Pset 3

3.(29-30).4

a)

$$\log(i^2) = \log(-1) = \ln 1 + i \cdot \pi = i \cdot \pi = 2(\ln 1 + i \cdot \pi/2) = 2 \log i$$

b)

$$\log(i^2) = \log(-1) = \ln 1 + i \cdot \pi = i \cdot \pi = 2(\ln 1 + i \cdot \pi/2) \neq 2(\ln 1 + i \cdot 5\pi/2) = 2 \log i$$

In part b, our branch is different.

3.(29-30).10

Method 1: check if the function satisfies Laplace's equation:

$$\begin{aligned} H(x, y) &= \ln(x^2 + y^2) \\ H_x &= \frac{2x}{x^2 + y^2} \\ H_{xx} &= -\frac{2(x^2 - y^2)}{(x^2 + y^2)^2} \\ H_y &= \frac{2y}{x^2 + y^2} \\ H_{yy} &= \frac{2(x^2 - y^2)}{(x^2 + y^2)^2} \\ H_{xx} + H_{yy} &= 0 \end{aligned}$$

Method 2: we know the component functions $u(x, y), v(x, y)$ of a function $f(z) = u + iv$ are harmonic in D if f is analytic in D. Since $\ln(x^2 + y^2)$ is the real component of any branch cut of $2 \log z$, and $\log z$ and consequently $2 \log z$ are analytic in the domain $r > 0$, aka not the origin, $\ln(x^2 + y^2)$ is holomorphc in the same domain.

3.(29-30).11

$$\begin{aligned} \Re(\log(z-1)) &= \Re(\ln|z-1| + i \arg(z-1)) \\ &= \ln|z-1| \\ &= \ln|(x-1) + iy| \\ &= \ln(((x-1)^2 + y^2)^{1/2}) \\ &= \frac{1}{2}\ln((x-1)^2 + y^2) \end{aligned}$$

$\log z$ is analytic everywhere where its input is non-zero, so $\log(z-1)$ is analytic on the domain $z \neq 1$, and since the components of an analytic function are harmonic, $\Re(\log(z-1))$ is harmonic and thus satisfies Laplace's equation by definition.

3.31.1

With the imposed restrictions $\Re(z_1), \Re(z_2) > 0$, the domain of the angle remains correct.

$$
\begin{aligned}
-\frac{\pi}{2} &< \Theta_1, \Theta_2 < \frac{\pi}{2} \\
-\pi &< \Theta_1 + \Theta_2 < \pi \\
\text{Log } z_1 + \text{Log } z_2 &= \ln r_1 + i\Theta_1 + \ln r_2 + i\Theta_2 \\
&= (\ln r_1 + \ln r_2) + (i\Theta_1 + i\Theta_2) \\
&= \ln(r_1 r_2) + i(\Theta_1 + \Theta_2) \\
&= \text{Log } (z_1 z_2)
\end{aligned}
$$

3.31.4

$$
\begin{aligned}
z_1 = i, z_2 &= -i \\
\text{Log}\left(\frac{z_1}{z_2}\right) &= \text{Log}(-1) \\
\text{Log}(i) - \text{Log}(-i) &= \frac{i\pi}{2} - \frac{-i\pi}{2} = i\pi
\end{aligned}
$$

The LHS is not defined because for an input of -1, $\Theta = \pi$ which is not in the domain of the principal branch.

3.31.6

$$
\begin{aligned}
\log(z^{1/n}) &= \log\left(\exp\left(\frac{1}{n}\log z\right)\right) \\
&= \log\left(r^{1/n}\exp\left[i\left(\frac{\Theta}{n} + \frac{2k\pi}{n}\right)\right]\right), k = 0, 1, 2, \ldots, n-1 \\
&= \ln(r^{1/n}) + i\left(\frac{\Theta}{n} + \frac{2k\pi}{n}\right) + i2\pi p, p = 0, \pm 1, \pm 2, \ldots \\
&= \frac{1}{n}\ln r + i\left(\frac{\Theta + 2k\pi + 2\pi pn}{n}\right) \\
&= \frac{1}{n}\ln r + i\left(\frac{\Theta + 2(pn+k)\pi}{n}\right)
\end{aligned}
$$

We also have that

$$
\frac{1}{n}\log z = \frac{1}{n}\ln r + i\frac{\Theta + 2q\pi}{n}, q = 0, \pm 1, \pm 2, \ldots
$$

