Debt Financing Investment in the Long Run*

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Abstract

Does a debt market make any difference for firm output in the long run? To answer this question firm investment under financial autarky is compared to investment when risk-free debt exists. Firm investment and debt decisions are proved to be unique, reflect a state dependent capital target, and there is a unique stationary distribution to which the model converges. Under many circumstances the firm borrows to the risk-free limit, in order to capture the full tax benefits. Both the capital target and the borrowing limit are state dependent. The capital target with a debt market is greater than under financial autarky primarily due to interest tax deductions; so even in the long run the firm never fully grows out of the impact of financial autarky. If there is no taxation, then there are conditions such that the firm may grow out of the absence of a debt market.

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1 Introduction

In some countries and during some historical time periods, firms have ready access to well-developed financial markets. In other countries and times, external financing is not readily available. When a firm has no access to external finance, any growth must be self-financed. Does this really matter? It is often thought that the absence of access to good financial markets is of critical importance for firms (King and Levine, 1993; Demirgüç-Kunt and Maksimovic, 1998), as are more moderate financing constraints (Myers, 1977; Fazzari et al., 1988). On the other hand, it is also often suggested that financial constraints are relatively unimportant in the long run, since firms may be able to grow out of the constraints (Moll, 2014).

To reconcile these perspectives requires an understanding of firm productivity and investment under financial autarky. However, the classical theory of production and investment assumes that capital is rented (Hall and Jorgenson, 1967), which is a type of financing. More recent investment studies introduce a range of frictions (Caballero, 1999; Bond and Van Reenen, 2007). So our understanding of this issue is limited by the absence of a well developed theory of the long run efficiency of production under financial autarky as a benchmark. Can the firm eventually achieve productive efficiency? Or does self-financing permanently limit the firm? Does a debt market alter the long run production by the firm?

In this paper first we study investment by a firm under financial autarky. Then we introduce risk-free saving and borrowing to establish the pure impact of financial market access. The model is quite traditional apart from the restricted access to external financing. The firm is trying to maximize the expected present value of dividends (i.e. profits), there is a linear flow budget constraint that imposes cash in equals cash out, there is a production function that depends on capital and a shock, capital depreciates, there are no capital adjustment costs, firm revenue is taxable with depreciation tax deductible. The productivity shock is given by a Markov process. When debt is available, it lasts for one period and interest is tax deductible.

Our first contribution is to show that under financial autarky the firm has a state dependent capital target. If the firm has enough revenue, it invests to achieve a capital target, and pays out the remainder as a dividend. If it does not have enough revenue to achieve the target, then all revenue is invested and no dividends are paid. The target depends on the current state of the shock, because that affects both the currently available revenue and the expected future value of the shocks and hence the future revenue. The solution to the firm problem exists and it is unique. Contrary to some assumptions in the literature, the target in our model is not a time-invariant constant.

Our second contribution is to prove the existence, and then to characterize the long run stationary distribution under financial autarky. The form of the stationary distribution depends on the form of the shock process. We prove that under reasonable conditions on the shock process (known as Harris ergoic), in the long run the model converges to the stationary distribution. We explicitly characterize the stationary distributions under several familiar shock processes including Martingales, random walks,and log normal iid shocks.

Our third contribution is the analysis of the impact of debt. The firm's objective function is linear. So when borrowing is feasible and desirable, the firm will do as much as possible. The upper limit is produced by our requirement that the debt is risk-free. Risk-free debt is a common assumption in the literature, for example Hennessy and Whited (2005). Risk-free debt is priced in the market as risk-free which simplifies the analysis. It avoids the significant complexities associated with default, the impact on the price of debt, and thus on feasible firm investment in the first place. Some of these complications are studied in Hennessy and Whited (2007) and Aguiar and Amador (2019).

When the debt market is available, the firm decisions have a similar form to the firm under financial autarky, in the sense that there is still a state-dependent capital target. However, the capital target in the model with debt is greater than the capital target under financial autarky. In this sense, the presence or absence of a debt market has a long run effect on firm capital. The firm under financial autarky does not merely grow out of the constraint created by the absence of a debt market.

Related literature

Studies that assume single period debt are widely used both when studying firm debt (Hennessy and Whited, 2005; Nikolov et al., 2021), and when studying sovereign debt (Eaton and Gersovitz, 1981). It helps simplify the analysis. Aguiar and Amador (2019) provides a particularly nice Markov equilibrium uniqueness proof for a sovereign debt model with risky one period debt.

Hennessy and Whited (2005) is particularly close to our paper in terms of model structure. Their primary focus is on providing a trade-off model that can match a number of empirical facts. Our primary focus is on understanding the role of debt in facilitating investment. This difference helps explain how our paper is different from theirs. They make the one period debt risk-free by having capital fire sales when necessary. In contrast, we permit future debt issues to help make current debt safe. Adding capital fire sales to our model would further relax the risk-free debt bound that we derive. Given their focus, Hennessy and Whited (2005) do not compare investment with a debt market, to investment under financial autarky.

Nikolov et al. (2021) are interested in comparing solutions to three related models. Their trade-off model is similar to our model. Their model differs from ours in that increasing or reducing capital entails adjustment costs, and debt is risky. In a similar model of sovereign debt, Aguiar and Amador (2019) point out that the investor break-even condition with risky debt may mean that the Blackwell (1965) discounting requirement on the original problem does not hold. So the original problem may not define a contraction. Nikolov et al. (2021) do not study potential concerns about existence or uniqueness of the solution for their trade-off model. They focus on comparing numerical solutions across models. So they do not have anything like our results on existence, uniqueness or convergence; nor do they have anything about investment under financial autarky. Their paper is very different from ours.

Multiple equilibria for firm debt models seem likely to be more common than generally recognized, given the literature on sovereign debt such as Cole and Kehoe (2000) and Aguiar and Amador (2019). Non-Markov equilibria as in Benzoni et al. (2020) and Malenko and Tsoy (2020) are also of potential interest, but beyond the scope of our paper. As in Aguiar et al. (2019) and DeMarzo and He (2021) we analyze Markov equilibria, and for that we prove uniqueness.

In many models of firm debt such as Modigliani and Miller (1958), Brennan and Schwartz (1978), and Leland (1994) it is assumed that the firm's cash flows are independent of the manner in which they are financed. By assumption such models cannot consider the connections between the firm's choice of debt and of investment. It is well established empirically that debt and capital are positively related, see Frank and Goyal (2009).

Rampini and Viswanathan (2020) study a model in which specific assets may be pledged as collateral for debt. In the model more financially constrained firms are more likely to pledge specific assets, and they find empirical support for this prediction. They are interested in comparing alternative forms of debt to each other. They do not consider investment under financial autarky as their purpose is very different from ours.

Production based capital structure models do connect debt to capital; for a review see Ai et al. (2021). Such models generally include complexities such as capital adjustment costs so that there is no analytic solution. The models are commonly solved numerically (Strebulaev and Whited, 2012; Nikolov et al., 2021). Papers like Nikolov et al. (2021) put real effort into finding a numerical solution, but do not show that the model itself has a unique solution.

Our analytic comparison of the firm under autarky to the firm that has access to a debt market, is new to the literature. We provide a characterization of the long run stationary distribution based on Harris recurrence following Asmussen (2003) and Meyn and Tweedie (2012); see also Stachurski (2009).

The Harris recurrence assumptions used to prove convergence to a unique stationary distribution are fairly mild and seem natural. So it is likely that this method of proving convergence to a unique long run stationary distribution will apply to a range of models in the literature. However, this approach will of course, not apply to all models. In particular, the distribution of possible states for the operating firm must be irreducible. Depending on details, models with actual firm bankruptcy such as Nikolov et al. (2021) may pose an under-appreciated problem. The unique stationary distribution for some models will simply say that the firm is eventually bankrupt for sure. Justifying parameter estimation of such a model using data on firms that are not yet bankrupt, may be a concern if the underlying justification for the estimation methods are based on large sample properties.

Our model is similar to the classic investment model by Hall and Jorgenson (1967). They assume that firms rent capital. Under complete markets and standard assumptions, it does not matter whether the firm owns or rents capital, see Miao (2014). But renting capital can be viewed as a type of financing. Since we are interested in capital choice under financial autarky, we assume throughout that the firm owns the capital it uses and there is no rental market. The resulting investment decisions are more complex than in Hall and Jorgenson (1967), and they depend on the firm's financial condition in our model.

Moll (2014) studies a model in which self-financing can undo capital misallocation due to financial frictions in the long run. In his model the persistence of the productivity shocks are important. In our model self financing does not undo the effect of a missing financial market, even in the long run. It has long been understood (Modigliani and Miller, 1963) that the corporate tax deductibility of interest payments makes debt financing attractive to firms. So from a tax perspective access to debt financing matters both in the short run and in the long run.

Empirical work on a range of firm decisions commonly study the speed of adjustment towards an assumed time-invariant target (see Kennan (1979); Adda and Cooper (2003); Flannery and Rangan (2006); King and Thomas (2006) and Cooper and Willis (2009)). Myers (1984) suggested that rapid target adjustment is a hallmark of the trade-off theory of firm capital structure. That helped stimulate studies of persistence and the speed of adjustment, such as Fama and French (2002), Lemmon et al. (2008), Faulkender et al. (2012) and DeAngelo and Roll (2015).

In our model the firm has a capital target. But that target depends on the current value of the shock. So it fluctuates. Shock-dependent targets may help to account for the common impression (Fama and French, 2002) that target-adjusting behavior by firms is extremely slow. Empirical evidence on investment was thought to be inconsistent with traditional target adjusting models. This helped motivate the introduction of explicit adjustment costs as in Hayashi (1982) and Caballero and Engel (1999). Shock-dependent targets may provide a useful alternative perspective, even without introducing standard adjustment costs.

Our result that the firm wishes to borrow as much as possible is similar in spirit to Modigliani and Miller (1963). Since our model has both tax and something like a bankruptcy avoidance condition, it can be interpreted as a version of the trade-off theory, see Kraus and Litzenberger (1973), Hennessy and Whited (2005) and Ai et al. (2021). Our comparison to financial autarky and the analysis of the long run distributions are new to the trade-off theory literature.

We conclude the introduction with some convention of the notation used in this paper. All random variables considered in the paper are discrete time Markov processes with stationary transitions, evaluated at a specific time. For example, $\{x_t, t \geq 0\}$ is a discrete time Markov process with a general state space $\mathbb{X}$. We describe the realization of the random variable x_t, by the deterministic value x. If the current state is $x_t = x$, the next period value is denoted $x' = x_{t+1}$. So, x is deterministic and x' is random variable.

Because the Markov process has a stationary transition, we usually do not write down the current time t. For example, the conditional expectation $\mathsf{E}_x g(x') \equiv \mathsf{E}\big[g(x_{t+1})\big|x_t = x\big] = \mathsf{E}\big[g(x_1)\big|x_0 = x\big]$ for any function g defined on $\mathbb{X}$. Throughout we assume without restating, that the function g must be measurable and the expectation is finite. When we consider the probability of x_t in a set A, we assume without restating that the set is measurable, i.e. a Borel measurable subset of $\mathbb{X}$.

2 Financial autarky

Consider the decisions of a firm in discrete time with periods $t = 0, 1, 2, \ldots$. The firm produces revenue using an increasing, concave function,

$$F(k, z) = zAk^{\alpha}, \tag{1}$$

that depends on both the capital k, and a random shock z, where $A > 0$ and $\alpha \in (0, 1)$. Capital is increased by purchasing more capital and reduced by depreciation. There is no second-hand capital market. Financial autarky means that there is no capital rental market, and no financial markets trading claims on the firm.

Shocks

The exogenous shock follows a Markov process $\{z_t, t \geq 0\}$ with a state space $\mathbb{Z} \subseteq \Re_+$, where $\Re_+$ is the set of all non-negative real numbers.

Assumption 1 *The exogenous shock $\{z_t, t \geq 0\}$ is a discrete-time Markov process with general state space $\mathbb{Z} \subseteq \Re_+$ and homogeneous transitions. The shock is assumed to be non-negative and has a finite upper bound $\bar{z}$; i.e.,* $\mathsf{P}(0 \leq z_t \leq \bar{z}) = 1$

Timing

The firm enters the period owning capital k. Nature determines the current period value of the shock z. The firm observes the shock and produces. The firm then picks investment i, dividends d, and pays taxes. The firm outside opportunity has a value of 0.

Firm problem

Physical capital can be increased by investing, i. Physical capital is reduced through depreciation at a rate $\delta \in (0, 1)$. The firm's discount factor is $\beta \in (0, 1)$. There is a tax on firm revenue at rate $\tau \in (0, 1)$. Capital depreciation is tax deductible which creates a benefit of, $\tau\delta k$. There are no capital adjustment costs. In each period, the firm is given the state (z, k), where z is the (current) exogenous state and k is the current capital level. Let $\mathbb{X} = \mathbb{Z} \times \Re_+$ denote the state space.

The firm chooses the dividend d and the investment i, under the flow budget constraint. With free disposal of resources the budget constraint says that cash in must be at least as great as cash out. Since resources are valuable, the firm never chooses to waste them. So the financing constraint says cash in = cash out, $(1 - \tau)F(k) + \delta k\tau = d + i$. Because the shock is strictly positive and so is the initial capital, it is never in the interest of the firm to shut down and get zero.

