Chapter 1

1.2.10

$$(x,y)(x,y)+(x,y)+(1,0)=(0,0)$$
$$(x^2-y^2,2xy)+(x,y)+(1,0)=(0,0)$$
$$\text{Imaginary Part: } 2xy+y=0$$
$$(2x+1)y=0$$
$$x=-\frac{1}{2} \text{ or } y=0$$
$$\text{Real Part: } x^2-y^2+x+1=0$$
$$y=0 \implies x^2+x+1=0$$
$$\text{No real solutions for } x \implies y\neq 0$$
$$x=-\frac{1}{2} \implies -y^2+\frac{3}{4}=0$$
$$y^2=\frac{3}{4}$$
$$y=\pm\frac{\sqrt{3}}{2}$$
$$z=\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$$

1.3.7

$$\begin{aligned}\frac{z_1 z}{z_2 z} &= \left(\frac{z_1}{z_2}\right)\left(\frac{z}{z}\right) \text{ by identity 8}\\ &= \frac{z_1}{z_2}\end{aligned}$$

1.4.5

a)

$-4i$ and $4i$ are complex numbers at coordinates $(0,-4)$ and $(0,4)$ on the complex plane, and we want the set of all points z such that $|z-4i|+|z+4i|$, or the sum of their distance to $(0,-4)$ and $(0,4)$ on the complex plane, is 10. That is the definition of an ellipse: the set of all points such that the sum of their distances to the two foci is constant.

b)

-1 and i correspond to coordinates $(-1,0)$ and $(0,1)$. If we consider the set of all points equidistant to both, we get the line $y=-x$, which goes through the origin and has slope -1. $|z-1|=|z+i|$ refers to points equidistant to the two aforementioned points, hence giving us what we want.

1.5.5

$$
\begin{aligned}
z_1 = x_1 + y_1 i, z_2 = x_2 + y_2 i \\
\left|\frac{z_1}{z_2}\right| &= \left|\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right| \\
&= \sqrt{\left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right)^2 + \left(\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right)^2} \\
&= \sqrt{\frac{(x_1x_2 + y_1y_2)^2 + (y_1x_2 - x_1y_2)^2}{(x_2^2 + y_2^2)^2}} \\
&= \sqrt{\frac{x_1^2x_2^2 + x_1^2y_2^2 + y_1^2x_2^2 + y_1^2y_2^2}{(x_2^2 + y_2^2)^2}} \\
&= \sqrt{\frac{x_1^2(x_2^2 + y_2^2) + y_1^2(x_2^2 + y_2^2)}{(x_2^2 + y_2^2)^2}} \\
&= \sqrt{\frac{x_1^2 + y_1^2}{x_2^2 + y_2^2}} \\
&= \frac{\sqrt{x_1^2 + y_1^2}}{\sqrt{x_2^2 + y_2^2}} \\
&= \frac{|z_1|}{|z_2|}
\end{aligned}
$$

1.5.11

a)

$$
\begin{gathered}
\text{Definition of Real: } y = 0 \\
z = x + yi = x \\
\bar{z} = x - yi = x
\end{gathered}
$$

b)

$$
\begin{gathered}
\text{Real: } y = 0 \implies z = \bar{z} \text{ per part a} \implies z^2 = \bar{z}^2 \\
\text{Pure Imaginary: } x = 0 \\
z = x + yi = yi \\
z^2 = -y^2 \\
\bar{z} = x - yi \\
\bar{z}^2 = (-yi)^2 = -y^2
\end{gathered}
$$

1.7.9

$$
\begin{gathered}
c_1 = r_1 e^{i\theta_1}, c_2 = r_2 e^{i\theta_2}, \bar{c}_2 = r_2 e^{-i\theta_2} \\
z_1 = c_1 c_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \\
z_2 = r_1 r_2 e^{i(\theta_1 - \theta_2)}
\end{gathered}
$$

Both have moduli $r_1 r_2$.

1.9.6

$$z^4 + 4 = 0$$
$$z^4 = -4$$
$$z = \sqrt{2}e^{i\frac{2k\pi+\pi}{4}}, k = 0, 1, 2, 3$$
$$z = 1 + i, 1 - i, -1 - i, -1 + i$$
$$(z - (1 + i))(z - (1 - i))(z - (-1 - i))(z - (-1 + i)) = (z^2 - 2z + 2)(z^2 + 2z + 2)$$

1.9.7

$$c^n = 1, c \neq 1$$
$$1 + c + c^2 + \cdots + c^{n-1} = \frac{1 - c^n}{1 - c} = 0 \text{ when } c \neq 1$$

Geometric intuition: space the roots of unity around a circle, think of them as vectors, their vector sum is the zero vector by symmetry.

1.10.2

a is closed

b is open

c is open

d is closed

e is neither: $z = 0$ and $z = 1 + i$ are both boundary points, the former is not contained while the latter is.

f is closed