Differential Geometry of Curves

An Elementary Introduction

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1) a) Parametrized Curves

DEFINITION 1 (Parametrized curve): Let n be a non-zero natural number in $\mathbb{N}_{\geqslant 1}$ and let $]a, b[$ be a non-degenerate open interval of $\mathbb{R}$.

- A **parametrized curve of** $]a, b[$ **into** $\mathbb{R}^n$ (or **parametrized curve**)is a continuous function $\vec{\gamma}$ in $\mathscr{C}\left(]a, b[, \mathbb{R}^n\right)$.
- A **differentiable curve of** $]a, b[$ **into** $\mathbb{R}^n$ (or **differentiable curve**) is a differentiable parametrized curve $\vec{\gamma}$ in $\mathscr{C}\left(]a, b[, \mathbb{R}^n\right)$.

DEFINITION 2 (Regular curve): Let n be a non-zero natural number in $\mathbb{N}_{\geqslant 1}$, let $]a, b[$ be a non-degenerate open interval of $\mathbb{R}$, let t be a real number in $]a, b[$ and let $\vec{\gamma}$ be a differentiable curve in $\mathscr{C}\left(]a, b[, \mathbb{R}^n\right)$.

- The differentiable curve $\vec{\gamma}$ is **regular at** t if and only if $\vec{\gamma}'(t) \neq 0$.
- The differentiable curve $\vec{\gamma}$ is **regular** if and only if, for all u in $]a, b[$, $\vec{\gamma}$ is regular at u.
- The differentiable curve $\vec{\gamma}$ is **singular at** t if and only if $\vec{\gamma}'(t) = 0$.
- The differentiable curve $\vec{\gamma}$ is **singular** if and only if $\vec{\gamma}$ is not regular.

1) a) Parametrized Curves

EXAMPLE 1 (Parametrized curves): Let a and b be strictly positive real numbers in $\mathbb{R}_{>0}$ and let $\vec{\boldsymbol{\alpha}}$, $\vec{\boldsymbol{\beta}}$, $\vec{\boldsymbol{\gamma}}$, $\vec{\boldsymbol{\delta}}$ and $\vec{\boldsymbol{\varepsilon}}$ be the functions defined by:

1. $\vec{\boldsymbol{\alpha}}\colon \mathbb{R} \to \mathbb{R}^3;\ t \mapsto (a\cos(t), a\sin(t), bt)$ **(Circular helix of radius a and pitch $2\pi b$)**
2. $\vec{\boldsymbol{\beta}}\colon \mathbb{R} \to \mathbb{R}^2;\ t \mapsto \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right)$ **(Cissoid of Diocles)**
3. $\vec{\boldsymbol{\gamma}}\colon]0,\pi[\to \mathbb{R}^2;\ t \mapsto \left(\sin(t), \cos(t) + \ln\left(\tan\left(\frac{t}{2}\right)\right)\right)$ **(Tractrix)**
4. $\vec{\boldsymbol{\delta}}\colon]-1,+\infty[\to \mathbb{R}^2;\ t \mapsto \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\right)$ **(Folium of Descartes)**
5. $\vec{\boldsymbol{\varepsilon}}\colon \mathbb{R} \to \mathbb{R}^2;\ t \mapsto \left(ae^{-bt}\cos(t), ae^{-bt}\sin(t)\right)$ **(Logarithmic spiral)**

Then all of $\vec{\boldsymbol{\alpha}}$, $\vec{\boldsymbol{\beta}}$, $\vec{\boldsymbol{\gamma}}$, $\vec{\boldsymbol{\delta}}$ and $\vec{\boldsymbol{\varepsilon}}$ are differentiable curves as they are differentiable on their domain. Moreover, only $\vec{\boldsymbol{\alpha}}$ and $\vec{\boldsymbol{\varepsilon}}$ are regular, as their derivatives are nowhere-vanishing on their domain, whereas $\vec{\boldsymbol{\beta}}$, $\vec{\boldsymbol{\gamma}}$ and $\vec{\boldsymbol{\delta}}$ are singular.

