Mathematical Biosciences 249 (2014) 102–109

Contents lists available at ScienceDirect

Mathematical Biosciences

journal homepage: www.elsevier.com/locate/mbs

A multi-stage compartmental model for HIV-infected individuals: II – Application to insurance functions and health-care costs

L. Billard [a,*], P.W.A. Dayananda [b]

[a] Department of Statistics, University of Georgia, Athens, GA, United States
[b] Department of Mathematics, University of St. Thomas, St. Paul, MN, United States

ARTICLE INFO

Article history:
Received 12 March 2012
Received in revised form 20 November 2013
Accepted 31 January 2014
Available online 11 February 2014

Keywords:
Compartmental models
Death from non-HIV/AIDS causes
Insurance premiums
Life annuity policies
Life expectancies
Health-care functions

ABSTRACT

Stochastic population processes have received a lot of attention over the years. One approach focuses on compartmental modeling. Billard and Dayananda (2012) [1] developed one such multi-stage model for epidemic processes in which the possibility that individuals can die at any stage from non-disease related causes was also included. This extra feature is of particular interest to the insurance and health-care industries among others especially when the epidemic is HIV/AIDS. Rather than working with numbers of individuals in each stage, they obtained distributional results dealing with the waiting time any one individual spent in each stage given the initial stage. In this work, the impact of the HIV/AIDS epidemic on several functions relevant to these industries (such as adjustments to premiums) is investigated. Theoretical results are derived, followed by a numerical study.

1. Introduction

Over the century since Ross [2] first developed his mathematical model for a malaria process, there has been an extensive array of models introduced for numerous population processes, some being deterministic such as Ross' model, some being stochastic such as Bartlett's [3] early stochastic models for evolutionary processes followed by Bailey's [4] simple stochastic epidemic process. As a general rule (though obviously not true universally), equations governing deterministic models are 'easier' to solve than the more mathematically intractable equations governing stochastic processes. This is particularly so for stochastic models with nonlinear transition rates/probabilities. Yet, these nonlinearities are quite endemic to epidemic/disease processes. Our focus is with stochastic epidemic models; and in particular the impact of the HIV/AIDS epidemic on insurance and health-care functions.

Typically, many researchers have established the relevant set of differential-difference equations describing a given stochastic process (be these epidemic, queuing, biological, etc. models), and then explored techniques to solve these (largely intractable) equations. In more recent times, other researchers have set up their processes in a compartmental modeling framework. An extensive, but by no means exhaustive, review of some of the many approaches adopted over the years is in [1].

Billard and Dayananda [1] considered a compartmental model with transitions from one compartment (stage) to another and in addition allowed for death at any stage to occur. While their general approach is applicable to many population processes, their work was motivated by the need to establish guidelines for the insurance industry in particular (with parallel applications to the health-care industry, among others) as to the impact of the HIV/AIDS epidemic on insurance functions. Therefore, in that work, several probabilities and waiting time distributions for the compartmental process were derived. In this work, we derive insurance functions of interest. For example, insurance companies are vitally interested in future payouts (when and how much) for policies issued to their clients. A major concern relates to the impact that the HIV/AIDS epidemic has on these payments. These in turn are related to the premium levels clients are required to pay for continuous t-year life annuity and insurance policies (e.g.) and their long term effect ($t \to \infty$). Clearly, these entities are impacted by whether or not a potential policy-holder is healthy (i.e., is uninfected) or not (i.e., is already infected with HIV) at the time the policy is issued. Life expectancies and how, if at all, these are affected by the HIV/AIDS process are other functions of interest to insurers.

Accordingly, we give first, in Section 2, a description of the basic compartmental model, along with a summary of the approach used

* Corresponding author. Tel.: +1 706 542 3281.
E-mail address: lynne@stat.uga.edu (L. Billard).

http://dx.doi.org/10.1016/j.mbs.2014.01.009
0025-5564/

in Billard and Dayananda [1] to obtain the necessary distributions etc. Then, in Section 3, we derive insurance functions, against the backdrop that policy holders can become infected with HIV and be diagnosed with AIDS and how this diagnosis impacts the insurance entities. Section 4 looks briefly at how the ideas can be extended to health-care functions. The special case of exponential waiting times (usually not applicable for HIV/AIDS but can be valid for other diseases) is included, in Section 5. How these functions are affected by changing values of the model parameters is investigated in Section 6. This numerical study includes consideration of the impact of HIV/AIDS treatment to those known to have the disease.

2. Compartmental model – waiting time approach

In general, a compartmental process can be modeled as a multistage process with $X_j(t)$ individuals in stage $j = 0, \ldots, J$ at time $t > 0$. In the framework of epidemic processes, stage $j = 0$ corresponds to individuals who are initially susceptibles (i.e., uninfected by whatever disease is under study). Stages $j = 1, \ldots, m+1$ correspond to different levels of infection through which individuals successively pass. (Processes which allow a return to an earlier stage are accommodated by interpreting passage to a given stage as the last such visit to that stage.) Ultimately, individuals can die when they move into stage $j = m+2$. In the context of HIV/AIDS, the stage $j = 1$ is when initially infected with the HIV virus, and $j = m+1$ corresponds to diagnosis with AIDS. Transition rates, and/or probabilities, of moving from stage to stage are functions of the number of individuals in each stage; most such functions are nonlinear and so produce differential-difference equations that are mathematically intractable.

Instead of writing the model in terms of the variables $X_j(t)$, Billard and Dayananda [1] focused on waiting time distributions to move across stages. They also allowed for individuals to die from non-disease (such as non-HIV/AIDS) causes at any time; i.e., individuals could move directly to stage $j = m+2$. This feature is missing from most models even though it readily exists in many realistic situations (especially in insurance and health-care among other applications).

To summarize the Billard and Dayananda [1] (B&D) approach, and using their notation in the sense they defined it, let us define V_j to be the time an individual stays in stage j until moving out into stage $(j+1)$, $j = 0, \ldots, m+1$. Let us further define U_j, $j = 0, \ldots, m+1$, as the time an individual stays in stage j until death, i.e., until he moves directly into stage $m+2$. It is assumed that V_j and U_j are independent random variables. Suppose at time t an individual is in stage $S(t)$. A key entity (see Sections 3–5) is the probability $q_{ij}(t)$ that an individual is in stage j at time t given that he started in stage, i.e.,

$$q_{ij}(t) = P\{S(t) = j | S(0) = i\}, \quad i \leqslant j,\ i,j = 0, \ldots, m+2,\ t > 0. \tag{1}$$

From B&D (Theorem 1, Eq. (34)), we have, for $i \leqslant j$, $i,j = 0, \ldots, m+2$,

$$q_{ij}(t) = [P(Y_{ij} > t | W_k > 0, \quad k = i, \ldots, j-1) - P(Y_{i,j-1} > t | W_k > 0, \quad k = i, \ldots, j-1)] \times P(W_k > 0, \quad k = i, \ldots, j-1), \quad t > 0, \tag{2}$$

where

$$W_j = U_j - V_j, \tag{3}$$

$$Y_{ij} = \sum_{k=i}^{j} H_k \tag{4}$$

with

$$H_i = \min(U_i, V_i). \tag{5}$$

Here, when an individual survives stage j (without dying) to move into stage $(j+1)$, $W_j > 0$; but when an individual died while in stage j, $W_j < 0$. The variable H_j is the actual time an individual spends in stage j, by moving to the next stage after time V_j or by dying and moving to stage $m+2$ after time U_j, whichever happens first. Thence, the variable Y_{ij} in (4) is the total time an individual who started in stage i spends in stages $i, \ldots, j$ before moving into stage $j+1$.

