Application of the Malliavin Calculus to the Black-Scholes Model

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1) Introduction

Introduction

Background:

- Named after Paul Malliavin (1925 - 2010).
- Also known as the **stochastic calculus of variations**.
- Originally introduced by Paul Malliavin in the 1970s in his proof of Hörmander's hypoellipticity theorem for partial differential equations.
- Notable mathematicians: Paul Malliavin, David Nualart and Eulalia Nualart.

Central Question:

- How can the Malliavin Calculus be applied to calculate and estimate the Greeks of certain option prices?

2) Elements of the Malliavin Calculus

2)a) The Malliavin Derivative

The Malliavin Derivative

In what follows, let:

- $T \in \mathbb{R}_{\geqslant 0}$;
- $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space;
- $(W_t)_{t\in[0,T]}$ be a $\mathbb{R}$-valued Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$;
- $\mathscr{F}$ be the σ-algebra generated by $(W_t)_{t\in[0,T]}$, and;
- $\mathscr{S}$ be the set of all random variables X of the form $X = F(W_{t_1}, \ldots, W_{t_n})$, for some $n \in \mathbb{N}_{\geqslant 1}$, some smooth function $F\colon \mathbb{R}^n \to \mathbb{R}$ and some $t_1, \ldots, t_n \in [0, T]$.

The Malliavin Derivative

Definition (Malliavin Derivative) [2]:

- Let $X \in \mathscr{S}$ and let $t \in \mathbb{R}$. Define the **Malliavin derivative** $D_t(X)$ of X by:

$$D_t(X) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F(W_{t_1}, \ldots, W_{t_n}) \bigg|_{x_i = W_{t_i}} \mathbf{1}_{[0,t_i]}(t)$$

- For $j \in \mathbb{N}_{\geqslant 1}$, define:

$$D^j_{t_1,\ldots,t_j} X = \left(D_{t_1}\left(\ldots\left(D_{t_j} X\right)\right)\right)$$

The Malliavin Derivative

Proposition [5, 7]:

Let $X, Y \in \mathscr{S}$, let $t, \lambda \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then:

1. $D_t(X + Y) = D_t(X) + D_t(Y)$ (Sum Rule)
2. $D_t(\lambda X) = \lambda D_t(X)$ (Scalar Product Rule)
3. $D_t(XY) = D_t(X)Y + XD_t(Y)$ (Product Rule Rule)
4. $D_t(f(X)) = f'(X)D_t(X)$ (Chain Rule)

Proof.

Omitted (refer to [6, Proposition 1.20, Page 28] or to [7, Proposition 2.1.2, Page 15]). ∎

The Malliavin Derivative

Definition [3]:

For $k \in \mathbb{N}_{\geqslant 1}$ and $p \in \mathbb{R}_{\geqslant 1}$, denote by $\mathbb{D}^{k,p}$ the closure of $\mathscr{S}$ with respect to the norm:

$$\|X\|_{k,p} = \left(\mathbb{E}\left[|X|^p + \left(\sum_{j=1}^{k} \int_{[0,T]^j} \left| D^j_{s_1,\ldots,s_j} X \right|^2 \mathrm{d}s_1 \ldots \mathrm{d}s_j \right)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}}$$

and let:

$$\|X\|_{0,p} = \left(\mathbb{E}\left[|F|^p \right] \right)^{\frac{1}{p}}$$

Define:

$$\mathbb{D}^{\infty} = \bigcap_{k \in \mathbb{N}_{\geqslant 1}} \bigcap_{p \in \mathbb{R}_{\geqslant 1}} \mathbb{D}^{k,p}$$

Examples

Let $s, t \in [0, T]$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function. Then:

- $D_s W_T = 1$
- $D_s W_t = \mathbf{1}_{[0,t]}(s)$
- $D_s f(W_t) = f'(W_t)\mathbf{1}_{[0,t]}(s)$
- $D_s \left(\int_0^T f(W_u)\,\mathrm{d}W_u \right) = \int_s^T f'(W_u)\,\mathrm{d}W_u + f(W_s)$

2)b) The Skorokhod Integral

The Skorokhod Integral

Definition [2, 6]:

The **Skorokhod integral** operator is the adjoint operator of D., denoted by D^*.

- Remark [7, Proposition 2.2.1, Page 17]: If $(u_t)_{t\in[0,T]}$ is stochastic process adapted to $\mathscr{F}$, then:

$$D^*(u) = \int_0^T u_t \, \mathrm{d}W_t$$

  that is, D^* coincides with the Itô integral of $(u_t)_{t\in[0,T]}$.

- Notation [2, Page 2]: We shall write:

$$\int_0^T u_t \, \mathrm{d}W_t \coloneqq D^*(u)$$

  even if $(u_t)_{t\in[0,T]}$ is not adapted to $\mathscr{F}$.

The Skorokhod Integral

Examples

Let $s, t \in [0, T]$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function. Then:

- $D^*(1) = W_T$
- $D^*\left(\mathbf{1}_{[0,t]}(s)\right) = W_t$
- $D^*\left(f'(W_t)\mathbf{1}_{[0,t]}(s)\right) = f(W_t)$
- $D^*\left(\int_s^T f'(W_u)\,\mathrm{d}W_u + f(W_s)\right) = \int_0^T f(W_u)\,\mathrm{d}W_u$

Proposition:

Let $X, Y \in \mathscr{S}$ and let $\lambda \in \mathbb{R}$. Then:

1. $D^*(X + Y) = D^*(X) + D^*(Y)$ (Sum Rule)
2. $D^*(\lambda X) = \lambda D^*(X)$ (Scalar Product Rule)

Proof.