The firm's problem is for all $x = (z, k) \in \mathbb{X}$,

$$\begin{aligned} V(z,k) &= \sup_{\{d,i\}} \quad d + \beta \mathsf{E}_z V(z', k'), && (2) \\ \text{subject to} & \quad (1-\tau)F(z,k) + \tau\delta k = d + i && (3) \\ & \quad k' = (1-\delta)k + i && (4) \\ & \quad d \geq 0, i \geq 0. && (5) \end{aligned}$$

To summarize the elements of the problem: The set of states is $\{z, k\}$. The set of feasible actions $\{d, i\}$ are determined by the constraints. Equation (3) shows that cash coming in to the firm must equal cash going out. The capital accumulation condition is given by (4). Because there are no financial markets, dividends and investment cannot be negative as shown in equation (5). The value function is $V(z, k)$ and the discount factor is β. The exogenous process satisfies Assumption 1.

Using equations (1) and (4), the firm's per-period net resources are,

$$R(z,k) = (1-\tau)zAk^{\alpha} + [1 - \delta(1-\tau)]k. \tag{6}$$

The firm resources consist of the after-tax revenue from production, and the capital that remains from the past period.

2.1 Optimal capital under financial autarky

Under financial autarky, investment in capital can only be paid for using revenue generated from production. The firm's investment and dividend choices depend on whether it has as much capital as it would prefer in light of the current state of the shock. The amount that the firm would prefer we call the firm's target capital, and it is given by

$$k^*(z) = \left(\frac{\alpha\beta A(1-\tau)\mathsf{E}_z\,(z')}{1 - \beta + \beta\delta(1-\tau)} \right)^{1/(1-\alpha)}. \tag{7}$$

If the firm condition is such that it has interior solutions for both investment and dividends, then the objective function is concave in i (and equivalently in k') so this target capital is found from the first order condition for capital. That is how (7) is obtained. Whether the firm is in that region depends on the amount of capital the firm already has, and on the realization of the shock. It is possible that the value of the shock is such that reaching the target capital is not feasible.

If the firm has too much capital to start with ($k^*(z) \leq (1-\delta)k$), then even if it invests nothing it will still have too much capital. We refer to this region as "excess capital". A firm that already has excess capital will not want to get still more.

At the other extreme, the amount of revenue that the firm has may not be sufficient to reach the target, even if all revenue is used to obtain capital ($R(z,k) \leq k^*(z)$). We refer to this region as inadequate capital. A firm that has inadequate capital will get as much capital as the budget permits.

Proposition 1 *Under Assumption 1, there exists a unique optimal policy solving the Bellman equation (2). The optimal policy depends on the current exogenous state z and the current capital k. Case (a) Excess capital. If $k^*(z) \leq (1-\delta)k$, then the firm invests nothing $i = 0$, and all revenue is paid as a dividend $d = (1-\tau)F(z,k) + \tau\delta k$. Case (b) Capital in range. If $(1-\delta)k < k^*(z) < R(z,k)$, then the firm invests enough to attain the capital target $i = k^*(z) - (1-\delta)k$, and the remaining revenue is paid as a dividend $d = R(z,k) - k^*(z)$. Case (c) Inadequate capital. If $R(z,k) \leq k^*(z)$, then the firm invests as much as is feasible acquiring capital $i = (1-\tau)F(k,z) + \tau\delta k$, and pays no dividends $d = 0$.*

A common approach to proving this would be to apply the Blackwell (1965) conditions as in Stokey and Lucas Jr. (1989). It would be assumed that the single period reward function is bounded under the uniform norm. However, the single period reward for the Bellman problem (2), which is given by the dividend d, is not obviously bounded if we do not assume the state space $\mathbb{X}$ is bounded. One possible approach is to use the flow budget constraint (3) and show by induction that the firm capital choices are bounded as in Harris (1987).

We take another approach by applying the sufficient conditions in Lippman (1975), Van Nunen and Wessels (1978) and Boyd III (1990). The idea is to change the norm and establish that the problem is suitably bounded under the change of norm. Then the Bellman equation defines a contraction mapping, and that is used to establish existence and uniqueness for the underlying problem.

Why is Proposition 1 correct? We identify the optimal policy in Proposition 1 together with the corresponding value function given by Corollary 1 as shown in the appendix. Here we explain the idea.

Start by assuming that we have an interior solution as in case (b), so the constraints $d \geq 0$ and $i \geq 0$ are not binding. We then show that the value function must take the form, $V(z, k) = R(z, k) + h(z)$, for some function $h(z)$. Substitute this value function into the Bellman equation (2). That gives,

$$h(z) = \max_{k'} \Big\{\beta \mathsf{E}_z \big\{(1-\tau)F(z', k') + [1-\delta(1-\tau)]k' + h(z')\big\} - k'\Big\}.$$

The objective function is concave in k'. So the optimal solution $k' = k^*(z)$ or $i = k^*(z) - (1-\delta)k$, provided the solution is interior. For the solution to be interior requires that $i > 0$ or equivalently $(1-\delta)k < k^*(z)$, and $d = R(z, k) - k^*(z) > 0$.

Next consider what happens if the firm has too much capital to start with as in case (a). So $(1-\delta)k \geq k^*(z)$. The concavity of the objective function implies the optimal investment is $k' = (1-\delta)k$ or $i = 0$. From the budget constraint and the fact that dividends are valuable, the optimal dividend $d = (1-\tau)F(z, k) + \tau\delta k$.

Now consider what happens when the firm does not have enough capital so that the resources at the firm's disposal are not enough to reach the capital target. This is case (c), written $R(z, k) \leq k^*(z)$. Now investment i is constrained by the upper bound $(1-\tau)F(z, k) + \tau\delta k$. This is due to the restriction that $d \geq 0$ from the budget flow constraint. The concavity of the objective function implies the optimal investment is achieved at its upper bound $i = (1-\tau)F(z, k) + \tau\delta k$. So dividend $d = 0$. This completes the three cases needed for the proof.

Direct calculations give the natural monotonicity properties for the target. Under financial autarky, the target capital level $k^*(z)$ is increasing in A, α and β and decreasing in τ and δ. Whether it is increasing or decreasing in z depends on whether $\mathsf{E}_z(z')$ is increasing or decreasing in z.

Discussion

The solution depends on the target capital given by equation (7). Proposition 1 shows that the solution to the firm problem under financial autarky depends on whether it is feasible for the firm to achieve the target amount of capital. If the firm is already over-capitalized it simply pays out the available revenue as dividends. If the firm can achieve the target then it does so, and it pays out the remaining available revenue as dividends. If the firm cannot achieve the target, then it invests as much as it can, and pays no dividends.

The meaning of the firm having a target for a choice variable differs across papers. In many papers a static frictionless model is used to determine a target value. Then fixed adjustment costs make instantaneous adjustment to that target excessively costly. In many papers the firm does not respond to small shocks unless it hits a boundary condition, see Fischer et al. (1989), and Caballero and Engel (1993) for instance. Adda and Cooper (2003) observes that such models are often used to motivate empirical studies. Relatively slow adjustment towards a target is often found. The existence and size of the theorized zones of inaction is not well established empirically, see Ai et al. (2021).

The target found in Proposition 1 does not involve fixed adjustment costs. The firm is reacting as best it can each period to the current value of the shock. Due to financial autarky, the available resources limit what the firm can do. Shocks alter the available resources.

According to Proposition 1 the capital target is not a constant number. It depends on the current realization of the shock. When the shock has different realizations, so too does the current value of the target.

As in Rampini and Viswanathan (2013) we assume that there are no capital adjustment costs; that facilitates analytic derivations. However, in calibration papers such as Nikolov et al. (2021) it is common to include adjustment costs. Does a solution still exist in that case? Following (Hayashi, 1982; Adda and Cooper, 2003; Nikolov et al., 2021), let the adjustment costs take the form of $\Psi(k_{i+1}, k_{it}) = \frac{1}{2}\psi\left(\frac{i_{it}}{k_{it}}\right)^2 k_{it}$, where $\psi > 0$ is a constant. This expression would be an extra cost term on the right hand side of equation (3). If that is done, the result for existence and the uniqueness of the optimal value function in Proposition 1 given in Lemma 4 will still apply. However, the simple form of our optimal policy function may no longer hold. The model can still be solved numerically. However the purpose of this paper is analytic, so we avoid doing that.

2.2 Stationary distribution under financial autarky

Proposition 1 establishes that there exist a unique equilibrium for the firm under financial autarky. This section characterizes the long run of the model. In particular, the purpose is to characterize the long run behavior of the state process $\{(z_t, k_t), t \geq 0\}$ under the optimal policy. If the limiting distribution exists, it must be the stationary distribution.

Definition 1 *A discrete time Markov process $\{x_t, t \geq 0\}$ is said to be in a stationary distribution, if $x_{t+1} \overset{d}{=} x_t$ for all $t \geq 0$, where "$\overset{d}{=}$" means "equal in distribution".*

Following Definition 1, when a discrete time Markov process $\{x_t, t \geq 0\}$ is in the stationary distribution, it remains invariant for all periods going forward in time. The limit of the distribution of x_t as $t \to \infty$, if exists, is the stationary distribution.

Suppose however, that x_t is not already in the stationary distribution. Then the distribution of x_t, as $t \to \infty$, may converge to the stationary distribution under some conditions. The purpose of this subsection is to show that under mild conditions this happens for the firm under financial autarky. Proposition 2 below provides a sufficient condition such that there is convergence in distribution. A process that converges in distribution is also called weakly convergent, see Stokey and Lucas Jr. (1989).

Trivial Case

Under financial autarky, if the firm has no capital and no method to obtain any capital, then it remains permanently with no capital. Due to production function (1), if existing capital is $k_t = 0$, then so is the production revenue y, and so $k_{t+1} = 0$. So if the firm starts at $k_0 = 0$, then $k_t = 0$ for all $t \geq 0$.

This observation extends to later periods as well. If there is a positive probability that $z = 0$ for any time period, then there will be a positive probability $k_t = 0$ for some k. Then $k = 0$ is the only stationary distribution. We will explicitly rule this out below.

In order to study the stationary distribution of the Markov process we use the notion of Harris ergodicity from the literature on Markov processes. We follow Asmussen (2003) and Meyn and Tweedie (2012) for the setup and definitions. They are briefly described here for convenience. Time is discrete. Consider a Markov process $\{x_t, t \geq 0\}$ with the stationary transition and a state space $\mathbb{X}$.

For any $R \subset \mathbb{X}$, define $\tau(R) = \inf\{t \geq 1 : x_t \in R\}$. The set $R \subset \mathbb{X}$ is said to be recurrent if $\mathsf{P}_x\left(\tau(R) < \infty\right) = 1$ for all $x \in \mathbb{X}$. The set $R \subset \mathbb{X}$ is said to be a regenerative set if it is recurrent and there exists a probability measure $\lambda(\cdot)$ such that for some $r > 0$ and $\epsilon > 0$, the following inequality holds,

$$\mathsf{P}_x\left(x_r \in B\right) \geq \epsilon\lambda(B) \quad \text{for all } x \in R \text{ and all set } B \subseteq \mathbb{X}. \tag{8}$$

For all states of the system and for any set in the state space that has a positive measure under $\lambda(\cdot)$, there is a positive probability that the process revisits that set.

The Markov process is said to be a Harris process, or to be Harris recurrent, if it has a regenerative set. It is said to be positive Harris recurrent if for the regenerative set R, $\mathsf{E}_x\tau(R) < \infty$ for all $x \in \mathbb{X}$.

In defining the regenerative set R, there may be many pairs of (r, ϵ) satisfying inequality (8). If the greatest common divisor of such r's is one, then the Markov process is said to be aperiodic. In other words, it has no regularly repeating cycles. A Markov process is said to be a Harris ergodic process, if it is aperiodic and positive Harris recurrent.

Harris recurrence allow us to deal with uncountable state spaces. If the Markov process has a countable state space and is irreducible, then Harris recurrence corresponds to recurrence; and positive Harris recurrence corresponds to positive recurrence. In a recurrent Markov process with a countable state space, any subset forms a regenerative set. This includes a subset consisting of a single state. The concept of aperiodicity is the same if one chooses the regenerative set being the set of a single state.

In section 2.3 we provide a series of widely used shock process assumptions that are Harris ergodic. Assumption 2 below says that the shock process facing the firm is Harris ergodic.

Stationary Distribution

In order to establish the uniqueness of the stationary distribution, we rule out a trivial stationary distribution by restricting attention to $k \in (0, \infty)$ only. Accordingly we make the following assumption.

Assumption 2 *The Markov process $\{z_t, t \geq 0\}$ is Harris ergodic. In addition, $\mathsf{P}(z_t \geq \underline{z}) = 1$ for some constant $\underline{z} > 0$.*

Using assumption 2 we have the basic proposition about the long run for the optimal firm under financial autarky.

Proposition 2 *Let $\{x_t, t \geq 0\}$, where $x_t = (z_t, k_t)$, be the state process under the optimal policy in Proposition 1. Consider x in the state space $\mathbb{Z} \times (0, \infty)$. Under Assumptions 1 and 2, (a) the state process $\{x_t, t \geq 0\}$ is Harris ergodic and has a unique stationary probability measure π. (b) x_t converges in distribution. This means that for any $x \in \mathbb{X}$, $\mathsf{P}_x(x_t \in B) \to \pi(B)$ as $t \to \infty$ for all Borel set $B \subseteq \mathbb{X}$.*

The key to the proof is to show that the state process is aperiodic and positive Harris recurrent. Once that is done then the proposition is due to Theorems VII.3.2, VII.3.5 and VII.3.6 in Asmussen (2003).