We need expressions for the probabilities $P(W_k > 0,\ k = i, \ldots, j-1)$ and the conditional probabilities $P(Y_{ij} > t | W_k > 0,\ k = i, \ldots, j-1)$ of (2). Given its importance, these are derived in B&D (Sections 3.2 and 3.4, respectively) explicitly for the 4-stage model ($m = 1$). In those derivations, susceptibles were modeled as becoming infected with HIV at a Poisson rate with mean λ; hence the waiting time random variable V_0 is exponentially distributed with parameter λ. The waiting time distribution V_1 corresponds to the incubation period between infection with HIV and diagnosis with AIDS; thus a Weibull distribution with parameters $(\alpha, \beta = 2)$ was used. The waiting time distribution V_2 corresponding to the time of diagnosis to death from AIDS was assumed to follow an exponential distribution with parameter θ. The waiting times for death from non-AIDS causes, U_j, were assumed to be exponentially distributed random variables with parameters μ_j, $j = 0, 1, 2$.

Thence, the survival probabilities $q_{ij}(t)$ can be obtained as in B&D (Section 3.5). The $q_{ij}(t)$ are summarized in Appendix A.1. As we shall see in Sections 3,4, these waiting time distributions are fundamental entities in the derivation of many insurance and health-care functions. While the derivations in Sections 3,4 focus on the four compartmental model ($m = 1$) with the numerical study of Section 6 including parameter values that easily accommodate the influence and impact of individuals undergoing HIV/AIDS treatment, an alternative model could include an additional stage (such as a "treatment" stage, so that $m = 2$). Another alternative model would be to adjust the current ($m = 1$) model's distributions appropriately so as to capture the effect of treatment. Thus, the model presented in the paper can be modified to accommodate the effects of drug therapies.

3. Insurance functions

There are numerous insurance functions of interest. Traditionally, insurance companies charge premiums, at a given percentage higher than the normal rates, to individuals or groups that are likely to be considered as having "higher risk" than normal. Since individuals exposed to HIV/AIDS are perceived to be in higher risk categories, insurers are particularly interested in those insurance functions relating to premium rates, expected payouts and life expectancies. Therefore, we limit our focus to but these illustrative few functions. The basic theory behind these entities can be found in any of a number of actuarial sources (e.g., Bowers et al. [5]).

3.1. Premiums for t-year pure endowment policy

Payouts and hence premium rates are typically expressed in terms of \$1 units. A \$1 (future) payout to an individual who has survived t units of time is currently worth $\${exp(-\delta t)}$ where δ is the force of interest. This \$1 is paid only if the individual has survived, with that person being in stage j, $j \leqslant m+1$, at time t. If the present expected value for an individual currently in stage i is denoted by $\bar{E}_i(t)$, then

$$\bar{E}_i(t) = \sum_{j=i}^{m+1} \quad (\text{present value of } \$1) q_{ij}(t).$$

Hence, the net single premium for a t-year pure endowment for someone in stage i when the policy is issued is

$$\bar{E}_i(t) = \sum_{j=i}^{m+1} e^{-\delta t} q_{ij}(t). \tag{6}$$

Therefore, by substituting for the $q_{ij}(t)$ from (A.1)–(A.9), the net single premium for a pure t-year term endowment policy, can be shown to equal, for $i = 0, 1, 2$, respectively,

$$\begin{aligned}\bar{E}_0(t) = {} & \frac{\lambda e^{-\delta t}}{(\lambda+\mu_0-\theta-\mu_2)}\left[e^{-(\theta+\mu_2)t} - \frac{(\theta+\mu_2-\mu_0)}{\lambda} e^{-(\lambda+\mu_0)t}\right. \\ & + (\mu_1-\theta-\mu_2)\{G(t,\mu_1,\lambda+\mu_0,\lambda+\mu_0) \\ & \left. - G(t,\mu_1,\theta+\mu_2,\theta+\mu_2)\}\right], \quad t>0; \end{aligned} \tag{7}$$

$$\bar{E}_1(t) = e^{-\delta t} + e^{-(\theta+\mu_2)t}, \quad t>0; \tag{8}$$

$$\bar{E}_2(t) = e^{-(\delta+\theta+\mu_2)t}, \quad t>0, \tag{9}$$

where

$$\begin{aligned} G(x,a,b,c) = {} & e^{-cx+(a-b)^2/4\alpha} \\ & \times \sqrt{\pi/\alpha}\left\{\Phi\left(x\sqrt{2\alpha}+\frac{a-b}{\sqrt{2\alpha}}\right) - \Phi\left(\frac{a-b}{\sqrt{2\alpha}}\right)\right\} \end{aligned} \tag{10}$$

with $\Phi(u)$ being the usual standard normal distribution function

$$\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2} dw.$$

3.2. Continuous t-year life annuity policy

A t-year continuous life annuity and insurance policy issued to an individual in stage i has a value of, for $t>0$,

$$\bar{a}_i(t) = \sum_{j=i}^{m+1} \int_0^t e^{-\delta u} q_{ij}(u) du. \tag{11}$$

This can be written as

$$\bar{a}_i(t) = \sum_{j=i}^{m+1} Q_{ij}^*(t) \tag{12}$$

where

$$Q_{ij}^*(t) = \int_0^t e^{-\delta u} q_{ij}(u) du. \tag{13}$$

Expressions for $Q_{ij}^*(t)$ can be derived from simple to algebraically tedious but routine derivations by substituting for $q_{ij}(t)$ given in Appendix A.1 and by utilizing B&D (Lemma 1 and integrals I_k, $k = 1, \ldots, 4$). Therefore, setting $\rho_1 = \delta+\lambda+\mu_0$ and $\rho_2 = \delta+\theta+\mu_2$, we can show that

$$Q_{00}^*(t) = [1-e^{-\rho_1 t}]/\rho_1; \tag{14}$$

$$Q_{01}^*(t) = (\lambda/\rho_1)[G(t,\delta+\mu_1,0,0) - G(t,\mu_1,\lambda+\mu_0,\rho_1)]; \tag{15}$$

$$\begin{aligned} Q_{02}^*(t) = {} & [\lambda/\{\rho_1\rho_2(\rho_1-\rho_2)\}][(\rho_1-\rho_2+\rho_2 e^{-\rho_1 t} - \rho_1 e^{-\rho_2 t} \\ & + \rho_1(\mu_1-\theta-\mu_2)G(t,\mu_1,\theta+\mu_2,\rho_2) \\ & - \rho_2(\mu_1-\lambda-\mu_0)G(t,\mu_1,\lambda+\mu_0,\rho_1) \\ & - (\rho_1-\rho_2)(\delta+\mu_1)G(t,\delta+\mu_1,0,0)]; \end{aligned} \tag{16}$$

