Topics:

- **Critical Points**
- **Second Derivative Test (D-Test) and Optimization**
- **Applications of Optimization**

Critical Points

- A **critical point** is a **point on the graph** where the **slope changes direction**, or where the first derivative of a function is **zero or undefined**.
- For **functions of two or more variables**, the concept is essentially the same, except for the fact that we are now working with **partial derivatives.**

Definition: Critical Point for Function of Two Variables

Let $z = f(x, y)$ be a function of **two variables** that is defined on an interval containing the point (x_0, y_0). The **point** is **called** a **critical point** of a **function of two variables** if one of the two following conditions holds:

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

   OR

2. Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ **is undefined**.

Example

Finding Critical Points

Find the critical points of each of the following functions:

a. $f(x, y) = \sqrt{4y^2 - 9x^2 + 24y + 36x + 36}$
b. $g(x, y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

Solution

a. First, we calculate $f_x(x, y)$ and $f_y(x, y)$:

$$\begin{aligned} f_x(x, y) &= \tfrac{1}{2}(-18x + 36)\left(4y^2 - 9x^2 + 24y + 36x + 36\right)^{-1/2} \\ &= \frac{-9x+18}{\sqrt{4y^2-9x^2+24y+36x+36}} \\ f_y(x, y) &= \tfrac{1}{2}(8y + 24)\left(4y^2 - 9x^2 + 24y + 36x + 36\right)^{-1/2} \\ &= \frac{4y+12}{\sqrt{4y^2-9x^2+24y+36x+36}}. \end{aligned}$$

Next, we set each of these expressions equal to zero:

$$\begin{aligned} \frac{-9x+18}{\sqrt{4y^2-9x^2+24y+36x+36}} &= 0 \\ \frac{4y+12}{\sqrt{4y^2-9x^2+24y+36x+36}} &= 0. \end{aligned}$$

Then, multiply each equation by its common denominator:

$$\begin{aligned} -9x + 18 &= 0 \\ 4y + 12 &= 0. \end{aligned}$$

Therefore, $x = 2$ and $y = -3$, so $(2, -3)$ is a critical point of f.
We must also check for the possibility that the denominator of each partial derivative can equal zero, thus causing the partial derivative not to exist. Since the denominator is the same in each partial derivative, we need only do this once:

$$4y^2 - 9x^2 + 24y + 36x + 36 = 0.$$

This equation represents a hyperbola. We should also note that the domain of f consists of points satisfying the inequality

$$4y^2 - 9x^2 + 24y + 36x + 36 \geq 0.$$

Therefore, any points on the hyperbola are not only critical points, they are also on the boundary of the domain. To put the hyperbola in standard form, we use the method of completing the square:

$$\begin{aligned}
4y^2 - 9x^2 + 24y + 36x + 36 &= 0 \\
4y^2 - 9x^2 + 24y + 36x &= -36 \\
4y^2 + 24y - 9x^2 + 36x &= -36 \\
4\left(y^2 + 6y\right) - 9\left(x^2 - 4x\right) &= -36 \\
4\left(y^2 + 6y + 9\right) - 9\left(x^2 - 4x + 4\right) &= -36 + 36 - 36 \\
4(y+3)^2 - 9(x-2)^2 &= -36.
\end{aligned}$$

Dividing both sides by -36 puts the equation in standard form:

$$\begin{aligned}
\frac{4(y+3)^2}{-36} - \frac{9(x-2)^2}{-36} &= 1 \\
\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} &= 1.
\end{aligned}$$

Notice that point $(2, -3)$ is the center of the hyperbola.

b. First, we calculate $g_x(x, y)$ and $g_y(x, y)$:

$$\begin{aligned} g_x(x, y) &= 2x + 2y + 4 \\ g_y(x, y) &= 2x - 8y - 6. \end{aligned}$$

Next, we set each of these expressions equal to zero, which gives a system of equations in x and y:

$$\begin{aligned} 2x + 2y + 4 &= 0 \\ 2x - 8y - 6 &= 0. \end{aligned}$$

Subtracting the second equation from the first gives $10y + 10 = 0$, so $y = -1$. Substituting this into the first equation gives $2x + 2(-1) + 4 = 0$, so $x = -1$. Therefore $(-1, -1)$ is a critical point of g (Figure 4.46). There are no points in $\mathbb{R}^2$ that make either partial derivative not exist.

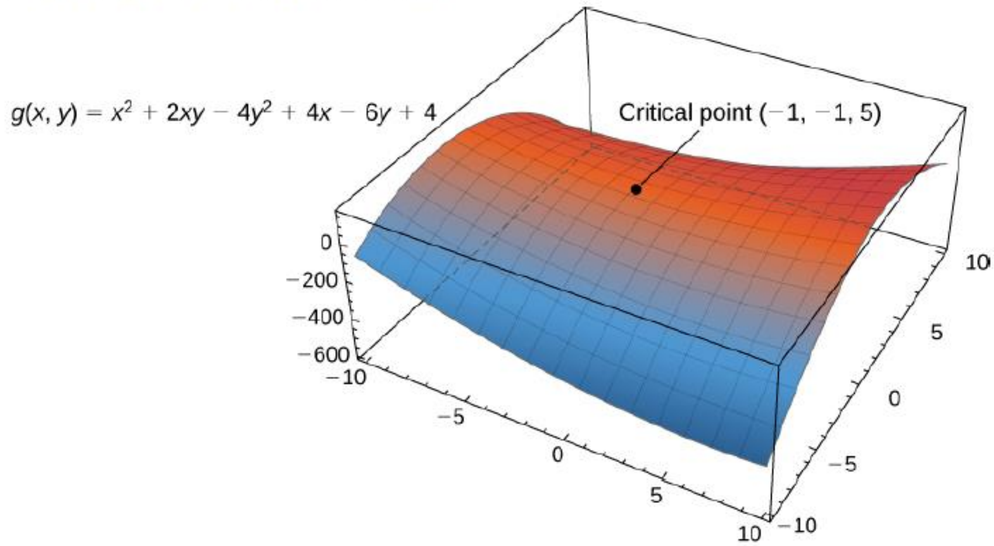


Figure 4.46 The function $g(x, y)$ has a critical point at $(-1, -1, 5)$.

Definition

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Then f has a *local maximum* at (x_0, y_0) if

$$f(x_0, y_0) \geq f(x, y)$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a *local maximum value*. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a *global maximum* (also called an *absolute maximum*) at (x_0, y_0).

The function f has a *local minimum* at (x_0, y_0) if

$$f(x_0, y_0) \leq f(x, y)$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a *local minimum value*. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a *global minimum* (also called an *absolute minimum*) at (x_0, y_0).

If $f(x_0, y_0)$ is either a local maximum or local minimum value, then it is called a *local extremum* (see the following figure).

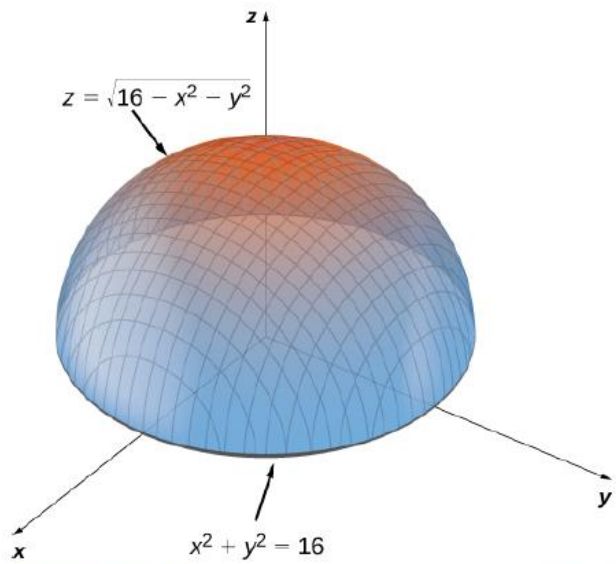


Figure 4.47 The graph of $z = \sqrt{16 - x^2 - y^2}$ has a maximum value when $(x, y) = (0, 0)$. It attains its minimum value at the boundary of its domain, which is the circle $x^2 + y^2 = 16$.

Fermat's Theorem for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Suppose f_x and f_y each exists at (x_0, y_0). If f has a local extremum at (x_0, y_0), then (x_0, y_0) is a critical point of f.

Second Derivative Test for Two Variables (D-Test)

Let $z = f(x, y)$ be a **function of two variables** for which the first- and second-order partial derivatives are continuous on some interval containing the **critical point**(s), (x_0, y_0).