1) b) Differentiation and Integration

Proposition 1 (Algebraic properties of derivatives and integrals of differentiable curves): Let n be a non-zero natural number in $\mathbb{N}_{\geqslant 1}$ and let $]a,b[$ be a non-degenerate open interval of $\mathbb{R}$, let λ be a real number in $\mathbb{R}$, let f be a differentiable function in $\mathscr{C}(]a,b[,\mathbb{R})$ and let $\vec{\boldsymbol{\alpha}}$, $\vec{\boldsymbol{\beta}}$ and $\vec{\boldsymbol{\gamma}}$ be differentiable curves in $\mathscr{C}(]a,b[,\mathbb{R}^n)$. Then:

- $\left(\vec{\boldsymbol{\alpha}}+\vec{\boldsymbol{\beta}}\right)' = \vec{\boldsymbol{\alpha}}' + \vec{\boldsymbol{\beta}}'$
- $\left(\lambda\cdot\vec{\boldsymbol{\alpha}}\right)' = \lambda\cdot\vec{\boldsymbol{\alpha}}'$
- $\left(f\cdot\vec{\boldsymbol{\alpha}}\right)' = f\cdot\vec{\boldsymbol{\alpha}}' + f'\cdot\vec{\boldsymbol{\alpha}}$
- $\left(\vec{\boldsymbol{\alpha}}\bullet\vec{\boldsymbol{\beta}}\right)' = \vec{\boldsymbol{\alpha}}'\bullet\vec{\boldsymbol{\beta}} + \vec{\boldsymbol{\alpha}}\bullet\vec{\boldsymbol{\beta}}'$
- If $n=3$, then $\left(\vec{\boldsymbol{\alpha}}\wedge\vec{\boldsymbol{\beta}}\right)' = \vec{\boldsymbol{\alpha}}'\wedge\vec{\boldsymbol{\beta}} + \vec{\boldsymbol{\alpha}}\wedge\vec{\boldsymbol{\beta}}'$.
- If $n=3$, then $\left[\vec{\boldsymbol{\alpha}},\vec{\boldsymbol{\beta}},\vec{\boldsymbol{\gamma}}\right]' = \left[\vec{\boldsymbol{\alpha}}',\vec{\boldsymbol{\beta}},\vec{\boldsymbol{\gamma}}\right] + \left[\vec{\boldsymbol{\alpha}},\vec{\boldsymbol{\beta}}',\vec{\boldsymbol{\gamma}}\right] + \left[\vec{\boldsymbol{\alpha}},\vec{\boldsymbol{\beta}},\vec{\boldsymbol{\gamma}}'\right]$.

If $\vec{\boldsymbol{\alpha}}$, $\vec{\boldsymbol{\beta}}$ and $\vec{\boldsymbol{\gamma}}$ are Riemann-integrable on $]a,b[$, then one has:

- $\displaystyle\int_{]a,b[}\left(\vec{\boldsymbol{\alpha}}+\vec{\boldsymbol{\beta}}\right) = \int_{]a,b[}\vec{\boldsymbol{\alpha}} + \int_{]a,b[}\vec{\boldsymbol{\beta}}$
- $\displaystyle\int_{]a,b[}\left(\lambda\cdot\vec{\boldsymbol{\alpha}}\right) = \lambda\int_{]a,b[}\vec{\boldsymbol{\alpha}}$

1) c) Arc-Length

DEFINITION 3: Let n be a non-zero natural number in $\mathbb{N}_{\geqslant 1}$, let $]a,b[$ be a non-degenerate open interval of $\mathbb{R}$, let t_0 be a real number in $[a,b]$ and let $\vec{\gamma}$ be a Riemann-integrable differentiable curve in $\mathscr{C}\left(]a,b[,\mathbb{R}^n\right)$

- The **speed** of $\vec{\gamma}$ is the function $\|\vec{\gamma}'\|$.
- The differentiable curve $\vec{\gamma}$ is **unit-speed** if and only if, for all t in $]a,b[$, one has $\|\vec{\gamma}'(t)\| = 1$.
- The **length** of $\vec{\gamma}$ is the real number denoted by $\ell(\vec{\gamma})$ and defined by:

$$\boxed{\ell(\vec{\gamma}) = \int_{]a,b[} \|\vec{\gamma}'\|}$$

- The **arc-length of $\vec{\gamma}$ measured from** t_0 (or **curvilinear abscissa of $\vec{\gamma}$ with origin** t_0), is the function denoted by s_{t_0} (or s) and defined by:

$$\boxed{s_{t_0} : \left| \begin{array}{ccc}]a,b[& \longrightarrow & \mathbb{R} \\ t & \longmapsto & \displaystyle\int_{t_0}^{t} \|\vec{\gamma}'(t)\| \,\mathrm{d}t \end{array} \right.}$$

2) Plane Curves

2) a) Curvature and Simple Closed Curves

DEFINITION 4 (Curvature of a plane curve): Let n be a non-zero natural number in $\mathbb{N}_{\geqslant 1}$, let $]a,b[$ be a non-degenerate interval of $\mathbb{R}$ and let $\vec{\gamma}$ be a twice differentiable curve in $\mathscr{C}\left(]a,b[,\mathbb{R}^2\right)$.