Omitted (the result is an immediate consequence of the definiton of an adjoint operator). ∎

2)c) Integration by Parts

Integration by Parts

Theorem (Duality Principle) [2, 6, 7]:

Let $X \in \mathbb{D}^{1,2}$ and let $(u_t)_{t\in[0,T]}$ be a stochastic process. Then:

$$\mathbb{E}\left[\int_0^T (D_t(X))u_t \,\mathrm{d}t\right] = \mathbb{E}\left[X D^*(u)\right]$$

Proof.

Ommitted (refer to [2, Appendix A, Page 18 for a heuristic derivation]).
■

Integration by Parts

Proposition [7]:

Let $X, Y \in \mathbb{D}^{1,2}$ and let $(u_t)_{t\in[0,T]}$ be a stochastic process. Then:

$$\mathbb{E}\left[Y D^*(Xu)\right] = \mathbb{E}\left[Y\left(X D^*(u) - \int_0^T D_t(Xu)\,\mathrm{d}t\right)\right]$$

Integration by Parts

Proof.

By the product rule and the duality principle, one has:

$$
\begin{aligned}
\mathbb{E}\left[YXD^*(u)\right] &= \mathbb{E}\left[\int_0^T D_t(YX)u_t\,\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T YD_t(X)u_t\,\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T D_t(Y)Xu_t\,\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T YD_t(X)u_t\,\mathrm{d}t\right] + \mathbb{E}\left[YD^*(Xu)\right]
\end{aligned}
$$

■

Integration by Parts

Corollary [7]:

Let $X \in \mathbb{D}^{1,2}$ and let $(u_t)_{t\in[0,T]}$ be a stochastic process. Then:

$$\int_0^T X u_t \,\mathrm{d}W_t = X \int_0^T u_t \,\mathrm{d}W_t - \int_0^T D_t(X) u_t \,\mathrm{d}t$$

Integration by Parts

Theorem (Integration By Parts) [1, 2, 6, 7]:

Let $X, Y \in \mathbb{D}^{1,2}$. Then, for any stochastic process $(u_t)_{t\in[0,T]}$ such that

$$\int_0^T D_t(X) u_t \,\mathrm{d}t \overset{\mathbb{P}\text{-a.s.}}{\neq} 0$$

and for any smooth function $f : \mathbb{R} \to \mathbb{R}$, one has:

$$\mathbb{E}\left[f'(X)Y\right] = \mathbb{E}\left[f(X)H(X,Y)\right]$$

where:

$$H(X,Y) = D^*\left(\frac{Y u_\cdot}{\int_0^T D_t(X) u_t \,\mathrm{d}t}\right)$$

Integration by Parts

Proof:

By the chain rule, one has:

$$\int_0^T D_t(f(X))u_t \, \mathrm{d}t = f'(X) \int_0^T D_t(X)u_t \, \mathrm{d}t$$

Integration by Parts

Proof:

Thus, by the duality principle, one has:

$$
\begin{aligned}
\mathbb{E}\left[f'(X)Y\right] &= \mathbb{E}\left[\frac{Y\int_0^T D_t(f(X))u_t\,\mathrm{d}t}{\int_0^T D_t(X)u_t\,\mathrm{d}t}\right] \\
&= \mathbb{E}\left[\int_0^T \frac{D_t(f(X))Yu_t}{\int_0^T D_s(X)u_s\,\mathrm{d}s}\,\mathrm{d}t\right] \\
&= \mathbb{E}\left[f(X)D^*\left(\frac{Yu.}{\int_0^T D_t(X)u_t\,\mathrm{d}t}\right)\right] \\
&= \mathbb{E}\left[f(X)H(X,Y)\right]
\end{aligned}
$$

■

Integration By Parts

Corollary [7]:

Let $X, Y \in \mathbb{D}^{1,2}$. If the function $t \mapsto D_t(X)$ is differentiable on $[0, T]$ and:

$$\int_0^T \frac{\partial D_t(X)}{\partial t} D_t(X)\,\mathrm{d}t \overset{\mathbb{P}\text{-a.s.}}{\neq} 0$$

then, for any smooth function $f\colon \mathbb{R} \to \mathbb{R}$, one has:

$$\mathbb{E}\left[f'(X)Y\right] = \mathbb{E}\left[f(X)D^*\left(\frac{2Y\frac{\partial D_\cdot(X)}{\partial \cdot}}{D_T(X)^2 - D_0(X)^2}\right)\right]$$

Integration by Parts

Proof:

Take $(u_t)_{t\in[0,T]}$ in the previous theorem to be defined by:

$$\forall t \in [0,T],\ \ u_t = \frac{\partial D_t(X)}{\partial t}$$

and compute $\int_0^T \frac{\partial D_t(X)}{\partial t} D_t(X)\,\mathrm{d}t$ as follows:

$$\int_0^T \frac{\partial D_t(X)}{\partial t} D_t(X)\,\mathrm{d}t = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2} D_t(X)^2\right)\mathrm{d}t = \frac{D_T(X)^2 - D_0(X)^2}{2}$$

∎

3) Application to the Black-Scholes Model

3)a) General Setting

General Setting

Let:

- $T, K \in \mathbb{R}_{\geqslant 0}$;
- $S_0, r \in \mathbb{R}_{>0}$;
- $\sigma \in \mathbb{R}$;
- $M \in \mathbb{N}_{\geqslant 1}$;
- $(W_t)_{t\in[0,T]}$ be a $\mathbb{R}$-valued Brownian motion;
- $(S_t)_{t\in[0,T]}$ be a $\mathbb{R}$-valued geometric Brownian motion satisfying the following:

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t\,,\ \ S_t = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t},\ \ t \in [0,T]$$

  and;

- $(V_t)_{t\in[0,T]}$ be a stochastic process such that $V_T = \Phi(S_T)$, for some smooth function $\Phi\colon \mathbb{R} \to \mathbb{R}$.

