Topic: Functions of Several Variables

The definition of a function of several variables (two or more variables) is very similar to the definition for a function of one variable. The main difference is that, instead of mapping values of one variable to values of another variable, we map ordered pairs of variables to another variable.

Definition: Function of two variables

A function of two variables $\boldsymbol{z = f(x, y)}$ maps each ordered pair $(\boldsymbol{x}, \boldsymbol{y})$ in a subset $\boldsymbol{D}$ of the real plane $\boldsymbol{R^2}$ to a unique real number $\boldsymbol{z}$. The set $\boldsymbol{D}$ is called the domain of the function. The range of $\boldsymbol{f}$ is the set of all real numbers $\boldsymbol{z}$ that has at least one ordered pair $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{D}$ such that $\boldsymbol{f(x, y) = z}$ as shown in figure below.

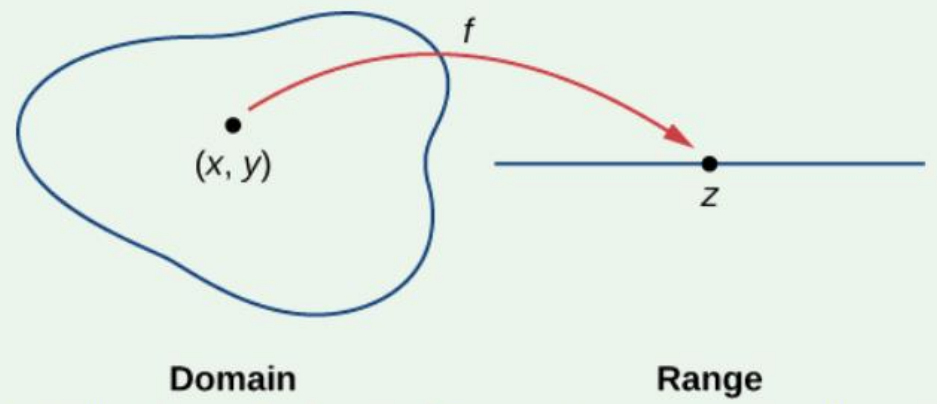


Figure 14.1.1: The domain of a function of two variables consists of ordered pairs (x, y).

An application problem regarding function of several variables can be of the following types:

1. Tabular Data Representation (using limit formula of partial derivative).
2. Algebraic function (solve the functions algebraically)
3. Comparative or complimentary commodities (Interpretation of the function depending upon commodity behaviour showing direct or inverse proportion).

Example 1:

Consider an **airline's ticket** pricing. To avoid flying planes with many empty seats, it sells some tickets at full price and some at a discount. For a particular route, the airline's revenue $\boldsymbol{R}$, earned in a given time period is determined by the number of full priced tickets, $\boldsymbol{x}$, and number of discounted tickets, $\boldsymbol{y}$, sold. We say that $\boldsymbol{R}$ is a function of $\boldsymbol{x},\ \boldsymbol{y}$ and we write:

$$\boldsymbol{R} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$$

where $\boldsymbol{R}$ is the dependent variable whereas, $\boldsymbol{x}$ and $\boldsymbol{y}$ are the independent variables.

- The collection of all possible inputs $(\boldsymbol{x}, \boldsymbol{y})$ is called domain of $\boldsymbol{f}$.
- We say a function is an increasing (or decreasing) function of one of its variables if the function increases (or decreases) as that variable increases while the other independent variables are held constant.

Consider the table below

y/x	**100**	**200**	**300**	**400**
200	75,000	110,000	145,000	180,000
400	115,000	150,000	185,000	220,000
600	155,000	190,000	225,000	260,000
800	195,000	230,000	265,000	300,000
1000	235,000	270,000	305,000	340,000

a) Find the value of $f(300, 600)$ and interpret it.
b) Is f an increasing or decreasing function of x?
c) Is f an increasing or decreasing function of y?

Solution:

(a) From the above table, we can write

$$\boldsymbol{f}(\mathbf{300}, \mathbf{600}) = \mathbf{225,000}$$

Interpretation: The revenue from $\mathbf{300}$ full price tickets and $\mathbf{600}$ discounted tickets is \$ $\mathbf{225,000}$.

