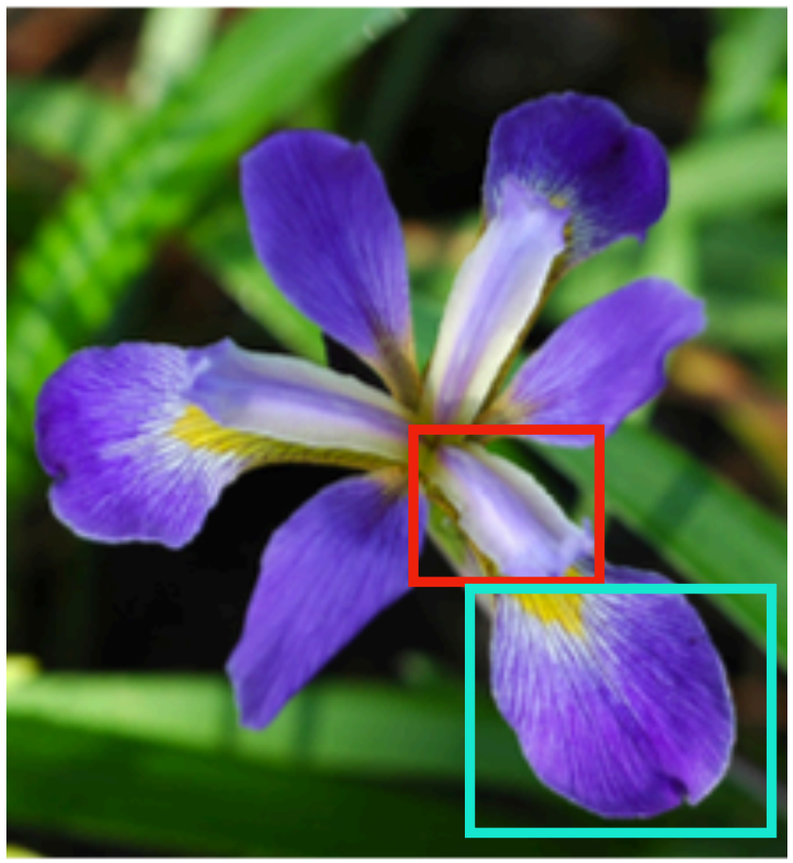
Classification and Naive Bayes

Foundations of Data Analysis

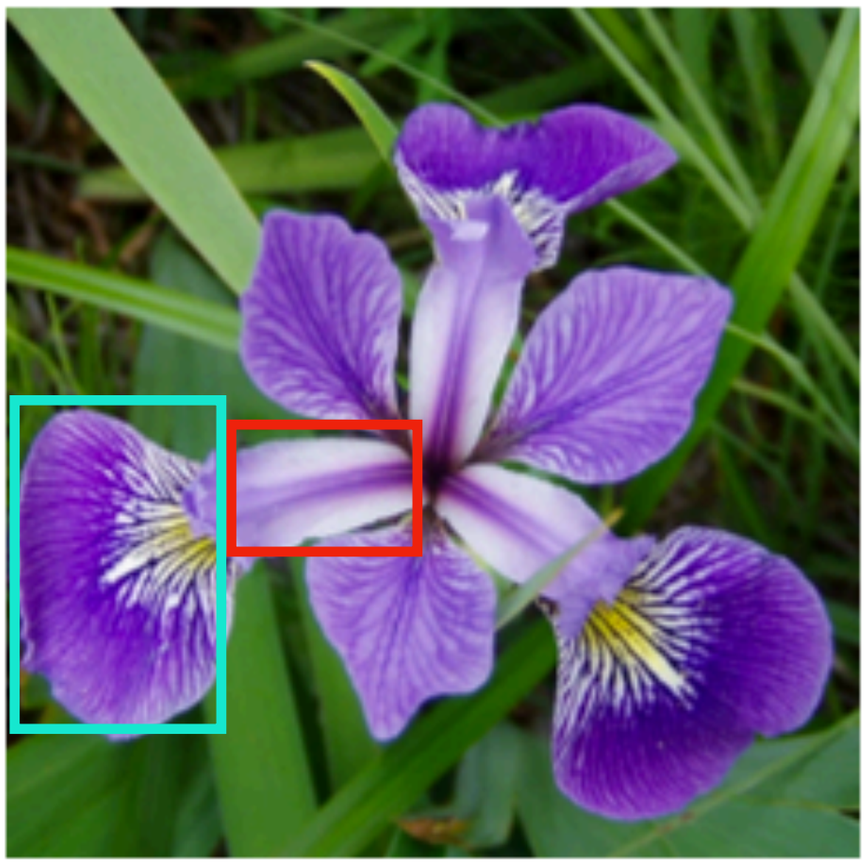
February 1, 2023

Naive Bayes Classifier

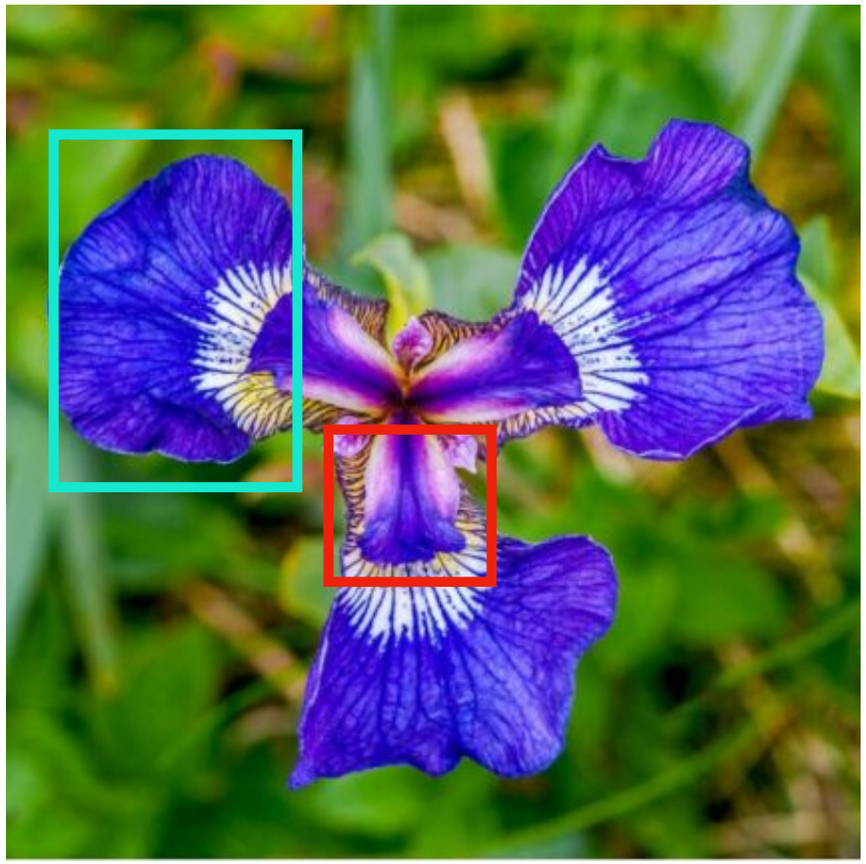
Iris virginica



Iris versicolor