$$Q_{11}^*(t) = G(t,\delta+\mu_1,0,0); \tag{17}$$

$$\begin{aligned} Q_{12}^*(t) = {} & (1/\rho_2)[1-e^{-\rho_2 t} + (\mu_1-\theta-\mu_2)G(t,\mu_1,\theta+\mu_2,\rho_2) \\ & - (\delta+\mu_1)G(t,\delta+\mu_1,0,0)]; \end{aligned} \tag{18}$$

$$Q_{22}^*(t) = 1 - (1/\rho_2)e^{-\rho_2 t}. \tag{19}$$

Thence, we can show that, for $i = 0$,

$$\begin{aligned} \bar{a}_0(t) = {} & [(\delta+\lambda+\mu_0)(\delta+\theta+\mu_2)]^{-1} \\ & \left\{(\delta+\lambda+\theta+\mu_2) + \frac{(\mu_1-\theta-\mu_2)}{(\lambda+\mu_0-\theta-\mu_2)} a_0(t)\right\}, \quad t>0, \end{aligned} \tag{20}$$

where

$$\begin{aligned} a_0(t) = {} & [(\theta+\mu_2-\mu_0)(\delta+\theta+\mu_2)e^{-(\delta+\lambda+\mu_0)t} - \lambda(\delta+\lambda \\ & + \mu_0)e^{-(\delta+\theta+\mu_2)t}]/(\mu_1-\theta-\mu_2) + \lambda(\delta+\lambda+\mu_0)G(t,\mu_1,\theta \\ & + \mu_2,\delta+\theta+\mu_2) - \lambda(\delta+\theta+\mu_2)G(t,\mu_1,\lambda+\mu_0,\delta+\lambda+\mu_0) \\ & - \lambda(\lambda+\mu_0-\theta-\mu_2)G(t,\mu_1+\delta,0,0); \end{aligned}$$

and, for $i = 1, 2$,

$$\begin{aligned} \bar{a}_1(t) = {} & \frac{1}{(\delta+\theta+\mu_2)}[1-e^{-(\delta+\theta+\mu_2)t} + (\mu_1-\theta-\mu_2) \\ & \times \{G(t,\mu_1,\theta+\mu_2,\delta+\theta+\mu_2) - G(t,\delta+\mu_1,0,0)\}], \quad t>0; \end{aligned} \tag{21}$$

$$\bar{a}_2(t) = \frac{1}{(\delta+\theta+\mu_2)}[1-e^{-(\delta+\theta+\mu_2)t}], \quad t>0. \tag{22}$$

The net single premium for these continuous t-year insurance policies can be obtained from

$$\bar{A}_i(t) = \sum_{j=i}^{m+1} \mu_j' \int_0^t e^{-\delta u} q_{ij}(u) du = \sum_{j=1}^{m+1} \mu_j' Q_{ij}^*(t) \tag{23}$$

where μ_j' is the death rate parameter for an individual in stage j, i.e., $\mu_0' \equiv \mu_0, \mu_1' \equiv \mu_1$ and $\mu_2' \equiv \theta+\mu_2$, and where the $Q_{ij}^*(t)$ are given in (14)–(19).

Hence, the net single premium for a t-year ($t>0$) insurance policy for an uninfected, i.e., in stage $i = 0$, individual when the policy is taken out is, from (23),

$$\begin{aligned} \bar{A}_0(t) = {} & [(\theta+\mu_2)/\rho_2]\left[1-e^{-\rho_2 t} + (\mu_1-\theta-\mu_2)G(t,\mu_1,\theta+\mu_2,\rho_2)\right. \\ & \left. + \frac{\delta(\mu_1-\theta-\mu_2)}{(\theta+\mu_2)} G(t,\delta+\mu_1,0,0)\right]; \end{aligned} \tag{24}$$

the premium necessary for an individual who is infected at the time the policy is issued is, for $i = 1$,

$$\begin{aligned} \bar{A}_1(t) = {} & \frac{(\theta+\mu_2)}{(\delta+\theta+\mu_2)}\left[1-e^{-(\delta+\theta+\mu_2)t} + (\mu_1-\theta-\mu_2)\right. \\ & \left. \times \left\{G(t,\mu_1,\theta+\mu_2,\delta+\theta+\mu_2) + \frac{\delta}{(\theta+\mu_2)} G(t,\delta+\mu_1,0,0)\right\}\right]; \end{aligned} \tag{25}$$

and for stage $i = 2$,

$$\bar{A}_2(t) = [(\theta+\mu_2/\rho_2][1-e^{-\rho_2 t}]. \tag{26}$$

It can easily be shown that the fundamental relationship

$$\bar{A}_i(t) = 1 - \delta\bar{a}_i(t), \quad t>0, \tag{27}$$

holds for all i and all time t.

Further, the net continuously paid premium per unit time for a t-year continuous term annuity and insurance policy issued to someone in stage i is

$$\bar{P}_i(t) = \bar{A}_i(t)/\bar{a}_i(t). \tag{28}$$

Given these expressions for the values of the policies written, $\bar{a}_i(t)$, and the total (single) premium values, $\bar{A}_i(t)$, a straightforward application of (28) allows calculation of the continuous premium per unit time $\bar{P}_i(t)$ required from an individual who was in stage i when the policy was issued. The details are omitted. Note that, because of (27), these $\bar{P}_i(t)$ can be expressed in terms of $\bar{a}_i(t)$ only.

Insurance companies are vitally interested in the long term; i.e., interest is centered on the $\bar{a}_i(t)$, $\bar{A}_i(t)$ and $\bar{P}_i(t)$, as $t \to \infty$. Taking this limit on each of these terms in turn, we can show that, if we write $\bar{a}_i(\infty) = \lim_{t\to\infty} \bar{a}_i(t)$ and likewise for $\bar{A}_i(\infty)$ and $\bar{P}_i(\infty)$, we have

$$\begin{aligned} \bar{a}_0(\infty) = {} & [(\delta+\lambda+\mu_0)(\delta+\theta+\mu_2)]^{-1}[\delta+\lambda+\theta+\mu_2 \\ & - \lambda(\mu_1-\theta-\mu_2)I(0,\delta+\mu_1,0)]; \end{aligned} \tag{29}$$

$$\bar{a}_1(\infty) = (\delta+\theta+\mu_2)^{-1}(\mu_1-\theta-\mu_2)I(0,\delta+\mu_1,0); \tag{30}$$

$$\bar{a}_2(\infty) = 1/(\delta+\theta+\mu_2); \tag{31}$$

$$\bar{A}_0(\infty) = [(\delta+\lambda+\mu_0)(\delta+\theta+\mu_2)]^{-1}[\mu_0(\delta+\theta+\mu_2)+\lambda(\theta+\mu_2) + (\mu_1-\theta-\mu_2)I(0,\delta+\mu_1,0)]; \tag{32}$$

$$\bar{A}_1(\infty) = (\delta+\theta+\mu_2)^{-1}[\theta+\mu_2+\delta(\mu_1-\theta-\mu_2)I(0,\delta+\mu_1,0)]; \tag{33}$$

$$\bar{A}_2(\infty) = (\theta+\mu_2)/(\delta+\theta+\mu_2) \tag{34}$$

where

$$I(t,a,b) = e^{a^2/4\alpha}\sqrt{\pi/\alpha}\left\{1-\Phi\left(t\sqrt{2\alpha}+\frac{a-b}{\sqrt{2\alpha}}\right)\right\}. \tag{35}$$

Details of these derivations are omitted. It can easily be shown that the fundamental relationship (27) holds for $t=\infty$, for each i value.

