Pset 2

2.13.1

$$
\begin{gathered}
z = x + iy \\
u = x^2 - y^2 \\
v = 2xy \\
1 \leq u, v \leq 2 \\
1 \leq x^2 - y^2, 2xy \leq 2
\end{gathered}
$$

We want the intersection of the area between the hyperbolas $x^2 - y^2 = 1$ and $x^2 - y^2 = 2$, and the area between the hyperbolas $xy = 1/2$ and $xy = 1$.

2.(14-17).7

$$
\begin{gathered}
\lim_{z \to z_0} f(z) = w_0 \\
\implies \forall\, \epsilon\, \exists\, \delta, |f(z) - w_0| < \epsilon, 0 < |z - z_0| < \delta \\
\text{By the inequality } ||z_1| - |z_2|| \leq |z_1 - z_2| \\
\implies ||f(z)| - |w_0|| \leq |f(z) - w_0| < \epsilon \\
||f(z)| - |w_0|| < \epsilon, 0 < |z - z_0| < \delta \\
\implies \lim_{z \to z_0} |f(z)| = |w_0|
\end{gathered}
$$

2.(14-17).10

a)

$$
\begin{gathered}
\lim_{z \to \infty} \frac{4z^2}{(z-1)^2} = 4 \\
\iff \lim_{z \to 0} \frac{4/z^2}{(1/z - 1)^2} = 4
\end{gathered}
$$

$$
\begin{aligned}
\lim_{z \to 0} \frac{4/z^2}{(1/z - 1)^2} &= \lim_{z \to 0} \frac{4/z^2}{1/z^2 - 2/z + 1} \\
&= \lim_{z \to 0} \frac{4}{1 - 2z + z^2}, z \neq 0 \\
&= \frac{4}{1 - 2 \cdot 0 + 0^2} = 4
\end{aligned}
$$

b)

$$
\begin{gathered}
\lim_{z \to 1} \frac{1}{(z-1)^3} = \infty \\
\iff \lim_{z \to 1} (z-1)^3 = 0 \\
\lim_{z \to 1} (z-1)^3 = (1-1)^3 = 0
\end{gathered}
$$

c)

$$
\begin{aligned}
\lim_{z \to \infty} \frac{z^2+1}{z-1} &= \infty \\
\iff \lim_{z \to 0} \frac{1/z-1}{1/z^2+1} &= 0 \\
\lim_{z \to 0} \frac{1/z-1}{1/z^2+1} &= \lim_{z \to 0} \frac{z-z^2}{1+z^2}, z \neq 0 \\
&= \frac{0-0^2}{1+0^2} = 0
\end{aligned}
$$

2.(14-17).13

Definition of bounded: every point of S lies inside some circle $|z| = R$.

Assume that there exists a neighborhood of ∞ that contains no points of S. The definition of a neighborhood of ∞ is some set $|z| > 1/\epsilon$ for some small positive number ϵ. Since no points of S lie in this neighborhood, all points of S must instead lie inside the set $|z| \leq 1/\epsilon$. This is the definition of a bounded set with $R = 1/\epsilon$, thus S is not unbounded. Each neighborhood of ∞ must thus contain a point in S for S to be unbounded.

2.(18-19).8

a)

$$
\begin{aligned}
f(z) &= \overline{z} \\
\frac{\Delta w}{\Delta z} &= \frac{\overline{z+\Delta z} - \overline{z}}{\Delta z} \\
&= \frac{\overline{z} + \overline{\Delta z} - \overline{z}}{\Delta z} \\
&= \frac{\overline{\Delta z}}{\Delta z}
\end{aligned}
$$

$$
\text{Approach from } (\Delta x, 0): \\
\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = 1
$$

$$
\text{Approach from } (0, \Delta y): \\
\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{-\Delta z}{\Delta z} = -1
$$

$1 \neq -1$, so no limit exists and thus the derivative fails to exist for any point z.

b)

$$
\begin{aligned}
f(z) &= \Re(z) \\
\frac{\Delta w}{\Delta z} &= \frac{\Re(z+\Delta z) - \Re(z)}{\Delta z} \\
&= \frac{\Re(z) + \Re(\Delta z) - \Re(z)}{\Delta z} \\
&= \frac{\Re(\Delta z)}{\Delta z}
\end{aligned}
$$

$$
\text{Approach from } (\Delta x, 0): \\
\lim_{\Delta z \to 0} \frac{\Re(\Delta z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = 1
$$

$$\text{Approach from } (0, \Delta y):$$
$$\lim_{\Delta z \to 0} \frac{\Re(\Delta z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{0}{\Delta z} = 0$$