(b) **Answer:** Increasing function
(c) **Answer:** Increasing function

Example 2:

A car rental company charges $ $\mathbf{40}$ **per day** & $\mathbf{15}\ \boldsymbol{cents}$ **per mile** for its cars.

a) Write a **formula** for the cost $\boldsymbol{C}$, of renting a car as a function of the number of days $\boldsymbol{x}$, and the number of **miles** driven $\boldsymbol{y}$.
b) If $\boldsymbol{C} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$. Find $\boldsymbol{f}(\mathbf{5}, \mathbf{300})$ and interpret it.
c) Explain the significance of $\boldsymbol{f}(\mathbf{3}, \boldsymbol{y})$ in terms of rental cost. Graph this function with $\boldsymbol{C}$ as a function of $\boldsymbol{y}$.
d) Explain the significance of $\boldsymbol{f}(\boldsymbol{x}, \mathbf{100})$ in terms of rental cost. Graph this function with $\boldsymbol{C}$ as a function of $\boldsymbol{x}$.

Solution:

Solution (a): Let us consider the variables $\boldsymbol{x}$ and $\boldsymbol{y}$ represents

$\boldsymbol{x}$ **: Per day charges of renting a car**

$\boldsymbol{y}$ **: Per mile charges of renting a car**

Since we know that

$$\mathbf{1\$ = 100}\ \boldsymbol{cents}$$

Hence, the cost of renting a car can be represented as a function of $\boldsymbol{x}$ and $\boldsymbol{y}$ as

$$\boldsymbol{C} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{40}\,\boldsymbol{x} + \mathbf{0.15}\,\boldsymbol{y} \_\_\_\_\_\_\_\_\_\_\_\_ (\mathbf{1})$$

Solution (b):

To find the value of $f(5, 300)$, put $\boldsymbol{x} = \mathbf{5}$, and $\boldsymbol{y} = \mathbf{300}$ in equation (1), we get

$$\boldsymbol{f}(\mathbf{5}, \mathbf{300}) = \mathbf{40}(\mathbf{5}) + \mathbf{0.15}(\mathbf{300}) = \mathbf{245}$$

It means if the car is rented for $\mathbf{5}$ **days** and is driven $\mathbf{300}$ **miles** then it costs $ $\mathbf{245}$.

Solution (c):

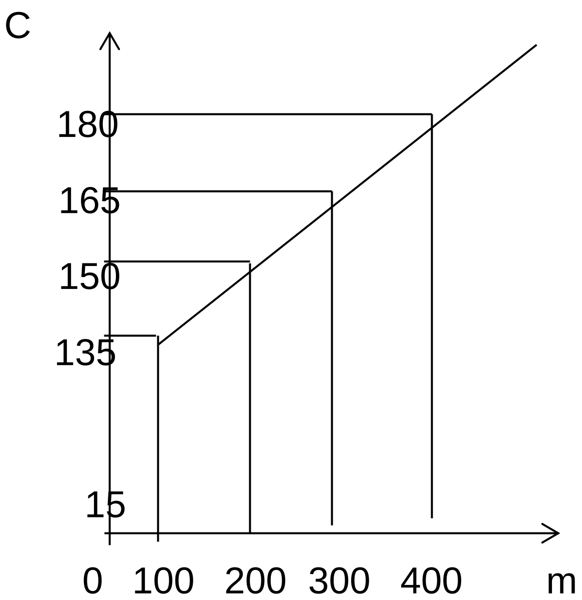
Since $\boldsymbol{f}(\mathbf{3}, \boldsymbol{y})$ is the value of $\boldsymbol{C}$ for various values of $\boldsymbol{y}$ with $\boldsymbol{x}$ fixed at $\mathbf{3}$ **days**.

In other words, it is the cost of renting a car for **three days** & driven for variable **number of miles**.

Since days are fixed, that is $x = 3$. Then

Miles (y)	$\boldsymbol{C = f(3, y) = 120 + 0.15y}$
100	$C = f(3{,}100) = 120 + 0.15(100) = 135$
200	$C = f(3{,}200) = 120 + 0.15(200) = 150$
300	$C = f(3{,}300) = 120 + 0.15(300) = 165$
400	$C = f(3{,}400) = 120 + 0.15(400) = 180$

Graph:



<u>Solution (d):</u>

Since $\boldsymbol{f(x, 100)}$ is the value of $\boldsymbol{C}$ for various values of $\boldsymbol{x}$ with $\boldsymbol{y}$ fixed at **100 miles**.

