Problem 1 (20 Points).

(a) What is the DFT of $(1, 0, 0, 0)$? What is the appropriate choice of ω in this case? What sequence is $(1, 0, 0, 0)$ the DFT of?

(b) Repeat for $(1, 0, 1, -1)$

Solution:

(a) We choose $\omega = e^{-i2\pi/4} = e^{-i\pi/2} = -i$. Then the DFT of $(1, 0, 0, 0)$ is

$$A_k = \sum_{t=0}^{3} a_t \omega^{kt} = \omega^0 = 1$$

So the DFT is $(1, 1, 1, 1)$.

Given the that the inverse DFT is just the DFT using ω^{-1}, all divided by n, we can immediately see that $(1/4, 1/4, 1/4, 1/4)$ is the sequence that gives $(1, 0, 0, 0)$ as the DFT

(b) We still use the same $\omega = -i$ since $n = 4$. Now,

$$A_k = \sum_{t=0}^{3} a_t \omega^{kt} = 1 + (-i)^{2k} - (-i)^{3k} = 1 + (-1)^k - i^k$$

This gives the sequence $(1, -i, 3, i)$

Again, given the similarity between the DFT and its inverse, it is easy to show that the DFT of $(1/4, i/4, 3/4, -i/4)$ gives $(1, 0, 1, -1)$

Problem 2 (30 Points).

(a) Say we want to multiply two polynomials $x+1$ and x^2+1 using the FFT. Choose an appropriate power of two, find the FFT of the two sequences, multiply the results componentwise, and compute the inverse FFT to get the final result.

(b) Repeat for the pair of polynomials $1 + x + 2x^2$ and $2 + 3x$.

Solution:

(a) We need n to be larger than the sum of the degrees, which is 3. Thus, we choose $n = 4$. The polynomials are thus represented by the sequences $(1, 1, 0, 0)$ and $(1, 0, 1, 0)$. Taking the FFT of each gives $(2, 1 - i, 0, 1 + i)$ and $(2, 0, 2, 0)$. Pointwise multiplying these sequences gives $(4, 0, 0, 0)$. Taking the inverse FFT then gives $(1, 1, 1, 1)$, so the final polynomial is $1 + x + x^2 + x^3$.

(b) Again, we need $n = 4$. The polynomials are thus represented by the sequences $(1, 1, 2, 0)$ and $(2, 3, 0, 0)$. Taking the FFT of each yields $(4, -1 - i, 2, -1 + i)$ and $(5, 2 - 3i, -1, 2 + 3i)$. Pointwise multiplying gives $(20, -5 + i, -2, -5 - i)$. Taking the inverse FFT gives $(2, 5, 7, 6)$, or $2 + 5x + 7x^2 + 6x^3$.

Problem 3 (20 Points).

(a) What is the sum of the nth roots of unity?

(b) If n is odd, what is the product of the nth root of unity?

(c) What if n is even?

Solution:

(a)

$$\sum_{i=0}^{n-1} \omega^i = \frac{1 - \omega^n}{1 - \omega} = \frac{1 - 0}{1 - \omega} = 0$$

We could also see this for even n as follows: we chose the nth roots of unity exactly because they come in plus/minus pairs. Adding each pair together thus gives 0, so adding all roots together gives 0

(b)

$$\prod_{i=0}^{n-1} \omega^i = \omega^{\sum_{i=0}^{n-1} i} = \omega^{\frac{n(n-1)}{2}}$$

Since n is odd, $n - 1$ is even, so $(n - 1)/2$ is an integer. Thus, this expressions becomes $(\omega^n)^{(n-1)/2} = 1^{(n-1)/2} = 1$

(c) Since n is even, $n/2$ is an integer, so the expression from part (b) becomes $\left(\omega^{n-1}\right)^{n/2} = \omega^{-n/2}$. Since ω is a primitive nth root of unity, $\left(\omega^{-n/2}\right)^2 = \omega^{-n} = 1$, but $\omega^{-n/2} \neq 1$. Therefore, $\omega^{-n/2} = -1$. Thus, the product of all the nth roots of unity is -1.

Problem 4 (30 Points). Let $(a_0, a_2, ..., a_{n-1})$ be a sequence, and let $(A_0, ..., A_{n-1})$ be its DFT.

(a) Suppose we construct a new input sequence

$$(a'_0, a'_1, ..., a'_{n-1}) = (a_k, a_{k+1}, ..., a_{n-1}, a_0, a_1, ..., a_{k-1})$$

obtained by rotating the original by k spots. What is the DFT of this sequence in terms of $(A_0, ..., A_{n-1})$, the DFT of the original sequence.

(b) What input sequence would yield the DFT

$$(A'_0, ..., A'_{n-1}) = (A_k, A_{k+1}, ..., A_{n-1}, A_0, ..., A_{k-1}) \ ?$$

(c) What is the DFT of

$$(a_{n-1}, a_{n-2}, ..., a_0)$$

in terms of the A_js?

Solution:

(a)

$$\begin{aligned} A'_r &= \sum_{t=0}^{n-1} a'_t \omega^{rt} = \sum_{t=0}^{n-k-1} a_{t+k} omega^{rt} + \sum_{t=n-k}^{n-1} a_{t+k-n} omega^{rt} \\ &= \sum_{j=k}^{n-1} a_j \omega^{r(j-k)} + \sum_{j=0}^{k-1} a_j \omega^{r(j+n-k)} \\ &= \omega^{-rk} \sum_{j=k}^{n-1} a_j \omega^{rj} + \omega^{rn} \omega^{-rk} \sum_{j=0}^{k-1} a_j \omega^{rj} \\ &= \omega^{-rk} \sum_{j=0}^{n-1} a_j \omega^{rj} = \omega^{-rk} A_r \end{aligned}$$

(b) Given the similarity between the forward and inverse DFT, using the same techniques as in part (a), we can get $a'_t = \omega^{tk} a_t$

(c)

$$\begin{aligned} A'_r &= \sum_{t=0}^{n-1} a'_t \omega^{rt} = \sum_{t=0}^{n-1} a_{n-1-t} \omega^{rt} \\ &= \sum_{j=0}^{n-1} a_j \omega^{r(n-1-j)} = \omega^{-r} \sum_{j=0}^{n-1} a_j \omega^{(-r)j} \\ &= \omega^{-r} A_{-r} \end{aligned}$$

This solution is acceptable, but technically, $-r$ is only in the range $[0, n-1]$ when $r = 0$. Recall from class that we can extend A_r to all integers r, with the property that $A_{r+n} = A_r$ for all r. Thus, for $r \neq 0$, we can take $A'_r = \omega^{-r} A_{n-r}$. Thus, the DFT looks like

$$(A_0, \omega^{-1} A_{n-1}, \omega^{-2} A_{n-2}, ..., \omega^{-(n-1)} A_1)$$

Total points: 100