The set of values covered are equal. Consider $q = pn + k$, where p is the uniquely defined quotient after dividing by n and k is the uniquely defined remainder. The possible values of k span the set of numbers $0, 1, \ldots, n-1$, and p can be any integer, which matches exactly with the set of possible values of p and k defined in the first expression. Thus:

$$
\log(z^{1/n}) = \frac{1}{n}\log z
$$

3.32.1

b)

$$
(-1)^{1/\pi} = \exp(1/\pi \cdot \log(-1)) = \exp(1/\pi \cdot (\ln 1 + i(\pi + 2n\pi))) = \exp(i(1+2n)) = e^{(2n+1)i}
$$

3.32.2

b)

$$\begin{aligned}
\text{P.V.}\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i} &= \exp\left(3\pi i\cdot \text{Log}\left(\frac{e}{2}(-1-\sqrt{3}i)\right)\right) \\
&= \exp\left(3\pi i\cdot (1-\frac{2\pi i}{3})\right) \\
&= \exp(2\pi^2 + i\cdot 3\pi) \\
&= \exp(2\pi^2)\exp(i\cdot 3\pi) \\
&= -\exp(2\pi^2)
\end{aligned}$$

3.32.3

$$\begin{aligned}
(-1+\sqrt{3}i)^{3/2} &= \exp\left(\frac{3}{2}\log(-1+\sqrt{3}i)\right) \\
&= \exp\left(\frac{3}{2}\cdot\log(2e^{i2\pi/3})\right) \\
&= \exp\left(\frac{3}{2}(\ln 2 + i(\frac{2\pi}{3}+2n\pi))\right) \\
&= \exp\left((\ln(2^{3/2}) + i(\pi+3n\pi))\right) \\
&= 2^{3/2}\cdot\exp(i\pi(3n+1)) \\
&= 2\sqrt{2}\cdot\exp(i\pi(3n+1))
\end{aligned}$$

Note that as we cycle through the values $n = 0, 1, 2, \ldots$, the angles take on values equivalent to $\pi, 0, \pi, 0, \pi, \ldots$, so we only get two unique values, namely $\pm 2\sqrt{2}$.

3.32.5

From Sec. 8:

$$\text{P.V. } z_0^{1/n} = \sqrt[n]{r_0}\exp(i\Theta_0), -\pi < \Theta_0 \le \pi$$

From Sec. 32:

$$\text{P.V. } z_0^{1/n} = \exp\left(\frac{1}{n}\text{Log } z_0\right) = \exp\left(\frac{1}{n}\ln r_0\right)\cdot\exp(i\Theta_0) = \sqrt[n]{r_0}\exp(i\Theta_0)$$

which is equivalent.

3.32.6

$$z^a = \exp(a\log z)$$
$$|z^a| = |\exp(a\log z)| = |\exp(a\ln|z|)\cdot\exp(ai\theta)| = |\exp(a\ln|z|)|\cdot|\exp(ai\theta)| = \exp(a\ln|z|) = |z|^a$$

3.33.5

a)

$$\cos 0 = 1$$
$$\cos(z-z) = \cos z\cos(-z) - \sin z\sin(-z) = \cos z\cos z + \sin z\sin z = \sin^2 z + \cos^2 z = 1$$

b)

$\sin z$ and $\cos z$ are entire functions, so $\sin^2 z$, $\cos^2 z$, and $f(z) = \sin^2 z + \cos^2 z - 1$ are entire, upholding part 1 of the lemma from Sec. 26. Consider the x-axis as our line segment of choice. Then $f(z) = f(x + 0i) = \sin^2 x + \cos^2 x - 1 = 0$ for all x, upholding part 2 of the lemma from Sec. 26. Since both parts of the lemma hold true, we can conclude that $f(z) \equiv 0$ in D, so $\sin^2 z + \cos^2 z = 1$.

3.33.11

a)

$$\begin{gathered}
|\sin z|^2 = \sin^2 x + \sinh^2 y \\
|\sin z|^2 \geq \sinh^2 y \\
|\sin z| \geq |\sinh y| \\
|\sin z|^2 \leq 1 + \sinh^2 y = \cosh^2 y \\
|\sin z| \leq |\cosh y| = \cosh y \\
\implies |\sinh y| \leq |\sin z| \leq \cosh y
\end{gathered}$$

b)

$$\begin{gathered}
|\cos z|^2 = \cos^2 x + \sinh^2 y \\
|\cos z|^2 \geq \sinh^2 y \\
|\cos z| \geq |\sinh y| \\
|\cos z|^2 \leq 1 + \sinh^2 y = \cosh^2 y \\
|\cos z| \leq \cosh y \\
\implies |\sinh y| \leq |\cos z| \leq \cosh y
\end{gathered}$$

