

Math Reference Sheet

For semi-frequent stuff that I don't remember off the top of my head.

BEST Theorem

Counts number of Eulerian circuits in directed graphs.

$$\text{ec}(G) = t_w(G) \prod_{v \in V} (\deg(v) - 1)!$$

$t_w(G)$ denotes the number of arborescences (directed trees pointing towards the root) rooted at any arbitrary node w and is calculated via Kirchhoff's matrix tree theorem.

Kirchhoff's Matrix Tree Theorem

Counts number of spanning trees of a graph. The algorithm builds the Laplacian matrix defined as the following for a simple graph:

$$L_{i,j} := \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

The answer is any cofactor of this matrix (e.g. the determinant after deleting the last row and column).

- Cayley's formula: the number of spanning trees of a complete graph of size n is n^{n-2}
- For multigraphs, $L_{i,j}$ equals $-m$, where m is the number of edges between i and j , self-loops are excluded
- For directed multigraphs, $L_{i,j}$ equals $-m$, where m is the number of edges from i to j , and $L_{i,i}$ equals the indegree of i minus the number of loops at i
- Removing the i th row and column and taking the determinant gives the number of oriented spanning trees rooted at (pointing towards) vertex i .

Stirling Numbers of the First Kind

Counts number of permutations of length n with k cycles.

$$\begin{aligned} dp[n+1][k] &= n \cdot dp[n][k] + dp[n][k-1] \\ dp[0][0] &= 1 \\ dp[n][0] &= dp[0][k] = 0 \end{aligned}$$

Explanation: if you have n elements split into k cycles and are inserting a new element, you can either create a new cycle or insert it directly behind any of the previous n elements in an existing cycle.

The generating function for signed Stirling numbers of the first kind can be computed for a fixed n in $\mathcal{O}(n \log^2 n)$:

$$\sum_{k=0}^n s(n, k) x^k = x(x-1) \dots (x-(n-1))$$

The unsigned Stirling numbers of the first kind are similar:

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} = x(x+1) \dots (x+n-1)$$

The unsigned Stirling numbers of the first kind can be computed for a fixed k in $\mathcal{O}(n \log n)$:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{k!} [x^n] \left(\sum_{n=1}^{\infty} \frac{x^n}{n} \right)^k$$

Stirling Numbers of the Second Kind

Counts number of ways to partition n labeled objects into k non-empty unlabeled subsets.

$$\begin{aligned} dp[n+1][k] &= k \cdot dp[n][k] + dp[n][k-1] \\ dp[0][0] &= 1 \\ dp[n][0] &= dp[0][k] = 0 \end{aligned}$$

Explanation: the $n+1$ th object is either a singleton or not. If it is a singleton, distribute the remaining n objects among $k-1$ groups, otherwise insert $n+1$ into one of the k groups and distribute the remaining n objects in k groups as well.

The generating function for Stirling numbers of the second kind can be computed for a fixed n in $\mathcal{O}(n \log n)$:

$$\sum_{k=0}^n S(n, k) x^k = \left(\sum_{i=0}^n \frac{(-1)^i}{i!} \right) \left(\sum_{i=0}^n \frac{i^n}{i!} \right) \mod x^{n+1}$$

They can also be computed for a fixed k in $\mathcal{O}(n \log n)$:

$$\begin{aligned} S(n, k) &= n! [x^n] \frac{(e^x - 1)^k}{k!} \\ e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

Partition Function

Counts number of ways to partition n into non-negative integer parts. Partition of n into k parts follows the following recurrence:

$$\begin{aligned} dp[n][k] &= dp[n-1][k-1] + dp[n-k][k] \\ dp[0][0] &= 1 \\ dp[n][0] &= dp[0][k] = 0 \end{aligned}$$

Explanation: there are two possibilities. If we include a 1 in the partition, we simply partition the remaining of $n-1$ into $k-1$ parts. Otherwise, each part has size greater than 1, so we subtract 1 from each part and solve recursively.

$p(n)$ denotes the number of partitions of n and its generating function can be computed in $\mathcal{O}(n \log n)$:

$$\sum_{n=0}^{\infty} p(n) x^n = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}$$

The denominator can be computed in $\mathcal{O}(n)$ time with the pentagonal number theorem:

$$\prod_{n=1}^{\infty} (1 - x^n) = 1 + \sum_{k=1}^{\infty} (-1)^k \left(x^{k(3k+1)/2} + x^{k(3k-1)/2} \right)$$

Derangements

Counts the number of permutations where $p_i \neq i$ for all i .

$$\begin{aligned} dp[n] &= (n - 1)(dp[n - 1] + dp[n - 2]) \\ dp[0] &= 1 \\ dp[1] &= 0 \end{aligned}$$

Explanation: With n elements, consider index 1 which may receive $n - 1$ different values. There are two cases. Either index 1 swaps with another index i , so we count the number of derangements among the remaining $n - 2$ indices. Alternatively, index 1 receives value i but index i does not receive value 1, so this is equivalent to counting the number of derangements of $n - 1$ indices as we can renumber value 1 as value i .

The probability of getting a derangement from a random shuffle is $\frac{1}{e}$, which means the number of shuffles needed to get a derangement is effectively constant.

Catalan Numbers

Shows up in numerous counting problems, such as number of correct bracket sequences of length $2n$, number of full binary trees with $n + 1$ leaves, etc.

$$\{1, 1, 2, 5, 14, 42, 132, 429, \dots\}$$

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

$$C_{n+1} = \sum_{i=0}^n C_i C_{n-i}$$

$$C_0 = 1$$

Proof for combinatorial definition of Catalan numbers is based on reflection argument for grid paths.