Topics:

- **Need For Gradient Vector**
- **Vector Differential Operator Del**
- **Gradient Vector of function**
- **Directional Derivative of function**
- **Maximum Directional Derivative**

- ## Why Do We Need the Gradient Vector?

1. **Precise Analysis of Change:** It provides a quantitative way to analyze how a function changes in multi-dimensional space.
2. **Efficient Problem-Solving:** The gradient is a critical tool for solving problems involving:
   - Optimizing cost functions in machine learning.
   - Finding shortest paths or directions of change in geographical or physical models.
3. **Geometrical Interpretation:** The relationship between the gradient and level surfaces helps in visualizing and solving equations involving contours or surfaces.

Vector Differential Operator Del

The vector differential operator **del**, written ∇, is defined by

For 2D: $\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath}$

For 3D: $\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$

- This vector operator possesses properties analogous to those of ordinary vectors.
- It is useful in defining three quantities which arise in practical applications and are known as the **gradient**, the **divergence** and the **curl**.
- The operator ∇ is also known as "**nabla**".

Gradient Vector of a Scalar Function

The Gradient: A Vector of

- **Definition:** The gradient of a scalar-valued function *f(x, y)* is a vector that points in the direction of the greatest rate of increase of the function at each point.

• Interpretation

The gradient vector ∇f at a point (x, y) has two key properties:

1. **Direction:** The direction of $\boldsymbol{\nabla f}$ **indicates** the direction in which the function $\boldsymbol{f}$ increases most rapidly.
2. **Magnitude of gradient function:** The magnitude of $\boldsymbol{\nabla f}$ represents the rate of change of $\boldsymbol{f}$ in that direction.

- If $\boldsymbol{f(x,y)}$ is a function of **two variables** $\boldsymbol{x}$ and $\boldsymbol{y}$, then the gradient of $\boldsymbol{f(x,y)}$ is a **vector function** denoted by $\boldsymbol{\nabla f(x,y)}$ and is defined as

$$\boldsymbol{\nabla f(x,y) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = f_x(x,y)\,\hat{\imath} + f_y(x,y)\,\hat{\jmath}}$$

- If $\boldsymbol{f(x,y,z)}$ is a function of **three variables** $\boldsymbol{x, y}$ and $\boldsymbol{z}$, then the gradient of $\boldsymbol{f(x,y,z)}$ is a **vector function** denoted by $\boldsymbol{\nabla f(x,y,z)}$ and is defined as

$$\boldsymbol{\nabla f(x,y,z) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} = f_x(x,y,z)\,\hat{\imath} + f_y(x,y,z)\,\hat{\jmath} + f_z(x,y,z)\,\hat{k}}$$

Example # 1

If $f(x, y) = \sin(x) + e^{xy}$, then find $\nabla f(x, y)$.

Solution:

The **gradient** of a **scalar function** $f(x, y)$ is given by:

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = f_x(x, y)\,\hat{\imath} + f_y(x, y)\,\hat{\jmath} \text{ -----------(1)}$$

Step 1: **Finding $\frac{\partial f}{\partial x}$:**

$$f_x(x, y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[\sin(x) + e^{xy}] = \frac{\partial}{\partial x}[\sin(x)] + \frac{\partial}{\partial x}[e^{xy}]$$

$$f_x(x, y) = \frac{\partial f}{\partial x} = \cos x + ye^{xy}$$

Step 2: **Finding $\frac{\partial f}{\partial y}$:**

$$f_y(x, y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\sin(x) + e^{xy}] = \frac{\partial}{\partial y}[\sin(x)] + \frac{\partial}{\partial x}[e^{xy}]$$

$$f_y(x, y) = 0 + xe^{xy}$$

$$f_y(x, y) = \frac{\partial f}{\partial y} = xe^{xy}$$

Step 3: **Putting values in the formula given in equation (1)**

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = f_x(x, y)\hat{\imath} + f_y(x, y)\hat{\jmath}$$

$$\nabla f(x, y) = (\cos x + ye^{xy})\hat{\imath} + (xe^{xy})\hat{\jmath}$$

Example # 2

If $\boldsymbol{f(x,y,z) = \cos(x) + e^{xy} + ln\,(z)}$, then find $\boldsymbol{\nabla f(x,y,z)}$.

Solution:

The **gradient** of a **scalar function** $\boldsymbol{f(x,y,z)}$ is given by:

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} = f_x(x,y,z)\,\hat{\imath} + f_y(x,y,z)\,\hat{\jmath} + f_z(x,y,z)\,\hat{k}$$

Step 1: **Finding** $\boldsymbol{\frac{\partial f}{\partial x}}$**:**

$$f_x(x,y,z) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[\cos(x) + e^{xy} + ln\,(z)]$$

$$f_x(x,y,z) = \frac{\partial}{\partial x}[\cos(x)] + \frac{\partial}{\partial x}[e^{xy}] + \frac{\partial}{\partial x}[ln\,(z)]$$

$$f_x(x,y,z) = \frac{\partial f}{\partial x} = -\sin(x) + ye^{xy} + 0$$

$$f_x(x,y,z) = \frac{\partial f}{\partial x} = -\sin(x) + ye^{xy}$$

Step 2: **Finding** $\boldsymbol{\frac{\partial f}{\partial y}}$**:**

$$f_y(x,y,z) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\cos(x) + e^{xy} + ln\,(z)]$$

$$f_y(x,y,z) = \frac{\partial}{\partial y}[\cos(x)] + \frac{\partial}{\partial x}[e^{xy}] + \frac{\partial}{\partial y}[ln\,(z)]$$

$$f_y(x,y,z) = 0 + xe^{xy} + 0$$

$$f_y(x,y,z) = \frac{\partial f}{\partial y} = xe^{xy}$$

Step 3: **Finding $\frac{\partial f}{\partial z}$:**

$$f_z(x, y, z) = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[\cos(x) + e^{xy} + ln\,(z)]$$

$$f_z(x, y, z) = \frac{\partial}{\partial z}[\cos(x)] + \frac{\partial}{\partial z}[e^{xy}] + \frac{\partial}{\partial z}[ln\,(z)]$$

$$f_z(x, y, z) = 0 + 0 + \frac{1}{z}$$

$$f_z(x, y, z) = \frac{\partial f}{\partial z} = \frac{1}{z}$$

Step 4: **Putting values in the gradient formula, we have**

$$\nabla f(x, y, z) = f_x(x, y, z)\hat{\imath} + f_y(x, y, z)\hat{\jmath} + f_z(x, y, z)\hat{k}$$

$$\nabla f(x, y, z) = [-\sin(x) + ye^{xy}]\hat{\imath} + [xe^{xy}]\,\hat{\jmath} + \left[\frac{1}{z}\right]\hat{k}$$

Example # 3

Find the gradient $\vec{\nabla} f(x,y)$ of each of the following functions:

a. $f(x,y) = x^2 - xy + 3y^2$
b. $f(x,y) = \sin 3x \cos 3y$

Solution

For both parts a. and b., we first calculate the partial derivatives f_x and f_y, then use Equation 12.

a. $f_x(x,y) = 2x - y$ and $f_y(x,y) = -x + 6y$, so

$$\begin{aligned}\vec{\nabla} f(x,y) &= f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} \\ &= (2x - y)\,\hat{\mathbf{i}} + (-x + 6y)\,\hat{\mathbf{j}}.\end{aligned}$$

b. $f_x(x,y) = 3\cos 3x \cos 3y$ and $f_y(x,y) = -3\sin 3x \sin 3y$, so

$$\begin{aligned}\vec{\nabla} f(x,y) &= f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} \\ &= (3\cos 3x \cos 3y)\,\hat{\mathbf{i}} - (3\sin 3x \sin 3y)\,\hat{\mathbf{j}}.\end{aligned}$$

Practice Question

Find the gradient $\vec{\nabla} f(x,y)$ of $f(x,y) = \dfrac{x^2 - 3y^2}{2x + y}$.

