Topics:

- **Introduction to Chain Rule**
- **Recalling of chain rule of Single variable Calculus**
- **Chain rule for two variables**
- **Chain rule for three variables**

Introduction:

In **single-variable calculus**, we found that one of the most useful **differentiation rules is the chain rule**, which allows us to find the **derivative of the composition of two functions**.

The same thing is true for **multivariable calculus**, but this time we have to deal with **more than one form of the chain rule**.

In this section, we study **extensions of the chain rule** and learn how to take derivatives of compositions of functions of more than one variable.

Recall:

In **single variable calculus**, when $\boldsymbol{w = w(x)}$ is a **differentiable function** of $\boldsymbol{x}$ and $\boldsymbol{x = x(t)}$ is **differentiable function** of $\boldsymbol{t}$, then $\boldsymbol{w}$ becomes the **differentiable function** of $\boldsymbol{t}$ and $\boldsymbol{\frac{dw}{dt}}$ can be calculated with **chain rule formula**

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$

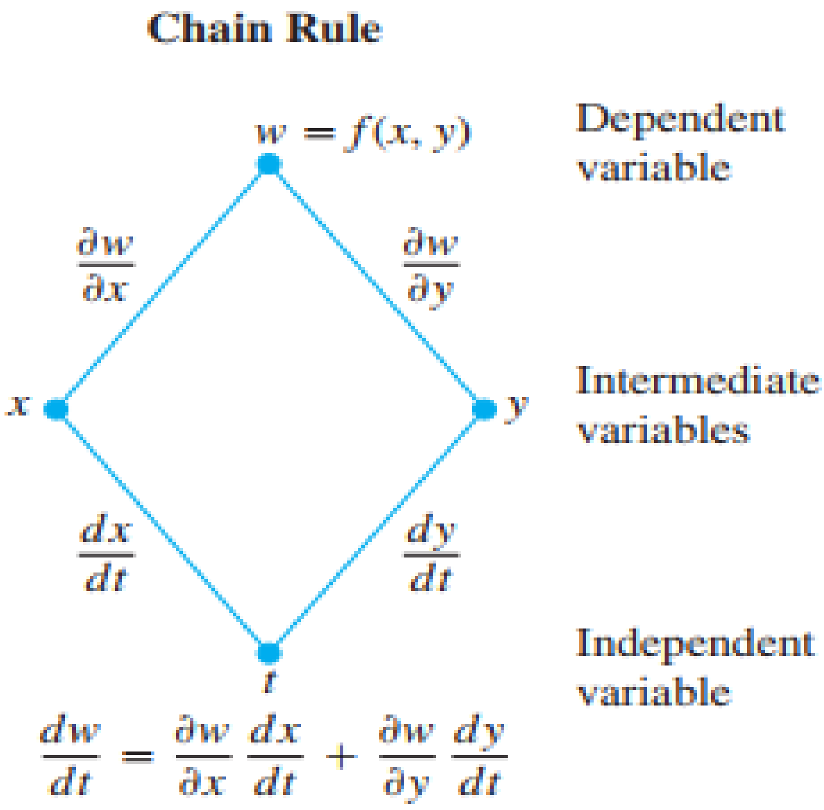
Chain Rule Case 1: Type I:

- **When w is a function of two variables x and y and then both x and y are function of one variable t**

If $w = w(x, y)$ has **continuous partial derivatives** w_x and w_y and if $x = x\,(t)$ and $y = y\,(t)$ are **differentiable functions** of t, then the **composite function** $w = w(x(t)\,, y(t))$ is a **differentiable function** of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

Tree Diagram:



Example 1:

Using the Chain Rule

Calculate dz/dt for each of the following functions:

a. $z = f(x, y) = 4x^2 + 3y^2, x = x(t) = \sin t, y = y(t) = \cos t$

b. $z = f(x, y) = \sqrt{x^2 - y^2}, x = x(t) = e^{2t}, y = y(t) = e^{-t}$

Solution

a. To use the chain rule, we need four quantities—$\partial z/\partial x$, $\partial z/\partial y$, dx/dt, and dy/dt:

$$\frac{\partial z}{\partial x} = 8x \qquad \frac{\partial z}{\partial y} = 6y$$
$$\frac{dx}{dt} = \cos t \qquad \frac{dy}{dt} = -\sin t$$

Now, we substitute each of these into Equation 4.29:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \\ &= (8x)(\cos t) + (6y)(-\sin t) \\ &= 8x \cos t - 6y \sin t.\end{aligned}$$

This answer has three variables in it. To reduce it to one variable, use the fact that $x(t) = \sin t$ and $y(t) = \cos t$. We obtain

$$\begin{aligned}\frac{dz}{dt} &= 8x \cos t - 6y \sin t \\ &= 8(\sin t)\cos t - 6(\cos t)\sin t \\ &= 2 \sin t \cos t.\end{aligned}$$

This derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x, y)$, then differentiating with respect to t:

$$\begin{aligned}z &= f(x, y) \\ &= f(x(t), y(t)) \\ &= 4(x(t))^2 + 3(y(t))^2 \\ &= 4\sin^2 t + 3\cos^2 t.\end{aligned}$$

Then

$$\begin{aligned}\frac{dz}{dt} &= 2(4 \sin t)(\cos t) + 2(3 \cos t)(-\sin t) \\ &= 8 \sin t \cos t - 6 \sin t \cos t \\ &= 2 \sin t \cos t,\end{aligned}$$

which is the same solution. However, it may not always be this easy to differentiate in this form.

b. To use the chain rule, we again need four quantities—$\partial z/\partial x$, $\partial z/dy$, dx/dt, and dy/dt:

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2-y^2}} \qquad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{x^2-y^2}}$$
$$\frac{dx}{dt} = 2e^{2t} \qquad \frac{dx}{dt} = -e^{-t}.$$

We substitute each of these into Equation 4.29:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\\ &= \left(\frac{x}{\sqrt{x^2-y^2}}\right)\left(2e^{2t}\right) + \left(\frac{-y}{\sqrt{x^2-y^2}}\right)\left(-e^{-t}\right)\\ &= \frac{2xe^{2t}-ye^{-t}}{\sqrt{x^2-y^2}}.\end{aligned}$$

To reduce this to one variable, we use the fact that $x(t) = e^{2t}$ and $y(t) = e^{-t}$. Therefore,

$$\begin{aligned}\frac{dz}{dt} &= \frac{2xe^{2t}+ye^{-t}}{\sqrt{x^2-y^2}}\\ &= \frac{2\left(e^{2t}\right)e^{2t}+\left(e^{-t}\right)e^{-t}}{\sqrt{e^{4t}-e^{-2t}}}\\ &= \frac{2e^{4t}+e^{-2t}}{\sqrt{e^{4t}-e^{-2t}}}.\end{aligned}$$

To eliminate negative exponents, we multiply the top by e^{2t} and the bottom by $\sqrt{e^{4t}}$:

$$\begin{aligned}\frac{dz}{dt} &= \frac{2e^{4t}+e^{-2t}}{\sqrt{e^{4t}-e^{-2t}}}\cdot\frac{e^{2t}}{\sqrt{e^{4t}}}\\ &= \frac{2e^{6t}+1}{\sqrt{e^{8t}-e^{2t}}}\\ &= \frac{2e^{6t}+1}{\sqrt{e^{2t}\left(e^{6t}-1\right)}}\\ &= \frac{2e^{6t}+1}{e^{t}\sqrt{e^{6t}-1}}.\end{aligned}$$

Again, this derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x, y)$, then differentiating with respect to t:

$$\begin{aligned} z &= f(x, y) \\ &= f(x(t), y(t)) \\ &= \sqrt{(x(t))^2 - (y(t))^2} \\ &= \sqrt{e^{4t} - e^{-2t}} \\ &= \left(e^{4t} - e^{-2t}\right)^{1/2}. \end{aligned}$$

Then

$$\begin{aligned} \frac{dz}{dt} &= \frac{1}{2}\left(e^{4t} - e^{-2t}\right)^{-1/2}\left(4e^{4t} + 2e^{-2t}\right) \\ &= \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}}. \end{aligned}$$

This is the same solution.

Example 2:

Use **chain rule** to find the **derivative** of

$$w = xy$$

with respect to variable t along the path

$$x = \cos(t) \qquad \text{and} \qquad y = \sin(t)$$

What is the value of derivative at $t = \frac{\pi}{2}$?

Solution:

Formula:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} . \frac{dx}{dt} + \frac{\partial w}{\partial y} . \frac{dy}{dt} \text{ --------(1)}$$

Step 1: Find partial derivatives of w w.r.t both x and y variables

$$w = xy$$

- Partially differentiate w.r.t x

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(xy) = y\frac{\partial}{\partial x}(x) = y(1) = \boldsymbol{y}$$

- Similarity, partially differentiate w.r.t $\boldsymbol{y}$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}(xy) = x\frac{\partial}{\partial y}(y) = x(1) = \boldsymbol{x}$$

Step 2: Find ordinary derivatives of $\boldsymbol{x}$ w.r.t variable $\boldsymbol{t}$

$$\boldsymbol{x = \cos(t)}$$

- Differentiate w.r.t $\boldsymbol{t}$

$$\frac{dx}{dt} = \frac{d}{dt}[\cos(t)] = -\sin(t)$$

Step 3: Find ordinary derivatives of $\boldsymbol{y}$ w.r.t variable $\boldsymbol{t}$

$$\boldsymbol{y = \sin(t)}$$

- Differentiate w.r.t $\boldsymbol{t}$

$$\frac{dy}{dt} = \frac{d}{dt}[\sin(t)] = \cos(t)$$

Step 4: Substitute values in formula (1)