It then follows that the long-term annual premium rates, $\bar{P}_i(\infty)$, satisfy, from (28),

$$\bar{P}_0(\infty) = [(\lambda+\mu_0)(\theta+\mu_2)+\delta\mu_0+\delta\lambda(\mu_1-\theta-\mu_2)I(0,\delta+\mu_1,0)] /[\delta+\lambda+\theta+\mu_2-\lambda(\mu_1-\theta-\mu_2)I(0,\delta+\mu_1,0)]; \tag{36}$$

$$\bar{P}_1(\infty) = [\theta+\mu_2+\delta(\mu_1-\theta-\mu_2)I(0,\delta+\mu_1,0)] /[1-(\mu_1-\theta-\mu_2)I(0,\delta+\mu_1,0)]; \tag{37}$$

$$\bar{P}_2(\infty) = (\theta+\mu_2). \tag{38}$$

3.3. Life expectancies

Finally, the life expectancy for someone presently in stage i is

$$e_i = \sum_{j=i}^{m+1}\mu_j'\int_0^\infty tq_{ij}(t)dt. \tag{39}$$

The life expectancies, given in (39), can be rewritten as

$$e_i = \sum_{j=i}^{m+1}\mu_j' Q_{ij} \tag{40}$$

where

$$Q_{ij} = \int_0^\infty tq_{ij}(t)dt. \tag{41}$$

Expressions for these integrals Q_{ij} are derived by utilizing extensively the integral I_5 of B&D (Appendix A.1). It can thence be shown that

$$Q_{00} = 1/(\lambda+\mu_0)^2; \tag{42}$$

$$Q_{01} = \frac{\lambda}{\lambda+\mu_0}\left[\frac{1}{2\alpha}\{1-\mu_1 I(0,\mu_1,0)\}+\left(\frac{1}{\lambda+\mu_0}\right)I(0,\mu_1,0)\right]; \tag{43}$$

$$Q_{02} = \frac{\lambda}{(\lambda+\mu_0)(\theta+\mu_2)}\left[\frac{\lambda+\mu_0+\theta+\mu_2}{(\lambda+\mu_0)(\theta+\mu_2)}-\frac{\mu_1}{2\alpha} + \left\{\frac{(\lambda+\mu_0)(\theta+\mu_2)-\mu_1(\lambda+\mu_0+\theta+\mu_2)}{(\lambda+\mu_0)(\theta+\mu_2)}+\frac{\mu_1^2}{2\alpha}\right\}I(0,\mu_1,0)\right]; \tag{44}$$

$$Q_{11} = \frac{1}{2\alpha}[1-\mu_1 I(0,\mu_1,0)]; \tag{45}$$

$$Q_{12} = \left(\frac{1}{\theta+\mu_2}\right)\left[1-\frac{\mu_1}{2\alpha}+\left\{\frac{\mu_1^2}{2\alpha}-\frac{(\mu_1-\theta-\mu_2)}{(\theta+\mu_2)}\right\}I(0,\mu_1,0)\right]; \tag{46}$$

$$Q_{22} = 1/(\theta+\mu_2)^2. \tag{47}$$

Substituting back into (39), we can show that

$$e_0 = 1+\frac{\mu_1}{(\theta+\mu_2)}I(0,\mu_1,0); \tag{48}$$

$$e_1 = \frac{1}{(\lambda+\mu_0)^2}\left[\mu_0+\lambda(\lambda+\mu_0+\theta+\mu_2) \times\frac{\lambda}{(\theta+\mu_2)}\{(\lambda+\mu_0)(\theta+\mu_2)-\mu_1(\lambda+\mu_0+\theta+\mu_2)\}I(0,\mu_1,0)\right]; \tag{49}$$

$$e_2 = (\theta+\mu_2)^{-1}. \tag{50}$$

4. Health-care costs

It is clear that some insurance entities are equivalent to various health-care counterparts. Thus, for example, the value of the t-year continuous term life annuity policy, $\bar{a}_i(t)$, of Eq. (11) could instead be the expected cost of health care for someone currently in stage i assuming cost is \$1 per unit time in the future.

These costs can be adjusted to allow for stage dependent costs. For example, the health cost for an individual currently in stage i (e.g., $i=0$, without HIV infection) who develops AIDS at a future time during the period $(0,t)$ could be adjusted to

$$H(t,i) = \int_0^t (c_{1i}+c_{2i}e^{h_i u})e^{-\delta u}q_{i,m+1}(u)du \tag{51}$$

where c_{1i}, c_{2i} and h_i are constants. In (51), it is assumed there is some fixed care cost c_{1i} for stage i (e.g., cost of basic medical tests or etc.) along with a variable cost $c_{2i}e^{h_i u}, 0<u<t$, related to the initial stage i, over time (e.g., continuing cost of treatment including medications, say). The term $e^{-\delta u}$ is used to discount the cost to time $t=0$ so that costs evaluated over time are comparable; this compares with the future \$1 value payout having a current value of $\$(e^{-\delta u})$. Clearly, interest centers on the overall time $0\leqslant u\leqslant t$ and (51) is concerned with the overall cost of going from the present stage i (at $t=0$) to the AIDS stage $i=m+1$ (at $t=t$). Other formulations can be validated; for example, $\{c_{1i}e^{h_i u}\}$ could be replaced by $(c_{1i}u)$. Or, interest may center on the care costs up to some other specific stage j, and not necessarily the $j=m+1$ stage of (51).

We can then show that, for the health-care cost function of (51),

$$H(t,0) = \sum_{k=1}^{2}\frac{c_{k0}\lambda}{(\lambda+\mu_0-\theta-\mu_2)} \times\left[\left(\frac{1}{\delta_{k0}+\theta+\mu_2}\right)\{e^{-(\delta_{k0}+\lambda+\mu_0)t}-e^{-(\delta_{k0}+\theta+\mu_2)t}\} +\left(\frac{\mu_1-\theta-\mu_2}{\delta_{k0}+\theta+\mu_2}\right)G(t,\mu_1,\theta+\mu_2,\delta_{k0}+\theta+\mu_2) -\left(\frac{\mu_1-\lambda-\mu_0}{\delta_{k0}+\lambda+\mu_0}\right)G(t,\mu_1,\lambda+\mu_0,\delta_{k0}+\lambda+\mu_0) +\frac{(\mu_1+\delta_{k0})(\lambda+\mu_0-\theta-\mu_2)}{(\delta_{k0}+\lambda+\mu_0)(\delta_{k0}+\theta+\mu_2)}G(t,\mu_1+\delta_{k0},0,0)\right]; \tag{52}$$

$$H(t,1) = \sum_{k=1}^{2}\frac{c_{k1}}{(\delta_{k1}+\theta+\mu_2)} \times[1-e^{-(\delta_{k1}+\theta+\mu_2)t}-(\mu_1+\delta_{k1})G(t,\mu_1+\delta_{k1},0,0) -(\mu_1-\theta-\mu_2)G(t,\mu_1,\theta+\mu_2,\delta_{k1}+\theta+\mu_2)]; \tag{53}$$

$$H(t,2) = \sum_{k=1}^{2}\frac{c_{k2}}{(\delta_{k2}+\theta+\mu_2)}\{1-e^{-(\delta_{k2}+\theta+\mu_2)t}\} \tag{54}$$

where $\delta_{1i} = \delta$ and $\delta_{2i} = \delta - h_i$, and where $G(t,a,b,c)$ was defined in (10). The details are omitted but the derivations use the integrals of B&D (Appendix A.1) repeatedly.