Iris setosa

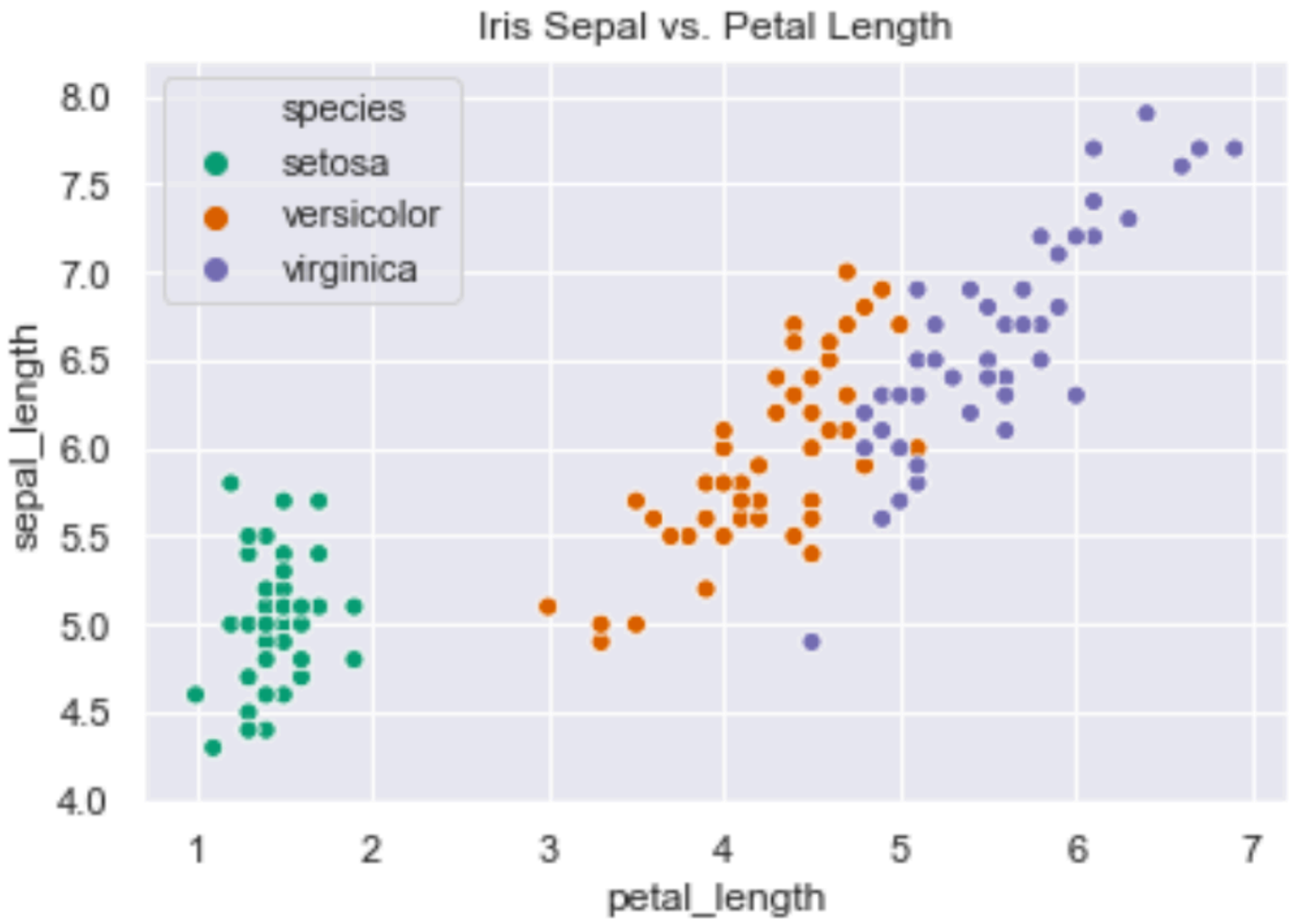


:Example regions of petal.

:Example regions of sepal.

Classification

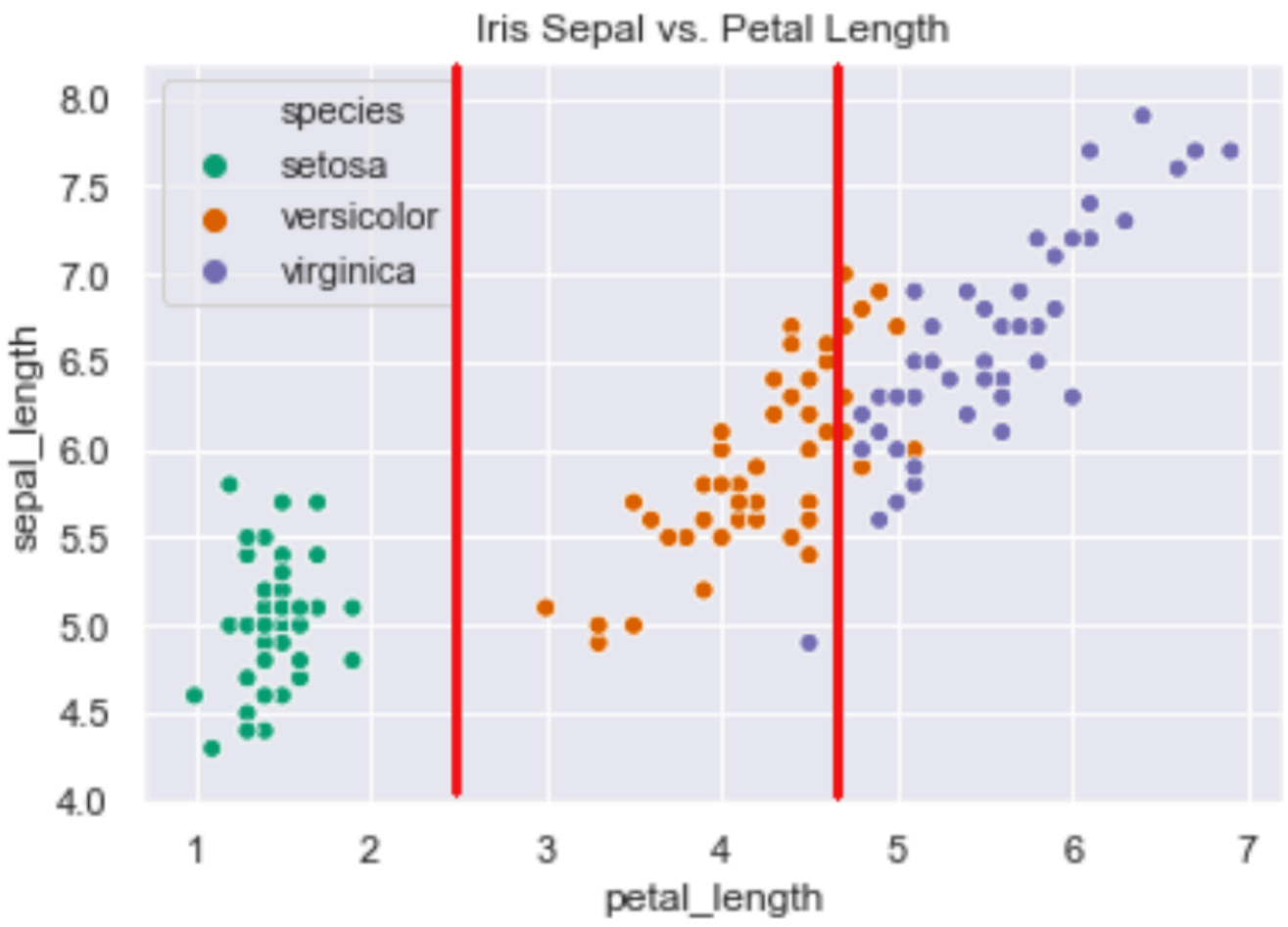
Say we want to automatically identify an iris species based on its petal and sepal length measurements.



