X is non-singular if $\forall x \in X$, $\dim m_x/m_x^2 = \dim X$.

Ex. Homogeneous spaces.

Def. An action of an alg. group G on a variety X is a regular map $a: G \times X \to X$ with the usual properties:

$$a(g_2, a(g_1, x)) = a(\mu(g_2, g_1), x),$$

where $\mu: G \times G \to G$ is group law.

Def The action is transitive if the orbit of any x is X.

Def. X is homogeneous (for G) if G acts transitively on X.

Example. G is homogeneous for G, w/ left-action. Never under conjugation.

Rmk. Homogeneous spaces are non-singular, since pts. of a homogeneous space are "indistinguishable", since g s.t. $g \cdot x = y$ is an iso $X \xrightarrow{\sim} X$ sending $x \to y$. So all points are sing. or none are. However, any variety is non-singular at some point.

Example. $PGL(n+1, k) = GL(n+1)/k^*$ is an affine algebraic group. Acts transitively on $\mathbb{P}^n_k$.

Rmk. The projective homogeneous spaces for linear algebraic groups $/k$ are the flag varieties.

Ex. $G(m,n)$

$\mathbb{F}l(m_1, \dots, m_r, n)$) homog. ~~sub~~spaces for $PGL(n)$

$\{\Lambda_1 \subset \dots \subset \Lambda_r \subset k^n\}$

$\dim \Lambda_i = m_i$.

Rmk. $G \times X \longrightarrow X$ homog. Then $X \cong G/H$, H is stab of a point X.

For $G = GL(n)$,

$$\begin{bmatrix} * & & \\ & \ddots & \\ * & & * \end{bmatrix} = B = \operatorname{stab} \langle e_0 \rangle \subset \langle e_0, e_1 \rangle$$

$$G/B = \{\Lambda_1, \Lambda_2 \subset \dots \subset \Lambda_n = k^n\}$$

full flag variety.

Rmk. An action of G on X is $G \subset \operatorname{Aut}(X)$.

Prop. $\operatorname{Aut}(\mathbb{P}^n_k) = PGL(n+1, k)$.

Pf. $g : \mathbb{P}^n_k \longrightarrow \mathbb{P}^n_k$ we saw that $g = [F_0 : \dots : F_n]$ F_i do not simult. vanish

If $F = \sum a_i y_i$, ~~then~~ ~~[illegible]~~ are —, then

$g \in PGL(n+1)$! $g \longleftrightarrow (a_{ij})$.

If $\deg F = d > 1$, g cannot be iso.

(consider $k(\mathbb{P}^n)^{g^*} \subset k(\mathbb{P}^n)$ degree d.

Ex. $\mathbb{P}^1 \to \mathbb{P}^1$

$(x:y) \quad (x^2, y^2)$

look at affines.

$(x:1) \to (x^2:1)$

$k \longrightarrow k \qquad\qquad k(y) \subset k(x)$

$y \longrightarrow x^2 = y \qquad\qquad y \longrightarrow x^2$

Ex. Degeneracy loci:

$$\mathrm{Mat}_n \supset X_{n-1} \supset \cdots \supset X_0 = (0)$$

$$\| $$

$$k^{n^2} \ni \{A = (a_{ij})\}$$

$$X_r = \{A \mid \mathrm{rk}(A) \le r\}$$

$$X_r = V(\det(M_{r+1}) \mid M_{r+1} \text{ is an } r+1 \text{ minor of } (x_{ij}) \}$$

Ex. $X_{n-1} = V(\Delta) \qquad \Delta = \det(x_{ij})$

Rmk. $\nabla(\Delta) = (\ldots, \ldots, \ldots)$

$$\Delta = \pm x_{ij} \det(M_{ij})$$

$$\frac{\partial \Delta}{\partial x_{ij}} = \pm \det(M_{ij}).$$

Similarly, $\dfrac{\partial \det(M_{r+1})}{\partial x_{ij}} = \pm \det(M_r).$

equations: $\det(M_{r+1})$

$$X_{r-1} \subseteq \operatorname{sing}(X_r)$$

and the Jacobson matrix is $\equiv 0$ on X_{r-1}

$\to \quad m_x/m_x^2 = k^{n^2}$ for $x \in X_{r-1} \subset X_r$.

To prove $X_r \setminus X_{r-1}$ is non-singular, consider

$$\begin{tikzcd}
p^{-1}(X_r \setminus X_{r-1}) \cap I_r := \{(A, \Lambda) \mid \Lambda \subset \ker(A)\} \arrow[d, "p"'] \arrow[dr, "\pi \text{ (vector bundle)}"] & \\
\{A \mid rk(A) \le r\} = X_r & G(n-r, n) \ni \Lambda.
\end{tikzcd}$$

$$X_r \setminus X_{r-1} \xrightarrow{1:1} p^{-1}(X_r \setminus X_{r-1}), \qquad A \mapsto \ker A = \Lambda$$

Consider: $\pi(\Lambda = \langle e_1, \ldots, e_{n-r} \rangle) = \{ A = \begin{pmatrix} \overbrace{\begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix}}^{n-r} & \overbrace{*}^{r} \end{pmatrix}$

$\cong k^{rn}$