Another formulation of total health care costs, for someone in stage i at time T, is given by

$$C(T,i) = h_i(1+\lambda_i)\int_T^\infty (t-T)e^{-\delta(t-T)}q_{i,i}(t-T)dt + \sum_{j=i+1}^{m+1}\int_T^\infty \left\{\frac{d}{dt}q_{ij}(t-T)\right\}e^{-\delta(t-T)}C(t,j)dt \quad (55)$$

for $i = 0,\ldots,m+1$. The first term in (55) represents the cost per unit time, $h_i(1+\lambda_i)$, of being in stage i at a future time t multiplied by the expected time of tenure in this i stage; as before, there is the need to adjust a future \$1 to the current \$($e^{-\delta u}$), $u > T$, so that the evaluated costs are comparable. The second term in (55) represents the differential cost of care for each future stage ($j = i+1,\ldots, m+1$). Thus, for example, the health-care costs for an individual who has just been diagnosed with AIDS is, from (55), $C(T,2)$, found to be

$$C(T,2) = \frac{h_2(1+\lambda_2)}{(\delta+\theta+\mu_2)^2}e^{-(\theta+\mu_2)T}, \quad T > 0. \quad (56)$$

For an individual found to be HIV infected at time T, the overall health-care cost is

$$C(T,1) = \frac{h_1(1+\lambda_1)}{2\lambda}\left[e^{-\mu_1 T-\lambda T^2} - e^{\delta T}I_\lambda(T,\delta+\mu_1,0)\right] + \frac{h_1(1+\lambda_2)e^{-(\theta+\mu_2)T}}{(\delta+\theta+\mu_2)(\delta+2\theta+2\mu_2)} \times \left[1-(\delta+\mu_1+\theta+\mu_2)I(0,\delta+\mu_1+\theta+\mu_2,0)\right] \quad (57)$$

where $I(x,a,b)$ is defined in (35) and where $I_\lambda(x,a,b)$ is the same as $I(x,a,b)$ but with α everywhere replaced by λ.

The range of possible functions of interest is large and depends on the cost structure per unit time assumed. In general these health care cost functions, as for the insurance functions, depend on knowledge of the conditional probabilities $q_{ij}(t)$ defined in (1).

5. Exponential waiting times for U_i, V_i

When all U_i and V_i, $i = 1,\ldots,m+1$, waiting times are exponentially distributed, $U_i \sim Exp(\mu_i)$ and $V_i \sim Exp(\lambda_i)$, the derivations for the conditional probabilities $q_{ij}(t)$ simplify. This exponential assumption does not hold for the HIV/AIDS process in general, although Bailey [6] in effect has proposed exponential times but with $m > 6$ or 7 as an estimated process; likewise, for other epidemic processes with nonlinear transition rates. There can be however other processes (especially those whose underlying transition rates are linear, e.g., in queues) for which this exponential assumption pertains.

We give without proof the basic result for $q_{ij}(t)$. Thus, we can show that

$$q_{ij}(t) = [F_{i,j-1}(t) - F_{i,j}(t)]\prod_{k=1}^{j-1}\frac{\lambda_k}{\lambda_k+\mu_k}, \quad i,j = 0,\ldots,m, \quad (58)$$

where

$$F_{ij}(t) = P(Y_{ij} \leqslant t) = 1 - \sum_{k=1}^{j}\left(\prod_{\substack{r=i\\ r\neq k}}^{j}\frac{\alpha_k}{\alpha_r-\alpha_k}\right)e^{-\alpha_k t},$$
$$j = i, i+1,\ldots,m,$$

with $\alpha_k = \mu_k + \lambda_k$, and $\prod_{\substack{r=i\\ r\neq k}}^{i}(\cdot) \equiv 1$ when $i = j = m+1$, and where

$$q_{m+1,m+1}(t) = e^{-\alpha_{m+1}t}. \quad (59)$$

An application of this model to some simple models for insurance functions can be found in Dayananda [7]. Derivation of various insurance functions and health-care costs can proceed along the lines outlined in Sections 3,4. Thus, for example, the expected health-care cost for someone currently in stage i can be found from (11). We obtain

$$\bar{a}_i(t) = \sum_{s=i}^{m+1}\left(\mu_s\prod_{r=i}^{s-1}\lambda_r\right)\sum_{k=i}^{s}\left\{\frac{1-e^{-(\delta+\alpha_k)t}}{\alpha_k(\delta+\alpha_k)}\right\}\prod_{\substack{r=i\\ r\neq k}}^{s}(\alpha_r-\alpha_k)^{-1}. \quad (60)$$

6. Numerical illustrations

Numerical work can be carried out to study the influence of model parameters on the insurance functions. We focus attention on two important insurance functions, viz., the value at time $t = 0$ of a t-year continuous life annuity for an insurance policy issued to someone in stage $i, \bar{a}_i(t)$, $i = 0,1,2$. We also look at the net single premiums for these policies when the policy-holder was infected at the time the policy was issued $\bar{A}_1(t)$ and their long term values as $t \to \infty$, as well as premium increases for uninfected individuals $\bar{A}_0(t)$. (Similar comparisons could be made for the other insurance functions and for health-care functions.).

Since the many insurance functions developed in Section 3 are expectations, they depend heavily on the probabilities $q_{ij}(t)$, $i < j = 0,1,2$. Hence, sensitivities to varying parameter values observed in these probabilities will also be reflected in the insurance functions. Billard and Dayananda [1] investigated the sensitivity over different values of the parameters $(\lambda,\alpha,\mu_0,\mu_1,\mu_2,\theta)$ on these $q_{ij}(t)$ terms; a summary of those findings is in Table 1. Thus, e.g., all applicable $q_{ij}(t)$ terms are sensitive to changes in values of α. As in [1], we use values for $(\lambda,\alpha,\mu_0,\mu_1,\mu_2,\theta)$ to reflect those known to pertain for these parameters. Thus, death rates for 20-, 30-, 40-, and 50-year-olds are known to produce values for μ_k, $k = 0,1,2$, that are on or around $\mu = .001, .0026, .0042, .0057$, respectively. Expected incubation times (from infection to diagnosis with AIDS) of about 4, 9.5, 15, and 28 years, translate into the parameter α taking values on or around $\alpha = .05, .009, .005, .001$, respectively; the expected time (θ^{-1}) from diagnosis to death from AIDS of 2, 10, 16, 20 years, equates to the parameter θ being on or around $\theta = .5, .1, .08, .05$, respectively; and λ values on or around $\lambda = .005, .01, .02, .05, .1$, correspond to 0.5, 1, 2, 5, and 10 susceptibles per 100 being infected with HIV, respectively. We note that an impact of HIV/AIDS drug treatments is that incubation times can be increased and/or the expected time from diagnosis to death from AIDS can be increased. Therefore, the numerical study herein includes parameter values that reflect the impact on individuals who are faithful to the treatment regime, viz., $\alpha = .001$ (which corresponds to an expected incubation period of 28 years), and $\theta = .05$ (for an expected time from diagnosis to death from AIDS of 20 years). For individuals who do not undertake treatment, these expected incubation times are closer to 4 years for young children and 9.5 years for adults (i.e., $\alpha = .05, .009$, respectively), and the expected time from diagnosis to death is closer to 2 years (i.e., $\theta = .5$). Thus, the range of values investigated run the gamut across the levels of the use of and/or adherence to treatment therapies.

