Linear Precoding for Space-Time Coded Systems With Known Fading Correlations

Hemanth Sampath, *Member, IEEE*, and Arogyaswami Paulraj, *Fellow, IEEE*

***Abstract*—We design an optimal linear precoder for a space-time coded system assuming knowledge of only the transmit antenna fading correlations. Assuming a flat fading channel and a maximum-likelihood receiver, we show that the linear precoder transmits power on the eigenmodes of the transmit antenna correlation matrix. The power allocation on the eigenmodes is a form of *waterpouring* policy. Simulation results are presented to show performance improvement on a space-time coded system.**

***Index Terms*—Fading correlations, linear precoding, MIMO, space-time coding.**

I. Introduction

SPACE-TIME coding [8], [9] is a powerful tool for achieving diversity and coding gain over multi-input multi-output (MIMO) fading channels. The code design criteria in [8], [9] assumes that the transmit and receive antennas are uncorrelated and each element of the MIMO channel matrix fades independently. This may not hold true in practice. For example, in outdoor wireless systems, the base-station (BS) antennas are placed high above the ground and close to each other. In such a scenario, the BS antennas are unobstructed and see no local scatterers leading to high correlation between the BS antennas [6]. Recent studies have shown that fading correlations reduce MIMO channel capacity and system performance [3], [4].

Prior work has suggested that considerable capacity and performance gains can be obtained by transmission on the eigenmodes of the transmit antenna correlation matrix [2]. In this letter, we propose an optimal linear precoder[1] that assumes knowledge of only the transmit antenna correlations and improves performance of a space-time coded system. Assuming a flat fading channel and a maximum likelihood receiver, we show that the optimum precoder forces transmission only on the nonzero eigenmodes of the transmit antenna correlation matrix. This is referred to as *eigenbeamforming*. The power allocation policy on the eigenmodes is given by a *waterpouring* solution.

The main advantage of the above-designed linear precoder is that it does not have to track fast-fading, but only the structure of slowly varying antenna correlations. The latter can be fed back to the transmitter using a low-rate feedback link.

Manuscript received December 14, 2001. The associate editor coordinating the review of this letter and approving it for publication was Dr. H. Jafarkhani.

H. Sampath is with Iospan Wireless Inc., San Jose, CA 94301 USA (e-mail: hemanth1@stanfordalumni.org).

A. Paulraj is with the Department of Electrical Engineering, Stanford University, Stanford, CA 94305 (e-mail: paulraj@rascals.stanford.edu).

Publisher Item Identifier S 1089-7798(02)05792-7.

[1]The linear precoder is defined as a matrix multiplier with complex elements that operates on the output of the space-time coded symbols.

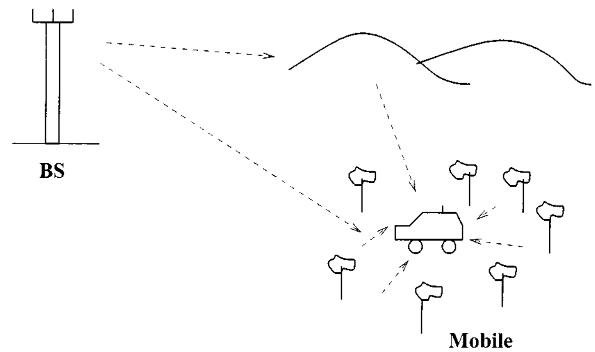

Fig. 1. Downlink wireless channel.

II. Channel Model

Let $\mathbf{H}(t)$ be the size $M_R \times M_T$ flat-fading channel matrix at time t, for a system with M_T base-station(BS) transmit antennas and M_R mobile receive antennas (see Fig. 1). The received signal at the mobile is a linear combination of several multipaths reflected from several local scatterers, leading to uncorrelated fading across the receive antennas. As a result, the rows of $\mathbf{H}(t)$ tend to be uncorrelated. However, the BS antennas are typically placed high above the ground and see no local scatterers. This leads to high antenna correlation at the BS. As a result, the columns of the $\mathbf{H}(t)$ matrix are correlated. In such an environment, the MIMO channel at a time instant t can be written as [3]:

$$\mathbf{H}(t) = \mathbf{H_w}(t)\mathbf{R}_{a,T}^{1/2} \tag{1}$$

where $\mathbf{R}_{a,T}$ is the $M_T \times M_T$ transmit antenna correlation matrix[2] and $\mathbf{H_w}$ is an $M_R \times M_T$ i.i.d complex matrix. Denote the i,jth element of $\mathbf{R}_{a,T}$ as $\rho_{i,j}$. The correlation coefficient between the BS antennas i and j is proportional to $\eta_{ij} \simeq \rho_{ij}/\sqrt{(\rho_{ii}\rho_{jj})}$.

