1 Basics, 8/22/16

We'll let k stand for a field, e.g. $(\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_q, \mathbb{C}(x_1, ..., x_n))$. Let $k[x_1, ..., x_n]$ be the polynomial ring in n generators (ignoring grading). Let $k[x_0, ..., x_n] = \bigoplus_{d=0}^{\infty} k[x_0, ..., x_n]_d$ be the graded ring in $n+1$ generators, where $k[x_0, ..., x_n]_d$ is the vector space of homogeneous polynomials of degree d.

As a convention, we'll write $S := k[x_0, ..., x_n]$, and $S_d := k[x_0, ..., x_n]_d$. Note that we have a map $\mu : S_d \times S_e \to S_{d+e}$ given by multiplication.

1.1 Affine Algebraic Geometry

In affine algebraic geometry, we study finitely generated modules over S (think of this as a local picture).

1.2 Projective algebraic geometry

Studies finitely generated graded modules over S (think of this as a global picture). This is analogous to vector bundles on compact manifolds.

1.3 Ideals in Polynomial Rings

Theorem 1.3.1 (Hilbert's Basis Theorem)**.** *If R is Noetherian, $R[x]$ is.*

Definition 1.3.2. *R is noetherian if every submodule $N \subset M$ of a finitely generated module M is also finitely generated. This is equivalent to all ideals being finitely generated.*

Given $f \in k[x_1, ..., x_n]$, consider $V(f) := \{\underline{a} = (a_1, ..., a_n) \in k^n | f(\underline{a}) = 0\}$, the hypersurface in k^n associated to f. Given an ideal $I \subset k[x_1, ..., x_n]$, then $V(I) = V(f_1) \cap \cdots \cap V(f_n)$ where $I = \langle f_1, ..., f_n \rangle$, by the Hilbert basis theorem.

Theorem 1.3.3 (Nullstellensatz)**.** *The map $m : k^n \to \mathrm{MSpec}(k[x_1, ..., x_n])$ given by $(a_1, ..., a_n) \to \langle x_1 - a_1, ..., x_n - a_n \rangle = \ker(k[x_1, ..., x_n] \to k, f \to f(\underline{a})) = \{f | f(\underline{a}) = 0\}$ is bijective, if $k = \bar{k}$.*

Remark 1. *When $n = 1$, $k[x]$ is a PID so any maximal $m = \langle f \rangle$. But k algebraically closed iff any f factors iff every max ideal is of the form $m = \langle x - a \rangle$.*

Corollary 1.3.4. *Let I be any proper ideal in S, then $V(I) \neq \emptyset$. In fact, there is a bijection between $V(I)$ and the set of maximal ideals containing I.*

Corollary 1.3.5. *Let $I = \langle f_1, ..., f_n \rangle$ and $V(I) = \emptyset$. Then $1 \in I$, so there exists g_i such that $1 = \sum f_i g_i$.*

Definition 1.3.6. *$X \subset k^n$ is algebraic if $X = V(I)$ for some I.*

Given an algebraic subset, define $I(X) := \{f | f(\underline{a}) = 0, \forall \underline{a} \in X\}$. This function satisfies $I(V(I)) = \sqrt{I}$.

Proof. Given $I = \langle f_1, ..., f_n \rangle$, $X = V(I)$. Take $g \in I(V(I))$. Consider $I = \langle f_1, ..., f_n, gx_{n+1} - 1 \rangle$, an ideal of $k[x_1, ..., x_{n+1}]$. It follows that $V(I) = \emptyset$, so $1 = \sum g_i f_i + h(gx_{n+1} - 1)$. Now take $g = x_{n+1}$, so that

$$1 = \sum g_i(x_1, ..., x_n, 1/g) f_i(x_1, ..., x_n).$$

Clearing denominators gives us g^N, for some N, as a linear combination of the f_i, so $g^N \in I$, i.e. $g \in \sqrt{I}$. □