We know that at (x_0, y_0),

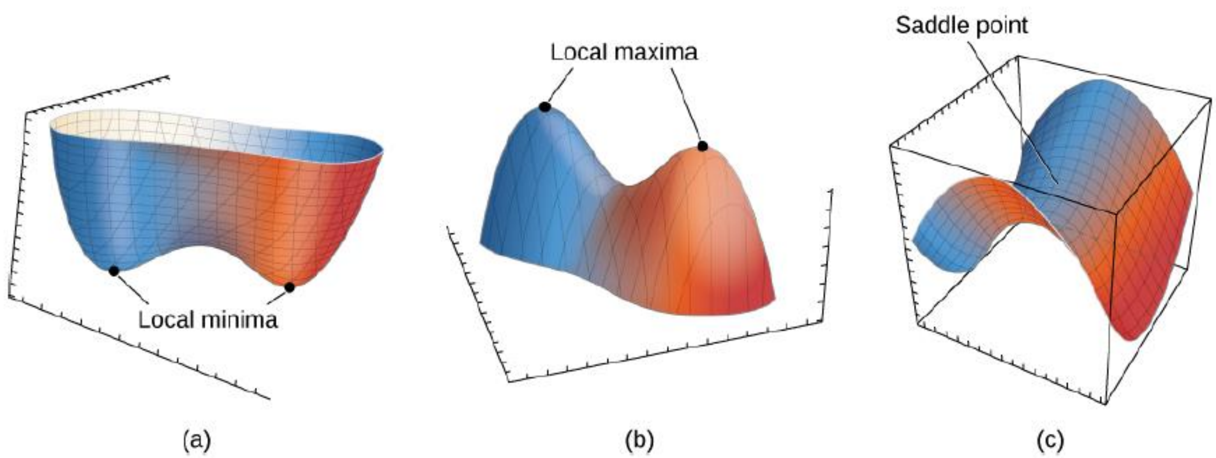
$$f_x(x_0, y_0) = 0 \qquad \text{and} \qquad f_y(x_0, y_0) = 0.$$

We define the D-test as,

$$D = f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - \left[f_{xy}(x_0, y_0) \right]^2$$

then:

1) If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then $f(x, y)$ has a ***local minima or relative minima*** at (x_0, y_0).
2) If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then $f(x, y)$ has a ***local maxima or relative maxima*** at (x_0, y_0).
3) If $D < 0$, then $f(x, y)$ has a **saddle point** (neither maximum nor minimum) at (x_0, y_0).
4) If $D = 0$, then test is **inconclusive**. (Test Fails)



Example

Using the Second Derivative Test

Find the critical points for each of the following functions, and use the second derivative test to find the local extrema:

a. $f(x, y) = 4x^2 + 9y^2 + 8x - 36y + 24$
b. $g(x, y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$

Solution

a. Step 1 of the problem-solving strategy involves finding the critical points of f. To do this, we first calculate $f_x(x, y)$ and $f_y(x, y)$, then set each of them equal to zero:

$$\begin{aligned} f_x(x, y) &= 8x + 8 \\ f_y(x, y) &= 18y - 36. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} 8x + 8 &= 0 \\ 18y - 36 &= 0. \end{aligned}$$

The solution to this system is $x = -1$ and $y = 2$. Therefore $(-1, 2)$ is a critical point of f.
Step 2 of the problem-solving strategy involves calculating D. To do this, we first calculate the second partial derivatives of f:

$$\begin{aligned} f_{xx}(x, y) &= 8 \\ f_{xy}(x, y) &= 0 \\ f_{yy}(x, y) &= 18. \end{aligned}$$

Therefore, $D = f_{xx}(-1, 2) f_{yy}(-1, 2) - \left(f_{xy}(-1, 2)\right)^2 = (8)(18) - (0)^2 = 144.$
Step 3 states to check the Second Derivative Test for Functions of Two Variables.. Since $D > 0$ and $f_{xx}(-1, 2) > 0$, this corresponds to case 1. Therefore, f has a local minimum at $(-1, 2)$ as shown in the following figure.

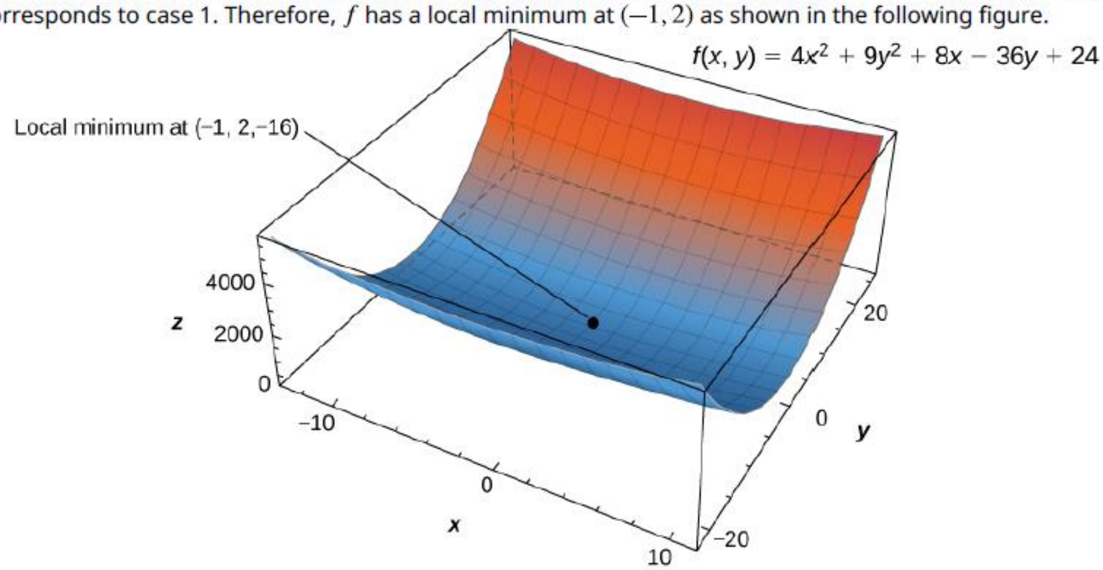


Figure 4.50 The function $f(x, y)$ has a local minimum at $(-1, 2, -16)$.

b. For step 1, we first calculate $g_x(x, y)$ and $g_y(x, y)$, then set each of them equal to zero:

$$\begin{aligned} g_x(x, y) &= x^2 + 2y - 6 \\ g_y(x, y) &= 2y + 2x - 3. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} x^2 + 2y - 6 &= 0 \\ 2y + 2x - 3 &= 0. \end{aligned}$$

To solve this system, first solve the second equation for y. This gives $y = \frac{3-2x}{2}$. Substituting this into the first equation gives

$$\begin{aligned} x^2 + 3 - 2x - 6 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0. \end{aligned}$$

Therefore, $x = -1$ or $x = 3$. Substituting these values into the equation $y = \frac{3-2x}{2}$ yields the critical points $\left(-1, \frac{5}{2}\right)$ and $\left(3, -\frac{3}{2}\right)$.

Step 2 involves calculating the second partial derivatives of g:

$$\begin{aligned} g_{xx}(x, y) &= 2x \\ g_{xy}(x, y) &= 2 \\ g_{yy}(x, y) &= 2. \end{aligned}$$

Then, we find a general formula for D:

$$\begin{aligned} D &= g_{xx}(x_0, y_0)\, g_{yy}(x_0, y_0) - \left(g_{xy}(x_0, y_0)\right)^2 \\ &= (2x_0)(2) - 2^2 \\ &= 4x_0 - 4. \end{aligned}$$

Next, we substitute each critical point into this formula:

$$\begin{aligned} D\left(-1, \tfrac{5}{2}\right) &= (2(-1))(2) - (2)^2 = -4 - 4 = -8 \\ D\left(3, -\tfrac{3}{2}\right) &= (2(3))(2) - (2)^2 = 12 - 4 = 8. \end{aligned}$$

In step 3, we note that, applying the Second Derivative Test for Functions of Two Variables. to point $\left(-1, \frac{5}{2}\right)$ leads to case 3, which means that $\left(-1, \frac{5}{2}\right)$ is a saddle point. Applying the theorem to point $\left(3, -\frac{3}{2}\right)$ leads to case 1, which means that $\left(3, -\frac{3}{2}\right)$ corresponds to a local minimum as shown in the following figure.

