$\mathbf{C}^*$-Algebras

An Elementary Introduction

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1) Fundamentals

1) a) Algebras - Definitions and Properties

Definition 1: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field, let A be a non-empty set, let $+$ and $\times$ be binary operations in $\mathscr{F}(A^2, A)$ and let $\cdot$ be a binary operation in $\mathscr{F}(\mathbb{K} \times A, A)$. The ordered quadruple $(A, +, \cdot, \times)$ is a $\mathbb{K}$-**algebra** if and only if all of the following axioms are satisfied:

1. The ordered triple $(A, +, \times)$ is a ring.
2. The ordered triple $(A, +, \cdot)$ is a $\mathbb{K}$-vector space.
3. $\forall (\lambda, \mu, x, y) \in \mathbb{K} \times \mathbb{K} \times A \times A, \quad (\lambda \cdot x) \times (\mu \cdot y) = (\lambda \times_{\mathbb{K}} \mu) \cdot (x \times y)$

Definition 2: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field and let $(A, +, \cdot, \times)$ be a $\mathbb{K}$-algebra. The **additive identity** of A is the element of A denoted by 0_A and defined by:

$$\forall x \in A, \quad 0_A + x = x + 0_A = x$$

Definition 3: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field. A $\mathbb{K}$-**algebra** $(A, +, \cdot, \times)$ is:

- **commutative** (or **Abelian**) if and only if the following condition is satisfied:

$$\boxed{\forall (x,y) \in A^2, \quad x \times y = y \times x}$$

- **unital** if and only if the following condition is satisfied:

$$\boxed{\exists e \in A \setminus \{0_A\} : \forall x \in A, \quad e \times x = x \times e = x}$$

- **a division** $\mathbb{K}$-**algebra** if and only if the following condition is satisfied:

$$\boxed{\forall (x,y) \in A^2, \quad x \times y = 0_A \implies 0_A \in \{x, y\}}$$

PROPOSITION 1: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field and let $(A, +, \cdot, \times)$ be a unital $\mathbb{K}$-algebra. If there exists an ordered pair (e, f) in $\left(A \setminus \{0_A\}\right)^2$ such that, for every ordered pair (x, y) in A^2, one has $e \times x = x \times e = x$ and $f \times y = y \times f = y$, then:

$$\boxed{e = f}$$

DEFINITION 4: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field and let $(A, +, \cdot, \times)$ be a unital $\mathbb{K}$-algebra. The **multiplicative identity** of $A \setminus \{0_A\}$ is the element of A denoted by 1_A and defined by:

$$\boxed{\forall x \in A, \quad 1_A \times x = x \times 1_A = x}$$

DEFINITION 5: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field, let $(A, +, \cdot, \times)$ be a unital $\mathbb{K}$-algebra and let (x, y) be an ordered pair in A^2.

- The element x of A is **invertible** if and only if the following condition is satisfied:

$$\boxed{\exists z \in A: \quad x \times z = z \times x = 1_A}$$

- The element y of A is a **multiplicative inverse** of x if and only if the following condition is satisfied:

$$\boxed{x \times y = y \times x = 1_A}$$

- The **set of invertible elements** of A is the set denoted by A^* and defined by:

$$\boxed{A^* = \left\{x \in A: \quad \exists z \in A: \quad x \times z = z \times x = 1_A\right\}}$$

PROPOSITION 2: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field, let $(A, +, \cdot, \times)$ be a unital $\mathbb{K}$-algebra, and let (x, y, z) be an ordered triple in $A^* \times A \times A$. If y and z are multiplicative inverses of x, then:

$$\boxed{y = z}$$

DEFINITION 6: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field, let $(A, +, \cdot, \times)$ be a unital $\mathbb{K}$-algebra and let x be an element of A^*. The **multiplicative inverse** of x is the element denoted by x^{-1} and defined by:

$$\boxed{x \times x^{-1} = x^{-1} \times x = 1_A}$$

1) a) Algebras - Subalgebras and Ideals

DEFINITION 7: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field, let $(A, +, \cdot, \times)$ be a $\mathbb{K}$-algebra and let B be a subset of A. The set B is a $\mathbb{K}$-**subalgebra** of $(A, +, \cdot, \times)$ if and only if all of the following conditions are satisfied:

1. The set B is a subring of $(A, +, \times)$.
2. The set B is a $\mathbb{K}$-vector subspace of $(A, +, \cdot)$.

Definition 8: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field, let $(A, +, \cdot, \times)$ be a $\mathbb{K}$-algebra and let I be a $\mathbb{K}$-vector subspace of $(A, +, \cdot, \times)$.

