Topics:

- **Second Order Partial Derivatives**
- **Higher Order Partial Derivatives**

Second & Higher Order Partial Derivatives

Second-order partial derivatives are derivatives of a function of multiple variables, where the differentiation is performed twice with respect to different variables. These derivatives provide insight into the function's curvature and how the variables interact.

We can differentiate a function more than once to obtain *higher-order* partials.

Second-Order Partials

Subscript Notation		*∂ Notation*	*In Words*
f_{xx}	$=$	$\frac{\partial^2}{\partial x^2} f$	Differentiate twice with respect to x
f_{yy}	$=$	$\frac{\partial^2}{\partial y^2} f$	Differentiate twice with respect to y
f_{xy}	$=$	$\frac{\partial^2}{\partial y\, \partial x} f$	Differentiate first with respect to x, then with respect to y
f_{yx}	$=$	$\frac{\partial^2}{\partial x\, \partial y} f$	Differentiate first with respect to y, then with respect to x

Each notation means differentiate first with respect to the variable *closest* to f.

Calculating a "second partial" such as f_{xy} is a two-step process: First calculate f_x, and then differentiate the *result* with respect to y.

How to Compute 2nd Order Derivatives

We will compute the **first order partial derivatives** initially.

Consider the function $f(x, y)$, then there will be **two first order partial derivatives**:

1) $f_x = \frac{\partial f}{\partial x}$

2) $f_y = \frac{\partial f}{\partial y}$

Now if we need to **compute** the **second order partial derivatives**:

We will use the solutions of the first order dervatives.

Since the function $f(x, y)$ was initially a two variable function thus we will further compute the second derivative partially with respect to x and y using the solutions of the first order dervatives.

Thus, we will have **four second order partial derivatives** for $f(x, y)$:

- $f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$

The **second derivative** with respect to x while holding y constant. This indicates the curvature of the function along the x-direction.

- $f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$

The second derivative with respect to y, holding x constant.

- $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ *and* $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

The **mixed second derivative**, where **differentiation** is **first done with respect to x** and **then y** (**or vice versa**). This **measures** how the **function changes as both variables change.**

Important Note:

- f_{xy} and f_{yx} are called the **mixed partial derivatives**.
- $f_{yx} = f_{xy}$ **, when function is continuous and twice differentiable.**
- The variable which appears first is generally the one you would want to differentiate with respect to first.

  For example,

  f_{yx} means that you will first differentiate with respect to y, and then will differentiate with respect to x.

Example 1:

Find all four second-order partials of $f(x, y) = x^4 + 2x^2y^2 + x^3y + y^4$.

Solution

First we calculate

$$f_x = 4x^3 + 4xy^2 + 3x^2y \quad \text{Partial of } f \text{ with respect to } x$$

Then from this we find f_{xx} and f_{xy}:

$$f_{xx} = 12x^2 + 4y^2 + 6xy \quad \text{Differentiating } f_x = 4x^3 + 4xy^2 + 3x^2y \text{ with respect to } x$$

$$f_{xy} = 8xy + 3x^2 \quad \text{Differentiating } f_x = 4x^3 + 4xy^2 + 3x^2y \text{ with respect to } y$$

Now, returning to the original function $f = x^4 + 2x^2y^2 + x^3y + y^4$, we calculate

$$f_y = 4x^2y + x^3 + 4y^3 \quad \text{Partial of } f \text{ with respect to } y$$

Then, from this,

$$f_{yy} = 4x^2 + 12y^2 \quad \text{Differentiating } f_y = 4x^2y + x^3 + 4y^3 \text{ with respect to } y$$

$$f_{yx} = 8xy + 3x^2 \quad \text{Differentiating } f_y = 4x^2y + x^3 + 4y^3 \text{ with respect to } x$$

Example 2:

Compute the 2nd order partial derivatives of

$$f(x, y) = xy^2 + 3x^2e^y$$

Solution:

1st Order Partial Derivatives: f_x and f_y

Finding partial derivative of f w.r.t x	Finding partial derivative of f w.r.t y
$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy^2 + 3x^2e^y)$ $\frac{\partial f}{\partial x} = y^2\frac{\partial}{\partial x}(x) + 3e^y\frac{\partial}{\partial x}(x^2)$ $\frac{\partial f}{\partial x} = y^2(1) + 3e^y(2x)$	$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xy^2 + 3x^2e^y)$ $\frac{\partial f}{\partial y} = x\frac{\partial}{\partial x}(y^2) + 3x^2\frac{\partial}{\partial x}(e^y)$ $\frac{\partial f}{\partial y} = x(2y) + 3e^yx^2$
This implies that $f_x = y^2 + 6e^yx$	**This implies that** $f_y = 2xy + 3e^yx^2$

