Multiple Integrals

Multiple integrals are integrals that involve two or more integrations of functions in the real domain. They are used in multivariable calculus.

What are multiple integrals used for?

- **Double integrals**: Used to find the volume under a surface and above the xy-plane
- **Triple integrals**: Used to find the volume under a surface in three dimensions

How to solve multiple integrals?

1. Start with the innermost integral and evaluate it first.

2. Evaluate the next integral, moving from inside to outside.

3. Evaluate the outermost integral last.

What are some examples of multiple integrals?

- **Double integrals**: The volume under the surface $z = f(x, y)$ and above the xy-plane
- **Triple integrals**: The volume under a surface in three dimensions
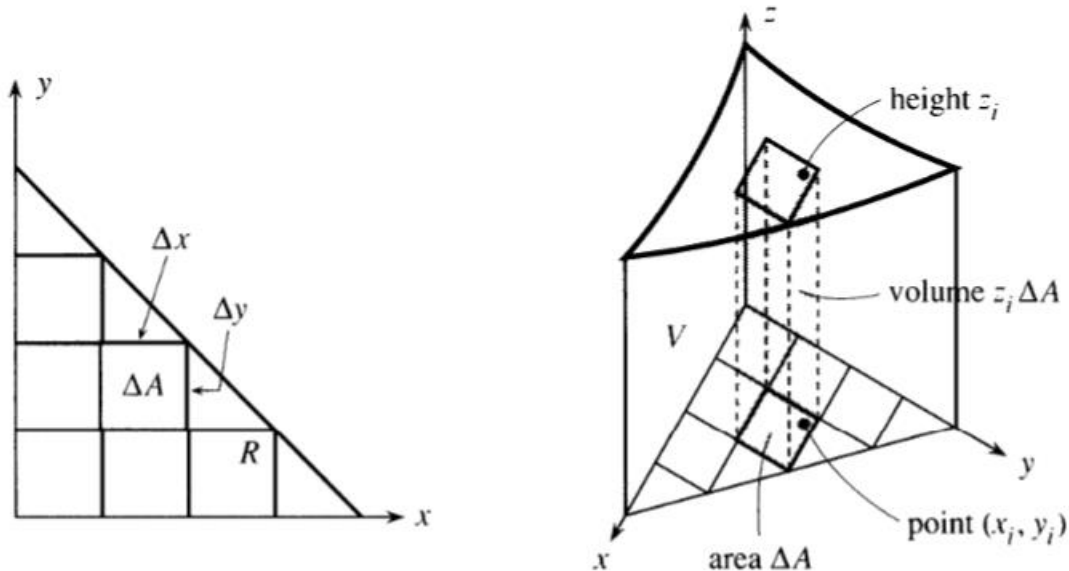
Volume under the surface $z = f(x, y)$

The graph of $z = f(x, y)$ is a curved surface above the xy plane. At the point

(x, y) in the plane, the height of the surface is z. (The surface is above the xy plane only when z is positive. Volumes below the plane come with minus signs, like areas below the x axis.) We begin by choosing a positive function-for example $z = 1 + x^2 + y^2$. The base of our solid is a region R in the xy plane. That region will be chopped into small rectangles (sides $\Delta x \; and \; \Delta y$). When R itself is the rectangle $0 \le x \le 1, 0 \le y \le 2$, the small pieces fit perfectly. For a triangle

or a circle, the rectangles miss part of **R**. But they do fit in the limit, and any region with a piecewise smooth boundary will be acceptable.