- The set I is a **left-ideal** of A if and only if the following condition is satisfied:

$$\boxed{\forall (a, b) \in A \times I, \quad x \times y \in I}$$

- The set I is a **right-ideal** of A if and only if the following condition is satisfied:

$$\boxed{\forall (a, b) \in A \times I, \quad y \times x \in I}$$

- The set I is a **bilateral ideal** (or **ideal**) of A if and only if the following condition is satisfied:

$$\boxed{\forall (a, b) \in A \times I, \quad \left(x \times y, y \times x\right) \in I}$$

DEFINITION 9: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field, let $(A, +, \cdot, \times)$ be a $\mathbb{K}$-algebra and let I be an ideal of A.

- The ideal I is **proper** if and only if the following condition is satisfied:

$$\boxed{I \neq A}$$

- The ideal I is **maximal** if and only if the following conditions are satisfied:

1. The ideal I is proper.
2. If J is a proper ideal of A, then $J \subseteq I$.

The **set of maximal ideals** of A is denoted by $\mathrm{M}(A)$.

DEFINITION 10: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field, let $(A, +, \cdot, \times)$ be a $\mathbb{K}$-algebra, let I be an ideal of A and let x be an element of A.

- The **coset** of x **modulo** I is the set denoted by $[x]_I$ and defined by:

$$\boxed{[x]_I = \{x + y \in A : \quad y \in I\}}$$

- The **quotient mapping modulo** I is function denoted by $[\cdot]_I$ and defined by:

$$\boxed{[\cdot]_I : \quad A \longrightarrow A/I; \quad x \longmapsto [x]_I}$$

- The **quotient algebra** of A **modulo** I is the set denoted by A/I and defined by:

$$\boxed{A/I = \{[x]_I : \quad x \in I\} = [A]_I}$$

PROPOSITION 3: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field, let $(A, +, \cdot, \times)$ be a $\mathbb{K}$-algebra, let I be an ideal of A and let $+_I$, $\cdot_I$ and $\times_I$ be the functions defined by:

$$\begin{array}{lcclccc} +_I : & A/I \times A/I & \longrightarrow & A/I; & \left([x]_I, [y]_I\right) & \longmapsto & [x+y]_I \\ \cdot_I : & \mathbb{K} \times A/I & \longrightarrow & A/I; & \left(\lambda, [x]_I\right) & \longmapsto & [\lambda \cdot x]_I \\ \times_I : & A/I \times A/I & \longrightarrow & A/I; & \left([x]_I, [y]_I\right) & \longmapsto & [x \times y]_I \end{array}$$

Then the ordered quadruple $\left(A/I, +_I, \cdot_I, \times_I\right)$ is a $\mathbb{K}$-algebra.

PROPOSITION 4: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field, let $(A, +, \cdot, \times)$ be a $\mathbb{K}$-algebra, let I be an ideal of A. Then I is maximal if and only if the ring $\left(A/I, +_I, \times_I\right)$ is a field.

1) b) Normed Algebras - Definitons and Properties

Definition 11: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times)$ be a $\mathbb{F}$-algebra and let $\|\cdot\|$ be a function in $\mathscr{F}(A, \mathbb{R}_+)$.

- The function $\|\cdot\|$ is a **norm** on $(A, +, \cdot, \times)$ if and only if all of the following conditions are satisfied:

  1. The ordered quadruple $(A, +, \cdot, \|\cdot\|)$ is a normed $\mathbb{F}$-vector space.
  2. $\forall (x, y) \in A^2, \quad \|x \times y\| \leqslant \|x\| \|y\|$

- The ordered quintuple $(A, +, \cdot, \times, \|\cdot\|)$ is a **normed $\mathbb{F}$-algebra** if and only if the function $\|\cdot\|$ is a norm on $(A, +, \cdot, \times)$.

- The ordered quintuple $(A, +, \cdot, \times, \|\cdot\|)$ is a **unital normed $\mathbb{F}$-algebra** if and only if the function $\|\cdot\|$ is a norm on $(A, +, \cdot, \times)$, the $\mathbb{F}$-algebra $(A, +, \cdot, \times)$ is unital and the following condition is satisfied:

$$\boxed{\|1_A\| = 1}$$

DEFINITION 12: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $(A, +, \cdot, \times, \|\cdot\|)$ be a normed $\mathbb{F}$-algebra. The **induced metric** of $\|\cdot\|$ is the function denoted by d_A and defined by:

$$d_A : \left| \begin{array}{ccc} A \times A & \longrightarrow & \mathbb{R}_+ \\ (x, y) & \longmapsto & \|y - x\| \end{array} \right.$$

THEOREM 1: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $(A, +, \cdot, \times, \|\cdot\|)$ be a normed $\mathbb{F}$-algebra. Then the ordered pair (A, d_A) is a metric space.