Aperiodicity is immediate from the aperiodicity of the exogenous state (Assumption 2). Positive Harris recurrence requires identification of a regenerative set R satisfying $\mathsf{E}_x\tau(R) < \infty$ for all $x \in R$. It follows from Assumptions 1 and 2 that the optimal target $k^*(z)$ is bounded. To be more specific, there exists two positive constants $\underline{k}$ and $\bar{k}$ such that $\underline{k} \leq k^*(z) \leq \bar{k}$ for all $z \in \mathbb{Z}$. Knowing that, we can show that $R = \{(z, k) \in \mathbb{X} : (1-\delta)k \leq \bar{k}\}$ is a desired regenerative set. Intuitively, if $x = (z, k) \notin R$, then the optimal policy implies that $k' = (1-\delta)k$. In word, the capital in the following period decreases by a constant rate $(1-\delta)$. So in a finite number of periods, the state process must belong to the set R and the time to reach it is bounded by the initial state. A formal proof can then be completed by choosing a probability measure λ defined as follows:

$$\lambda(B) = \inf_{x \in R} \mathsf{P}_x\left(x' \in B \cap R\right) \qquad \text{for any set } B \subseteq \mathbb{X},$$

and showing that the inequality (8) holds with $r = 1$ and $\mathsf{E}_x \tau(R) < \infty$.

Now we know that there is a unique stationary distribution to which the system converges. What can be said about the form of that distribution? According to Proposition 1, optimal capital can be written as follows,

$$k' = R(z,k)1_{\{R(z,k)<k^*(z)\}} + k^*(z)1_{\{(1-\delta)k \le k^*(z) \le R(z,k)\}} + (1-\delta)k1_{\{k^*(z)<(1-\delta)k\}}. \tag{9}$$

Suppose that $\{(z_t, k_t), t \ge 0\}$ is in a stationary distribution. Equality (9) becomes,

$$k_t \overset{d}{=} R(z_t,k_t)1_{\{R(z_t,k_t)<k^*(z_t)\}} + k^*(z_t)1_{\{(1-\delta)k_t \le k^*(z_t) \le R(z_t,k_t)\}} + (1-\delta)k_t1_{\{k^*(z_t)<(1-\delta)k_t\}}. \tag{10}$$

Assume $k_t > 0$. It is clear that $k_t \overset{d}{=} (1-\delta)k_t$ can never happen. The firm purchases capital up to the target, but not beyond. If the capital starts beyond the target the firm allows it to erode back to the target. So capital beyond the target is not part of a stationary distribution. The capital level must be no more than the optimal target level. All the states belonging to the event $\{k^*(z) < (1-\delta)k\}$ are transient states. So the last term in equation (10) does not happen in the stationary distribution. The probability of the event $\{k^*(z_t) < (1-\delta)k_t\}$ must be zero under the stationary distribution.

Lemma 1 *Suppose that the firm is following an optimal policy as given by Proposition 1. Then the state process $\{(z_t, k_t), t \ge 0\}$ satisfies,*

(a) *If $\{(z_t, k_t)\}$ is in a stationary distribution, then it must satisfy*

$$k_t \overset{d}{=} R(z_t,k_t)1_{\{R(z_t,k_t)<k^*(z_t)\}} + k^*(z_t)1_{\{(1-\delta)k_t \le k^*(z_t) \le R(z_t,k_t)\}}. \tag{11}$$

(b) *Suppose that exogenous process $\{z_t, t \ge 0\}$ is in a stationary distribution. Suppose that k_t, together with z_t, satisfies the equality (11). Then the pair $\{(z_t, k_t), t \ge 0\}$ is in a stationary distribution.*

Part (a) follows immediately from equality (10). Part (b) follows from the definition of a stationary distribution.

In a stationary distribution the states can be partitioned relative to the capital target. In the high region Z_H the firm can reach the capital target. In the low region Z_L, the firm can only reach a lower capital target given by

$$k(z) = \left(\frac{zA}{\delta}\right)^{1/(1-\alpha)}. \tag{12}$$

These sets of states are defined by

$$Z_L = \{z \in \mathbb{Z} : k(z) < k^*(z)\} \quad \text{and} \quad Z_H = \{z \in \mathbb{Z} : k(z) \geq k^*(z)\}.$$

Note that Z_L and Z_H do form a partition for the exogenous state space $\mathbb{Z}$.

Proposition 3 *Suppose that $\{(z_t, k_t), t \geq 0\}$ is the state process generated by the firm following the optimal policy in Proposition 1. Suppose that the exogenous state $\{z_t, t \geq 0\}$ follows a stationary distribution. Then this distribution, together with the following conditional distribution,*

$$\mathsf{P}\left(k_t = k(z)\middle| z_t = z\right) = 1 \textit{ for } z \in Z_L, \textit{ and } \mathsf{P}\big(k_t = k^*(z)\big| z_t = z\big) = 1 \textit{ for } z \in Z_H,$$

gives a stationary distribution for the state process. Under the conditions in Proposition 2, they jointly give the unique stationary distribution with $\mathsf{P}(k_t > 0) > 0$.

Why is Proposition 3 true? To establish this, verify that the distribution satisfies the condition (11) in Lemma 1. The argument starts with the observation that due to the right-hand side of the equality (11), the stationary distribution consists of two cases depending on whether $\{R(z_t, k_t) < k^*(z_t)\}$ or $\{(1-\delta)k_t \leq k^*(z_t) \leq R(z_t, k_t)\}$.

When the firm cannot attain the capital target, $k_t \overset{d}{=} R(z_t, k_t)$. It is immediate that $k_t = k(z_t)$ with $k(\cdot)$ defined by (12) meets this requirement. In this case, the condition $\{R(z_t, k_t) < k^*(z_t)\}$ can be written as $\{R(z, k(z)) < k^*(z)\}$. Tedious but routine mathematical steps, then establish that $Z_L = \{z : R\left(z, k(z)\right) < k^*(z)\}$.

When the firm can attain the capital target, equality (11) implies $k_t \overset{d}{=} k^*(z_t)$. Then the condition can be written $\{(1-\delta)k^*(z_t) \leq k^*(z_t) \leq R(z_t, k^*(z_t)\}$, or $\{k^*(z) \leq R(z, k^*(z)\}$. This is due to the fact that $\{(1-\delta)k^*(z) \leq k^*(z)\}$ obviously holds and hence is dropped. Again, tedious but routine mathematical steps, show that that $Z_H = \left\{z \in \mathbb{Z} : k^*(z) \leq R(z, k^*(z))\right\}$, thus completing the argument. A more detailed formal proof is provided in the appendix.

This proposition shows that in the stationary distribution, the capital level is completely determined by the exogenous state variable. The exogenous state can be classified into two sets Z_L and Z_H. Alternative expressions for these two sets are

$$Z_L = \{z : R\left(z, k(z)\right) < k^*(z)\} \text{ and } Z_H = \left\{z \in \mathbb{Z} : k^*(z) \leq R(z, k^*(z))\right\}.$$

So the set Z_L is the set that includes all the exogenous state where the revenue $R\left(z, k(z)\right)$ is strictly less than the target capital, and the set Z_H is the set that includes all the exogenous states where the revenue $R(z, k^*(z))$ is greater than or equal to the target capital $k^*(z)$.

When the exogenous state is in set Z_L, the capital level is at a level $k(z)$, which is lower than the capital target level. The corresponding production level is therefore also lower than the target. When the exogenous state is in set Z_H, the capital level is at a level $k^*(z)$, which is the capital target level. The corresponding production level is therefore at the target level.

The set Z_L is decreasing in τ, increasing in δ and β, and it is invariant in A and α.

Under the stationary distribution the expectations are obtained by integrating over the ranges in the stationary distribution. Let the stationary state process $\{(z_t, k_t), t \geq 0\}$ be determined by the optimal policy as described in Proposition 1. Let G denote the cumulative stationary distribution function of the exogenous state. Expected capital is given by

$$\mathsf{E}k_t = \int_{z \in Z_L} k(z) dG(z) + \int_{z \in Z_H} k^*(z) dG(z). \tag{13}$$

Expected production is given by

$$\mathsf{E}F(z_t, k_t) = \int_{z \in Z_L} zA\left[k(z)\right]^{\alpha} dG(z) + \int_{z \in Z_H} zA\left[k^*(z)\right]^{\alpha} dG(z). \tag{14}$$

Expected revenue is given by

$$\mathsf{E}R(z_t, k_t) = (1-\tau)\mathsf{E}F(z_t, k_t) + [1 - \delta(1-\tau)]\,\mathsf{E}k_t. \tag{15}$$

Discussion

Assumption 1 is a *sufficient* condition for the existence and uniqueness of the optimal investment policy. The additional assumption 2 is a *sufficient* condition for the existence and the uniqueness of the stationary distribution for the state under the optimal investment policy. The state consists of the exogenous state and the capital level.

When these sufficient conditions fail, the conclusions of both Proposition 1 and Proposition 2 may still apply. In particular, explicit forms of the optimal policy and the stationary distribution derived here may hold for a much more general class of the exogenous state processes. To be specific, the boundedness assumption may be relaxed to a process that decays at a rate faster than the exponential rate. Other relaxations may also be possible.

As mentioned before, the firm has no capital adjustment costs. Suppose that the quadratic adjustment costs are added, i.e. $\Psi(k_{i+1}, k_{it}) = \frac{1}{2}\psi\left(\frac{i_{it}}{k_{it}}\right)^2 k_{it}$, where $\psi > 0$ is a constant. We can still show the existence and the uniqueness of the optimal solution. The first statement in Proposition 1 will still apply. However, we cannot show that the optimal policy has the specific structure described by Proposition 1. It seems unlikely to be true. Accordingly, our approach to obtaining the stationary distribution would not apply as is.

2.3 Specific shock processes

Martingale

When the exogenous Markov state $\{z_t, t \geq 0\}$ is a Martingale, we have $\mathsf{E}_z(z') = z$. Given the productivity today (zA), the productivity tomorrow ($z'A$) is the same as today in expectation. A special case is when the exogenous state is deterministic.

In this case, the capital target is given by

$$k^*(z) = \left(\frac{\alpha\beta A(1-\tau)z}{1-\beta+\beta\delta(1-\tau)}\right)^{1/(1-\alpha)}.$$

Then the set Z_L is determined by

$$\left(\frac{zA}{\delta}\right)^{1/(1-\alpha)} < \left(\frac{\alpha\beta A(1-\tau)z}{1-\beta+\beta\delta(1-\tau)}\right)^{1/(1-\alpha)},$$

which gives

$$1-\beta+\beta\delta(1-\tau) < \alpha\beta\delta(1-\tau) \text{ or } 1-\beta+\beta\delta(1-\tau)(1-\alpha) < 0.$$

This can never happen; so, $Z_L = \emptyset$. Hence, when the exogenous Markov state is a martingale, the capital level is always at the target in the stationary distribution.

Random walk with bounds.

Here it is assumed that the exogenous shock follows a simple random walk. The system takes values in $\{\underline{z}, \underline{z}+1, \ldots, \underline{z}+N\}$, where $\underline{z}$ is a positive number. Note that we restrict the random walk to be bounded below by a positive number so that the random shock is always positive.

Let $u_i = \underline{z} + i$ and $p_{ij} = \mathsf{P}\big(z' = u_j | z = u_i\big)$, $i, j = 0, \ldots, N$. The transition probabilities of the Markov chain $\{z_t, \geq 0\}$ are given by: $p_{i,i+1} = p$ ($i = 0, \ldots, N-1$), $p_{NN} = p$; $p_{i,i-1} = q$ ($i = 1, \ldots, N$), $p_{00} = q$, where $p + q = 1$ and $0 < p < 1$. The stationary distribution $\pi_i = \mathsf{P}\big(z = u_i\big)$ is given by

$$\pi_0 = \frac{1-\frac{p}{q}}{1-\left(\frac{p}{q}\right)^{N+1}} \qquad \text{and} \qquad \pi_i = \pi_0\left(\frac{p}{q}\right)^i, \quad i = 1, \ldots, N.$$

We note that $\mathsf{E}\big(z'|z = u_i\big) = u_i + p1_{\{i<N\}} - q1_{\{i>0\}}$, $i = 0, \ldots, N$. In this case, the target capital level has an explicit form,

$$k_i^* \equiv k^*(u_i) = \left(\frac{\alpha\beta A(1-\tau)\left(u_i + p1_{\{i<N\}} - q1_{\{i>0\}}\right)}{1-\beta+\beta\delta(1-\tau)} \right)^{1/(1-\alpha)}, \quad i = 0, \dots, N,$$

where we note that $k_i^* = k_1^*$ for all $i = 1, \dots, N-1$.

In this case, the set Z_L is determined by

$$\left(\frac{u_i A}{\delta}\right)^{1/(1-\alpha)} < \left(\frac{\alpha\beta A(1-\tau)\left(u_i + p1_{\{i<N\}} - q1_{\{i>0\}}\right)}{1-\beta+\beta\delta(1-\tau)} \right)^{1/(1-\alpha)},$$

which is equivalent to

$$\left[1-\beta+\beta\delta(1-\tau)(1-\alpha)\right]u_i < \alpha\beta\delta(1-\tau)\left(p1_{\{i<N\}} - q1_{\{i>0\}}\right). \tag{16}$$

We first note the two boundary cases. When $i = N$, the above inequality cannot hold since its right-hand-side is negative for $i = N$. This means that when the exogenous state is at its upper bound u_N, the capital must be at its target level. When $i = 0$, the above inequality takes the form,

$$\left[1-\beta+\beta\delta(1-\tau)(1-\alpha)\right]u_0 < \alpha\beta\delta(1-\tau)p.$$

If it holds, then the capital at u_0 (the lower bound of the exogenous state) will be at the level $(u_0 A/\delta)^{1/(1-\alpha)}$, lower than its target level k_0^*; otherwise, the capital is at its target level k_0^*.