In other words, it is the cost of renting a car for **various number of days** & driven for **100 miles**.

Draw the graph by yourself.

Example 3:

The total sales of a product $\boldsymbol{S}$ can be expressed as a **function** of the price $\boldsymbol{P}$ charged for the product and amount $\boldsymbol{a}$ spent on advertisement, so $\boldsymbol{S} = \boldsymbol{f}(\boldsymbol{p}, \boldsymbol{a})$.

a) Do you expect $\boldsymbol{f}$ to be an **increasing or decreasing function** of $\boldsymbol{P}$?
b) Do you expect $\boldsymbol{f}$ to be an **increasing or decreasing function** of $\boldsymbol{a}$?

Solution:

Since given that $\boldsymbol{S} = \boldsymbol{f}(\boldsymbol{p}, \boldsymbol{a})$

Solution (a):

Let suppose we fixed some amount for advertisement, i.e., $\boldsymbol{a} = \mathbf{20}\,\$$, then $\boldsymbol{S} = \boldsymbol{f}(\boldsymbol{p}, \$\,\mathbf{20})$. That is, $\boldsymbol{S}$ is a function of $\boldsymbol{P}$, i.e., $\boldsymbol{a} =$ advertisement amount is fixed.

- Now there is an **inverse relationship** between **sales and price. That is**
  - If the **price** of product **increases**, its **sale decreases**

    P↑ ⇔ S↓

  - If the **price** of product **decreases**, its **sale increases**

    P↓ ⇔ S↑

So, $\boldsymbol{f}$ is **decreasing function** of $\boldsymbol{P}$.

Solution (b):

Let suppose we fixed the price of product, i.e., $\boldsymbol{p} = \mathbf{20}\,\$$, then $\boldsymbol{S} = \boldsymbol{f}(\$\,\mathbf{20}, \boldsymbol{a})$. That is, $\boldsymbol{S}$ is a function of **advertisement** $\boldsymbol{a}$, i.e., $\boldsymbol{p} =$ price of product is fixed.

- Now, there is **direct relationship** between **advertisement and sales. That is**
  - If the **advertisement** of product **increases**, its **sale increases**

a↑ ⇔ S↑

- If the **advertisement** of product **decreases**, its **sale decreases**

a↓ ⇔ S↓

So, f is **increasing function** of a.

Practice Questions for Students:

Exercise: 9.1. Q # 3-5, Q # 10-11 from Applied Calculus Book By Hughes Hallet 4th Edition.

Partial Derivatives

Example 1:

Consider the airline's ticket pricing example having **revenue function**

$$R = f(x, y)$$

where

R = Revenue of Airline Company,

x = Number of **full price** tickets, and

y = Number of **discounted price** tickets.

Such that the revenue function is given as

$$R = f(x, y) = 350x + 200y$$

Fixing one variable (say we fix variable y)

Suppose we **fix** the **number of discounted price tickets** at $y = 10$, we have a function **one variable,** that is

$$R = f(x, 10) = g(x) = 350\,x + 2000$$

The **rate of change of revenue** with respect to x is given by

$$g'(x) = \frac{d}{dx}(350\,x + 2000) = \frac{d}{dx}(350\,x) + \frac{d}{dx}(2000) = 350$$

<u>Interpretation:</u>

- This tells us that if we **increase** the **number of full price tickets** by **one unit** then the **revenue** of airlines is **increased** by \$ $\mathbf{350}$ while the **number of discounted tickets** is **fixed** at $\mathbf{10}$.
- We call $\boldsymbol{g'(x)}$ *the partial derivative of* $\boldsymbol{R}$ *with respect to* $\boldsymbol{x}$ *at point* $(\boldsymbol{x}, \mathbf{10})$.