Question What is the volume above **R** and below the graph of $z = f(x, y)$?
Answer: It is a double integral-the integral of $f(x, y)$ over R.
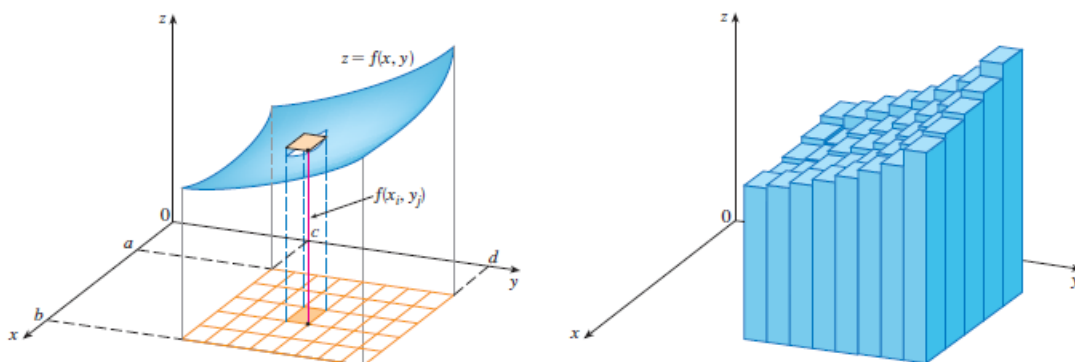

Evaluating Double Integrals over Rectangular Region:

If $f(x, y) \geq 0$, the **volume** of the solid that lies under the surface $z = f(x, y)$ and above the rectangle R is

$$V = \iint\limits_{R} f(x, y) \, dA$$

If f has both positive and negative values on **R**, then a positive value for the double integral of f over **R** means that there is more volume above **R** than below, a negative value for the double integral means that there is more volume below **R** than above, and a value of zero means that the volume above **R** is the same as the volume below **R**.

The partial derivatives of a function $f(x, y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other variable. Let us consider the reverse of this process, *partial integration*. The symbols

$$\int_a^b f(x, y)\, dx \quad \text{and} \quad \int_c^d f(x, y)\, dy$$

denote *partial definite integrals*; the first integral, called the *partial definite integral with respect to x*, is evaluated by holding y fixed and integrating with respect to x, and the second integral, called the *partial definite integral with respect to y*, is evaluated by holding x fixed and integrating with respect to y. As the following example shows, the partial definite integral with respect to x is a function of y, and the partial definite integral with respect to y is a function of x.

If the case when the region $R = \{(x, y): a \leq x \leq b, c \leq y \leq d\}$ is a rectangular region, the double integral can be evaluated as:

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

or

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

This two-stage integration process is called *iterated* (or *repeated*) *integration*.

We introduce the following notation

Solved Examples

EXAMPLE 1 $A = \int_{y=0}^{2} (1 + x^2 + y^2) dy = \left[y + x^2 y + \frac{y^3}{3} \right]_{y=0}^{y=2} = 2 + 2x^2 + \frac{8}{3}.$

This is the reverse of a partial derivative! The integral of $x^2 dy$, with x constant, is $x^2 y$. This "partial integral" is actually called an **inner integral**. After substituting the limits $y = 2$ and $y = 0$ and subtracting, we have the area $A(x) = 2 + 2x^2 + \frac{8}{3}$. Now the **outer integral** adds slices to find the volume $\int A(x) \, dx$. The answer is a *number*:

$$\text{volume} = \int_{x=0}^{1} \left(2 + 2x^2 + \frac{8}{3} \right) dx = \left[2x + \frac{2x^3}{3} + \frac{8}{3}x \right]_{0}^{1} = 2 + \frac{2}{3} + \frac{8}{3} = \frac{16}{3}.$$
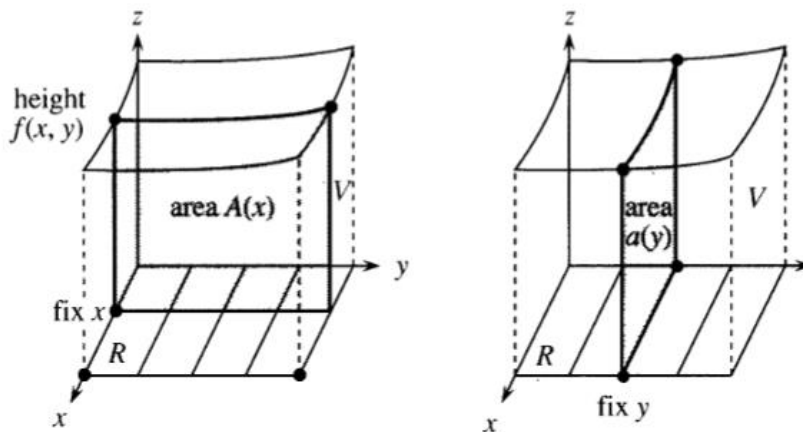