Next, consider the non-boundary state ($i = 1, \dots, N-1$). In this case, the inequality (16) becomes

$$\left[1-\beta+\beta\delta(1-\tau)(1-\alpha)\right]u_i < \alpha\beta\delta(1-\tau)\left(p-q\right),$$

where the right-hand side is a constant and the left-hand-side increases in i. If the above does not hold for $i = 1$, then all the values of capital in the stationary distribution must be at their target levels (k_i^*). If the above holds for $i = N-1$, then all the stationary distribution capitals must be at their corresponding levels, $(u_i A/\delta)^{1/(1-\alpha)}$, lower than their corresponding target levels. In the more general case, there exists an index i^* such that in the stationary distribution capital is at $((u_i A/\delta)^{1/(1-\alpha)})$ lower than the target capital level for $i < i^*$ and the stationary distribution capital is at the target level (k_i^*) for $i \geq i^*$.

Random walk without bounds

Consider an unbounded random walk in the sense that $N = \infty$. This is unbounded. It is necessary that we assume $p < q$, otherwise, there is no stationary distribution. In this case,

$$\pi_i = \left(1 - \frac{p}{q}\right)\left(\frac{p}{q}\right)^i, \qquad i = 0, 1, \ldots;$$

then $\mathsf{E}\big(z' = u_i\big) = u_i + p - q1_{\{i>0\}}$.

The special case of the stationary distribution is when the capital level is always at its target, is given by

$$\mathsf{P}\big(z = u_i, k = k_i^*\big) = \pi_i \quad \text{or} \quad \mathsf{P}\big(k = k_i^* | z = u_i\big) = 1, \qquad i = 0, \ldots, N,$$

where

$$k_i^* \equiv \left(\frac{\alpha\beta A(1-\tau)\left(u_i + p - q1_{\{i>0\}}\right)}{1 - \beta + \beta\delta(1-\tau)}\right)^{1/(1-\alpha)}, \quad i = 0, 1, \ldots.$$

Independent exogenous shocks with the same mean

In this section assume that the exogenous shock is a sequence of independent random variables with a common mean. Note that an iid sequence is a special case of this. Let $\tilde{z}$ be the expected value of the exogenous state. Then, the capital target,

$$k^* = \left(\frac{\alpha\beta A(1-\tau)\tilde{z}}{1 - \beta + \beta\delta(1-\tau)}\right)^{1/(1-\alpha)},$$

does not vary with the exogenous state.

The set Z_L is determined by the inequality,

$$\left(\frac{zA}{\delta}\right)^{1/(1-\alpha)} < \left(\frac{\alpha\beta A(1-\tau)\tilde{z}}{1-\beta+\beta\delta(1-\tau)}\right)^{1/(1-\alpha)},$$

or equivalent to

$$\big[1-\beta+\beta\delta(1-\tau)\big]z < \alpha\beta\delta(1-\tau)\tilde{z}.$$

Let

$$z^* = \frac{\alpha\beta\delta(1-\tau)\tilde{z}}{1-\beta+\beta\delta(1-\tau)}$$

Then it is clear that the above inequality holds when $z < z^*$ and does not hold when $z \geq z^*$.

The capital follows a threshold policy with the threshold z^*. When $z < z^*$, the capital level is at $(zA/\delta)^{1/(1-\alpha)}$ which is lower than the target level. When $z \geq z^*$ capital is at the target level k^*.

It is possible that the set Z_L is an empty set when z^* is a lower bound of the state space $\mathbb{Z}$. Then the target k^* is always reached.

AR-1 with normal iid shocks

Consider the Markov process defined by $\log(z_{t+1}) = \nu \log(z_t) + \epsilon_t$ with $\{\epsilon_t\}$ iid Normally distributed with mean 0 and standard deviation σ. Assume $\nu \in (0,1)$. Then z_t follows a geometric Normal distribution. This is a particularly popular shock process in applied papers.

In this case, $\mathsf{E}_z(z') = e^{\sigma^2/2} z^\nu$ and

$$k^*(z) = \left(\frac{\alpha\beta A(1-\tau)e^{\sigma^2/2}}{1-\beta+\beta\delta(1-\tau)}\right)^{1/(1-\alpha)} z^{\nu/(1-\alpha)}.$$

The set Z_L is determined by

$$\left(\frac{zA}{\delta}\right)^{1/(1-\alpha)} < \left(\frac{\alpha\beta A(1-\tau)e^{\sigma^2/2}}{1-\beta+\beta\delta(1-\tau)}\right)^{1/(1-\alpha)} z^{\nu/(1-\alpha)},$$

or equivalently,

$$\big[1-\beta+\beta\delta(1-\tau)\big]z^{1-\nu} < \alpha\beta\delta(1-\tau)e^{\sigma^2/2}.$$

Let

$$z^* = \left(\frac{\alpha\beta\delta(1-\tau)e^{\sigma^2/2}}{1-\beta+\beta\delta(1-\tau)}\right)^{1/(1-\nu)}.$$

Then it is clear that the above inequality holds when $z < z^*$ and does not hold when $z \geq z^*$.

The capital follows a threshold policy with the threshold z^*. When $z < z^*$, the capital is at $(zA/\delta)^{1/(1-\alpha)}$, which is lower than the target level. When $z \geq z^*$, the capital is at the target level $k^*(z)$.

The threshold z^* increases in σ, ν, α and β, decreases in τ, and is invariant in A.

3 Debt market

In this section the firm has access to a debt market in which it can save or borrow. The key question is whether the existence of the debt market results in the same optimal choice of capital in the stationary distribution. A debt market increases the dimension of the problem. Does it affect optimal capital in the long run?

The debt market permits the buying and selling of one period risk-free bonds. A bond is bought or sold in the current period at a market price of q. The owner of that bond is paid $\$1$ at maturity next period. The interest rate is $1-q$. The number of bonds due in the current period is denoted b and the number of bonds being issued in the current period is denoted b'. For a firm these numbers can be positive (debt) or negative (saving).

Interest received on firm saving is taxable at the same rate as ordinary firm revenue, τ. Interest paid on firm borrowing is tax deducible. As in Strebulaev and Whited (2012) the interest tax deduction benefit is received when the bond is issued. With this tax structure, the cash received by the firm from issuing b' bonds is $qb' + \tau(1-q)b'$.

Firm budget constraint

We now add the debt market to the firm's budget constraint. As before, the flow budget says cash in = cash out.

$$\underbrace{(1-\tau)F(k)}_{\substack{\text{After-tax}\\ \text{revenue}}} + \underbrace{b'q}_{\substack{\text{new bond}\\ \text{issue revenue}}} + \underbrace{\delta k\tau}_{\substack{\text{depreciation}\\ \text{tax benefit}}} + \underbrace{b'(1-q)\tau}_{\substack{\text{Interest}\\ \text{tax benefit}\\ \text{from new bonds}}} = \underbrace{d}_{\text{dividends}} + \underbrace{i}_{\text{investment}} + \underbrace{b}_{\substack{\text{old bond}\\ \text{repayment}}} \tag{17}$$

Risk-free debt

It is assumed that all debt is required to be risk-free. This avoids the need to model bankruptcy, and it simplifies the pricing of bonds. With risk-free debt, the price of the debt is determined by the risk-free interest rate. Let $\rho \in (0,1)$ be the risk free interest rate. Because the debt is risk free, $q = 1/(1+\rho)$ is the market price of a one period bond.

Recall that the revenue the firm obtains by issuing a risk-free bond is $q + \tau(1-q)$, the money raised from the market plus the tax benefit. We can group these as,

$$q + \tau(1-q) = \frac{1+\tau\rho}{1+\rho}.$$

The firm budget in equation (17) can then be written as $(1-\tau)F(z,k) + \tau\delta k + \frac{1+\tau\rho}{1+\rho}b' = d + i + b$.

How can we be sure that the debt is risk free? For simplicity saving is assumed to be deposited in a risk free bank account. To ensure that firm borrowing is risk free is more complex. Start with the firm budget equation (17). Recall that cash in must be at least as great as cash out due to free disposal. Minimize the non-debt cash out by setting $d = i = 0$. That gives,

$$(1-\tau)F(z,k) + \tau\delta k + [q + \tau(1-q)]b' \geq b. \tag{18}$$

The current exogenous state z was not known when the older bond b was issued. To ensure that the firm borrowing is risk free regardless the exogenous state, replace the current after-tax revenue by an observable lower bound; the worst case after-tax revenue. Let $\underline{z} \geq 0$ be the largest lower bound for the exogenous state, so that $\mathsf{P}_z(z' \geq \underline{z}) = 1$ for all $z \in \mathbb{Z}$. Then $F(\underline{z}, k)$ was a known lower bound for the production in the previous period when the older bond b was issued. So condition (18) can be replaced by

$$(1-\tau)F(\underline{z},k)+\tau\delta k+[q+\tau(1-q)]b'\geq b. \tag{19}$$

Inequality (19) always holds if the firm chooses new bond b' to be large enough and the price at which the firm sells that debt remains positive. However that is not reasonable. We need to rule out Ponzi schemes; and even before that, if the debt is too great it will be risky. We need to have an upper bound on the bond issue in any period that is consistent with the bond being free of risk.

The bond issued in the previous period b will be paid in the current period. The cash flow into the firm in the current period depends on the capital level in the current period k. Hence we assume that the bound for the previous period bond is a function of k. Similarly, the bond issued in the current period b' will be paid in the next period, and the cash flow into the firm in the next period depends on the capital level at the next period k'. So correspondingly it is natural to assume that the bound for the current bond is a function of capital.

The next step is to find a function $\bar{b}(k)$ such that the debt obligation, inequality (19) is feasible. Setting the boundary at zero debt would always leave the firm with no risk of defaulting on debt. But clearly a much more permissive bound is of greater interest. In order to pay for the current debt obligation the firm has the available after-tax revenue as well as potentially newly issued debt. Of course the new debt that is used to make the old debt payment, must itself also be risk-free. Hennessy and Whited (2005) side step this issue by having capital fire-sales used to make the debt risk-free instead of more subsequent borrowing.

Set $b=\bar{b}(k)$ as the worst case in the sense that the old bond is at the upper bound. Assume that the firm does no investing $i=0$, so that it has as much resources as possible to pay for the old debt. Then $k'=(1-\delta)k$ and $b'=\bar{b}((1-\delta)k)$. Inequality (19) becomes,

$$(1-\tau)F(\underline{z},k)+\tau\delta k+\frac{1+\tau\rho}{1+\rho}\bar{b}((1-\delta)k)\geq\bar{b}(k). \tag{20}$$

If the function $\bar{b}(\cdot)$ satisfies inequality (20), and $b_t\leq\bar{b}(k_{t+1})$ for all $t\geq 0$, then the bond is risk free.

To ensure that the debt is risk-free therefore, we need to find a function $\bar{b}(\cdot)$ satisfying inequality (20). We want to exclude Ponzi-like schemes in which debt is supported by nothing other than more future debt. But we do not want to prevent new debt from playing any role in supporting old debt, as is sometimes assumed for simplicity. According to the evidence provided by Lian and Ma (2021) for US non-financial firms, the dominant form of debt is backed by the firm's cash flows from operations. Benmelech et al. (2020) also find that lending secured by assets is declining. So we consider a debt limit that takes the following form,

$$\bar{b}(k) = \theta[(1-\tau)F(\underline{z}, k) + \tau\delta k]. \tag{21}$$

Will this work? Substitute the conjecture for $\bar{b}(k)$ into inequality (20) to get,

$$(1-\theta)\left[(1-\tau)F(\underline{z}, k) + \tau\delta k\right] + \frac{1+\tau\rho}{1+\rho}\theta\left[(1-\tau)F(\underline{z}, (1-\delta)k) + \tau\delta(1-\delta)k\right] \geq 0.$$

Collecting the terms and dividing both sides by k^α, it can also be expressed as,

$$\left[(1-\theta) + \frac{1+\tau\rho}{1+\rho}\theta(1-\delta)^\alpha\right](1-\tau)\underline{z}A + \left[(1-\theta) + \frac{1+\tau\rho}{1+\rho}\theta(1-\delta)\right]\tau\delta k^{1-\alpha} \geq 0. \tag{22}$$

For the above inequality to hold for all k, we must have

$$(1-\theta) + \frac{1+\tau\rho}{1+\rho}\theta(1-\delta)^\alpha \geq 0 \quad \text{and} \quad (1-\theta) + \frac{1+\tau\rho}{1+\rho}\theta(1-\delta) \geq 0.$$

The first of these inequalities follows by setting $k = 0$. The second inequality follows by considering $k \to \infty$. These two inequalities are also sufficient for the inequality (22). Note that,

$$(1-\theta) + \frac{1+\tau\rho}{1+\rho}\theta(1-\delta)^\alpha \geq (1-\theta) + \frac{1+\tau\rho}{1+\rho}\theta(1-\delta).$$

Accordingly the necessary and sufficient condition becomes

$$(1-\theta) + \frac{1+\tau\rho}{1+\rho}\theta(1-\delta) \geq 0.$$

That is equivalent to,

$$\left[1 - \frac{(1+\tau\rho)(1-\delta)}{1+\rho}\right]\theta \leq 1.$$

So we see that θ has the form,

$$\theta = \frac{1+\rho}{(1-\tau)\rho + \delta(1+\tau\rho)}, \tag{23}$$

is the largest θ such that the above inequality holds. In other words, θ given by equation (23) gives the most permissive debt bound $\bar{b}(k)$, of the form given by equation (21). This establishes inequality (20), and provides a risk-free debt bound that is greater than zero. We summarize this as Lemma 2.