<u>Example 2:</u>

<u>Find</u> the **rate of change** of **revenue** $\boldsymbol{R}$ as $\boldsymbol{x}$ increases with $\boldsymbol{y}$ fixed at $\boldsymbol{y} = \mathbf{10}$ such that the revenue function is

$$\boldsymbol{R = f(x, y) = 350x + 200y}$$

<u>Solution:</u>

The given revenue function is

$$\boldsymbol{R = f(x, y) = 350x + 200y}$$

Since $\boldsymbol{y} = \mathbf{10}$ is fixed, then

$$\boldsymbol{R = f(x, 10) = 350x + 200(10)}$$

$$\boldsymbol{R = f(x, 10) = 350x + 2000}$$

$$\boldsymbol{\frac{\partial R}{\partial x} = f_x(x, 10) = \frac{\partial}{\partial x}(350\,x + 2000)}$$

$$\boldsymbol{\frac{\partial R}{\partial x} = f_x(x, 10) = 350}$$

<u>Example 3:</u>

<u>Find</u> the **rate of change** of **revenue** $\boldsymbol{R}$ as $\boldsymbol{y}$ increases with $\boldsymbol{x}$ **fixed** at $\boldsymbol{x} = \mathbf{20}$.

<u>Solution:</u>

The given revenue function is

$$\boldsymbol{R = f(x, y) = 350x + 200y}$$

Since $\boldsymbol{x} = \mathbf{20}$ is fixed, then

$$R = f(20, y) = 350\,(20) + 200y$$

$$R = f(20, y) = 7000 + 200y$$

$$\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{y}} = \boldsymbol{f_y(20, y)} = \frac{\partial}{\partial \boldsymbol{y}}(\mathbf{7000} + \mathbf{200}\,\boldsymbol{y})$$

$$\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{y}} = \boldsymbol{f_y(20, y)} = \frac{\partial}{\partial \boldsymbol{y}}(\mathbf{7000}) + \frac{\partial}{\partial \boldsymbol{y}}(\mathbf{200}\,\boldsymbol{y})$$

$$\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{y}} = \boldsymbol{f_y(20, y)} = \mathbf{0} + \mathbf{200}\frac{\partial}{\partial \boldsymbol{y}}(\boldsymbol{y})$$

$$\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{y}} = \boldsymbol{f_y(20, y)} = \mathbf{200(1)}$$

$$\boxed{\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{y}} = \boldsymbol{f_y(20, y)} = \mathbf{200}}$$

Recall:

The **derivative** of f with respect to x is defined as:

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Partial Derivatives for function of two variables $f(x, y)$

- The **partial derivative** of f with respect to x with y **fixed** or **constant** is defined as:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \quad - - - (1)$$

- The **partial derivative** of f with respect to y with x **fixed** or **constant** is defined as:

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k} \quad - - - (2)$$

At any arbitrary point $(x, y) = (a, b)$, the formulas in equations **(1)** and **(2)** can be written as:

- The **partial derivative** of f with respect to x at (a, b) with y **fixed** or **constant** is defined as:

$$\frac{\partial f}{\partial x}\Big|_{(x,y)=(a,b)} = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} - - - (3)$$

- The **partial derivative** of f with respect to y at (a, b) with x **fixed** or **constant** is defined as:

$$\frac{\partial f}{\partial y}\Big|_{(x,y)=(a,b)} = \lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k} - - - (4)$$

Alternative Notations for Partial Derivatives:

If $z = f(x, y)$

$$f_x(x, y) = \frac{\partial z}{\partial x} \qquad \& \qquad f_y(x, y) = \frac{\partial z}{\partial y}$$

At any arbitrary point $(x, y) = (a, b)$

$$f_x(a, b) = \frac{\partial z}{\partial x}\Big|_{(a,b)} \quad \& \quad f_y(a, b) = \frac{\partial z}{\partial y}\Big|_{(a,b)}$$

Example 4:

An experiment done on rats to measure the **toxicity of formaldehyde** yielded the data shown in the given table.

Concentration c (ppm)	Time t (in months)												
$c\backslash t$	**0**	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**
0	100	100	100	100	100	100	100	100	100	100	99	97	95
2	100	100	100	100	100	100	100	100	99	98	97	95	92
6	100	100	100	99	99	98	96	96	95	93	90	86	80
15	100	100	100	99	99	99	99	96	93	82	70	58	36

The values in the table show the percent $\boldsymbol{P}$ of rats that survived an exposure with **concentration** $\boldsymbol{c}$ (in parts per million) after time, $\boldsymbol{t}$ month, so, $\boldsymbol{P = f(t, c)}$.