- If $\vec{\gamma}$ is unit-speed, the **curvature** of $\vec{\gamma}$ is the function denoted by κ and defined by:

$$\boxed{\kappa = \|\vec{\gamma}''\|}$$

- If $\vec{\gamma}$ is not unit-speed, the **curvature** of $\vec{\gamma}$ is the curvature of its arc-length parametrization.

DEFINITION 5 (Simple and closed plane curves): Let $]a,b[$ be a non-degenerate interval of $\mathbb{R}$ and let $\vec{\gamma}$ parametrized curve in $\mathscr{C}\left(]a,b[,\mathbb{R}^2\right)$.

- The parametrized curve $\vec{\gamma}$ is **simple** if and only if $\vec{\gamma}$ is injective.
- The parametrized curve $\vec{\gamma}$ is **closed** if and only if the following condition is satisfied:

$$\boxed{\lim_{\substack{t\to a\\ a<t}}\left(\vec{\gamma}(t)\right) = \lim_{\substack{t\to b\\ t<b}}\left(\vec{\gamma}(t)\right) \in \mathbb{R}^2}$$

2) b) The Isoperimetric Inequality

THEOREM 1 (The isoperimetric inequality): Let $]a,b[$ be a non-degenerate interval of $\mathbb{R}$ and let $\vec{\boldsymbol{\gamma}}$ be a simple closed differentiable curve in $\mathscr{C}\left(]a,b[,\mathbb{R}^2\right)$ and let $\mathscr{A}(\vec{\boldsymbol{\gamma}})$ be the area of $\mathrm{int}\left(\vec{\boldsymbol{\gamma}}\right)$. Then one has:

$$\boxed{\mathscr{A}(\vec{\boldsymbol{\gamma}}) \leqslant \frac{\ell(\vec{\boldsymbol{\gamma}})^2}{4\pi}}$$

with equality if and only if $\vec{\boldsymbol{\gamma}}(]a,b[)$ is a circle in $\mathbb{R}^2$.

2) c) The Fundamental Theorem of Plane Curves

THEOREM 2 (The fundamental theorem of plane curves): Let $]a,b[$ be a non-degenerate open interval of $\mathbb{R}$ and let κ be a nowhere-vanishing differentiable function in $\mathscr{C}\left(]a,b[,\mathbb{R}_{>0}\right)$. Then:

1. There exists a regular differentiable curve $\vec{\boldsymbol{\gamma}}$ in $\mathscr{C}\left(]a,b[,\mathbb{R}^2\right)$ such that κ is the curvature of $\vec{\boldsymbol{\gamma}}$.
2. If $\vec{\boldsymbol{\gamma}}$ and $\vec{\boldsymbol{\rho}}$ are regular differentiable curves in $\mathscr{C}\left(]a,b[,\mathbb{R}^2\right)$ such that κ is the curvature of both $\vec{\boldsymbol{\gamma}}$ and $\vec{\boldsymbol{\rho}}$, then there exists a 2-square matrix $\mathbf{R}$ in $SO_2(\mathbb{R})$ and a vector $\vec{\mathbf{t}}$ in $\mathbb{R}^2$ such that:

$$\boxed{\vec{\boldsymbol{\rho}} = \mathbf{R} \times \vec{\boldsymbol{\gamma}} + \vec{\mathbf{t}}}$$

3) Space Curves

3) a) The Frenet Frame

DEFINITION 6 (Tangent, normal and binormal vectors of a unit-speed curve): Let $]a,b[$ be a non-degenerate open interval of $\mathbb{R}$ and let $\vec{\gamma}$ be a thrice differentiable unit speed curve in $\mathscr{C}\left(]a,b[,\mathbb{R}^3\right)$.