Fig. 14.2 A slice of V at a fixed x has area $A(x) = \int f(x, y) dy$.

2. Evaluate the integral $\int_{0}^{3} \int_{0}^{2} (4 - y^2) \, dy \, dx.$

$$= \int_{0}^{3} \left(\int_{0}^{2} 4 dy - \int_{0}^{2} y^2 dy \right) dx$$

$$= \int_{0}^{3} \left[|4y|_{0}^{2} - \frac{|y^3|}{3} \Big|_{0}^{2} \right] dx$$

$$= \int_{0}^{3} \left[(4(2) - 4(0)) - \left(\frac{2^3}{3} - \frac{0^3}{3} \right) \right] dx$$

$$= \int_0^3 \left[\frac{24-8}{3}\right] dx = \int_0^3 \frac{16}{3} dx$$

$$= \left|\frac{16}{3}x\right|_0^3 = \frac{16}{3}(3) - \frac{16}{3}(0) = 16$$

3. Find the value of $\int_0^3 \int_{-2}^0 (x^2y - 2xy)dydx$.

$$= \int_0^3 \left[\int_{-2}^0 x^2 y\, dy - \int_{-2}^0 2xy\, dy\right] dx$$

$$= \int_0^3 \left[x^2 \left.\frac{y^2}{2}\right|_{-2}^0 - 2x \left.\frac{y^2}{2}\right|_{-2}^0\right] dx =$$

$$= \int_0^3 \left[x^2 \left(\frac{0^2}{2} - \frac{(-2)^2}{2}\right) - 2x \left(\frac{0^2}{2} - \frac{(-2)^2}{2}\right)\right] dx$$

$$= \int_0^3 x^2 \left(-\frac{4}{2}\right) - 2x \left(-\frac{4}{2}\right) dx = \int_0^3 (-2x^2 + 4x)\, dx$$

$$= \left|-\frac{2x^3}{3}\right|_0^3 + \left|\frac{4x^2}{2}\right|_0^3 = -\frac{2}{3}[3^3 - 0^3] + 2[3^3 - 0^3]$$

$$= -\frac{2}{3}(27) + 2(9) = -18 + 18 = 0$$

4. Solve $\int_\pi^{2\pi} \int_0^\pi (\sin x + \cos y) dx dy.$

$$\int_\pi^{2\pi} \left[\int_0^\pi \sin x \, dx + \int_0^\pi \cos y \, dx \right] dy = \int_\pi^{2\pi} \left[|-\cos x|_0^\pi + \cos y \left(|x|_0^\pi \right) \right] dy$$

$$= \int_\pi^{2\pi} \left[-(\cos \pi - \cos 0) + \cos y \, (\pi - 0) \right] dy$$

$$= \int_\pi^{2\pi} \left[-(-1 - 1) + \pi \cos y \right] dy = \int_\pi^{2\pi} \left[2 + \pi \cos y \right] dy$$

$$= |2y|_\pi^{2\pi} + \pi |\sin y|_\pi^{2\pi} = 2(2\pi - \pi) + \pi(\sin 2\pi - \sin \pi) = 2\pi + 0 = 2\pi$$

Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$ (b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

SOLUTION

(a) Working from the inside out, we first evaluate $\int_1^2 x^2 y \, dy$. Regarding x as a constant, we obtain

$$\int_1^2 x^2 y \, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right) = \tfrac{3}{2} x^2$$