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{dw}{dt} = [y][-\sin(t)] + [x][\cos(t)]$$

$$\frac{dw}{dt} = -y\sin(t) + x\cos(t)$$

Re-arranging,

$$\frac{dw}{dt} = x\cos(t) - y\sin(t)$$

Substitute values of x and y, we have

$$\frac{dw}{dt} = \left(\cos(t)\right)\left(\cos(t)\right) - \left(\sin(t)\right)\left(\sin(t)\right)$$

$$\frac{dw}{dt} = \cos^2(t) - \sin^2(t)$$

Since

$$\cos(2t) = \cos^2(t) - \sin^2(t)$$

Therefore,

$$\frac{dw}{dt} = \cos(2t)$$

Step 5: Find the value of derivative at given point $t = \frac{\pi}{2}$

At the given value of t,

$$\left[\frac{dw}{dt}\right]_{t=\frac{\pi}{2}} = \cos\left(2.\left(\frac{\pi}{2}\right)\right) = \cos(\pi) = -1$$

Hence

$$\left[\frac{dw}{dt}\right]_{t=\frac{\pi}{2}} = -1$$

Practice question with answer:

Calculate dz/dt given the following functions. Express the final answer in terms of t.

$$\begin{aligned} z &= f(x,y) = x^2 - 3xy + 2y^2, \\ x &= x(t) = 3\sin 2t, \\ y &= y(t) = 4\cos 2t \end{aligned}$$

Hint
Answer

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= (2x - 3y)(6\cos 2t) + (-3x + 4y)(-8\sin 2t) \\ &= -92\sin 2t\cos 2t - 72(\cos^2 2t - \sin^2 2t) \\ &= -46\sin 4t - 72\cos 4t. \end{aligned}$$

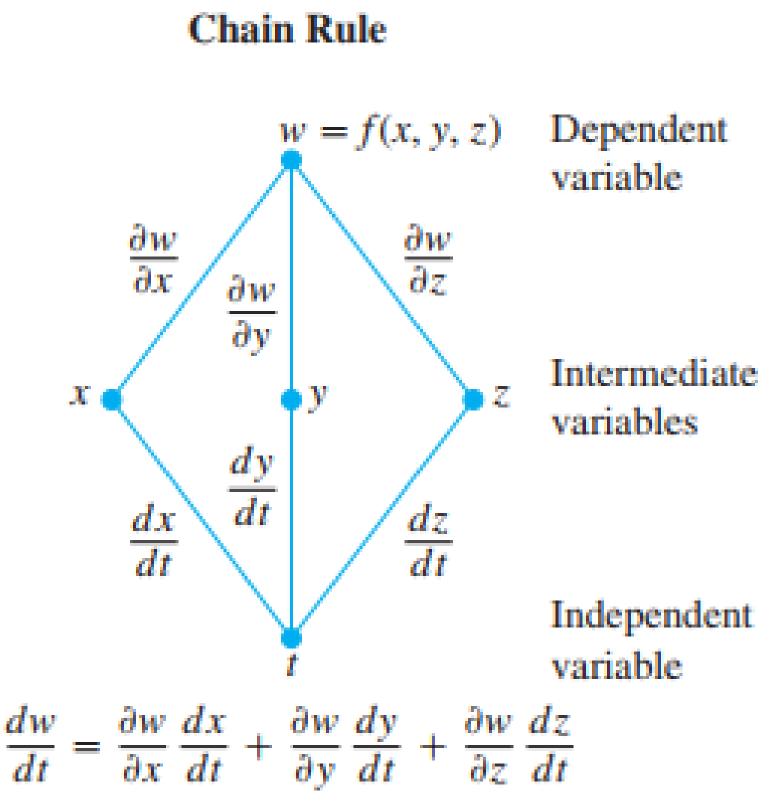
Chain Rule Case 1: Type II:

- **When w is a function of three variables x, y and z and then these three variables x, y and z are functions of single variable t**

If $w = w(x, y, z)$ is a **differentiable** function of three variables x, y, z and these three variables x, y, z are **differentiable functions** of variable t, then w is a **differentiable function** of t.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Tree Diagram:



Example 1: Changes in Function's values along a Helix

If

$$w = xy + z$$

and

$$x = \cos(t), \quad y = \sin(t), \qquad z = t$$

then find the derivative $\frac{dw}{dt}$. In this example the values of w are **changing along the path** of a **helix**. What is the value of derivative at $t = 0$?