This is a famous data set in machine learning / statistics, from Ronald Fisher in 1936!

A Classifier is a *Decision Rule*

$x =$ 'petal length', $c =$ 'species'



if $x < 2.5$: $c =$ 'setosa'

if $2.5 < x < 4.7$: $c =$ 'versicolor'

if $4.7 < x$: $c =$ 'virginica'

Classification Task

Training:

Learn a decision rule, based on training data, to predict a class C from features X.

Testing:

Use trained classifier to predict unknown class C^* from features of new testing data, X^*.

Important! Training and testing data should be completely separate!

Probabilistic Classifier

Features X and class C are random variables.

Learn a probability distribution from the training data:

$$P(C \mid X)$$

Example:

An iris test point X^* might give something like this:

C^*	setosa	versicolor	virginica
$P(C^* \mid X^*)$	0.80	0.15	0.05

Bayes' Rule for Classification

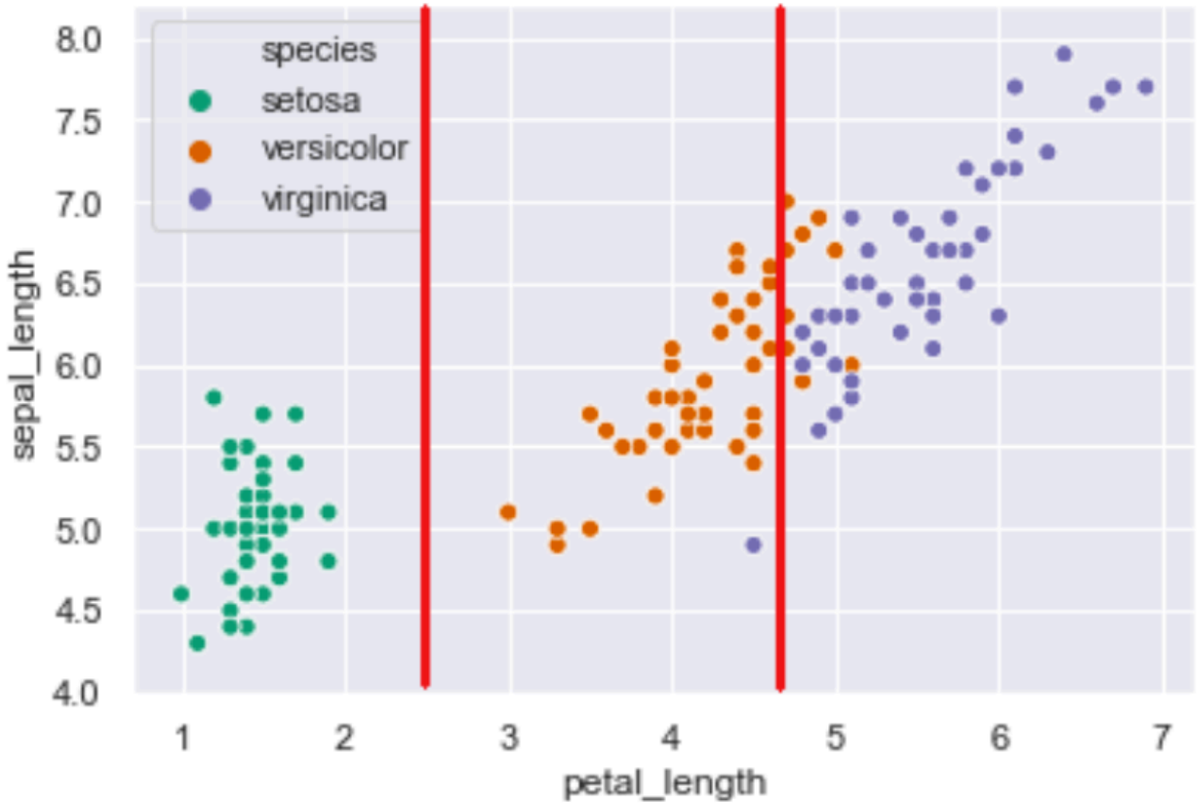
$$P(C \mid X) = \frac{P(X \mid C)P(C)}{P(X)}$$

- $P(X \mid C)$ **Likelihood** - learned from data

  The probability of random observed data conditional on particular model parameters.

Likelihood Function

$x =$ 'petal length', $c =$ 'species'



$$P(X \mid C)$$

if $c =$ 'setosa', what is the probability to assign x to this class c ?

Bayes' Rule for Classification

$$P(C \mid X) = \frac{P(X \mid C)P(C)}{P(X)}$$

- $P(X \mid C)$ **Likelihood** - learned from data
- $P(C)$ **Prior** - determined beforehand