2nd Order Partial Derivatives:

Finding f_{xx} and f_{yx}

Finding partial derivative of f_x w.r.t x	Finding partial derivative of f_y w.r.t x
$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(y^2 + 6e^y x)$ $f_{xx} = 0 + 6e^y \frac{\partial}{\partial x}(x)$	$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2xy + 3e^y x^2)$ $f_{yx} = 2y\frac{\partial}{\partial x}(x) + 3e^y \frac{\partial}{\partial x}(x^2)$
This implies that $f_{xx} = 6e^y$	**This implies that** $f_{yx} = 2y + 6xe^y$

Similarly, finding f_{xy} and f_{yy}:

Finding partial derivative of f_x w.r.t y	Finding partial derivative of f_y w.r.t y
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(y^2 + 6e^y x)$ $f_{xy} = \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(6xe^y)$	$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(2xy + 3e^y x^2)$ $f_{yy} = 2x\frac{\partial}{\partial y}(y) + 3x^2 \frac{\partial}{\partial y}(e^y)$
This implies that $f_{xy} = 2y + 6xe^y$	**This implies that** $f_{yy} = 2x + 3x^2 e^y$

PRACTICE EXERCISES

1) **Find** the **second order partial derivatives** of the following**:**

a) $g(x,y,z) = \dfrac{x^2\,y - 4\,x\,z + y^2}{xyz}$

b) $h(x,y,z) = \sin(x^2 y - z) + \cos\,(x^2 - yz)$

2) **Verify** that

$$u(x,y,t) = 5\,sin(3\pi x)\,sin(4\pi y)\,cos(10\pi t)$$

is a solution to the **wave equation**

$$u_{tt} = 4(u_{xx} + u_{yy}).$$

3) **Verify** that

$$u(x,y,t) = 2sin\left(\frac{x}{3}\right)sin\left(\frac{y}{4}\right)\,e^{-\frac{25\,t}{16}}$$

is a solution to the heat equation

$$u_t = 9(u_{xx} + u_{yy})$$

27–32. For each function, find the second-order partials **a.** f_{xx}, **b.** f_{xy}, **c.** f_{yx}, and **d.** f_{yy}.

27. $f(x,y) = 5x^3 - 2x^2y^3 + 3y^4$

28. $f(x,y) = 4x^2 - 3x^3y^2 + 5y^5$

29. $f(x,y) = 9x^{1/3}y^{2/3} - 4xy^3$

30. $f(x,y) = 32x^{1/4}y^{3/4} - 5x^3y$

31. $f(x,y) = ye^x - x\ln y$

32. $f(x,y) = y\ln x + xe^y$

33–34. For each function, calculate the third-order partials **a.** f_{xxy}, **b.** f_{xyx}, and **c.** f_{yxx}.

33. $f(x,y) = x^4y^3 - e^{2x}$ **34.** $f(x,y) = x^3y^4 - e^{2y}$

Example:

Consider the function

$$f(x,y,z) = z\, ln[x^2 y\, cos(z)]$$

then find $f_x(x,y,z)$, $f_y(x,y,z)$, $f_z(x,y,z)$, $f_{xx}(x,y,z)$, $f_{yy}(x,y,z)$, $f_{zz}(x,y,z)$.
Also verify that

$$\left(-\frac{x^2}{z}\right) f_{xx}(x,y,z) + \left(\frac{y^2}{z}\right) f_{yy}(x,y,z) + \left(\frac{1}{z\sec^2(z) + 2\tan(z)}\right) f_{zz}(x,y,z) = 0$$

Solution:

Sol:

$$f_x = z \cdot \frac{1}{x^{\cancel{2}}\cancel{y}\,\cancel{\cos(z)}} \cdot 2\cancel{x}\,\cancel{y}\,\cancel{\cos(z)}$$

$$\boxed{f_x = \frac{2z}{x}} = 2z \cdot x^{-1}$$

$$\Rightarrow \boxed{f_{xx} = -\frac{2z}{x^2}}$$

And

$$f_y = z \cdot \frac{1}{\cancel{x^2}y\,\cancel{\cos(z)}} \cdot \cancel{x^2}\,\cancel{\cos(z)}$$

$$\boxed{f_y = \frac{z}{y}}$$

$$\Rightarrow \boxed{f_{yy} = -\frac{z}{y^2}}$$

And

$$f_z = z \cdot \frac{1}{\cancel{x^2}\cancel{y}\cos(z)} \cdot \cancel{x^2}\cancel{y} \cdot (-\sin(z)) + \ln[x^2 y \cos(z)] \cdot (1)$$