Solution:

Formula:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt} \text{--------}(1)$$

Step 1: Find partial derivatives of w w.r.t x, y and z variables

$$w = xy + z$$

- Partially differentiate w.r.t x

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(xy + z) = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(z) = y\frac{\partial}{\partial x}(x) + 0 = y(1) = \boldsymbol{y}$$

- Similarity, partially differentiate w.r.t y

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}(xy + z) = \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial y}(z) = x\frac{\partial}{\partial y}(y) + 0 = x(1) = \boldsymbol{x}$$

- Similarity, partially differentiate w.r.t z

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}(xy + z) = \frac{\partial}{\partial z}(xy) + \frac{\partial}{\partial z}(z) = 0 + 1 = \mathbf{1}$$

Step 2: Find ordinary derivatives of x w.r.t variable t

$$x = \cos(t)$$

- Differentiate w.r.t t

$$\frac{dx}{dt} = \frac{d}{dt}\left[\cos(t)\right] = -\sin(t)$$

Step 3: Find ordinary derivatives of y w.r.t variable t

$$y = \sin(t)$$

- Differentiate w.r.t t

$$\frac{dy}{dt} = \frac{d}{dt}\left[\sin(t)\right] = \cos(t)$$

Step 4: Find ordinary derivatives of y w.r.t variable t

$$z = t$$

- Differentiate w.r.t t

$$\frac{dz}{dt} = \frac{d}{dt}\left(t\right) = 1$$

Step 5: Substitute values in formula (1)

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}.\frac{dx}{dt} + \frac{\partial w}{\partial y}.\frac{dy}{dt} + \frac{\partial w}{\partial z}.\frac{dz}{dt}$$

$$\frac{dw}{dt} = [y][-\sin(t)] + [x][\cos(t)] + [1][1]$$

$$\frac{dw}{dt} = -y\sin\left(t\right) + x\cos\left(t\right) + 1$$

Re-arranging,

$$\frac{dw}{dt} = x\cos(t) - y\sin(t) + 1$$

Substitute values of x, y and z, we have

$$\frac{dw}{dt} = (\cos(t))(\cos(t)) - (\sin(t))(\sin(t)) + 1$$

$$\frac{dw}{dt} = \cos^2(t) - \sin^2(t) + 1$$

$$\frac{dw}{dt} = \cos(2t) + 1$$

Step 6: Find the value of derivative at given point $t = 0$

$$\left[\frac{dw}{dt}\right]_{t=0} = \cos(2.(0)) + 1 = \cos(0) + 1$$

$$\left[\frac{dw}{dt}\right]_{t=0} = 1 + 1 = 2$$

Hence

$$\left[\frac{dw}{dt}\right]_{t=0} = 2$$

Example 2 $\frac{dz}{dx}$ for $z = x\ln(xy) + y^3$, $y = \cos(x^2+1)$

Hide Solution ▼

We'll just plug into the formula.

$$\begin{aligned}\frac{dz}{dx} &= \left(\ln(xy) + x\frac{y}{xy}\right) + \left(x\frac{x}{xy} + 3y^2\right)\left(-2x\sin(x^2+1)\right)\\ &= \ln(x\cos(x^2+1)) + 1 - 2x\sin(x^2+1)\left(\frac{x}{\cos(x^2+1)} + 3\cos^2(x^2+1)\right)\\ &= \ln(x\cos(x^2+1)) + 1 - 2x^2\tan(x^2+1) - 6x\sin(x^2+1)\cos^2(x^2+1)\end{aligned}$$

Example 3:

If

$$z = xe^{xy}$$

and

$$x = x(t) = t^2 \text{ and } y = y(t) = \frac{1}{t} = t^{-1}$$

then find the derivative $\frac{dz}{dt}$.

Solution:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\\ &= \left(\mathbf{e}^{xy} + yx\mathbf{e}^{xy}\right)(2t) + x^2\mathbf{e}^{xy}\left(-t^{-2}\right)\\ &= 2t\left(\mathbf{e}^{xy} + yx\mathbf{e}^{xy}\right) - t^{-2}x^2\mathbf{e}^{xy}\end{aligned}$$

Substituting values of $\boldsymbol{x}$ and $\boldsymbol{y}$, we have

$$\frac{dz}{dt} = 2t\left(\mathbf{e}^{t} + t\mathbf{e}^{t}\right) - t^{-2}t^{4}\mathbf{e}^{t} = 2t\mathbf{e}^{t} + t^{2}\mathbf{e}^{t}$$

Practice Questions for students:

Thomas Calculus: 12th Edition, Ex# 14.4, Q # 1 – 6.

Chain Rule: One Independent Variable

a) Express $\boldsymbol{\frac{dw}{dt}}$ as a function of $\boldsymbol{t}$ by using **chain rule**.

b) Evaluate $\boldsymbol{\frac{dw}{dt}}$ at the given value of $\boldsymbol{t}$.