Prior

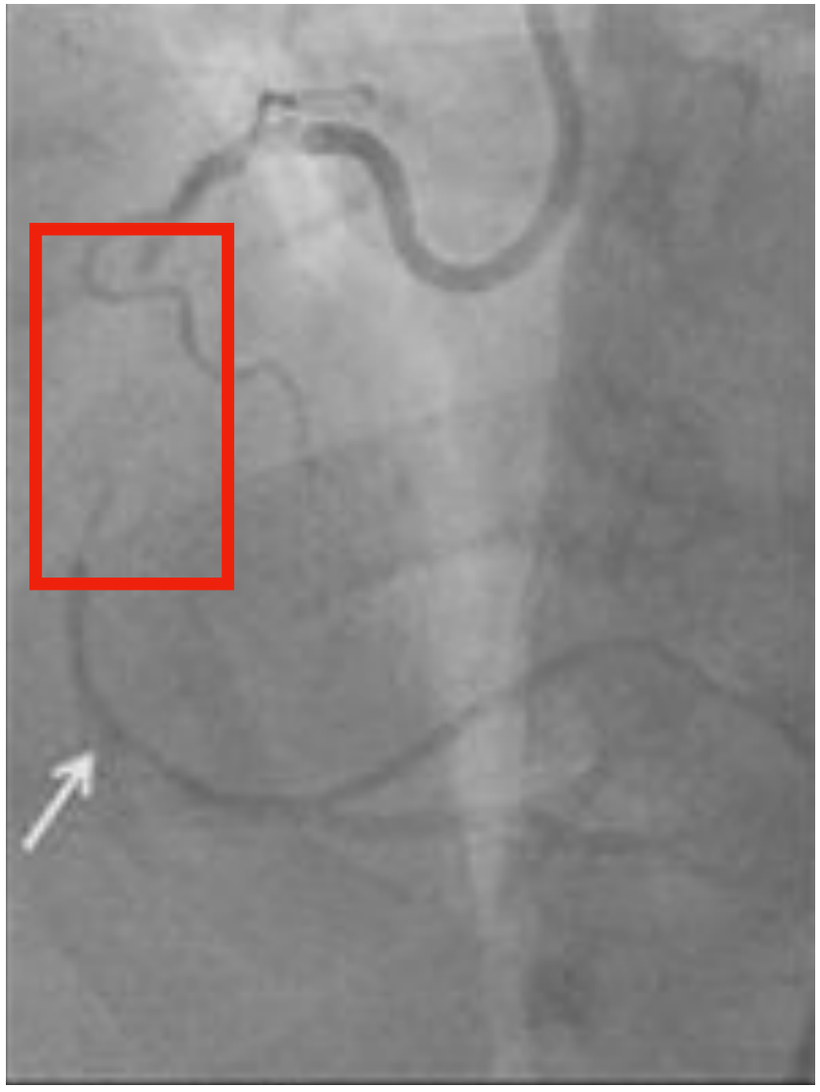
$$P(C \mid X) = \frac{P(X \mid C)P(C)}{P(X)}$$

- $P(X \mid C)$ **Likelihood** - learned from data
- $P(C)$ **Prior** - determined beforehand

Probability of one's beliefs about this object before evidences are taken into account.

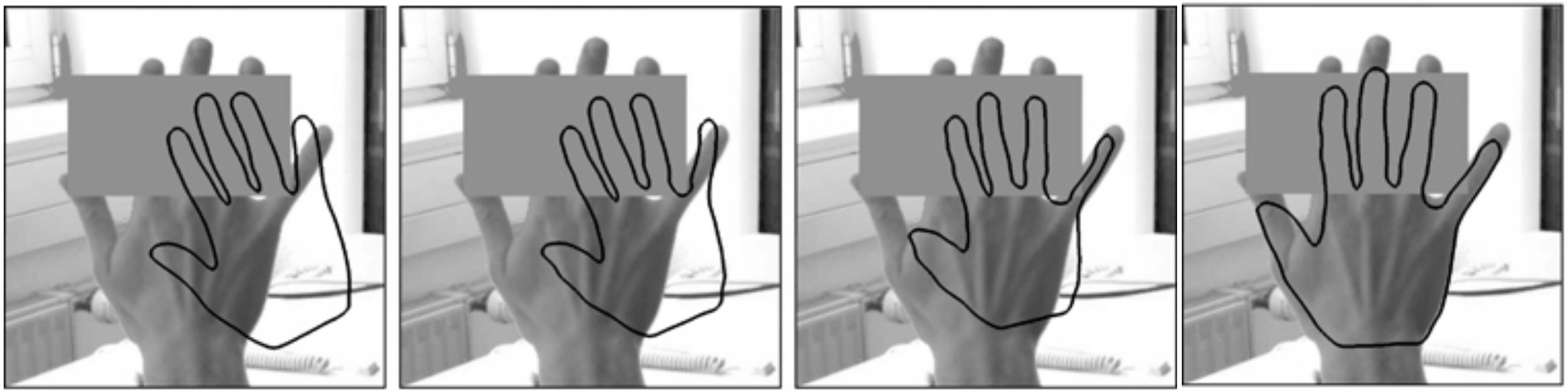
Prior

Normal

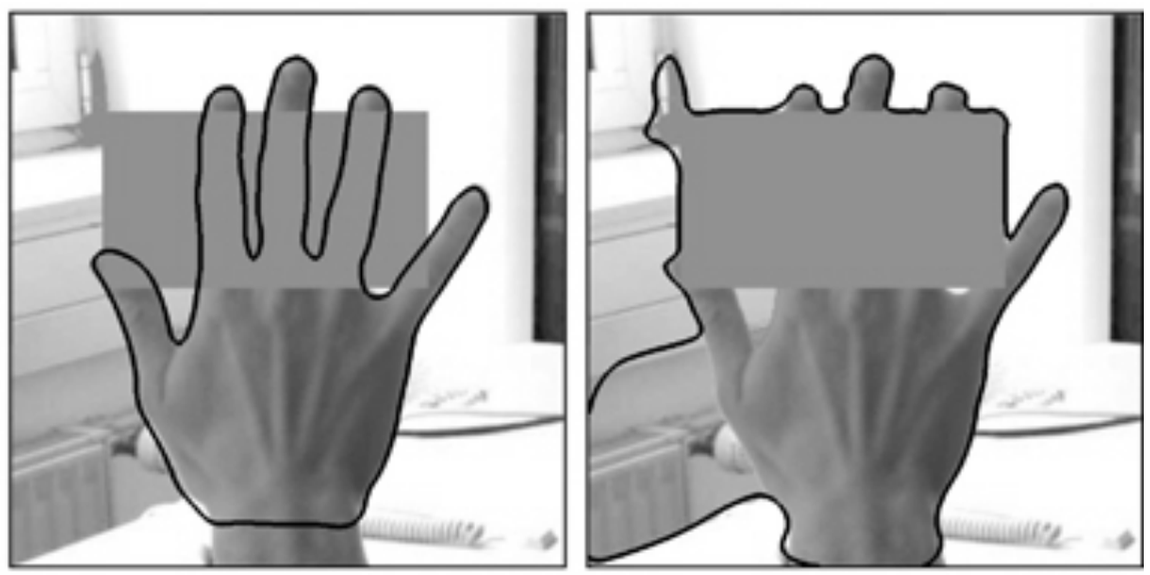


Coronary artery blockage

Prior



Initial Step 1 Step 2 Step 3



Final with prior Final w/ prior

Bayes' Rule for Classification

$$P(C \mid X) = \frac{P(X \mid C)P(C)}{P(X)}$$

- $P(X \mid C)$ **Likelihood** - learned from data
- $P(C)$ **Prior** - determined beforehand
- $P(X)$ **Evidence** - not needed for decision

The revised/updated probability of after considering new data. It is treated as a constant after giving a training data x.