Hint

Answer

$$\vec{\nabla} f(x,y) = \frac{2x^2 + 2xy + 6y^2}{(2x+y)^2}\,\hat{\mathbf{i}} - \frac{x^2 + 12xy + 3y^2}{(2x+y)^2}\,\hat{\mathbf{j}}$$

Example # 4

Find the gradient $\vec{\nabla} f(x,y,z)$ of each of the following functions:

a. $f(x,y,z)=5x^2-2xy+y^2-4yz+z^2+3xz$
b. $f(x,y,z)=e^{-2z}\sin 2x\cos 2y$

Solution:

For both parts a. and b., we first calculate the partial derivatives f_x, f_y, and f_z, then use Equation 26.

a. $f_x(x,y,z)=10x-2y+3z$, $f_y(x,y,z)=-2x+2y-4z$, and $f_z(x,y,z)=3x-4y+2z$, so

$$\begin{aligned}\vec{\nabla} f(x,y,z) &= f_x(x,y,z)\,\hat{\mathbf{i}}+f_y(x,y,z)\,\hat{\mathbf{j}}+f_z(x,y,z)\,\hat{\mathbf{k}}\\ &=(10x-2y+3z)\,\hat{\mathbf{i}}+(-2x+2y-4z)\,\hat{\mathbf{j}}+(3x-4y+2z)\,\hat{\mathbf{k}}.\end{aligned}$$

b. $f_x(x,y,z)=2e^{-2z}\cos 2x\cos 2y$, $f_y(x,y,z)=-2e^{-2z}\sin 2x\sin 2y$, and $f_z(x,y,z)=-2e^{-2z}\sin 2x\cos 2y$, so

$$\begin{aligned}\vec{\nabla} f(x,y,z) &= f_x(x,y,z)\,\hat{\mathbf{i}}+f_y(x,y,z)\,\hat{\mathbf{j}}+f_z(x,y,z)\,\hat{\mathbf{k}}\\ &=(2e^{-2z}\cos 2x\cos 2y)\,\hat{\mathbf{i}}+(-2e^{-2z}\sin 2x\sin 2y)\,\hat{\mathbf{j}}+(-2e^{-2z}\sin 2x\cos 2y)\,\hat{\mathbf{k}}\\ &=2e^{-2z}(\cos 2x\cos 2y\,\hat{\mathbf{i}}-\sin 2x\sin 2y\,\hat{\mathbf{j}}-\sin 2x\cos 2y\,\hat{\mathbf{k}}).\end{aligned}$$

Practice Question

Find the gradient $\vec{\nabla} f(x,y,z)$ of $f(x,y,z)=\dfrac{x^2-3y^2+z^2}{2}$ and find its gradient vector at the point $(-1,2,3)$.

Answer

$$\begin{aligned}\vec{\nabla} f(x,y,z) &= x\,\hat{\mathbf{i}}-3y\,\hat{\mathbf{j}}+z\,\hat{\mathbf{k}}\\ \vec{\nabla} f(-1,2,3) &= -\hat{\mathbf{i}}-6\,\hat{\mathbf{j}}+3\,\hat{\mathbf{k}}\end{aligned}$$

Finding Gradients in Three Dimensions

Find the gradient $\nabla f(x,y,z)$ of each of the following functions:

a. $f(x,y,z)=5x^2-2xy+y^2-4yz+z^2+3xz$
b. $f(x,y,z)=e^{-2z}\sin 2x\cos 2y$

The Relationship Between Gradient and Directional Derivatives

The gradient and directional derivatives are fundamental concepts in multivariable calculus that provide insights into the behavior of functions in multiple dimensions. The gradient and directional derivatives are closely related:

- **Maximum Rate of Change:** The maximum value of the directional derivative $\boldsymbol{D_{\vec{u}}f}$ occurs when $\boldsymbol{\hat{u}}$ is parallel to the gradient vector $\boldsymbol{\nabla f}$. In other words, the function increases most rapidly in the direction of its gradient.
- **Minimum Rate of Change:** The minimum value of $\boldsymbol{D_{\vec{u}}f}$ occurs when $\boldsymbol{\hat{u}}$ is antiparallel to $\boldsymbol{\nabla f}$.
  This means the function decreases most rapidly in the direction opposite to the gradient.
- **No Change:** If $\boldsymbol{\hat{u}}$ is perpendicular to $\boldsymbol{\nabla f}$, then $\boldsymbol{D_{\vec{u}}f = 0}$.
  This indicates that the function does not change in the direction of $\mathbf{u}$.

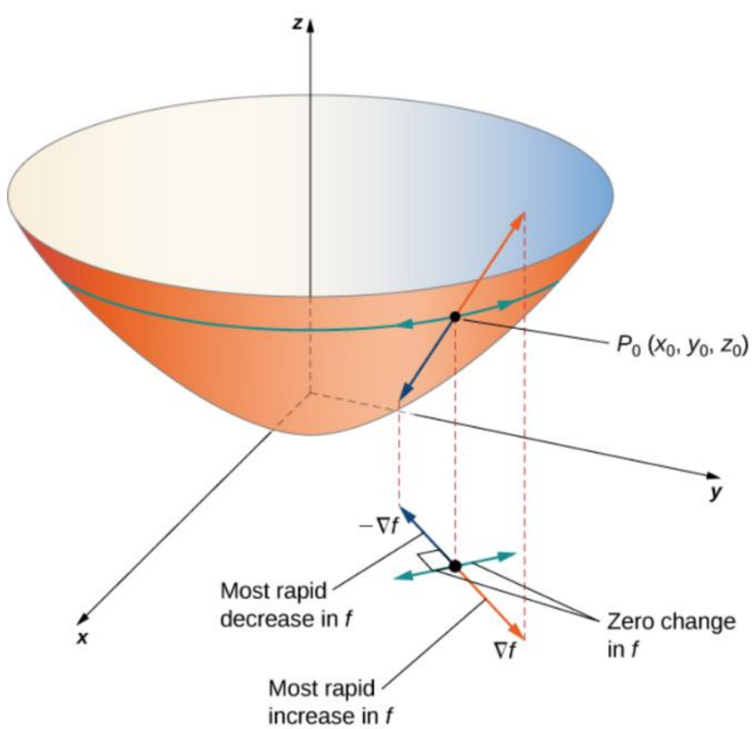

Figure 3: The gradient indicates the maximum and minimum values of the directional derivative at a point.