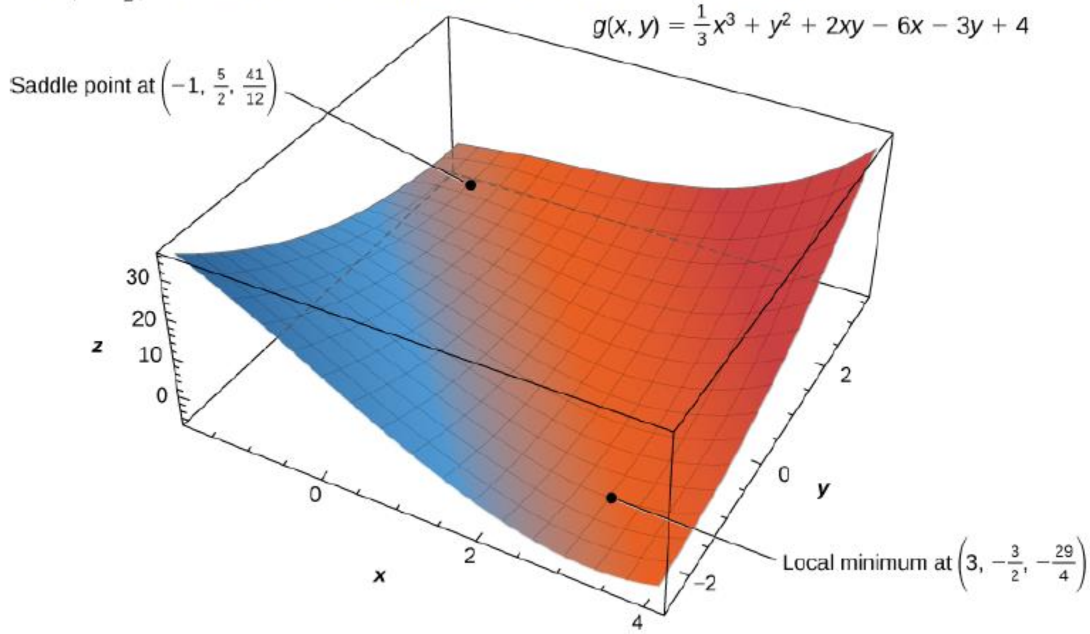


Figure 4.51 The function $g(x, y)$ has a local minimum and a saddle point.

OPTIMIZATION

To optimize a function means to find the largest or smallest value of the function.

If the function represents profit, we may want to find the conditions that maximize profit. On the other hand, if the function represents cost, we may we may want to find the conditions that minimize cost.

For the procedure of optimization we will follow the following steps in calculation:

1. **Find 2nd order Derivatives of the given function.**
2. **Find Critical Points**
3. **Apply 2nd Derivative Test (D-Test) for each critical point.**
4. **Conclusion**

Recall: Second Derivative Test for Functions of Two Variables

Suppose (x_o, y_o) is a critical point where $f_x(x_o, y_o) = f_y(x_o, y_o) = 0$.

Let

$$\boldsymbol{D = f_{xx}(x_o, y_o) f_{yy}(x_o, y_o) - \left(f_{xy}(x_o, y_o)\right)^2}$$

- If $\boldsymbol{D} > 0$ & $\boldsymbol{f_{xx}(x_o, y_o)} > 0$, then f has a local minima at $(\boldsymbol{x_o, y_o})$.
- If $\boldsymbol{D} > 0$ & $\boldsymbol{f_{xx}(x_o, y_o)} < 0$, then f has a local maxima at $(\boldsymbol{x_o, y_o})$.
- If $\boldsymbol{D} < 0$ then $\boldsymbol{f}$ has neither maxima nor minima at $(\boldsymbol{x_o, y_o})$. It is saddle point.
- If $\boldsymbol{D = 0}$ then test is inconclusive. (Test fails).

Example 1:

Optimize the given function by determining if it has a local maxima, local minima or neither.

$$\boldsymbol{f(x,y) = x^2 - 2x + y^2 - 4y + 5}$$

Solution:

The given function is

$$f(x,y) = x^2 - 2x + y^2 - 4y + 5$$

Find $\frac{\partial f}{\partial x}$

Partially differentiate $\boldsymbol{f(x,y)}$ w.r.t variable $\boldsymbol{x}$, we have

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 - 2x + y^2 - 4y + 5)$$

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial x}(4y) + \frac{\partial}{\partial x}(5)$$

$$f_x(x,y) = \frac{\partial f}{\partial x} = 2x - 2 + 0 + 0 + 0 = 2x - 2$$

$$\boxed{f_x(x,y) = \frac{\partial f}{\partial x} = 2x - 2}$$

Find $\frac{\partial f}{\partial y}$

Partially differentiate $\boldsymbol{f(x,y)}$ w.r.t variable $\boldsymbol{y}$, we have

$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 - 2x + y^2 - 4y + 5)$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial y}(2x) + \frac{\partial}{\partial y}(y^2) - \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial y}(5)$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = 0 + 0 + 2y - 4 + 0 = 2y - 4$$

$$\boldsymbol{f_y(x,y) = \frac{\partial f}{\partial y} = 2y - 4}$$

Find the critical point

To find critical point, put

$$\frac{\partial f}{\partial x} = f_x(x_0, y_0) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = f_y(x_0, y_0) = 0$$

This implies that

$f_x(x_0, y_0) = 0$	$f_y(x_0, y_0) = 0$
$\Rightarrow 2x - 2 = 0$	$\Rightarrow 2y - 4 = 0$
$\Rightarrow 2x = 2$	$\Rightarrow 2y = 4$
$\Rightarrow x = \frac{2}{2} = 1$	$\Rightarrow y = \frac{4}{2} = 2$
$\Rightarrow x = 1$	$\Rightarrow y = 2$

Hence, the critical point is

$$(x, y) = (1, 2)$$

Find Second Derivatives

Find $f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2}$

Partially differentiate $f_x(x,y)$ w.r.t variable x, we have

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2x - 2)$$

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = 2\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial x}(2) = 2(1) - 0 = 2$$

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = 2$$

Find $f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2}$

Partially differentiate $f_y(x,y)$ w.r.t variable y, we have

$$f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(2y - 4)$$

$$f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = 2\frac{\partial}{\partial y}(y) - \frac{\partial}{\partial y}(4) = 2(1) - 0 = 2$$

$$\boxed{f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = 2}$$

Find $f_{xy}(x,y) = \frac{\partial^2 f}{\partial y\,\partial x}$

Partially differentiate $f_x(x,y)$ w.r.t variable y, we have

$$f_{xy}(x,y) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}(2x - 2)$$

$$f_{xy}(x,y) = \frac{\partial^2 f}{\partial y\,\partial x} = 2\frac{\partial}{\partial y}(x) - \frac{\partial}{\partial y}(2) = 2(0) - 0 = 0$$

$$\boxed{f_{xy}(x,y) = \frac{\partial^2 f}{\partial y\,\partial x} = 0}$$

Find Second Derivative value at critical point(s)

As there is one critical point that is obtained $(x,y) = (1,2)$, so

$f_{xx}(x,y) = 2 \quad \Rightarrow \quad f_{xx}(1,2) = 2 > 0$

$f_{yy}(x,y) = 2 \quad \Rightarrow \quad f_{yy}(1,2) = 2$

$f_{xy}(x,y) = 0 \quad \Rightarrow \quad f_{xy}(1,2) = 0$

Apply Second Derivative Test

$$D = f_{xx}(1,2)\, f_{yy}(1,2) - \left[\, f_{xy}(1,2)\,\right]^2$$

Putting values in the formula, we get

$$D = [2][2] - [\,0\,]^2$$

$$\Rightarrow D = 4 - 0$$

$$\Rightarrow D = 4 > 0$$

Check the function has local maxima/minima or neither

Since $D = 4 > 0$ **and** $f_{xx}(1,2) = 2 > 0$, so by using **2nd derivative test**, the given function $f(x,y)$ has a **local minima** at **critical point** $(1,2)$**.**

Example 2:

Determine whether the function has a local maxima, local minima or neither.

$$f(x,y) = y^3 - 3xy + 6x$$

Solution:

The given function is

$$f(x,y) = y^3 - 3xy + 6x$$

Step 1: Find $\frac{\partial f}{\partial x}$

Partially differentiate $f(x,y)$ w.r.t variable x, we have

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(y^3 - 3xy + 6x)$$

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(y^3) - \frac{\partial}{\partial x}(3xy) + \frac{\partial}{\partial x}(6x)$$

$$f_x(x,y) = \frac{\partial f}{\partial x} = 0 - 3y(1) + 6 = -3y + 6$$

$$f_x(x,y) = \frac{\partial f}{\partial x} = -3y + 6 \text{ -------------(1)}$$

Step 2: Find $\frac{\partial f}{\partial y}$

Partially differentiate $f(x,y)$ w.r.t variable y, we have

$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y^3 - 3xy + 6x)$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y^3) - \frac{\partial}{\partial y}(3xy) + \frac{\partial}{\partial y}(6x)$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = 3y^2 - 3x(1) + 0 = 3y^2 - 3x$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = 3y^2 - 3x \text{ -------------(2)}$$

