Topic: Computing Partial Derivatives Algebraically

First Order Partial Derivatives

Introduction

Functions of several variables have *several* derivatives, called **partial derivatives,** one for each variable. In this section you will learn how to calculate and interpret these derivatives.

Partial Derivatives

Before defining partial derivatives, we review the rules governing derivatives and constants.

$$\frac{d}{dx}c = 0$$

For a constant *standing alone,* the derivative is zero.

$$\frac{d}{dx}(cx^3) = c \cdot 3x^2$$

Carry along the constant

Derivative of x^3

For a constant *multiplying* a function, the constant is carried along.

These rules will be very useful in this section.

A function $f(x, y)$ has two *partial derivatives,* one with respect to x and the other with respect to y.

Partial Derivatives

$$\frac{\partial}{\partial x} f(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Partial derivative of f with respect to x (x is changed by h; y is held constant)

$$\frac{\partial}{\partial y} f(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Partial derivative of f with respect to y (y is changed by h; x is held constant)

(provided that the limits exist).

Partial derivatives are written with a "curly" ∂, $\partial/\partial x$ instead of d/dx, and are often called "partials." (The Greek letter ∂ is a lowercase delta, equivalent to our letter d,

Subscript Notation for Partial Derivatives

Partial derivatives are often denoted by subscripts: a subscript x means the partial with respect to x, and a subscript y means the partial with respect to y.*

Subscript Notation		***∂ Notation***	
$f_x(x, y)$	$=$	$\frac{\partial}{\partial x} f(x, y)$	f_x means the partial of f with respect to x
$f_y(x, y)$	$=$	$\frac{\partial}{\partial y} f(x, y)$	f_y means the partial of f with respect to y

In other words, it means that,

$$\frac{\partial}{\partial x} f(x, y) = \begin{pmatrix}\text{Derivative of } f \text{ with respect} \\ \text{to } x \text{ with } y \text{ held constant}\end{pmatrix}$$

$$\frac{\partial}{\partial y} f(x, y) = \begin{pmatrix}\text{Derivative of } f \text{ with respect} \\ \text{to } y \text{ with } x \text{ held constant}\end{pmatrix}$$

Students have to solve questions with complete methodology and solution steps.

Example: **Find the first order partial derivatives of**

$$f(x,y) = x^2 + 5y^2$$

Solution:

Partially differentiate $f(x,y)$ w.r.t variable x, we have

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 5y^2)$$

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(5y^2)$$

$$f_x(x,y) = \frac{\partial f}{\partial x} = 2x + 0 = 2x$$

Note: As we find partial derivative with respect to x, therefore y will be considered as constant.

$$\boxed{f_x(x,y) = \frac{\partial f}{\partial x} = 2x}$$

Similarly, partially differentiate $f(x,y)$ w.r.t variable y, we have

$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 5y^2)$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(5y^2)$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = 0 + 5\frac{\partial}{\partial y}(y^2) = 5(2y) = 10y$$

Note: As we find partial derivative with respect to y, therefore x will be considered as constant.

$$\boxed{f_y(x,y) = \frac{\partial f}{\partial y} = 10y}$$

❖ Examples

EXAMPLE 1 **FINDING A PARTIAL DERIVATIVE WITH RESPECT TO *X***

Find $\frac{\partial}{\partial x} x^3 y^4$.

Solution

The $\partial/\partial x$ means differentiate with respect to x, holding y (and therefore y^4) constant. We therefore differentiate the x^3 and carry along the "constant" y^4:

Derivative of x^3

$$\frac{\partial}{\partial x} x^3 y^4 = 3x^2 y^4$$

Carry along the "constant" y^4

$\partial/\partial x$ means differentiate with respect to x, treating y (and therefore y^4) as a constant

EXAMPLE 2 **FINDING A PARTIAL WITH RESPECT TO *Y***

Find $\frac{\partial}{\partial y} x^3 y^4$.

