Pset 5

4.(47-48).1

a)

$$\int_C \frac{e^{-z}dz}{z-\pi i/2} = 2\pi i \cdot e^{-\pi i/2} = 2\pi i \cdot -i = 2\pi$$

b)

$$\int_C \frac{\cos z}{z(z^2+8)}dz = 2\pi i \cdot \frac{\cos 0}{0^2+8} = \frac{\pi i}{4}$$

c)

$$\int_C \frac{zdz}{2z+1} = 2\pi i \cdot \frac{z}{2}\Big|_{z=-1/2} = 2\pi i \cdot -\frac{1}{4} = -\frac{\pi i}{2}$$

d)

$$\int_C \frac{\cosh z}{z^4}dz = \frac{2\pi i}{3!}f^{(3)}(0) = \frac{\pi i}{3} \cdot \sinh 0 = 0$$

e)

$$\int_C \frac{\tan(z/2)}{(z-x_0)^2}dz = 2\pi i \cdot f'(x_0) = 2\pi i \cdot \frac{1}{2}\sec^2(x_0/2) = \pi i \sec^2(x_0/2)$$

4.(47-48).5

Case 1: z_0 is inside C.

$$\int_C \frac{f'(z)dz}{z-z_0} = 2\pi i \cdot f'(z_0)$$
$$\int_C \frac{f(z)dz}{(z-z_0)^2} = \frac{2\pi i}{1!}f'(z_0) = 2\pi i \cdot f'(z_0)$$

Case 2: z_0 is outside C.

For all points on and inside the contour (which no longer includes z_0), $\frac{f'(z)}{z-z_0}$ and $\frac{f(z)}{(z-z_0)^2}$ are analytic, so by the Cauchy-Goursat theorem the integral of both functions evaluate to 0.

4.(47-48).8

a)

$$P_n(z) = \frac{1}{n!2^n}\frac{d^n}{dz^n}(z^2-1)^n = \frac{1}{n!2^n}\frac{d^n}{dz^n}\left(\sum_{k=0}^{n}\binom{n}{k}z^{2k}(-1)^{n-k}\right)$$

The highest exponent of z inside the summation is z^{2n} occurring when $k=n$, and after taking the nth derivative the term gets reduced to order z^n, confirming that the function is a polynomial of degree n.

b)

$$P_n(z) = \frac{1}{n!2^n}\frac{d^n}{dz^n}(z^2-1)^n = \frac{1}{n!2^n}\frac{n!}{2\pi i}\int_C \frac{(s^2-1)^n}{(s-z)^{n+1}}ds = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s-z)^{n+1}}ds$$

c)

When $z = 1$:

$$\frac{(s^2-1)^n}{(s-1)^{n+1}} = \frac{(s-1)^n(s+1)^n}{(s-1)^{n+1}} = \frac{(s+1)^n}{s-1}$$

Evaluating $P_n(z)$ at 1 and -1:

$$P_n(1) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s-1)^{n+1}}ds = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s+1)^n}{s-1}ds = \frac{1}{2^{n+1}\pi i}\cdot 2\pi i(1+1)^n = \frac{2\pi i\cdot 2^n}{2^{n+1}\pi i} = 1$$
$$\frac{(s^2-1)^n}{(s-(-1))^{n+1}} = \frac{(s-1)^n(s+1)^n}{(s+1)^{n+1}} = \frac{(s-1)^n}{s+1}$$
$$P_n(-1) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s-(-1))^{n+1}}ds = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s-1)^n}{s+1}ds = \frac{1}{2^{n+1}\pi i}\cdot 2\pi i(-1-1)^n = \frac{2\pi i(-2)^n}{2^{n+1}\pi i} = (-1)^n$$

5.(51-52).1

Way 1:

$$\begin{aligned} z_n &= -2 + i\frac{(-1)^n}{n^2} \\ x_n &= -2 \\ y_n &= \frac{(-1)^n}{n^2} \text{ which converges to } 0 \\ \implies \lim_{n\to\infty} z_n &= -2 + i\cdot 0 = -2 \end{aligned}$$

Way 2:

$$|z_n - (-2)| = \left| i\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2}$$

For any $\epsilon > 0$, there exists a positive integer n_0 (say $\frac{1}{\sqrt{\epsilon}}$) such that $\frac{1}{n^2} < \epsilon$ when $n > n_0$.