In equation (23) recall that ρ, τ, δ are all positive numbers. Then due to the fact that $\delta < 1$, we see that $\theta > 1$. The firm can borrow more than it will generate as revenue in the worst case next period, due to the ability to do subsequent borrowing.

Lemma 2 *Let*

$$\bar{b}(k) = \theta\left[(1-\tau)F(\underline{z}, k) + \tau\delta k\right], \tag{24}$$

with θ given by (23). For any current state (z, k, b) satisfying $z \geq \underline{z}$, $k \geq 0$ and $b \leq \bar{b}(k)$, there always exists a feasible policy (b', d, i) satisfying $b' \leq \bar{b}(k')$, the budget balance equation (17), and the nonnegativity constraints $d \geq 0$ and $i \geq 0$.

Tax makes debt beneficial

Since the debt will be risk-free, when a bond is issued the promised payment next period will take place for sure. From the perspective of the current period that is discounted. We can define a net revenue from issuing a bond as

$$\nu \equiv \frac{1+\tau\rho}{1+\rho} - \beta. \tag{25}$$

This nets out $\beta \in (0, 1)$ which is the one-period discounting factor.

If $\beta = 1/(1+\rho)$ then $\nu > 0$. In a model that determines a market clearing risk-free rate of return endogenously, β equals $1/(1+\rho)$ would be a natural equilibrium condition. We do not have a market clearing condition for risk-free debt since we are studying a partial equilibrium firm model. However, it might still be thought to be close, so $\nu > 0$ seems reasonable.

If $\nu > 0$ the firm wants to borrow as much as is feasible. Feasibility is restricted by the risk-free borrowing constraint. Recall that $b' < 0$ represent saving by the firm. Due to taxation, the firm does not want to save.

If $\nu < 0$, then the firm would have an incentive to avoid debt and to save as much as possible. This can be inferred from the proof of Proposition 4. To avoid this, we assume throughout that $\nu > 0$.

Firm problem

The firm's problem with a debt market is defined on a state space given by $(z, k, b) \in \mathbb{X}$ where, $\mathbb{X} = \{(z, k, b) \in \mathbb{Z} \times \Re_+ \times \Re : b \leq \bar{b}(k)\}$. The firm problem is,

$$V(z,k,b) = \sup_{\{b',d,i\}} d + \beta \mathsf{E}_z V(z',k',b') \tag{26}$$

$$\text{subject to} \quad (1-\tau)F(z,k) + \tau\delta k + \frac{1+\tau\rho}{1+\rho} b' = d + i + b \tag{27}$$

$$k' = (1-\delta)k + i \tag{28}$$

$$d \geq 0 \quad \text{and} \quad i \geq 0 \tag{29}$$

$$b' \leq \bar{b}(k') \tag{30}$$

The only extra complication in this problem is the risk-free single period debt with interest payments being tax deductible. Due to Lemma 2 it can be shown by induction that equation (30) implies that $b_t \leq \bar{b}(k_t)$ for all $t \geq 0$.

3.1 Optimal capital with a debt market

Here we establish existence and uniqueness of the firm decisions when risk-free borrowing is allowed.

It is again possible for the firm to have so much capital already (excess capital) that it does not want to buy more. It may be able to reach the capital target (in range). It is also possible that the firm does not have enough revenue or enough risk-free borrowing capacity to reach the capital target (inadequate capital). It is this last case that creates the most complexity.

As in the analysis of financial autarky, it is natural to start with the capital target which assumes that the firm is at an interior solution. Then the first order condition for capital gives the target $k_d^*(z)$ as follows.

$$k_d^*(z) = \left(\frac{\alpha A(1-\tau)\left[\beta \mathsf{E}_z\left(z'\right) + \nu\theta\underline{\mathsf{z}}\right]}{1-\beta\left[1-\delta(1-\tau)\right] - \nu\theta\tau\delta} \right)^{1/(1-\alpha)}. \tag{31}$$

So there are three cases to be distinguished.

In the first case, the firm starts with excess capital, $k_d^*(z) \leq (1-\delta)k$. The firm issues as much risk-free debt as it can in order to take advantage of the tax benefit, $b' = \bar{b}\left((1-\delta)k\right)$. The firm invests nothing in capital. It just pays out the resources as dividends,

$$d = R(z,k) + \frac{1+\tau\rho}{1+\rho}\bar{b}\left((1-\delta)k\right) - (1-\delta)k - b \geq 0. \tag{32}$$

Nonnegativity is ensured by $b \leq \bar{b}(k)$.

In the second case, the firm does not have excess capital, $k_d^*(z) > (1-\delta)k$ and

$$R(z,k) + \frac{1+\tau\rho}{1+\rho}\bar{b}\left(k_d^*(z)\right) - k_d^*(z) - b \geq 0. \tag{33}$$

The firm will again issue as much risk-free debt as it can,

$$b' = \bar{b}\left(k_d^*(z)\right). \tag{34}$$

It uses the resources to invest enough so that it reaches the capital target, $i = k_d^*(z) - (1-\delta)k$. The remainder is paid out as a dividend,

$$d = R(z,k) + \frac{1+\tau\rho}{1+\rho}\bar{b}\left(k_d^*(z)\right) - k_d^*(z) - b. \tag{35}$$

In the third case, the firm does not have excess capital and even with maximum risk-free borrowing it cannot reach the capital target. This can be expressed as,

$$R(z,k)+\frac{1+\tau\rho}{1+\rho}\bar{b}\left(k_d^*(z)\right)-k_d^*(z)-b<0, \tag{36}$$

The firm will pay no dividends. Optimal debt b' and investment i are determined by the following maximization problem,

$$\begin{aligned} \max_{b'\leq\bar{b}((1-\delta)k+i),i\geq 0} \quad & \mathsf{E}_z V(z',(1-\delta)k+i,b') \\ \text{subject to} \quad & (1-\tau)F(z,k)+\tau\delta k+\frac{1+\tau\rho}{1+\rho}b'-b=i. \end{aligned} \tag{37}$$

The objective function in (37) is the firm's optimal value function. Proposition 4 below guarantees that it is uniquely defined. However, we do not have an explicit functional form for it. We denote the optimal debt $b'=\psi(z,k,b)$, the optimal investment $i=\phi(z,k,b)-(1-\delta)k$, and the corresponding optimal capital level $k'=\phi(z,k,b)$. The optimal dividend is zero. Under the condition (36), we can show by contradiction that

$$\phi(z,k,b)<k_d^*(z) \quad \text{and} \quad \psi(z,k,b)<\bar{b}(k_d^*(z))$$

must hold. The value function is increasing in k' and decreasing in b'. We have not been able to establish the relative strengths and interactions of these opposing forces on firm value.

The three cases exhaust all possible cases. This is because inequality (36) implies $k_d^*(z)>(1-\delta)$. To see this,

$$\begin{aligned} k_d^*(z) &> R(z,k)+\frac{1+\tau\rho}{1+\rho}\bar{b}\left(k_d^*(z)\right)-b \\ &\geq R(z,k)+\frac{1+\tau\rho}{1+\rho}\bar{b}\left(k_d^*(z)\right)-\bar{b}(k) \\ &= (1-\tau)F(z)+\tau\delta k+\frac{1+\tau\rho}{1+\rho}\bar{b}\left(k_d^*(z)\right)-\bar{b}(k)+(1-\delta)k \\ &\geq (1-\delta)k. \end{aligned}$$

The first inequality follows from inequality (36). The second is from the debt bound $b\leq$

$\bar{b}(k)$. The last inequality is due to equation (20).

Proposition 4 *Under Assumption 1, there exists a unique optimal policy solving the Bellman equation (26). The optimal policy depends on the current exogenous state z, the current debt b, and the current capital k. Case (a) Excess capital. If $k_d^*(z) \leq (1-\delta)k$ the firm has excess capital. It invests nothing, $i = 0$; issues the maximum debt, $b' = \bar{b}\left((1-\delta)k\right)$; the remaining resources are paid as a dividend, shown in equation (32). Case (b) Capital in range. For the capital to be in range, $k_d^*(z) > (1-\delta)k$ and equation (33) hold. The firm invests enough to attain the capital target, $i = k_d^*(z) - (1-\delta)k$; it issues the maximum debt $b' = \bar{b}\left(k_d^*(z)\right)$; and the remaining revenue is paid as a dividend, shown in equation (35). Case (c) Inadequate capital. For the capital to be inadequate, condition (36) holds. The the firm pays no dividend ($d = 0$). The optimal debt $b' = \psi(z,k,b)$ and optimal investment $i = \phi(z,k,b) - (1-\delta)k$ are determined by (37). The new capital level $k' = \phi(z,k,b) < k_d^*(z)$ and the new bond level $b' \leq \bar{b}(k_d^*(z))$.*

The proof of this proposition is similar to the proof of Proposition 1 and is provided in the appendix.

In case (b) the firm has less capital than the target, and it has enough resources to reach that target. Due to the interest tax benefit, it issues debt right to the risk-free limit. So the firm invests in acquiring capital, it issues debt, and it also pays dividends.

In case (c) the firm has less capital than the target, but it does not have enough resources to reach that target. Due to the interest tax benefit and the production benefit from the extra capital that the debt permits, it issues debt. So the firm invests in acquiring capital, it issues debt, but it does not pay dividends.

Discussion

Issuing debt means that money is coming in to the firm. Paying dividends means that money is flowing out of the firm. Some papers assume that money flows in one direction or the other but not in both directions at the same time. This is particularly common in papers that interpret what we call dividends, as being share repurchases. It can be described as 'debt financed dividends' (Farre-Mensa et al., 2020), which makes it sound odd. In case (b) of Proposition 4 both are taking place together. This is because even a firm with adequate internal resources, will wish to take advantage of the interest tax benefits, as in Modigliani and Miller (1963). Farre-Mensa et al. (2020) report that more than 30% of dividends and share repurchases are externally financed in this sense, mostly by debt.

The risk-free limit in Lemma 2 is based on the form given in equation (21). This form seems reasonable, tractable, and broadly consistent with the evidence that firms about 80% of debt is backed by the on going cash flows, see Lian and Ma (2021). However, other ideas might generate different risk-free borrowing limits. Hennessy and Whited (2005) assume that asset fire-sales are used to ensure that the debt is risk free. Asset-based lending accounts for about 20% of firm debt according to Lian and Ma (2021). Asset fire sales can be added to our risk-free debt restriction if it is assumed that such a market exists. If so then the right hand side of equation (24) would be larger, and so $\bar{b}(k)$ would also be larger.

In Proposition 4 case (c) the firm cannot reach the capital target $k_d^*(z)$. Does it go right to the risk-free bond limit? The answer is not obvious. The value function is the objective function for the maximization problem given by (37)). V increases in the capital level k' and decreases in the bond b'. Due to the constraint that the budget must balance, increasing bonds b' allows the firm to increase the capital level $k' = (1-\delta)k + i$. The former has a negative impact on firm value. The latter has a positive impact on firm value. How they net out is unclear. We do not know in general if optimal bond b' is at an interior solution, or right at the upper bound $\bar{b}(k')$.

The firm has an optimal capital investment structure similar to that of the financial autarky model. The firm has an optimal investment target level. Due to the ability to issue debt, the firm has more cash flow. The firm uses the additional cash flow for capital investment. Hence the firm has a higher capital target level, $k_d^*(z) \geq k^*(z)$.

Hennessy and Whited (2005) makes a different assumption than we do in order to make the debt risk free. They allow a fire sale of assets to be used to cover the bond. They do not allow the firm to issue new bonds to cover the old bond repayment. If we ignore the fire sale term and the corporate tax term in their inequality (12), it becomes $p' \leq \pi(k', \underline{z})$; the bond payment less than the lower bound of the future revenue. In our notation, this means $b' \leq R(\underline{z}, k')$. Suppose we let $b' = 0$ and $\pi(k, z) = (1-\tau)F(z, k) + \tau\delta k$; then our inequality (19) gives $b \leq \pi(k, \underline{z})$. Next replace b by b' and k by k'. That gives $b' \leq \pi(k', \underline{z})$, which is the same as their inequality (12) albeit without the fire sale and the corporate tax effects. A key difference between our model and theirs, is that we allow greater scope for future debt while they allow for asset fire sales. Of course, due to our focus we study autarky and the stationary state which they do not; while they study numerical solutions to the model which we do not.

According to Myers (1984) the existence of a leverage target is a critical characteristic of the trade-off theory of capital structure. We have seen that even without a debt market, the firm can have a physical capital target. When a debt market is open, the firm again has a physical capital target. It does play a critical role in the firm's debt choices. So a type of finance targeting does emerge that seems similar to the financing behavior described in DeAngelo and Roll (2015), Frank and Shen (2019), and DeMarzo and He (2021).