DEFINITION 13: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $(A, +, \cdot, \times, \|\cdot\|)$ be a normed $\mathbb{F}$-algebra.

- The normed $\mathbb{F}$-algebra $(A, +, \cdot, \times, \|\cdot\|)$ is a **Banach $\mathbb{F}$-algebra** if and only if the ordered pair (A, d_A) is a complete metric space.

DEFINITION 14: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $(A, +, \cdot, \times, \|\cdot\|)$ be a Banach $\mathbb{F}$-algebra.

- The Banach $\mathbb{F}$-algebra $(A, +, \cdot, \times, \|\cdot\|)$ is a **unital** if and only if the $\mathbb{F}$-algebra $(A, +, \cdot, \times)$ is unital.
- The Banach $\mathbb{F}$-algebra $(A, +, \cdot, \times, \|\cdot\|)$ is a **Banach division $\mathbb{F}$-algebra** if and only if the $\mathbb{F}$-algebra $(A, +, \cdot, \times)$ is a division $\mathbb{F}$-algebra.

DEFINITION 15: Let n be a natural number in $\mathbb{N}$, Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times)$ be a unital $\mathbb{F}$-algebra and let x be an element of A. The ***n*-th power** of x is the element of A denoted x^n and defined by:

$$\boxed{x^n = \begin{cases} 1_A, & \text{if } n = 0. \\ x \times x^{n-1}, & \text{if } n > 0. \end{cases}}$$

THEOREM 2 (Neumann Series): Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times, \|\cdot\|)$ be a unital Banach $\mathbb{F}$-algebra and let x be an element of A such that $\|x\| < 1$ and $1_A - x \in A^*$. Then:

$$\boxed{(1_A - x)^{-1} = \sum_{k=0}^{+\infty} x^n}$$

PROPOSITION 5: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times, \|\cdot\|)$ be a unital normed $\mathbb{F}$-algebra. Then:

1. The set A^* is d_A-open in A.
2. The function $A^* \to A^*$; $x \mapsto x^{-1}$ is a homeomorphism (with respect to d_A).

PROPOSITION 6: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times, \|\cdot\|)$ be a unital Banach $\mathbb{F}$-algebra and let I be a subset of A. Then:

1. If I is a proper ideal of A, then so is $\operatorname{clos}_{d_A}(I)$.
2. If I is a maximal ideal of A, then I is closed.

THEOREM 3: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times, \|\cdot\|)$ be a Banach $\mathbb{F}$-algebra, let I be a closed ideal of A and let $\|\cdot\|_I$ be the function defined by:

$$\|\cdot\|_I: \quad A/I \quad \longrightarrow \quad \mathbb{R}_+; \quad [x]_I \quad \longmapsto \quad \inf_{y \in I}\left(\|x+y\|\right)$$

Then:

1. The ordered quintuple $(A/I, +_I, \cdot_I, \times_I, \|\cdot\|_I)$ is a Banach $\mathbb{F}$-algebra.
2. If $(A, +, \cdot, \times)$ is unital, then so is $(A/I, +_I, \cdot_I, \times_I)$ and $1_{A/I} = [1_A]_I$.

1) b) Normed Algebras - The Spectrum

DEFINITION 16: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times)$ be a unital $\mathbb{F}$-algebra and let x be an element of A.

- The **spectrum** of x is the set denoted by $\sigma(x)$ and defined by:

$$\boxed{\sigma(x) = \left\{\lambda \in \mathbb{F} : \quad x - \lambda \cdot 1_A \notin A^*\right\}}$$

- The **resolvent set** of x is the set denoted by $\rho(x)$ and defined by:

$$\boxed{\rho(x) = \mathbb{F} \setminus \sigma(x)}$$

DEFINITION 17: Let U be an open set in $\mathbb{C}$, let $(A, +, \cdot, \times, \|\cdot\|)$ be a Banach $\mathbb{C}$-algebra and let f be a function in $\mathscr{F}(U, A)$. The function f is **holomorphic** if and only if, for every complex number z in U, the limit:

$$\boxed{\lim_{w \to z} \left(\frac{f(w) - f(z)}{w - z} \right)}$$

exists in A.

THEOREM 4: Let $(A, +, \cdot, \times, \|\cdot\|)$ be a unital Banach $\mathbb{C}$-algebra and let x be an element of A. Then the function f defined by:

$$\boxed{f : \left| \begin{array}{ccc} \rho(x) & \longrightarrow & A^* \\ z & \longmapsto & (x - z \cdot 1_A)^{-1} \end{array} \right.}$$

is holomorphic.

THEOREM 5: Let $(A, +, \cdot, \times, \|\cdot\|)$ be a unital Banach $\mathbb{C}$-algebra and let x be an element of A. Then $\sigma(x)$ is non-empty and compact.

