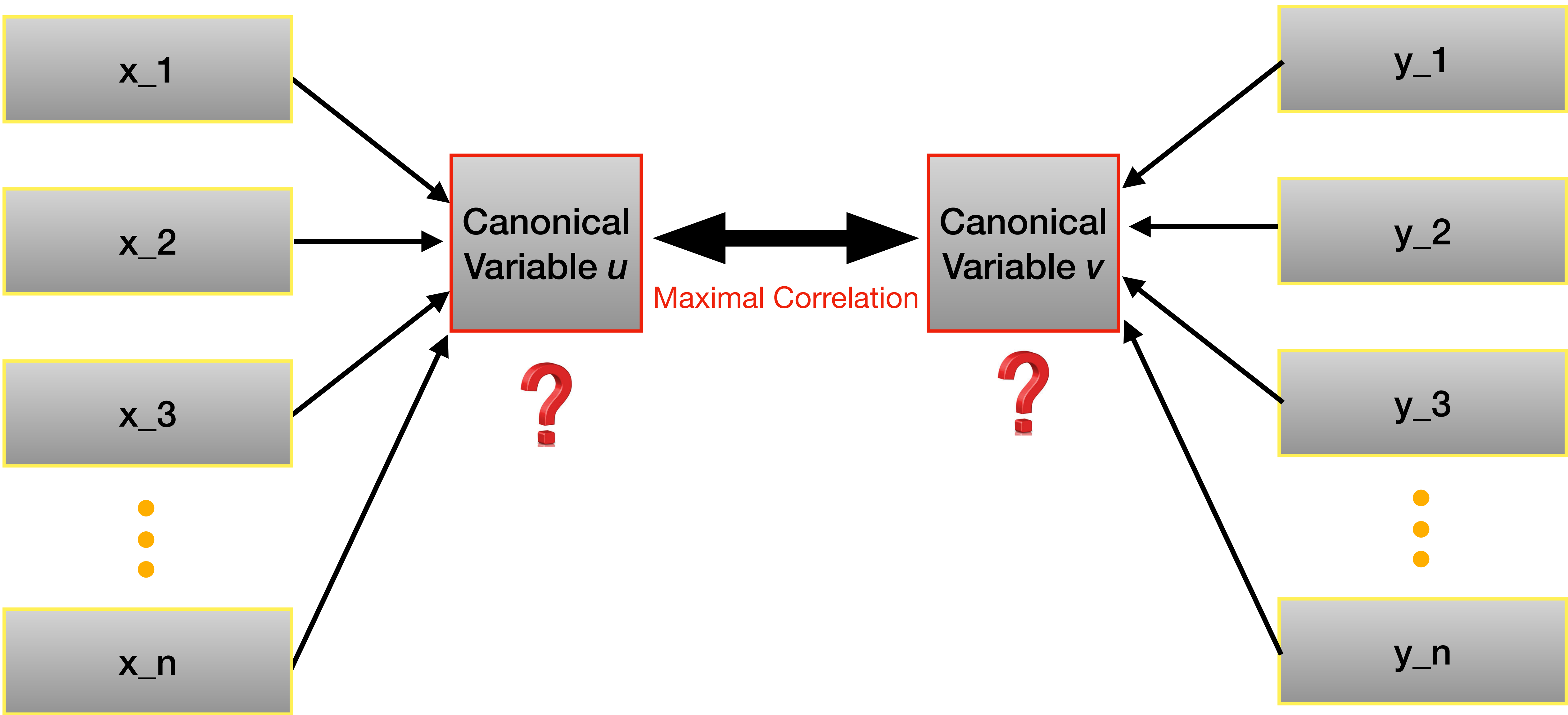
Find a single dimension in X and single dimension in Y that are maximally correlated.

CCA allows you to analyze correlations between two datasets.

CCA VS. PCA

- CCA focuses on finding linear combinations that account for the most correlation in two datasets.
- PCA focuses on finding linear combinations that account for the most variance in one data set.

CCA



x_1
x_2
x_3
x_n
Canonical Variable u
Maximal Correlation
Canonical Variable v
y_1
y_2
y_3
y_n

Math of CCA

Unit vector in X data: $u \in \mathbb{R}^{d_X}, \quad \|u\| = 1$

Unit vector in Y data: $v \in \mathbb{R}^{d_Y}, \quad \|v\| = 1$

Projected data:

$$\langle x_i - \mu_X, u \rangle, \qquad \langle y_i - \mu_Y, v \rangle$$

Or, using centered data matrices, $\tilde{X}, \tilde{Y}$:

$$\tilde{X}u, \qquad \tilde{Y}v$$

Maximize correlation in projected data:

$$(u', v') = \underset{u,v}{\arg\max}\ \mathrm{corr}(\tilde{X}u, \tilde{Y}v)$$

Math of CCA

Let $\Sigma_{XX} = \mathrm{cov}(X, X)$, $\Sigma_{YY} = \mathrm{cov}(Y, Y)$,
$\Sigma_{XY} = \mathrm{cov}(X, Y)$, $\Sigma_{YX} = \mathrm{cov}(Y, X)$