Step 3: Find the critical point

To find critical point, put

$\frac{\partial f}{\partial x} = f_x(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y} = f_y(x_0, y_0) = 0$

This implies that

$f_x(x_0, y_0) = 0$	$f_y(x_0, y_0) = 0$
$\Rightarrow -3y + 6 = 0$	$\Rightarrow 3y^2 - 3x = 0$
$\Rightarrow -3y = -6$	$\Rightarrow -3x = -3y^2$
$\Rightarrow y = \frac{-6}{-3} = 2$	$\Rightarrow x = \frac{-3y^2}{-3} = y^2$
$\Rightarrow y = 2$ -------------(3)	$\Rightarrow x = y^2$ -------------(4)

	From equation **(3)**, substitute value of $\boldsymbol{y}$ in equation **(4)**, we get $\Rightarrow \boldsymbol{x} = (\mathbf{2})^{\mathbf{2}} = \mathbf{4}$ $\Rightarrow \boldsymbol{x} = \mathbf{4}$ **-------- (5)**

Hence,

$$\boldsymbol{x} = \mathbf{4}, \quad \boldsymbol{y} = \mathbf{2}$$

So, the critical point is

$$(\boldsymbol{x}, \boldsymbol{y}) = (\mathbf{4}, \mathbf{2})$$

Step 4: Find Second Derivatives

Find $f_{xx}(x, y) = \frac{\partial^2 f}{\partial x^2}$

Partially differentiate $\boldsymbol{f_x(x, y)}$ w.r.t variable $\boldsymbol{x}$, we have

$$f_{xx}(x, y) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(-3y + 6)$$

$$f_{xx}(x, y) = \frac{\partial^2 f}{\partial x^2} = -3\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial x}(6) = -3(0) + 0 = 0$$

$$f_{xx}(x, y) = \frac{\partial^2 f}{\partial x^2} = 0$$

Find $f_{yy}(x, y) = \frac{\partial^2 f}{\partial y^2}$

Partially differentiate $\boldsymbol{f_y(x, y)}$ w.r.t variable $\boldsymbol{y}$, we have

$$f_{yy}(x, y) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(3y^2 - 3x)$$

$$f_{yy}(x, y) = \frac{\partial^2 f}{\partial y^2} = 3\frac{\partial}{\partial y}(y^2) - 3\frac{\partial}{\partial y}(x) = 3(2y) - 3(0) = 6y$$

$$f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = 6y$$

Find $f_{xy}(x,y) = \frac{\partial^2 f}{\partial y\,\partial x}$

Partially differentiate $f_x(x,y)$ w.r.t variable y, we have

$$f_{xy}(x,y) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}(-3y+6)$$

$$f_{xy}(x,y) = \frac{\partial^2 f}{\partial y\,\partial x} = -3\frac{\partial}{\partial y}(y) + \frac{\partial}{\partial y}(6) = -3(1) + 0 = -3$$

$$f_{xy}(x,y) = \frac{\partial^2 f}{\partial y\,\partial x} = -3$$

Step 5: Find Second Derivative value at critical point(s)

As there is one critical point that is obtained $(x,y) = (4,2)$, so

$f_{xx}(x,y) = 0 \qquad \Rightarrow f_{xx}(4,2) = 0$

$f_{yy}(x,y) = 6y \qquad \Rightarrow f_{yy}(4,2) = 6(2) = 12$

$f_{xy}(x,y) = -3 \qquad \Rightarrow f_{xy}(4,2) = -3$

Step 6: Apply Second Derivative Test

$$D = f_{xx}(4,2)\,f_{yy}(4,2) - \left[\,f_{xy}(4,2)\,\right]^2$$

Putting values in the formula, we get

$$D = [0][12] - [-3\,]^2$$

$$\Rightarrow D = 0 - 9$$

$$\Rightarrow D = -9 < 0$$

Step 7: Check the function has local maxima/minimum or neither

Since $\boldsymbol{D = -9 < 0}$, so by using 2nd **derivative test**, the given function $\boldsymbol{f(x, y)}$ has neither **local maxima nor local minima** at **critical point** $(\mathbf{4}, \mathbf{2})$. The function $\boldsymbol{f(x, y)}$ has **saddle point** at **critical point** $(\mathbf{2}, -\mathbf{1})$.

Example:

Determine whether the function has a local maxima, local minima or neither.

$$\boldsymbol{f(x, y) = x^3 + y^3 - 3x^2 - 3y + 10}$$

Solution:

The given function is

$$\boldsymbol{f(x, y) = x^3 + y^3 - 3x^2 - 3y + 10}$$

Step 1: Find $\frac{\partial f}{\partial x}$

Partially differentiate $\boldsymbol{f(x, y)}$ w.r.t variable $\boldsymbol{x}$, we have

$$\boldsymbol{f_x(x, y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3 + y^3 - 3x^2 - 3y + 10)}$$

$$\boldsymbol{f_x(x, y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(y^3) - \frac{\partial}{\partial x}(3x^2) - \frac{\partial}{\partial x}(3y) + \frac{\partial}{\partial x}(10)}$$

$$\boldsymbol{f_x(x, y) = \frac{\partial f}{\partial x} = 3x^2 + 0 - 3(2x) - 0 + 0 = 3x^2 - 6x}$$

$$\boxed{\boldsymbol{f_x(x, y) = \frac{\partial f}{\partial x} = 3x^2 - 6x}}$$

Step 2: Find $\frac{\partial f}{\partial y}$

Partially differentiate $\boldsymbol{f(x,y)}$ w.r.t variable $\boldsymbol{y}$, we have

$$f_y(x,y)=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(x^3+y^3-3x^2-3y+10)$$

$$f_y(x,y)=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(x^3)+\frac{\partial}{\partial y}(y^3)-\frac{\partial}{\partial y}(3x^2)-\frac{\partial}{\partial y}(3y)+\frac{\partial}{\partial y}(10)$$

$$f_y(x,y)=\frac{\partial f}{\partial y}=0+3y^2-0-3+0=3y^2-3$$

$$\boxed{f_y(x,y)=\frac{\partial f}{\partial y}=3y^2-3}$$

Step 3: Find the critical point

To find critical point, put

$$\frac{\partial f}{\partial x}=f_x(x_0,y_0)=0 \qquad \text{and} \qquad \frac{\partial f}{\partial y}=f_y(x_0,y_0)=0$$

This implies that

$f_x(x_0,y_0)=0$	$f_y(x_0,y_0)=0$
$\Rightarrow 3x^2-6x=0$	$\Rightarrow 3y^2-3=0$
$\Rightarrow 3x(x-2)=0$	$\Rightarrow 3(y^2-1)=0$
$\Rightarrow x(x-2)=0$	$\Rightarrow y^2-1=0$
$\Rightarrow x=0,\quad x-2=0$	$\Rightarrow (y+1)(y-1)=0$
$\Rightarrow x=0,\quad x=2$	$\Rightarrow y=-1,\quad y=1$

Since we obtained the values of $\boldsymbol{x}$ and $\boldsymbol{y}$ as below

$$x=0,\quad x=2\,, \qquad \text{and} \quad y=-1,\quad y=1$$

- Now, fix first value of $\boldsymbol{x}$ and make ordered pairs will all values of $\boldsymbol{y}$, that is

$$(0,-1),\qquad (0,1)$$

- Now, fix second value of $\boldsymbol{x}$ and make ordered pairs will all values of $\boldsymbol{y}$, that is

$$(2,-1), \qquad (2,1)$$

Hence, we obtain the four (4) critical points. The critical points are as follows:

1. $(x,y) = (0,-1)$
2. $(x,y) = (0,1)$
3. $(x,y) = (2,-1)$
4. $(x,y) = (2,1)$

Note: In this case we get 4 critical points. It means that, for every critical point, you have to find that where function has local maxima, local minima, or neither separately, *i.e. we compute D-Test for all critical points separately.*

