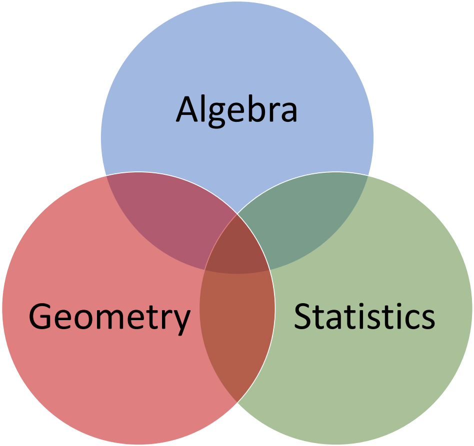
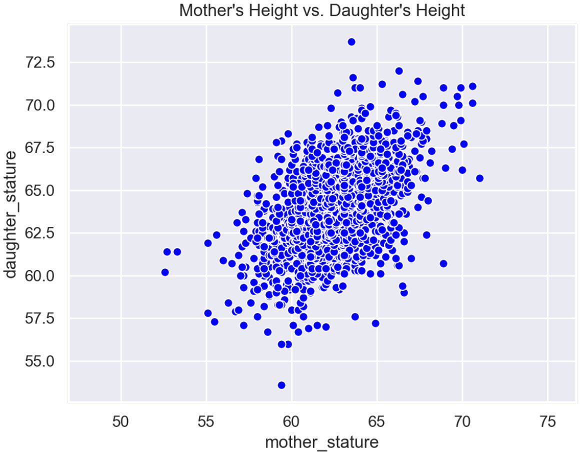
Linear Regression

Foundations of Data Analysis

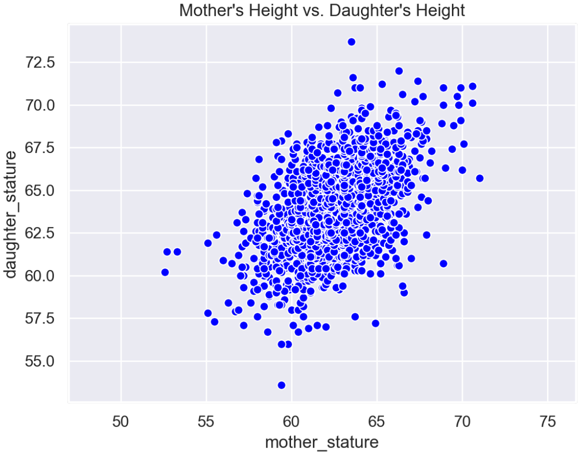
February 20, 2020


Algebra
Geometry
Statistics

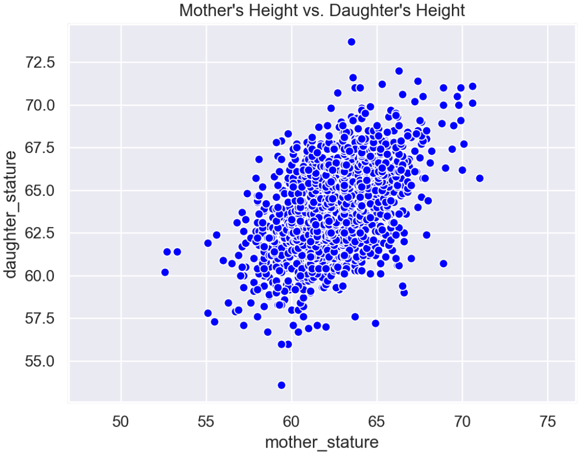
Is there a relationship between the heights of mothers and their daughters?



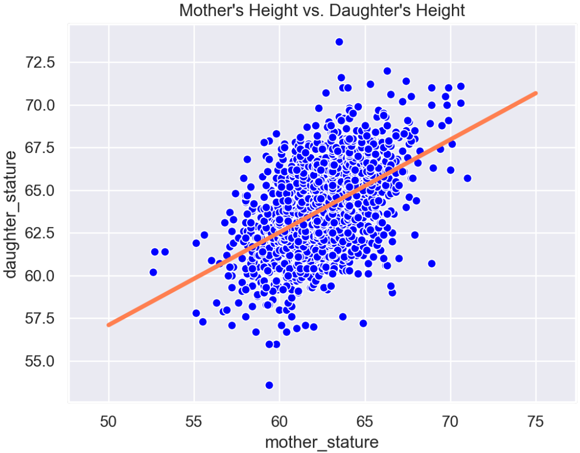
If you know a mother's height, can you predict her daughter's height with any accuracy?



Linear regression is a tool for answering these types of questions.



