Cartier divisors

$$\left\{ (U_\alpha ; \phi_\alpha) \;\middle|\; \begin{array}{l} X = \bigcup U_\alpha \\ \phi_\alpha \in k(X)^* \end{array} \quad \begin{array}{c} \phi_\alpha \cdot \phi_\beta^{-1} \in \mathcal{O}_X(U_\alpha \cap U_\beta) \\ \forall \alpha, \beta \end{array} \right\}$$

Idea. $(U_\alpha ; \phi_\alpha) \sim (U_\alpha ; \phi_\alpha \cdot \psi_\alpha)$

$$\psi_\alpha \in \mathcal{O}_X(U_\alpha)$$

Refinements. $(U_\alpha, \phi_\alpha) \sim (V_\beta, \psi_\beta)$

if $V_\beta \subset U_\alpha$

$\psi_\beta \approx \phi_\alpha$ (??

Assume X is normal. (Key: $\mathcal{O}_{X,Z}$ are DVR]

$$\mathrm{div} : \mathrm{CDiv}(X) \longrightarrow \mathrm{Div}(X)$$

claim: Independent of α

$$(U_\alpha ; \phi_\alpha) \longrightarrow \sum_{\substack{Z \subset X \\ Z \cap U_\alpha \neq \emptyset}} \mathrm{ord}_Z(\phi_\alpha) \cdot Z$$

Claim. div is injective. $(U_\alpha ; \phi_\alpha) \in \ker \mathrm{div}$

$\iff \forall \alpha, \phi_\alpha$ is unit in each $\mathcal{O}_{U_\alpha, Z}$

$\iff \phi_\alpha \in \mathcal{O}_X^*(U_\alpha)$

$\iff (U_\alpha ; \phi_\alpha) \sim (X, 1) = id$

<u>Prop.</u> If each $\mathcal{O}_{X,x}$ is a UFD (in particular, this holds if X is non-singular), then

$$\mathrm{div} : \mathrm{CDiv}(X) \twoheadrightarrow \mathrm{Div}(X) .$$

<u>Pf.</u> Given $Z \subset X$, construct a Cartier divisor $(U_\alpha, \phi_\alpha) = D$ s.t. $\mathrm{div}(D) = Z$. For $U = X \setminus Z$,

- $(U, 1)$
- $\forall\, x \in X$, let $\phi_x \in \mathcal{O}_{X,x}$ generate the ideal corresponding to Z.

(UFD $\Rightarrow$ every height 1 prime is principal)

$f_x = \phi_x$ is defined, <u>and</u> generated $I(Z)$ on some affine nbhd $x \in U_x$

$$D = (U, 1) \cup (U_x, f_x)$$

<u>Def.</u> Two Cartier divisors $D_1 = (U_\alpha, \phi_\alpha)$, $D_2 = (U_\alpha, \psi_\alpha)$ are rationally equiv if $\exists\, \psi \in K(X)^*$ s.t.

$$(U_\alpha, \phi_\alpha \cdot \psi) \underset{\substack{\uparrow \\ \text{as Cartier divisors}}}{=} (U_\alpha, \psi_\alpha)$$

Then $$\underline{\mathrm{div}} : \mathrm{CDiv}(X) / \substack{\text{rational} \\ \text{equivalence}} \longrightarrow \overset{\mathrm{Cl}(X)}{\overset{\shortparallel}{\mathrm{Div}(X)}} / \substack{\text{linear} \\ \text{equivalence}}$$

(if X is normal).

<u>Eg.</u> $\mathbb{P}^n$; cover $U_0, \dots, U_n$

Fix $\ell \in k[x_0, \dots, x_n]_1 \setminus \{0\}$ $\quad (V(\ell) = H)$

$D = \{(U_i, \frac{\ell}{x_i})\}$ is a Cartier divisor.

$\operatorname{div}(D) = V(\ell) = H$.

Each $\varphi \in K(\mathbb{P}^n) \rightsquigarrow$ rationally equivalent Cartier divisor

$$\operatorname{div}(U_i, \frac{\ell}{x_i}\varphi) = H + \operatorname{div}\varphi$$

<u>Def.</u> $D = (U_\alpha, \phi_\alpha)$ is <u>effective</u> if $\phi_\alpha \in \mathcal{O}_X(U_\alpha)$ (regular) for all α.

$\varphi \rightsquigarrow$ <u>effective</u> rationally equiv divisor

$\iff \frac{\ell}{x_i}\varphi$ are <u>regular</u> on U_i $\forall i$.

$\iff \varphi = \frac{\ell'}{\underset{\uparrow \text{fixed}}{\ell}}$ for any linear ℓ'

$$(U_i, \frac{\ell'}{x_i}) \underset{\uparrow \text{CDiv}}{=} (U_i, \frac{\ell''}{x_i})$$

$\iff \ell' = c \cdot \ell''$

Thm. Among effective CDiv, the equivalence classes are projective spaces (finite dim if X is projective).

Pf. Fix $D_0 = (U_\alpha ; f_\alpha)$; $f_\alpha \in \mathcal{O}_X(U_\alpha)$

$$f_\alpha / f_\beta \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$$

$D \sim D_0 \iff D = (U_\alpha ; f_\alpha \cdot \varphi)$ for some $\varphi \in k(X)^*$

and D is effective $\iff f_\alpha \varphi \in \mathcal{O}_X(U_\alpha)\ \forall \alpha$.

$$(u_i, \frac{\ell}{x_i}) \qquad \varphi = \frac{\ell'}{\ell}$$

$$H \rightsquigarrow \underbrace{(u_i, \frac{\ell'}{x_i})}_{H'}$$

Obs. Given two such rational functions $\varphi_1, \varphi_2 \in k(X)^*$ then $\varphi_1 + \varphi_2$ also satisfies

$$(U_\alpha, \underset{\mathcal{O}_X(U_\alpha)}{\underset{\cap}{f_\alpha(\varphi_1 + \varphi_2)}})$$

"Complete Linear Series"

TODO: Prove that equiv classes

$|D_0| = \{$ rat equiv eff Cartier divs $\}$

$$= \{ \varphi \in k(X)^* \mid f_\alpha \varphi \in \mathcal{O}_X(U_\alpha) \ \forall \alpha \} / k^*$$

are fin dimensional.

<u>Rmk</u>. If $\Phi : X \longrightarrow \mathbb{P}^n$ is a regular map,
proj. variety.

then we can <u>pull back</u> the Cartier divisors.

$H = (U_i, \frac{\ell}{x_i})$ provided that $\Phi(X) \not\subset H$.

$\rightsquigarrow (\Phi^{-1}(U_i), \Phi^*(\frac{\ell}{x_i}))$

$\Phi^*(\frac{\ell}{x_i}) \in k(X)^*$

(Cartier divisor on X)

Moreover the projective space of hyperplanes in $\mathbb{P}^n$ $(= (\mathbb{P}^n)^\vee)$

$\longleftrightarrow |H| \ni H'$

pulls back Φ to a proj. subspace

$(\mathbb{P}^n)^\vee \subset D$.