1. $w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t; \quad t = \pi$

2. $w = x^2 + y^2, \quad x = \cos t + \sin t, \quad y = \cos t - \sin t; \quad t = 0$

3. $w = \frac{x}{z} + \frac{y}{z}, \quad x = \cos^2 t, \quad y = \sin^2 t, \quad z = 1/t; \quad t = 3$

4. $w = \ln(x^2 + y^2 + z^2), \quad x = \cos t, \quad y = \sin t, \quad z = 4\sqrt{t}; \quad t = 3$

5. $w = 2ye^x - \ln z, \quad x = \ln(t^2 + 1), \quad y = \tan^{-1} t, \quad z = e^t; \quad t = 1$

6. $w = z - \sin xy, \quad x = t, \quad y = \ln t, \quad z = e^{t-1}; \quad t = 1$

Chain Rule: Functions defined on surfaces:

If we are interested in the temperature $\boldsymbol{w = w(x, y, z)}$ on a globe in space, we might prefer to think of $\boldsymbol{x, y}$ and $\boldsymbol{z}$ as functions of variables $\boldsymbol{r}$ and $\boldsymbol{s}$ that gives points' longitudes & latitudes. If $\boldsymbol{x = x(r, s)}$, $\boldsymbol{y = y(r, s)}$ and $\boldsymbol{z = z(r, s)}$, we could then express the temperature as a function of r and s with the composite function.

$$\boldsymbol{w = w\big(x(r, s), y(r, s), z(r, s)\big)}$$

Under the right conditions, w could have partial derivatives with respect to both $\boldsymbol{r}$ and $\boldsymbol{s}$ that could be calculated in the following way.

Chain Rule Case 2: Type I:

- **When w is a function of two variables x and y and then these two variables x and y are further functions of two variables r and s.**

Suppose that $w = w(x, y)$; and $x = x(r, s)$ and $y = y(r, s)$. If all the three functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s}$$

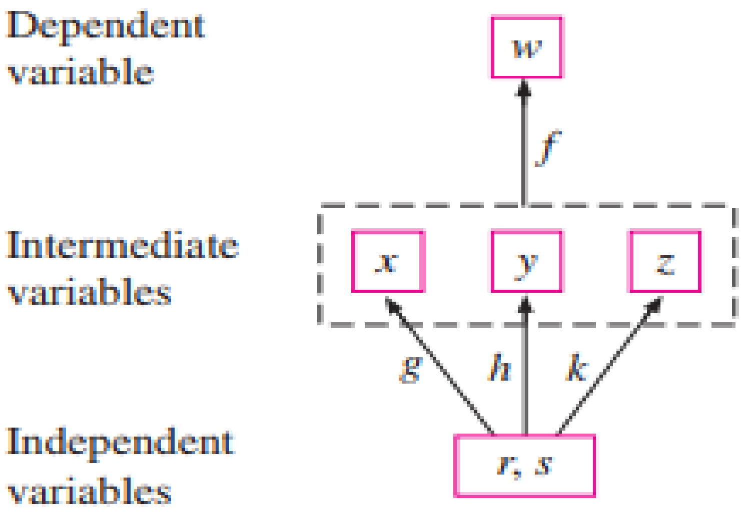
Chain Rule Case 2: Type II:

- **When w is a function of three variables x, y and z and then these three variables x, y and z are further functions of two variables r and s.**

Suppose that $w = w(x, y, z)$; and $x = x(r, s)$, $y = y(r, s)$ and $z = z(r, s)$. If all the four functions are **differentiable**, then w has **partial derivatives** with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

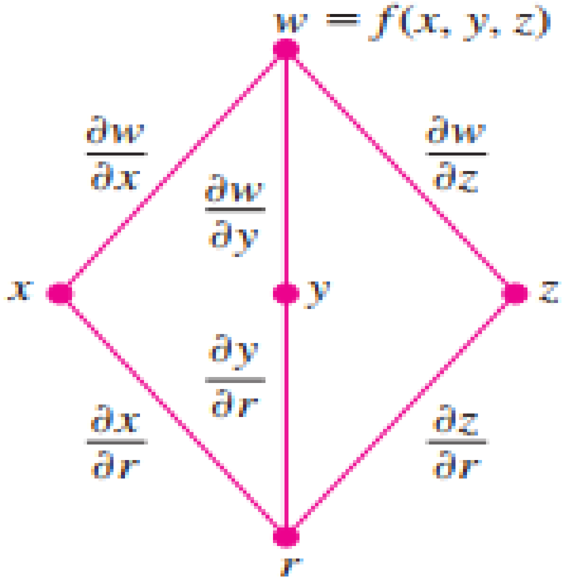
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$