DEFINITION 18: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times, \|\cdot\|)$ be a unital Banach $\mathbb{F}$-algebra and let x be an element of A. The **spectral radius** of x is the positive real number denoted by $r(x)$ and defined by:

$$\boxed{r(x) = \sup_{\lambda \in \sigma(x)} (|\lambda|)}$$

THEOREM 6 (Spectral radius formula): Let $(A,+,\cdot,\times,\|\cdot\|)$ be a unital Banach $\mathbb{C}$-algebra and let x be an element of A. Then:

$$\boxed{r(x)=\lim_{n\to+\infty}\left(\sqrt[n]{\|x^n\|}\right)=\inf_{n\in\mathbb{N}}\left(\sqrt[n]{\|x^n\|}\right)}$$

COROLLARY 1: Let $(A,+,\cdot,\times,\|\cdot\|)$ be a unital Banach $\mathbb{C}$-algebra and let x be an element of A. Then:

$$\boxed{\lim_{n\to+\infty}\left(\|x^n\|\right)=0 \iff r(x)<1}$$

THEOREM 7 (Spectral mapping theorem): Let $(A,+,\cdot,\times,\|\cdot\|)$ be a unital Banach $\mathbb{C}$-algebra, let x be an element of A and let p be a complex polynomial. Then:

$$\boxed{\sigma\left(p(x)\right)=p(\sigma(x))}$$

Theorem 8 (Gelfand-Mazur theorem): Let $(A,+,\cdot,\times,\|\cdot\|)$ be a Banach division $\mathbb{C}$-algebra. Then:

$$\boxed{A \cong \mathbb{C}}$$

Theorem 9 (Gelfand-Mazur-Kaplansky theorem): Let $(A,+,\cdot,\times,\|\cdot\|)$ be a Banach division $\mathbb{R}$-algebra. Then exactly one of the following propositions holds true:

1. $\boxed{A \cong \mathbb{R}}$
2. $\boxed{A \cong \mathbb{C}}$
3. $\boxed{A \cong \mathbb{H}}$

THEOREM 10: Let $(A,+,\cdot,\times,\|\cdot\|)$ be a unital Banach $\mathbb{C}$-algebra and let λ be a complex number in $\mathbb{C}^*$. Then:

$$\boxed{\forall (x,y)\in A^2,\ \ x\times y - y\times x \neq \lambda\cdot 1_A}$$

1) b) Normed Algebras - The Exponential

Definition 19: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times, \|\cdot\|)$ be a unital Banach $\mathbb{F}$-algebra and let x be an element of A. The **exponential** of x is the element of A denoted by $\exp(x)$ and defined by:

$$\boxed{\exp(x) = \sum_{n=0}^{+\infty} \frac{1}{n!} \cdot x^n}$$

Proposition 7: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times, \|\cdot\|)$ be a unital Banach $\mathbb{F}$-algebra and let (x, y) be an ordered pair in A^2. Then:

1. $\boxed{\|\exp(x)\| \leqslant e^{\|x\|}}$
2. $\boxed{x \times y = y \times x \implies \exp(x+y) = \exp(x)\exp(y)}$
3. $\boxed{\exp(x) \in A^*}$
4. $\boxed{\exp(x)^{-1} = \exp(-x)}$

COROLLARY 2: Let $(A, +, \cdot, \times, \|\cdot\|)$ be a non-commutative unital Banach $\mathbb{C}$-algebra. Then there exists an ordered pair (x, y) in $A \times A^*$ such that:

$$\boxed{\|x\| \neq \left\|y \times x \times y^{-1}\right\|}$$

1) b) Normed Algebras - Homomorphisms and Characters

DEFINITION 20: Let $(\mathbb{K},+_{\mathbb{K}},\times_{\mathbb{K}})$ be a field, let $(A,+_A,\cdot_A,\times_A)$ and $(B,+_B,\cdot_B,\times_B)$ be $\mathbb{K}$-algebras and let φ be a function in $\mathscr{F}(A,B)$.

- The function φ is a $\mathbb{K}$**-algebra homomorphism** of A into B if and only if all of the following conditions are satisfied:
  1. The function φ is a $\mathbb{K}$-linear mapping of $(A,+_A,\cdot_A)$ into $(B,+_B,\cdot_B)$.
  2. The function φ is a ring homomorphism of $(A,+_A,\times_A)$ into $(B,+_B,\times_B)$.
- The function φ is a $\mathbb{K}$**-algebra isomorphism** of A onto B if and only if φ is a bijective $\mathbb{K}$-algebra homomorphism of A into B.