Directional Derivative

- The **directional derivative** of a function $\boldsymbol{f}$ measures the rate of change of $\boldsymbol{f}$ in the direction of a given vector $\vec{u}$. The result of the directional derivative is a scalar (number).

- Let $\boldsymbol{f(x,y)}$ be a **differentiable function** at $(\mathbf{a},\mathbf{b})$ and let $\boldsymbol{\hat{u}}$ be a **unit vector** in the $\boldsymbol{xy}$-plane. Then the **directional derivative of $\boldsymbol{f}$** at point $(\boldsymbol{a},\boldsymbol{b})$ **in the direction of $\boldsymbol{\hat{u}}$** is

$$\boldsymbol{D_{\vec{u}}\, f(\mathrm{a},\mathrm{b}) = \nabla f(a,b)\cdot \hat{u}}$$

where $\cdot$ represents the dot product and we already know that $\boldsymbol{\nabla f(a,b)}$ gradient of $\boldsymbol{f}$ at $(\boldsymbol{a},\boldsymbol{b})$ *i.e.*

$$\boldsymbol{\nabla f(a,b) = f_x(a,b)\hat{\imath} + f_y(a,b)\hat{\jmath}}$$

- Let $\boldsymbol{f(x,y,z)}$ be a **differentiable function** at $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ and let $\boldsymbol{\hat{u}}$ be a **unit vector** in the $\boldsymbol{xy}$-plane. Then the **directional derivative** of $\boldsymbol{f}$ at point $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ in the direction of $\boldsymbol{\vec{u}}$ is

$$\boldsymbol{D_{\vec{u}}\, f(\mathrm{a},\mathrm{b},\mathrm{c}) = \nabla f(a,b,c)\cdot \hat{u}}$$

Where the gradient at (a,b,c) is given by,

$\boldsymbol{\nabla f(a,b,c) = f_x(a,b,c)\hat{\imath} + f_y(a,b,c)\hat{\jmath} + f_z(a,b,c)\hat{k}}$.

- **Interpretation:** The directional derivative $\boldsymbol{D_{\vec{u}}f}$ tells us how fast the function f is changing when we move in the direction of $\boldsymbol{\hat{u}}$.

- **Note:** In each question we have to verify that the vector given is a unit vector. If the $|\vec{u}| = 1$ then it is already a unit vector.

- If $|\vec{u}| \neq 1$, then we convert the vector into unit vector $\boldsymbol{\hat{u} = \dfrac{\vec{u}}{|\vec{u}|}}$.

WHAT DO THE VALUES OF DIRECTIONAL DERIVATIVE INDICATE?

Positive Directional Derivative:

- **Increase:** A positive directional derivative means that the function is increasing as you move in the direction of the given vector.
- **Visual:** Imagine walking uphill. The slope in the direction you're walking is positive.

Negative Directional Derivative:

- **Decrease:** A negative directional derivative means that the function is decreasing as you move in the direction of the given vector.
- **Visual:** Imagine walking downhill. The slope in the direction you're walking is negative.

Zero Directional Derivative:

- **No Change:** A zero directional derivative means that the function is neither increasing nor decreasing in the direction of the given vector.
- **Visual:** Imagine walking on a flat surface. There's no change in elevation.

Example 1

Find the **directional derivative** of the function $\boldsymbol{f(x,y) = x^2y^3 - 4y}$ at the **point** $\boldsymbol{(2,-1)}$ **in the direction of the vector** $\boldsymbol{\vec{u} = 2\hat{i} + 5\hat{j}}$.

Solution:

We have to find the directional derivative of the function $\boldsymbol{f(x,y)}$.

Formula for the directional derivative:

$$\boldsymbol{D_{\vec{u}} f(2,-1) = \nabla f(2,-1) \cdot \hat{u}} \text{ ------------ } \mathbf{(1)}$$

Step 1: Find the gradient of $\boldsymbol{f(x,y)}$

Formula for finding gradient:

$$\boldsymbol{\nabla f(x,y) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = f_x(x,y)\,\hat{\imath} + f_y(x,y)\,\hat{\jmath}}$$

Given function is

$$\boldsymbol{f(x,y) = x^2y^3 - 4y}$$

Finding $\boldsymbol{\frac{\partial f}{\partial x}}$:

$$\boldsymbol{f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[x^2y^3 - 4y] = \frac{\partial}{\partial x}[x^2y^3] - \frac{\partial}{\partial x}[4y]}$$

$$\boldsymbol{f_x(x,y) = 2xy^3 - 0}$$

$$\boldsymbol{f_x(x,y) = \frac{\partial f}{\partial x} = 2xy^3}$$

Finding $\boldsymbol{\frac{\partial f}{\partial y}}$:

$$\boldsymbol{f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^2y^3 - 4y] = \frac{\partial}{\partial y}[x^2y^3] - \frac{\partial}{\partial y}[4y]}$$

$$\boldsymbol{f_y(x,y) = 3x^2y^2 - 4(1)}$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = 3x^2y^2 - 4$$

Putting values in the formula

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = f_x(x,y)\,\hat{\imath} + f_y(x,y)\,\hat{\jmath}$$

$$\nabla f(x,y) = [2xy^3]\hat{\imath} + [3x^2y^2 - 4]\hat{\jmath}$$

Step 2: Find the gradient of $f(x,y)$ at point $(2,-1)$

The gradient $\nabla f(x,y)$ at point $(2,-1)$ is given as

$$\nabla f(2,-1) = [2(2)(-1)^3]\hat{\imath} + [3(2)^2(-1)^2 - 4]\,\hat{\jmath}$$

$$\boxed{\nabla f(2,-1) = -4\,\hat{\imath} + 8\,\hat{\jmath} \text{ -------------------(2)}}$$

Step 3: Find the unit vector

The unit vector $\widehat{u}$ of the vector $\vec{u}$ is

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|}$$

$$|\vec{u}\,| = \sqrt{(2)^2 + (5)^2} = \sqrt{4+25} = \sqrt{29}$$

$$\boxed{\hat{u} = \frac{2\hat{\imath}+5\hat{\jmath}}{\sqrt{29}} = \frac{2}{\sqrt{29}}\hat{\imath} + \frac{5}{\sqrt{29}}\hat{\jmath} \text{ ---------------(3)}}$$

Step 4: Find the directional derivative

The formula for directional derivative is given in equation (1) as

$$D_{\vec{u}}\, f(2,-1) = \nabla f(2,-1) \cdot \hat{u}$$

Putting values of equation **(2)** and **(3)**, we have

$$D_{\vec{u}}\, f(2,-1) = (-4\hat{\imath} + 8\hat{\jmath}) \cdot \left(\frac{2}{\sqrt{29}}\hat{\imath} + \frac{5}{\sqrt{29}}\hat{\jmath}\right)$$

$$D_{\vec{u}}\, f(2,-1) = (-4)\left(\frac{2}{\sqrt{29}}\right) + (8)\left(\frac{5}{\sqrt{29}}\right)$$

$$D_{\vec{u}}\, f(2,-1) = \frac{-8+40}{\sqrt{29}}$$

$$\boldsymbol{D_{\vec{u}}\, f(2,-1) = \frac{32}{\sqrt{29}}}$$

which is the required solution.