3.2 Stationary distribution with a debt market

There are similarities between the stationary distribution under financial autarky and when there is a risk-free debt market. Of course the dimensionality increased so there are added considerations.

Proposition 5 *Let $\{x_t, t \geq 0\}$, where $x_t = (z_t, k_t, b_t)$, be the state process under the optimal policy in Proposition 4. Consider x in the state space $\mathbb{X} \equiv \mathbb{Z} \times (0, \infty) \times \Re$. Under Assumptions 1 and 2, (a) the state process $\{x_t, t \geq 0\}$ is Harris ergodic and has a unique stationary probability measure π. (b) x_t converges in distribution; specifically, for any $x \in \mathbb{X}$, $\mathsf{P}_x(x_t \in B) \to \pi(B)$ as $t \to \infty$ for all $B \subseteq \mathbb{X}$.*

The proof of this proposition is almost the same as the proof for Proposition 2 with the details included in the appendix.

Similar to the autarky model, case (a) in Proposition 4 happens with zero probability in the stationary distribution. Conditional on the exogenous state z, the stationary distribution of the state in case (b) is characterized by the capital target $k_d^*(z)$ and the debt upper bound $\bar{b}(k_d^*(z))$.

Similar to financial autarky a critical role is played by the capital target. With a debt market the optimal capital and the optimal debt for the next period only depend on the current exogenous state z. Neither depend on the current capital level k nor on the current debt b.

With a debt market case (c) is more complicated. The optimal capital $k' = \phi(z, k, b)$ and the optimal debt $b' = \phi(z, k, b)$ for the next period may depend on the current capital level k and the current debt b, as well as the current exogenous state z. Under the stationary distribution, we have $k_{t+1} \overset{d}{=} k_t$ and $b_{t+1} \overset{d}{=} b_t$. Heuristically, let $k = \phi(z, k, b)$ and $b = \phi(z, k, b)$, and solve for $k = k_d(z)$ and $b = b_d(z)$.[1] Then $k_d(z)$ is the target capital and $b_d(z)$ to be the target debt in the stationary distribution.

Similarly, we introduce a partition for $\mathbb{Z}$

$$Z_{H,d} = \left\{ z \in \mathbb{Z} : R\left(z, k_d^*(z)\right) + \frac{1+\tau\rho}{1+\rho} \bar{b}\left(k_d^*(z)\right) - k_d^*(z) - \bar{b}\left(k_d^*(z)\right) \geq 0 \right\}, \tag{38}$$

$$Z_{L,d} = \left\{ z \in \mathbb{Z} : R\left(z, k_d^*(z)\right) + \frac{1+\tau\rho}{1+\rho} \bar{b}\left(k_d^*(z)\right) - k_d^*(z) - \bar{b}\left(k_d^*(z)\right) < 0 \right\}. \tag{39}$$

Next we define,

$$\tilde{Z}_{L,d} = \left\{ z \in \mathbb{Z} : R\left(z, k_d(z)\right) + \frac{1+\tau\rho}{1+\rho} \bar{b}\left(k_d^*(z)\right) - k_d(z) - b_d(z) < 0 \right\}. \tag{40}$$

The stationary distribution is,

Proposition 6 *Suppose that: the state process $\{(z_t, k_t, b_t), t \geq 0\}$ satisfies the optimal policy in Proposition 4, and that $\{z_t\}$ follows a stationary distribution. Then the exogenous state distribution, together with the following conditional distribution,*

$$\mathsf{P}\Big(k_t = k_d(z), b_t = b_d(z) \Big| z_t = z\Big) = 1 \qquad \text{for } z \in Z_{L,d} \cap \tilde{Z}_{L,d},$$

$$\mathsf{P}\Big(k_t = k_d^*(z), b_t = \bar{b}\left(k_d^*(z)\right) \Big| z_t = z\Big) = 1 \qquad \text{for } z \in Z_{H,d},$$

gives a stationary distribution for the state process.

This proposition may have just provided a partial description for the stationary distri-

[1] A formal proof is still to be added to the appendix.

bution. Specifically, we have not yet been able to establish that the set $Z_{L,d} = \tilde{Z}_{L,d}$. If it is not true, there exists an exogenous state $z \in Z_{L,d} \setminus \tilde{Z}_{L,d}$; then the stationary distribution of (k_t, b_t) conditional on $z_t = z$ is not included in the proposition.

The proof of this proposition is similar to the proof of Proposition 3. It corresponds to the case in which $z \in Z_H$ in that Proposition. Inequality (38) corresponds case (b) (capital in range) in Proposition 4. In the stationary distribution, case (a) (excess capital) has probability zero. Case (c) (adequate capital) is a difficult case where we do not have an explicit optimal policy. We do not have explicit expressions for the lower capital target $k_d(z)$ and the lower debt target $b_d(z)$.

Using the definitions of $R(z,k)$ and $\bar{b}(k)$, condition (38) can be written as

$$k_d^*(z) \leq \left(\frac{A\left[(1+\rho)z - \rho\theta(1-\tau)\underline{z}\right]}{\delta[1+\rho+\rho\theta\tau]} \right)^{1/(1-\alpha)} . \tag{41}$$

4 How important is the debt market?

This section compares production under autarky to production with a debt market. We will show that if interest is tax deductible then even in the long run the models are not equivalent. If there is no taxation, more restricted comparisons can also be made. In some cases the debt market is unimportant and the models are equivalent. But generally we find that they are not equivalent.

Proposition 7 *The optimal capital target under autarky is lower than the optimal capital target with a debt market. In other words $k^*(z) \leq k_d^*(z)$.*

It is of particular interesting whether the debt market is still important when the corporate tax rate $\tau = 0$. Even when $\tau = 0$, the debt limit $\bar{b}(k) = \theta F(\underline{z}, k)$ can still be positive provided that the production has a positive lower bound. In this case, the firm's optimal policy still issues a positive debt at least when there is excess capital or the capital is in range. That is, the capital market is still active. In particular, the optimal capital target in this case is larger than the optimal capital target for the autarky economy,

$$k_d^*(z) = \left(\frac{\alpha A\left[\beta \mathsf{E}_z\left(z'\right) + \nu\theta\underline{\mathrm{z}}\right]}{1-\beta\left(1-\delta\right)}\right)^{1/(1-\alpha)} \quad \geq \quad k^*(z) = \left(\frac{\alpha\beta A \mathsf{E}_z\left(z'\right)}{1-\beta(1-\delta)}\right)^{1/(1-\alpha)},$$

with $\theta = (1+\rho)/(\delta+\rho)$ and $\nu = \left[1-\beta(1+\rho)\right]/(1+\rho)$.

In terms of the stationary or limiting distributions, we compare the special case when both the autarky model and the debt model have explicit expressions of the stationary distributions. Suppose that

$$k^*(z) \leq R(z, k^*(z)) \quad \text{and} \quad R\left(z, k_d^*(z)\right) \geq k_d^*(z) + \frac{\rho}{1+\rho}\bar{b}\left(k_d^*(z)\right)$$

for all $z \in \mathbb{Z}$. Then, it follows Proposition 3 (corresponding to the case when $Z_H = \mathbb{Z}$) and Proposition 6 that for all $z \in \mathbb{Z}$,

$$\mathsf{P}\big(k_t = k^*(z)\big|z_t = z\big) = 1 \quad \text{and} \quad \mathsf{P}\big(k_{d,t} = k_d^*(z), b_{d,t} = \bar{b}\left(k_d^*(z)\right)\big|z_t = z\big) = 1,$$

where z_t is assumed following the stationary distribution of the exogenous state process $\{z_t\}$. Note that k_t is the optimal capital level of the autarky model following its stationary distribution and $(k_{d,t}, b_{d,t})$ are the optimal capital level and optimal debt under the stationary distribution. Clearly, k_t is stochastically dominated by $k_{d,t}$.

4.1 Some interpretations

Hennessy and Whited (2005) study a model with similarities to ours. To make debt risk-free they have capital fire sales and they do not permit debt roll over to be used to ensure that the debt is risk-free. That restricts the role of debt. They took a useful approach to interpreting some of their expressions. In this subsection we follow their method.

The maximand of the Bellman equation (26) in view of the budget constraint (27) is

$$\tilde{V}(z', k', b'; z, k, b) = (1-\tau)F(z,k) + \left[1-\delta(1-\tau)\right]k + \frac{1+\tau\rho}{1+\rho}b' - b - k' + \beta\mathsf{E}_z V(z', k', b').$$

Suppose that the firm is in range of the capital target and it has an interior solution with both dividends and investment positive, with $d > 0$ and $i > 0$. Then,

$$V(z, b, k) = (1 - \tau)F(z, k) + [1 - \delta(1 - \tau)]\, k.$$

In this case,

$$\frac{\partial \tilde{V}(z', k', b'; z, k, b)}{\partial b'} = \underbrace{\frac{1 + \tau\rho}{1 + \rho}}_{\substack{\text{marginal benefit} \\ \text{of debt}}} - \underbrace{\beta}_{\substack{\text{expected discounted} \\ \text{marginal cost of} \\ \text{servicing debt}}}.$$

The marginal benefit from increasing the debt is $\frac{1+\tau\rho}{1+\rho}$ and the expected discounted marginal cost of serving the debt is β.

We first consider optimal financial policy in the sense of Hennessy and Whited (2005). Suppose that the firm does not have enough capital already to be in range of the capital target, so it pays no dividends $d = 0$. The maximand becomes

$$\tilde{V}(z', k', b'; z, k, b) = \beta \mathsf{E}_z V(z', k', b').$$

In this case, we have

$$k' = (1 - \tau)F(z, k) + [1 - \delta(1 - \tau)]\, k + \frac{1 + \tau\rho}{1 + \rho} b' - b$$

Then assuming the maximand is suitably differentiable,

$$\frac{\partial \tilde{V}(z', k', b'; z, k, b)}{\partial b'} = \underbrace{\beta \frac{1 + \tau\rho}{1 + \rho} \cdot \frac{\partial V(z', k', b')}{\partial k'}}_{\substack{\text{expected discounted} \\ \text{marginal benefit} \\ \text{of increasing debt}}} + \underbrace{\beta \cdot \frac{\partial V(z', k', b')}{\partial b'}}_{\substack{\text{expected discounted} \\ \text{marginal cost} \\ \text{of increased debt}}}.$$

In this case the increase in debt has a benefit consisting of an interaction between the tax benefit and the increased productive capital benefit. It has a discounted cost reflecting the fact that it will need to be repaid.

Next we consider optimal real investment policy in the sense of Hennessy and Whited (2005). Suppose that the firm is in range of the capital target and it has an interior solution, so that $d > 0$ and $i > 0$. Then,

$$\frac{\partial \tilde{V}(z', k', b'; z, k, b)}{\partial k'} = -1 + \beta \mathsf{E}_z \left\{(1-\tau)F_2'(z', k') + 1 - \delta(1-\tau)\right\}.$$

So, the expected discounted marginal gain from increasing the capital stock is

$$\beta \mathsf{E}_z \left\{(1-\tau)F_2'(z', k') + 1 - \delta(1-\tau)\right\} = \alpha\beta(1-\tau)\mathsf{E}_z\left(z'\right)\left(k'\right)^{\alpha-1} + \beta\left[1 - \delta(1-\tau)\right].$$

When $d = 0$,

$$\frac{\partial \tilde{V}(z', k', b'; z, k, b)}{\partial k'} = \underbrace{\beta \cdot \frac{\partial V(z', k', b')}{\partial k'}}_{\substack{\text{expected discounted} \\ \text{marginal benefit} \\ \text{of capital}}} + \underbrace{\beta \frac{1+\rho}{1+\tau\rho} \cdot \frac{\partial V(z', k', b')}{\partial b'}}_{\substack{\text{expected discounted} \\ \text{marginal cost} \\ \text{of capital}}}.$$

This shows that an increase in capital has two effects. There is the direct production benefit, but there is also the cost associated with the increased debt needed to acquire that capital. When the firm has dividends of zero, the cost of acquiring the capital reflects the fact that the marginal unit of capital must be paid for through the use of external debt rather than from internally generated revenue.

5 Conclusion

This paper is a study of the capital investment decisions of a firm under financial autarky and how it compares to investment by a firm that has access to a debt market. Profits are taxable, and interest payments are tax deductible. Debt is restricted to being risk-free, but new debt issues can be used to help cover old debt. The risk-free limit on debt is created by a cash flow lending requirement which Lian and Ma (2021) show is very common.

The requirement that debt be risk-free is important for the stationary distribution. If the debt is risky, it would be necessary to specify details of how bankruptcy operates. In many models bankruptcy means liquidation of the firm. In that case, if the firm that has a positive probability of bankruptcy, then the stationary distribution is that the firm is liquidated and does not operate. The decisions of a not-yet bankrupt firm are for a transitory state. We avoid this as do Hennessy and Whited (2005), by requiring the debt to be risk-free.

The firm has a state-dependent capital target, and when risk-free debt is permitted, it is used to help achieve that target. In the long run, the firm with access to a debt market has a greater capital target than does the firm under financial autarky. So even in the long run, access to a debt market matters for the amount of capital used by the firm; and the firm does not just 'grow out of the constraint' created by the absence of a debt market.

6 Appendix

6.1 A Preliminary Result

Define a transition operator P for the Markov process $\{z_t, t \geq 0\}$. For any function f defined on the domain of the process $\{z_t, t \geq 0\}$, $\mathsf{P}f$ is a function defined on the same domain such that $(\mathsf{P}f)(z) = \mathsf{E}_z f(z')$. If $\{z_t, t \geq 0\}$ is a discrete time Markov chain taking a finite number of values with a transition matrix $P = (p_{ij})$, then the operator P is simply the probability transition matrix for the Markov chain.