5.(51-52).6

$$\begin{aligned} z_n &= x_n + iy_n \\ \sum_{n=1}^{\infty} x_n &= S_x \\ \sum_{n=1}^{\infty} y_n &= S_y \\ \sum_{n=1}^{\infty} z_n &= S = S_x + iS_y \\ \sum_{n=1}^{\infty} -y_n &= -S_y \\ \sum_{n=1}^{\infty} \overline{z_n} &= \sum_{n=1}^{\infty} x_n + i\sum_{n=1}^{\infty} -y_n = S_x - iS_y = \overline{S} \end{aligned}$$

5.(53-54).11

a)

$$\frac{e^z}{z^2} = \frac{1}{z^2}\left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right) = \frac{1}{z^2}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\dots\right) = \frac{1}{z^2}+\frac{1}{z}+\frac{1}{2!}+\frac{z}{3!}+\frac{z^2}{4!}+\dots$$

b)

$$\frac{\sin(z^2)}{z^4} = \frac{1}{z^4}\left(\sum_{n=0}^{\infty}(-1)^n\frac{(z^2)^{2n+1}}{(2n+1)!}\right) = \frac{1}{z^4}\left(z^2-\frac{z^6}{3!}+\frac{z^{10}}{5!}-\frac{z^{14}}{7!}\right) = \frac{1}{z^2}-\frac{z^2}{3!}+\frac{z^6}{5!}-\frac{z^{10}}{7!}+\dots$$

5.(53-54).1

$$\cosh z = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}$$
$$z\cosh(z^2) = z\sum_{n=0}^{\infty}\frac{(z^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{z^{4n+1}}{(2n)!}$$

5.(55-56).4

Case 1: $0 < |z| < 1$

$$f(z) = \frac{1}{z^2}\frac{1}{1-z} = \frac{1}{z^2}\sum_{n=0}^{\infty}z^n = \sum_{n=0}^{\infty}z^{n-2} = \frac{1}{z^2}+\frac{1}{z}+\sum_{n=0}^{\infty}z^n$$

Case 2: $1 < |z| < \infty \implies 0 < |1/z| < 1$

$$f(z) = \frac{1}{z^2}\frac{1/z}{1/z-1} = -\frac{1}{z^3}\sum_{n=0}^{\infty}(1/z)^n = -\sum_{n=0}^{\infty}\frac{1}{z^{n+3}} = -\sum_{n=3}^{\infty}\frac{1}{z^n}$$

5.(55-56).8

a)

$$\frac{a}{z-a}\cdot\frac{1/z}{1/z} = \frac{a}{z}\cdot\frac{1}{1-a/z} = \frac{a}{z}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n = \sum_{n=0}^{\infty}\frac{a^{n+1}}{z^{n+1}} = \sum_{n=1}^{\infty}\frac{a^n}{z^n}$$

The transformation to an infinite series is valid because $|a| < |z| \implies |a/z| < 1$.

b)

Plug in $z = e^{i\theta}$:

$$\frac{a}{e^{i\theta}-a} = \sum_{n=1}^{\infty}\frac{a^n}{(e^{i\theta})^n}$$
$$\frac{a}{e^{i\theta}-a} = \frac{a}{\cos\theta+i\sin\theta-a}\cdot\frac{\cos\theta-a-i\sin\theta}{\cos\theta-a-i\sin\theta} = \frac{a\cos\theta-a^2-ia\sin\theta}{(\cos\theta-a)^2+\sin^2\theta} = \frac{a\cos\theta-a^2-ia\sin\theta}{1-2a\cos\theta+a^2}$$
$$\sum_{n=1}^{\infty}\frac{a^n}{(e^{i\theta})^n} = \sum_{n=1}^{\infty}a^n(e^{-in\theta}) = \sum_{n=1}^{\infty}a^n(\cos(-n\theta)+i\sin(-n\theta)) = \sum_{n=1}^{\infty}a^n\cos n\theta - i\sum_{n=1}^{\infty}a^n\sin n\theta$$

Equate real and imaginary parts:

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2}$$

$$\sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2}$$