Accordingly, we display in Fig. 1(a) the plots for the continuous time annuity for a susceptible (i.e., someone who has not been infected with the HIV virus at the time the policy was taken out) $\bar{a}_0(t)$ of (20), for changing infection rates $\lambda = (.005, .01, .02, .05, .1)$ for $\delta = .05, \theta = .08, \alpha = .005, \mu_0 = .0026, \mu_1 = .0042$ and $\mu_2 = .0057$. It is clear from these plots that the different infection rates have no impact on this annuity (until at least 20–30 years, with only small differences in impact for longer time frames). It is also the case that this annuity is essentially the same for changing

Table 1
Sensitivities of probabilities $\{q_{ij},\ i<j=0,1,2\}$ to parameters values.

	q_{01}	q_{02}	q_{03}	q_{12}	q_{13}	q_{23}
λ	Sensitive	Sensitive	Sensitive	N/A	N/A	N/A
α	Sensitive	Sensitive	Sensitive	Sensitive	Sensitive	N/A
μ_0	Insensitive	Insensitive	Insensitive	N/A	N/A	N/A
μ_1	Insensitive	Insensitive	Insensitive	Insensitive	Insensitive	N/A
μ_2	N/A	Insensitive	Insensitive	Insensitive	Insensitive	Insensitive
θ	N/A	Sensitive	Sensitive	Sensitive	Sensitive	Sensitive

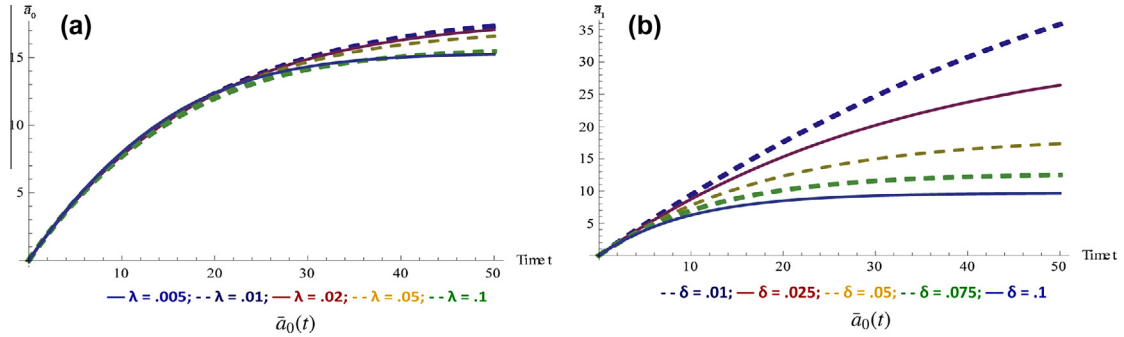

Fig. 1. (a). Annuity rates for uninfectives $\bar{a}_0(t)$ by time as infection rate λ varies. (b). Annuity rates for uninfectives $\bar{a}_0(t)$ by time as interest rate δ varies.

Table 2
$\bar{a}_0(\infty)$ and $\bar{a}_1(\infty)$ for some (δ,α) values.

Interest Rate δ		(a) $\bar{a}_0(\infty)-\alpha^\dagger$					(b) $\bar{a}_1(\infty)-\alpha\dagger$				
		.05	.02	.009	.005	.001	.05	.02	.009	.005	.001
.01	(1%)	60.71	61.21	61.86	62.49	65.12	13.71	15.50	17.78	20.01	29.30
.025	(2.5%)	32.47	32.69	32.95	33.19	34.06	11.75	13.16	14.86	16.45	22.15
.05	(5%)	18.18	18.26	18.36	18.44	18.68	9.46	10.44	11.55	12.49	15.28
.075	(7.5%)	12.58	12.62	12.67	12.70	12.79	7.89	8.60	9.35	9.95	11.44
.1	(10%)	9.61	9.63	9.67	9.67	9.71	6.75	7.29	7.81	8.20	9.06

† Mean time infection to diagnosis of 4–28 years.

$\alpha,\theta,\mu_0,\mu_1$ and μ_2 values (plots not shown). This is not surprising for the $\mu_k,\ k=0,1,2$, values as Billard and Dayananda [1] has already shown that the functions $q_{ij}(t)$ (used in the derivation of $\bar{a}_i(t)$, see (11)) were not sensitive to changes in these death rates from non-HIV/AIDS related causes. Collectively, then, for a given interest rate δ, the impact on annuities for someone who has not been infected at the time of taking out the policy for at least a 20–30 year term is negligible.

There is an impact, as would be expected, on annuity values as the interest rate δ varies; see Fig. 1(b) where $\delta=(.01,.025,.05,.075,.1)$ for $\alpha=.005,\lambda=.005,\theta=.08,\mu_0=.0026,\mu_1=.0042$ and $\mu_2=.0057$. This result is also evident when studying the long term annuity values as $t\to\infty$. Table 2(a) provides these long term values as δ,α change over $(.01,.025,.05,.075,.1)\times(.05,.02,.009,.005,.001)$. Thus, as the mean incubation period increases (i.e., α decreases), the long term annuity increases; while as the interest rate δ increases, the long term annuity decreases as expected. Fig. 2 shows plots for the continuous time annuity, $\bar{a}_1(t)$, for someone already infected with HIV initially as θ,δ and α vary for $\mu_1=.0026$ and $\mu_2=.0057$. Thus, Fig. 2(a) takes $\theta=(.5,.1,.08,.05)$ (i.e., reflecting increases in the expected time from diagnosis with AIDS to death) for interest rate $\delta=.05$ (i.e., 5%) and $\alpha=.009$ (i.e., incubation period of 9.5 years). We first observe there is no significant difference in these annuities for a 6–8 year policy ($6\leqslant t^*\leqslant 8$ approximately). When $\theta=.5$ (i.e., the expected time to death from AIDS diagnosis is 2 years), the present value of $\bar{a}_1(t)$ is perforce low with a limiting value of $\bar{a}_1(\infty)=8.15$; whereas when $\theta=.05$ (i.e., the expected time to death from AIDS diagnosis is 20 years), a higher present value pertains with $\bar{a}_1(\infty)=12.98$. Plots for $\delta=(.01,.025,.05,.075,.1)$ for $\theta=.08$ and $\alpha=.009$ are shown in Fig. 2(b). There are no real differences across δ for a 3–4 year policy (t^* near 3.5). After that as δ increases, then the present values $\bar{a}_1(t)$ must decrease for any t value. In this case, $\bar{a}_1(\infty)=11.66$ when the interest rate is 5%. The $\alpha=(.05,.02,.009,.005,.001)$ plots when $\theta=.08$ and $\delta=.05$ are displayed in Fig. 2(c). Again for a 6–8 year policy, there is no significant difference in the annuity $\bar{a}_1(t)$ for different α values. Since annuity values represent the present values of future payments of \$1, they are expected to increase as the respective θ,δ,α values decrease; see Figs. 2(a)-(c). All increase with time with the higher annuities occurring for lower α values since the later payouts will therefore be higher. Thus we have $\bar{a}_1(\infty)=9.51$ when $\alpha=.05$ (i.e., when the mean incubation period is 4 years) compared with $\bar{a}_1(\infty)=15.579$ when $\alpha=.001$ (i.e., when the mean incubation period is 28 years). Table 2(b) gives these long term results for $\bar{a}_1(\infty)$ for several (δ,α) values. Table 3(a) displays the corresponding results for several (δ,θ) values.