Example 2

If the function is $\boldsymbol{f(x,y) = \frac{y^2}{x^2}}$, a point $\boldsymbol{P(1,2)}$ and a vector $\vec{\mathbf{u}} = \frac{2}{3}\hat{\boldsymbol{\imath}} + \frac{\sqrt{5}}{3}\hat{\boldsymbol{\jmath}}$, where $\vec{\boldsymbol{u}}$ is a **unit vector**, then

a) **Find** the **gradient** of $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$.
b) **Find** the **gradient** of $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ at point $\boldsymbol{P(1,2)}$.
c) **Find** the **rate of change of** $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ in the **direction** of $\vec{\mathbf{u}}$ at point $\boldsymbol{P}(\mathbf{1},\mathbf{2})$.

Solution:

Part (a): Find the gradient of $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$

Formula for finding gradient:

$$\boldsymbol{\nabla f(x,y) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = f_x(x,y)\,\hat{\imath} + f_y(x,y)\,\hat{\jmath}}$$

Given function is

$$\boldsymbol{f(x,y) = \frac{y^2}{x^2}}$$

Finding $\frac{\partial f}{\partial x}$:

$$\boldsymbol{f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\frac{y^2}{x^2}\right] = y^2\frac{\partial}{\partial x}[x^{-2}]}$$

$$\boldsymbol{f_x(x,y) = y^2[-2x^{-3}]}$$

$$\boldsymbol{f_x(x,y) = \frac{\partial f}{\partial x} = -\frac{2y^2}{x^3}}$$

Finding $\frac{\partial f}{\partial y}$:

$$\boldsymbol{f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\frac{y^2}{x^2}\right] = \frac{1}{x^2}\frac{\partial}{\partial y}[y^2]}$$

$$\boldsymbol{f_y(x,y) = \frac{1}{x^2}[2y]}$$

$$\boldsymbol{f_y(x,y) = \frac{\partial f}{\partial y} = \frac{2y}{x^2}}$$

Putting values in the formula

$$\boldsymbol{\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = f_x(x,y)\,\hat{\imath} + f_y(x,y)\,\hat{\jmath}}$$

$$\boldsymbol{\nabla f(x,y) = \left[-\frac{2y^2}{x^3}\right]\hat{\imath} + \left[\frac{2y}{x^2}\right]\hat{\jmath}}$$

<u>Part (b)</u>: Find the gradient of $f(x,y)$ at point $(1,2)$

The gradient $\boldsymbol{\nabla f(x,y)}$ at point $(\boldsymbol{2,-1})$ is given as

$$\boldsymbol{\nabla f(1,2) = \left[-\frac{2(2)^2}{(1)^3}\right]\hat{\imath} + \left[\frac{2(2)}{(1)^2}\right]\hat{\jmath}}$$

$$\boxed{\boldsymbol{\nabla f(1,2) = -8\,\hat{\imath} + 4\,\hat{\jmath}} \text{ --------}\mathbf{(1)}}$$

Part (c): Find the **rate of change of** $\boldsymbol{f}$ in the **direction** of $\vec{\mathbf{u}}$ at point $\boldsymbol{P}$.

The formula for rate of change is given as

$$\boldsymbol{D_{\vec{u}}\, f(1,2) = \nabla f(1,2) \cdot \hat{u}} \text{ ------------}\mathbf{(A)}$$

Since, it is mentioned that the given vector is a unit vector, so

$$\boxed{\boldsymbol{\hat{u} = \vec{u} = \frac{2}{3}\,\hat{\imath} + \frac{\sqrt{5}}{3}\,\hat{\jmath}} \text{ ---------------}\mathbf{(2)}}$$

Putting values of equation **(1)** and **(2)** in equation **(A)**, we have

$$D_{\vec{u}}\, f(1,2) = (-8\hat{\imath} + 4\hat{\jmath}) \cdot \left(\frac{2}{3}\hat{\imath} + \frac{\sqrt{5}}{3}\hat{\jmath}\right)$$

$$D_{\vec{u}}\, f(1,2) = (-8)\left(\frac{2}{3}\right) + (4)\left(\frac{\sqrt{5}}{3}\right)$$

$$D_{\vec{u}}\, f(1,2) = -\frac{16}{3} + \frac{4\sqrt{5}}{3}$$

$$\boldsymbol{D_{\vec{u}}\, f(1,2) = \frac{-16 + 4\sqrt{5}}{3}}$$

which is the required solution.

Example 3

<u>Find</u> the **directional derivative** of the function $\boldsymbol{g(r,s) = tan^{-1}(rs)}$ at the **point** $(\mathbf{1},\mathbf{2})$ **in the direction of the vector** $\vec{\boldsymbol{v}} = \mathbf{5}\,\hat{\boldsymbol{\imath}} + \mathbf{10}\,\hat{\boldsymbol{\jmath}}$.

<u>Solution</u>:

Remember the formula from Calculus,

$$\frac{d}{dx}\left[tan^{-1}(\,f(x)\,)\right] = \frac{1}{1+[f(x)]^2} \times \frac{d}{dx}[f(x)]$$

<u>To Find:</u>

We have to find the **directional derivative** of the function $\boldsymbol{g(r,s)}$.

Formula for the directional derivative:

$$D_{\vec{v}}\, g(1,2) = \nabla g(1,2) \cdot \hat{v} \text{ ------------ } (1)$$

<u>Step 1</u>: Find the gradient of $\boldsymbol{g(r,s)}$

Formula for finding gradient:

$$\nabla g(r,s) = \frac{\partial g}{\partial r}\hat{\imath} + \frac{\partial g}{\partial s}\hat{\jmath} = g_r(r,s)\,\hat{\imath} + g_s(r,s)\,\hat{\jmath}$$

