1 Motivation, 8/24/16

Lie groups are differentiable manifolds with a group structure. Tangent space of a Lie group has additional structure, which we try to use to study the group action. Let G, H be Lie groups. Then we have

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d] & H \arrow[d] \\ T_1(G) \arrow[r, "T_1(\varphi)"] & T_1(H) \end{tikzcd},$$

where T_1 is the tangent space functor.

Recall that in the previous lecture, we consided the group $\mathrm{GL}(n, \mathbb{R})$. This is an open subspace of $\mathrm{M}(n, \mathbb{R})$ with $\dim \mathrm{GL}(n, \mathbb{R}) = \dim \mathrm{M}(n, \mathbb{R}) = n^2$. Now take a matrix A, and consider the curve passing through the identity with tangent vector at $t = 0$ equal to A: $t \to I + tA$. Thus we can identify tangent vectors with elements of $\mathrm{GL}(n, \mathbb{R})$.

If we look at the multiplication map m, we can look at

$$T_{(I,I)}(m) : T_1(\mathrm{GL}(n, \mathbb{R})) \times T_1(\mathrm{GL}(n, \mathbb{R})) \to T_1(\mathrm{GL}(n, \mathbb{R})).$$

Note that we can still speak of the functor T_1 since a product of manifolds is a manifold. Now consider the curve $t \to (I + tA, I + tB)$, which has tangent vector (A, B). Then $m(I + tA, I + tB) = I + t(A + B) + t^2AB$. Hence the differential of m is $A + B$, and $T_1((A, B)) = A + B$, which is not very useful.

Take $A \in \mathrm{GL}(n, \mathbb{R})$, and consider the map $B \to ABA^{-1}$. Let $\mathrm{Int}(A)$ be this map, and let's calculate $T_1(\mathrm{Int}(A))$. If we look at

$$\mathrm{Int}(A)(I + tB) = I + tABA^{-1}.$$

The differential is given by the same formula as the map, however they are different since the domain/ranges aren't the same.

In some sense, the group operation is preserved, since we can tell whether the group is abelian. Now look at $\mathrm{Int}(I + tA)(B) = (B + tAB)(I + tA)^{-1}$. Recall that the geometric series formula holds for matrices (for some values of t), so we obtain the expansion $B + tAB - tBA + t^2(\cdots)$. So the differential is a map $(A, B) \to [A, B] = AB - BA$. This gives us an algebra structure, i.e. a Lie algebra.

Definition 1.0.1. *A Lie algebra L over k is a vector space with a Lie bracket $L \times L \to L$ satisfying anticommutativity and the Jacobi identity,*

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

Anticommutativity implies that $[A, A] = 0$, unless the characteristic is positive (which is the case for algebraic groups).

Theorem 1.0.2. *Every finite-dimensional Lie algebra is isomorphic to a subalgebra of a matrix algebra.*

Some remarks about solvable and nilpotent Lie algebras. Comment about the Lie algebra functor, which induces the diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d] & H \arrow[d] \\ L(G) \arrow[r, "L(\varphi)"] & L(H) \end{tikzcd},$$

where $L(\varphi) := T_1(\varphi)$.

Lie theory is not really able to see components. For instance, $\mathrm{GL}(n, \mathbb{R})$ have two components, and by definition, only one of these has the identity. So Lie algebra cannot "see" the components, and so it gives us something for connected groups, e.g. $T_1(S)^1 = T_1(\mathbb{R}) = \mathbb{R}$. However, it will be able to tell us something about the group's universal cover.

2 Overview, 8/26/16

Recall we constructed the Lie group functor $G \to L(G)$ in the case of $G = \mathrm{GL}(n, \mathbb{R})$ by looking at an inner automorphism.

Now assume G is connected. If G is nice enough, you can construct a universal covering space $\hat{G}$, and there is the covering map $p : \hat{G} \to G$. We will show that p is a Lie group morphism, and that $L(p)$ is a Lie algebra isomorphism.

We will show that there is an equivalence of categories between Lie algebras and connected, simply-connected Lie groups. Take a connected Lie group G, $\dim G = 1$, so that $\dim L(G) = 1$ and $L(G) \cong \mathbb{R}$ and $[\cdot, \cdot] = 0$ and $(\mathbb{R}, +) = H$ has $L(H) = \mathbb{R}$. Any such G has $\mathbb{R}$ as a covering space.Say we have a covering map $c : \mathbb{R} \to C$. Then $\ker c$ is closed and a discrete subgroup so $G \cong \mathbb{Z}$ or 0. So G is either $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$.

G is a connected Lie group if and only if $L(G)$ is abelian, so the Lie bracket is identically zero. The rest is incomprehensible since I was sleepy.