Step 4: Find Second Derivatives

Find $f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2}$

Partially differentiate $f_x(x,y)$ w.r.t variable x, we have

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 6x)$$

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = 3\frac{\partial}{\partial x}(x^2) - 6\frac{\partial}{\partial x}(x) = 3(2x) - 6(1) = 6x - 6$$

$$\boxed{f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = 6x - 6}$$

Find $f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2}$

Partially differentiate $f_y(x,y)$ w.r.t variable y, we have

$$f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(3y^2 - 3)$$

$$f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = 3\frac{\partial}{\partial y}(y^2) - \frac{\partial}{\partial y}(3) = 3(2y) - 0 = 6y$$

$$\boxed{f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = 6y}$$

Find $f_{xy}(x,y) = \frac{\partial^2 f}{\partial y\, \partial x}$

Partially differentiate $f_x(x,y)$ w.r.t variable y, we have

$$f_{xy}(x,y) = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial}{\partial y}(3x^2 - 6x)$$

$$f_{xy}(x,y) = \frac{\partial^2 f}{\partial y\, \partial x} = 3\frac{\partial}{\partial y}(x^2) - 6\frac{\partial}{\partial y}(x) = 3(0) - 6(0) = 0$$

$$\boxed{f_{xy}(x,y) = \frac{\partial^2 f}{\partial y\, \partial x} = 0}$$

Step 5: Find Second Derivative value at critical point(s)

$$f_{xx}(x,y) = 6x - 6$$

$$f_{yy}(x,y) = 6y$$

$$f_{xy}(x,y) = 0$$

For critical point $(x,y) = (0,-1)$

$f_{xx}(x,y) = 6x - 6 \qquad \Rightarrow f_{xx}(0,-1) = 6(0) - 6 = 0 - 6 = -6 < 0$

$f_{yy}(x,y) = 6y \qquad \Rightarrow f_{yy}(0,-1) = 6(-1) = -6$

$f_{xy}(x,y) = 0 \qquad \Rightarrow f_{xy}(0,-1) = 0$

For critical point $(x, y) = (0, 1)$

$f_{xx}(x, y) = 6x - 6 \qquad \Rightarrow f_{xx}(0, 1) = 6(0) - 6 = 0 - 6 = -6 < 0$

$f_{yy}(x, y) = 6y \qquad \Rightarrow f_{yy}(0, 1) = 6(1) = 6$

$f_{xy}(x, y) = 0 \qquad \Rightarrow f_{xy}(0, 1) = 0$

For critical point $(x, y) = (2, -1)$

$f_{xx}(x, y) = 6x - 6 \qquad \Rightarrow f_{xx}(2, -1) = 6(2) - 6 = 12 - 6 = 6 > 0$

$f_{yy}(x, y) = 6y \qquad \Rightarrow f_{yy}(2, -1) = 6(-1) = -6$

$f_{xy}(x, y) = 0 \qquad \Rightarrow f_{xy}(2, -1) = 0$

For critical point $(x, y) = (2, 1)$

$f_{xx}(x, y) = 6x - 6 \qquad \Rightarrow f_{xx}(2, 1) = 6(2) - 6 = 12 - 6 = 6 > 0$

$f_{yy}(x, y) = 6y \qquad \Rightarrow f_{yy}(2, 1) = 6(1) = 6$

$f_{xy}(x, y) = 0 \qquad \Rightarrow f_{xy}(2, 1) = 0$

Step 6: Apply Second Derivative Test

For critical point $(x, y) = (0, -1)$

- $f_{xx}(0, -1) = -6 < 0$
- $f_{yy}(0, -1) = -6$
- $f_{xy}(0, -1) = 0$

Since

$$D = f_{xx}(0, -1)\, f_{yy}(0, -1) - \left[\, f_{xy}(0, -1)\, \right]^2$$

Putting values in the formula, we get

$$D = [-6][-6] - [\, 0\,]^2$$

$$\Rightarrow D = 36 - 0$$

$$\Rightarrow D = 36 > 0$$

For critical point $(x, y) = (0, 1)$

- $f_{xx}(0,1) = -6 < 0$
- $f_{yy}(0,1) = 6$
- $f_{xy}(0,1) = 0$

Since

$$D = f_{xx}(0,1)\, f_{yy}(0,1) - \left[\, f_{xy}(0,1)\,\right]^2$$

Putting values in the formula, we get

$$D = [-6][6] - [\,0\,]^2$$

$$\Rightarrow D = -36 - 0$$

$$\Rightarrow D = -36 < 0$$

For critical point $(x, y) = (2, -1)$

- $f_{xx}(2,-1) = 6 > 0$
- $f_{yy}(2,-1) = -6$
- $f_{xy}(2,-1) = 0$

Since

$$D = f_{xx}(2,-1)\, f_{yy}(2,-1) - \left[\, f_{xy}(2,-1)\,\right]^2$$

Putting values in the formula, we get

$$D = [6][-6] - [\,0\,]^2$$

$$\Rightarrow D = -36 - 0$$

$$\Rightarrow D = -36 < 0$$

For critical point $(x, y) = (2, 1)$

- $f_{xx}(2,1) = 6 > 0$
- $f_{yy}(2,1) = 6$
- $f_{xy}(2,1) = 0$

Since

$$D = f_{xx}(2,1)\, f_{yy}(2,1) - \left[\, f_{xy}(2,1)\, \right]^2$$

Putting values in the formula, we get

$$D = [6][6] - [\ 0\ ]^2$$

$$\Rightarrow D = 36 - 0$$

$$\Rightarrow D = 36 > 0$$

Step 7: Check the function has local maxima/minimum or neither

For critical point $(x, y) = (0, -1)$

Since $D = 36 > 0$ **and** $f_{xx}(0, -1) = -6 < 0$, so by using **2nd derivative test**, the given function $f(x, y)$ has a **local maxima** at **critical point** $(0, -1)$**.**

For critical point $(x, y) = (0, 1)$

Since $D = -36 < 0$**,** so by using **2nd derivative test**, the given function $f(x, y)$ has neither **local maxima nor local minima** at **critical point** $(0, 1)$**.** The function $f(x, y)$ has **saddle point** at **critical point** $(0, 1)$**.**

For critical point $(x, y) = (2, -1)$

Since $\boldsymbol{D = -36 < 0}$**,** so by using **2nd derivative test**, the given function $f(x, y)$ has neither **local maxima nor local minima** at **critical point** $(2, -1)$**.** The function $f(x, y)$ has **saddle point** at **critical point** $(2, -1)$**.**

For critical point $(x, y) = (2, 1)$

Since $\boldsymbol{D = 36 > 0}$ **and** $f_{xx}(2, 1) = 6 > 0$, so by using **2nd derivative test**, the given function $f(x, y)$ has a **local minima** at **critical point** $(2, 1)$**.**

Conclusion:

$(x, y) = (0, -1)$ $\Rightarrow$ **Function has Local Maxima.**

$(x, y) = (0, 1)$ $\Rightarrow$ **Function has Saddle Point.**

$(x, y) = (2, -1)$ $\Rightarrow$ **Function has Saddle Point.**

$(x, y) = (2, 1)$ $\Rightarrow$ **Function has Local Minima.**

Example:

In regions with severe winter weather, the **wind-chill index** is often used to describe the **apparent severity of the cold**. This index W is a **subjective temperature** that depends on the **actual temperature** T and the **wind speed** v. So W is a function of T and v, and we can write

$$W = f(T, v)$$

The wind-chill index W compiled by the **National Weather Service of the US** and the **Meteorological Service of Canada** is given in the form of the function.

$$W = f(T, v) = 400 - 3T^2 - 4T + 2Tv - 5v^2 + 48v$$

a) **Find** the **optimal wind-chill index** for the given function.
b) **Find** the **optimal value of wind chill index.**

Solution:

$$W = f(T, V) = 400 - 3T^2 - 4T + 2TV - 5V^2 + 48V$$

$$W_T = f_T = 0 - 6T - 4 + 2V - 0 + 0$$

$$\boxed{W_T = f_T = -6T + 2V - 4}$$

$$W_V = f_V = 0 - 0 - 0 + 2T - 10V + 48$$

$$\boxed{W_V = f_V = 2T - 10V + 48}$$

put

$W_T = 0$

$$-6T + 2V - 4 = 0$$
$$-2(3T - V + 2) = 0$$
$$\Rightarrow 3T - V + 2 = 0$$
$$\Rightarrow \boxed{3T - V = -2} \quad - (1)$$

$W_V = 0$

$$2T - 10V + 48 = 0$$
$$2(T - 5V + 24) = 0$$
$$\Rightarrow T - 5V + 24 = 0$$
$$\Rightarrow \boxed{T - 5V = -24} \quad - (2)$$

$(5 \times eq(1)) - eq(2)$

$$15T - \cancel{5V} = -10$$
$$\underline{\pm T \mp \cancel{5V} = \mp 24}$$
$$14T = 14$$
$$\Rightarrow \boxed{T = 1}$$

put in eq (1)

$$3T - V = -2$$
$$3(1) - V = -2$$
$$3 - V = -2$$
$$3 + 2 = V$$
$$\boxed{V = 5}$$

$\therefore$ So

$\therefore$ $T = 1$, $V = 5$ are critical points

Now

$W_T = f_T = -6T + 2V - 4$

$W_{TT} = f_{TT} = -6 + 0 - 0$

$\boxed{W_{TT} = f_{TT} = -6} \Rightarrow \boxed{W_{TT}(1,5) = -6}$

And

$W_{TV} = 0 + 2 - 0$

$\boxed{W_{TV} = 2} \Rightarrow \boxed{W_{TV}(1,5) = 2}$

Also

$W_V = f_V = 2T - 10V + 48$

$W_{VV} = f_{VV} = 0 - 10 + 0$

$\boxed{W_{VV} = f_{VV} = -10} \Rightarrow \boxed{W_{VV}(1,5) = -10}$

Now

$D = W_{TT}(1,5) \cdot W_{VV}(1,5) - W_{TV}^2(1,5)$

$D = (-6) \cdot (-10) - (2)^2$

$D = 60 - 4$

$\boxed{D = 56 > 0}$

As $D > 0$ and $W_{TT} < 0$, So 'W' has local Maximum at $(1,5)$.