III. System Model

Consider the downlink system model assuming a flat fading channel (see Fig. 2). Let the length of the space-time codeword be N time-symbols. At time instant n, the space-time encoder takes in a set of input bits and creates a $B \times 1$ output code symbol vector $\mathbf{c}(n) = \text{vec}[c_1(n), c_2(n), \ldots, c_B(n)]$, where $B \leq M_T$. A time sequence of N code symbol vectors $\mathbf{c}(Nt), \mathbf{c}(Nt-1), \ldots, \mathbf{c}(Nt-N+1)$ form a $B \times N$ space-time codeword, $\mathbf{s}(t) = [\mathbf{c}(Nt)\mathbf{c}(Nt-1)\mathbf{c}(Nt-N+1)]$. The $B \times N$ space-time

[2]In practice, the transmit antenna correlation matrix can be calculated as: $\mathbf{R}_{a,T} = E_t[\mathbf{H}^*(t)\mathbf{H}(t)]$, where $(\cdot)^*$ denotes the complex-conjugate transpose, and E_t denotes the expectation over time.

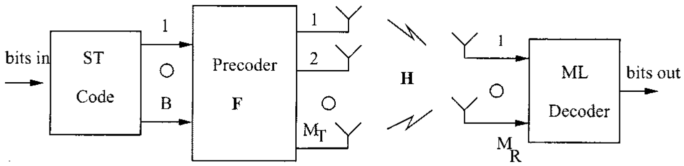

Fig. 2. Linear precoding for a space-time coded system.

codeword is then processed by the $M_T \times B$ precoder matrix $(\mathbf{F})$ and launched into the MIMO channel using M_T independent transmit antennas. Note that the precoder $(\mathbf{F})$ can add redundancy of $M_T - B$ and provide additional diversity gains. The system equation is thus given as (see Fig. 2)

$$\mathbf{x}(t) = \mathbf{H}(t)\mathbf{F}\mathbf{s}(t) + \mathbf{n}(t) \tag{2}$$

where $\mathbf{s}(t)$ is the $B \times N$ codeword matrix, $\mathbf{F}$ is the $M_T \times B$ precoder matrix, $\mathbf{H}(t)$ is the $M_R \times M_T$ channel matrix, $\mathbf{x}(t)$ is the $M_R \times N$ received signal matrix, and $\mathbf{n}(t)$ is the $M_R \times N$ noise matrix. We assume for simplicity of analysis that $\mathbf{R_{nn}}(t) = E_t[\mathbf{n}(t)\mathbf{n}^*(t)] \simeq \mathbf{R_{nn}} = \sigma^2\mathbf{I}$. Substituting (1) in (2), we get the system equation as

$$\mathbf{x}(t) = \mathbf{H_w}(t)\mathbf{R}_{a,T}^{1/2}\mathbf{F}\mathbf{s}(t) + \mathbf{n}(t). \tag{3}$$

We will assume a block-fading model used in [9] wherein $\mathbf{H_w}(t)$ remains constant over the N symbol periods (spanning a space-time codeword) and then changes in an independent fashion to a new realization. Note that $\mathbf{R}_{a,T}$ and hence $\mathbf{F}$ will be stationary over time.