Given function is

$$g(r,s) = tan^{-1}(rs)$$

Finding $\frac{\partial g}{\partial r}$:

$$g_r(r,s) = \frac{\partial g}{\partial r} = \frac{\partial}{\partial r}\left[tan^{-1}(rs)\right]$$

$$g_r(r,s) = \frac{\partial g}{\partial r} = \frac{1}{1+(rs)^2} \times \frac{\partial}{\partial r}[rs]$$

$$g_r(r,s) = \frac{\partial g}{\partial r} = \frac{1}{1+r^2s^2} \times\ s \cdot \frac{\partial}{\partial r}[r]$$

$$g_r(r,s) = \frac{\partial g}{\partial r} = \frac{1}{1+r^2s^2} \times\ s \cdot [1]$$

$$\boxed{g_r(r,s) = \frac{\partial g}{\partial r} = \frac{s}{1+r^2s^2}}$$

Finding $\frac{\partial g}{\partial s}$:

$$g_s(r,s) = \frac{\partial g}{\partial s} = \frac{\partial}{\partial s}[tan^{-1}(rs)]$$

$$g_s(r,s) = \frac{\partial g}{\partial s} = \frac{1}{1+(rs)^2} \times \frac{\partial}{\partial s}[rs]$$

$$g_s(r,s) = \frac{\partial g}{\partial s} = \frac{1}{1+r^2s^2} \times\ r \cdot \frac{\partial}{\partial s}[s]$$

$$g_s(r,s) = \frac{\partial g}{\partial r} = \frac{1}{1+r^2s^2} \times\ r \cdot [1]$$

$$\boxed{g_s(r,s) = \frac{\partial g}{\partial s} = \frac{r}{1+r^2s^2}}$$

Putting values in the formula

$$\nabla g(r,s) = \frac{\partial g}{\partial r}\hat{\imath} + \frac{\partial g}{\partial s}\hat{\jmath} = g_r(r,s)\ \hat{\imath} + g_s(r,s)\ \hat{\jmath}$$

$$\nabla g(r,s) = \left[\frac{s}{1+r^2s^2}\right]\hat{\imath} + \left[\frac{r}{1+r^2s^2}\right]\hat{\jmath}$$

<u>Step 2</u>: Find the gradient of $g(r,s)$ at point $(1,2)$

As the gradient is

$$\nabla g(r,s) = \left[\frac{s}{1+r^2s^2}\right]\hat{\imath} + \left[\frac{r}{1+r^2s^2}\right]\hat{\jmath}$$

The gradient $\nabla g(r,s)$ at point $(1,2)$ is given as

$$\nabla \boldsymbol{g}(\mathbf{1},\mathbf{2}) = \left[\frac{\mathbf{2}}{\mathbf{1}+(\mathbf{1})^{\mathbf{2}}(\mathbf{2})^{\mathbf{2}}}\right]\hat{\boldsymbol{\imath}} + \left[\frac{\mathbf{1}}{\mathbf{1}+(\mathbf{1})^{\mathbf{2}}(\mathbf{2})^{\mathbf{2}}}\right]\hat{\boldsymbol{\jmath}}$$

$$\nabla \boldsymbol{g}(\mathbf{1},\mathbf{2}) = \left[\frac{\mathbf{2}}{\mathbf{1}+\mathbf{4}}\right]\hat{\boldsymbol{\imath}} + \left[\frac{\mathbf{1}}{\mathbf{1}+\mathbf{4}}\right]\hat{\boldsymbol{\jmath}}$$

$$\boxed{\nabla \boldsymbol{g}(\mathbf{1},\mathbf{2}) = \left[\frac{\mathbf{2}}{\mathbf{5}}\right]\hat{\boldsymbol{\imath}} + \left[\frac{\mathbf{1}}{\mathbf{5}}\right]\hat{\boldsymbol{\jmath}} \text{ ------------(2)}}$$

Step 3: Find the unit vector

The unit vector $\widehat{\boldsymbol{v}}$ of the vector $\vec{\boldsymbol{v}}$ is

$$\widehat{\boldsymbol{v}} = \frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|} \text{ -----------(3)}$$

$$|\vec{\boldsymbol{v}}\,| = \sqrt{(\mathbf{5})^{\mathbf{2}} + (\mathbf{10})^{\mathbf{2}}} = \sqrt{\mathbf{25}+\mathbf{100}} = \sqrt{\mathbf{125}}$$

$$|\vec{\boldsymbol{v}}\,| = \sqrt{\mathbf{25}\times\mathbf{5}} = \mathbf{5}\sqrt{\mathbf{5}}$$

Putting values in the formula (3), we have

$$\widehat{\boldsymbol{v}} = \frac{\mathbf{5}\,\hat{\boldsymbol{\imath}} + \mathbf{10}\,\hat{\boldsymbol{\jmath}}}{\mathbf{5}\sqrt{\mathbf{5}}} = \frac{\mathbf{5}}{\mathbf{5}\sqrt{\mathbf{5}}}\hat{\boldsymbol{\imath}} + \frac{\mathbf{10}}{\mathbf{5}\sqrt{\mathbf{5}}}\hat{\boldsymbol{\jmath}} = \frac{\mathbf{1}}{\sqrt{\mathbf{5}}}\hat{\boldsymbol{\imath}} + \frac{\mathbf{2}}{\sqrt{\mathbf{5}}}\hat{\boldsymbol{\jmath}}$$

$$\boxed{\widehat{\boldsymbol{v}} = \frac{\mathbf{5}\,\hat{\boldsymbol{\imath}} + \mathbf{10}\,\hat{\boldsymbol{\jmath}}}{\mathbf{5}\sqrt{\mathbf{5}}} = \frac{\mathbf{1}}{\sqrt{\mathbf{5}}}\hat{\boldsymbol{\imath}} + \frac{\mathbf{2}}{\sqrt{\mathbf{5}}}\hat{\boldsymbol{\jmath}} \text{ ---------------(4)}}$$

Step 4: Find the directional derivative

The formula for directional derivative is given in equation (1) as

$$\boldsymbol{D}_{\vec{\boldsymbol{v}}}\,\boldsymbol{g}(\mathbf{1},\mathbf{2}) = \nabla \boldsymbol{f}(\mathbf{1},\mathbf{2}) \cdot \widehat{\boldsymbol{v}}$$

Putting values of equation **(2)** and **(4)**, we have

$$D_{\vec{v}}\,g(1,2) = \left(\frac{2}{5}\hat{\imath} + \frac{1}{5}\hat{\jmath}\right) \cdot \left(\frac{1}{\sqrt{5}}\hat{\imath} + \frac{2}{\sqrt{5}}\hat{\jmath}\right)$$

$$D_{\vec{v}}\, g(1,2) = \left(\frac{2}{5}\right)\left(\frac{1}{\sqrt{5}}\right) + \left(\frac{1}{5}\right)\left(\frac{2}{\sqrt{5}}\right)$$

$$D_{\vec{v}}\, g(1,2) = \frac{2}{5\sqrt{5}} + \frac{2}{5\sqrt{5}}$$

$$D_{\vec{v}}\, g(1,2) = \frac{2+2}{5\sqrt{5}}$$

$$D_{\vec{v}}\, g(1,2) = \boldsymbol{\frac{4}{5\sqrt{5}}}$$

which is the required solution.

Example 4

Find the **directional derivative** of the function

$$\boldsymbol{h(r,s,t) = \ln\,(3r + 6s + 9t)}$$

at the **point** $(\mathbf{1,1,1})$ **in the direction of the vector**

$$\boldsymbol{\vec{u} = 4\,\hat{i} + 12\,\hat{j} + 6\,\hat{k}}$$

Solution:

Note: Formula

$$\boldsymbol{\frac{d}{dx}[ln\,(\,f(x)\,)] = \frac{1}{f(x)} \times \frac{d}{dx}[f(x)]}$$

To Find:

We have to find the **directional derivative** of the function $\boldsymbol{h(r,s,t)}$.

