Let X be a topological space, and consider the category Ab_X of sheaves of groups on X.

Claim. Ab_X has enough injectives, i.e. for any $v \in Ob(Ab_X)$ there is an injective resolution

$$0 \to v \to I^0 \to I^1 \to \cdots$$

Recall that the global sections functor is left-exact, so we have

$$0 \to 0 \to \Gamma(X, I^0) \to \Gamma(X, I^1) \to \cdots$$

Def. For $p \in \mathbb{Z}$, let $H^p(X, V) := H^p(\Gamma(X, I^\bullet))$, the p-th cohomology group.

Th'm If X is noetherian, $H^p(X, V) = 0$ for $p > \dim X$.

Def. Suppose we have $0 \to V \to A^0 \to A^1 \to \cdots$, we say that A^i is Γ-acyclic if $H^p(X, A^i) = 0$, $p > 0$.

Def. $\mathcal{F}$ is a flasque sheaf if $\mathcal{F}(X) \twoheadrightarrow \mathcal{F}(U)$ for every open $U \subset X$.

Claim. Injective sheaves are flasque.

Pf. Given a sheaf $\mathcal{V}$, form $D(\mathcal{V})$, the sheaf of <u>discontinuous sections</u> by

$$D(\mathcal{V})(U): \underset{x}{\overset{U}{\cup}} \longrightarrow s_x^{\in \mathcal{V}_x}$$

where $\mathcal{V}_x = \varinjlim_{x \in U} \mathcal{V}(U)$.

<u>Claim</u>. $D(\mathcal{V})$ is flasque.

Moreover, we have $0 \longrightarrow \mathcal{V} \longrightarrow D(\mathcal{V})$ sending, given an open $U \subset X$, $s \longrightarrow (x \to s_x)$ where $x \in U$ and $s_x \in \mathcal{V}_x$.

This embeds any sheaf into a <u>flasque</u> one.
Then we have a split,

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "id"'] & F \arrow[dl, dotted] \\ & I & \end{tikzcd}$$

hence $F = I \oplus G$. Since F is flasque,

$$\begin{array}{ccccc} F(X) & = & I(X) & \oplus & G(X) \\ \downarrow & & \downarrow & & \downarrow \\ F(U) & = & I(U) & \oplus & G(U) \end{array}$$

So injective $\Rightarrow$ flasque.

Now let X be an alg. variety, and $\mathcal{O}_X$ the structure sheaf.

Let $\mathcal{M}(\mathcal{O}_X)$ be the category of sheaves of $\mathcal{O}_X$-modules. Let $\mathcal{V}$ be a sheaf. Then for any $U \subset X$ open, $\mathcal{V}(U)$ is an $\mathcal{O}_X(U)$-module.

Let $\rho^U_V : \mathcal{V}(U) \to \mathcal{V}(V)$ be restriction, and $r^U_V : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ be restriction. Want these to be compatible, i.e.

$$\rho^U_V(f \cdot s) = r^U_V(f) \cdot \rho^U_V(s).$$

Moreover, note that we have a forgetful functor $\mathrm{For} : \mathcal{M}(\mathcal{O}_X) \to Ab_X$. Notice that $H^p(X, \mathcal{V}) = H^p(X, \mathrm{For}(\mathcal{V}))$.

<u>Claim</u>. $\mathcal{M}(\mathcal{O}_X)$ has enough injectives.

<u>Pf</u> (Sketch)

Take $\mathcal{V}$ a sheaf of $\mathcal{O}_X$-modules. Then $\mathcal{V}_x$ is an $\mathcal{O}_{X,x}$-module, and we know we have

$$0 \to \mathcal{V}_x \to I_x$$

$U \longrightarrow$ discontinuous sections w/ values in I_x, $x \in U$ is an injective obj in $\mathcal{M}(\mathcal{O}_X)$.

<u>Note</u>: Injectives are not necessarily preserved under functors!

<u>Define</u> $\bar{H}^p(X, \mathcal{V}) = H^p(\Gamma(X, \mathcal{I}^\bullet))$

$\Gamma(X, \mathcal{O}_X)$- modules

$\mathcal{I}^i$ are flasque, so Γ-acyclic, so $\bar{H}^p$ is isomorphic to "real" cohomologies.

Now assume X is an alg. variety, and $\mathcal{O}_X$ its structure sheaf. There is a finite covering by open affines.

<u>Def.</u> An $\mathcal{O}_X$-module $\mathcal{V}$ is <u>quasi-coherent</u> if for any $x \in X$, there is an open $U \subset X$ such that

$$\mathcal{O}_X(U)^{(J)} \to \mathcal{O}_X(U)^{(I)} \to \mathcal{V}|_U \to 0$$

is exact, where $\mathcal{O}_X(U)^{(K)} = \bigoplus_K \mathcal{O}_X(U)$, so that locally $\mathcal{V}$ is a cokernel.

<u>Def.</u> $\mathcal{M}_{qc}(\mathcal{O}_X)$ – full subcategory of q.c. $\mathcal{O}_X$-modules in $\mathcal{M}(\mathcal{O}_X)$.

<u>Th'm</u> $\mathcal{M}_{qc}(\mathcal{O}_X)$ is an abelian category.

Then if we have a morphism $\mathcal{V} \to \mathcal{U}$, ker and coker are quasi-coherent.

Th'm A (Serre) Let X be an affine variety. Then for any quasi-coherent $\mathcal{O}_X$-module V, $H^p(X, V) = 0$, $p \geq 0$.

This is equivalent to saying $\Gamma : \mathcal{M}_{qc}(\mathcal{O}_X) \to \mathcal{M}(R(X))$ is exact.

Th'm B (Serre) Let X be an affine variety and V q.c. Then $\Gamma(X, V) = 0 \Rightarrow V = 0$.

This says that $\mathcal{M}_{qc}(\mathcal{O}_X) \xrightarrow{\Gamma} \mathcal{M}(R(X))$ is an equivalence of categories.