$$f_z = -z \cdot \tan(z) + \ln[x^2 y \cos(z)] = \ln[x^2 y \cos(z)] - z\tan(z)$$

$$f_{zz} = \frac{-\cancel{x^2}\cancel{y}\sin z}{\cancel{x^2}\cancel{y}\cos z} - [z \cdot \sec^2(z) + \tan(z) \cdot (1)]$$

$$f_{zz} = -\frac{\sin z}{\cos z} - z\sec^2(z) - \tan z$$

$$f_{zz} = -\tan z - z\sec^2 z - \tan z$$

$$\boxed{f_{zz} = -2\tan(z) - z \cdot \sec^2(z)}$$

$$\Rightarrow \boxed{f_{zz} = -(2\tan(z) + z\sec^2(z))}$$

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$$\left(-\frac{x^2}{z}\right) f_{xx} = \left(-\frac{\cancel{x^2}}{\cancel{z}}\right)\left(\frac{-2\cancel{z}}{\cancel{x^2}}\right) = 2$$

$$\left(\frac{y^2}{z}\right) f_{yy} = \left(\frac{\cancel{y^2}}{\cancel{z}}\right)\left(-\frac{\cancel{z}}{\cancel{y^2}}\right) = -1$$

$$\frac{1}{2\tan z + z\sec^2(z)} \cdot f_{zz} = \frac{-1}{\cancel{(2\tan z + z\sec^2 z)}} \cdot \cancel{(2\tan(z) + z\sec^2(z))} = -1$$

$$2 - 1 - 1 = 2 - 2 = 0$$

Hence verified.

Higher Order Partial Derivatives

In multivariate calculus, higher-order partial derivatives extend the concept of first-order partial derivatives to describe the rate of change of a function with respect to its variables multiple times. They provide deeper insights into the behavior of functions, especially in applications like optimization, physics, and machine learning. The concept of partial derivatives was first formalized by mathematicians such as Leonhard Euler and Joseph-Louis Lagrange in the 18th century.

What Are Higher-Order Partial Derivatives?

Higher-order partial derivatives are derivatives of a function taken more than once, either with respect to the same variable or with respect to different variables.

For example:

- **Second-order partial derivatives:** These are obtained by differentiating a first-order partial derivative again.
    - **Examples:** $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}$
- **Third-order and higher derivatives:** These are obtained by continuing the differentiation process.
    - **Examples:** $\frac{\partial^3 f}{\partial x^3}, \frac{\partial^3 f}{\partial x^2 \partial y}$

How to Continue Taking Derivatives?

1. **Start with the First Derivative:** Compute the first partial derivatives of the function with respect to each variable.
2. **Move to Higher Orders:** Treat each first derivative as a new function and differentiate again. Repeat this process according to the desired order of differentiation.
3. **Respect the Variable Order:** When working with mixed derivatives, ensure you follow the order specified.

   For example:

   $\frac{\partial^3 f}{\partial x^2 \partial y}$ means differentiating twice with respect to $\boldsymbol{x}$, then once with respect to $\boldsymbol{y}$.

4. **Check Continuity for Clairaut's Theorem:** If all the mixed partial derivatives up to a certain order are **continuous**, the **order of differentiation** does not matter:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

How Does It Work for Continuous Functions?

For functions that are sufficiently smooth (i.e., their derivatives exist and are continuous up to the required order):

- The process of differentiation can be repeated as many times as needed, following the same principles as for first-order derivatives.
- The results of higher-order derivatives describe more variation in behavior of the function, such as curvature or changes in slope in multiple directions.

- For continuous functions we can make interchanges within the differentiation order.
  For instance, if $\boldsymbol{f(x,y) = 4x^3y^2}$ we can verify that $\boldsymbol{f_{xyy} = f_{yxy}}$.

For example:

- $\frac{\partial^2 f}{\partial x^2}$ gives the **concavity (maxima/minima)** in the $\boldsymbol{x}$-direction.
- $\frac{\partial^2 f}{\partial x \partial y}$ describes how the **rate of change** in the $\boldsymbol{x}$-direction is affected by changes in $\boldsymbol{y}$.

Step-by-Step Examples

Example 1: Algebraic Function

Find the **third-order partial derivatives** of $\boldsymbol{f(x,y) = x^3y^2 + 2xy}$.