PROPOSITION 8: Let $(\mathbb{K},+_{\mathbb{K}},\times_{\mathbb{K}})$ be a field, let $(A,+_A,\cdot_A,\times_A)$ and $(B,+_B,\cdot_B,\times_B)$ be $\mathbb{K}$-algebras and let φ be a $\mathbb{K}$-algebra homomorphism of A into B. Then:

1. The set $\operatorname{im}(\varphi)$ is a $\mathbb{K}$-subalgebra of $(B,+_B,\cdot_B,\times_B)$.
2. The set $\ker(\varphi)$ is an ideal of A.

Definition 21: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field, let $(A, +_A, \cdot_A, \times_A)$ and $(B, +_B, \cdot_B, \times_B)$ be unital $\mathbb{K}$-algebras and let φ be a $\mathbb{K}$-algebra homomorphism of A into B. The $\mathbb{K}$-algebra homomorphism φ is a **unital** if and only if the following condition is satisfied:

$$\boxed{\varphi(1_A) = 1_B}$$

Proposition 9: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field, let $(A, +_A, \cdot_A, \times_A)$ and $(B, +_B, \cdot_B, \times_B)$ be unital $\mathbb{K}$-algebras, let φ be a unital $\mathbb{K}$-algebra homomorphism of A into B and let x be an element of A^*. Then:

1. $\boxed{\varphi(x) \in B^*}$
2. $\boxed{\varphi\left(x^{-1}\right) = \varphi(x)^{-1}}$

Proposition 10: Let $(\mathbb{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ be a field, let $(A, +, \cdot, \times)$ be a $\mathbb{K}$-algebra and let I be an ideal of A.

1. Then the quotient mapping modulo I $[\cdot]_I$ is a $\mathbb{K}$-algebra homomorphism of A into A/I.
2. If $(A, +, \cdot, \times)$ is unital, then so is $[\cdot]_I$.

DEFINITION 22: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $(A, +_A, \cdot_A, \times_A, \|\cdot\|_A)$ and $(B, +_B, \cdot_B, \times_B, \|\cdot\|_B)$ be Banach $\mathbb{F}$-algebras and let φ be a $\mathbb{F}$-algebra homomorphism of A into B.

- The **operator norm** of φ is the positive real number denoted by $\|\varphi\|$ and defined by:

$$\boxed{\|\varphi\| = \sup_{\substack{x \in A \\ \|x\| \leqslant 1}} \left(\|\varphi(x)\|_B\right)}$$

  if it exists in $\mathbb{R}_+$ (in which case φ is **bounded**).

- The function φ is an **isometry** of A into B if and only if the following condition is satisfied:

$$\boxed{\forall x \in A, \quad \|\varphi(x)\|_B = \|x\|_A}$$

- The function φ is an **isometric isomorphism** of A onto B if and only if φ is a bijective isometry of A into B.

PROPOSITION 11: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times, \|\cdot\|)$ be a Banach $\mathbb{F}$-algebra and let I be an ideal of A. Then:

$$\boxed{\|[\cdot]_I\| \leqslant 1}$$

Definition 23: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times, \|\cdot\|)$ be a Banach $\mathbb{F}$-algebra and let χ be a $\mathbb{K}$-algebra homomorphism of A into $\mathbb{F}$. The $\mathbb{K}$-algebra homomorphism χ is a **character** of A if and only if the following condition is satisfied:

$$\boxed{\chi(A) \neq \{0_A\}}$$

The **Gelfand spectrum** of A is the set of all characters of A, denoted by $\Sigma(A)$.

Proposition 12: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times, \|\cdot\|)$ be a unital Banach $\mathbb{F}$-algebra, let χ be character of A and let x be an element of A^*. Then:

1. $\boxed{\chi(1_A) = 1}$
2. $\boxed{\chi(x) \neq 0}$
3. $\boxed{\chi\left(x^{-1}\right) \neq \dfrac{1}{\chi(x)}}$

THEOREM 11: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times, \|\cdot\|)$ be a commutative Banach $\mathbb{F}$-algebra and let χ be a character of A. Then χ is bounded and:

$$\boxed{\|\chi\| = 1}$$

THEOREM 12: Let $(A, +, \cdot, \times, \|\cdot\|)$ be a commutative and unital Banach $\mathbb{C}$-algebra. Then the function κ defined by:

$$\boxed{\kappa: \quad \Sigma(A) \quad \longrightarrow \quad \mathrm{M}(A); \quad \chi \quad \longmapsto \quad \ker(\chi)}$$

is a bijection.

THEOREM 13: Let $(A, +, \cdot, \times, \|\cdot\|)$ be a commutative and unital Banach $\mathbb{C}$-algebra and let x be an element of A. Then:

$$\boxed{\sigma(x) = \left\{\chi(x) \in A: \quad \chi \in \Sigma(A)\right\}}$$

Definition 24: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times, \|\cdot\|)$ be a commutative Banach $\mathbb{F}$-algebra and let x be an element of A.

