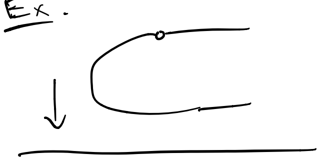
Def. A map $\Phi: X \longrightarrow Y$ of affine varieties is <u>finite</u> if $\Phi^*: k[Y] \longrightarrow k[X]$ is integral.

Th'm. If Φ is finite, then

(i) $\Phi^{-1}(y)$ is finite $\forall y \in Y$.

(ii) Φ maps closed sets to closed sets.

Ex.



$V(x-y^2) \setminus \{(1,1)\}$

This map satisfies (i), (ii), however Φ is not finite.

Def. A map $\Phi: X \longrightarrow Y$ is ~~affine~~

(a) <u>affine</u> if $\Phi^{-1}(U) = V$ is affine when U is.

(b) <u>finite</u> if affine and each $\Phi|_V : V \longrightarrow U$ is finite.

<u>Problem</u>. This definition is <u>impossible</u> to check in practice.

<u>Th'm</u>. It suffices to check affine/finite-ness over a single open affine cover of Y.

<u>Criterion</u>. Given a variety V, suppose there are regular functions $h_1, \dots, h_n, g_1, \dots, g_n$ on V s.t.

- $\sum h_i g_i = 1$.
- $U_{h_i} := V \setminus V(h_i)$ is affine cover of V

Then V is affine.

Pf. To prove

$$\Gamma(V, \mathcal{O}_V) = k[U_{h_1}] \cap \dots \cap k[U_{h_k}]$$

is finitely generated as a k-algebra.

Choose generators for $k[U_{h_i}] = k[x_{i_1}, \dots, x_{i m_i}]/P_i$

Rmk. For each $U_{h_i} \cap U_{h_j}$ is affine, because

$$U(h_j) \cap U(h_i) = U_{h_j} \setminus V(h_i)$$

as a basic open, i.e. $(U_{h_j})_{h_i}$.

Therefore. if $\phi \in k[U_i]$, then $\phi \in k[U_i \cap U_j]$

$= k[(U_{h_j})_{h_i}] = k[U_j^{-1}][h_i^{-1}]$, $\forall j$. So $\exists n_j$ s.t.

$$\phi \cdot h_i^{n_j} \in k[U_{h_i}] \quad \forall i.$$

If $n > \max\{n_j\}$, then

$$\phi h_i^n \in k[U_j] \quad \forall j$$
$$= \Gamma(V, \mathcal{O}_V) = A$$

Conclusion. $k[U_i] = \Gamma(V, \mathcal{O}_V)[h_i^{-1}]$ $\forall i$.

In particular, each $x_{i,l} h_i^{n_{i,l}} \in A$ for all i, l.

Let $n = \max\{n_{i,l}\}$. Then $x_{i,l} h_i^n \in A$ $\forall i, l$.

Claim. $\{x_{i,l} h_i^n, h_i, g_i\}$ generate A as a k-algebra.

Suppose $a \in A$. Then $a \in k[U_i]$ $\forall i$, so

$$a = p_i(\bar{x}_{i,1}, \ldots, \bar{x}_{i,m_i}) \quad \forall i.$$

Final obs. $\exists N$, $ah_i^N = p_i h_i^N$ is a poly in the $x_{i,e} h_i^n$, h_i $\forall i$.

Finally, use $\sum g_i h_i = 1 \Rightarrow \underbrace{a\left(\sum g_i h_i\right)^{(N-1)(n+1)}}_{\text{poly in } \underbrace{ah_i^N}_{\text{poly in } x_{i,e} h_i^N}, g_i, h_i} = a.$

So $k[y_{i,e}, z_i, w_i] \twoheadrightarrow A$.

Apply to our situation. Let V be a variety.

$$\begin{array}{ccc} & V & & U \\ & \cap & & \cap \\ \Phi : & X & \longrightarrow & Y \\ & & & \cap \\ & & & \bigcup U_i \end{array}$$

To apply the criterion, need to know:

$$\underset{\text{variety}}{Y} \supset U \cap U_i \overset{\exists ?}{=} \bigcup U_{ie}$$

$$U_{ie} \begin{array}{l} \subset U \setminus V(g_e) \\ \subset U_i \setminus V(g_e') \end{array}$$

Then $U = \bigcup (U \cap U_i) = \bigcup \underset{\substack{\text{basic opens} \\ \text{for } U \,\&\, U_i}}{U_{ie}}$

<u>Observation 1</u>. Because $U_{i\ell} \subset U_i$ is basic open, it follows that $\Phi^{-1}(U_{i\ell}) \subset \Phi^{-1}(U_i) = V_i$

$$\underset{\text{affine}}{\uparrow} \qquad\qquad \underset{\text{affine}}{\uparrow}$$

$$\Phi^{-1}(U_{i\ell}) = V_i \setminus V_i(g'_{i\ell})$$

Suppose $U_{i\ell} = U \setminus V(h_{i\ell})$. Then $\exists$ $g_{i\ell}$ s.t.

$$\sum h_{i\ell} g_{i\ell} = 1.$$

and $V_{i\ell} \subset V$ are affine basic opens

$$V_{i\ell} = V - V(\Phi^*(h_{i\ell})).$$

Now use the criterion.