General Setting

Consider an options contract written on an asset whose price follows the dynamics of $(S_t)_{t\in[0,T]}$. In particular, let T be the contract's maturity date, K be the strike price, S_0 be the initial asset price, r be the risk-free interest rate, σ be the volatility and V_T be option's payoff at maturity. Should the asset price be discretely monitored, suppose that it is monitored at M distinct and evenly separated points in time from the beginning of the options contract until its maturity date T. In what follows, the following sensitivities of the options contract's price - also known as *Greeks* - will be calculated:

- $\Delta := \dfrac{\partial V_0}{\partial S_0}$ (Delta)
- $\mathcal{V} := \dfrac{\partial V_0}{\partial \sigma}$ (Vega)
- $\Gamma := \dfrac{\partial^2 V_0}{\partial S_0^2}$ (Gamma)

General Setting

Under the no-arbitrage principle, we have $V_0 = e^{-rT}\mathbb{E}[V_T]$. Moreover, the following cases will be presented:

- $V_T = (S_T - K)^+$ (European Call)
- $V_T = (\frac{1}{T}\int_0^T S_t \,\mathrm{d}t - K)^+$ (Asian Fixed Strike Call)

3)b) Calculating the Greeks of a European Call Option

Analytic Expression

Theorem (Analytic Formula for the Greeks of a European Call Option) [2, 7]:

For a European call option on an asset whose price follows the dynamics of $(S_t)_{t\in[0,T]}$, one has:

1. $\Delta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{x^2}{2}} \, \mathrm{d}x$
2. $\mathcal{V} = S_0 e^{-\frac{d^2}{2}} \sqrt{\frac{T}{2\pi}}$
3. $\Gamma = \frac{e^{-\frac{d^2}{2}}}{S_0 \sigma \sqrt{2\pi T}}$

where:

$$d = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$

Analytic Expression

Proof.

Omitted (refer to [2, Pages 10-12] for the proof). ■

Computation via Malliavin Calculus

Lemma [2, 7]:

In the Black-Scholes model, one has:

1. $\int_0^T D_u(S_T)\,\mathrm{d}u = \sigma T S_T$
2. $D^*(S_T) = S_T(W_T - \sigma T)$
3. $D^*(W_T) = W_T^2 - T$
4. $D^*\left(\frac{S_T}{\int_0^T D_u(S_T)\mathrm{d}u}\right) = \frac{W_T}{\sigma T}$
5. $D^*\left(\frac{S_T(W_T-\sigma T)}{\int_0^T D_u(S_T)\mathrm{d}u}\right) = \frac{W_T^2-T}{\sigma T} - W_T$

Computation via Malliavin Calculus

Proof.

One has:

1. $\int_0^T D_u(S_T)\,\mathrm{d}u = \int_0^T D_u\left(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_T}\right)\mathrm{d}u = \int_0^T \sigma S_T\,\mathrm{d}u = \sigma T S_T$
2. $D^*(S_T) = \int_0^T S_T\,\mathrm{d}W_t = S_T\int_0^T \mathrm{d}W_t - \int_0^T D_t(S_T)\,\mathrm{d}t = S_T(W_T-\sigma T)$
3. $D^*(W_T) = W_T\int_0^T \mathrm{d}W_T - \int_0^T D_t(W_T)\,\mathrm{d}t = W_T^2 - T$
4. $D^*\left(\frac{S_T}{\int_0^T D_u(S_T)\mathrm{d}u}\right) = D^*\left(\frac{S_T}{\sigma T S_T}\right) = D^*\left(\frac{1}{\sigma T}\right) = \frac{W_T}{\sigma T}$
5. $D^*\left(\frac{S_T(W_T-\sigma T)}{\int_0^T D_u(S_T)\mathrm{d}u}\right) = D^*\left(\frac{W_T}{\sigma T}-1\right) = \frac{W_T^2-T}{\sigma T} - W_T$

∎

Computation via Malliavin Calculus

Proposition (Δ for a European Call Option) [7]:

For a European call option on an asset whose price follows the dynamics of $(S_t)_{t\in[0,T]}$, one has:

$$\Delta = \frac{e^{-rT}}{\sigma T S_0}\mathbb{E}\left[\Phi(S_T)W_T\right]$$

where $\Phi(x) = (x-K)^+$.

Computation via Malliavin Calculus

Proof.

By the chain rule and the integration by parts formula, one has:

$$\Delta = \frac{\partial V_0}{\partial S_0} = \frac{\partial}{\partial S_0}\left(e^{-rT}\mathbb{E}[V_T]\right) = e^{-rT}\mathbb{E}\left[\frac{\partial \Phi(S_T)}{\partial S_0}\right]$$

$$= e^{-rT}\mathbb{E}\left[\Phi'(S_T)\frac{\partial S_T}{\partial S_0}\right] = e^{-rT}\mathbb{E}\left[\Phi'(S_T)\frac{\partial\left(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T}\right)}{\partial S_0}\right] = \frac{e^{-rT}}{S_0}\mathbb{E}\left[\Phi'(S_T)S_T\right]$$

$$= \frac{e^{-rT}}{S_0}\mathbb{E}\left[\Phi(S_T)D^*\left(\frac{S_T}{\int_0^T D_u(S_T)\,\mathrm{d}u}\right)\right] = \frac{e^{-rT}}{S_0}\mathbb{E}\left[\Phi(S_T)D^*\left(\frac{S_T}{\sigma T S_T}\right)\right]$$

$$= \frac{e^{-rT}}{S_0}\mathbb{E}\left[\Phi(S_T)D^*\left(\frac{1}{\sigma T}\right)\right] = \frac{e^{-rT}}{\sigma T S_0}\mathbb{E}\left[\Phi(S_T)W_T\right]$$