Solution

$$\frac{\partial}{\partial y} x^3 y^4 = x^3 4 y^3 \quad = \quad 4x^3 y^3$$

Carry along the "constant" x^3

Derivative of y^4

Writing the 4 first

$\partial/\partial y$ means differentiate with respect to y, treating x (and therefore x^3) as a constant

Practice Question for Students:

EXAMPLE 3 **FINDING A PARTIAL DERIVATIVE**

Find $\frac{\partial}{\partial x} y^4$.

Solution

$$\frac{\partial}{\partial x} y^4 = 0$$

Partial with respect to x

Function of y alone

The derivative of a constant is zero (since $\partial/\partial x$ means hold y constant)

Some Rules of Partial Derivatives:

Partial Derivative Rules

Same as ordinary derivatives, partial derivatives follow some rule like product rule, quotient rule, chain rule etc.

Product Rule

If u = f(x,y).g(x,y), then,

$$u_x = \frac{\partial u}{\partial x} = g(x,y)\frac{\partial f}{\partial x} + f(x,y)\frac{\partial g}{\partial x}$$

$$And, u_y = \frac{\partial u}{\partial y} = g(x,y)\frac{\partial f}{\partial y} + f(x,y)\frac{\partial g}{\partial y}$$

Quotient Rule

If u = f(x,y)/g(x,y), where g(x,y) ≠ 0, then;

$$u_x = \frac{g(x,y)\frac{\partial f}{\partial x} - f(x,y)\frac{\partial g}{\partial x}}{[g(x,y)]^2}$$

$$\text{And } u_y = \frac{g(x,y)\frac{\partial f}{\partial y} - f(x,y)\frac{\partial g}{\partial y}}{[g(x,y)]^2}$$

- If $u = [f(x, y)]^n$ then, the partial derivative of u with respect to x and y are respectively defined as

$$u_x = n\ \{f(x, y)\}^n\ \frac{\partial f}{\partial x}$$

$$u_y = n\ \{f(x, y)\}^n\ \frac{\partial f}{\partial y}$$

- If $u = \ln\big(f(x, y)\big)$ then, the partial derivative of u with respect to x and y are respectively defined as

$$u_x = \frac{1}{f(x, y)} \frac{\partial f}{\partial x}$$

$$u_y = \frac{1}{f(x, y)} \frac{\partial f}{\partial y}$$

- If $u = sin(f(x, y))$ then, the partial derivative of u with respect to x and y defined as

$$u_x = cos\big(f(x, y)\big) \frac{\partial f}{\partial x}$$

$$u_y = cos\big(f(x, y)\big) \frac{\partial f}{\partial y}$$

- If $u = cos(f(x, y))$ then, the partial derivative of u with respect to x and y defined as

$$u_x = -sin\big(f(x, y)\big) \frac{\partial f}{\partial x}$$

$$u_y = -sin\big(f(x, y)\big) \frac{\partial f}{\partial y}$$

EXAMPLE 5 USING SUBSCRIPT NOTATION

Find $f_x(x, y)$ if $f(x, y) = 5x^4 - 2x^2y^3 - 4y^2$.

Solution

$$f_x(x, y) = 20x^3 - 4xy^3$$

Differentiating with respect to x, holding y constant

EXAMPLE 6 FINDING A PARTIAL INVOLVING LOGS AND EXPONENTIALS

Find both partials of $f = e^x \ln y$.

Solution

$$f_x = e^x \ln y$$

The derivative of e^x is e^x (times the "constant" $\ln y$)

$$f_y = e^x \frac{1}{y}$$

The derivative of $\ln y$ is $\frac{1}{y}$ (times the "constant" e^x)

EXAMPLE 7 FINDING A PARTIAL OF A FUNCTION TO A POWER

Find f_y if $f = (xy^2 + 1)^4$.