- The **tangent vector** of $\vec{\gamma}$ is the function denoted by $\hat{\mathbf{T}}$ and defined by:

$$\boxed{\hat{\mathbf{T}} = \vec{\gamma}'}$$

- The **normal vector** of $\vec{\gamma}$ is the function denoted by $\hat{\mathbf{N}}$ and defined by:

$$\boxed{\hat{\mathbf{N}} = \frac{\vec{\gamma}''}{\kappa}}$$

  wherever κ is non-vanishing.

- The **binormal vector** of $\vec{\gamma}$ is the function denoted by $\hat{\mathbf{B}}$ and defined by:

$$\boxed{\hat{\mathbf{B}} = \hat{\mathbf{T}} \wedge \hat{\mathbf{N}}}$$

- The **torsion** of $\vec{\gamma}$ is the function denoted by τ and defined by:

$$\boxed{\tau = -\hat{\mathbf{B}}' \bullet \hat{\mathbf{N}}}$$

3) a) The Frenet Frame

PROPOSITION 2 (Orthonormality of the tangent, normal and binormal vectors of a unit speed curve): Let $]a,b[$ be a non-degenerate open interval of $\mathbb{R}$ and let $\vec{\boldsymbol{\gamma}}$ be a thrice differentiable unit speed curve in $\mathscr{C}\left(]a,b[,\mathbb{R}^3\right)$. Then, for every real number t in $]a,b[$ such that $\kappa(t)\neq 0$, the set $\{\hat{\mathbf{T}},\hat{\mathbf{N}},\hat{\mathbf{B}}\}$ is an orthonormal basis of $\mathbb{R}^3$.

3) a) The Frenet Frame

Definition 7 (Tangent, normal and binormal vectors of a regular curve): Let $]a,b[$ be a non-degenerate open interval of $\mathbb{R}$ and let $\vec{\gamma}$ be a thrice differentiable regular curve in $\mathscr{C}\left(]a,b[,\mathbb{R}^3\right)$ such that κ is nowhere vanishing on $]a,b[$.

- The **tangent vector**, **normal vector**, **binormal vector** and **torsion** of $\vec{\gamma}$ are the tangent vector, normal vector, binormal vector and torsion of its arc-length parametrization.
- The **Frenet frame** of $\vec{\gamma}$ is the ordered quadruple $\left(\vec{\gamma},\hat{\mathbf{T}},\hat{\mathbf{N}},\hat{\mathbf{B}}\right)$
- The **osculating plane** of $\vec{\gamma}$ is the function denoted by $\mathscr{P}_\Gamma$ and defined by:

$$\boxed{\mathscr{P}_\Gamma = \vec{\gamma} + \operatorname{span}_{\mathbb{R}}\left(\{\hat{\mathbf{T}},\hat{\mathbf{N}}\}\right)}$$

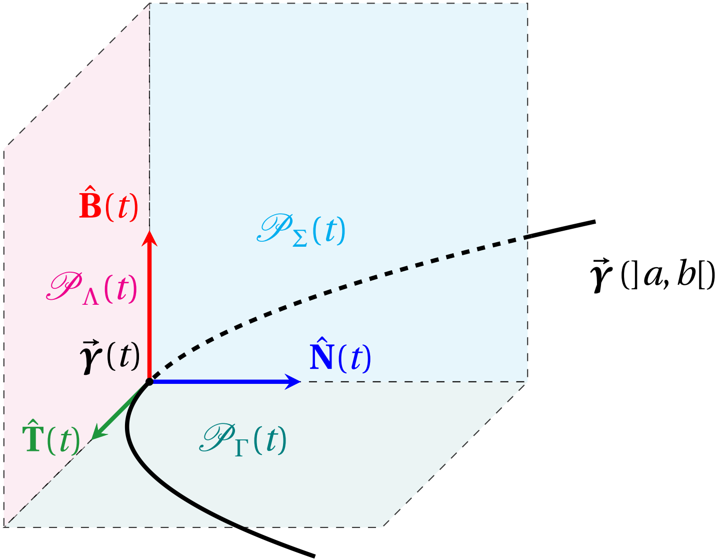
- The **normal plane** of $\vec{\gamma}$ is the function denoted by $\mathscr{P}_\Sigma$ and defined by:

$$\boxed{\mathscr{P}_\Sigma = \vec{\gamma} + \operatorname{span}_{\mathbb{R}}\left(\{\hat{\mathbf{N}},\hat{\mathbf{B}}\}\right)}$$