Since the $\bar{a}_i(t)$ represent the amount needed now for a future payout at time t, then we expect that a policy for an individual who is infection free will need a larger payout than one for someone who is now infected (and so has a lower life expectancy). Therefore, e.g., when $\delta=.025$ and $\alpha=.02$, we see from Table 2, that $\bar{a}_0(\infty)=32.69>13.16=\bar{a}_1(\infty)$. Plots for $\bar{a}_2(t)$ as δ,θ and μ_2 change are similar to those for $\bar{a}_1(t)$ in shape and trends but with

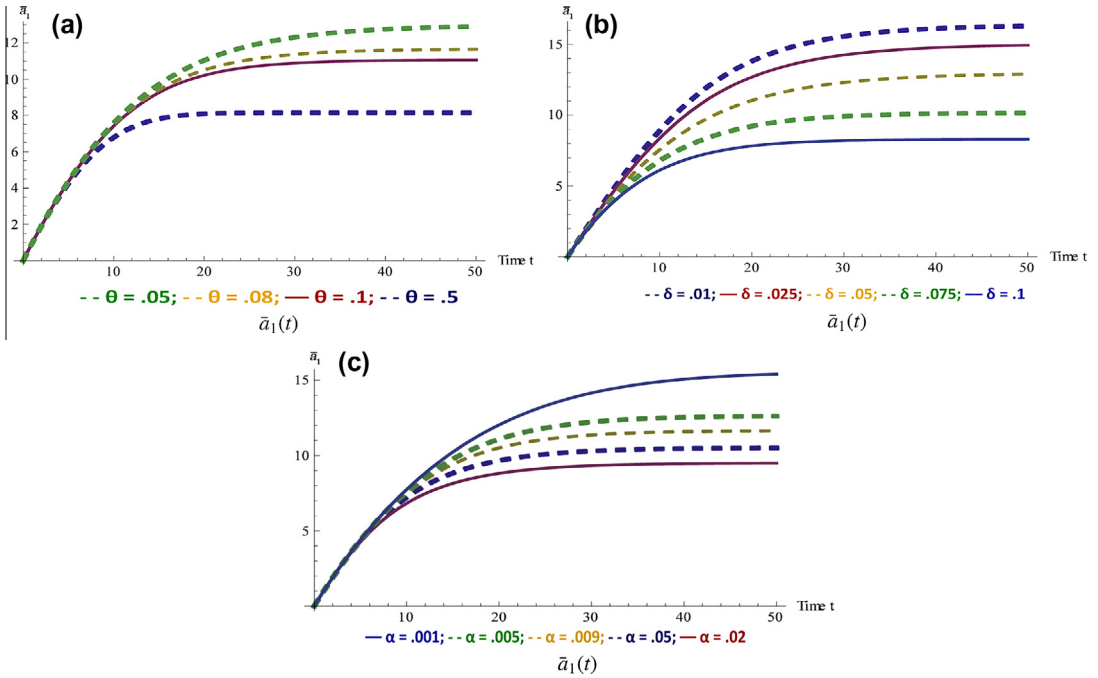

Fig. 2. (a). Annuity rates for infectives $\bar{a}_1(t)$ by time as expected time from diagnosis to death θ^{-1} varies. (b). Annuity rates for infectives $\bar{a}_1(t)$ by time as interest rate δ varies. (c). Annuity rates for infectives $\bar{a}_1(t)$ by time as expected incubation time (via α) varies.

Table 3
$\bar{a}_1(\infty)$ and $\bar{a}_2(\infty)$for some (δ, θ) values.

Interest Rate δ		(a) $\bar{a}_1(\infty) - \theta^{\dagger}$				(a) $\bar{a}_2(\infty) - \theta\dagger$			
		.5	.1	.08	.05	.5	.1	.08	.05
.01	(1%)	5.66	12.00	13.71	18.22	1.94	8.64	10.45	15.22
.025	(2.5%)	5.37	10.55	11.75	14.74	1.88	7.65	9.03	12.39
.05	(5%)	4.94	8.69	9.46	11.15	1.80	6.42	7.37	9.46
.075	(7.5%)	4.56	7.38	7.89	8.94	1.72	5.53	6.22	7.65
.1	(10%)	4.22	6.39	6.75	7.45	1.65	4.86	5.39	6.42

† Mean time to death after diagnosis 2–20 years.

lower annuity values; see, e.g., Fig. 3 which shows the changes as $\theta = (.05, .08, .1, .5)$ for $\delta = .05$ and $\mu_2 = .0057$. This is also reflected by a comparison of the annuity values for the long term, $\bar{a}_2(\infty)$, displayed in Table 3(b) with those for $\bar{a}_1(\infty)$ in Table 3(a), for $\mu_2 = .0057$. For example, when $\delta = .025$ and $\theta = .08$, we see that $\bar{a}_1(\infty) = 11.75 > 9.03 = \bar{a}_2(\infty)$.

Now consider the influence of different parameters on $\bar{A}_1(t)$, the single premium for an infected individual taking out a $1 policy.

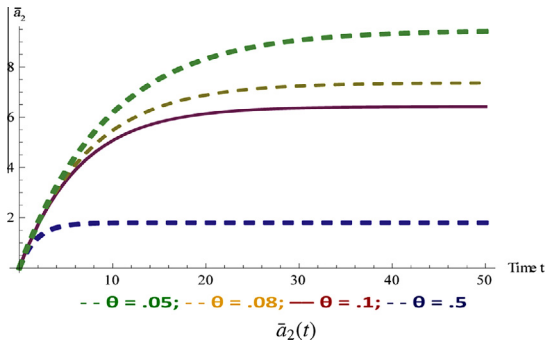

Fig. 3. Annuity rates for diagnosed $\bar{a}_2(t)$ by time as expected time from diagnosis to death θ^{-1} varies.