IV. Optimal Linear Precoder

Let $\mathbf{s}^k(t)$ be the $B \times N$ transmitted space-time codeword at time t. If the ML decoder (at the receiver) chooses the nearest distinct $B \times N$ codeword $\mathbf{s}^l(t)$ instead of $\mathbf{s}^k(t)$, the code error matrix can be written as $\tilde{\mathbf{E}}(k,l,t) := [\mathbf{s}^k(t) - \mathbf{s}^l(t)]$. Maximum diversity is obtained if the matrix set $\tilde{\mathbf{E}}(k,l,t)$ is full-rank for all distinct k,l [9]. The coding gain is dominated by the quantity, $\min_{k,l,t}\det[\tilde{\mathbf{E}}(k,l,t)\tilde{\mathbf{E}}^*(k,l,t)]$, where det is the determinant [9]. The performance of the space-time code is hence dictated by the so-called minimum distance code error matrix, $\mathbf{E} := \arg\min_{\tilde{\mathbf{E}}(k,l,t)}\det[\tilde{\mathbf{E}}(k,l,t)\tilde{\mathbf{E}}^*(k,l,t)]$.

We now design the optimal linear precoder given such a (fixed and known) space-time encoder. The *effective* minimum distance error matrix (with the inclusion of a linear precoder) becomes $\bar{\mathbf{E}} = \mathbf{FE}$. Following the proof steps in [9], the pairwise error probability (PEP) between $\mathbf{s}^k(t)$ and $\mathbf{s}^l(t)$ can be upper-bounded by $P(\mathbf{s}^k(t) \to \mathbf{s}^l(t)) \le e^{-d^2_{\min}(t)/2}$ where we have $d^2_{\min}(t) = (1/\sigma^2)\|\mathbf{H_w}(t)\mathbf{R}_{a,T}^{1/2}\bar{\mathbf{E}}\|_F^2 = (1/\sigma^2)\|\mathbf{H_w}(t)\mathbf{R}_{a,T}^{1/2}\mathbf{FE}\|_F^2$ and where $\|\cdot\|_F$ is the Frobenius norm. Let $\mathcal{D} = (1/\sigma^2)\mathbf{R}_{a,T}^{1/2}\mathbf{FEE}^*\mathbf{F}^*\mathbf{R}_{a,T}^{*1/2}$. Define the eigenvalue decomposition (EVD): $\mathcal{D} = \mathbf{V_d\Lambda_dV_d^*}$, where $\mathbf{V_d}$ is a $M_T \times M_T$ orthonormal eigenmatrix and $\mathbf{\Lambda_d}$ is a $M_T \times M_T$ diagonal matrix with diagonal elements, $\lambda_{d,i}$, for $i = 1, 2, \dots M_T$.

An upper-bound on the *average* PEP can be obtained by taking the expectation of the PEP w.r.t $\mathbf{H_w}$. Following the proof steps in [9], the average PEP can be written as: $\bar{P}(\mathbf{s}^k \to \mathbf{s}^l) \le [\prod_{i=1}^{M_T}(1/(1+\lambda_{d,i}))]^{M_R}$. We now note that $\prod_{i=1}^{M_T}(1+\lambda_{d,i}) = \det(\mathbf{I} + \mathcal{D})$. Hence, we have that $\bar{P}(\mathbf{s}^k \to \mathbf{s}^l) \le [\det(\mathbf{I}+\mathcal{D})]^{-M_R} = [\det(\mathbf{I} + (\mathbf{R}_{a,T}^{1/2}\mathbf{FEE}^*\mathbf{F}^*\mathbf{R}_{a,T}^{*1/2}/\sigma^2))]^{-M_R}$.

The optimal $\mathbf{F}$ that minimize the average PEP can hence be obtained by solving the following optimization problem:

$$\max_{\mathbf{F}} J = \det\left(\mathbf{I} + \frac{1}{\sigma^2}\mathbf{R}_{a,T}^{1/2}\mathbf{FEE}^*\mathbf{F}^*\mathbf{R}_{a,T}^{*1/2}\right) \tag{4}$$

$$\text{subject to:} \quad \mathrm{Tr}(\mathbf{FF}^*) = p_0 \tag{5}$$

where (5) constrains the total power across M_T transmit antennas to p_0. To solve the above optimization problem, we will assume for simplicity of analysis that $\mathrm{rank}(\mathbf{R}_{a,T}) = B$. Based on our solution, we will describe how to handle the case when $\mathrm{rank}(\mathbf{R}_{a,T}) \ne B$. Let us define the EVD: $\mathbf{EE}^* = \mathbf{V_e\Lambda_eV_e^*}$ where $\mathbf{V_e}$ is the $B \times B$ orthonormal eigenmatrix and $\mathbf{\Lambda_e}$ is the $B \times B$ diagonal matrix of eigenvalues, $\lambda_{e,i}$, for $i = 1, 2 \dots, B$. Let us also define the singular value decomposition (SVD)