Solution:

Step 1: First Order Partial Derivatives

$$\frac{\partial f}{\partial x} = 3x^2y^2 + 2y$$
$$\frac{\partial f}{\partial y} = 2x^3y + 2x$$

Step 2: Second Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = 6xy^2$$
$$\frac{\partial^2 f}{\partial y^2} = 2x^3$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 6x^2y + 2$$

Step 3: Third Order Partial Derivatives

$$\frac{\partial^3 f}{\partial x^3} = 6y^2$$
$$\frac{\partial^3 f}{\partial x^2 \partial y} = 6y + 6xy$$
$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial^3 f}{\partial y \partial x^2} = 6x$$

Example 2: Exponential and Trigonometric Function

Find the **second and third-order partial derivatives** of

$$\boldsymbol{g(x, y) = e^{x} \sin(y)}.$$

Solution:

Step 1: First Order Partial Derivatives

$$\boldsymbol{\frac{\partial g}{\partial x} = e^{x} \sin(y)}$$

$$\boldsymbol{\frac{\partial g}{\partial y} = e^{x} cos(y)}$$

Step 2: Second Order Partial Derivatives

$$\boldsymbol{\frac{\partial^2 g}{\partial x^2} = e^{x} sin(y)}$$
$$\boldsymbol{\frac{\partial^2 g}{\partial y^2} = -e^{x} sin(y)}$$
$$\boldsymbol{\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x} = e^{x} cos(y)}$$

Step 3: Third Order Partial Derivatives

$$\frac{\partial^3 g}{\partial x^3} = e^x sin(y)$$

$$\frac{\partial^3 g}{\partial x^2 \partial y} = \frac{\partial^3 g}{\partial x \partial y^2} = -e^x cos(y)$$

$$\frac{\partial^3 g}{\partial y^3} = -e^x cos(y)$$

Summary:

To compute higher-order partial derivatives:

1. Start by computing first-order partial derivatives.
2. Use these results iteratively to find second, third, or higher-order derivatives.
3. Ensure the function is continuous for applying Clairaut's theorem, which guarantees the equality of mixed derivatives when applicable.

Higher-order derivatives help capture complex behaviors in multivariable functions, making them essential tools in advanced mathematical and computational applications.

Practice Questions

Find the **indicated derivative** for each of the following functions.

1. Find f_{xxyzz} for $f(x, y, z) = z^3 y^2 ln(x)$

   Answer: $f_{xxyzz} = -\frac{12zy}{x^2}$

2. Find $\frac{\partial^3 f}{\partial y \partial x^2}$ for $f(x, y) = e^{xy}$

   Answer: $\frac{\partial^3 f}{\partial y \partial x^2} = 2ye^{xy} + xy^2 e^{xy}$

3. Compute **all second-order partial derivatives** of

$$h(x, y) = x^2 y + ln(y).$$

   Answers:

   $\frac{\partial h}{\partial x} = 2xy, \quad \frac{\partial h}{\partial y} = x^2 + \frac{1}{y}$

   $\frac{\partial^2 h}{\partial x^2} = 2y$

   $\frac{\partial^2 h}{\partial y^2} = -\frac{1}{y^2}$

   $\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x} = 2x$

4. Find $\frac{\partial^3 f}{\partial x \partial y^2}$ **for** $f(x, y) = xcos(y)$.

   Answer: $\frac{\partial^3 f}{\partial x \partial y^2} = -xcos(y)$

5. Verify that the **second order mixed derivatives** are equal for $k(x, y) = x^2 y^3 + e^y \sin(x)$.

Answer: Mixed derivatives are equal.

$$\frac{\partial^2 k}{\partial x \partial y} = \frac{\partial^2 k}{\partial y \partial x} = 3x^2y^2 + e^y cos(x)$$

6. **Determine** the **third-order partial derivative** $\frac{\partial^3 m}{\partial y^3}$ for

$$m(x, y) = tan(xy)$$

Answer:

$$\frac{\partial^3 m}{\partial y^3} = 3x^3 tan^2(xy) sec^2(xy)$$

Practice Questions:

Calculate the 2nd order partial derivatives of f.

i. $f(x, y) = x^2y$
ii. $f(x, y) = x^2 + 2xy + y^2$
iii. $f(x, y) = xe^y$
iv. $f(x, y) = \frac{2x}{y}$
v. $f(x, y) = 5 + x^2y^2$
vi. $f(x, y) = e^{yx}$
vii. $f(x, y) = 100e^{xy}$
viii. $f(x, y) = 5xe^{-2y}$
ix. $f(x, y) = \sin(xy)$
x. $f(x, y) = x^2y + \cos y + y \sin x$
xi. $f(x, y) = \ln(x + y)$
xii. $f(x, y) = xe^y + y + 1$