- The **rectifying plane** of $\vec{\gamma}$ is the function denoted by $\mathscr{P}_\Lambda$ and defined by:

$$\boxed{\mathscr{P}_\Lambda = \vec{\gamma} + \operatorname{span}_{\mathbb{R}}\left(\{\hat{\mathbf{T}},\hat{\mathbf{B}}\}\right)}$$

3) a) The Frenet Frame



3) a) The Frenet Frame

THEOREM 3 (Explicit expression of the Frenet frame, the curvature and torsion of a regular differentiable curve): Let $]a,b[$ be a non-degenerate open interval of $\mathbb{R}$ and let $\vec{\gamma}$ be a thrice differentiable regular curve in $\mathscr{C}\left(]a,b[,\mathbb{R}^3\right)$ such that $\vec{\gamma}' \wedge \vec{\gamma}''$ is nowhere vanishing on $]a,b[$. Then one has:

$$\boxed{\kappa = \frac{\|\vec{\gamma}' \wedge \vec{\gamma}''\|}{\|\vec{\gamma}'\|^3}} \qquad \boxed{\tau = \frac{[\vec{\gamma}', \vec{\gamma}'', \vec{\gamma}''']}{\|\vec{\gamma}' \wedge \vec{\gamma}''\|^2}}$$

$$\boxed{\hat{\mathbf{T}} = \frac{\vec{\gamma}'}{\|\vec{\gamma}'\|}} \qquad \boxed{\hat{\mathbf{N}} = \frac{\vec{\gamma}' \wedge (\vec{\gamma}'' \wedge \vec{\gamma}')}{\|\vec{\gamma}'\| \|\vec{\gamma}'' \wedge \vec{\gamma}'\|}} \qquad \boxed{\hat{\mathbf{B}} = \frac{\vec{\gamma}' \wedge \vec{\gamma}''}{\|\vec{\gamma}' \wedge \vec{\gamma}''\|}}$$

3) a) The Frenet Frame

THEOREM 4 (The Frenet-Serret equations): Let $]a,b[$ be a non-degenerate open interval of $\mathbb{R}$ and let $\vec{\boldsymbol{\gamma}}$ be a thrice differentiable regular curve in $\mathscr{C}\left(]a,b[,\mathbb{R}^3\right)$ such that $\vec{\boldsymbol{\gamma}}' \wedge \vec{\boldsymbol{\gamma}}''$ is nowhere vanishing on $]a,b[$. Then one has:

$$\boxed{\begin{bmatrix} \hat{\mathbf{T}}' \\ \hat{\mathbf{N}}' \\ \hat{\mathbf{B}}' \end{bmatrix} = \|\vec{\boldsymbol{\gamma}}'\| \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{N}} \\ \hat{\mathbf{B}} \end{bmatrix}}$$

3) b) The Fundamental Theorem of Space Curves

THEOREM 5 (The fundamental theorem of space curves): Let $]a,b[$ be a non-degenerate open interval of $\mathbb{R}$, let κ be a nowhere-vanishing differentiable function in $\mathscr{C}\left(]a,b[,\mathbb{R}_{>0}\right)$ and let τ be a nowhere-vanishing differentiable function in $\mathscr{C}\left(]a,b[,\mathbb{R}_{\neq 0}\right)$ Then:

1. There exists a regular differentiable curve $\vec{\boldsymbol{\gamma}}$ in $\mathscr{C}\left(]a,b[,\mathbb{R}^3\right)$ such that κ is the curvature of $\vec{\boldsymbol{\gamma}}$ and τ is the torsion of $\vec{\boldsymbol{\gamma}}$.
2. If $\vec{\boldsymbol{\gamma}}$ and $\vec{\boldsymbol{\rho}}$ are regular differentiable curves in $\mathscr{C}\left(]a,b[,\mathbb{R}^3\right)$ such that κ is the curvature of both $\vec{\boldsymbol{\gamma}}$ and $\vec{\boldsymbol{\rho}}$ and τ is the torsion of both $\vec{\boldsymbol{\gamma}}$ and $\vec{\boldsymbol{\rho}}$, then there exists a 3-square matrix $\mathbf{R}$ in $\mathrm{SO}_3(\mathbb{R})$ and a vector $\vec{\mathbf{t}}$ in $\mathbb{R}^3$ such that:

$$\boxed{\vec{\boldsymbol{\rho}} = \mathbf{R} \times \vec{\boldsymbol{\gamma}} + \vec{\mathbf{t}}}$$

Bibliography

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