$1 \neq 0$, so no limit exists and thus the derivative fails to exist for any point z.

c)

$$\begin{aligned} f(z) &= \Im(z) \\ \frac{\Delta w}{\Delta z} &= \frac{\Im(z+\Delta z) - \Im(z)}{\Delta z} \\ &= \frac{\Im(z) + \Im(\Delta z) - \Im(z)}{\Delta z} \\ &= \frac{\Im(\Delta z)}{\Delta z} \end{aligned}$$

$$\text{Approach from } (\Delta x, 0):$$
$$\lim_{\Delta z \to 0} \frac{\Im(\Delta z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{0}{\Delta z} = 0$$
$$\text{Approach from } (0, \Delta y):$$
$$\lim_{\Delta z \to 0} \frac{\Im(\Delta z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = 1$$

$0 \neq 1$, so no limit exists and thus the derivative fails to exist for any point z.

2.(18-19).9

$$\begin{aligned} f(z) &= \frac{\overline{z}^2}{z}, z \neq 0 \\ \frac{\Delta w}{\Delta z} &= \frac{\frac{\overline{z+\Delta z}^2}{z+\Delta z} - \frac{\overline{z}^2}{z}}{\Delta z} \\ &= \frac{\overline{(z+\Delta z)}^2}{(z+\Delta z)\Delta z} - \frac{\overline{z}^2}{z\Delta z} \end{aligned}$$

$$\text{Approach from } (\Delta x, 0) \text{ or } (0, \Delta y), z = 0:$$
$$\lim_{\Delta z \to 0} \frac{\overline{(z+\Delta z)}^2}{(z+\Delta z)\Delta z} - \lim_{\Delta z \to 0} \frac{\overline{z}^2}{z\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{(\Delta z)}^2}{(\Delta z)^2} - 0 = \lim_{\Delta z \to 0} \frac{(\Delta z)^2}{(\Delta z)^2} = 1$$
$$\text{Approach from } (\Delta x, \Delta x), z = 0:$$
$$(\Delta x + i\Delta x)^2 = (\Delta x)^2 + 2i(\Delta x)^2 - (\Delta x)^2 = 2i(\Delta x)^2$$
$$\lim_{\Delta z \to 0} \frac{\overline{(z+\Delta z)}^2}{(z+\Delta z)\Delta z} - \lim_{\Delta z \to 0} \frac{\overline{z}^2}{z\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{(\Delta z)}^2}{(\Delta z)^2} - 0 = \lim_{\Delta z \to 0} \frac{-2i(\Delta x)^2}{2i(\Delta x)^2} = -1$$

$1 \neq -1$, so no limit exists and thus the derivative fails to exist at $z = 0$.

2.(20-22).1

a)

$$\begin{aligned} f(z) = \overline{z} = x - iy \\ u_x = 1, v_y = -1, u_x \neq v_y \end{aligned}$$

Since the Cauchy-Riemann equations are never satisfied, $f'(z)$ cannot possibly exist for any z.

b)

$$
\begin{gathered}
f(z) = z - \overline{z} = (x - x) + i(y - (-y)) = 0 + i \cdot 2y \\
u_x = 0, v_y = 2, u_x \neq v_y
\end{gathered}
$$

Since the Cauchy-Riemann equations are never satisfied, $f'(z)$ cannot possibly exist for any z.

c)

$$
\begin{gathered}
f(z) = 2x + ixy^2 \\
u_x = 2, v_y = 2xy, u_x \neq v_y
\end{gathered}
$$

Since the Cauchy-Riemann equations are never satisfied, $f'(z)$ cannot possibly exist for any z.

d)

$$
\begin{gathered}
f(z) = e^x e^{-iy} = e^x(\cos(-y) + i\sin(-y)) = e^x \cos y - i \cdot e^x \sin y \\
u_y = -e^x \sin y, v_x = -e^x \sin y, u_y \neq -v_x
\end{gathered}
$$

Since the Cauchy-Riemann equations are never satisfied, $f'(z)$ cannot possibly exist for any z.