Estimate $\boldsymbol{f_t\,(18, 6)}$ & $\boldsymbol{f_c\,(18, 6)}$ from the table and interpret your answers.

Solution:

Step 1: Calculate $\boldsymbol{f_t\,(18, 6)}$

$$f_t\,(18,6) = \frac{\partial f}{\partial t}\Big|_{(18,6)} \approx \frac{f(t\,+\,h, c) - f(t, c)}{h}$$

$$f_t\,(18,6) = \frac{\partial f}{\partial t}\Big|_{(18,6)} \approx \frac{[f(18\,+\,2, 6) - \,f(18, 6)]}{2}$$

$$f_t\,(18,6) = \frac{\partial f}{\partial t}\Big|_{(18,6)} \approx \frac{[f(20, 6) - \,f(18, 6)]}{2}$$

Putting values from the table, we have

$$f_t\,(18,6) = \frac{\partial f}{\partial t}\Big|_{(18,6)} \approx \frac{(90-93)}{(20-18)}$$

$$f_t\,(18,6) = \frac{\partial f}{\partial t}\Big|_{(18,6)} = -1.5\ \%\ \text{per month.}$$

<u>Interpretation:</u> This partial derivative tells us that after **18 months** of exposure to formaldehyde at a concentration of **6 ppm**, $\boldsymbol{P}$ **decreases** by $\boldsymbol{1.5}\%$ for every additional month of exposure.

Step 2: Calculate $\boldsymbol{f_c\,(18,6)}$

$$f_c\,(18,6) = \frac{\partial f}{\partial t}\Big|_{(18,6)} \approx \frac{[f(t, c\ +\ k) - \ f(t,c)]}{k}$$

$$\approx \frac{[f(18, 6\ +\ 9) - \ f(18,6)]}{9}$$

$$\approx \frac{[f(18, 15) - \ f(18,6)]}{9}$$

$$\approx \frac{(82 - 93)}{(15 - 6)} = -1.22\ \%\ \text{per ppm}$$

<u>Interpretation:</u> It means that after **$\mathbf{18}$ months** of exposure to formaldehyde at a concentration of **$\mathbf{6}$ ppm**, **$\boldsymbol{P}$ decreases** by $\mathbf{1.22}\%$ for every additional ppm of concentration.

Example 5:

The following table gives the number of calories burned per minute $B = f(w,s)$ of a cyclist as a function of the person's weight w and speed s.

a) Is f_w positive or negative?
b) Is f_s positive or negative?
c) What do your answers tell us about the effect of weight and speed on calories burned per minute?
d) Estimate $f_w\ (160,10)$and interpret your answer.
e) Estimate $f_s\ (160,10)$ and interpret your answer.

w / s	8 mph	9 mph	10 mph	11 mph
120 lbs	4.2	5.8	7.4	8.9
140lbs	5.1	6.7	8.3	9.9
160 lbs	6.1	7.7	9.2	10.8
180 lbs	7.0	8.6	10.2	11.7
200 lbs	7.9	9.5	11.1	12.6

Solution:

(a) f_w measures how the calories burned change as weight increases while keeping speed constant.
From the table, we can see that f_w is positive since B increases as w increases, when s held constant.

(b) f_s measures how the calories burned change as speed increases while keeping weight constant.
f_s is also positive since B increases as s increases, when w held constant.

(c)What do these answers tell us about the effect of weight and speed on calories burned per minute?

Weight (w):

Heavier cyclists burn more calories per minute at a fixed speed.

Speed (s):

Cyclists burn more calories per minute when cycling faster at a fixed weight.

(d) Partial Derivative Estimation $f_w\,(160, 10)$:

$\boldsymbol{f_w\,(160, 10)}$**:** Rate of change in calories burned with respect to weight at $\boldsymbol{w = 160}$ lbs and $\boldsymbol{s = 10}$ mph.

$$f_w\,(160, 10) = \frac{\partial f(160, 10)}{\partial w} = \frac{[f(160 + 20, 10) - f(160, 10)]}{180 - 160}$$
$$= \frac{(10.2 - 9.2)}{(180 - 160)} = 0.05$$

Interpretation:

It means if we change the person's weight by $\boldsymbol{1}$ lbs then the number of calories burned per minute is increased by $\boldsymbol{0.05}$.