- The **Gelfand transform** of x is the function denoted by $\hat{x}$ and defined by:

$$\boxed{\hat{x}: \quad \Sigma(A) \longrightarrow \mathbb{F}; \quad \chi \longmapsto \chi(x)}$$

- The **Gelfand transform** of A is the function denoted by γ_A and defined by:

$$\boxed{\gamma_A: \quad A \longrightarrow \mathscr{F}(\Sigma(A), \mathbb{F}); \quad x \longmapsto \hat{x}}$$

THEOREM 14 (Gelfand representation theorem): Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times, \|\cdot\|)$ be a commutative Banach $\mathbb{F}$-algebra and let x be an element of A. Then:

1. In the weak-* topology of $\Sigma(A)$ inherited from $A^\vee$, one has:

$$\boxed{\gamma_A(x) \in \mathscr{C}_0\left(\Sigma(A)\right)}$$

2. $\boxed{\gamma \text{ is a continuous homomorphism of } A \text{ into } \mathscr{C}_0\left(\Sigma(A)\right).}$

3. $\boxed{\|\gamma\| \leqslant 1}$

4. $\boxed{\|\hat{x}\| = r(x)}$

5. If $\mathbb{F} = \mathbb{C}$ and $(A, +, \cdot, \times, \|\cdot\|)$ is unital, then:

$$\boxed{\ker(\gamma) = \bigcap_{I \in \mathrm{M}(A)} I}$$

THEOREM 15 (Gleason-Kahane-Żelazko theorem): Let $(A, +, \cdot, \times, \|\cdot\|)$ be a unital Banach $\mathbb{C}$-algebra and let φ be a $\mathbb{C}$-linear functional on A. Then:

$$\boxed{\varphi \in \Sigma(A) \iff \begin{cases} \varphi(1_A) = 1 \\ 0 \notin \varphi\left(A^*\right) \end{cases}}$$

1) c) Involutive Algebras

Definition 25: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times)$ be a $\mathbb{F}$-algebra and let $*$ be a unary operation in $\mathscr{F}(A, A)$ (also denoted by *).

- The function $*$ is an **involution** on $(A, +, \cdot, \times)$ if and only if all of the following axioms are satisfied:

  1. $\boxed{\forall x \in A,\ \left(x^*\right)^* = x}$
  2. $\boxed{\forall (x, y) \in A^2,\ (x+y)^* = x^* + y^*}$
  3. $\boxed{\forall (\lambda, x) \in \mathbb{F} \times A,\ (\lambda \cdot x)^* = \overline{\lambda} \cdot x}$
  4. $\boxed{\forall (x, y) \in A^2,\ (x \times y)^* = y^* \times x^*}$

- The ordered quintuple $(A, +, \cdot, \times, *)$ is a ***-$\mathbb{F}$-algebra** (or **involutive $\mathbb{F}$-algebra**) if and only if the function $*$ is an involution on $(A, +, \cdot, \times)$.

PROPOSITION 13: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $(A, +, \cdot, \times, *)$ be a *-$\mathbb{F}$-algebra. If $(A, +, \cdot, \times)$ is unital, then:

$$\boxed{1_A^* = 1_A}$$

DEFINITION 26: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times, \|\cdot\|)$ be a Banach $\mathbb{F}$-algebra and let $*$ be a unary operation in $\mathscr{F}(A, A)$.

- The ordered sextuple $(A, +, \cdot, \times, \|\cdot\|, *)$ is a **normed $*$-$\mathbb{F}$-algebra** if and only if the function $*$ is an involution on $(A, +, \cdot, \times)$.
- The ordered sextuple $(A, +, \cdot, \times, \|\cdot\|, *)$ is a **Banach $*$-$\mathbb{F}$-algebra** if and only if the ordered quintuple $(A, +, \cdot, \times, \|\cdot\|)$ is a Banach $\mathbb{F}$-algebra and the function $*$ is an involution on $(A, +, \cdot, \times)$

2) **C***-Algebras

2) a) Definitions and Examples

DEFINITION 27: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $(A, +, \cdot, \times, \|\cdot\|, *)$ be a Banach $*$-$\mathbb{F}$-algebra. The Banach $*$-$\mathbb{F}$-algebra $(A, +, \cdot, \times, \|\cdot\|, *)$ is a **$\mathbf{C}^*$-algebra** if and only if the following condition is satisfied:

$$\boxed{\forall x \in A, \quad \left\|x^* \times x\right\| = \|x\|^2}$$

PROPOSITION 14: Let $(A, +, \cdot, \times, \|\cdot\|, *)$ be a $\mathbf{C}^*$-algebra and let x be an element of A. Then:

1. $\boxed{\left\|x^*\right\| = \|x\|}$
2. The function $*$ is a homeomorphism.
3. $$\|x\| = \sup_{\substack{y \in A \\ \|y\| \leqslant 1}} \left(\left\|x \times y\right\|\right) = \sup_{\substack{y \in A \\ \|y\| \leqslant 1}} \left(\left\|y \times x\right\|\right)$$
4. If $(A, +, \cdot, \times)$ is unital, then $\|1_A\| = 1$.