$$\mathbf{R}_{a,T}^{1/2} = (\mathbf{U_r} \quad \tilde{\mathbf{U}}_\mathbf{r})\begin{pmatrix}\mathbf{\Lambda_r} & 0\\ 0 & \tilde{\mathbf{\Lambda}}_\mathbf{r}\end{pmatrix}(\mathbf{V_r} \quad \tilde{\mathbf{V}}_\mathbf{r})^*$$

where $\mathbf{U_r}$ and $\mathbf{V_r}$ are $M_T \times B$ orthogonal eigenmatrices which form a basis for the range space of $\mathbf{R}_{a,T}^{1/2}$, $\mathbf{\Lambda_r}$ is a diagonal matrix containing the B nonzero singular-values arranged in a decreasing order from top-left to bottom-right; $\tilde{\mathbf{\Lambda}}_\mathbf{r}$ contains the zero singular-values; $\tilde{\mathbf{V}}_\mathbf{r}$ and $\tilde{\mathbf{U}}_\mathbf{r}$ are $M_T \times (M_T - B)$ orthogonal eigenmatrices which constitute a basis for the null-space of $\mathbf{R}_{a,T}^{1/2}$.

Theorem 1: The solution of the optimization problem posed in (4) is given as

$$\begin{aligned}\mathbf{F} &= \mathbf{V_r\Phi_fV_e^*}\\ \mathbf{\Phi_f}^2 &= \left(\gamma\mathbf{I} - \mathbf{\Lambda_r^{-2}\Lambda_e^{-1}}\right)_+\end{aligned} \tag{6}$$

where the $(\cdot)_+$ sign means $\max(\cdot, 0)$ and $\gamma > 0$ is a constant that is computed from the trace constraint.

Proof: See Appendix. ■

Based on the above theorem, a few observations are now in order. First, our analysis assumed that $\mathrm{rank}(\mathbf{R}_{a,T}) = B$. For the case when $\mathrm{rank}(\mathbf{R}_{a,T}) \ne B$, we suggest that the precoder should pour power on the strongest B eigenvectors of the transmit antenna correlation matrix. This is because any other choice of B eigenvectors will lead to a lower cost (J). Next, the power allocation on the eigenmodes of $\mathbf{R}_{a,T}$ is given by the *waterpouring* policy, dependent on the eigenvalue distribution of $\mathbf{E}$ and $\mathbf{R}_{a,T}$. Finally, orthogonal space-time codes designed for i.i.d channels typically have $\mathbf{EE}^* = \beta\mathbf{I}$, in which case $\mathbf{\Lambda_e} = \beta\mathbf{I}$ and $\mathbf{V_e} = \mathbf{I}$, where β is a scalar. In such a scenario, the optimal precoder can be thought of as a statistical *eigenbeamformer*. The rotation matrix $\mathbf{V_r}$ ensures that the optimal precoder pours power only on the eigenmodes of $\mathbf{R}_{a,T}$.

V. Simulations

For our simulations, we consider a MIMO system with $M_R = 2$ receive antennas and $M_T = \{3, 4\}$ transmit antennas, employing a rate $3/4$ space-time block code [8] with $B = 3$ and $\mathbf{EE}^* = \beta\mathbf{I}$.

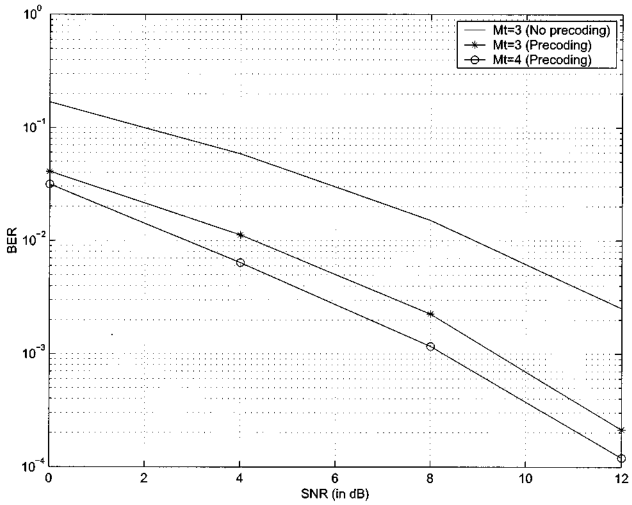

Fig. 3. Precoding gain for rate $3/4$ space-time coded system with $\eta = 0.7$.

