A proper algebraic group is abelian.

Pf. Consider $c : G \times G \longrightarrow G$

$c(g,h) = ghg^{-1}$

Think of this as a family $c_h : G \longrightarrow G$.
$g \longrightarrow ghg^{-1}$

Then $h \in Z(G) \iff c_h : G \to G$ is constant. Use

Lemma (Chevalley). Let X be proper and $\Phi : X \times Y \longrightarrow Z$ be a regular map which we think of as a family of regular maps $\Phi_y : X \to Z$. If $\Phi_{y_0}(X) = z_0$ is constant, then $\Phi_y(X) = z$ are constant maps $\forall y \in Y$.

Reference. Shafarevich.

Pf. Consider the graph of Φ: $\Gamma = \Gamma(\Phi) = \{(x, y, \Phi(x,y)) \subset X \times Y \times Z\}$.

Γ is closed because Z is separated.

Consider $\pi : \Gamma \longrightarrow Y \times Z$, $p : \pi(\Gamma) \longrightarrow Y$.

Goal. Show all fibers of π all have dimension $= \dim X$.

Why? Take (y, z). $\pi^{-1}(y,z) = \{(x,y,z) \mid \Phi_y(x) = z\}$. So fiber is portion of X sitting over z. If fiber is closed and Goal is satisfied,

But. $\Gamma \simeq X \times Y$, so $\dim \Gamma = \dim X + \dim Y$.

Rmk. Since X is proper, $\pi(\Gamma) \subset Y \times Z$ is closed and irreducible, so it's a closed subvariety.

Notice that $p(\pi(\Gamma)) \longrightarrow Y$ is surjective. So $\dim(\pi(\Gamma)) \geq \dim Y$.

Suffices to show $\dim(\pi(\Gamma)) = \dim Y$. For this, consider the map $p : \underset{(y_0, z_0)}{\overset{\psi}{\pi(\Gamma)}} \longrightarrow Y$. Then $p^{-1}(y_0) = (y_0, z_0)$.

A(φ), $(y_0, z) \in p^{-1}(y_0)$ iff $\exists x,\ \Phi(x, y_0) = z$, which can't happen.

<u>Claim.</u> $\dim(\pi(\Gamma)) = \dim(\Gamma)$ by upper semi-continuity of dim.

<u>Rmk.</u> Consider $\Phi : k \times k \longrightarrow xy$. This violates conditions about the theorem.

<u>We'll see.</u>

Alg. grps are non-singular varieties.

<u>Ex.</u> (Projective Gp) Elliptic Curve

$$C = V(y^2 - x(x-1)(x-\lambda)) \subset k^2$$

$$x \neq 0, 1$$

$$E = \overline{C} = V(y^2 z - x(x-z)(x-\lambda z)) = C \cup (0:1:0) \subset \mathbb{P}_k^2$$

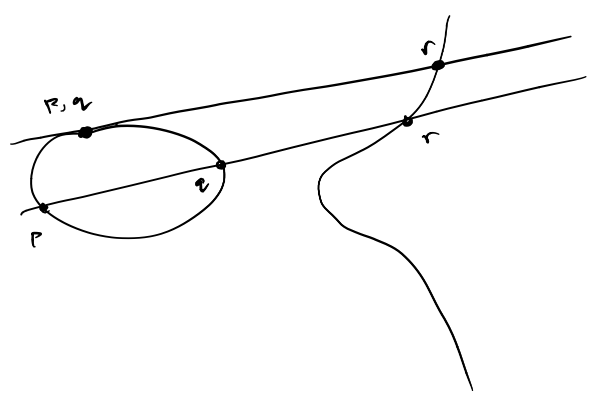
Claim. E is an alg. grp w/ $(0:1:0) = e \in E$.

$$p+q+r = e \iff p,q,r \text{ are } \underline{\text{collinear}}$$
$$(\text{in } \mathbb{P}^2_k)$$

i.e. p,q,r are the roots of a homogeneous cubic polynomial obtained by restricting F to $\ell \subset \mathbb{P}^2_k$.

$$\Phi : \mathbb{P}^1_k \xrightarrow{\sim} \ell \subset \mathbb{P}^2_k$$
$$\Phi(s:t) = (\ell_1(s,t) : \ell_2(s,t) : \ell_3(s,t))$$
$$F(\ell_1,\ell_2,\ell_3) = F(s,t) \leftarrow \text{cubic homog.}$$

$\Rightarrow$ 3 roots, counting mult.



The line at ∞

$$\ell = \{z=0\} \subset \mathbb{P}^2_k$$
$$\Phi(s:t) = (s:t:0)$$
$$F(s:t:0) = -s^3 \Rightarrow e+e+e = e.$$

Define. $-p$ (for $p \in E$) by

$$p + (-p) + e = e$$

i.e. if $p = (x, y)$, then $-p = (x, -y)$.

Define. $p+q$ via $p + q + \underset{\substack{\text{3rd pt of} \\ \overline{pq} \cap E}}{r} = e \qquad p+q = -r.$

Rmk. For $E_i \subset \mathbb{P}^2$, can form $\prod^n E_i \subset (\mathbb{P}^2)^n$, and almost every algebraic group is a "deformation" of these.

Local Properties of Varieties

$X \in \mathrm{Var}_k$ $\shortparallel$ $\bigcup U_i$	Consider the local rings $\forall x \in X$ $\mathcal{O}_{X,x} = \{\phi \in k(X) \mid \phi(x) \text{ is defined}\}$. This is a local ring w/ max ideal $m_x = \{\phi(x) = 0\}$, and has residue field $k := \mathcal{O}_{X,x}/m_x$. Frac. field $k(X)$.

For $Z \subset X$ closed subvariety, $\mathcal{O}_{X,Z} = \bigcup_{x \in Z} \mathcal{O}_{X,x} \subset k(X)$. max ideal $m_Z = \{\phi \mid \phi(Z) = 0\}$. This is a local ring, but of smaller dimension, since res. field is

$$k(Z) = \mathcal{O}_{X,Z}/m_Z.$$

Now $\dim(\mathcal{O}_{X,Z}) = \dim(X) - \dim(Z)$. For Z of codim 1...