Solution

$$f_y = 4(xy^2 + 1)^3(x2y)$$

Partial of the inside with respect to y

Using the Generalized Power Rule (the derivative of f^n is $nf^{n-1}f'$, but with f' meaning a *partial*)

$$= 8xy(xy^2 + 1)^3$$

Simplifying

EXAMPLE 8 FINDING A PARTIAL OF A QUOTIENT

Find $\frac{\partial g}{\partial x}$ if $g = \frac{xy}{x^2 + y^2}$.

Solution

Partial of the top with respect to x

Partial of the bottom with respect to x

$$\frac{\partial g}{\partial x} = \frac{(x^2 + y^2)y - 2x \cdot xy}{(x^2 + y^2)^2}$$ Using the Quotient Rule

Bottom squared

$$= \frac{x^2y + y^3 - 2x^2y}{(x^2 + y^2)^2} = \frac{y^3 - x^2y}{(x^2 + y^2)^2}$$ Simplifying

EXAMPLE 9 FINDING A PARTIAL OF THE LOGARITHM OF A FUNCTION

Find $f_x(x, y)$ if $f(x, y) = \ln(x^2 + y^2)$.

Solution

Partial of the bottom with respect to x

$$f_x(x, y) = \frac{2x}{x^2 + y^2}$$ Derivative of $\ln f$ is $\frac{f'}{f}$

An expression such as $f_x(2, 5)$, which involves both differentiation and evaluation, means *first differentiate and then evaluate.**

EXAMPLE 10 EVALUATING A PARTIAL DERIVATIVE

Find $f_y(1, 3)$ if $f(x, y) = e^{x^2+y^2}$.

Solution

$$f_y(x, y) = e^{x^2+y^2}(2y)$$ Derivative of e^f is $e^f \cdot f'$

Partial of the exponent with respect to y

$$f_y(1, 3) = e^{1^2+3^2}(2 \cdot 3)$$ Evaluating at $x = 1$ and $y = 3$

$$= 6e^{10}$$ Simplifying

EXAMPLE 11 **FINDING A PARTIAL OF A FUNCTION OF THREE VARIABLES**

$$\frac{\partial}{\partial x}(x^3 \underbrace{y^4 z^5}_{\text{Hold constant}}) = \underbrace{3x^2}_{\text{Derivative of } x^3} y^4 z^5$$

$\partial/\partial x$ means differentiate with respect to x, holding y and z constant

<u>Note:</u> In case of three variables, we will take the partial derivative according to one variable and hold the other two variables constant.

EXAMPLE 12 **EVALUATING A PARTIAL IN THREE VARIABLES**

Find $f_z(1, 1, 1)$ if $f(x, y, z) = e^{x^2+y^2+z^2}$.

Solution

$$f_z(x, y, z) = e^{x^2+y^2+z^2}(2z) \qquad \text{Partial with respect to } z$$

$$= 2ze^{x^2+y^2+z^2} \qquad \text{Writing the } 2z \text{ first}$$

$$f_z(1, 1, 1) = 2 \cdot 1 \cdot e^{1^2+1^2+1^2} = 2e^3 \qquad \text{Evaluating}$$

<u>Note:</u> If it is mentioned to compute partial derivatives for any two variable function or a three variables function, we will compute all the partial derivatives.

Solved Questions:

❖ **Find the first order partial derivatives of each of the following functions:**

I. $\boldsymbol{f(x,y) = f(x,y) = 3x + e^{-5y}}$

(Note: Two terms but each has only one variable)

Solution:

Partially differentiate $\boldsymbol{f(x,y)}$ w.r.t variable $\boldsymbol{x}$, we have

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(3x + e^{-5y})$$

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial x}(e^{-5y})$$

$$f_x(x,y) = \frac{\partial f}{\partial x} = 3 + 0 = 3$$

$$\boxed{f_x(x,y) = \frac{\partial f}{\partial x} = 3}$$

Similarly, Partially differentiate $\boldsymbol{f(x,y)}$ w.r.t variable $\boldsymbol{y}$, we have