2.(20-22).4

b)

For all z in the domain of definition, the function is defined in its neighborhood, the first-order partial derivatives exist everywhere in that neighborhood, the function is continuous at the point, and the Cauchy-Riemann equations are satisfied:

$$
\begin{gathered}
f(z) = \sqrt{r}e^{i\theta/2} = \sqrt{r}\cos(\theta/2) + i \cdot \sqrt{r}\sin(\theta/2) \\
u_r = \frac{1}{2}r^{-1/2}\cos(\theta/2) \\
ru_r = \frac{1}{2}\sqrt{r}\cos(\theta/2) \\
v_\theta = \sqrt{r} \cdot \frac{1}{2}\cos(\theta/2) \\
ru_r = v_\theta \\
u_\theta = -\frac{1}{2}\sqrt{r}\sin(\theta/2) \\
v_r = \frac{1}{2}r^{-1/2}\sin(\theta/2) \\
-rv_r = -\frac{1}{2}\sqrt{r}\sin(\theta/2) \\
u_\theta = -rv_r
\end{gathered}
$$

Thus, the function is differentiable in its domain of definition.

$$
\begin{aligned}
f'(z) &= e^{-i\theta}(u_r + iv_r) \\
&= e^{-i\theta}\left(\frac{1}{2}r^{-1/2}\cos(\theta/2) + i \cdot \frac{1}{2}r^{-1/2}\sin(\theta/2)\right) \\
&= \frac{1}{2\sqrt{r}e^{i\theta}} \cdot e^{i\theta/2} \\
&= \frac{1}{2\sqrt{r}e^{i\theta/2}} \\
&= \frac{1}{2f(z)}
\end{aligned}
$$

2.(20-22).6

$$f(z) = 0, z = 0$$
$$u_x = v_y = u_y = -v_x = 0$$

The Cauchy-Riemann equations are satisfied at the origin, yet in Exercise 9, Sec. 19, we proved $f'(0)$ fails to exist. This is because z_0 satisfying the Cauchy-Riemann equations alone is insufficient to ensure the existence of $f'(z_0)$.

2.(20-22).9

a)

$$\begin{aligned} f'(z_0) &= \frac{-i}{z_0}(u_\theta + iv_\theta) \\ &= \frac{-i}{re^{i\theta}}(-rv_r + iru_r) \\ &= \frac{v_r}{e^{i\theta}}i + \frac{u_r}{e^{i\theta}} \\ &= e^{-i\theta}(u_r + iv_r) \end{aligned}$$

b)

$$\begin{aligned} f(z) = 1/z = 1/re^{i\theta} &= \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta) \\ u_r &= -\frac{1}{r^2}\cos\theta \\ v_r &= \frac{1}{r^2}\sin\theta \end{aligned}$$

$$f'(z) = e^{-i\theta}(-\frac{1}{r^2}\cos\theta + i\cdot\frac{1}{r^2}\sin\theta) = -\frac{e^{-i\theta}}{r^2}\cdot e^{-i\theta} = -\frac{1}{r^2e^{i2\theta}} = -\frac{1}{(re^{i\theta})^2} = -\frac{1}{z^2}$$

2.(23-24).1

a)

Existence of neighborhood and continuity conditions are satisfied.

$$f(z) = (3x + y) + i(3y - x)$$
$$u_x = 3, v_y = 3, u_x = v_y$$
$$u_y = 1, v_x = -1, u_y = -v_x$$

The Cauchy-Riemann equations are also satisfied, thus $f(z)$ is analytic everywhere, and thus $f(z)$ is entire.

2.(23-24).2

a)

$$f(z) = xy + iy$$
$$u_x = y, v_y = 1, u_x \neq v_y$$

Since the Cauchy-Riemann equations are never satisfied, $f(z)$ is not differentiable anywhere and thus not analytic anywhere.

2.(23-24).4

a)

$$f(z) = \frac{2z+1}{z(z^2+1)}$$
$$z(z^2+1) = 0 \implies z = 0, \pm i$$

$z = 0, \pm i$ are singular points. The function is analytic everywhere except at those points because at the aforementioned points, $f(z)$ is not defined.

2.(23-24).5

$f(z)$ is defined, has partial derivatives, and is continuous everywhere.

$$f(z) = 2z - 2 + i = (2x-2) + i(2y+1)$$
$$u_x = 2, v_y = 2, u_x = v_y$$
$$u_y = 0, v_x = 0, u_y = -v_x$$

Thus, $f(z)$ is analytic everywhere.