Naive Bayes

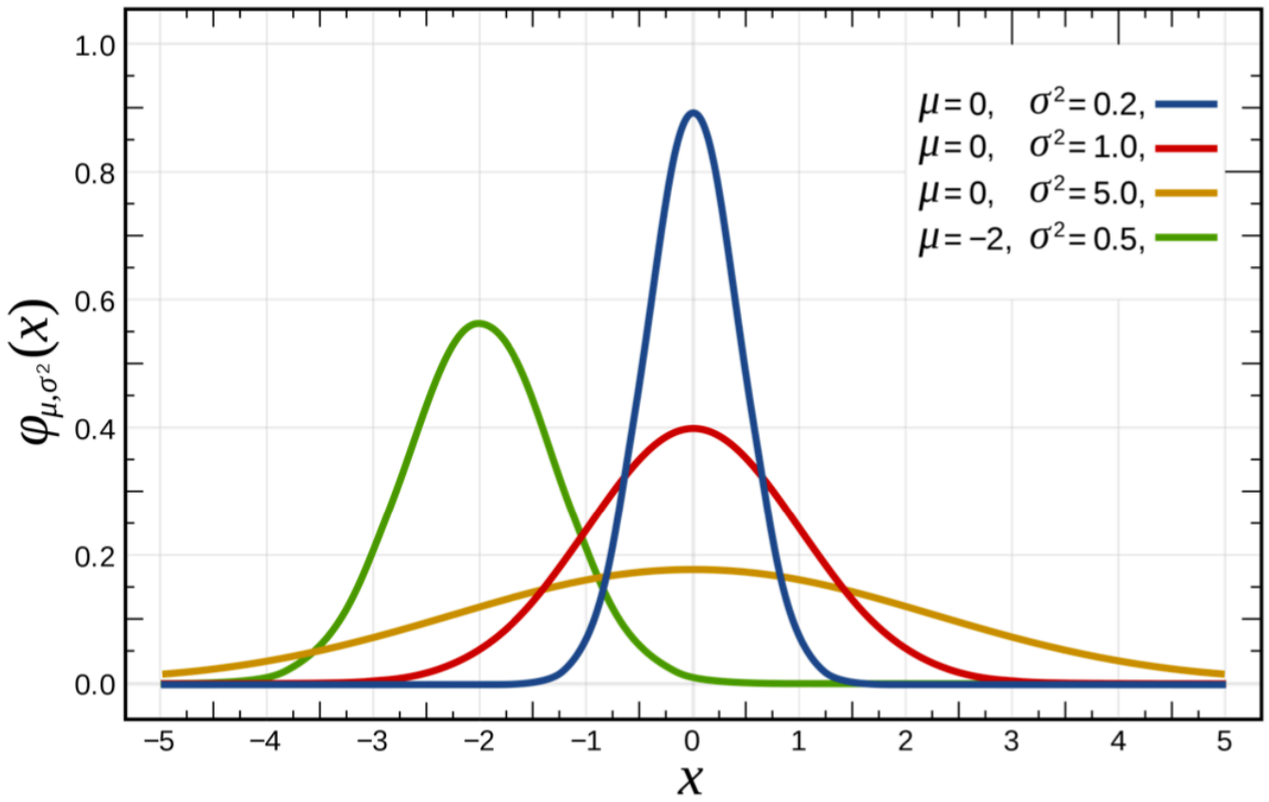
Multidimensional features $X = (X_1, X_2, \cdots, X_d)$.

"Naive" Assumption:

Assume features X_i are independent, given the class C:

$$P(X \mid C) = P(X_1 \mid C) \times P(X_2 \mid C) \times \cdots \times P(X_d \mid C).$$

Gaussian or Normal Distribution



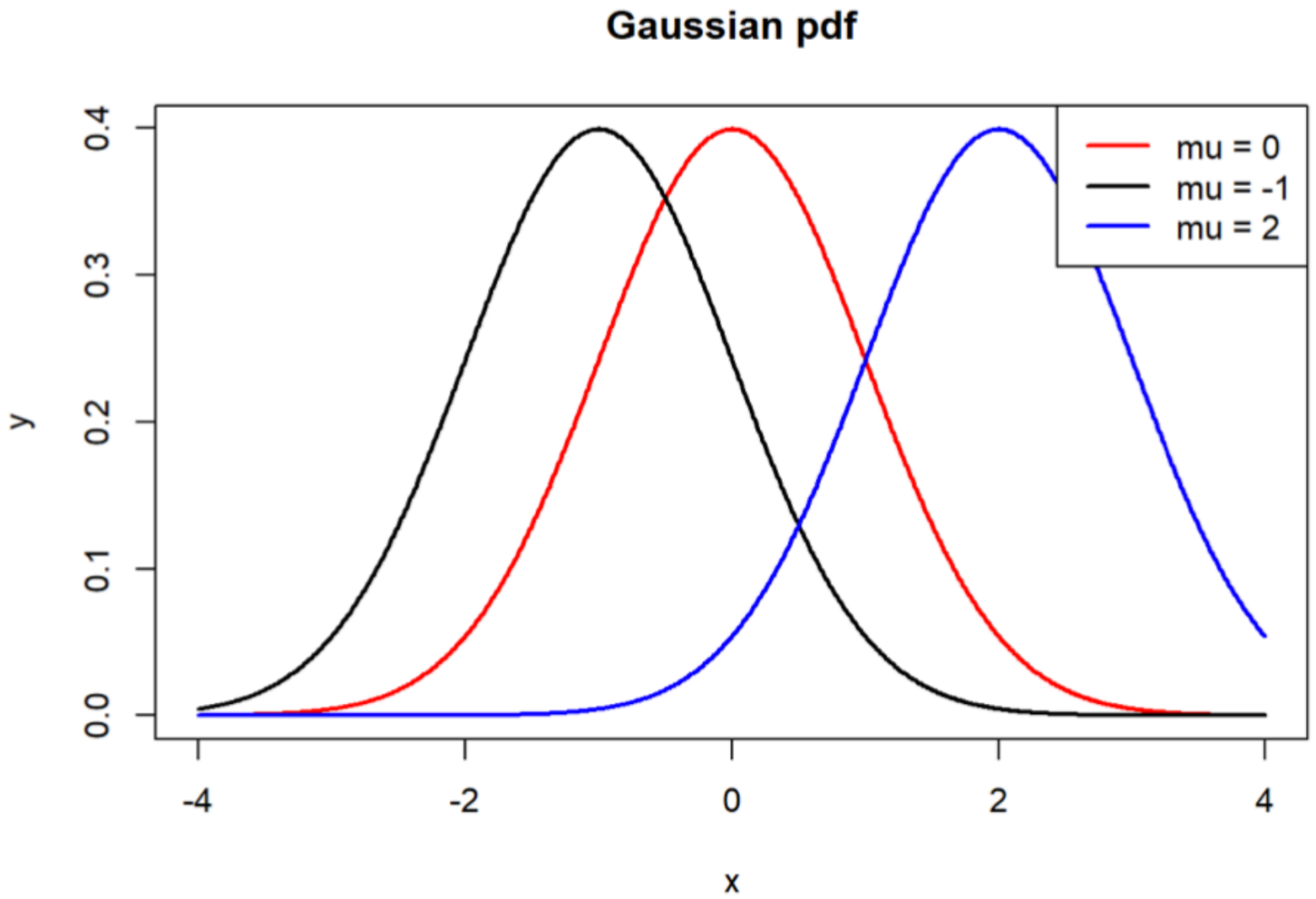
Probability density function (pdf):

