Math Reference Sheet

For semi-frequent stuff that I don't remember off the top of my head.

BEST Theorem

Counts number of Eulerian circuits in directed graphs.

$$\mathrm{ec}(G) = t_w(G) \prod_{v \in V} (\deg(v) - 1)!$$

$t_w(G)$ denotes the number of arborescences (directed trees pointing towards the root) rooted at any arbitrary node w and is calculated via Kirchhoff's matrix tree theorem.

Kirchhoff's Matrix Tree Theorem

Counts number of spanning trees of a graph. The algorithm builds the Laplacian matrix defined as the following for a simple graph:

$$L_{i,j} := \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

The answer is any cofactor of this matrix (e.g. the determinant after deleting the last row and column).

- Cayley's formula: the number of spanning trees of a complete graph of size n is n^{n-2}
- For multigraphs, $L_{i,j}$ equals $-m$, where m is the number of edges between i and j, self-loops are excluded
- For directed multigraphs, $L_{i,j}$ equals $-m$, where m is the number of edges from i to j, and $L_{i,i}$ equals the indegree of i minus the number of loops at i
- Removing the ith row and column and taking the determinant gives the number of oriented spanning trees rooted at (pointing towards) vertex i.

Stirling Numbers of the First Kind

Counts number of permutations of length n with k cycles.

$$dp[n+1][k] = n \cdot dp[n][k] + dp[n][k-1]$$
$$dp[0][0] = 1$$
$$dp[n][0] = dp[0][k] = 0$$

Explanation: if you have n elements split into k cycles and are inserting a new element, you can either create a new cycle or insert it directly behind any of the previous n elements in an existing cycle.

The generating function for signed Stirling numbers of the first kind can be computed for a fixed n in $\mathcal{O}(n \log^2 n)$:

$$\sum_{k=0}^{n} s(n,k) x^k = x(x-1)\ldots(x-(n-1))$$

The unsigned Stirling numbers of the first kind are similar:

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = x(x+1)\ldots(x+n-1)$$

The unsigned Stirling numbers of the first kind can be computed for a fixed k in $\mathcal{O}(n \log n)$:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{k!}[x^n] \left(\sum_{n=1}^{\infty} \frac{x^n}{n} \right)^k$$

Stirling Numbers of the Second Kind

Counts number of ways to partition n labeled objects into k non-empty unlabeled subsets.

$$dp[n+1][k] = k \cdot dp[n][k] + dp[n][k-1]$$
$$dp[0][0] = 1$$
$$dp[n][0] = dp[0][k] = 0$$

Explanation: the $n+1$th object is either a singleton or not. If it is a singleton, distribute the remaining n objects among $k-1$ groups, otherwise insert $n+1$ into one of the k groups and distribute the remaining n objects in k groups as well.

The generating function for Stirling numbers of the second kind can be computed for a fixed n in $\mathcal{O}(n \log n)$:

$$\sum_{k=0}^{n} S(n,k)x^k = \left(\sum_{i=0}^{n} \frac{(-1)^i}{i!} \right) \left(\sum_{i=0}^{n} \frac{i^n}{i!} \right) \mod x^{n+1}$$

They can also be computed for a fixed k in $\mathcal{O}(n \log n)$:

$$S(n,k) = n![x^n]\frac{(e^x-1)^k}{k!}$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Partition Function

Counts number of ways to partition n into non-negative integer parts. Partition of n into k parts follows the following recurrence:

$$dp[n][k] = dp[n-1][k-1] + dp[n-k][k]$$
$$dp[0][0] = 1$$
$$dp[n][0] = dp[0][k] = 0$$

Explanation: there are two possibilities. If we include a 1 in the partition, we simply partition the remaining of $n-1$ into $k-1$ parts. Otherwise, each part has size greater than 1, so we subtract 1 from each part and solve recursively.

$p(n)$ denotes the number of partitions of n and its generating function can be computed in $\mathcal{O}(n \log n)$:

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

The denominator can be computed in $\mathcal{O}(n)$ time with the pentagonal number theorem:

$$\prod_{n=1}^{\infty}(1-x^n) = 1 + \sum_{k=1}^{\infty}(-1)^k\left(x^{k(3k+1)/2} + x^{k(3k-1)/2}\right)$$

Derangements

Counts the number of permutations where $p_i \neq i$ for all i.

$$dp[n] = (n-1)(dp[n-1] + dp[n-2])$$
$$dp[0] = 1$$
$$dp[1] = 0$$

Explanation: With n elements, consider index 1 which may receive $n-1$ different values. There are two cases. Either index 1 swaps with another index i, so we count the number of derangements among the remaining $n-2$ indices. Alternatively, index 1 receives value i but index i does not receive value 1, so this is equivalent to counting the number of derangements of $n-1$ indices as we can renumber value 1 as value i.

The probability of getting a derangement from a random shuffle is $\frac{1}{e}$, which means the number of shuffles needed to get a derangement is effectively constant.

Catalan Numbers

Shows up in numerous counting problems, such as number of correct bracket sequences of length $2n$, number of full binary trees with $n+1$ leaves, etc.

$$\{1, 1, 2, 5, 14, 42, 132, 429, \ldots\}$$
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$
$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$
$$C_0 = 1$$

Proof for combinatorial definition of Catalan numbers is based on reflection argument for grid paths.

Linear Diophantine Equations

Solve equations of the form

$$ax + by = c$$

Let $\gcd(a, b) = g$. The equation has a solution iff g divides c. We can find a solution to the equation $ax_g + by_g = g$ with the Extended Euclidean algorithm (in this book). After that, an initial solution to our original equation is

$$x_0 = x_g \cdot \frac{c}{g}, y_0 = y_g \cdot \frac{c}{g}$$

All solutions can be enumerated from the initial solution with

$$x = x_0 + k \cdot \frac{b}{g}, y = y_0 - k \cdot \frac{a}{g}$$