Notes: Conditional Probability

Foundations of Data Analysis

February 11, 2021

Review of "English translation" for events:

- $A \cap B =$ "both events A and B happen"
- $A \cup B =$ "either event A or B (or both) happens"
- $A^c =$ "event A does not happen"

Set Theory Rules: (try drawing Venn diagrams of these)

- Definition of set difference: $A - B = A \cap B^c$ "event A happens, but B does not"
- Associative Law:
$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$
- Commutative Law:
$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$
- Distributive Law:
$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
- DeMorgan's Law:
$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

Counting:

- **Number of permutations of n items: (a.k.a. number of unique orderings)** $n! = n \times (n-1) \times (n-2) \times \cdots \times 2$
- **Number of ways to select k items out of n choices: (here order does not matter, just which k items you select)** $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Probability Rules:

- **Equally likely outcomes:** $P(A) = \frac{|A|}{|\Omega|}$
- **Inclusion-Exclusion Rule:** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- **Complement Rule:** $P(A^c) = 1 - P(A)$
- **Difference Rule:** $P(A - B) = P(A) - P(A \cap B)$

Exercise: Try deriving these rules from the definition of a probability function. Draw a Venn diagram to convince yourself they work.

Conditional Probability:

$$P(A \mid B) = \text{“the probability of event } A \text{ given that we know } B \text{ happened”}$$

Formula:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

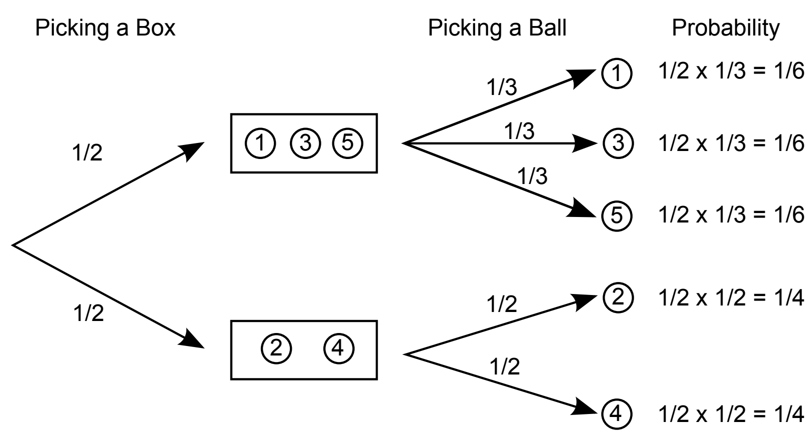
Multiplication Rule:

$$P(A \cap B) = P(A \mid B)P(B)$$

Tree diagrams to compute “two stage” probabilities (B = first stage, A = second stage):

1. First branch computes probability of first stage: $P(B)$
2. Second branch computes probability of second stage, given the first: $P(A \mid B)$
3. Multiply probabilities along a path to get final probabilities $P(A \cap B)$

Example: You are given two boxes with balls numbered 1 - 5. One box contains balls 1, 3, 5, and the other contains balls 2 and 4. You first pick a box at random, then pick a ball from that box at random. What is the probability that you pick a 2?



Exercise: You are analyzing the effectiveness of online advertising for a company that sells widgets. The company finds that 50% of traffic to their website comes from clicks of online ads. In addition, 20% of visitors to their website both had clicked an online ad and purchased a widget. If a person clicks on the company’s ad, what is the probability that they will purchase a widget?

Exercise: In Charlottesville the sky is overcast on about 40% of days. If it is overcast, there is a 25% chance that it will also be windy. What is the probability that it is both overcast and windy?

Sampling without replacement:
I have a box with 10 red balls and 10 green balls. I draw 2 balls from the box without replacing them. What is the probability that I get 2 red balls?

Define the events: $R1 =$ "first ball red" and $R2 =$ "second ball red", and use the product rule:

$$P(R1 \cap R2) = P(R1)P(R2 \mid R1) = \frac{1}{2} \times \frac{9}{19} = \frac{9}{38} \approx 0.24$$

If I draw 3 balls without replacement, what is the probability that they are all red?

$$\begin{aligned}
P(R1 \cap R2 \cap R3) &= P(R1 \cap R2)P(R3 \mid R1 \cap R2) && \text{Multiplication rule for } (R1 \cap R2) \cap R3 \\
&= P(R1)P(R2 \mid R1)P(R3 \mid R1 \cap R2) && \text{Multiplication rule for } R1 \cap R2 \\
&= \frac{1}{2} \times \frac{9}{19} \times \frac{8}{18} = \frac{18}{171} \approx 0.11
\end{aligned}$$