$\therefore$ New function value is

$\therefore W = f(T,V) = 400 - 3T^2 - 4T + 2TV - 5V^2 + 48V$ Page #9

$\therefore W = f(1,5) = 400 - 3 - 4 + 10 - 125 + 240$

$W = f(1,5) = 650 - 132$

$\boxed{W = f(1,5) = 518}$

Practice Questions

Question 1:

The terminal velocity (meters/second) that a two-stage rocket achieves is a function of the amount of fuel $\boldsymbol{x}$ and $\boldsymbol{y}$ (measured in liters) loaded into the two stages such that

$$f(x,y) = x^3y + 12x^2 - 8y$$

Minimize the total quantity of fuel required to achieve a specified terminal velocity.

Question 2:

A company designing a video streaming application that relies on two key parameters, denoted as $\boldsymbol{x}$ and $\boldsymbol{y}$, to control the video quality and buffer size. Let the function $\boldsymbol{f(x,y)}$ represent the user wait time.

$$f(x,y) = 9x^3 + \frac{y^3}{3} - 4xy$$

Find the **optimal solutions** and **minimizes** the user wait in the video streaming application.

Practice Questions:

Determine whether the following functions have a local maxima, local minima or neither.

1) $f(x, y) = x^2 + 4x + y^2$
2) $f(x, y) = x^2 + xy + 3y$
3) $f(x, y) = x^2 + y^2 + 6x - 10y + 8$
4) $f(x, y) = x^2 - 2xy + 3y^2 - 8y$
5) $f(x, y) = x^3 - 3x + y^3 - 3y$
6) $f(x, y) = x^3 + y^2 - 3x^2 + 10y + 6$
7) $f(x, y) = x^3 + y^3 - 6y^2 - 3x + 9$
8) $f(x, y) = x^3 + y^3 - 3x^2 - 3y + 10$
9) $f(x, y) = 400 - 3x^2 - 4x + 2xy - 5y^2 + 48y$

Optimization Using Gradient-Based Algorithms in PYTHON

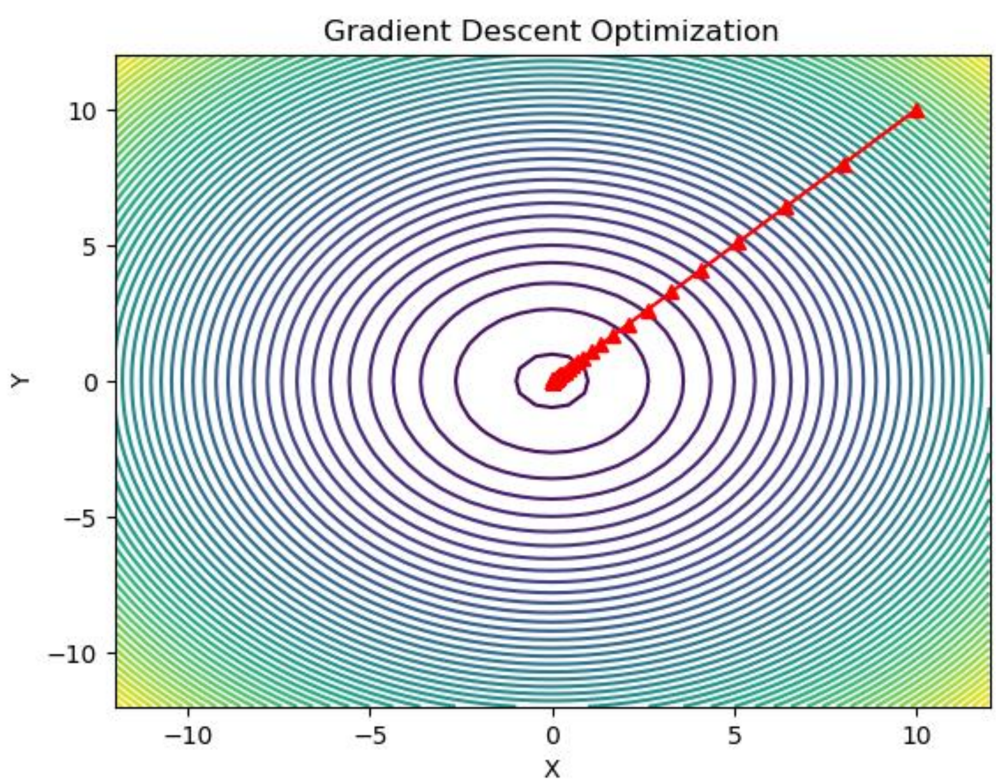
Example: **Optimize** the **cost function** $\boldsymbol{f(x, y) = x^2 + y^2 + 5}$ using **gradient descent**.

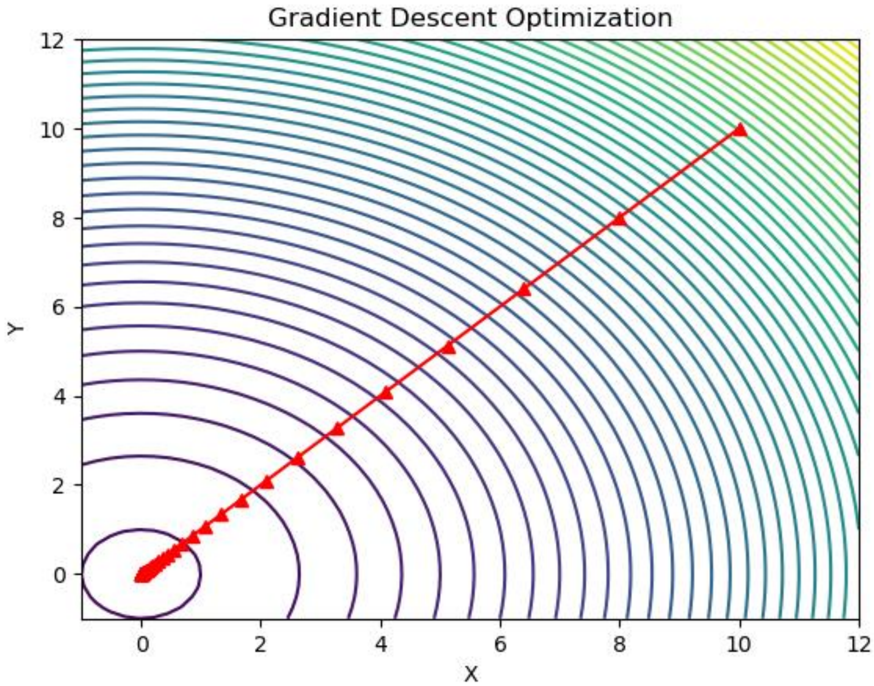
Python Code:

```
import numpy as np
import matplotlib.pyplot as plt
# Define a multivariate function (e.g., a simple quadratic function)
def cost_function(x, y):
    return x**2 + y**2 + 5
# Partial derivatives of the cost function
def gradient(x, y):
    df_dx = 2 * x
    df_dy = 2 * y
    return np.array([df_dx, df_dy])
# Gradient Descent Algorithm
def gradient_descent(starting_point, learning_rate, iterations):
    point = np.array(starting_point, dtype=float)
    path = [point.copy()]
    for _ in range(iterations):
        grad = gradient(*point)
        point -= learning_rate * grad
        path.append(point.copy())
    return np.array(path)
# Parameters
starting_point = [10, 10]
learning_rate = 0.1
```

```
iterations = 50
# Execute gradient descent
path = gradient_descent(starting_point, learning_rate, iterations)
# Plotting
x_vals = np.linspace(-12, 12, 100)
y_vals = np.linspace(-12, 12, 100)
X, Y = np.meshgrid(x_vals, y_vals)
Z = cost_function(X, Y)
plt.contour(X, Y, Z, levels=50)
plt.plot(path[:, 0], path[:, 1], marker='o', color='red')
plt.title('Gradient Descent Optimization')
plt.xlabel('X')
plt.ylabel('Y')
plt.show()
```

Output:





Note: Now find the critical points of the given exercises and verify using the gradient descent code.