Example 1: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Then all of the following are $\mathbf{C}^*$-algebras:

1. $(\mathbb{F}, +, \cdot, \times, |\cdot|, \bar{\cdot})$.
2. $(\mathbb{F}^n, +, \cdot, \times, \|\cdot\|, \bar{\cdot})$, for any non-zero natural number n.
3. $\left(\mathscr{M}_{n,n}(\mathbb{F}), +, \cdot, \times, \sqrt{\operatorname{tr}\left(\cdot^{\mathsf{H}} \times \cdot\right)}, \cdot^{\mathsf{H}}\right)$ for any non-zero natural number n.
4. $\left(\mathscr{C}\left([a,b], \mathbb{F}\right), +, \cdot, \times, \sup_{[a,b]}\left(|\cdot|\right), \bar{\cdot}\right)$.
5. $\left(\mathrm{L}^\infty(X, \mathscr{A}, \mu), +, \cdot, \times, \sup_X\left(|\cdot|\right), \bar{\cdot}\right)$ for any measure space $(X, \mathscr{A}, \mu)$.
6. $\left(\mathscr{C}_0\left(X, \mathbb{F}\right), +, \cdot, \times, \sup_X\left(|\cdot|\right), \bar{\cdot}\right)$ for any locally compact Hausdorff topological space (X, τ).
   $\left(\mathscr{B}(V), +, \cdot, \circ, \sup\limits_{\substack{x \in V \\ \|x\| \leqslant 1}}\left(\|\cdot(x)\|\right), *\right)$ for any $\mathbb{F}$-Hilbert space $(V, +, \cdot, \langle\cdot,\cdot\rangle)$.

DEFINITION 28: Let $(A,+,\cdot,\times,\|\cdot\|,*)$ be a $\mathbf{C}^*$-algebra and let x be an element of A.

- The element x is **Hermitian** (or **self-adjoint**) if and only if the following condition is satisfied:

$$\boxed{x = x^*}$$

- The element x is **normal** if and only if the following condition is satisfied:

$$\boxed{x \times x^* = x^* \times x}$$

- The element x is **unitary** if and only if $(A,+,\cdot,\times)$ is unital and the following condition is satisfied:

$$\boxed{x \times x^* = x^* \times x = 1_A}$$

THEOREM 16: Let $(A,+,\cdot,\times,\|\cdot\|,*)$ be a $\mathbf{C}^*$-algebra and let x be an element of A. If x is normal, then:

$$\boxed{\|x\| = r(x)}$$

COROLLARY 3: Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(A, +, \cdot, \times, *)$ be a *-$\mathbb{F}$-algebra and let $\|\cdot\|$ be a norm on $(A, +.\cdot, \times)$ such that $(A, +, \cdot, \times, \|\cdot\|, *)$ is a $\mathbf{C}^*$-algebra. Then:

$$\boxed{\forall x \in A, \quad \|x\| = \sqrt{r(x^* \times x)}}$$

2) b) The Gelfand-Naimark Theorem

DEFINITION 29: Let $(A, +_A, \cdot_A, \times_A, \|\cdot\|_A, *_A)$ and $(B, +_B, \cdot_B, \times_B, \|\cdot\|_B, *_B)$ and let φ be a function in $\mathscr{F}(A, B)$.

- The function φ is a $\mathbf{C^*}$ **-homomorphism** of A into B if and only if φ is an isometry and a homomorphism of A into B and the following condition is satisfied:

$$\boxed{\forall x \in A,\ \ \varphi(x^{*_A}) = \varphi(x)^{*_B}}$$

- The function φ is a $\mathbf{C^*}$ **-isomomorphism** of A onto B if and only if φ is a bijective $\mathbf{C^*}$ -homomorphism of A into B.

THEOREM 17: Let $(A, +_A, \cdot_A, \times_A, \|\cdot\|_A, *_A)$ and $(B, +_B, \cdot_B, \times_B, \|\cdot\|_B, *_B)$ and let φ be a $\mathbf{C^*}$ -homomorphism of A into B. Then φ is bounded and:

$$\boxed{\|\varphi\| \leqslant 1}$$

PROPOSITION 15: Let $(A, +, \cdot, \times, \|\cdot\|, *)$ be a $\mathbf{C}^*$-algebra and let (x, χ) be an ordered pair in $A \times \Sigma(A)$. Then:

1. $\boxed{x = x^* \implies \chi(x) \in \mathbb{R}}$
2. $\boxed{\chi(x^* \times x) \in \mathbb{R}_{\geqslant 0}}$
3. If $(A, +, \cdot, \times)$ is unital and x is unitary, then $\left|\chi(x)\right| = 1$.