2 Overview, 8/24/16

Definition 2.0.1. *$J \subset k[x_1, ..., x_n]$ is geometric ideal if $J = I(S)$ for some $S \subset k^n$.*

The nullstellensatz tells us that J is geometric if and only if J is a radical ideal, giving us a bijection between

1. algebraic sets of k^n
2. geometric ideals of $k[x_1, ..., x_n]$
3. reduced quotient k-algebra R with $k[x_1, ..., x_n] \twoheadrightarrow R$ (meaning no nilpotents)

The first bijection is inclusion reversing. We also have a bijection between

1. affine varieties
2. prime ideals
3. k-algebra quotient domains A with $k[x_1, ..., x_n] \twoheadrightarrow A =: k[X]$.

From the original nullstellensatz, we have a bijection between

1. points $x \in k^n$
2. max ideals
3. field quotients $k[x_1, ..., x_n] \twoheadrightarrow k$

Definition 2.0.2. *If $X \subset k^n$ is a variety, then*

- $k(X) :=$ *field of fractions of* $k[X]$
- $\dim X := \text{tr.deg.}(k(X)/k)$

There can't really be a good definition of dimension due to components.

2.1 Cubics in the plane

Example. Consider cubics in the plane in W-normal form: $y^2 - (x - r_1)(x - r_2)(x - r_3)$. For example, $f(x, y) = y^2 - x^3$. Then $X = V(f)$ is irreducible by Eisenstein's criterion (?). We have an embedding $k[X] = k[x, y]\langle y^2 - x^3\rangle$ into $k[t]$ by sending $x, y \to t^2, t^3$, i.e. $k[X] \cong k[t^2, t^3]$. However, $k(X) = k(t)$ is the field of rational functions in one variable. Geometric intuition: f has well-defined tangent lines except at the origin.

Let $g(x, y) = y^2 - x^2(x + 1)$, and $Y = V(g)$. We have an embedding $k[Y] \hookrightarrow k[t]$ by $x, y \to t^2 - 1, t(t^2 - 1)$. Hence $k[Y] \cong k[t^2 - 1, t(t^2 - 1)]$. Again, $k(Y) \cong k(t)$. A commutative algebraic intuition for this is that $k[X]$ and $k[Y]$ are not integrally closed, and $k[t]$ is actually their integral closure.

If we let $h_\lambda := y^2 - x(x + 1)(x - \lambda)$, for $\lambda \neq 0, -1$. Let $Z_\lambda := V(h_\lambda)$, then it turns out that $k(Z_\lambda) \ncong k(t)$. We will actually prove this by constructing differentials on this curve. Moreover, up to a finite group acting on the λ, $k(Z_{\lambda_1}) \ncong k(Z_{\lambda_2})$ if $\lambda_1 \neq \lambda_2$.

Remark 2. *It turns out h_λ has no common zeros with its partial derivatives. If they did, calculations can show that $\lambda = 0, -1$, a contradiction. Hence by the implicit function theore, h_λ is a manifold. It turns out that Z_λ being a manifold is related to the fact that $k(Z_\lambda) \neq k(t)$.*

At first glance, this isn't entirely true though: $W = V(y - x^3)$ is also a manifold and $k(W) \cong k(X)$. However, it turns out that W has a singular point at infinity.

Definition 2.1.1. *Let $X = V(P)$, some prime ideal P, $P = \langle f_1, ..., f_m\rangle$. If $\underline{a} = (a_1, ..., a_n)$, $J = \left(\dfrac{\partial f_i}{\partial x_j}\right)$. We say X is* <u>*non- singular*</u> *at $\underline{a}$ if $J(\underline{a})$ has rank $n - \dim X$.*

Theorem 2.1.2. *At all $x \in X$, $\operatorname{rank} J \leq n - \dim X$. Equality holds away from an algebraic subset $Y \subset X$, which has measure 0, a consequence of the Zariski topology.*

2.2 Birational geometry

In birational geometry, we study the classification of fields K of finite transcendence degree over $k = \bar{k}$. The idea is that if K has transcendence degree n, meaning there are $x_1, ..., x_n$ such that $K/k(x_1, ..., x_n)$ is a finite field extension. Consider the ring of integers of both fields, and let the ring of integers of the larger field be A.