Applications of Optimization

Example 1:

A **manufacturing company** produces **two products** which are sold in **two separate markets**. The **company's economists** analyze the two markets and determine that the two quantities q_1 and q_2 demanded by the consumers and **prices** P_1 and P_2 (in $) of each item are related by the equation

Price of 1st product = $p_1(q_1, q_2) = 600 - 0.3\, q_1$

Price of 2nd product = $p_2(q_1, q_2) = 500 - 0.2\, q_2$

If the **price** for either item **increases**, the **demand** for it **decreases**. The **company's total production cost** is given by

$$Cost\ function = C(q_1, q_2) = 16 + 1.2\, q_1 + 1.5\, q_2 + 0.2\, q_1\, q_2$$

a) If the company wants to **maximize** its **total profit**, **how much** of **each product should it produce**?
b) What is the **maximum profit**?

Solution:

Given information:

Price of 1st product: p_1

Price of 2nd product: p_2

Quantity produced of 1st product: q_1

Quantity produced of 2nd product: q_2

$Price\ of\ product\ 1 = p_1(q_1, q_2) = 600 - 0.3\, q_1$ -------- (1)

$Price\ of\ product\ 2 = p_2(q_1, q_2) = 500 - 0.2\, q_2$ -------- (2)

$Cost\ function = C = 16 + 1.2\, q_1 + 1.5\, q_2 + 0.2\, q_1\, q_2$ -------- (3)

Since we know that

$$Profit = P = Revenue - Cost \text{ -------(4)}$$

and

$$Revenue\ function = R = Price \times quantity \text{ -------(5)}$$

Since, in this case, the **company** produces **two products** which are sold in **two separate markets**. This implies that

$$Revenue = R(q_1, q_2) = p_1 q_1 + p_2 q_2 \text{ -------(6)}$$

Putting equation (1) and (2) in equation (4), we get

$$R(q_1, q_2) = (600 - 0.3q_1)q_1 + (500 - 0.2q_2)q_2$$

$$R(q_1, q_2) = 600q_1 - 0.3q_1^2 + 500q_2 - 0.2q_2^2 \text{ -------(7)}$$

So, putting equation (3) and (7) in equation (4), we get

$$\text{Profit} = \text{P} = \text{Revenue} - \text{Cost}$$

$$P = [600q_1 - 0.3q_1^2 + 500q_2 - 0.2q_2^2] - [16 + 1.2q_1 + 1.5q_2 + 0.2q_1q_2]$$

$$P = 600q_1 - 0.3q_1^2 + 500q_2 - 0.2q_2^2 - 16 - 1.2q_1 - 1.5q_2 - 0.2q_1q_2$$

$$\boxed{P(q_1, q_2) = -0.3q_1^2 - 0.2q_2^2 + 598.8q_1 + 498.5q_2 - 0.2q_1q_2 - 16 \text{ ----(8)}}$$

Note: This is your simple function. You can apply 2nd derivative test on this profit function given in equation (8).

Find $\frac{\partial P}{\partial q_1}$

Partially differentiate $P(q_1, q_2)$ w.r.t variable q_1, we have

$$P_{q_1}(q_1, q_2) = \frac{\partial P}{\partial q_1} = \frac{\partial}{\partial q_1}(-0.3q_1^2 - 0.2q_2^2 + 598.8q_1 + 498.5q_2 - 0.2q_1q_2 - 16)$$

$$P_{q_1}(q_1, q_2) = \frac{\partial P}{\partial q_1} = -0.3(2q_1) - 0 + 598.8(1) + 0 - 0.2q_2(1) - 0$$

$$P_{q_1}(q_1, q_2) = \frac{\partial P}{\partial q_1} = -0.6\,q_1 + 598.8 - 0.2q_2$$

$$P_{q_1}(q_1, q_2) = \frac{\partial P}{\partial q_1} = -0.6\,q_1 - 0.2\,q_2 + 598.8$$

$$\boxed{P_{q_1}(q_1, q_2) = \frac{\partial P}{\partial q_1} = -0.6\,q_1 - 0.2\,q_2 + 598.8 \text{----------}(9)}$$

Find $\frac{\partial P}{\partial q_2}$

Partially differentiate $P(q_1, q_2)$ w.r.t variable q_2, we have

$$P_{q_2}(q_1, q_2) = \frac{\partial P}{\partial q_2} = \frac{\partial}{\partial q_2}(-0.3q_1^2 - 0.2q_2^2 + 598.8q_1 + 498.5q_2 - 0.2q_1q_2 - 16)$$

$$P_{q_2}(q_1, q_2) = \frac{\partial P}{\partial q_2} = 0 - 0.2(2q_2) + 0 + 498.5(1) - 0.2q_1(1) - 0$$

$$P_{q_2}(q_1, q_2) = \frac{\partial P}{\partial q_2} = -0.4\,q_2 + 498.5 - 0.2q_1$$

$$P_{q_2}(q_1, q_2) = \frac{\partial P}{\partial q_2} = -0.2\,q_1 - 0.4\,q_2 + 498.5$$

$$\boxed{P_{q_2}(q_1, q_2) = \frac{\partial P}{\partial q_2} = -0.2\,q_1 - 0.4\,q_2 + 498.5 \text{----------}(10)}$$

Find the critical point

To find critical point, put

$$P_{q_1}(q_1, q_2) = \frac{\partial P}{\partial q_1} = 0 \qquad \text{and} \qquad P_{q_2}(q_1, q_2) = \frac{\partial P}{\partial q_2} = 0$$

This implies that

$P_{q_1}(q_1, q_2) = 0$	$P_{q_2}(q_1, q_2) = 0$
$\Rightarrow -0.6\,q_1 - 0.2\,q_2 + 598.8 = 0$	$\Rightarrow -0.2\,q_1 - 0.4\,q_2 + 498.5 = 0$
$\Rightarrow -0.6\,q_1 - 0.2\,q_2 = -598.8$	$\Rightarrow -0.2\,q_1 - 0.4\,q_2 = -498.5$
$\Rightarrow 0.6\,q_1 + 0.2\,q_2 = 598.8$ -------(11)	$\Rightarrow 0.2\,q_1 + 0.4\,q_2 = 498.5$----------(12)

We obtained two equations in two variables. These are two simultaneous equations.

$$0.6\,q_1 + 0.2\,q_2 = 598.8 \text{ -------(11)}$$

$$0.2\,q_1 + 0.4\,q_2 = 498.5\text{----------(12)}$$

Multiply equation (12) by "**3**" & subtract from equation (11), we get

$eq(11) \Rightarrow 0.6\,q_1 + 0.2\,q_2 = 598.8$

$[eq(12) \times 3] \Rightarrow 0.6\,q_1 + 1.2\,q_2 = 1495.5$

$[eq(12) \times 3] - eq(11) \Rightarrow 0.6\,q_1 + 1.2\,q_2 - [0.6\,q_1 + 0.2\,q_2] = 1495.5 - 598.8$

$\Rightarrow 0.6\,q_1 + 1.2\,q_2 - 0.6\,q_1 - 0.2\,q_2 = 1495.5 - 598.8$

$\Rightarrow 1.2\,q_2 - 0.2\,q_2 = 897$

$\Rightarrow q_2 = 897$ ---------(13)

Putting value of equation (13) in equation (11), we get

$0.6\,q_1 + 0.2\,q_2 = 598.8$

$\Rightarrow 0.6\,q_1 + 0.2\,(897) = 598.8$

$\Rightarrow 0.6\,q_1 + 179.4 = 598.8$

$\Rightarrow 0.6\,q_1 = 598.8 - 179.4$

$\Rightarrow 0.6\,q_1 = 419.4$

$$\Rightarrow q_1 = \frac{419.4}{0.6}$$

$$\Rightarrow q_1 = 699 \text{ ---------(14)}$$

Hence,

- $q_1 = 699$
- $q_2 = 897$

So, the critical point is

$$(q_1, q_2) = (699, 897)$$

Step 4: Find Second Derivatives

Find $P_{q_1q_1}(q_1, q_2) = \frac{\partial^2 P}{\partial {q_1}^2}$

Partially differentiate $P_{q_1}(q_1, q_2)$ w.r.t variable q_1, we have

$$P_{q_1q_1}(q_1, q_2) = \frac{\partial^2 P}{\partial {q_1}^2} = \frac{\partial}{\partial q_1}(-0.6\, q_1 - 0.2\, q_2 + 598.8)$$

$$P_{q_1q_1}(q_1, q_2) = \frac{\partial^2 P}{\partial {q_1}^2} = -0.6(1) - 0 + 0 = -0.6$$