Formula for the directional derivative:

$$\boldsymbol{D_{\vec{v}}\, h(1,1,1) = \nabla g(1,1,1) \cdot \hat{v}} \text{ ------------ } \mathbf{(1)}$$

Step 1: Find the gradient of $g(r,s)$

Formula for finding gradient:

$$\nabla h(r,s,t) = \frac{\partial h}{\partial r}\hat{\imath} + \frac{\partial h}{\partial s}\hat{\jmath} + \frac{\partial h}{\partial t}\hat{k} = h_r(r,s,t)\,\hat{\imath} + h_s(r,s,t)\,\hat{\jmath} + h_t(r,s,t)\,\hat{k}$$

Given function is

$$h(r,s,t) = \ln(3r + 6s + 9t)$$

Finding $\frac{\partial h}{\partial r}$:

$$h_r(r,s,t) = \frac{\partial h}{\partial r} = \frac{\partial}{\partial r}[\ln(3r + 6s + 9t)]$$

$$h_r(r,s,t) = \frac{\partial h}{\partial r} = \frac{1}{3r + 6s + 9t} \times \frac{\partial}{\partial r}[3r + 6s + 9t]$$

$$h_r(r,s,t) = \frac{\partial h}{\partial r} = \frac{1}{3r + 6s + 9t} \times \left[\frac{\partial}{\partial r}(3r) + \frac{\partial}{\partial r}(6s) + \frac{\partial}{\partial r}(9t)\right]$$

$$h_r(r,s,t) = \frac{\partial h}{\partial r} = \frac{1}{3(r + 2s + 3t)} \times \left[3\frac{\partial}{\partial r}(r) + 0 + 0\right]$$

$$h_r(r,s,t) = \frac{\partial h}{\partial r} = \frac{1}{3(r + 2s + 3t)} \times [3(1)]$$

$$h_r(r,s,t) = \frac{\partial h}{\partial r} = \frac{3}{3(r + 2s + 3t)}$$

$$\boxed{h_r(r,s,t) = \frac{\partial h}{\partial r} = \frac{1}{r + 2s + 3t}}$$

Finding $\frac{\partial h}{\partial s}$:

$$h_s(r,s,t) = \frac{\partial h}{\partial s} = \frac{\partial}{\partial s}[\ln(3r + 6s + 9t)]$$

$$h_s(r,s,t) = \frac{\partial h}{\partial s} = \frac{1}{3r+6s+9t} \times \frac{\partial}{\partial s}[3r+6s+9t]$$

$$h_s(r,s,t) = \frac{\partial h}{\partial s} = \frac{1}{3r+6s+9t} \times \left[\frac{\partial}{\partial s}(3r) + \frac{\partial}{\partial s}(6s) + \frac{\partial}{\partial s}(9t)\right]$$

$$h_s(r,s,t) = \frac{\partial h}{\partial s} = \frac{1}{3(r+2s+3t)} \times \left[0 + 6\frac{\partial}{\partial s}(s) + 0\right]$$

$$h_s(r,s,t) = \frac{\partial h}{\partial s} = \frac{1}{3(r+2s+3t)} \times [6(1)]$$

$$h_s(r,s,t) = \frac{\partial h}{\partial s} = \frac{6}{3(r+2s+3t)}$$

$$h_s(r,s,t) = \frac{\partial h}{\partial s} = \frac{2}{r+2s+3t}$$

Finding $\frac{\partial h}{\partial t}$:

$$h_t(r,s,t) = \frac{\partial h}{\partial t} = \frac{\partial}{\partial t}[\ln(3r+6s+9t)]$$

$$h_t(r,s,t) = \frac{\partial h}{\partial t} = \frac{1}{3r+6s+9t} \times \frac{\partial}{\partial t}[3r+6s+9t]$$

$$h_t(r,s,t) = \frac{\partial h}{\partial t} = \frac{1}{3r+6s+9t} \times \left[\frac{\partial}{\partial t}(3r) + \frac{\partial}{\partial t}(6s) + \frac{\partial}{\partial t}(9t)\right]$$

$$h_t(r,s,t) = \frac{\partial h}{\partial t} = \frac{1}{3(r+2s+3t)} \times \left[0 + 0 + 9\frac{\partial}{\partial t}(t)\right]$$

$$h_t(r,s,t) = \frac{\partial h}{\partial t} = \frac{1}{3(r+2s+3t)} \times [9(1)]$$

$$h_t(r,s,t) = \frac{\partial h}{\partial t} = \frac{9}{3(r+2s+3t)}$$

$$h_t(r,s,t)=\frac{\partial h}{\partial t}=\frac{3}{r+2s+3t}$$

Putting values in the formula

$$\nabla h(r,s,t)=\frac{\partial h}{\partial r}\hat{\imath}+\frac{\partial h}{\partial s}\hat{\jmath}+\frac{\partial h}{\partial t}\hat{k}=h_r(r,s,t)\,\hat{\imath}+h_s(r,s,t)\,\hat{\jmath}+h_t(r,s,t)\,\hat{k}$$

$$\nabla h(r,s,t)=\left[\frac{1}{r+2s+3t}\right]\hat{\imath}+\left[\frac{2}{r+2s+3t}\right]\hat{\jmath}+\left[\frac{3}{r+2s+3t}\right]\hat{k}$$

Step 2: Find the gradient of $h(r,s,t)$ at point $(1,1,1)$

As the gradient is

$$\nabla h(r,s,t)=\left[\frac{1}{r+2s+3t}\right]\hat{\imath}+\left[\frac{2}{r+2s+3t}\right]\hat{\jmath}+\left[\frac{3}{r+2s+3t}\right]\hat{k}$$

The gradient $\nabla h(r,s,t)$ at point $(1,2)$ is given as

$$\nabla h(r,s,t)=\left[\frac{1}{1+2(1)+3(1)}\right]\hat{\imath}+\left[\frac{2}{1+2(1)+3(1)}\right]\hat{\jmath}+\left[\frac{3}{1+2(1)+3(1)}\right]\hat{k}$$

$$\nabla h(1,1,1)=\left[\frac{1}{6}\right]\hat{\imath}+\left[\frac{2}{6}\right]\hat{\jmath}+\left[\frac{3}{6}\right]\hat{k}$$

$$\boxed{\nabla h(1,1,1)=\left[\frac{1}{6}\right]\hat{\imath}+\left[\frac{2}{6}\right]\hat{\jmath}+\left[\frac{3}{6}\right]\hat{k}\ \text{---------(2)}}$$

or

$$\boxed{\nabla h(1,1,1)=\left[\frac{1}{6}\right]\hat{\imath}+\left[\frac{1}{3}\right]\hat{\jmath}+\left[\frac{1}{2}\right]\hat{k}\ \text{------------(2)}}$$