Theorem 2.2.1. *A is a finitely generated k-algebra.*

It turns out that A corresponds to a normal variety $X \subset k^N$ (a normal affine model of K). The question then arises: can you find $x_1, ..., x_n \in K$ such that the model X (and points at infinity) is non- singular. Hironaka answered this question affirmatively for characteristic zero, and it is unknown in characteristic p.

3 Graded Modules, 8/26/16

3.1 Basics

Definition 3.1.1. *A module M over S is graded if $M = \oplus_{d\in\mathbb{Z}} M_d$ as a direct sum of vector spaces over k, with the action of S on M respecting the grading, i.e. $S_d \times M_e \to M_{d+e}$.*

Graded homomorphisms of graded modules is an S-module homomorphism $\phi : M \to N$ such that $\phi(M_d) \subset N_d$.

Example. If M is a finitely generated as an S-module, then M is generated by finitely many homogenous elements.

Example. An ideal $I \subset S$ is a graded submodule if and only if I is a homogenous ideal (i.e. generated by homogenous polynomials).

Remark 3. *Graded S-modules with graded homomorphisms form an abelian category.*

Remark 4. *As before, we have bijections between homogenous ideals $I \subset S$ and graded quotient k-algebras R such that $S \twoheadrightarrow R$.*

3.2 Fundamental Example and Syzygy Theorem

A fundamental example are twisted free S-modules of rank 1,

$$S(e) := \bigoplus_{d\in\mathbb{Z}} k[x_0, ..., x_n]_{d+e} =: \bigoplus S(e)_d. = \bigoplus_{d\geq -e} S_{d+e}.$$

Lemma 3.2.1. *The set of graded homomorphisms $\phi : S(-e) \to S(-f)$ is in bijection with $k[x_0, ..., x_n]_{e-f}$ (map associated with f, the image of $1 \in S(-e)_e = k$).*

Proof. Clear. $\square$

Let $M = \langle m_{d_1}, ..., m_{d_r} \rangle$ be a finitely generated graded module generated by m_{d_i}. Construct the exact sequence

$$0 \to k_0 \to \bigoplus_{i=1}^{r} S(-d_{i,0}) \twoheadrightarrow < \to 0,$$

which sends $1 \to m_{d_i} \in M_{d_i}$. If we iterate this process, we obtain a free resolution of M, which terminates by the Hilbert syzygy theorem.

Example. Let $k = S/\langle x_0, ..., x_n\rangle$. Then

$$0 \to S(-n-1) \to \cdots \to \bigoplus^{\binom{n+1}{2}} S(-2) \to \bigoplus^{n+1} S(-1) \to S \to k \to 0.$$

Example $S = k[x_0, ..., x_3]$. Then we have

$$0 \to \bigoplus^{2} S(-3) \to \bigoplus^{3} S(-2) \to S \twoheadrightarrow \bigoplus_{d=0}^{\infty} R_{3d} \to 0,$$

which sends $1, x_0, x_1, x_2, x_3 \to 1, s^3, s^2t, st^2, t^3$.

3.3 Final Hilbert Theorem

Theorem 3.3.1. *If M is a finitely generated graded S-module, then the function $h(d) := \dim M_d$ is a polynomial function for all $d >> 0$ of degree $\leq n$.*

Proof. True for $S(e)$, since $\dim S(e)_d = \binom{d+e+n}{n}$. By the exact sequence above, we have $\dim M_d = \sum_i (-1)^i \dim F_{i_d j}$ where $F_j = S(-d_{i,j})$. $\square$

As before, we have a bijection between graded quotient k-algebras and projective varieties.

Definition 3.3.2. *Let* $\dim X := \deg h(d)$ *for a projective variety X.*

The challenge is to reconcile this with dimension of affine varieties.