1 Basics, 8/22/16

We'll let k stand for a field, e.g. $(\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_q, \mathbb{C}(x_1, ..., x_n))$. Let $k[x_1, ..., x_n]$ be the polynomial ring in n generators (ignoring grading). Let $k[x_0, ..., x_n] = \bigoplus_{d=0}^{\infty} k[x_0, ..., x_n]_d$ be the graded ring in $n+1$ generators, where $k[x_0, ..., x_n]_d$ is the vector space of homogeneous polynomials of degree d.

As a convention, we'll write $S := k[x_0, ..., x_n]$, and $S_d := k[x_0, ..., x_n]_d$. Note that we have a map $\mu : S_d \times S_e \to S_{d+e}$ given by multiplication.

1.1 Affine Algebraic Geometry

In affine algebraic geometry, we study finitely generated modules over S (think of this as a local picture).

1.2 Projective algebraic geometry

Studies finitely generated graded modules over S (think of this as a global picture). This is analogous to vector bundles on compact manifolds.

1.3 Ideals in Polynomial Rings

Theorem 1.3.1 (Hilbert's Basis Theorem)**.** *If R is Noetherian, $R[x]$ is.*

Definition 1.3.2. *R is noetherian if every submodule $N \subset M$ of a finitely generated module M is also finitely generated. This is equivalent to all ideals being finitely generated.*

Given $f \in k[x_1, ..., x_n]$, consider $V(f) := \{\underline{a} = (a_1, ..., a_n) \in k^n | f(\underline{a}) = 0\}$, the hypersurface in k^n associated to f. Given an ideal $I \subset k[x_1, ..., x_n]$, then $V(I) = V(f_1) \cap \cdots \cap V(f_n)$ where $I = \langle f_1, ..., f_n \rangle$, by the Hilbert basis theorem.

Theorem 1.3.3 (Nullstellensatz)**.** *The map $m : k^n \to \mathrm{MSpec}(k[x_1, ..., x_n])$ given by $(a_1, ..., a_n) \to \langle x_1 - a_1, ..., x_n - a_n \rangle = \ker(k[x_1, ..., x_n] \to k, f \to f(\underline{a})) = \{f | f(\underline{a}) = 0\}$ is bijective, if $k = \bar{k}$.*

Remark 1. *When $n = 1$, $k[x]$ is a PID so any maximal $m = \langle f \rangle$. But k algebraically closed iff any f factors iff every max ideal is of the form $m = \langle x - a \rangle$.*

Corollary 1.3.4. *Let I be any proper ideal in S, then $V(I) \neq \emptyset$. In fact, there is a bijection between $V(I)$ and the set of maximal ideals containing I.*

Corollary 1.3.5. *Let $I = \langle f_1, ..., f_n \rangle$ and $V(I) = \emptyset$. Then $1 \in I$, so there exists g_i such that $1 = \sum f_i g_i$.*

Definition 1.3.6. *$X \subset k^n$ is* <u>*algebraic*</u> *if $X = V(I)$ for some I.*

Given an algebraic subset, define $I(X) := \{f | f(\underline{a}) = 0, \forall \underline{a} \in X\}$. This function satisfies $I(V(I)) = \sqrt{I}$.

Proof. Given $I = \langle f_1, ..., f_n \rangle$, $X = V(I)$. Take $g \in I(V(I))$. Consider $I = \langle f_1, ..., f_n, gx_{n+1} - 1\rangle$, an ideal of $k[x_1, ..., x_{n+1}]$. It follows that $V(I) = \emptyset$, so $1 = \sum g_i f_i + h(gx_{n+1} - 1)$. Now take $g = x_{n+1}$, so that

$$1 = \sum g_i(x_1, ..., x_n, 1/g) f_i(x_1, ..., x_n).$$

Clearing denominators gives us g^N, for some N, as a linear combination of the f_i, so $g^N \in I$, i.e. $g \in \sqrt{I}$. $\square$

2 Overview, 8/24/16

Definition 2.0.1. *$J \subset k[x_1, ..., x_n]$ is geometric ideal if $J = I(S)$ for some $S \subset k^n$.*

The nullstellensatz tells us that J is geometric if and only if J is a radical ideal, giving us a bijection between

1. algebraic sets of k^n
2. geometric ideals of $k[x_1, ..., x_n]$
3. reduced quotient k-algebra R with $k[x_1, ..., x_n] \twoheadrightarrow R$ (meaning no nilpotents)

The first bijection is inclusion reversing. We also have a bijection between

1. affine varieties
2. prime ideals
3. k-algebra quotient domains A with $k[x_1, ..., x_n] \twoheadrightarrow A =: k[X]$.