Step 3: Find the unit vector

The unit vector $\hat{u}$ of the vector $\vec{u}$ is

$$\hat{u}=\frac{\vec{u}}{|\vec{u}|}\ \text{-----------(3)}$$

$$|\vec{u}|=\sqrt{(4)^2+(12)^2+(6)^2}=\sqrt{16+144+36}$$

$$|\vec{u}| = \sqrt{196}$$

$$|\vec{u}| = 14$$

Putting values in the formula (3), we have

$$\hat{u} = \frac{4\,\hat{i} \; + \; 12\,\hat{j} \; + \; 6\,\hat{k}}{5\sqrt{5}}$$

$$\hat{u} = \frac{4}{14}\hat{i} + \frac{12}{14}\hat{j} + \frac{6}{14}\hat{k}$$

$$\boxed{\hat{u} = \frac{2}{7}\hat{i} + \frac{6}{7}\hat{j} + \frac{3}{7}\hat{k} \text{ ---------------(4)}}$$

Step 4: Find the directional derivative

The formula for directional derivative is given in equation (1) as

$$D_{\vec{u}}\, h(1, 1, 1) = \nabla h(1, 1, 1) \cdot \hat{u}$$

Putting values of equation **(2)** and **(4)**, we have

$$D_{\vec{u}}\, h(1,1,1) = \left(\left[\frac{1}{6}\right]\hat{i} + \left[\frac{1}{3}\right]\hat{j} + \left[\frac{1}{2}\right]\hat{k}\right) \cdot \left(\frac{2}{7}\hat{i} + \frac{6}{7}\hat{j} + \frac{3}{7}\hat{k}\right)$$

$$D_{\vec{u}}\, h(1,1,1) = \left(\frac{1}{6}\right)\left(\frac{2}{7}\right) + \left(\frac{1}{3}\right)\left(\frac{6}{7}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{7}\right)$$

$$D_{\vec{u}}\, h(1,1,1) = \left(\frac{1}{3}\right)\left(\frac{1}{7}\right) + \left(\frac{2}{7}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{7}\right)$$

$$D_{\vec{u}}\, h(1,1,1) = \frac{1}{21} + \frac{2}{7} + \frac{3}{14}$$

$$D_{\vec{u}}\, h(1,1,1) = \frac{2 + 12 + 9}{42}$$

$$D_{\vec{u}}\, h(1,1,1) = \frac{\mathbf{23}}{\mathbf{42}} = \mathbf{0.548}$$

which is the required solution.

Practice Problems

(a) Find the gradient of f.
(b) Evaluate the gradient at the point P.
(c) Find the rate of change of f at P in the direction of the vector $\mathbf{u}$.

7. $f(x, y) = \sin(2x + 3y)$, $P(-6, 4)$, $\mathbf{u} = \frac{1}{2}\left(\sqrt{3}\,\mathbf{i} - \mathbf{j}\right)$

8. $f(x, y) = y^2/x$, $P(1, 2)$, $\mathbf{u} = \frac{1}{3}\left(2\,\mathbf{i} + \sqrt{5}\,\mathbf{j}\right)$

9. $f(x, y, z) = x^2yz - xyz^3$, $P(2, -1, 1)$, $\mathbf{u} = \left\langle 0, \frac{4}{5}, -\frac{3}{5} \right\rangle$

10. $f(x, y, z) = y^2e^{xyz}$, $P(0, 1, -1)$, $\mathbf{u} = \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle$

11–17 Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

11. $f(x, y) = e^x \sin y$, $(0, \pi/3)$, $\mathbf{v} = \langle -6, 8 \rangle$

12. $f(x, y) = \dfrac{x}{x^2 + y^2}$, $(1, 2)$, $\mathbf{v} = \langle 3, 5 \rangle$

13. $g(p, q) = p^4 - p^2q^3$, $(2, 1)$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$

14. $g(r, s) = \tan^{-1}(rs)$, $(1, 2)$, $\mathbf{v} = 5\mathbf{i} + 10\mathbf{j}$

15. $f(x, y, z) = xe^y + ye^z + ze^x$, $(0, 0, 0)$, $\mathbf{v} = \langle 5, 1, -2 \rangle$

16. $f(x, y, z) = \sqrt{xyz}$, $(3, 2, 6)$, $\mathbf{v} = \langle -1, -2, 2 \rangle$

17. $h(r, s, t) = \ln(3r + 6s + 9t)$, $(1, 1, 1)$, $\mathbf{v} = 4\mathbf{i} + 12\mathbf{j} + 6\mathbf{k}$

Calculate $D_{\vec{\mathbf{v}}} f(x,y,z)$ and $D_{\vec{\mathbf{v}}} f(0,-2,5)$ in the direction of $\vec{\mathbf{v}} = -3\,\hat{\mathbf{i}} + 12\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}$ for the function

$$f(x,y,z) = 3x^2 + xy - 2y^2 + 4yz - z^2 + 2xz.$$

Hint
Answer

$$D_{\vec{\mathbf{v}}} f(x,y,z) = -\frac{3}{13}(6x+y+2z) + \frac{12}{13}(x-4y+4z) - \frac{4}{13}(2x+4y-2z)$$

$$D_{\vec{\mathbf{v}}} f(0,-2,5) = \frac{384}{13}$$

Practice Question

Finding a Directional Derivative in Three Dimensions

Calculate $D_{\mathbf{u}} f(1,-2,3)$ in the direction of $\mathrm{v} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ for the function

$$f(x,y,z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz.$$

Calculate $D_{\mathbf{u}} f(x,y,z)$ and $D_{\mathbf{u}} f(0,-2,5)$ in the direction of $\mathrm{v} = -3\mathbf{i} + 12\mathbf{j} - 4\mathbf{k}$ for the function $f(x,y,z) = 3x^2 + xy - 2y^2 + 4yz - z^2 + 2xz.$

Example:

Q5. (8 marks) Consider the function $f(x,y,z) = 2y - \sqrt{x^2+z^2}$, $P(-3,1,4)$, and $\vec{a} = 2\,\hat{i} - 2\,\hat{j} - \hat{k}$, then

(a) Find the gradient $\nabla f(x,y,z)$ of $f(x,y,z)$.
(Hint: $\nabla f(x,y,z) = \frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j} + \frac{\partial f}{\partial z}\,\hat{k}$)

(b) Find the gradient of $f(x,y,z)$ at point P.

(c) Find the rate of change of $f(x,y,z)$ in the direction of $\vec{b}$ (Hint: $\vec{b} = \frac{\vec{a}}{|\vec{a}|}$).

