Pset 4

4.41.3

$$
\begin{gathered}
|e^z - \bar{z}| \leq |e^z| + |\bar{z}| \\
x \leq 0 \,\forall\, z \text{ on the contour } \implies |e^z| = e^x \leq 1 \\
\text{Furthest point on contour is } z = -4 + 0i \implies |\bar{z}| = |z| \leq 4 \\
M = 1 + 4 = 5 \\
L = 3 + 4 + 5 = 12 \\
\left| \int_C (e^z - \bar{z}) dz \right| \leq ML = 5 \cdot 12 = 60
\end{gathered}
$$

4.41.6

Because $f(z)$ is analytic in the disk $|z| \leq 1$, it is also continuous and thus bounded by some M in that same disk.

$$
\begin{gathered}
z^{-1/2} = \exp\left(-\frac{1}{2}\log z\right) = \exp\left(-\frac{1}{2}(\ln r + i\theta)\right) = r^{-1/2}\exp\left(-\frac{i\theta}{2}\right) \\
|z^{-1/2} f(z)| \leq |z^{-1/2}||f(z)| \leq \rho^{-1/2} \cdot M \\
L = 2\pi\rho \\
\left| \int_{C_p} z^{-1/2} f(z) dz \right| \leq \rho^{-1/2} \cdot M \cdot 2\pi\rho = 2\pi M \sqrt{\rho}
\end{gathered}
$$

4.(42-43).3

For any $f(z) = (z - z_0)^{n-1}, n \leq 0$, there exists an antiderivative $F(z) = \frac{(z-z_0)^n}{n}$. Thus, statement 1 of the theorem from Sec. 42 is satisfied, also implying that statement 3 is true, so the integral of the function around any closed contour C_0 is 0.

4.(42-43).5

Consider the branch $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$. The antiderivative using this branch evaluated at the endpoints is

$$
\begin{aligned}
\frac{z^{i+1}}{i+1}\Big|_{-1}^{1} &= \frac{1}{i+1}(1^{i+1} - (-1)^{i+1}) \\
&= \frac{1}{i+1}(\exp((i+1)\log 1) - \exp((i+1)\log(-1))) \\
&= \frac{1}{i+1}(\exp((i+1)(\ln 1 + i \cdot 0)) - \exp((i+1)(\ln 1 + i \cdot \pi))) \\
&= \frac{1}{i+1}(1 - e^{-\pi}e^{i\pi}) \\
&= \frac{1 + e^{-\pi}}{i+1} \\
&= \frac{1 + e^{-\pi}}{1+i} \cdot \frac{1-i}{1-i} = \frac{1+e^{-\pi}}{2}(1-i)
\end{aligned}
$$

4.(44-46).2

a)

$f(z) = \frac{1}{3z^2+1}$ is analytic everywhere except at $z = \pm\frac{i}{\sqrt{3}}$, which is inside the region enclosed by C_2, so Corollary 2 of Sec. 46 is satisfied.

b)

$f(z) = \frac{z+2}{sin(z/2)}$ is analytic everywhere except at $z = 2n\pi$. For $n = 0$, $z = 0$ lies inside the region enclosed by C_2, and for all other values of n, $z = 2n\pi$ lies outside the region enclosed by C_1, so Corollary 2 of Sec. 46 is satisfied.

c)

$f(z) = \frac{z}{1-e^z}$ is analytic everywhere except at $z = 2n\pi i$. For $n = 0$, $z = 0$ lies inside the region enclosed by C_2, and for all other values of n, $z = 2n\pi i$ lies outside the region enclosed by C_1, so Corollary 2 of Sec. 46 is satisfied.

4.(44-46).5

Since $f(z)$ is entire, by Cauchy-Goursat we know

$$\int_{C_3-C_1} f(z)dz = 0$$

since $C_3 - C_1$ is closed and not self-intersecting.

$$\int_{C_3-C_1} f(z)dz = \int_{C_3} f(z)dz - \int_{C_1} f(z)dz = 0 \implies \int_{C_1} f(z)dz = \int_{C_3} f(z)dz$$

Also consider $C_2 + C_3$ which is a simple closed contour, thus also satisfying Cauchy-Goursat.

$$\int_{C_2+C_3} f(z)dz = \int_{C_2} f(z)dz + \int_{C_3} f(z)dz = 0 \implies \int_{C_2} f(z)dz = -\int_{C_3} f(z)dz$$

Now we consider the closed contour $C = C_1 + C_2 = C_1 - C_3 + C_2 + C_3$:

$$\int_C f(z)dz = \int_{C_1-C_3} f(z)dz + \int_{C_2+C_3} f(z)dz = 0 + 0 = 0$$

since $C_1 - C_3$ is the same contour as $C_3 - C_1$, just of opposite orientation.