Given the relationship (27), these plots (not shown) behave similarly to those for $\bar{a}_1(t)$, except for a "y-axis" scale change. The effect of the multiplier δ is that there is an immediate impact even at small time points (e.g., $t^* = 1$–2 years) rather than the $t^* = 6$–8 years observed for $\bar{a}_1(t)$ in Fig. 2. Note that the longer a person is expected to live, then the higher is the value of the single premium of the policy. Table 4 provides the limiting values as $t \to \infty$ for combinations of $(\delta = (.01, .025, .05, .075, .1)$ and $\alpha = (.05, .02, .009, .005, .001)$, which values can be compared with those in Table 2(b) for $\bar{a}_1(\infty)$.

Finally, in Table 5, we give the percentage increase in single premiums for interest rates $\delta = (.03, .04, .05)$ needed for an individual

Table 4
$\bar{A}_1(\infty)$ for some (δ, α) values.

Interest Rate δ		$\alpha^{\dagger}$				
		.05	.02	.009	.005	.001
.01	(1%)	0.86	0.85	0.82	0.80	0.71
.025	(2.5%)	0.71	0.67	0.63	0.59	0.45
.05	(5%)	0.53	0.48	0.42	0.32	0.24
.075	(7.5%)	0.41	0.35	0.30	0.25	0.14
.1	(10%)	0.33	0.27	0.22	0.18	0.09

† Mean time infection to diagnosis of 4–28 years.

Table 5
Increase in $(\bar{A}_0)$ for some (δ, λ) values.

Interest Rate δ		$\lambda^{\dagger}$					
		.020	.021	.022	.023	.024	.025
.03	(3%)	13.8	16.9	19.9	22.3	25.6	27.4
.04	(4%)	29.2	33.1	36.8	40.4	43.8	47.2
.05	(5%)	53.4	58.2	62.9	67.5	71.9	76.3

† 2–2.5 per 100 susceptibles become infected.

who is exposed to HIV at age 20, with $\alpha = .01$ and $\mu_0 = .0019$, for the case that $\lambda = (.020, .021, .022, .023, .024, .025)$. Thus, e.g., when the rate of susceptibles becoming infected is 2 per 100 ($\lambda = .02$) when the interest rate is 3% ($\delta = .03$), the increase in the single premium $(\bar{A}_0)$ is 13.8% over what is expected were there no exposure to HIV (as calculated from the Institute of Actuaries tables [8]). From these comparisons, it is clear that the impact of the presence of HIV is substantial on premium rates.

7. Conclusion

Insurance companies and healthcare organizations are particularly interested in the impact of high risk individuals and groups when it comes to determining the premium rates and related functions for life policies and healthcare costs. In this paper, we have taken fundamental probabilities in a compartmental model developed in [1] to derive a number of basic insurance functions, viz., premium rates, annuities, and life expectancies, for individuals exposed to HIV/AIDS. In this work, HIV/AIDS development and interest rates have been considered to be a continuous time process. The numerical study on a range of parameter values which covers infected and/or diagnosed individuals who are fully faithful to their treatment regimes to those who do not undertake treatment therapies was conducted.

Thus, it is seen that for a given interest rate, there is a negligible impact on a person uninfected when taking out a 20–30 year term policy. On the other hand, as interest rates increase, the long term annuity decreases. For someone infected when taking out a policy, there is no real effect on annuities for the first 6–8 years; also as interest rates increase, there is no effect for a 3–4 year policy, but there is a decrease in annuities for longer term policies. The overall conclusion is that the impact on premium rates (and hence also on other insurance functions) is substantial for individuals infected with HIV. Similar conclusions can be made for correspondingly equivalent health-care functions.

Appendix A

A.1. Probabilities $q_{ij}(t)$

The set of probabilities $q_{ij}(t)$, $i \leqslant j = 0, 1, 2, 3$, derived in [1], are given by:

$$q_{00}(t) = e^{-(\lambda+\mu_0)t}, \quad t > 0; \tag{A.1}$$

$$q_{01}(t) = \lambda G(t, \mu_1, \lambda+\mu_0, \lambda+\mu_0), \quad t > 0; \tag{A.2}$$

$$\begin{aligned} q_{02}(t) = \frac{\lambda}{(\lambda+\mu_0-\theta-\mu_2)}\big[e^{-(\theta+\mu_2)t} - e^{-(\lambda+\mu_0)t} \\ - (\mu_1-\theta-\mu_2)G(t, \mu_1, \theta+\mu_2, \theta+\mu_2) \\ + (\mu_1-\lambda-\mu_0)G(t, \mu_1, \lambda+\mu_0, \lambda+\mu_0)\big], \quad t > 0; \end{aligned} \tag{A.3}$$

$$\begin{aligned} q_{03}(t) = 1 - \frac{\lambda}{(\lambda+\mu_0-\theta-\mu_2)} \\ \left[e^{-(\theta+\mu_2)t} - \frac{(\theta+\mu_2-\mu_0)}{\lambda}e^{-(\lambda+\mu_0)t}\right. \\ + (\mu_1-\theta-\mu_2)\{G(t, \mu_1, \lambda+\mu_0, \lambda+\mu_0) \\ \left. - (t, \mu_1, \theta+\mu_2, \theta+\mu_2)\}\right], \quad t > 0; \end{aligned} \tag{A.4}$$

$$q_{11}(t) = e^{-\mu_1 t - \alpha t^2}, \quad t > 0; \tag{A.5}$$

$$\begin{aligned} q_{12}(t) = e^{-(\theta+\mu_2)t} - e^{-\mu_1 t - \alpha t^2} \\ - (\mu_1-\theta-\mu_2)G(t, \mu_1, \theta+\mu_2, \theta+\mu_2), \quad t > 0; \end{aligned} \tag{A.6}$$

$$\begin{aligned} q_{13}(t) = 1 - e^{-(\theta+\mu_2)t} \\ + (\mu_1-\theta-\mu_2)G(t, \mu_1, \theta+\mu_2, \theta+\mu_2), \quad t > 0; \end{aligned} \tag{A.7}$$

$$q_{22}(t) = e^{-(\theta+\mu_2)t}, \quad t > 0; \tag{A.8}$$

$$q_{23}(t) = 1 - e^{-(\theta+\mu_2)t}, \quad t > 0. \tag{A.9}$$

References

[1] L. Billard, D.W.A. Dayananda, A multi-stage compartmental model for HIV-infected individuals: I – waiting time approach, Math. Biosci. (2013), http://dx.doi.org/10.1016/j.mbs.2013.08.011.
[2] R. Ross, The Prevention of Malaria, second ed., Murray, London, 1911.
[3] M.S. Bartlett, Some evolutionary stochastic processes, J. R. Stat. Soc. B 11 (1949) 211.
[4] N.T.J. Bailey, A simple stochastic epidemic, Biometrika 37 (1950) 193.
[5] N.L. Bowers, H.U. Gerber, J.C. Hickman, D.A. Jones, C.J. Nesbitt, Actuarial Mathematics, second ed., The Society of Actuaries, Schaumburg, Illinois, 1997.
[6] N.T.J. Bailey, Application of stochastic epidemic modelling in the public health control of HIV/AIDS, Lect. Notes Biostat. 86 (1990) 14.
[7] P.W.A. Dayananda, A simplified method of evaluating life insurance functions for HIV risk groups, Aust. Actuari. J. 6 (2000) 97.
[8] Institute of Actuaries, Formulae and Tables for Actuarial Examinations, Alden Press, Oxford, 1988.