Fig. 3 illustrates the performance improvement of the precoder on such a system. We assume $p_0 = 1$, 4-PSK transmit symbol constellation, and an antenna correlation coefficient of $\eta = n_{ij}\,(i \neq j) = 0.7$. Our Monte Carlo simulations are averaged over 100 000 channel realizations, wherein the channel is constant across a space-time block code, and changes in an independent fashion from block to block. The matrix $\mathbf{R}_{a,T}$ is constant throughout the simulation. It can be seen that at a BER of 10^{-2}, we obtain a precoding gain of approx 4.7 dB, over an nonprecoded system. This can be explained by the fact that the precoder obtains array gain by beamforming on the eigenvectors with the highest gains. For $M_T = 4$, the precoder obtains an additional 1 dB of array gain. Computer simulations showed that as $\eta \rightarrow 1$, the precoder obtains a maximum of $10\log_{10}(M_T)$ dB of array gain over an nonprecoded system by transmitting on the dominant eigenvector. As $\eta \rightarrow 0$, the channel becomes i.i.d, and the linear precoder pours power uniformly in all directions. In such a case, we see no performance improvement over a nonprecoded system.

VI. Conclusion

Recent studies have shown that fading correlations reduce the MIMO system performance. In this paper, we designed a linear precoder for a space-time coded system, assuming knowledge of only the transmit antenna correlations. Simulation results show precoding gains of up to $10\log_{10}(M_T)$ dB over a nonprecoded system for high transmit antenna correlations. Linear precoding can be implemented in wireless systems with minimal feedback and hardware overhead.

Appendix

Proof: Using the EVD of $\mathbf{R}_{a,T}^{1/2}$ and $\mathbf{E}$, the cost function in (4) can be rewritten as

$$\begin{aligned} J =& \det(\mathbf{I} + (1/\sigma^2)\mathbf{U_r\Lambda_r V_r^* F V_e \Lambda_e V_e^* F^* V_r \Lambda_r U_r^*}) \\ =& \det(\mathbf{I} + (1/\sigma^2)\mathbf{\Lambda_r V_r^* F V_e \Lambda_e V_e^* F^* V_r \Lambda_r}). \end{aligned}$$

We can write without any loss of generality that $\mathbf{F} = [\mathbf{V_r}\tilde{\mathbf{V_r}}]\bar{\mathbf{\Phi}}\mathbf{V_e}^*$, where $\bar{\mathbf{\Phi}}$ is an $M_T \times B$ matrix. Using the above expression for $\mathbf{F}$ and the fact that $\{\mathbf{V_r}^*\mathbf{V_r} = \mathbf{I}, \mathbf{V_e}^*\mathbf{V_e} = \mathbf{I}, \mathbf{V_r}^*\tilde{\mathbf{V_r}} = \mathbf{0}\}$, we get $J = \det(\mathbf{I} + (1/\sigma^2)\mathbf{\Lambda_r}[\mathbf{I}_{B\times B}\mathbf{0}_{B\times(M_T-B)}]\bar{\mathbf{\Phi}}\Lambda_e\bar{\mathbf{\Phi}}^*[\mathbf{I}_{B\times B}\mathbf{0}_{B\times(M_T-B)}]^*\mathbf{\Lambda_r})$. In this expression for J, we assume without any loss of generality that $\bar{\mathbf{\Phi}} = [\mathbf{\Phi}_f^*\mathbf{0}_{B\times(M_T-B)}]^*$, where $\mathbf{\Phi_f}$ is a $B \times B$ matrix, to get $J = \det(\mathbf{I} + (1/\sigma^2)\mathbf{\Lambda_r\Phi_f\Lambda_e\Phi_f^*\Lambda_r})$. Using the *Hadamard inequality*, we can write

$$J \leq \prod_{i=1}^{B}\left(1 + \frac{1}{\sigma^2}(\mathbf{\Lambda_r\Phi_f\Lambda_e\Phi_f^*\Lambda_r})_{ii}\right). \tag{7}$$