When $x > 1$, $\Re(2z - 2 + i) = 2x - 2 > 2 \cdot 1 - 2 = 0$, meaning $r\cos\theta > 0$ and thus $-\pi < \theta < \pi$, so the image of $f(z)$ in the half plane is within the domain of definition of $g(w)$. It is also known that the composition of two analytic functions is analytic if the image of the inner function is contained in the domain of definition of the outer function, so $G(z) = g(f(z))$ is analytic in the half plane.

$$G'(z) = G'(f(z))f'(z) = \frac{1}{2 \cdot g(2z-2+i)} \cdot 2 = \frac{1}{g(2z-2+i)}$$

2.(23-24).7

a)

If $f(z)$ is real-valued in D, then $v = 0, v_x = v_y = 0$. Being analytic implies it satisfies Cauchy-Riemann, so $u_x = v_y = 0$. It is also known that $f'(z) = u_x + iv_x = 0 + i0 = 0$, which means $f(z)$ is constant throughout D.

b)

We know $|f(z)|^2 = f(z) \cdot \overline{f(z)}$. Since $|f(z)|$ is constant, say $|f(z)| = c$, we get that $\overline{f(z)} = c^2/f(z)$. If $c = 0$, then $\overline{f(z)} = 0$ is analytic, and the result from Example 3, Sec. 24 says if $f(z)$ and $\overline{f(z)}$ are both analytic, $f(z)$ is constant throughout D. If $c \neq 0$, then $\overline{f(z)}$ is still analytic because the quotient of two analytic functions is analytic, so Example 3, Sec. 24 applies once again.

2.25.1

a)

$$u(x,y) = 2x(1-y)$$
$$u_{xx} = 0, u_{yy} = 0, u_{xx} + u_{yy} = 0 \implies u(x,y) \text{ is harmonic}$$
$$u_x = 2 - 2y, u_y = -2x$$
$$u_x = v_y, u_y = -v_x$$
$$v(x,y) = 2y - y^2 + x^2$$
$$v_{xx} = 2, v_{yy} = -2, v_{xx} + v_{yy} = 0 \implies v(x,y) \text{ is harmonic}$$

$v(x,y)$ is a harmonic conjugate of u.

2.25.2

Suppose that $V = v + c(x, y)$:

$$u_x = v_y = V_y, u_y = -v_x = -V_x$$
$$v_{xx} + v_{yy} = V_{xx} + V_{yy} = 0$$
$$v_y = V_y = v_y + c_y \implies c_y = 0 \implies c(x, y) \text{ contains no } y\text{'s}$$
$$v_x = V_x = v_x + c_x \implies c_x = 0 \implies c(x, y) \text{ contains no } x\text{'s}$$

Thus, $c(x, y)$ must be a constant.

2.25.7

$u(x, y)$ and $v(x, y)$ are constant, meaning their derivatives are zero:

$$\nabla u = \langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \rangle = \langle 0, 0 \rangle$$
$$\nabla v = \langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \rangle = \langle 0, 0 \rangle$$
$$\nabla u \cdot \nabla v = 0 \implies \text{perpendicular}$$

2.25.8

$$f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + i \cdot 2xy$$

When you plot $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$, you indeed get two hyperbolas that look as shown in the diagram.

3.28.3

$$f(z) = e^{\bar{z}} = e^x e^{-iy} = e^x(\cos y - i \sin y)$$
$$u_x = e^x \cos y, v_y = -e^x \cos y, u_x \neq v_y$$

Since the Cauchy-Riemann equations are never satisfied, $f(z)$ is not differentiable and thus not analytic anywhere.

3.28.5

$$|e^{2z+i}| = |e^{2x} e^{i(2y+1)}| = e^{2x}$$
$$|e^{iz^2}| = |e^{-2xy} e^{i(x^2-y^2)}| = e^{-2xy}$$
$$|e^{2z+i} + e^{iz^2}| \leq |e^{2z+i}| + |e^{iz^2}| = e^{2x} + e^{-2xy}$$

3.28.10

a)

If e^z is real, that implies $e^z = e^x e^{iy}, e^{iy} = \pm 1$, as e^z must lie on the real axis. This means $y \in \{0 + n \cdot 2\pi\} \cup \{\pi + n \cdot 2\pi\}$, or $y \in \{0 + n\pi\}$, and $y = \Im(z)$

b)

$y \in \pi/2 + n\pi$, as that's the necessary angle to get e^z onto the imaginary axis.

3.28.13

Because of the nature of e^z giving itself when differentiating, and $\cos z$ and $\sin z$ giving signed versions of each other upon differentiating, and we get several other chain rule terms that end up canceling, we get $U_{xx} + U_{yy} = V_{xx} + V_{yy} = 0$ and the Cauchy-Riemann equations satisfied, so the two functions are harmonic and V is a harmonic conjugate of U.