∎

Computation via Malliavin Calculus

Proposition ($\mathcal{V}$ for a European Call Option) [7]:

For a European call option on an asset whose price follows the dynamics of $(S_t)_{t\in[0,T]}$, one has:

$$\mathcal{V} = e^{-rT}\mathbb{E}\left[\Phi(S_T)\left(\frac{W_T^2 - T}{\sigma T} - W_T\right)\right]$$

where $\Phi(x) = (x - K)^+$.

Computation via Malliavin Calculus

Proof.

By the chain rule and the integration by parts formula, one has:

$$
\begin{aligned}
\mathcal{V} = \frac{\partial V_0}{\partial \sigma} &= \frac{\partial}{\partial \sigma}\left(e^{-rT}\mathbb{E}[V_T]\right) = e^{-rT}\mathbb{E}\left[\frac{\partial \Phi(S_T)}{\partial \sigma}\right] \\
&= e^{-rT}\mathbb{E}\left[\Phi'(S_T)\frac{\partial S_T}{\partial \sigma}\right] = e^{-rT}\mathbb{E}\left[\Phi'(S_T)\frac{\partial\left(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T}\right)}{\partial \sigma}\right] \\
&= e^{-rT}\mathbb{E}\left[\Phi'(S_T)S_T(W_T-\sigma T)\right] = e^{-rT}\mathbb{E}\left[\Phi(S_T)D^*\left(\frac{S_T(W_T-\sigma T)}{\int_0^T D_u(S_T)\,\mathrm{d}u}\right)\right] \\
&= e^{-rT}\mathbb{E}\left[\Phi(S_T)D^*\left(\frac{S_T(W_T-\sigma T)}{\sigma T S_T}\right)\right] \\
&= e^{-rT}\mathbb{E}\left[\Phi(S_T)D^*\left(\frac{W_T}{\sigma T}-1\right)\right] = e^{-rT}\mathbb{E}\left[\Phi(S_T)\left(\frac{W_T^2-T}{\sigma T}-W_T\right)\right]
\end{aligned}
$$

■

Computation via Malliavin Calculus

Proposition (Γ for a European Call Option) [7]:

For a European call option on an asset whose price follows the dynamics of $(S_t)_{t\in[0,T]}$, one has:

$$\Gamma = \frac{e^{-rT}}{\sigma T S_0^2}\mathbb{E}\left[\Phi(S_T)\left(\frac{W_T^2 - T}{\sigma T} - W_T\right)\right]\left(= \frac{\mathcal{V}}{\sigma T S_0^2}\right)$$

where $\Phi(x) = (x-K)^+$.

Computation via Malliavin Calculus

Proof.

By the chain rule and the integration by parts formula, one has:

$$\Gamma = \frac{\partial^2 V_0}{\partial S_0^2} = \frac{\partial^2}{\partial S_0^2}\left(e^{-rT}\mathbb{E}\left[V_T\right]\right) = e^{-rT}\mathbb{E}\left[\frac{\partial^2 \Phi(S_T)}{\partial S_0^2}\right] = e^{-rT}\mathbb{E}\left[\frac{\partial}{\partial S_0}\left(\Phi'(S_T)e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T}\right)\right]$$

$$= e^{-rT}\mathbb{E}\left[\Phi''(S_T)\left(e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T}\right)^2\right] = \frac{e^{-rT}}{S_0^2}\mathbb{E}\left[\Phi''(S_T)S_T^2\right]$$

$$= \frac{e^{-rT}}{S_0^2}\mathbb{E}\left[\Phi'(S_T)D^*\left(\frac{S_T^2}{\int_0^T D_u(S_T)\,\mathrm{d}u}\right)\right] = \frac{e^{-rT}}{\sigma T S_0^2}\mathbb{E}\left[\Phi'(S_T)D^*\left(S_T\right)\right]$$

$$= \frac{e^{-rT}}{\sigma T S_0^2}\mathbb{E}\left[\Phi'(S_T)S_T(W_T-\sigma T)\right] = \frac{e^{-rT}}{\sigma T S_0^2}\mathbb{E}\left[\Phi(S_T)D^*\left(\frac{S_T(W_T-\sigma T)}{\int_0^T D_u(S_T)\,\mathrm{d}u}\right)\right]$$

$$= \frac{e^{-rT}}{\sigma T S_0^2}\mathbb{E}\left[\Phi(S_T)D^*\left(\frac{W_T}{\sigma T}-1\right)\right] = \frac{e^{-rT}}{\sigma T S_0^2}\mathbb{E}\left[\Phi(S_T)\left(\frac{W_T^2-T}{\sigma T}-W_T\right)\right]$$

■

3)c) Calculating the Greeks for an Asian Fixed Strike Call Option

Computation via Malliavin Calculus

Lemma [7]:

Let $\overline{S}_T = \int_0^T S_t \, \mathrm{d}t$ and let $t \in [0, T]$. Then, in the Black-Scholes model, one has:

1. $D_t\left(\overline{S}_T\right) = \sigma \int_t^T S_u \, \mathrm{d}u$
2. $\frac{\partial}{\partial t}\left(D_t\left(\overline{S}_T\right)\right) = -\sigma S_t$
3. $\int_0^T S_t D_t\left(\overline{S}_T\right) \mathrm{d}t = \frac{\sigma}{2}\overline{S}_T^2$
4. $D^*(S_.) = \frac{S_T - S_0 - r\overline{S}_T}{\sigma}$
5. $D^*\left(\frac{(S_T - S_0)S_.}{\overline{S}_T^2}\right) = \frac{S_T - S_0}{\sigma \overline{S}_T^2}\left(S_T - S_0 - r\overline{S}_T\right) - \frac{\sigma S_0}{\overline{S}_T}$

Proof.