$$\boxed{P_{q_1q_1}(q_1, q_2) = \frac{\partial^2 P}{\partial {q_1}^2} = -0.6}$$

Find $P_{q_2q_2}(q_1, q_2) = \frac{\partial^2 P}{\partial {q_2}^2}$

Partially differentiate $P_{q_2}(q_1, q_2)$ w.r.t variable q_2, we have

$$P_{q_2q_2}(q_1, q_2) = \frac{\partial^2 P}{\partial {q_2}^2} = \frac{\partial}{\partial q_2}(-0.2\, q_1 - 0.4\, q_2 + 498.5)$$

$$P_{q_2q_2}(q_1, q_2) = \frac{\partial^2 P}{\partial {q_2}^2} = 0 - 0.4(1) = -0.4$$

$$\boxed{P_{q_2q_2}(q_1,q_2) = \frac{\partial^2 P}{\partial {q_2}^2} = -0.4}$$

Find $P_{q_1q_2}(q_1,q_2) = \frac{\partial^2 P}{\partial q_2 \partial q_1}$

Partially differentiate $P_{q_1}(q_1,q_2)$ w.r.t variable q_2, we have

$$P_{q_1q_2}(q_1,q_2) = \frac{\partial^2 P}{\partial q_2 \partial q_1} = \frac{\partial}{\partial q_2}(-0.6\, q_1 - 0.2\, q_2 + 598.8)$$

$$P_{q_1q_2}(q_1,q_2) = \frac{\partial^2 P}{\partial q_2 \partial q_1} = 0 - 0.2(1) = -0.2$$

$$\boxed{P_{q_1q_2}(q_1,q_2) = \frac{\partial^2 P}{\partial q_2 \partial q_1} = -0.2}$$

Step 5: Find Second Derivative value at critical point(s)

$$P_{q_1q_1}(q_1,q_2) = \frac{\partial^2 P}{\partial {q_1}^2} = -0.6$$

$$P_{q_2q_2}(q_1,q_2) = \frac{\partial^2 P}{\partial {q_2}^2} = -0.4$$

$$P_{q_1q_2}(q_1,q_2) = \frac{\partial^2 P}{\partial q_2 \partial q_1} = -0.2$$

For critical point $(q_1, q_2) = (699, 897)$

$P_{q_1q_1}(q_1,q_2) = -0.6 \quad \Rightarrow \quad P_{q_1q_1}(699, 897) = -0.6 < 0$

$P_{q_2q_2}(q_1,q_2) = -0.4 \quad \Rightarrow \quad P_{q_2q_2}(699, 897) = -0.4$

$P_{q_1q_2}(q_1,q_2) = -0.2 \quad \Rightarrow \quad P_{q_1q_2}(699, 897) = -0.2$

Step 6: Apply Second Derivative Test

$$D = P_{q_1q_1}(699, 897)\, P_{q_2q_2}(699, 897) - \left[\, P_{q_1q_2}(699, 897)\, \right]^2$$

Putting values in the formula, we get

$$D = [-0.6][-0.4] - [-0.2\,]^2$$

$$\Rightarrow D = 0.24 - 0.04$$

$$\Rightarrow D = 0.2 > 0$$

Check the function has local maxima/minimum or neither

Since $D = 0.2 > 0$ **and** $P_{q_1 q_1}(699, 897) = -0.6 < 0$, so by using **2nd derivative test**, the given function $P(q_1, q_2)$ has a **local maxima** at **critical point** $(699, 897)$**. This means that** the company should produce 699 units of product q_1 and 897 units of q_2.

Find the Maximum Profit

Since the profit function is

$$P(q_1, q_2) = -0.3q_1^2 - 0.2q_2^2 + 598.8q_1 + 498.5q_2 - 0.2q_1q_2 - 16$$

Substituting the values $q_1 = 699$ **and** $q_2 = 897$ **in above function, we get**

$$P(q_1, q_2) = -0.3(699)^2 - 0.2(897)^2 + 598.8(699) + 498.5(897) - 0.2(699)(897) - 16$$

$$\Rightarrow P(q_1, q_2) = -146580.3 - 160921.8 + 418561.2 + 447154.5 - 125400.6 - 16$$

$$\Rightarrow P(q_1, q_2) = 432797$$

Hence, the **maximum profit** would be

Maximum Profit = $P(699, 897) = \$\,432,797.$

Example:

A company operates two plants which manufacture the same item and whose total functions are

$$C_1 = 8.5 + 0.03q_1^2 \quad \text{and} \quad C_2 = 5.2 + 0.04q_2^2$$

where q_1 and q_2 are the quantities produced by each plant. The company is a monopoly. The total quantity demanded, $q = q_1 + q_2$, is related to the price, P, by

$$P = 60 - 0.04q.$$

How much should each plant produce in order to maximize the company's profit?

Solution:

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

As $$\text{Revenue} = \text{Price} \times \text{Quantity}$$

$$= P \times q = (60 - 0.04q) \times (q)$$

$$= 60q - 0.04q^2$$

$$= 60(q_1 + q_2) - 0.04(q_1 + q_2)^2$$

$$= 60q_1 + 60q_2 - 0.04(q_1^2 + q_2^2 + 2q_1q_2)$$

$$\text{Revenue} = 60q_1 + 60q_2 - 0.04q_1^2 - 0.04q_2^2 - 0.08q_1q_2$$

$$\text{Profit} = 60q_1 + 60q_2 - 0.04q_1^2 - 0.04q_2^2 - 0.08q_1q_2 - (8.5 + 0.03q_1^2 + 5.2 + 0.04q_2^2)$$

$$= 60q_1 + 60q_2 - 0.04q_1^2 - 0.04q_2^2 - 0.08q_1q_2 - 13.7 - 0.03q_1^2 - 0.04q_2^2$$

$$f(q_1, q_2) = -0.07q_1^2 - 0.08q_2^2 - 0.08q_1q_2 + 60q_1 + 60q_2 - 13.7$$

$$f_{q_1} = -0.14q_1 - 0.08q_2 + 60$$

$$f_{q_2} = -0.16q_2 - 0.08q_1 + 60$$

$f_{q_1} = f_{q_2} = 0$ implies

$$-0.14q_1 - 0.08q_2 + 60 = 0$$

$$-0.16q_2 - 0.08q_1 + 60 = 0$$

$$0.14q_1 + 0.08q_2 = 60 \dots (1)$$

$$0.08q_1 + 0.16q_2 = 60 \dots (2)$$

Multiply equation (1) by 0.08 & equation (2) by 0.14 then subtract it from equation (1).

$$0.0112q_1 + 0.0064q_2 = 4.8$$

$$\underline{\pm 0.0112q_1 \pm 0.0224q_2 = \pm 8.4}$$

$$-0.016q_2 = -3.6$$

$$q_2 = 225$$

Put in (1)

$$0.14q_1 + 0.08(225) = 60$$

$$0.14q_1 = 60 - 18$$

$$q_1 = \frac{42}{0.14} = 300$$

$$q_1 = 300$$

So $(q_1, q_2) = (300, 225)$ is a critical point of f.

$$f_{q_1q_1} = -0.14 \qquad f_{q_1q_2} = -0.08 \qquad f_{q_2q_2} = -0.16$$

$$D = f_{q_1q_1} * f_{q_2q_2} - \left(f_{q_1q_2}\right)^2 = (-0.14) * (-0.16) - (-0.8)^2$$
$$= -0.0224 - 0.0064 = 0.016 > 0$$

As, $D > 0$ and $f_{q_1q_1} = -0.14 < 0$ so, f has a local maxima at (300, 225)

The plant should produce 300 units of q_1 and 225 units of q_2.

Textbook Practice Problems

Applied Calculus (4ht Edition) by Huges Hallet **Ex. 9.5: 18 – 21**.

Practice Problems

Question: An automobile manufacturer sells cars in America and Europe, charging different prices in the two markets. The **price function for cars sold in America** is $\boldsymbol{p = 20 - 0.2x}$ thousand dollars (for $\boldsymbol{0 \le x \le 100}$), and the **price function for cars sold in Europe** is $\boldsymbol{q = 16 - 0.1y}$ thousand dollars (for $\boldsymbol{0 \le y \le 160}$), where $\boldsymbol{x}$ is the quantity of cars sold per day in America and $\boldsymbol{y}$ is the quantity of cars sold per day in Europe, respectively. The **company's cost function** in per thousand dollars is $\boldsymbol{C = 20 + 4(x + y)}$.

a) **Find** the **company's profit function**. [*Hint:* Profit is revenue from America plus revenue from Europe minus costs, where each revenue is price times quantity.]

b) **Find how many cars should be sold** in each market to **maximize profit**. Also find the **price for each market**.