Lemma 3 *The equation*

$$f(z,k) = \ell(z)k^\alpha + mk + \beta \mathsf{E}_z f(z', (1-\delta)k)$$

has a solution given by

$$f(z,k) = \left(\left[\mathsf{I} - \beta(1-\delta)^\alpha \mathsf{P} \right]^{-1} \ell \right)(z) k^\alpha + \frac{m}{1-\beta(1-\delta)} k. \tag{42}$$

Proof Let us hypothesize the function takes the form, $f(z,k) = h(z)k^\alpha + gk$ for some constant g and function $h(z)$. Substituting it into the equation, we have

$$h(z)k^\alpha + gk = \ell(z)k^\alpha + mk + \beta \mathsf{E}_z \left[h(z')((1-\delta)k)^\alpha + g(1-\delta)k \right].$$

Equating the coefficient for k in the above yields,

$$g = \frac{m}{1-\beta(1-\delta)};$$

and equating the coefficient for k^α in the above yields,

$$h(z) = \ell(z) + \beta(1-\delta)^\alpha \mathsf{E}_z h(z').$$

Writing the above as a functional equality, we have

$$h = \ell + \beta(1-\delta)^{\alpha}\mathsf{P}h.$$

Solving h from the above, we have

$$h = [\mathsf{I} - \beta(1-\delta)^{\alpha}\mathsf{P}]^{-1}\,\ell.$$

Hence, the equality (42) gives an solution. $\square$

6.2 Proof of Proposition 1

We first prove the existence and the uniqueness of the optimal value function with Lemma 4 below. Then, we find the optimal policy, together with the optimal value function, with the latter stated as Corollary 1 below.

Lemma 4 *There exists an optimal policy to the Bellman equation (2) and the optimal value function is unique.*

One approach to establishing that T is a contraction under the uniform norm would be to show the dividend d is bounded. This can can be done using an inductive approach in Harris (1987). We take a different approach based on changing the measure.

Proof Define T for any bounded continuous function f on $\mathbb{Z} \times \Re_+$ as follows,

$$\begin{aligned}
(\mathsf{T}f)(z,k) \quad &= \quad \sup_{d,i}\Big\{d + \beta \mathsf{E}_z V(z',k')\Big\}, \\
&\quad\quad \text{subject to:} \\
&\quad\quad (1-\tau)F(z,k) + \tau\delta k = d + i \\
&\quad\quad k' = (1-\delta)k + i \\
&\quad\quad d \geq 0, i \geq 0.
\end{aligned}$$

Let

$$r(z,k;i) = (1-\tau)F(z,k) + \tau\delta k - i.$$

We will use the sufficient condition proposition in Van Nunen and Wessels (1978) (which simplifies Lippman (1975)) to establish the existence and the uniqueness. Specifically, it is sufficient to find a function $\phi(z,k)$ such that

$$0 \leq r(z,k;i) \leq \phi(z,k), \tag{43}$$

$$\beta \mathsf{E}_z \phi(z',k') \leq \rho\phi(z,k) \quad \text{for some } \rho \in (\beta,1). \tag{44}$$

Let

$$\phi(z,k) = [(1-\tau)\bar{z}A + \tau\delta]\,k + \eta.$$

We show that there exists an η such that both conditions (43) and (44) hold.

From its definition,

$$
\begin{aligned}
0 \le r(z,k;i) &\le (1-\tau)\bar{z}Ak^{\alpha} + \tau\delta k \\
&\le [(1-\tau)\bar{z}A + \tau\delta]k + \eta_1,
\end{aligned} \tag{45}
$$

where we used $k^{\alpha} \le k+1$ to obtain the last inequality, and $\eta_1 = (1-\tau)\bar{z}A$. Choosing any $\eta \ge (1-\tau)\bar{z}A$ would satisfy condition (43).

In view of $k' = (1-\delta)k + i$ and $i \le (1-\tau)\bar{z}Ak^{\alpha} + \tau\delta k$ (which follows from $r(z,k;i) \ge 0$), it follows from the inequality (45),

$$
\begin{aligned}
\phi(z',k') &= [(1-\tau)\bar{z}A + \tau\delta]\, k' + \eta \\
&\le [(1-\tau)\bar{z}A + \tau\delta] \left\{ (1-\tau)\bar{z}Ak^{\alpha} + [1-\delta(1-\tau)]k + \eta_1 \right\} + \eta.
\end{aligned}
$$

Choose any $\rho \in (\beta, 1)$ and let $1+\epsilon = \rho/\beta$. The condition (43) would clearly hold if we can find η such that $\phi(z',k') \le (1+\epsilon)\phi(z,k)$. In view of the above inequality, it suffices to have

$$
\epsilon\eta\, [(1-\tau)\bar{z}A + \tau\delta] \left\{ [\epsilon + \delta(1-\tau)]k - (1-\tau)\bar{z}Ak^{\alpha} - \eta_1 \right\} \ge 0.
$$

It is elementary to show that

$$
m(\epsilon) = \min_{k\ge 0} \left\{ [\epsilon + \delta(1-\tau)]k - (1-\tau)\bar{z}Ak^{\alpha} - \eta_1, \right\},
$$

is a finite negative number. Hence, the above inequality holds if $\eta \ge -m(\epsilon)\, [(1-\tau)\bar{z}A + \tau\delta] / \epsilon$. The lemma is proved by choosing

$$
\eta = \max \left\{ (1-\tau)\bar{z}A, -\frac{m(\epsilon)\, [(1-\tau)\bar{z}A + \tau\delta]}{\epsilon} \right\}.
$$

□

Corollary 1 *The value function depends on the amount of capital the firm has relative to the target as in proposition 1. The corresponding cases are,*

(a) Excess capital. *With high enough capital, the optimal value function is given by*

$$
V(z,k) = (1-\tau)Ak^{\alpha} \left([\mathsf{I} - \beta(1-\delta)^{\alpha}\mathsf{P}]^{-1} e \right)(z) + \frac{\tau\delta}{1-\beta(1-\delta)}k, \tag{46}
$$

where e is an identity function with $e(z) = z$ for all z.

(b) Capital in range. *With capital in range, the optimal value function is given by*

$$
V(z,k) = R(z,k) + (1-\alpha)\beta(1-\tau) \left((I - \beta\mathsf{P})^{-1}\phi \right)(z), \tag{47}
$$

where $\phi(z) = \mathsf{E}_z F(z', k^*(z))$ is the expected production for the next period under the target capital level.

(c) Inadequate capital. *With inadequate capital, the optimal value function satisfies*

$$V(z,k) = \beta \mathsf{E}_z V\left(z', R(z,k)\right). \tag{48}$$

Proof (of Proposition 1 and Corollary 1) In view of Lemma 4, it is sufficient to find an optimal policy together with a corresponding value function. It depends on the amount of capital that the firm has. We start with the interior solution given by case b.

Case b. Assume that the Bellman equation (2) has an interior solution. So the constraints $d \geq 0$ and $i \geq 0$ are not binding. If so, then we can rewrite the Bellman equation (2) by solving the dividend d from the flow budget constraint (43) and substituting it into the equation. Thus we get,

$$V(z,k) = \sup_{k'} \left\{R(z,k) - k' + \beta \mathsf{E}_z V(z',k')\right\} = R(z,k) + \sup_{k'} \left\{\beta \mathsf{E}_z V(z',k') - k'\right\}. \tag{49}$$

Note the last term does not depend on k. Hence, for some function $h(z)$ we have

$$V(z,k) = R(z,k) + h(z). \tag{50}$$

Substitute the value function (50) into the equality (49) to get,

$$h(z) = \sup_{k'} \Big\{\beta \mathsf{E}_z \big\{(1-\tau) F(z',k') + [1 - \delta(1-\tau)]k' + h(z')\big\} - k'\Big\}. \tag{51}$$

Recall that the function $F(z,k) = zAk^{\alpha}$. The above objective function is concave in k'. So the corresponding the first-order condition,

$$\beta \mathsf{E}_z \big\{\alpha z' A (1-\tau)(k')^{\alpha-1} + 1 - \delta(1-\tau)\big\} - 1 = 0,$$

gives the optimal solution $k' = k^*(z)$ or $i = k^*(z) - (1-\delta)k$, where $k^*(z)$ is given by (7). This justifies the capital target.

So far we have assumed that there is an interior solution. The next task is to verify the conditions so that this actually holds so case b) is applicable.

For $i = k^*(z) - (1-\delta)k$ to be interior, we require that $i > 0$ or equivalently $(1-\delta)k < k^*(z)$. For d to be interior, we require $d > 0$, which is given by

$$d = R(z,k) - k^*(z) > 0. \tag{52}$$

Substitute $k' = k^*(z)$ into (51) to get,

$$h(z) = (1-\alpha)\beta(1-\tau)\mathsf{E}_z F(z', k^*(z)) + \beta \mathsf{E}_z h(z').$$

This can also be written as,

$$h = (1-\alpha)\beta(1-\tau)\phi + \beta \mathsf{P} h.$$

Hence, $h = (1-\alpha)\beta(1-\tau)(I - \beta \mathsf{P})^{-1}\phi$. Equality (47) follows from the equality (50). This establishes case b).

Case a. With excess capital, $(1-\delta)k \geq k^*(z)$. Under financial autarky, there is no rental or resale market for capital. Optimal investment is $k' = (1-\delta)k$ or $i = 0$ follows because the objective function is concave in k'. The dividend $d = (1-\tau)F(z,k) + \tau\delta k \geq 0$ follows directly. When $i = 0$ is the optimal solution in the Bellman equation (2), we get

$$V(z,k) = (1-\tau)F(z,k) + \tau\delta k + \beta \mathsf{E}_z V(z', (1-\delta)k).$$

Lemma 3 shows the value function (46) solves the above equation. This establishes case a).

Case c. With inadequate capital inequality (52) does not hold. Now investment i is constrained by the upper bound $(1-\tau)F(z,k) + \tau\delta k$. In the absence of a capital rental market or a debt market, the firm has no market mechanism to relax this constraint.

This bound is no greater than $k^*(z) - (1-\delta)k$ due to the inequality $d \geq 0$. Since the objective function is concave in i, optimal investment is achieved at its upper bound $i = (1-\tau)F(z,k) + \tau\delta k$. So dividend $d = 0$. Let $i = (1-\tau)F(z,k) + \tau\delta k$ be the optimal solution in the Bellman equation (2). We get the equation (48). This establishes case c).

$\square$

6.3 Proof of Proposition 2

Proof We first note that by Assumptions 1 and 2, there exists two positive constants $\underline{k}$ and $\bar{k}$ such that $\underline{k} \leq k^*(z) \leq \bar{k}$ for all $z \in \mathbb{Z}$.

Choose $R = \{(z,k) \in \mathbb{X} : (1-\delta)k \leq \bar{k}\}$. Consider the Markov chain starting from any given $x_0 = x = (z,k) \in \mathbb{X}$. Then there must be a finite $t \geq 1$ such that $x_t \in R$; otherwise, $(1-\delta)k_t > \bar{k} \geq k^*(z_t)$, and hence, $k_t = (1-\delta)^{t-1}k_1$ and $(1-\delta)k_t > \bar{k}$ for all $t \geq 1$. We reach a contradiction. This shows that starting at $x_0 = x$, $\tau(R) \leq t$ must hold for some finite t; hence, $\mathsf{P}_x(\tau(R) < \infty) = 1$ for all $x \in \mathbb{X}$. In fact, the above argument actually shows $\tau(R)$ is bounded and hence $\mathsf{E}_x\tau(R) < \infty$ for all $x \in \mathbb{X}$.

Next, we show that the set R is a regeneration set. To see this, we define a probability measure λ as follows:

$$\lambda(B) = \inf_{x \in R} \mathsf{P}_x\left(x' \in B \cap R\right) \qquad \text{for any } B \subseteq \mathbb{X}.$$

To show $\lambda(\cdot)$ is a probability measure, it is sufficient to show that $\lambda(R) = 1$, or equivalently $\mathsf{P}_x\left(x' \in R\right) = 1$ for all $x \in R$. The latter follows from the fact $x \in R$ implies $x' \in R$, due to the optimal policy (9) (which gives $k' \leq \max\{k^*(z), (1-\delta)k\}$) and the definition of the set R.

To complete the proof that the set R is a regeneration set, we show that for all $x \in R$ and for all $B \subseteq \mathbb{X}$, we have $\mathsf{P}_x(x' \in B) \geq \lambda(B)$. The fact that $x \in R$ implies $x' \in R$ implies that if $B_1 \subseteq \mathbb{X} \setminus R$, then $\mathsf{P}_x(x' \in B_1) = 0$ for all $x \in R$. Hence,

$$\mathsf{P}_x(x' \in B) = \mathsf{P}_x(x' \in B \cap R) \geq \lambda(B).$$

Then it follows from Theorem VII.3.2 and VII.3.5 in Asmussen (2003) that the process is Harris recurrent and has a stationary measure unique up to a multiplicative constant.

Since $\mathsf{E}_x\tau(R) < \infty$ for all $x \in \mathbb{X}$, we further know that the process is positive Harris recurrent and has a unique stationary probability measure.