$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(3x + e^{-5y})$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(3x) + \frac{\partial}{\partial y}(e^{-5y})$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = 0 + e^{-5y}(-5)$$

$$\boxed{f_y(x,y) = \frac{\partial f}{\partial y} = -5e^{-5y}}$$

II. $\boldsymbol{f(u,v) = u^2 e^{2v}}$

Solution:

Partially differentiate $\boldsymbol{f(u,v)}$ w.r.t variable $\boldsymbol{u}$, we have

$$\boldsymbol{f_u\,(u,v)} = \frac{\partial f}{\partial u} = \frac{\partial}{\partial u}\,(u^2\, e^{2v}\,)$$

$$\boldsymbol{f_u\,(u,v)} = \; e^{2v}\;\frac{\partial}{\partial u}\,(u^2\,)$$

$$\boldsymbol{f_u\,(u,v)} = \; e^2 v\,(2u)$$

$$\boxed{\boldsymbol{f_u\,(u,v) = 2ue^{2v}}}$$

Similarly, Partially differentiate $\boldsymbol{f(u,v)}$ w.r.t variable $\boldsymbol{v}$, we have

$$\boldsymbol{f_v(u,v)} = \frac{\partial f}{\partial v} = \frac{\partial}{\partial v}(u^2 e^{2v})$$

$$\boldsymbol{f_v(u,v)} = \; u^2\frac{\partial}{\partial v}(e^{2v})$$

$$\boldsymbol{f_v(u,v)} = \; u^2\;(2e^{2v})$$

$$\boxed{\boldsymbol{f_v(u,v) = 2u^2 e^{2v}}}$$

III. $\boldsymbol{f(x, y) = tan^{-1}(xy)}$

Solution:

f_x:

Treat y as a constant

Apply the chain rule: $\frac{1}{(xy)^2+1}\frac{\partial}{\partial x}(xy)$

$$= \frac{1}{(xy)^2+1}\frac{\partial}{\partial x}(xy)$$

$$\frac{\partial}{\partial x}(xy) = y$$

$$= \frac{1}{(xy)^2+1}y$$

Simplify $\frac{1}{(xy)^2+1}y$: $\frac{y}{x^2y^2+1}$

$$= \frac{y}{x^2y^2+1}$$

f_y:

Treat x as a constant

Apply the chain rule: $\frac{1}{(xy)^2+1}\frac{\partial}{\partial y}(xy)$

$$= \frac{1}{(xy)^2+1}\frac{\partial}{\partial y}(xy)$$

$$\frac{\partial}{\partial y}(xy) = x$$

$$= \frac{1}{(xy)^2+1}x$$

Simplify $\frac{1}{(xy)^2+1}x$: $\frac{x}{x^2y^2+1}$

$$= \frac{x}{x^2y^2+1}$$

IV. $f(x, y) = sin(2xy)$

Solution:

f_x:

$$\frac{\partial}{\partial y}\left(\sin(2xy)\right)$$

Treat x as a constant

Apply the chain rule: $\cos(2xy)\frac{\partial}{\partial y}(2xy)$

$$= \cos(2xy)\frac{\partial}{\partial y}(2xy)$$

$$\frac{\partial}{\partial y}(2xy) = 2x$$

$$= \cos(2xy) \cdot 2x$$

f_y:

$$\frac{\partial}{\partial x}\left(\sin(2x)y\right)$$

Treat y as a constant

Take the constant out: $(a \cdot f)' = a \cdot f'$

$$= y\frac{\partial}{\partial x}\left(\sin(2x)\right)$$

Apply the chain rule: $\cos(2x)\frac{\partial}{\partial x}(2x)$

$$= \cos(2x)\frac{\partial}{\partial x}(2x)$$

$$\frac{\partial}{\partial x}(2x) = 2$$

$$= y\cos(2x) \cdot 2$$

V. $\boldsymbol{f(x,y) = \cos(3x - 5y)}$

Solution:

f_x:

Solution steps

$\frac{\partial}{\partial x}(\cos(3x-5y))$

Treat y as a constant

Apply the chain rule: $-\sin(3x-5y)\frac{\partial}{\partial x}(3x-5y)$

$= -\sin(3x-5y)\frac{\partial}{\partial x}(3x-5y)$

$\frac{\partial}{\partial x}(3x-5y) = 3$

$= -\sin(3x-5y)\cdot 3$

f_y:

$\frac{\partial}{\partial y}(\cos(3x-5y))$

Treat x as a constant

Apply the chain rule: $-\sin(3x-5y)\frac{\partial}{\partial y}(3x-5y)$

$= -\sin(3x-5y)\frac{\partial}{\partial y}(3x-5y)$

$\frac{\partial}{\partial y}(3x-5y) = -5$

$= -\sin(3x-5y)(-5)$

Simplify

$= 5\sin(3x-5y)$

COMPUTING THE PARTIAL DERIVATIVES AT A GIVEN SET OF VALUES.

- **Find $f_x(3,2)$ and $f_y(3,2)$ for $f(x,y) = x^2 + 5y^2$.**

Solution:

$f_x(x,y) = 2x$

$f_x(3,2) = 2(3) = 6$

$f_y(x,y) = 10y$

$f_y(3,2) = 10(2) = 20$

WORD PROBLEMS INVOLVING PARTIAL DERIVATIVES AND THEIR INTERPRETATION

EXAMPLE 13 **INTERPRETING THE PARTIALS OF A COBB–DOUGLAS PRODUCTION FUNCTION**

Find and interpret $P_L(120, 200)$ and $P_K(120, 200)$ for the Cobb–Douglas function $P(L, K) = 20L^{0.6}K^{0.4}$.

Solution

$$P_L = 12L^{-0.4}K^{0.4}$$

Partial with respect to L (the 12 is 20 times 0.6)

$$P_L(120, 200) = 12(120)^{-0.4}(200)^{0.4} \approx 14.7$$

Substituting $L = 120$ and $K = 200$, and evaluating using a calculator

Interpretation: $P_L = 14.7$ means that production increases by about 14.7 units for each additional unit of labor (when $L = 120$ and $K = 200$). This is called the *marginal productivity of labor.*

$$P_K = 8L^{0.6}K^{-0.6}$$

Partial with respect to K (the 8 is 20 times 0.4)

$$P_K(120, 200) = 8(120)^{0.6}(200)^{-0.6} \approx 5.9$$

Substituting $L = 120$ and $K = 200$, and evaluating using a calculator

Interpretation: $P_K = 5.9$ means that production increases by about 5.9 units for each additional unit of capital (when $L = 120$ and $K = 200$). This is called the *marginal productivity of capital.*

These numbers show that to increase production, additional units of labor are more than twice as effective as additional units of capital (at the levels $L = 120$ and $K = 200$).

Question: **Let's consider a small printing business where N is the number of workers; v is the value of raw material (in thousand dollars) and P is the production, measured in thousands of pages per day.**

$$P = f(N, v) = 2N^{0.6}v^{0.4}$$

a) **If this company has a labor force of 100 workers and 200 units worth of equipments. What is the production of company?**
b) **Find $f_N(100, 200)$ and $f_v(100, 200)$. Interpret your answers in terms of production.**

Solution:

a) Given that

Number of workers $= N = 100$,

Value of raw material $= v = 200$

Then production of company is computed as

$$P = f(100, 200) = 2(100)^{0.6}(200)^{0.4}$$

$$\boxed{P = 2639}$$

Interpretation: The company produces **2639** thousand pages per day are produced when **100** workers are working with raw material worth $**200, 000**.