THEOREM 18: Let $(A, +, \cdot, \times, \|\cdot\|, *)$ be a $\mathbf{C}^*$-algebra and let x be an element of A. Then:

1. $\boxed{x = x^* \implies \sigma(x) \subseteq \mathbb{R}}$
2. $\boxed{\sigma\left(x^* \times x\right) \subseteq \mathbb{R}_{\geqslant 0}}$
3. If $(A, +, \cdot, \times)$ is unital and x is unitary, then $\sigma(x) \subseteq \partial\mathbb{D}$.

THEOREM 19: Let $(A,+,\cdot,\times,\|\cdot\|,*)$ be a unital $\mathbf{C}^*$-algebra (over the field $\mathbb{C}$) and let x be an element of A. Then:

1. $\boxed{\exp(x^*) = \exp(x)^*}$
2. If x is Hermitian, then $\exp(i\cdot x)$ is unitary.

THEOREM 20: Let $(A,+,\cdot,\times,\|\cdot\|,*)$ be a commutative $\mathbf{C}^*$-algebra. Then the Gelfand transform γ_A of A is a $\mathbf{C}^*$-isomorphism of A onto $\mathscr{C}_0(\Sigma(A))$:

$$\boxed{A \cong \mathscr{C}_0(\Sigma(A))}$$

THEOREM 21 (Gelfand-Naimark theorem): For every commutative $\mathbf{C}^*$-algebra $(A,+,\cdot,\times,\|\cdot\|,*)$, there exists a locally compact Hausdorff topological space (X,τ) and a $\mathbf{C}^*$-isomorphism of A onto X:

$$\boxed{A \cong X}$$

2) c) Positive Elements and Positive Linear Functionals

Definition 30: Let $(A, +, \cdot, \times, \|\cdot\|, *)$ be a $\mathbf{C}^*$-algebra and let x be an element of A. The element x of A is **positive** if and only if x is Hermitian and the following condition is satisfied:

$$\boxed{\sigma(x) \subseteq \mathbb{R}_{\geqslant 0}}$$

.

Proposition 16: Let $(A, +, \cdot, \times, \|\cdot\|, *)$ be a $\mathbf{C}^*$ algebra and let x be an element of A. The the following are equivalent:

1. x is positive.
2. $\exists y \in A: \quad (x, y) = \left(y^2, y^*\right)$
3. $\exists y \in A: \quad x = y^* \times y$

DEFINITION 31: Let $(A,+,\cdot,\times,\|\cdot\|,*)$ be a $\mathbf{C}^*$ algebra and let φ be a linear functional in $A^\vee$.

- The functional φ is **positive** if and only if the following condition is satisfied:

$$\boxed{\forall x \in A,\ \ \varphi(x^* \times x) \geqslant 0}$$

- The functional φ is a **state** on A if and only if the following condition is satisfied:

$$\boxed{\|\varphi\| = 1}$$

PROPOSITION 17: Let $(A,+,\cdot,\times,\|\cdot\|,*)$ be a $\mathbf{C}^*$ algebra and let x be a normal element of A. Then:

1. If $(A,+,\cdot,\times)$ is commutative, then every character χ of A is a state on A.
2. For every complex number λ in $\sigma(x)$, there exists a state φ on A such that $\varphi(x) = \lambda$.

THEOREM 22 (Cauchy-Schwarz inequality): Let $(A, +, \cdot, \times, \|\cdot\|, *)$ be a $\mathbf{C}^*$-algebra, let φ be a positive linear functional in $A^{\vee}$ and let (x, y) be an ordered pair in A^2. Then:

$$\boxed{\left|\varphi\left(y^* \times x\right)\right|^2 \leqslant \varphi\left(x^* \times x\right)\varphi\left(y^* \times y\right)}$$

2) d) The Gelfand-Naimark-Segal Theorem

THEOREM 23 (Gelfand-Naimark-Segal Theorem): For every $\mathbf{C}^*$-algebra $(A, +, \cdot, \times, \|\cdot\|, *)$, there exists a Hilbert space $(V, +, \cdot, \langle \cdot, \cdot \rangle)$, a closed $\mathbf{C}^*$-subalgebra $\mathscr{L}$ of $\mathscr{B}(V)$ and a $\mathbf{C}^*$-isomorphism of A onto $\mathscr{L}$:

$$\boxed{A \cong \mathscr{L} \subseteq \mathscr{B}(V)}$$

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