3.33.13

$$\begin{gathered}
\sin \bar{z} = \sin(x - iy) = \sin x \cosh(-y) + i \cos x \sinh(-y) = \sin x \cosh y - i \cos x \sinh y = u + iv \\
u_x = \cos x \cosh y, v_y = -\cos x \cosh y \\
u_x = v_y \implies \cos x \cosh y = 0 \implies \cos x = 0 \implies x = \frac{\pi}{2} + n\pi \\
u_y = v_x = \sin x \sinh y \\
u_y = -v_x \implies \sin x \sinh y = 0 \implies \sin x = 0 \lor \sinh y = 0 \\
\text{For } x = \frac{\pi}{2} + n\pi, \sin x \neq 0 \implies \sinh y = 0 \implies y = 0
\end{gathered}$$

Cauchy-Riemann only holds for $z = \frac{\pi}{2} + n\pi$, which are points spaced in discrete intervals, so there is no continuous neighborhood around a point for which the function is analytic.

$$\begin{gathered}
\cos \bar{z} = \cos(x - iy) = \cos x \cosh(-y) - i \sin x \sinh(-y) = \cos x \cosh y + i \sin x \sinh y = u + iv \\
u_x = -\sin x \cosh y, v_y = \sin x \cosh y \\
u_x = v_y \implies \sin x \cosh y = 0 \implies \sin x = 0 \implies x = n\pi \\
u_y = v_x = \cos x \sinh y \\
u_y = -v_x \implies \cos x \sinh y = 0 \implies \cos x = 0 \lor \sinh y = 0 \\
\text{For } x = n\pi, \cos x \neq 0 \implies \sinh y = 0 \implies y = 0
\end{gathered}$$

Cauchy-Riemann only holds for $z = n\pi$, which are points spaced in discrete intervals, so there is no continuous neighborhood around a point for which the function is analytic.

3.33.17

$$
\begin{gathered}
\sin z = \sin x \cosh y + i \cos x \sinh y \\
\sin x \cosh y = \cosh 4 \\
\cos x \sinh y = 0 \implies \cos x = 0 \lor \sinh y = 0 \\
\sinh y = 0 \implies y = 0 \implies \cosh y = 1 \implies \sin x = \cosh 4 \\
\text{Since } -1 \leq \sin x \leq 1 \text{ and } \cosh 4 \approx 27.3\text{, this is not possible.} \\
\cos x = 0 \implies x = \frac{\pi}{2} + n\pi \implies \sin x = (-1)^n \\
n \text{ is odd} \implies \cosh y = -\cosh 4 \\
\text{Since } \cosh y \geq 1\text{, this is not possible.} \\
n \text{ is even} \implies \cosh y = \cosh 4 \implies y = \pm 4 \\
z = \left(\frac{\pi}{2} + 2n\pi\right) \pm 4i, n = 0, \pm 1, \pm 2, \ldots
\end{gathered}
$$

3.34.6

a)

$$
\begin{gathered}
|\cosh z|^2 = \sinh^2 x + \cos^2 y \geq \sinh^2 x \\
|\cosh z| \geq |\sinh x| \\
|\cosh z|^2 \leq \sinh^2 x + 1 = \cosh^2 x \\
|\cosh z| \leq \cosh x \\
|\sinh x| \leq |\cosh z| \leq \cosh x
\end{gathered}
$$

b)

$$
\begin{gathered}
|\sinh y| \leq |\cos z| \leq \cosh y \\
\text{Consider } \cos(iz) = \cosh z \\
i(x + iy) = -y + ix \\
|\sinh x| \leq |\cosh z| \leq \cosh x
\end{gathered}
$$

3.34.12

$$
\overline{\tanh z} = \overline{\left(\frac{\sinh z}{\cosh z}\right)} = \frac{\overline{\sinh z}}{\overline{\cosh z}} = \frac{\sinh \bar{z}}{\cosh \bar{z}} = \tanh \bar{z}
$$

3.34.13

a)

Consider $f(z) = \cosh^2 z - \sinh^2 z - 1$, which is analytic everywhere. By picking the x-axis for our line segment in D and noting that $\cosh^2 x - \sinh^2 x = 1$, we conclude from the Sec. 26 lemma that $f(z) \equiv 0$ in D, so $\cosh^2 z - \sinh^2 z = 1$.

b)

Consider $f(z) = \sinh z + \cosh z - e^z$, which is analytic everywhere. By picking the x-axis for our line segment in D and noting that $\sinh x + \cosh x = e^x$, we conclude from the Sec. 26 lemma that $f(z) \equiv 0$ in D, so $\sinh z + \cosh z = e^z$.