$$p(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

Notation: $X \sim N(\mu, \sigma^2)$

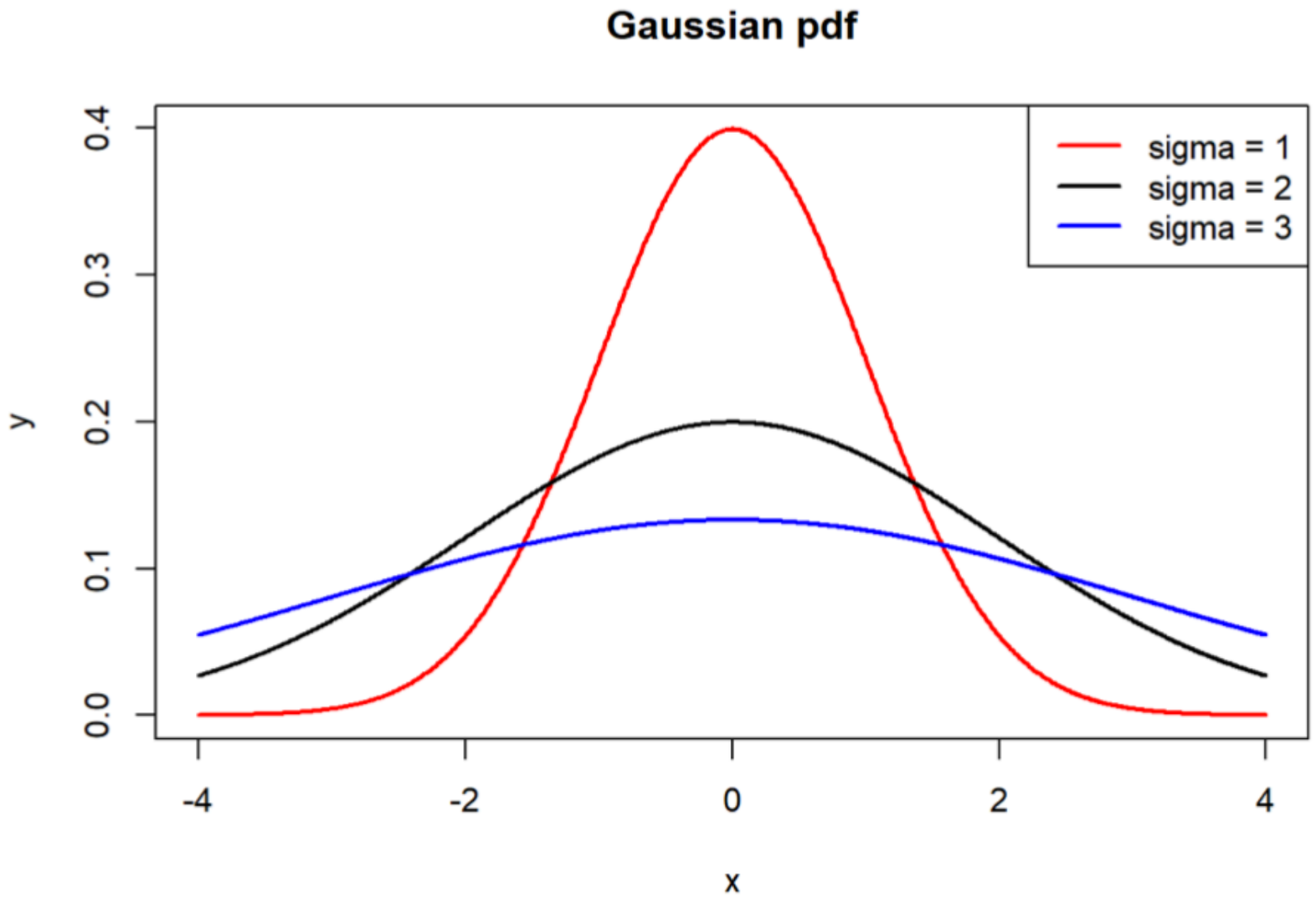
Mean, μ, and variance, σ^2, are parameters.