Omitted (refer to [7, Pages 30-34]) for the proof. ∎

Calculating the Greeks for an Asian Fixed Strike Call Option

Proposition (Δ for an Asian Fixed Strike Call Option) [7]:

Let $\overline{S}_T = \int_0^T S_t \,\mathrm{d}t$. Then, for an Asian fixed strike call option on an asset whose price follows the dynamics of $(S_t)_{t\in[0,T]}$, one has:

$$\Delta = \frac{e^{-rT}}{S_0\sigma^2}\mathbb{E}\left[\Phi\left(\overline{S}_T\right)\left(\frac{2(S_T - S_0)}{\overline{S}_T} + \sigma^2 - 2r\right)\right]$$

where $\Phi(x) = (\frac{1}{T}x - K)^+$.

Calculating the Greeks for an Asian Fixed Strike Call Option

Proof.

By the chain rule and integration by parts formula, one has:

$$
\begin{aligned}
\Delta &= \frac{\partial V_0}{\partial S_0} = \frac{\partial}{\partial S_0}\left(e^{-rT}\mathbb{E}\left[V_T\right]\right) = e^{-rT}\mathbb{E}\left[\frac{\partial \Phi(\overline{S}_T)}{\partial S_0}\right] = e^{-rT}\mathbb{E}\left[\Phi'(\overline{S}_T)\frac{\partial \overline{S}_T}{\partial S_0}\right] \\
&= e^{-rT}\mathbb{E}\left[\Phi'(\overline{S}_T)\int_0^T \frac{\partial \overline{S}_T}{\partial S_0}\,\mathrm{d}t\right] = \frac{e^{-rT}}{S_0}\mathbb{E}\left[\Phi'(\overline{S}_T)\overline{S}_T\right] \\
&= \frac{e^{-rT}}{S_0}\mathbb{E}\left[\Phi(\overline{S}_T)D^*\left(\frac{2\overline{S}_T\frac{\partial}{\partial\cdot}D_\cdot(\overline{S}_T)}{D_T(\overline{S}_T)^2 - D_0(\overline{S}_T)^2}\right)\right] = \frac{e^{-rT}}{S_0}\mathbb{E}\left[\Phi(\overline{S}_T)D^*\left(\frac{2\overline{S}_T(-\sigma S_\cdot)}{-(\sigma\overline{S}_T)^2}\right)\right] \\
&= \frac{e^{-rT}}{S_0}\mathbb{E}\left[\Phi(\overline{S}_T)D^*\left(\frac{2S_\cdot}{\sigma\overline{S}_T}\right)\right] = \frac{2e^{-rT}}{\sigma S_0}\mathbb{E}\left[\Phi(\overline{S}_T)\left(\frac{D^*(S_\cdot)}{\overline{S}_T} - \int_0^T S_t D_t\left(\frac{1}{\overline{S}_T}\right)\mathrm{d}t\right)\right] \\
&= \frac{2e^{-rT}}{\sigma S_0}\mathbb{E}\left[\Phi(\overline{S}_T)\left(\frac{S_T - S_0 - r\overline{S}_T}{\sigma\overline{S}_T} + \frac{1}{\overline{S}_T^2}\int_0^T S_t D_t(\overline{S}_T)\,\mathrm{d}t\right)\right] \\
&= \frac{e^{-rT}}{S_0\sigma^2}\mathbb{E}\left[\Phi\left(\overline{S}_T\right)\left(\frac{2(S_T - S_0)}{\overline{S}_T} + \sigma^2 - 2r\right)\right]
\end{aligned}
$$

■

Calculating the Greeks for an Asian Fixed Strike Call Option

Proposition (Γ for an Asian Fixed Strike Call Option) [7]:

Let $\overline{S}_T = \int_0^T S_t \, \mathrm{d}t$. Then, for an Asian fixed strike call option on an asset whose price follows the dynamics of $(S_t)_{t \in [0,T]}$, one has:

$$\Gamma = \frac{4e^{-rT}}{S_0^2 \sigma^3} \mathbb{E} \left[\Phi \left(\overline{S}_T \right) \left(\frac{(S_T - S_0)^2 - (S_T - S_0) r \overline{S}_T}{\sigma \overline{S}_T^2} - \frac{\sigma S_0}{\overline{S}_T} \right) \right] - \frac{2r}{S_0 \sigma^2} \Delta$$

Calculating the Greeks for an Asian Fixed Strike Call Option

Proof.