3.34.14

Both $\sinh z$ and e^z are entire, and a composition of two analytic functions is analytic on the same domain, so $\sinh(e^z)$ is also entire.

$$\begin{aligned}e^z = e^x e^{iy} &= e^x(\cos y + i\sin y)\\ \Re(\sinh z) &= \sinh x \cos y\\ \Re(\sinh(e^z)) &= \sinh(e^x \cos y)\cos(e^x \sin y)\end{aligned}$$

We know this function must be harmonic everywhere because it is a component function of a complex function that is analytic everywhere.

3.34.16

$$\begin{gathered}\cosh z = \cosh x \cos y + i \sinh x \sin y\\ \cosh x \cos y = -2\\ \sinh x \sin y = 0 \implies \sinh x = 0 \vee \sin y = 0\\ \sinh x = 0 \implies x = 0 \implies \cos y = -2 \text{ which is not possible}\\ \sin y = 0 \implies y = n\pi \implies \cos y = (-1)^n\\ n \text{ is odd} \implies \cosh x = 2 \implies x = \pm\ln(2+\sqrt{3})\\ n \text{ is even} \implies \cosh x = -2 \text{ which is not possible}\\ z = \pm\ln(2+\sqrt{3}) + (2n+1)\pi i\end{gathered}$$

3.35.3

$$\begin{aligned}\cos z &= \sqrt{2}\\ z &= \cos^{-1}\sqrt{2}\\ &= -i\log[\sqrt{2} + i(1-(\sqrt{2})^2)^{1/2}]\\ &= -i\log[\sqrt{2}\pm 1]\\ &= -i(\ln(\sqrt{2}\pm 1) + i\arg(\sqrt{2}\pm 1))\\ &= -i\ln(\sqrt{2}\pm 1)\end{aligned}$$

3.35.4

$$\begin{aligned}\sin^{-1} z &= -i\log[iz + (1-z^2)^{1/2}]\\ \frac{d}{dz}\sin^{-1} z &= -i\cdot\frac{1}{iz+(1-z^2)^{1/2}}\cdot\left(i + \frac{1}{2}(1-z^2)^{-1/2}\cdot(-2z)\right)\\ &= -\frac{i}{iz+(1-z^2)^{1/2}}\cdot\left(i - \frac{z}{(1-z^2)^{1/2}}\right)\\ &= -\frac{-1-\frac{iz}{(1-z^2)^{1/2}}}{iz+(1-z^2)^{1/2}}\cdot\frac{iz-(1-z^2)^{1/2}}{iz-(1-z^2)^{1/2}}\\ &= \frac{-\frac{1}{(1-z^2)^{1/2}}}{-1}\\ &= \frac{1}{(1-z^2)^{1/2}}\end{aligned}$$

4.(36-37).2

a)

$$
\begin{aligned}
\int_1^2 \left(\frac{1}{t} - i\right)^2 dt &= \int_1^2 \left(\frac{1}{t^2} - \frac{2i}{t} - 1\right) dt \\
&= \left(-\frac{1}{t} - 2i\ln t - t\right)\Big|_1^2 \\
&= \left(-\frac{1}{2} - 2i\ln 2 - 2\right) - (-1 - 2i\ln 1 - 1) \\
&= -\frac{1}{2} - 2i\ln 2 = -\frac{1}{2} - i\ln 4
\end{aligned}
$$

4.(36-37).4

$$
\begin{aligned}
\int_0^\pi e^{(1+i)x}\, dx &= \frac{e^{(1+i)x}}{1+i}\Big|_0^\pi \\
&= \frac{e^{(1+i)\pi}}{1+i} - \frac{1}{1+i} \\
&= \frac{-e^\pi - 1}{1+i} \cdot \frac{1-i}{1-i} \\
&= \frac{(-1-e^\pi) + i(1+e^\pi)}{2}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\pi e^x \cos x dx &= -\frac{1+e^\pi}{2} \\
\int_0^\pi e^x \sin x dx &= \frac{1+e^\pi}{2}
\end{aligned}
$$

4.(36-37).5

Consider $w(t) = e^{it}, a = 0, b = 2\pi$

$$
\int_0^{2\pi} e^{it} dt = \frac{1}{i} e^{it}\Big|_0^{2\pi} = 0
$$

If it was true that there exists some $c \in (0, 2\pi)$ such that the mean value theorem held, then

$$
\begin{aligned}
0 = w(c)(b-a) &= e^{ic}(2\pi - 0) \\
e^{ic} &= 0
\end{aligned}
$$

But $e^z \neq 0$ for all z, so no such c can exist.