The upper bound in (7) is achieved and maximized if we force $\mathbf{\Phi_f}$ to be diagonal, in which case the cost function reduces to $J = \det(\mathbf{I} + \mathbf{\Lambda_r^2\Lambda_e\Phi_f^2}) = \prod_{i=1}^{B}[1 + \lambda_{e,i}\lambda_{r,i}^2\phi_{f,i}^2]$.

Let $(1/\bar{N})\sum_{i=1}^{\bar{N}}((1/\lambda_{e,i}\lambda_{r,i}^2) + \phi_{f,i}^2) = \gamma$, for any $\bar{N} \leq B$. It now holds that

$$\begin{aligned} &\prod_{i=1}^{\bar{N}}(1 + \lambda_{e,i}\lambda_{r,i}^2\phi_{f,i}^2)^{1/\bar{N}} \\ &\quad = \prod_{i=1}^{\bar{N}}(\lambda_{e,i}\lambda_{r,i}^2)^{1/\bar{N}}\prod_{i=1}^{\bar{N}}((1/\lambda_{e,i}\lambda_{r,i}^2) + \phi_{f,i}^2)^{1/\bar{N}} \\ &\quad \leq \prod_{i=1}^{\bar{N}}(\lambda_{e,i}\lambda_{r,i}^2)^{1/\bar{N}}\gamma. \end{aligned}$$

This upper bound is achieved and maximized if and only if $((1/\lambda_{e,i}\lambda_{r,i}^2) + \phi_{f,i}^2) = \gamma$ for all $i = 1, 2, \ldots \bar{N}$. Equivalently, we have that $\mathbf{\Phi_f}^2 = (\gamma\mathbf{I} - \Lambda_r^{-2}\Lambda_e^{-1})_+$ where $(\cdot)_+$ means $\max(\cdot, 0)$. This concludes our proof. ■

References

[1] A. M. Tehrani, R. Negi, and J. Cioffi, "Space-time coding and transmission optimization for wireless channels," in *Proc. 32nd Asilomar Conf. on Signals, Systems and Computers*, vol. 2, 1998, pp. 1798–1802.

[2] C. Brunner, J. Hammerschmidt, and J. Nosek, "Downlink eigenbeamforming in WCDMA," presented at the Eur. Wireless 2000, Dresden, Germany, Sept. 2000.

[3] D. Shiu, J. G. Foschini, M. Gans, and J. M. Kahn, "Fading correlation and its effect on the capacity of multi-element antenna systems," *IEEE Trans. Commun.*, vol. 48, pp. 502–513, Mar. 2000.

[4] H. Bolcskei and A. J. Paulraj, "Performance of space-time codes in the presence of spatial fading correlation," in *Thirty-Fourth Asilomar Conf. on Signals, Systems and Computers*, vol. 1, 2000, pp. 687–693.

[5] H. Sampath and A. Paulraj, "Linear precoding for space-time coded systems with known fading correlations," in *Proc. 35th Asilomar Conf. on Signals, Systems and Computers*, vol. 1, 2001, pp. 246–251.

[6] J. Salz and J. Winters, "Effect of fading correlation on adaptive arrays in digital mobile radio," *IEEE Trans. Veh. Technol.*, vol. 43, pp. 1049–1057, Nov. 1994.

[7] M. Katz and J. Ylitalo, "Extension of space-time coding to beamforming WCDMA base stations," *Proc. 51st IEEE Vehicular Technology Conf.*, vol. 2, pp. 1230–1234, May 2000.

[8] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, "Space-time block coding for wireless communications: Performance results," *IEEE J. Seect.. Areas Commun.*, vol. 17, pp. 451–460, Mar. 1999.

[9] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communication: Performance criterion and code construction," *IEEE Trans. Inform. Theory*, vol. 44, pp. 744–765, Mar. 1998.