It models the relationship as a straight line.



Regression Setup

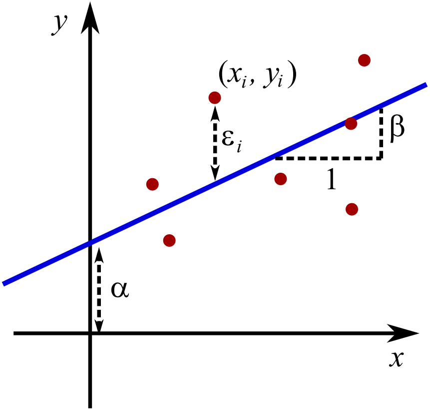
When we are given real-valued data in pairs:

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \in \mathbb{R}^2$$

Example:
x_i is the height of the ith mother
y_i is the height of the ith mother's daughter

Linear Regression

Model the data as a line:



$$y_i = \alpha + \beta x_i + \epsilon_i$$

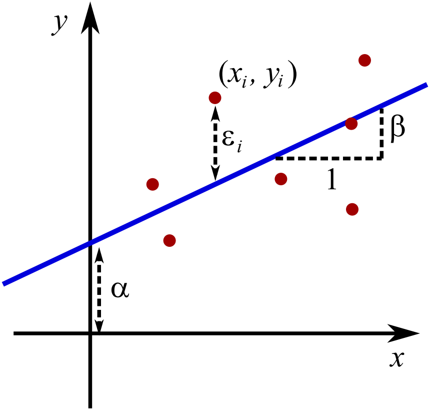
α : intercept

β : slope

ϵ_i : error

Geometry: Least Squares

We want to fit a line as close to the data as possible, which means we want to **minimize the errors**, ϵ_i.



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We want to minimize the **sum-of-squared errors (SSE)**:

$$\mathrm{SSE}(\alpha, \beta) = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2$$

Least Squares: Step 1

Center the data by removing the mean:

$$\tilde{y}_i = y_i - \bar{y}$$

$$\tilde{x}_i = x_i - \bar{x}$$

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We'll first get a solution: $\tilde{y} = \alpha + \beta\tilde{x}$, then shift it back to the original (uncentered) data at the end

Least Squares: Step 2

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$$0 = \frac{\partial}{\partial \alpha} \mathrm{SSE}(\alpha, \beta)$$

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Using $\sum \tilde{y}_i = \sum \tilde{x}_i = 0$, we get

$$\hat{\alpha} = 0$$

Least Squares: Step 3

With $\alpha = 0$, we are left with

$$\tilde{y}_i = \beta \tilde{x}_i + \epsilon_i$$

Or, in vector notation:

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_n \end{bmatrix} = \beta \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

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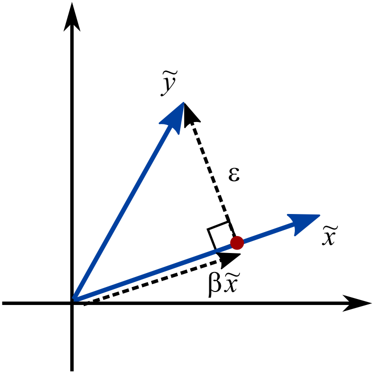

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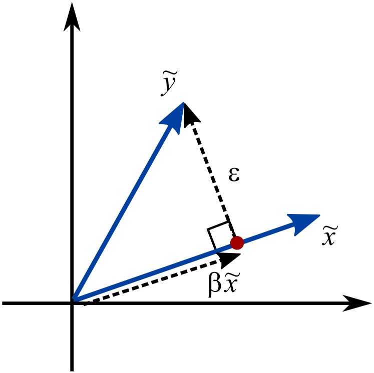

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Solution is $\hat{\beta} = \frac{\langle \tilde{x}, \tilde{y} \rangle}{\|\tilde{x}\|^2}$

Shifting Back to Uncentered Data

So far, we have:

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So, for the uncentered data, $\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$

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To make linear regression probabilistic, we model the errors as Gaussian:

$$\epsilon_i \sim N(0, \sigma^2)$$

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The likelihood is

$$L(\alpha, \beta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\epsilon_i^2}{2\sigma^2}\right)$$

Probability: Maximum Likelihood

The log-likelihood is then

$$\log L(\alpha, \beta) = -\frac{1}{2\sigma^2} \sum_{i=1}^{n} \epsilon_i^2 + \text{const.}$$

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Maximizing this is equaivalent to minimizing SSE!

$$\max \log L = \min \sum \epsilon_i^2 = \min \text{SSE}$$