By the chain rule and integration by parts formula, one has:

$$\Gamma = \frac{\partial^2 V_0}{\partial S_0^2} = \frac{\partial^2}{\partial S_0^2}\left(e^{-rT}\mathbb{E}\left[V_T\right]\right) = e^{-rT}\mathbb{E}\left[\frac{\partial^2 \Phi(\overline{S}_T)}{\partial S_0^2}\right] = \frac{e^{-rT}}{S_0^2}\mathbb{E}\left[\Phi''(\overline{S}_T)\overline{S}_T^2\right]$$

$$= \frac{e^{-rT}}{S_0^2}\mathbb{E}\left[\Phi'(\overline{S}_T)D^*\left(\frac{2\overline{S}_T^2\frac{\partial}{\partial\cdot}D_\cdot(\overline{S}_T)}{D_T(\overline{S}_T)^2 - D_0(\overline{S}_T)^2}\right)\right] = \frac{e^{-rT}}{S_0^2}\mathbb{E}\left[\Phi'(\overline{S}_T)D^*\left(\frac{2\overline{S}_T^2(-\sigma S_\cdot)}{-(\sigma\overline{S}_T)^2}\right)\right]$$

$$= \frac{2e^{-rT}}{\sigma S_0^2}\mathbb{E}\left[\Phi'(\overline{S}_T)D^*\left(S_\cdot\right)\right] = \frac{2e^{-rT}}{\sigma^2 S_0^2}\mathbb{E}\left[\Phi'(\overline{S}_T)\left(S_T - S_0\right)\right] - \frac{2e^{-rT}}{\sigma^2 S_0^2}\mathbb{E}\left[\Phi'(\overline{S}_T)\left(r\overline{S}_T\right)\right]$$

$$= \frac{2e^{-rT}}{\sigma^2 S_0^2}\mathbb{E}\left[\Phi(\overline{S}_T)D^*\left(\frac{2(S_T - S_0)(-\sigma S_\cdot)}{-(\sigma\overline{S}_T)^2}\right)\right] - \frac{2r}{\sigma^2 S_0}\Delta$$

$$= \frac{4e^{-rT}}{\sigma^3 S_0^2}\mathbb{E}\left[\Phi(\overline{S}_T)\left(\frac{S_T - S_0}{\sigma\overline{S}_T^2}\left(S_T - S_0 - r\overline{S}_T\right) - \frac{\sigma S_0}{\overline{S}_T}\right)\right] - \frac{2r}{\sigma^2 S_0}\Delta$$

$$= \frac{4e^{-rT}}{S_0^2\sigma^3}\mathbb{E}\left[\Phi\left(\overline{S}_T\right)\left(\frac{(S_T - S_0)^2 - (S_T - S_0)r\overline{S}_T}{\sigma\overline{S}_T^2} - \frac{\sigma S_0}{\overline{S}_T}\right)\right] - \frac{2r}{S_0\sigma^2}\Delta$$

∎

3)d) Monte-Carlo Simulations for a European Call Option

Monte-Carlo Simulations for a European Call Option

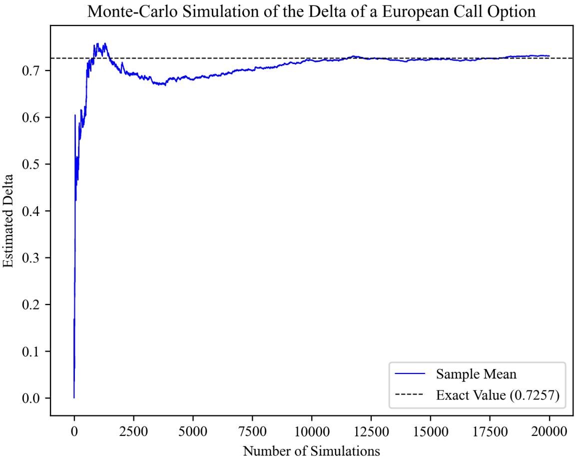

Figure 1: Monte-Carlo simulation of the Delta for a European call option with $(r, \sigma, S_0, K, T) = (0.1, 0.2, 100, 100, 1)$ in 20000 simulations

Monte-Carlo Simulations for a European Call Option

```python
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from scipy.stats import norm

def MonteCarloEuropeanCallDelta(r,sigma,S0,K,T,N):
    d = (np.log((S0) / (K)) + (r + (sigma**2 / 2)) * T) / (sigma * np.sqrt(T))
    SAMPLE = []
    PATH = []
    for i in range(N):
        W = np.random.normal(0, np.sqrt(T))
        S = S0*np.exp((r - (sigma**2) / 2)*T + sigma*W)
        V = max(S - K,0)
        SAMPLE.append(np.exp(-r*T)*V*W / (S0*T*sigma))
        PATH.append(sum(SAMPLE) / len(SAMPLE))
    plt.rcParams["font.family"] = "Times New Roman"
    pd.Series(PATH).plot(linewidth = 0.75, color = "b")
    plt.axhline(norm.cdf(d), color = "black", linestyle = "--", linewidth = 0.75)
    plt.xlabel("Number of Simulations")
    plt.ylabel("Estimated Delta")
    plt.legend(["Sample Mean", "Exact Value (" + str(round(norm.cdf(d),4)) + ")"])
    plt.title("Monte-Carlo Simulation of the Delta of a European Call Option")
    plt.rcParams['figure.dpi'] = 300
    plt.rcParams['savefig.dpi'] = 300

MonteCarloEuropeanCallDelta(0.1,0.2,100,100,1,20000)
```

Python implementation of the Monte-Carlo simulation in Figure 1

Monte-Carlo Simulations for a European Call Option

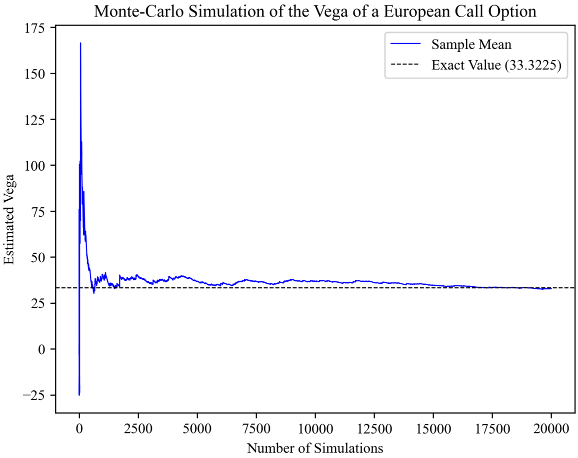

Figure 2: Monte-Carlo simulation of the Vega for a European call option with $(r, \sigma, S_0, K, T) = (0.1, 0.2, 100, 100, 1)$ in 20000 simulations.