(e) Partial Derivative Estimation $f_s\,(160, 10)$:

$\boldsymbol{f_s\,(160, 10)}$**:** Rate of change in calories burned with respect to speed at $\boldsymbol{w = 160}$ lbs and $\boldsymbol{s = 10}$ mph.

$$f_s\,(160, 10) = \frac{\partial f(160, 10)}{\partial s} = \frac{[f(160, 11) - f(160, 10)]}{11 - 10}$$
$$= \frac{(10.8 - 9.2)}{11 - 10} = 1.6$$

Interpretation:

It means if we change the speed of the person by $\boldsymbol{1}$ mph then the number of calories burned per minute is increased by $\boldsymbol{1.6}$ units.

Example:

The demand for coffee 'Q', in pounds sold per week is a function of the price of coffee, 'c', in dollars per pound & the price of tea, 't' in dollars per pound, so, $Q = f(c, t)$.

a) Do you expect f_c to be **positive or negative**? What about f_t? **Explain**.
b) **Interpret** each of the following statement in terms of the demand for coffee: $f(3, 2) = 780$, $f_c(3, 2) = -60$, $f_t(3, 2) = 20$.

Solution:

The function $Q = f(c, t)$ describes the **weekly demand for coffee** Q, in pounds, as a function of:

- c: the **price of coffee** (dollars per pound),
- t: the **price of tea** (dollars per pound).

f_c (Partial derivative f with respect to c):

f_c **measures** the **rate of change of coffee demand** with respect to its **price**.

As the **price of coffee increases**, the **demand for coffee usually decreases** (**Law of demand**).

$f_c < 0$ **(Negative)**.

- If the **price of coffee increases**, the **demand for coffee usually decreases, and the rate of change of coffee demand** with respect to its price **decreases**.

  c↑ ⇔ Q↓

- If the **price of coffee decreases**, the **demand for coffee usually increases, and the rate of change of coffee demand** with respect to its price **increase**.

  c↓ ⇔ Q↑

The above statements represent the Inverse relationship and $\frac{\partial f}{\partial c}$ is negative.

So, $\boldsymbol{f}$ is **decreasing function** of $\boldsymbol{c}$.

$\boldsymbol{f_t}$ (partial derivative with respect to $\boldsymbol{t}$):

$\boldsymbol{f_t}$ **measures** the **rate of change of coffee demand** with respect to the **price of tea**.

Tea is likely a substitute for coffee.

As the **price of tea increases**, consumers might **switch to coffee**, **increasing its demand**.

$\boldsymbol{f_t > 0}$ **(Positive)**.

- If the **price of tea increases**, consumers might **switch to coffee**, **increasing its demand,** and the **rate of change of coffee demand** with respect to **price of tea is also increases**.

  t↑ ⇔ Q↑

- If the **price of tea decreases**, consumers will be interested to have tea instead of coffee, which lead to the decease **the demand of coffee,** and the **rate of change of coffee demand** with respect to **price of tea is also decreases**.

  t↓ ⇔ Q↓

The above statements represent the **direct relationship** and $\frac{\partial f}{\partial t}$ is negative.

So, $\boldsymbol{f}$ is **increasing function** of $\boldsymbol{t}$.

Interpretations:

- $\boldsymbol{f(3,2) = 780}$

Interpretation: If the price of coffee is $\$\boldsymbol{3}$ and price of tea is $\$\boldsymbol{2}$ then we can expect $\boldsymbol{780}$ pounds of **coffee** to be sold each week.

- $\boldsymbol{f_c(3,2) = \frac{\partial f}{\partial c}(3,2) = -60}$

Interpretation: If we **change or increase** the price of coffee by $\boldsymbol{1}$ unit then the **demand of coffee is decreased** by $\boldsymbol{60}$ units.

- $\boldsymbol{f_t(3,2) = \frac{\partial f}{\partial t}(3,2) = 20}$

<u>Interpretation:</u> If we **increase the price of tea** by $\mathbf{1}$ unit then the **demand for coffee is increased** by $\mathbf{20}$ units.

<u>Practice Questions for Students:</u>

Exercise: 9.2. Q # 3-5, from Applied Calculus Book By Hughes Hallet 4th Edition.