Gaussian μ Parameter



Shifts the pdf, shape stays the same

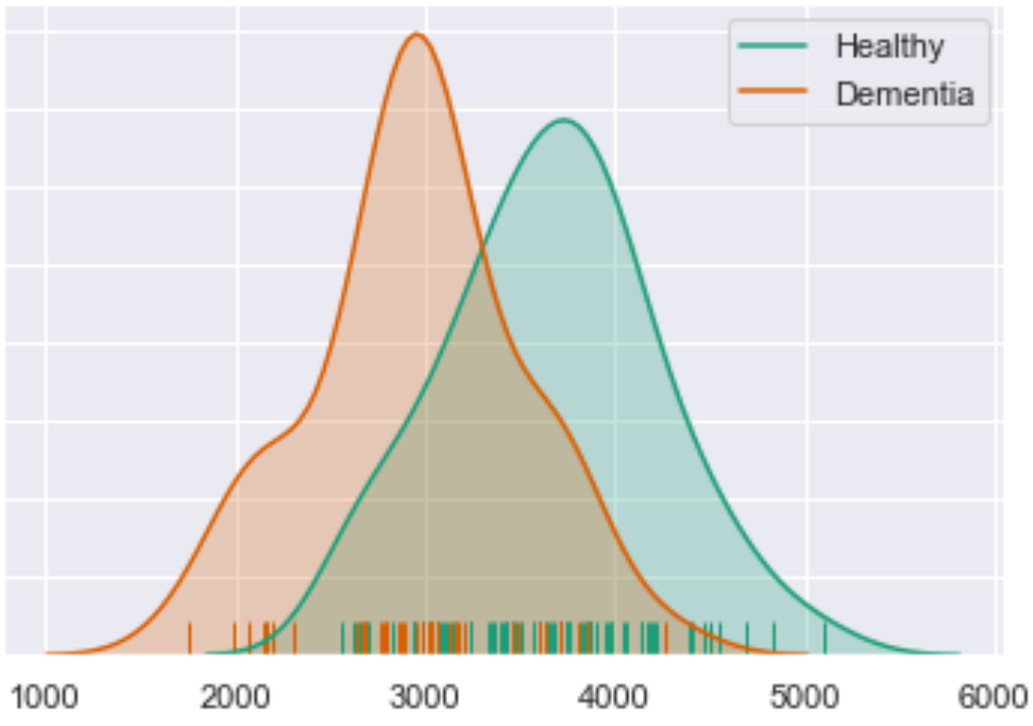
Gaussian σ Parameter



Stretches/shrinks the pdf, position stays the same

Gaussian or Normal Distribution

Density function of hippocampus volumes between healthy vs. dementia



Left Hippo Volume

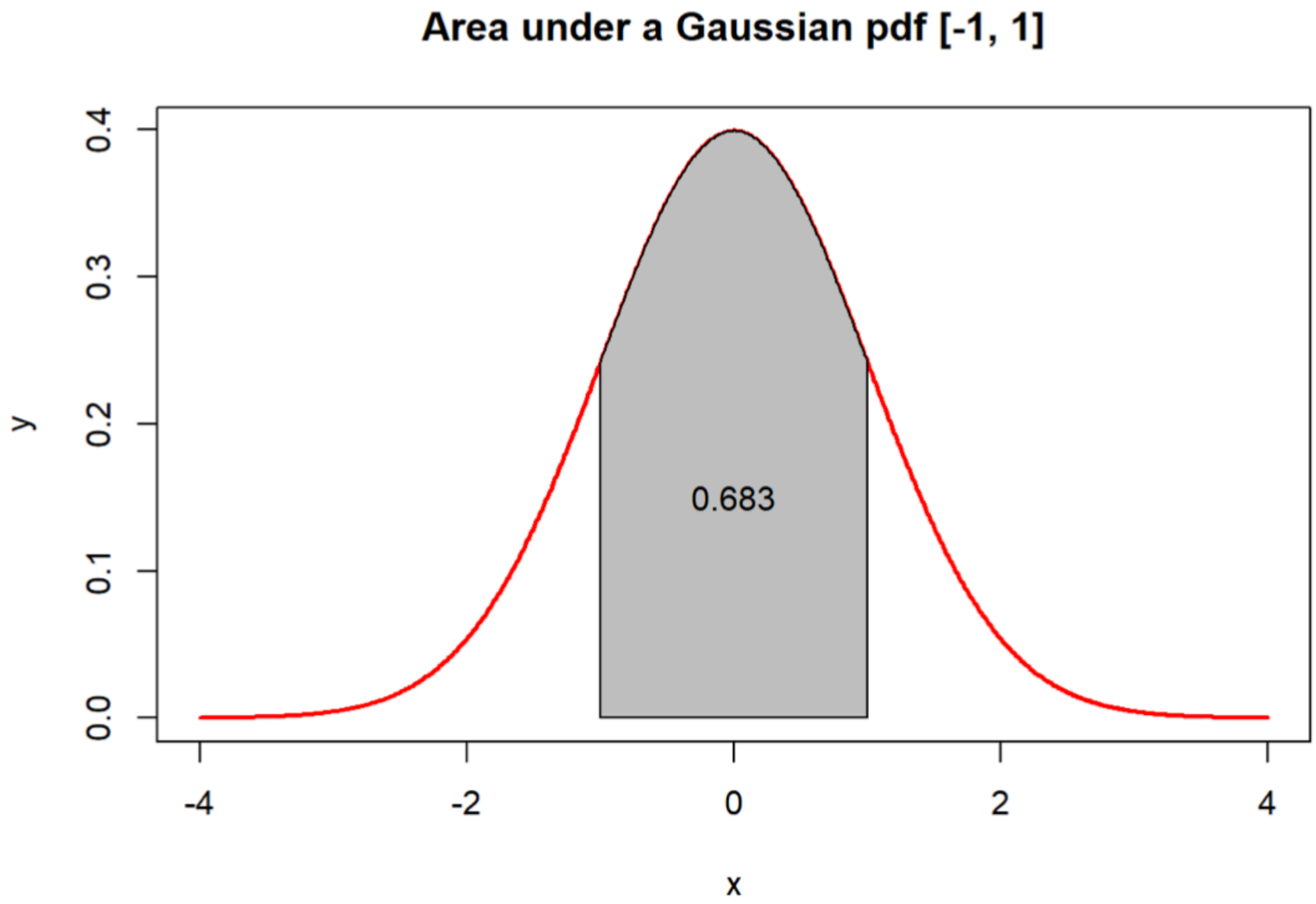
Right Hippo Volume

Probabilities of Continuous Random Variables

Probability is given by area under the pdf:

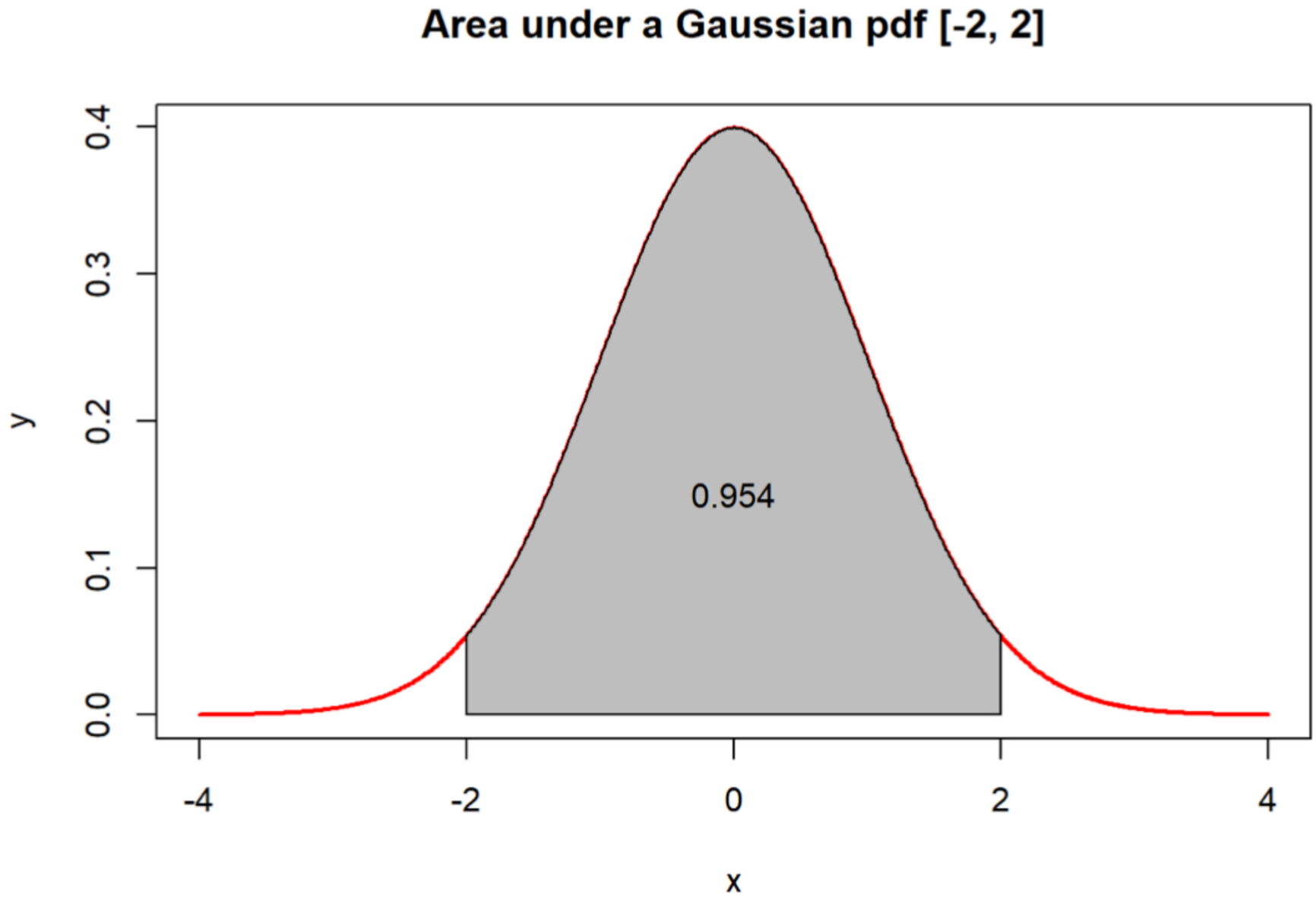
$$P(a < x < b) = \int_a^b p(x)dx$$

Gaussian Area



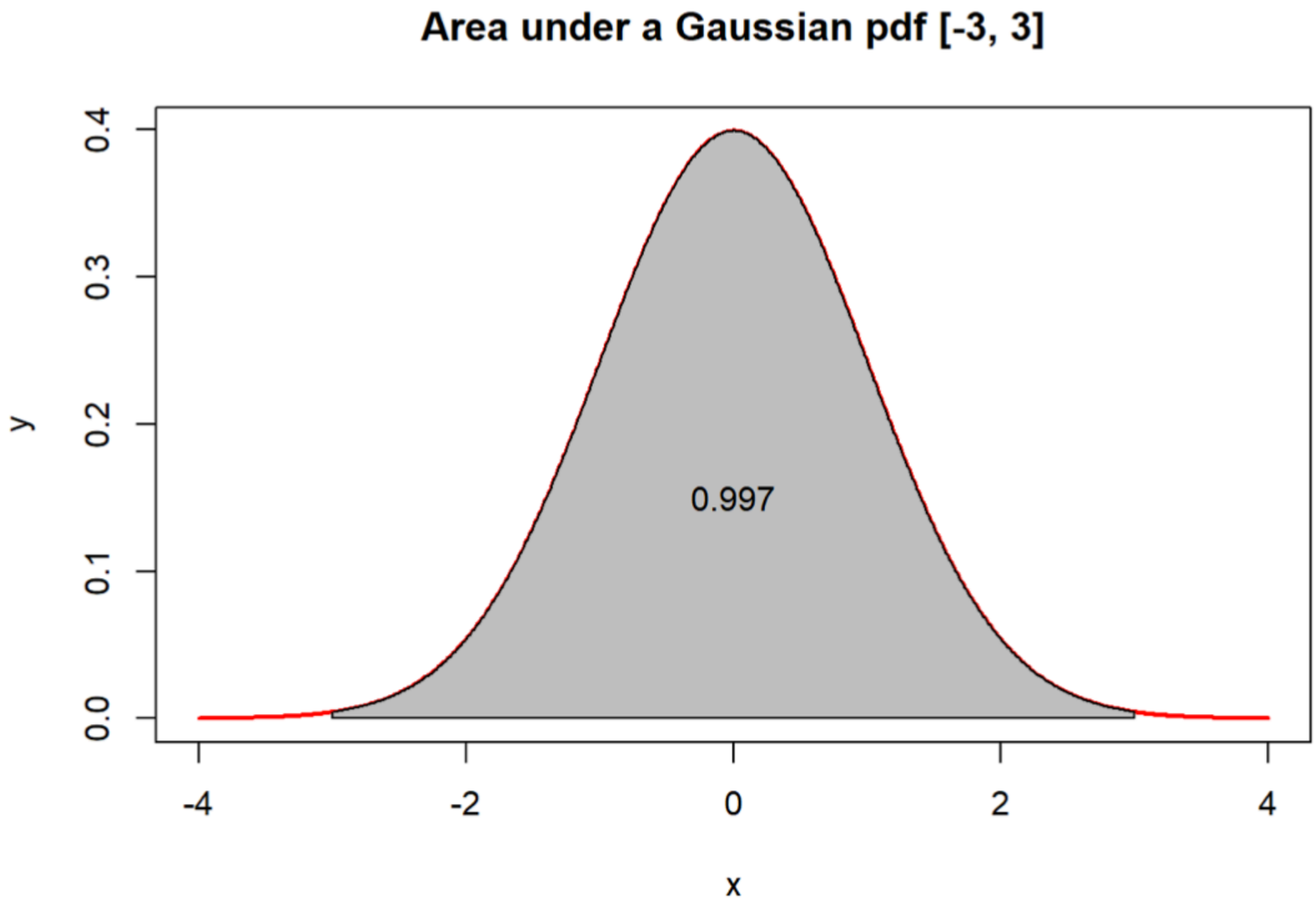
Units of horizontal axis are σ

Gaussian Area



Units of horizontal axis are σ

Gaussian Area



Units of horizontal axis are σ

Gaussian Naive Bayes

Likelihood is Gaussian pdf:

$$p(x \mid C = c_k) = \frac{1}{\sqrt{2\pi}\sigma_k} \exp\left(-\frac{(x - \mu_k)^2}{2\sigma_k^2}\right)$$

- The Gaussian depends on the class

$$C \in \{c_1, c_2, \cdots, c_k\}$$

- Each class needs a mean, μ_k, and a variance, σ_k^2

How to “Train” a Gaussian Distribution

For each feature in your data, given training data:

$(x_1, x_2, \cdots, x_n)$ all from the $k-$th class

Set parameters:

- Mean: $\hat{\mu}_k = \frac{1}{n}\sum_{i=1}^{n} x_i$
- Variance: $\hat{\sigma}_k^2 = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \hat{\mu}_k)^2$

Evidence Calculation

If we have K classes $C \in \{c_1, c_2, \cdots, c_K\}$:

$$p(x) = \sum_{k=1}^{K} p(X \mid C = c_k) P(C = c_k)$$

Using **Total Probability.**

For the case that we have two classes:

$$p(x) = p(x \mid C = c_1) P(C = c_1) + p(x \mid C = c_2) P(C = c_2)$$

Choosing a Prior

How to set $P(C = c_k)$?

- Equally likely: $P(C = c_k) = 1/K$
- Frequency of classes in training data
- Derive from previous experiments or knowledge

Putting It All Together

- Pick a prior: $P(C = c_k)$
- Train your Gaussians on training data: μ_k, σ_k^2
- For each test data point, $x^* = (x_1^*, x_2^*, \cdots, x_d^*)$, compute the likelihood:

$$p(x^* \mid C = c_k) = p(x_1^* \mid C = c_k) \times p(x_2^* \mid C = c_k) \times \cdots p(x_d^* \mid C = c_k)$$

- Compute the class probabilities:

$$P(C = c_k \mid x^*) = \frac{p(x^* \mid C = c_k) P(C = c_k)}{p(x^*)}$$

- Classify x^* as the class c_k with highest probability