Thus the function A in the preceding discussion is given by $A(x) = \tfrac{3}{2} x^2$ in this example. We now integrate this function of x from 0 to 3:

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 \left[\int_1^2 x^2 y \, dy \right] dx$$

$$= \int_0^3 \tfrac{3}{2} x^2 \, dx = \frac{x^3}{2} \Big]_0^3 = \frac{27}{2}$$

(b) Here we first integrate with respect to x:

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 \left[\int_0^3 x^2 y \, dx \right] dy = \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy$$

$$= \int_1^2 9y \, dy = 9 \frac{y^2}{2} \Big]_1^2 = \frac{27}{2}$$ ■

Notice that we obtained the same answer whether we integrated with respect to y or x first. In general, it turns out that the two iterated integrals in Equations and are always equal; that is, **the order of integration does not matter.**

$$\int_c^d \int_a^b f(x, y) \, dx \, dy = \int_c^d \left[\int_a^b f(x, y) \, dx \right] dy$$

$$\int_a^b \int_c^d f(x, y) \, dy \, dx = \int_a^b \left[\int_c^d f(x, y) \, dy \right] dx$$

Question: Find the value of $\int_{-1}^{0} \int_{-1}^{1} \left(7x^2 + \frac{y^8}{5} - 1 \right) dy\, dx.$

(do it yourself)

✚ **How to define integration over a region by using intervals?**

Example 5:

Find the volume of the solid lying under the surface

$$f(x, y) = 1 - 6x^2 y$$

and over the region R: $0 \le x \le 2, \quad -1 \le y \le 1.$

Solution:

$$\boldsymbol{Volume} = \iint_R f(x, y)\, dA$$

$$= \int_0^2 \int_{-1}^1 (1 - 6x^2 y)\, dy\, dx$$

$$= \int_0^2 \left(\int_{-1}^1 1\, dy - 6x^2 \int_{-1}^1 y\, dy \right) dx$$

$$= \int_0^2 \left(|y|_{-1}^1 - 6x^2 \left. \frac{y^2}{2} \right|_{-1}^1 \right) dx = \int_0^2 \left((1 - (-1)) - 6x^2 \left[\frac{1^2}{2} - \frac{(-1)^2}{2} \right] \right) dx$$

$$= \int_0^2 (2 - 3x^2 [1 - 1])\, dx = \int_0^2 (2 - 3x^2 (0))\, dx = \int_0^2 2\, dx = |2x|_0^2 = 2(2) - 0$$

$$= 4$$

6. Find the volume of the solid lying under the surface

$$f(x, y) = x + y + 1$$

and over the region $R: \ -1 \le x \le 1, \quad -1 \le y \le 0.$

Solution: $Volume = \iint_R f(x, y) dA$

$$= \int_{-1}^{1} \int_{-1}^{0} (x + y + 1) dy dx$$

$$= \int_{-1}^{1} \left(\int_{-1}^{0} x dy + \int_{-1}^{0} y dy + \int_{-1}^{0} dy \right) dx$$

$$= \int_{-1}^{1} \left(x|y|_{-1}^{0} + \left. \frac{y^2}{2} \right|_{-1}^{0} + |y|_{-1}^{0} \right) dx$$

$$= \int_{-1}^{1} \left(x(0 - (-1)) + \left[\frac{0^2}{2} - \frac{(-1)^2}{2} \right] + (0 - (-1)) \right) dx$$

$$= \int_{-1}^{1} \left(x - \frac{1}{2} + 1 \right) dx = \int_{-1}^{1} \left(x + \frac{1}{2} \right) dx$$

$$= \int_{-1}^{1} x dx + \int_{-1}^{1} \frac{1}{2} dx = |x^2|_{-1}^{1} + \frac{1}{2} |x|_{-1}^{1}$$

$$= (1^2 - (-1)^2) + \frac{1}{2}(1 - (-1)) = 1 - 1 + \frac{1}{2}(2)$$

$$= 1$$

Practice Problems:

Evaluate the given double integrals.