Solution:

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Q#5

$$f(x,y,z) = 2y - \sqrt{x^2+z^2} \quad , \quad P(-3,1,4) \; , \; \vec{a} = 2\hat{i} - 2\hat{j} - \hat{k}$$

Sol:

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \quad \text{—} ①$$

Now

$f(x,y,z) = 2y - \sqrt{x^2+z^2}$

$$f_x = 0 - \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+z^2}}$$

$$\boxed{f_x = -\frac{x}{\sqrt{x^2+z^2}}}$$

$f(x,y,z) = 2y - \sqrt{x^2+z^2}$

$$f_y = 2 - 0$$

$$\boxed{f_y = 2}$$

$f(x,y,z) = 2y - \sqrt{x^2+z^2}$

$$f_z = 0 - \frac{\cancel{2}z}{\cancel{2}\sqrt{x^2+z^2}}$$

$$\boxed{f_z = \frac{-z}{\sqrt{x^2+z^2}}}$$

Now

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$\nabla f = -\frac{x}{\sqrt{x^2+z^2}}\hat{i} + (2)\hat{j} + \frac{-z}{\sqrt{x^2+z^2}}\hat{k}$$

$$\boxed{\nabla f = -\frac{x}{\sqrt{x^2+z^2}}\hat{i} + 2\hat{j} - \frac{z}{\sqrt{x^2+z^2}}\hat{k}}$$

(b) $P(x,y,z) = P(-3,1,4)$

$$\nabla f\Big|_{P(-3,1,4)} = \frac{-(-3)}{\sqrt{(-3)^2+(4)^2}}\hat{i} + 2\hat{j} - \frac{4}{\sqrt{(-3)^2+(4)^2}}\hat{k}$$

$$= \frac{3}{\sqrt{9+16}}\hat{i} + 2\hat{j} - \frac{4}{\sqrt{9+16}}\hat{k}$$

$$= \frac{3}{\sqrt{25}}\hat{i} + 2\hat{j} - \frac{4}{\sqrt{25}}\hat{k}$$

$$\boxed{\nabla f\Big|_{P(-3,1,4)} = \frac{3}{5}\hat{i} + 2\hat{j} - \frac{4}{5}\hat{k}}$$

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$\therefore$ (c) $\vec{b} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{2\hat{i} - 2\hat{j} - \hat{k}}{\sqrt{(2)^2 + (-2)^2 + (-1)^2}}$

$= \dfrac{2\hat{i} - 2\hat{j} - \hat{k}}{\sqrt{4+4+1}} = \dfrac{2\hat{i} - 2\hat{j} - \hat{k}}{\sqrt{9}} = \dfrac{2\hat{i} - 2\hat{j} - \hat{k}}{3}$

$\boxed{\vec{b} = \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}}$

Now

$D_{\vec{b}} f(-3,1,4) = \nabla f(-3,1,4) \cdot \vec{b}$

$= \left(\frac{3}{5}\hat{i} + 2\hat{j} - \frac{4}{5}\hat{k}\right) \cdot \left(\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}\right)$

$= \left(\frac{\cancel{3}}{5}\right)\left(\frac{2}{\cancel{3}}\right) + (2)\left(-\frac{2}{3}\right) + \left(-\frac{4}{5}\right)\left(-\frac{1}{3}\right)$

$= \frac{2}{5} - \frac{4}{3} + \frac{4}{15}$

$= \dfrac{6 - 20 + 4}{15}$

$= -\dfrac{\cancel{10}^{12}}{\cancel{15}_{3}} = -\frac{2}{3}$

$\boxed{D_{\vec{b}} f(-3,1,4) = -\frac{2}{3}}$

Practice Question 1:

Q4. (8 marks) Consider the function $f(x,y,z) = 2z^3 - 3(x^2+y^2)z + zx\ln(xy)$, $P(1,1,-2)$, and $\vec{v} = 3\,\hat{\imath} + 6\,\hat{\jmath} - 2\hat{k}$, then

(a) Find the gradient $\nabla f(x,y,z)$ of $f(x,y,z)$.

(b) Find the gradient of $f(x,y,z)$ at point P.

(c) Find the rate of change of $f(x,y,z)$ in the direction of $\vec{u}$. (Hint: $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$).

Practice Question 2:

Q # 5 (7+2+6=15 marks): A skier is on a mountain with equation

$$\boldsymbol{h(x,y) = 10 - 4x^2 - 3y^2}$$

where $\boldsymbol{h}$ denotes height. The skier is skiing in the direction indicated by a vector

$$\boldsymbol{\vec{u} = 3\,\hat{\imath} - 5\,\hat{\jmath}}$$

(a) Find the steepest possible path $\boldsymbol{\nabla h(x,y)}$ of the skier which he wants to ski downhill.

(b) Find the steepest possible path $\boldsymbol{\nabla h(x,y)}$ for the skier at the point $\boldsymbol{P(2,1)}$.

(c) Find the direction of the skier begins skiing $\boldsymbol{D_{\vec{u}}\, h(2,1)}$ in the direction of $\boldsymbol{\vec{u}}$**.**

Example: Finding a Maximum Directional Derivative

Find the direction for which the directional derivative of $f(x, y) = 3x^2 - 4xy + 2y^2$ at $(-2, 3)$ is a maximum. What is the maximum value?

Solution:

The maximum value of the directional derivative occurs when $\vec{\nabla} f$ and the unit vector point in the same direction. Therefore, we start by calculating $\vec{\nabla} f(x, y)$:

$$f_x(x, y) = 6x - 4y \text{ and } f_y(x, y) = -4x + 4y$$

so

$$\vec{\nabla} f(x, y) = f_x(x, y)\,\hat{\mathbf{i}} + f_y(x, y)\,\hat{\mathbf{j}} = (6x - 4y)\,\hat{\mathbf{i}} + (-4x + 4y)\,\hat{\mathbf{j}}.$$

Next, we evaluate the gradient at $(-2, 3)$:

$$\vec{\nabla} f(-2, 3) = (6(-2) - 4(3))\,\hat{\mathbf{i}} + (-4(-2) + 4(3))\,\hat{\mathbf{j}} = -24\,\hat{\mathbf{i}} + 20\,\hat{\mathbf{j}}.$$

The gradient vector gives the direction of the maximum value of the directional derivative.

The maximum value of the directional derivative at $(-2, 3)$ is $\|\vec{\nabla} f(-2, 3)\| = 4\sqrt{61}$ (see the Figure 4).

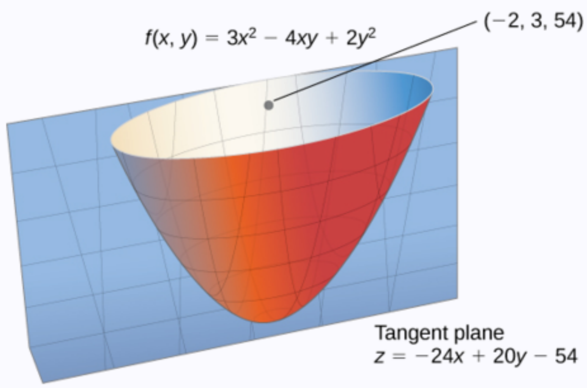

Figure 4: The maximum value of the directional derivative at $(-2, 3)$ is in the direction of the gradient.

Practice Question:

Find the direction for which the directional derivative of $g(x, y) = 4x - xy + 2y^2$ at $(-2, 3)$ is a maximum. What is the maximum value?

Hint
Answer

The gradient of g at $(-2, 3)$ is $\vec{\nabla} g(-2, 3) = \hat{\mathbf{i}} + 14\,\hat{\mathbf{j}}$. This gives the direction of the maximum value of the directional derivative at the point $(-2, 3)$.

The maximum value of the directional derivative is $\|\vec{\nabla} g(-2, 3)\| = \sqrt{197}$.