$$w = f(g(r, s), h(r, s), k(r, s))$$

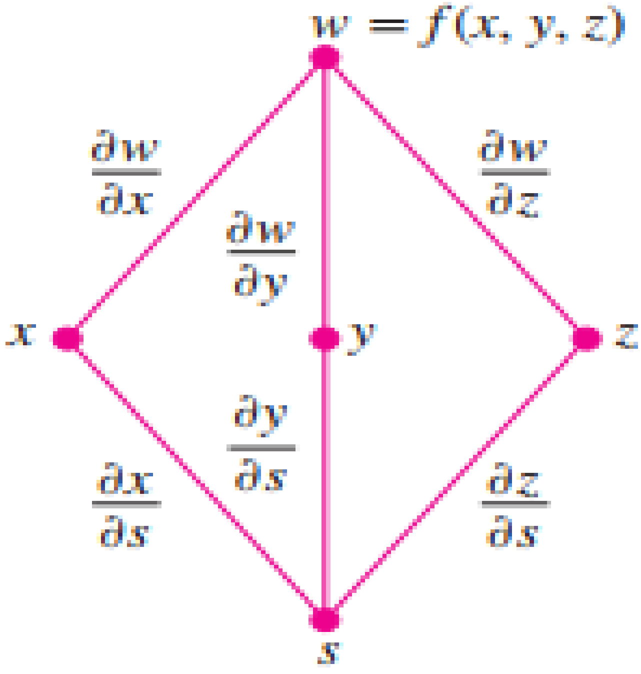
(a)

Tree Diagram:



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

(b)



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

(c)

Example 1:

Calculate $\partial w/\partial u$ and $\partial w/\partial v$ using the following functions:

$$\begin{aligned} w &= f(x, y, z) = 3x^2 - 2xy + 4z^2 \\ x &= x(u, v) = e^u \sin v \\ y &= y(u, v) = e^u \cos v \\ z &= z(u, v) = e^u. \end{aligned}$$

Solution:

The formulas for $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ are

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u}$$

and

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v}.$$

Therefore, there are nine different partial derivatives that need to be calculated and substituted. We need to calculate each of them:

$$\begin{aligned} \frac{\partial w}{\partial x} &= 6x - 2y & \frac{\partial w}{\partial y} &= -2x & \frac{\partial w}{\partial z} &= 8z \\ \frac{\partial x}{\partial u} &= e^u \sin v & \frac{\partial y}{\partial u} &= e^u \cos v & \frac{\partial z}{\partial u} &= e^u \\ \frac{\partial x}{\partial v} &= e^u \cos v & \frac{\partial y}{\partial v} &= -e^u \sin v & \frac{\partial z}{\partial v} &= 0. \end{aligned}$$

Now, we substitute each of them into the first formula to calculate $\boldsymbol{\frac{\partial w}{\partial u}}$

$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u} \\ &= (6x - 2y)\, e^u \sin v - 2xe^u \cos v + 8ze^u, \end{aligned}$$

Then substitute $\boldsymbol{x(u, v) = e^u \sin(v)}$, $\boldsymbol{y(u, v) = e^u \cos(v)}$ and $\boldsymbol{z(u, v) = e^u}$ into this equation:

$$\begin{aligned} \frac{\partial w}{\partial u} &= (6x - 2y)\, e^u \sin v - 2xe^u \cos v + 8ze^u \\ &= (6e^u \sin v - 2e^u \cos v)\, e^u \sin v - 2\,(e^u \sin v)\, e^u \cos v + 8e^{2u} \\ &= 6e^{2u} \sin^2 v - 4e^{2u} \sin v \cos v + 8e^{2u} \\ &= 2e^{2u} \left(3 \sin^2 v - 2 \sin v \cos v + 4\right). \end{aligned}$$

Next, we calculate $\frac{\partial w}{\partial v}$

$$\begin{aligned}\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v} \\ &= (6x-2y)\,e^{u}\cos v - 2x\,(-e^{u}\sin v) + 8z\,(0)\,,\end{aligned}$$

Then substitute $\boldsymbol{x(u,v) = e^{u}\sin(v)}$, $\boldsymbol{y(u,v) = e^{u}\cos(v)}$ and $\boldsymbol{z(u,v) = e^{u}}$ into this equation:

$$\begin{aligned}\frac{\partial w}{\partial v} &= (6x-2y)\,e^{u}\cos v - 2x\,(-e^{u}\sin v) \\ &= (6e^{u}\sin v - 2e^{u}\cos v)\,e^{u}\cos v + 2\,(e^{u}\sin v)\,(e^{u}\sin v) \\ &= 2e^{2u}\sin^2 v + 6e^{2u}\sin v\cos v - 2e^{2u}\cos^2 v \\ &= 2e^{2u}\left(\sin^2 v + \sin v\cos v - \cos^2 v\right).\end{aligned}$$

Example 2:

Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of $\boldsymbol{r}$ and $\boldsymbol{s}$, if

$$w = x + 2y + z^2$$

and

$$x = \frac{r}{s}\,; \quad y = r^2 + \ln(s)\,; \quad z = 2r$$

Solution:

Step 1: Recall the chain rule

The chain rule is applied here because w depends on r and s indirectly through x, y, and z.