Part b) follows immediately from Theorem VII.3.6 in Asmussen (2003). $\square$

6.4 Proof of Proposition 3

Proof In view of Lemma 1, it is sufficient to show that the equality (11) holds. When $z \in Z_L$, the equalities $z_t = z$ and $k_t = (zA/\delta)^{1/(1-\alpha)}$ imply that $R(z_t, k_t) = (zA/\delta)^{1/(1-\alpha)}$, which is strictly less than $k^*(z)$. Hence,

$$\begin{aligned} & \mathsf{P}\left(R(z_t,k_t)1_{\{R(z_t,k_t)<k^*(z_t)\}} + k^*(z_t)1_{\{(1-\delta)k\le k^*(z_t)\le R(z_t,k_t)\}} = k(z)\big|z_t = z\right) \\ = & \mathsf{P}\left(R(z,k_t) = k(z)\big|z_t = z\right) = \mathsf{P}\left(k_t = k(z)\big|z_t = z\right), \end{aligned}$$

which establishes for the equality (11). When $z \in Z_H$, the equalities $z_t = z$ and $k_t = k^*(z)$ imply $k^*(z) \le R(z, k_t)$. Hence,

$$\begin{aligned} & \mathsf{P}\left(R(z_t,k_t)1_{\{R(z_t,k_t)<k^*(z_t)\}} + k^*(z_t)1_{\{(1-\delta)k_t\le k^*(z_t)\le R(z_t,k_t)\}} = k^*(z)\Big|z_t = z\right) \\ = & \mathsf{P}\left(k^*(z)1_{\{(1-\delta)k_t\le k^*(z)\le R(z,k_t)\}} = k^*(z)\Big|z_t = z\right) \\ = & \mathsf{P}\left((1-\delta)k_t \le k^*(z) \le R(z,k_t)\big|z_t = z\right) \\ \ge & \mathsf{P}\left(k_t = k^*(z), (1-\delta)k_t \le k^*(z) \le R(z,k_t)\big|z_t = z\right) \\ = & \mathsf{P}\left(k_t = k^*(z), (1-\delta)k^*(z) \le k^*(z) \le R(z,k^*(z))\big|z_t = z\right) \\ = & \mathsf{P}\left(k_t = k^*(z)\big|z_t = z\right) = 1. \end{aligned}$$

This establishes the equality (11) for $z \in Z_H$. $\square$

6.5 Proof of Lemma 2

Proof If we choose $i = 0$ (and hence $k' = (1-\delta)k$), $b' = \bar{b}(k')$ and

$$d = (1-\tau)F(z,k) + \tau\delta k + \frac{1+\tau\rho}{1+\rho}\bar{b}((1-\delta)k) - b,$$

then the chosen triplet (b', d, i) clearly satisfies the budget balance equation. The inequality (20) implies that $d \ge 0$. Hence, the triplet gives a feasible policy. $\square$

6.6 Proof of Proposition 4

Proof The Bellman equation (26) has a unique solution. The proof is similar to the case without the saving. Suppose that the solution is an interior such that $d > 0$ and $i > 0$. In this case, the equality (26) in view of the budget constraint (27) can be written as follows,

$$V(z,k,b) = (1-\tau)F(z,k) + [1-\delta(1-\tau)]k - b + h(z), \text{where} \quad (53)$$

$$h(z) = \max_{k';b'\le\bar{b}(k')} \left\{ \frac{1+\tau\rho}{1+\rho}b' - k' + \beta \mathsf{E}_z V(z',k',b') \right\}. \quad (54)$$

Substituting (53) into (54) yields,

$$\begin{aligned} h(z) &= \beta \mathsf{E}_z h(z') + \max_{k'} \Big\{ \beta(1-\tau)\mathsf{E}_z F(z,k') - \big\{1-\beta[1-\delta(1-\tau)]\big\}k' + \max_{b'\le\bar{b}(k')} \nu b' \Big\} \\ &= \beta \mathsf{E}_z h(z') + \sup_{k'} \Big\{ \beta(1-\tau)\mathsf{E}_z F(z,k') - \big\{1-\beta[1-\delta(1-\tau)]\big\}k' + \nu\bar{b}(k') \Big\}, \end{aligned} \quad (55)$$

where ν is defined in (25). The last equality follows $b' = \bar{b}(k')$ due to $\nu > 0$.

The first-order condition gives the optimal capital $k' = k_d^*(z)$ in (31).

To ensure an interior solution, we require $k_d^*(z) > (1-\delta)k$ and the condition (33).

Suppose that $k_d^*(z) \le (1-\delta)k$. Since the maximand in (55) is concave in k', we must have $k' = (1-\delta)k$. In this case,

$$R(z,k) + \frac{1+\tau\rho}{1+\rho}\bar{b}\left((1-\delta)k\right) - (1-\delta)k - b \ge 0.$$

This is due to our choice of the upper bound function $\bar{b}$ and $b \le \bar{b}(k)$.

Finally, suppose that the condition (33) does not hold or equivalently, condition (36) holds. The optimal policy must be have $d = 0$.

The Bellman problem (26) becomes (37). This completes the proof. □

6.7 Proof of Proposition 5

Proof The proof of this proposition is almost the same as the proof of Proposition 2. The main difference is that the state space $\mathbb{X}$ includes three components (z,k,b) (vs. two components (z,k) in the previous proof). In this case, we choose $R = \{(z,k,b) \in \mathbb{X} : (1-\delta)k \le \bar{k}\}$. The rest of the proof is essentially the same. □

6.8 Proof of Proposition 6

Proof The proof is very similar to the proof of Proposition 3. So, we sketch the proof leaving out some details.

First, we establish the results similar to those in Lemma 1. Let $\{(b_{t+1}, d_t, i_t)\}$ be the optimal policy in Proposition 4 and let $k_{t+1} = (1-\delta)k_t + i_t$. Define three indicator functions,

$$\begin{aligned} \mathsf{I}_a(z,k,b) &= 1_{\{k_d^*(z)\le(1-\delta)k\}}, \\ \mathsf{I}_b(z,k,b) &= 1_{\{(1-\delta)k<k_d^*(z)\}\cap\{R(z,k)+\frac{1+\tau\rho}{1+\rho}\bar{b}\left(k_d^*(z)\right)-k_d^*(z)-b\ge 0\}}, \\ \mathsf{I}_c(z,k,b) &= 1_{\{R(z,k)+\frac{1+\tau\rho}{1+\rho}\bar{b}\left(k_d^*(z)\right)-k_d^*(z)-b<0\}}. \end{aligned}$$

Then, in view of Proposition 4, we can write

$$
\begin{aligned}
k_{t+1} &= (1-\delta)k_t \mathsf{I}_a(z_t,k_t,b_t) + k_d^*(z_t)\mathsf{I}_b(z_t,k_t,b_t) + \phi(z_t,k_t,b_t)\mathsf{I}_c(z_t,k_t,b_t),\\
b_{t+1} &= \bar{b}((1-\delta)k_t)\mathsf{I}_a(z_t,k_t,b_t) + \bar{b}\left(k_d^*(z_t)\right)\mathsf{I}_b(z_t,k_t,b_t) + \psi(z_t,k_t,b_t)\mathsf{I}_c(z_t,k_t,b_t),
\end{aligned}
$$

where the pair, $b' = \psi(z,k,b)$ and $i = \phi(z,k,b) - (1-\delta)k$, is the optimal solution to the maximization problem (37).

Consider the stationary distribution so we assume $(z_{t+1}, b_{t+1}, k_{t+1}) \stackrel{d}{=} (z_t, b_t, k_t)$. Then, we can show that $\{(z_t, b_t, k_t)\}$ is a steady state process if and only if

$$
\begin{aligned}
k_t &\stackrel{d}{=} k_d^*(z_t)\mathsf{I}_b(z_t,k_t,b_t) + \phi(z_t,k_t,b_t)\mathsf{I}_c(z_t,k_t,b_t), && (56)\\
b_t &\stackrel{d}{=} \bar{b}\left(k_d^*(z_t)\right)\mathsf{I}_b(z_t,k_t,b_t) + \psi(z_t,k_t,b_t)\mathsf{I}_c(z_t,k_t,b_t), && (57)
\end{aligned}
$$

where we note that $\mathsf{I}_a(z_t,k_t,b_t) = 0$ in stationary distribution because $k_t \stackrel{d}{=} (1-\delta)k_t$ is not possible.

First we consider $z \in Z_{L,d} \cap \tilde{Z}_{L,d}$ (i.e., both (41) and (40) hold).

$$
\begin{aligned}
& \mathsf{P}\Big(k_d^*(z_t)\mathsf{I}_b(z_t,k_t,b_t) + \phi(z_t,k_t,b_t)\mathsf{I}_c(z_t,k_t,b_t) = k_d(z) \text{ and} \\
& \qquad \bar{b}\left(k_d^*(z_t)\right)\mathsf{I}_b(z_t,k_t,b_t) + \psi(z_t,k_t,b_t)\mathsf{I}_c(z_t,k_t,b_t) = b_d(z)\Big| z_t = z\Big)\\
= \; & \mathsf{P}\Big(\phi(z,k_t,b_t)\mathsf{I}_c(z,k_t,b_t) = k_d(z) \text{ and } \psi(z,k_t,b_t)\mathsf{I}_c(z,k_t,b_t) = b_d(z)\Big| z_t = z\Big)\\
= \; & \mathsf{P}\Big(\mathsf{I}_c(z, k_d(z) = 1, b_d(z), k_t = k_d(z) \text{ and } b_t = b_d(z)\Big| z_t = z\Big)\\
= \; & \mathsf{P}\Big(k_t = k_d(z) \text{ and } b_t = b_d(z)\Big| z_t = z\Big),
\end{aligned}
$$

where the first equality holds since $k_d^*(z) > k_d(z)$ and $\bar{b}(k_d^*(z)) > b_d(z)$ for $z \in Z_{L,d}$, the second equality holds since the pair, $k_t = k_d(z)$ and $b_t = b_d(z)$, is the unique solution to the pair of equations $\phi(z_t,k_t,b_t) = k_d(z)$ and $\psi(z_t,k_t,b_t) = b_d(z)$, and the last equality holds since $z \in \tilde{Z}_{L,d}$. This establishes the equalities (56) and (57) for all $z_t = z \in Z_{L,d} \cap \tilde{Z}_{L,d}$.

Next we consider $z \in Z_{H,d}$ (i.e., the inequality (38) holds),

$$\begin{aligned}
& \mathsf{P}\Big(k_d^*(z_t)\mathsf{I}_b(z_t,k_t,b_t) + \phi(z_t,k_t,b_t)\mathsf{I}_c(z_t,k_t,b_t) = k_d^*(z) \text{ and} \\
& \qquad\qquad \bar{b}\left(k_d^*(z_t)\right)\mathsf{I}_b(z_t,k_t,b_t) + \psi(z_t,k_t,b_t)\mathsf{I}_c(z_t,k_t,b_t) = \bar{b}\big(k_d^*(z)\big)\Big| z_t = z\Big) \\
= \; & \mathsf{P}\Big(k_d^*(z_t)\mathsf{I}_b(z_t,k_t,b_t) = k_d^*(z) \text{ and } \bar{b}\left(k_d^*(z_t)\right)\mathsf{I}_b(z_t,k_t,b_t) = \bar{b}\big(k_d^*(z)\big)\Big| z_t = z\Big) \\
= \; & \mathsf{P}\Big(k_d^*(z_t) = k_d^*(z), \bar{b}\left(k_d^*(z_t)\right) = \bar{b}\big(k_d^*(z)\big), \\
& \qquad\qquad (1-\delta)k_t < k_d^*(z), R(z,k_t) + \frac{1+\tau\rho}{1+\rho}\bar{b}\left(k_d^*(z)\right) - k_d^*(z) - b_t \geq 0 \Big| z_t = z\Big) \\
= \; & \mathsf{P}\Big((1-\delta)k_t < k_d^*(z), R(z,k_t) + \frac{1+\tau\rho}{1+\rho}\bar{b}\left(k_d^*(z)\right) - k_d^*(z) - b_t \geq 0 \Big| z_t = z\Big) \\
\geq \; & \mathsf{P}\Big(k_t = k_d^*(z), b_t = \bar{b}\big(k_d^*(z)\big), \\
& \qquad\qquad (1-\delta)k_t < k_d^*(z), R(z,k_t) + \frac{1+\tau\rho}{1+\rho}\bar{b}\left(k_d^*(z)\right) - k_d^*(z) - b_t \geq 0 \Big| z_t = z\Big) \\
= \; & \mathsf{P}\Big(k_t = k_d^*(z), b_t = \bar{b}\big(k_d^*(z)\big), (1-\delta)k_d^*(z) < k^*(z), \\
& \qquad\qquad R(z,k_d^*(z)) + \frac{1+\tau\rho}{1+\rho}\bar{b}\left(k_d^*(z)\right) - k_d^*(z) - \bar{b}\big(k_d^*(z)\big) \geq 0 \Big| z_t = z\Big) \\
= \; & \mathsf{P}\left(k_t = k_d^*(z), b_t = \bar{b}\big(k_d^*(z)\big) \Big| z_t = z\right) = 1,
\end{aligned}$$

where the second equality holds since the indicator function $\mathsf{I}_b(z_t,k_t,b_t) = 1$ must hold (as $k_d^*(z) > 0$), and the second to the last equality holds because $z \in Z_{H,d}$. This establishes the equalities (56) and (57) for all $z_t = z \in Z_{H,d}$.

The uniqueness statement follows from a proof similar to that of Proposition 2. □

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