1. $\displaystyle\int_0^1\int_0^2 (x+3)\,dy\,dx$

2. $\displaystyle\int_1^3\int_{-1}^1 (2x-4y)\,dy\,dx$

3. $\displaystyle\int_2^4\int_0^1 x^2y\,dx\,dy$

4. $\displaystyle\int_{-2}^0\int_{-1}^2 (x^2+y^2)\,dx\,dy$

5. $\displaystyle\int_0^{\ln 3}\int_0^{\ln 2} e^{x+y}\,dy\,dx$

6. $\displaystyle\int_0^2\int_0^1 y\sin x\,dy\,dx$

7. $\displaystyle\int_{-1}^0\int_2^5 dx\,dy$

8. $\displaystyle\int_4^6\int_{-3}^7 dy\,dx$

Use double integral to find the volume.

29. The volume under the plane $z = 2x + y$ and over the rectangle $R = \{(x, y) : 3 \le x \le 5, 1 \le y \le 2\}$.

30. The volume under the surface $z = 3x^3 + 3x^2y$ and over the rectangle $R = \{(x, y) : 1 \le x \le 3, 0 \le y \le 2\}$.

31. The volume of the solid enclosed by the surface $z = x^2$ and the planes $x = 0$, $x = 2$, $y = 3$, $y = 0$, and $z = 0$.

Evaluating Double Integrals over Nonrectangular Region

■ **Double Integrals over More General Regions**

What happens if we need to integrate a function $f(x, y)$ over a region D that is not a rectangle? Suppose, for instance, that the domain D of f lies between the graphs of two continuous functions of x:

$$D = \{(x, y) \mid a \leqslant x \leqslant b, \ g_1(x) \leqslant y \leqslant g_2(x)\}$$

The double integral of f over D,it can be defined by a limit similar to the one we have done for numerical limits,

$$\iint\limits_{D} f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx$$

_____(7)

In the case when the region R is a nonrectangular region, the limits of integration in the inner integral are **not constants** and the double integral can be of **two types**:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \right] dx$$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) \, dx \, dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y) \, dx \right] dy$$

Examples:

1. Find the volume of the prism whose base is the triangle in the xy-plane formed by the x-axis and the lines $y = x$ & $x = 1$ and whose top lies in the plane

$$z = f(x,y) = 3 - x - y.$$

Solution: $\qquad\qquad Volume = \iint_R f(x,y)dA$

$$= \int_0^1 \int_0^x (3 - x - y)\,dy\,dx$$

$$= \int_0^1 \left(\int_0^x 3dy - x \int_0^x dy - \int_0^x ydy \right) dx$$

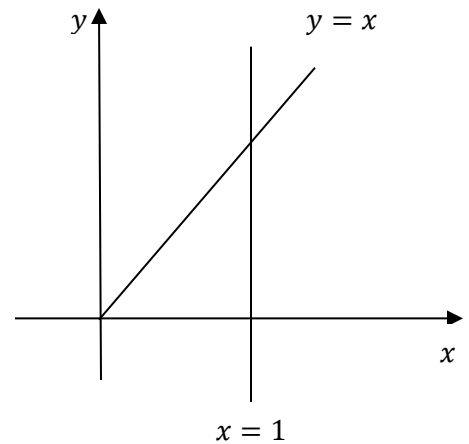$$= \int_0^1 \left(|3y|_0^x - x|y|_0^x - \left|\frac{y^2}{2}\right|_0^x \right) dx$$

$$= \int_0^1 \left([3(x) - 3(0)] - x(x - 0) - \left|\frac{x^2}{2} - \frac{0^2}{2}\right|_0^x \right) dx$$