From the original nullstellensatz, we have a bijection between

1. points $x \in k^n$
2. max ideals
3. field quotients $k[x_1, ..., x_n] \twoheadrightarrow k$

Definition 2.0.2. *If $X \subset k^n$ is a variety, then*

- $k(X) :=$ *field of fractions of* $k[X]$
- $\dim X := \text{tr.deg.}(k(X)/k)$

There can't really be a good definition of dimension due to components.

2.1 Cubics in the plane

Example. Consider cubics in the plane in W-normal form: $y^2 - (x - r_1)(x - r_2)(x - r_3)$. For example, $f(x, y) = y^2 - x^3$. Then $X = V(f)$ is irreducible by Eisenstein's criterion (?). We have an embedding $k[X] = k[x, y]/\langle y^2 - x^3\rangle$ into $k[t]$ by sending $x, y \to t^2, t^3$, i.e. $k[X] \cong k[t^2, t^3]$. However, $k(X) = k(t)$ is the field of rational functions in one variable. Geometric intuition: f has well-defined tangent lines except at the origin.

Let $g(x, y) = y^2 - x^2(x + 1)$, and $Y = V(g)$. We have an embedding $k[Y] \hookrightarrow k[t]$ by $x, y \to t^2 - 1, t(t^2 - 1)$. Hence $k[Y] \cong k[t^2 - 1, t(t^2 - 1)]$. Again, $k(Y) \cong k(t)$. A commutative algebraic intuition for this is that $k[X]$ and $k[Y]$ are not integrally closed, and $k[t]$ is actually their integral closure.

If we let $h_\lambda := y^2 - x(x + 1)(x - \lambda)$, for $\lambda \neq 0, -1$. Let $Z_\lambda := V(h_\lambda)$, then it turns out that $k(Z_\lambda) \ncong k(t)$. We will actually prove this by constructing differentials on this curve. Moreover, up to a finite group acting on the λ, $k(Z_{\lambda_1}) \ncong k(Z_{\lambda_2})$ if $\lambda_1 \neq \lambda_2$.

Remark 2. *It turns out h_λ has no common zeros with its partial derivatives. If they did, calculations can show that $\lambda = 0, -1$, a contradiction. Hence by the implicit function theore, h_λ is a manifold. It turns out that Z_λ being a manifold is related to the fact that $k(Z_\lambda) \neq k(t)$.*

At first glance, this isn't entirely true though: $W = V(y - x^3)$ is also a manifold and $k(W) \cong k(X)$. However, it turns out that W has a singular point at infinity.

Definition 2.1.1. *Let $X = V(P)$, some prime ideal P, $P = \langle f_1, ..., f_m\rangle$. If $\underline{a} = (a_1, ..., a_n)$, $J = \left(\dfrac{\partial f_i}{\partial x_j}\right)$. We say X is non-singular at $\underline{a}$ if $J(\underline{a})$ has rank $n - \dim X$.*

Theorem 2.1.2. *At all $x \in X$, $\text{rank} J \leq n - \dim X$. Equality holds away from an algebraic subset $Y \subset X$, which has measure 0, a consequence of the Zariski topology.*

2.2 Birational geometry

In birational geometry, we study the classification of fields K of finite transcendence degree over $k = \bar{k}$. The idea is that if K has transcendence degree n, meaning there are $x_1, ..., x_n$ such that $K/k(x_1, ..., x_n)$ is a finite field extension. Consider the ring of integers of both fields, and let the ring of integers of the larger field be A.

Theorem 2.2.1. *A is a finitely generated k-algebra.*

It turns out that A corresponds to a normal variety $X \subset k^N$ (a normal affine model of K). The question then arises: can you find $x_1, ..., x_n \in K$ such that the model X (and points at infinity) is non-singular. Hironaka answered this question affirmatively for characteristic zero, and it is unknown in characteristic p.