Goal: find u, v that maximize the correlation:

$$\rho = \frac{u^T \Sigma_{XY} v}{\sqrt{u^T \Sigma_{XX} u}\sqrt{v^T \Sigma_{YY} v}}$$

CCA Solution

u is the eigenvector with largest eigenvalue of

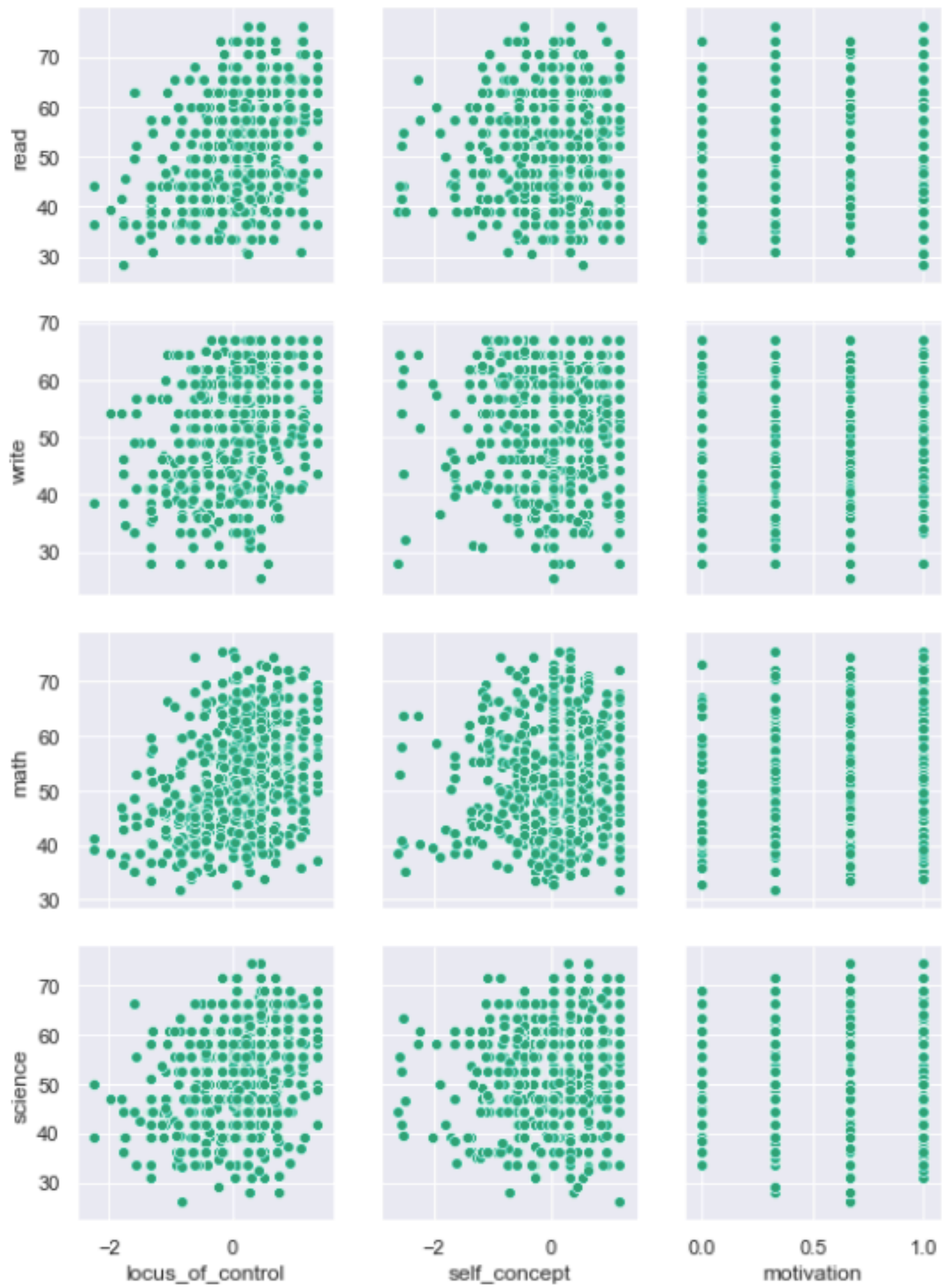
$$\Sigma_{XX}^{-1}\Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}$$

v is the eigenvector with largest eigenvalue of

$$\Sigma_{YY}^{-1}\Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}$$

Just like PCA, we can then proceed to find the dimensions with the **second most** correlation, etc.

Example: Psych vs. Test



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Example: Psych vs. Test

Psych Canonical Components:

	0	1
Control	0.876809	-0.429422
Self	-0.174754	-0.496948
Motivation	0.447959	0.755662

- 0: First canonical component
- 1: Second canonical component

Academic Canonical Components:

	0	1
Read	0.617204	0.012375
Write	0.743148	0.676109
Math	0.253335	0.021393
Science	-0.051115	-0.926878

First Canonical Correlation = 0.4464364824283061
Second Canonical Correlation = 0.15335902492287964

Example: Psych vs. Test