$$= \int_0^1 \left(3x - x^2 - \frac{x^2}{2} \right) dx = \int_0^1 \left(3x - \left[\frac{2x^2 + x^2}{2}\right] \right) dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = 3 \int_0^1 xdx - \frac{3}{2} \int_0^1 x^2\,dx$$

$$= 3 \left|\frac{x^2}{2}\right|_0^1 - \frac{3}{2}\left|\frac{x^3}{3}\right|_0^1 = \frac{3}{2}(1^2 - 0) - \frac{3}{6}(1^3 - 0)$$
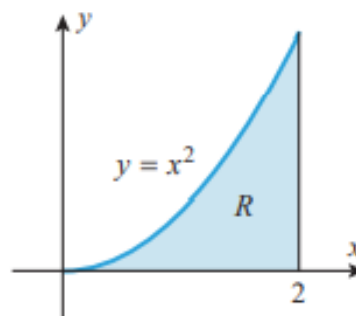
$$= \frac{3}{2} - \frac{1}{2} = \frac{2}{2} = 1$$

Graph (right side):

$y = x$; $x = 1$; axes y (vertical), x (horizontal)

2. Evaluate

$$\iint_R (x + y)\, dA$$

where R is the region enclosed by the parabola $y = x^2$, the line $x = 2$ and the x-axis.

Solution:

$$\iint\limits_R (x + y)\, dA$$

$$= \int_0^2 \int_0^{x^2} (x + y)\, dy\, dx$$

$$= \int_0^2 \left(\int_0^{x^2} x\, dy + \int_0^{x^2} y\, dy \right) dx$$

$$= \int_0^2 \left(|xy|_0^{x^2} + \left.\frac{y^2}{2}\right|_0^{x^2} \right) dx$$

$$= \int_0^2 \left(x^3 - \frac{x^4}{2} \right) dx$$

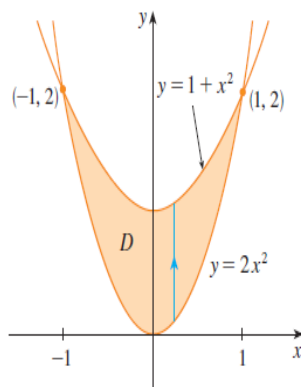$$= \left.\frac{x^4}{4}\right|_0^2 - \frac{1}{2}\left.\frac{x^5}{5}\right|_0^2 = \frac{4}{5}$$

EXAMPLE 3 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

SOLUTION The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$. The region D is sketched in Figure 8 and we can write

$$D = \{(x, y) \mid -1 \leqslant x \leqslant 1,\ 2x^2 \leqslant y \leqslant 1 + x^2\}$$

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 7 gives

$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

$$= \int_{-1}^{1} \left[xy + y^2 \right]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} \left[x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2 \right] dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

$$= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x \Bigg]_{-1}^{1} = \frac{32}{15}$$

FIGURE 8

NOTE When we set up a double integral as in Example 3, it is essential to draw a diagram. Often it is helpful to draw a vertical arrow as in Figure 8. Then the limits of integration for the *inner* integral can be read from the diagram as follows: The arrow starts at the lower boundary $y = g_1(x)$, which gives the lower limit in the integral, and the arrow ends at the upper boundary $y = g_2(x)$, which gives the upper limit of integration.

Practice Questions:

Evaluate the double integrals.

1. $\displaystyle\int_0^1 \int_{x^2}^x xy^2 \, dy \, dx$

2. $\displaystyle\int_1^{3/2} \int_y^{3-y} y \, dx \, dy$

3. $\displaystyle\int_0^3 \int_0^{\sqrt{9-y^2}} y \, dx \, dy$

4. $\displaystyle\int_{1/4}^1 \int_{x^2}^x \sqrt{\frac{x}{y}} \, dy \, dx$

Applications of Multiple Integrals

Area of Bounded Regions in the Plane:

The area of closed bounded region R is

$$A = \iint_R dA.$$

Examples:

Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

Solution:

$$\textbf{Area} = \iint\limits_{R} d\boldsymbol{A} = \int\limits_{0}^{4} \int\limits_{\frac{x^2}{2}}^{2x} \boldsymbol{dy dx}$$

$$= \int\limits_{0}^{4} |y|_{\frac{x^2}{2}}^{2x} dx = \int\limits_{0}^{4} \left(2x - \frac{x^2}{2} \right) dx$$

$$= \left| x^2 - \frac{x^3}{3} \right|_{0}^{4} = 16 - \frac{16}{3} = \frac{16}{3}$$

Density of Thin Plate:

Definition: Suppose that we have a thin plate, so thin that it's practically 2-dimensional. **The density of this plate is defined as the mass per unit area.**

So, $$mass = density \times area$$

Examples:

A thin plate covers the triangular region bounded by x-axis & the lines
$$x = 1 \quad \& \quad y = 2x \text{ in the first quadrant.}$$
The plate's density at the point (x, y) is $f(x, y) = 6x + 6y + 6$.
Find the plate's mass.

Solution:
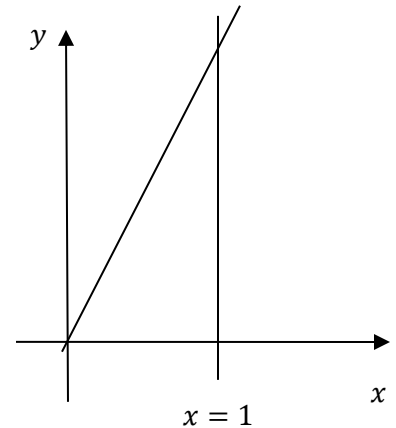
$$Mass = \iint\limits_R f(x, y) dA$$

$$= \int_0^1 \int_0^{2x} (6x + 6y + 6) dy dx = \int_0^1 \left(\int_0^{2x} 6x dy + \int_0^{2x} 6y dy + \int_0^{2x} 6 dy \right) dx \qquad y = 2x$$

$$= \int_0^1 |6xy + 6y^2 + 6y|_0^{2x} dx = \int_0^1 (12x^2 + 12x^2 + 12x) dx$$

$$= \int_0^1 (24x^2 + 12x) dx = \int_0^1 24x^2 dx + \int_0^1 12x dx$$

$$= 24 \left|\frac{x^3}{3}\right|_0^1 + 12 \left|\frac{x^2}{2}\right|_0^1 = 24 \left(\frac{1^3}{3} - \frac{0^3}{3}\right) + 12 \left(\frac{1^2}{2} - \frac{0^2}{2}\right)$$

$$= \frac{24}{3} + \frac{12}{2} = 8 + 6 = 14$$

y

$x = 1$

x

2. Find the mass M of a metal plate R bounded by $y = x$ & $y = x^2$ with density given by $f(x,y) = 1 + xy$

Solution: $M = \int_0^1 \int_{x^2}^x f(x,y)\,dy\,dx$

$$M = \int_0^1 \int_{x^2}^x (1 + xy)\,dy\,dx$$

$$= \int_0^1 \left(\int_{x^2}^x 1\,dy + \int_{x^2}^x xy\,dy \right) dx = \int_0^1 \left[|y|_{x^2}^x + x \left.\frac{|y^2|}{2}\right|_{x^2}^x \right] dx$$

$$= \int_0^1 \left[(x - x^2) + x \left(\frac{x^2}{2} - \frac{x^4}{2} \right) \right] dx = \int_0^1 \left[(x - x^2) + \left(\frac{x^3}{2} - \frac{x^5}{2} \right) \right] dx$$

$$= \left.\frac{|x^2|}{2}\right|_0^1 - \left.\frac{|x^3|}{3}\right|_0^1 + \frac{1}{2}\left.\frac{|x^4|}{4}\right|_0^1 - \frac{1}{2}\left.\frac{|x^6|}{6}\right|_0^1$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{8} - \frac{1}{12}$$

$$= \frac{5}{24}$$