- For $\frac{\partial w}{\partial r}$:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}.$$

- For $\frac{\partial w}{\partial s}$:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Step 2: Compute intermediate derivatives

(a) Partial derivatives of w with respect to x, y, and z:

From $w = x + 2y + z^2$:

$$\frac{\partial w}{\partial x} = 1, \quad \frac{\partial w}{\partial y} = 2, \quad \frac{\partial w}{\partial z} = 2z.$$

(b) Partial derivatives of x, y, and z with respect to r and s:

1. $x = \frac{r}{s}$:

$$\frac{\partial x}{\partial r} = \frac{1}{s}, \quad \frac{\partial x}{\partial s} = -\frac{r}{s^2}.$$

2. $y = r^2 + \ln(s)$:

$$\frac{\partial y}{\partial r} = 2r, \quad \frac{\partial y}{\partial s} = \frac{1}{s}.$$

3. $z = 2r$:

$$\frac{\partial z}{\partial r} = 2, \quad \frac{\partial z}{\partial s} = 0.$$

Step 3: Compute $\frac{\partial w}{\partial r}$

Using the chain rule:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}.$$

Substitute the values:

$$\frac{\partial w}{\partial r} = (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2).$$

Since $z = 2r$, substitute z:

$$\frac{\partial w}{\partial r} = \frac{1}{s} + 4r + 2(2r)(2).$$

Simplify:

$$\frac{\partial w}{\partial r} = \frac{1}{s} + 4r + 8r = \frac{1}{s} + 12r.$$

Step 4: Compute $\frac{\partial w}{\partial s}$

Using the chain rule:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Substitute the values:

$$\frac{\partial w}{\partial s} = (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0).$$

Since $\frac{\partial z}{\partial s} = 0$, the last term is zero:

$$\frac{\partial w}{\partial s} = -\frac{r}{s^2} + \frac{2}{s}.$$

Final Answer:

$$\frac{\partial w}{\partial r}=\frac{1}{s}+12r,\quad \frac{\partial w}{\partial s}=\frac{2}{s}-\frac{r}{s^2}.$$

Example 3 Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for $z=\mathbf{e}^{2r}\sin(3\theta)$, $r=st-t^2$, $\theta=\sqrt{s^2+t^2}$.

Hide Solution

Here is the chain rule for $\frac{\partial z}{\partial s}$.

$$\begin{aligned}\frac{\partial z}{\partial s}&=\left(2\mathbf{e}^{2r}\sin(3\theta)\right)(t)+\left(3\mathbf{e}^{2r}\cos(3\theta)\right)\frac{s}{\sqrt{s^2+t^2}}\\&=t\left(2\mathbf{e}^{2(st-t^2)}\sin\left(3\sqrt{s^2+t^2}\right)\right)+\frac{3s\mathbf{e}^{2(st-t^2)}\cos\left(3\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}\end{aligned}$$

Now the chain rule for $\frac{\partial z}{\partial t}$.

$$\begin{aligned}\frac{\partial z}{\partial t}&=\left(2\mathbf{e}^{2r}\sin(3\theta)\right)(s-2t)+\left(3\mathbf{e}^{2r}\cos(3\theta)\right)\frac{t}{\sqrt{s^2+t^2}}\\&=(s-2t)\left(2\mathbf{e}^{2(st-t^2)}\sin\left(3\sqrt{s^2+t^2}\right)\right)+\frac{3t\mathbf{e}^{2(st-t^2)}\cos\left(3\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}\end{aligned}$$

Example 4:

Express $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial r}$ in terms of s and r, if

$$w = x^2 + y^2$$

and

$$x = r - s,\quad y = r + s$$

Solution:

Step 1: Understand the given information

- We are given $w = x^2 + y^2$.
- The variables x and y are expressed in terms of r and s as:

$$x = r - s, \quad y = r + s.$$

- We need to compute the partial derivatives of w with respect to r and s: $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$.