Monte-Carlo Simulations for a European Call Option

```python
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from scipy.stats import norm

def MonteCarloEuropeanCallVega(r,sigma,S0,K,T,N):
    d = (np.log((S0) / (K)) + (r + (sigma**2 / 2)) * T) / (sigma * np.sqrt(T))
    SAMPLE = []
    PATH = []
    for i in range(N):
        W = np.random.normal(0, np.sqrt(T))
        S = S0*np.exp((r - (sigma**2) / 2)*T + sigma*W)
        V = max(S - K,0)
        SAMPLE.append(np.exp(-r*T) * V * (((W**2 - T) / (sigma * T)) - T))
        PATH.append(sum(SAMPLE) / len(SAMPLE))
    plt.rcParams["font.family"] = "Times New Roman"
    pd.Series(PATH).plot(linewidth = 0.75, color = "b")
    plt.axhline(S0*np.exp(-(d**2) / 2)*np.sqrt((T) / (2*np.pi)), color = "black", linestyle = "--", linewidth = 0.75)
    plt.xlabel("Number of Simulations")
    plt.ylabel("Estimated Vega")
    plt.legend(["Sample Mean", "Exact Value (" + str(round(S0*np.exp(-(d**2) / 2)*np.sqrt((T) / (2*np.pi)),4)) + ")"])
    plt.title("Monte-Carlo Simulation of the Vega of a European Call Option")
    plt.rcParams['figure.dpi'] = 300
    plt.rcParams['savefig.dpi'] = 300

MonteCarloEuropeanCallVega(0.1,0.2,100,100,1,20000)
```

Python implementation of the Monte-Carlo simulation in Figure 2

Monte-Carlo Simulations for a European Call Option

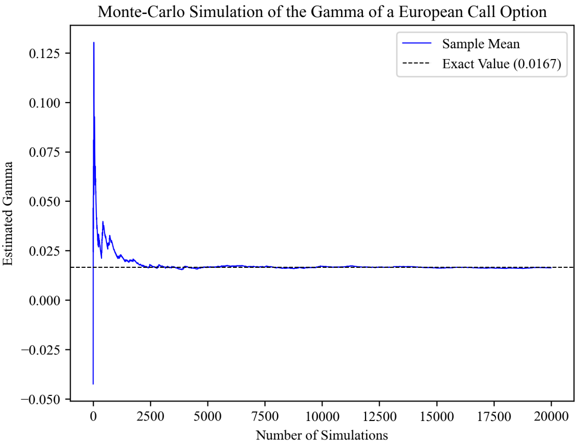

Figure 3: Monte-Carlo Simulation of the Gamma for a European call option with $(r, \sigma, S_0, K, T) = (0.1, 0.2, 100, 100, 1)$ in 20000 simulations.

Monte-Carlo Simulations for a European Call Option

```python
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from scipy.stats import norm

def MonteCarloEuropeanCallGamma(r,sigma,S0,K,T,N):
    d = (np.log((S0) / (K)) + (r + (sigma**2 / 2)) * T) / (sigma * np.sqrt(T))
    SAMPLE = []
    PATH = []
    for i in range(N):
        W = np.random.normal(0, np.sqrt(T))
        S = S0*np.exp((r - (sigma**2) / 2)*T + sigma*W)
        V = max(S - K,0)
        SAMPLE.append(((np.exp(-r*T)) / (sigma * T * S0**2)) * V * (((W**2 - T) / (sigma * T)) - T))
        PATH.append(sum(SAMPLE) / len(SAMPLE))
    end = time.time()
    plt.rcParams["font.family"] = "Times New Roman"
    pd.Series(PATH).plot(linewidth = 0.75, color = "b")
    plt.axhline((np.exp(-(d**2) / 2)) / (S0*sigma*np.sqrt(2*np.pi*T)), color = "black", linestyle = "--", linewidth = 0.75)
    plt.xlabel("Number of Simulations")
    plt.ylabel("Estimated Gamma")
    plt.legend(["Sample Mean", "Exact Value (" + str(round((np.exp(-(d**2) / 2)) / (S0*sigma*np.sqrt(2*np.pi*T)),4)) + ")"])
    plt.title("Monte-Carlo Simulation of the Gamma of a European Call Option")
    plt.rcParams['figure.dpi'] = 300
    plt.rcParams['savefig.dpi'] = 300

MonteCarloEuropeanCallGamma(0.1,0.2,100,100,1,20000)
```

Python implementation of the Monte Carlo simulation in Figure 3

3)e) Monte-Carlo Simulations for an Asian Fixed Strike Call Option

Monte-Carlo Simulations for an Asian Fixed Strike Call Option

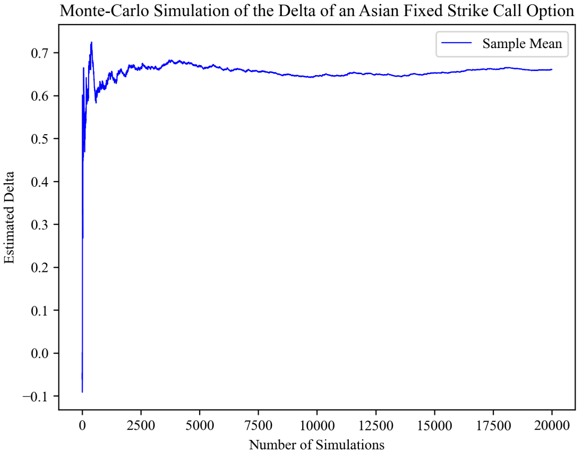

Figure 4: Monte-Carlo Simulation of the Delta for an Asian fixed strike call option with $(r, \sigma, S_0, K, T, M) = (0.1, 0.2, 100, 100, 1, 252)$ in 20000 simulations.

