

1 Motivation, 8/24/16

Lie groups are differentiable manifolds with a group structure. Tangent space of a Lie group has additional structure, which we try to use to study the group action. Let G, H be Lie groups. Then we have

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ \downarrow & & \downarrow \\ T_1(G) & \xrightarrow{T_1(\varphi)} & T_1(H) \end{array},$$

where T_1 is the tangent space functor.

Recall that in the previous lecture, we considered the group $\mathrm{GL}(n, \mathbb{R})$. This is an open subspace of $M(n, \mathbb{R})$ with $\dim \mathrm{GL}(n, \mathbb{R}) = \dim M(n, \mathbb{R}) = n^2$. Now take a matrix A , and consider the curve passing through the identity with tangent vector at $t = 0$ equal to A : $t \rightarrow I + tA$. Thus we can identify tangent vectors with elements of $\mathrm{GL}(n, \mathbb{R})$.

If we look at the multiplication map m , we can look at

$$T_{(I,I)}(m) : T_1(\mathrm{GL}(n, \mathbb{R})) \times T_1(\mathrm{GL}(n, \mathbb{R})) \rightarrow T_1(\mathrm{GL}(n, \mathbb{R})).$$

Note that we can still speak of the functor T_1 since a product of manifolds is a manifold. Now consider the curve $t \rightarrow (I + tA, I + tB)$, which has tangent vector (A, B) . Then $m(I + tA, I + tB) = I + t(A + B) + t^2AB$. Hence the differential of m is $A + B$, and $T_1((A, B)) = A + B$, which is not very useful.

Take $A \in \mathrm{GL}(n, \mathbb{R})$, and consider the map $B \rightarrow ABA^{-1}$. Let $\mathrm{Int}(A)$ be this map, and let's calculate $T_1(\mathrm{Int}(A))$. If we look at

$$\mathrm{Int}(A)(I + tB) = I + tABA^{-1}.$$

The differential is given by the same formula as the map, however they are different since the domain/ranges aren't the same.

In some sense, the group operation is preserved, since we can tell whether the group is abelian. Now look at $\mathrm{Int}(I + tA)(B) = (B + tAB)(I + tA)^{-1}$. Recall that the geometric series formula holds for matrices (for some values of t), so we obtain the expansion $B + tAB - tBA + t^2(\dots)$. So the differential is a map $(A, B) \rightarrow [A, B] = AB - BA$. This gives us an algebra structure, i.e. a Lie algebra.

Definition 1.0.1. A Lie algebra L over k is a vector space with a Lie bracket $L \times L \rightarrow L$ satisfying anticommutativity and the Jacobi identity,

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

Anticommutativity implies that $[A, A] = 0$, unless the characteristic is positive (which is the case for algebraic groups).

Theorem 1.0.2. *Every finite-dimensional Lie algebra is isomorphic to a subalgebra of a matrix algebra.*

Some remarks about solvable and nilpotent Lie algebras. Comment about the Lie algebra functor, which induces the diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ \downarrow & & \downarrow \\ L(G) & \xrightarrow{L(\varphi)} & L(H) \end{array},$$

where $L(\varphi) := T_1(\varphi)$.

Lie theory is not really able to see components. For instance, $\mathrm{GL}(n, \mathbb{R})$ have two components, and by definition, only one of these has the identity. So Lie algebra cannot “see” the components, and so it gives us something for connected groups, e.g. $T_1(S)^1 = T_1(\mathbb{R}) = \mathbb{R}$. However, it will be able to tell us something about the group’s universal cover.