Step 2: Apply the chain rule

The function w depends on r and s through x and y. Therefore, we use the chain rule:

For $\frac{\partial w}{\partial r}$:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}.$$

For $\frac{\partial w}{\partial s}$:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}.$$

Step 3: Compute intermediate derivatives

Derivatives of w with respect to x and y:

Since $w = x^2 + y^2$,

$$\frac{\partial w}{\partial x} = 2x, \quad \frac{\partial w}{\partial y} = 2y.$$

Derivatives of x and y with respect to r and s:

Using $x = r - s$ and $y = r + s$:

$$\frac{\partial x}{\partial r} = 1, \quad \frac{\partial x}{\partial s} = -1,$$

$$\frac{\partial y}{\partial r} = 1, \quad \frac{\partial y}{\partial s} = 1.$$

Step 4: Compute $\frac{\partial w}{\partial r}$

Using the chain rule:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}.$$

Substitute the values:

$$\frac{\partial w}{\partial r} = (2x)(1) + (2y)(1).$$

Substitute $x = r - s$ and $y = r + s$:

$$\frac{\partial w}{\partial r} = 2(r - s) + 2(r + s).$$

Simplify:

$$\frac{\partial w}{\partial r} = 2r - 2s + 2r + 2s = 4r.$$

Step 5: Compute $\frac{\partial w}{\partial s}$

Using the chain rule:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}.$$

Substitute the values:

$$\frac{\partial w}{\partial s} = (2x)(-1) + (2y)(1).$$

Substitute $x = r - s$ and $y = r + s$:

$$\frac{\partial w}{\partial s} = (2(r - s))(-1) + (2(r + s))(1).$$

Simplify:

$$\frac{\partial w}{\partial s} = -2(r - s) + 2(r + s).$$

Distribute and combine terms:

$$\frac{\partial w}{\partial s} = -2r + 2s + 2r + 2s = 4s.$$

Final Answer:

$$\frac{\partial w}{\partial r} = 4r, \quad \frac{\partial w}{\partial s} = 4s.$$

Example

Let $z = e^{x^2 y}$, where
$x(u,v) = \sqrt{uv}$ and
$y(u,v) = 1/v$. Then

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= \left(2xye^{x^2 y}\right)\left(\frac{\sqrt{v}}{2\sqrt{u}}\right) + \left(x^2 e^{x^2 y}\right)(0) \\
&= 2\sqrt{uv} \cdot \frac{1}{v} e^{(\sqrt{uv})^2 \cdot \frac{1}{v}} \cdot \frac{\sqrt{v}}{2\sqrt{u}} + (\sqrt{uv})^2 \cdot e^{(\sqrt{uv})^2 \cdot \frac{1}{v}} \cdot (0) \\
&= e^u + 0 \\
&= e^u \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= \left(2xye^{x^2 y}\right)\left(\frac{\sqrt{u}}{2\sqrt{v}}\right) + \left(x^2 e^{x^2 y}\right)\left(-\frac{1}{v^2}\right) \\
&= 2\sqrt{uv} \cdot \frac{1}{v} e^{(\sqrt{uv})^2 \cdot \frac{1}{v}} \cdot \frac{\sqrt{u}}{2\sqrt{v}} + (\sqrt{uv})^2 e^{(\sqrt{uv})^2 \cdot \frac{1}{v}} \cdot \left(-\frac{1}{v^2}\right) \\
&= \frac{u}{v} e^u - \frac{u}{v} e^u \\
&= 0.
\end{aligned}$$

Practice Questions for Chain Rule:

(a) Express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as a function of u and v by using chain rule.

(b) Evaluate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at the given point (u, v).

1. $z = 4e^x \ln y, \; x = \ln[u\cos(v)], \; y = u\sin(v); \quad (u, v) = \left(2, \frac{\pi}{4}\right)$
2. $z = \tan^{-1}\left(\frac{x}{y}\right), \; x = u\cos(v), \quad y = u\sin(v); \quad (u, v) = \left(1.3, \frac{\pi}{6}\right)$
3.

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ given the following functions:

$$z = f(x, y) = \frac{2x - y}{x + 3y}, \quad x(u, v) = e^{2u}\cos 3v, \quad y(u, v) = e^{2u}\sin 3v.$$

Hint
Answer

$$\frac{\partial z}{\partial u} = 0, \quad \frac{\partial z}{\partial v} = \frac{-21}{(3\sin 3v + \cos 3v)^2}$$

Chain Rule Case 3: Type I:

- **When w is a function of one variable x and then the variable x is further function of two variables r and s.**

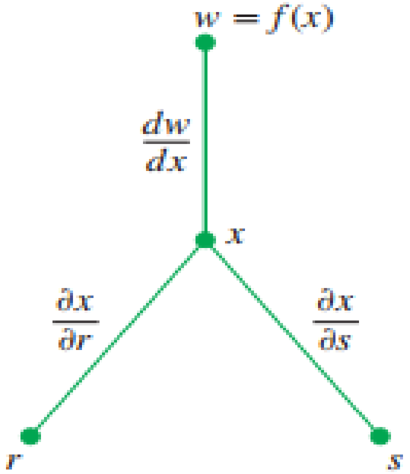
If $w = w(x)$ and $x = x(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx} \cdot \frac{\partial x}{\partial r}$$

and

$$\frac{\partial w}{\partial s} = \frac{dw}{dx} \cdot \frac{\partial x}{\partial s}$$

Chain Rule



$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$