b) To find $\boldsymbol{f_N(100, 200)}$**,** we treat $\boldsymbol{v}$ as a constant,

$$\frac{\partial f}{\partial N} = \frac{\partial}{\partial N}(2N^{0.6}v^{0.4})$$

$$\frac{\partial f}{\partial N} = 2v^{0.4}\frac{\partial}{\partial N}(N^{0.6}) = 2v^{0.4}(0.6)N^{0.6-1}(1)$$

$$\boxed{\boldsymbol{\frac{\partial f}{\partial N} = 1.2v^{0.4}N^{-0.4}}}$$

At point $(\mathbf{100}, \mathbf{200})$, we get $\frac{\partial f}{\partial N}(100{,}200) = 1.2(200)^{0.6}\, 100^{-0.4}$

$$\boxed{\boldsymbol{\frac{\partial f}{\partial N}(100, 200) = 1.583}}$$

<u>Interpretation:</u>

Production increases at a rate of approximately 1.6 if the number of workers is increased by 1 more than the current workforce of 100 workers working constantly using raw material worth $200000.

This tells us if we have $200,000 worth raw material and increase the no of worker by 1 from 100 to 101 the productions output will go up by 1.58 units or 1580 units by per pages.

Now, to find $\boldsymbol{f_v}$ $(\mathbf{100}, \mathbf{200})$**, we treat** $\boldsymbol{N}$ **as constant and partially differentiate** $\boldsymbol{f}$ **with respect to** $\boldsymbol{v}$**.**

$$\frac{\partial f}{\partial v} = \frac{\partial}{\partial v}(2N^{0.6}v^{0.4})$$

$$\frac{\partial f}{\partial v} = 2N^{0.6}(0.4)v^{0.4-1}(1)$$

$$\frac{\partial f}{\partial v} = 0.8N^{0.6}v^{-0.6}$$

At point (100,200), we get

$$\frac{\partial f}{\partial V}(100{,}200) = \ 0.8\ (100)^{0.6}\ (200)^{-0.6}$$

$$\boldsymbol{\frac{\partial f}{\partial V}(100, 200) = 0.53\ thousand\ pages\ per\ of\ equipment}$$

Interpretation: This tells us that if we have $\mathbf{100}$ workers and increase the value of raw materials by $\mathbf{1}$ unit from \$$\mathbf{200,000}$ to \$$\mathbf{201,000}$ the production goes up by about $\mathbf{0.53}$ units or **530 pages per day**.

Practice Questions for Students:

Calculate the 1st order partial derivative of the following:

1) $f(x,y) = x^2 - xy + y^2$

2) $f(x,y) = (x^2 - 1)(y + 2)$

3) $f(x,y) = (xy - 1)^2$

4) $f(x,y) = \sqrt{x^2 + y^2}$

5) $f(x,y) = \frac{\mathbf{1}}{\boldsymbol{x+y}}$

6) $f(x,y) = \boldsymbol{e^{x+y+1}}$

7) $f(x,y) = \boldsymbol{e^{-x}} \sin(x + y)$

8) $f(x,y) = \ln(x + y)$

9) $f(x,y) = \boldsymbol{e^{xy}} \ln(y)$

10) $f(x,y) = \frac{\boldsymbol{x}}{\boldsymbol{x^2+y^2}}$

11) $f(x,y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$

12) $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

13) $h(x,y) = x\,e^y + y + 1$

14) $g(x,y) = x^2y + \cos(y) + y\,sin(x)$

15) $f(x,y,z) = 1 + xy^2 - 2z^2$

16) $f(x,y,z) = xy + yz + zx$

17) $f(x,y,z) = x - \sqrt{y^2 + z^2}$

18) $f(x,y,z) = (x^2 + y^2 + z^2)^{\frac{-1}{2}}$

19) $f(x,y,z) = \ln(x + 2y + 3z)$

20) $f(x,y,z) = e^{-(x^2+y^2+z^2)}$

21) $f(x,y,z) = e^{-xyz}$

22) $f(x,y,z) = y\,z\ln(xy)$