Monte-Carlo Simulations for an Asian Fixed Strike Call Option

```python
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from scipy.stats import norm

def MonteCarloAsianFixedStrikeCallDelta(r,sigma,S0,K,T,M,N):
    dt = T / M
    SAMPLE = []
    PATH = []
    S = np.zeros(M + 1)
    S[0] = S0
    for i in range(N):
        for j in range(1,M+1):
            S[j] = S[j-1]*np.exp((r - (sigma**2)/2)*dt + sigma*np.sqrt(dt)*np.random.normal(0,1))
        STBar = np.mean(S[1:])
        VT = max(STBar - K, 0)
        SAMPLE.append(((np.exp(-r*T)) / (S0*sigma**2)) * (VT*(((2*(S[-1]-S0)))/(STBar)+sigma**2-2*r)))
        PATH.append(sum(SAMPLE) / len(SAMPLE))
    end = time.time()
    plt.rcParams["font.family"] = "Times New Roman"
    pd.Series(PATH).plot(linewidth = 0.75, color = "b")
    plt.xlabel("Number of Simulations")
    plt.ylabel("Estimated Delta")
    plt.legend(["Sample Mean"])
    plt.title("Monte-Carlo Simulation of the Delta of an Asian Fixed Strike Call Option")
    plt.rcParams['figure.dpi'] = 300
    plt.rcParams['savefig.dpi'] = 300

MonteCarloAsianFixedStrikeCallDelta(0.1,0.2,100,100,1,252,20000)
```

Python implementation of the Monte Carlo simulation in Figure 4

Monte-Carlo Simulations for an Asian Fixed Strike Call Option

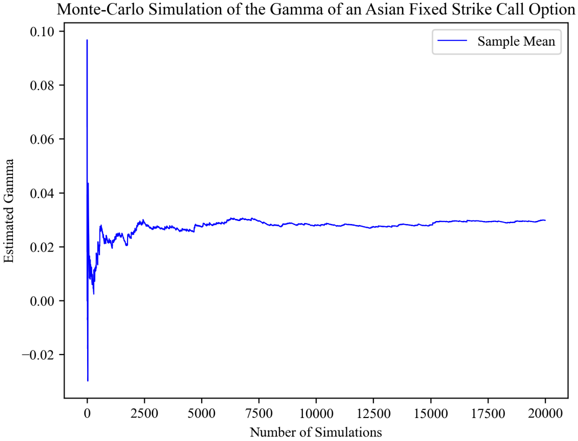

Figure 5: Monte-Carlo Simulation of the Gamma for an Asian fixed strike call option with $(r, \sigma, S_0, K, T, M) = (0.1, 0.2, 100, 100, 1, 252)$ in 20000 simulations.

Monte-Carlo Simulations for an Asian Fixed Strike Call Option

```python
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from scipy.stats import norm

def MonteCarloAsianFixedStrikeCallGamma(r,sigma,S0,K,T,M,N):
    dt = T / M
    SAMPLE = []
    PATH = []
    S = np.zeros(M + 1)
    S[0] = S0
    for i in range(N):
        for j in range(1,M+1):
            S[j] = S[j-1]*np.exp((r - (sigma**2)/2)*dt + sigma*np.sqrt(dt)*np.random.
      normal(0,1))
        STBar = np.mean(S[1:])
        VT = max(STBar - K, 0)
        Delta = ((np.exp(-r*T)) / (S0*sigma**2)) * (VT*(((2*(S[-1]-S0)))/(STBar)+sigma
      **2-2*r))
        SAMPLE.append(((4*np.exp(-r*T)) / ((S0**2) * (sigma**3)))*(VT*((((S[-1] - S0)**2
      - (S[-1]-S0)*r*STBar) / (sigma*(STBar**2))) - ((sigma*S0) / (STBar)))) - ((2*r) / (
      S0* sigma**2))*Delta)
        PATH.append(sum(SAMPLE) / len(SAMPLE))
    plt.rcParams["font.family"] = "Times New Roman"
    pd.Series(PATH).plot(linewidth = 0.75, color = "b")
    plt.xlabel("Number of Simulations")
    plt.ylabel("Estimated Gamma")
    plt.legend(["Sample Mean"])
    plt.title("Monte-Carlo Simulation of the Gamma of an Asian Fixed Strike Call Option")
    plt.rcParams['figure.dpi'] = 300
    plt.rcParams['savefig.dpi'] = 300

MonteCarloAsianFixedStrikeCallGamma(0.1,0.2,100,100,1,252,20000)
```

Python implementation of the Monte Carlo simulation in Figure 5

4) Conclusion

Conclusion

In conclusion:

- The Malliavin Calculus can be applied to calculate the Greeks of certain options by way of the Duality Principle and the Integration by Parts formula. The quantities can then be estimated using the Monte-Carlo simulations.

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