4.38.2

$$\phi(y) = \arctan \frac{y}{\sqrt{4-y^2}}$$

$$\begin{aligned}
z[\phi(y)] &= 2\exp\left(i \arctan \frac{y}{\sqrt{4-y^2}}\right) \\
&= 2\left(\cos\arctan \frac{y}{\sqrt{4-y^2}} + i \sin\arctan \frac{y}{\sqrt{4-y^2}}\right) \\
&= 2\left(\frac{1}{\sqrt{1+\left(\frac{y}{\sqrt{4-y^2}}\right)^2}} + i\frac{\frac{y}{\sqrt{4-y^2}}}{\sqrt{1+\left(\frac{y}{\sqrt{4-y^2}}\right)^2}}\right) \\
&= 2\left(\frac{\sqrt{4-y^2}}{2} + \frac{iy}{2}\right) \\
&= \sqrt{4-y^2} + iy = Z(y)
\end{aligned}$$

$$\frac{d}{dy}\arctan \frac{y}{\sqrt{4-y^2}} = \frac{1}{\sqrt{4-y^2}} > 0$$

4.38.5

$$\begin{gathered}
w(t) = u[x(t), y(t)] + iv[x(t), y(t)] \\
w'(t) = \frac{du}{dt} + i\frac{dv}{dt} = u_x x' + u_y y' + i(v_x x' + v_y y') \\
\text{Apply Cauchy-Riemann:} \\
f'(z_0) = u_x + iv_x \text{ (equation 6 of Sec. 21)} \\
u_x x' + u_y y' + i(v_x x' + v_y y') = x'(u_x + iv_x) + y'(u_y + iv_y) = x' f'(z) + y' f'(z) = f'(z(t)) z'(t)
\end{gathered}$$

4.38.6

a)

$$\begin{gathered}
y(x) = 0 \\
x = 0 \implies y(x) = 0 \implies z = 0 \text{ is one intersection point} \\
x > 0 \implies x^3 \sin\left(\frac{\pi}{x}\right) = 0 \implies \sin\left(\frac{\pi}{x}\right) = 0 \implies \frac{\pi}{x} = n\pi \implies x = \frac{1}{n} \implies z = \frac{1}{n}
\end{gathered}$$

b)

$$\begin{gathered}
0 \le \left|x^3 \sin\left(\frac{\pi}{x}\right)\right| \le x^3 \\
\implies \lim_{x\to 0} x^3 \sin\left(\frac{\pi}{x}\right) = 0 \text{ by squeeze theorem} \\
\implies \text{continuous at } x = 0 \\
y'(x) = 3x^2 \sin\left(\frac{\pi}{x}\right) - \pi x \cos\left(\frac{\pi}{x}\right) \\
0 \le |y'(x)| \le \pi x + 3x^2 \text{ for } x > 0 \\
\implies \lim_{x\to 0} y'(x) = 0 \\
\implies \text{continuous at } x = 0
\end{gathered}$$

4.(39-40).9

$$\int_C f(z)dz = \int_0^{2\pi} f(Re^{i\theta})Rie^{i\theta}d\theta$$
$$\int_{C_0} f(z - z_0)dz = \int_0^{2\pi} f(z_0 + Re^{i\theta} - z_0)Rie^{i\theta}d\theta = \int_0^{2\pi} f(Re^{i\theta})Rie^{i\theta}d\theta$$

The two are equal.

4.(39-40).10

a)

$$\int_{C_0} \frac{dz}{z - z_0} = \int_{-\pi}^{\pi} \frac{Rie^{i\theta}d\theta}{Re^{i\theta}} = i\int_{-\pi}^{\pi} d\theta = 2\pi i$$

b)

$$\int_{C_0} (z - z_0)^{n-1}dz = \int_{-\pi}^{\pi} (Re^{i\theta})^{n-1}Rie^{i\theta}d\theta = R^n i\int_{-\pi}^{\pi} e^{in\theta}d\theta = \frac{R^n}{n}(e^{in\theta})\Big|_{-\pi}^{\pi} = \frac{2iR^n\sin(n\pi)}{n} = 0$$

4.(39-40).11

No steps in 4.(39-40).10 relied on n being an integer, so we can apply the exact same steps to get

$$\int_{C_0} (z - z_0)^{a-1}dz = \frac{2iR^a\sin(a\pi)}{a}$$