Example 11

From https://www.stat.cmu.edu/~genovese/class/iprob-S06/notes/generating-functions.pdf

Goal: Compute $\sum_{k=1}^{n} k^2$

Definitions/Notation:

Let $G(z) = g_0 + g_1 z + g_2 z^2 + g_3 z^3 \cdots = \sum_{n \geq 0} g_n z^n$

The coefficient of the nth term is denoted as $g_n = [z^n]G(z)$

Right-shifting: $S^k G(z) = z^k G(z) = \sum_{n \geq 0} g_n z^{n+k}$

Differentiating: $D^k G(z) = G^{(k)}(z) = \sum_{n \geq 0} n^{\underline{k}} g_n z^n$

where $n^{\underline{k}}$ is the falling factorial, $n^{\underline{k}} = n \cdot (n-1)^{\underline{(k-1)}}, n^{\underline{0}} = 1$

Partial Summation:

$$\begin{aligned}
\frac{G(z)}{1-z} &= \frac{\sum_{n \geq 0} g_n z^n}{\sum_{n \geq 0} z^n} \\
&= (g_0 + g_1 z + g_2 z^2 + g_3 z^3 + \ldots)(1 + z + z^2 + z^3 + \ldots) \\
&= (g_0) + (g_1 z + g_0 z) + (g_2 z^2 + g_1 z^2 + g_0 z^2) + \ldots \\
&= \sum_{n \geq 0} \left(\sum_{k \leq n} g_k \right) z^n
\end{aligned}$$

Computing the Goal:

Consider $(SD)^2 \frac{1}{1-z}$:

$$\begin{aligned}
(SD)G(z) = S(DG(z)) = S\left(\sum_{n \geq 1} n g_n z^{n-1}\right) &= \sum_{n \geq 1} n g_n z^n \\
(SD)^2 G(z) = SD((SD)G(z)) = SD\left(\sum_{n \geq 1} n g_n z^n\right) = S\left(\sum_{n \geq 1} n^2 g_n z^{n-1}\right) &= \sum_{n \geq 1} n^2 g_n z^n \\
(SD)^2 \frac{1}{1-z} &= \sum_{n \geq 1} n^2 z^n
\end{aligned}$$

If we divide that result by $1-z$, then by partial summation we get:

$$\frac{1}{1-z} \cdot \left((SD)^2 \frac{1}{1-z} \right) = \sum_{n \geq 1} \left(\sum_{k \leq n} k^2 \right) z^n$$

(I write $n \geq 1$ instead of $n \geq 0$ since g_0 is 0 anyways.)

So our answer is:

$$\sum_{k=1}^{n} k^2 = [z^n] \left(\frac{1}{1-z} \cdot \left((SD)^2 \frac{1}{1-z} \right) \right)$$

To simplify that inner expression, we keep them in fractional form instead of writing them out as series.

$$D\left(\frac{1}{1-z}\right) = \frac{1}{(1-z)^2}$$
$$SD\left(\frac{1}{1-z}\right) = \frac{z}{(1-z)^2}$$
$$DSD\left(\frac{1}{1-z}\right) = \frac{1+z}{(1-z)^3}$$
$$(SD)^2\left(\frac{1}{1-z}\right) = SDSD\left(\frac{1}{1-z}\right) = \frac{z^2+z}{(1-z)^3}$$
$$\frac{1}{1-z}\cdot\left((SD)^2\frac{1}{1-z}\right) = \frac{z^2+z}{(1-z)^4} = \frac{z^2}{(1-z)^4} + \frac{z}{(1-z)^4}$$

Evaluate $1/(1-z)^4 = (1-z)^{-4}$ with binomial theorem (the negative variant):

$$(1+z)^{-n} = \sum_{k\geq 0}\binom{n+k-1}{k}(-1)^k z^k$$
$$(1-z)^{-4} = \sum_{n\geq 0}\binom{4+n-1}{n}(-1)^n(-z)^n = \sum_{n\geq 0}\binom{n+3}{n}(-1)^n(-1)^n z^n = \sum_{n\geq 0}\binom{n+3}{3}z^n$$

So the RHS of our original equation becomes:

$$\frac{z^2}{(1-z)^4} + \frac{z}{(1-z)^4} = \sum_{n\geq 0}\binom{n+3}{3}z^{n+2} + \sum_{n\geq 0}\binom{n+3}{3}z^{n+1} = \sum_{n\geq 2}\binom{n+1}{3}z^n + \sum_{n\geq 1}\binom{n+2}{3}z^n$$

and $[z^n]$ of the RHS is

$$\binom{n+1}{3} + \binom{n+2}{3} = \frac{(n+1)n(n-1)}{3\cdot 2\cdot 1} + \frac{(n+2)(n+1)n}{3\cdot 2\cdot 1} = \frac{n(n+1)(n-1+n+2)}{6} = \frac{n(n+1)(2n+1)}{6}$$

Note for $n=1$, we get just the coefficient of the second sum on the RHS, or $\binom{1+2}{3} = \binom{3}{3} = 1 = \sum_{k=1}^{n} k^2$, and $\frac{1(1+1)(2\cdot 1+1)}{6} = 1$, so the formula checks out.