X is non-singular at $x \in X$ if $\mathcal{O}_{X,x}$ is a regular local ring, i.e.

$$\dim_k m_x/m_x^2 = \dim \mathcal{O}_{X,x} \quad (\text{Krull})$$

$$= \text{tr. deg. } k(X)/k$$

<u>Rmk.</u> Non-singular $\Rightarrow$ normal

<u>Def.</u> m_x/m_x^2 is called the Zariski cotangent space.

<u>Observation.</u> Natural map $d: \mathcal{O}_{X,x} \longrightarrow m_x/m_x^2$

$$\phi \longrightarrow [\phi - \phi(x)].$$

d is k-linear, but not $\mathcal{O}_{X,x}$ linear. Satisfies

$$d(\phi\psi) = [\phi\psi - \phi(x)\psi(x)]$$

[Use the fact that this is equiv. class].

A set of elements

$u_1, \ldots, u_n \in m_x$ is a system of parameters

if $\langle du_1, \ldots, du_n \rangle = m_x/m_x^2$.

<u>Nakayama's Lemma.</u> $u_1, \ldots, u_n$ is a system of parameters $\Leftrightarrow \langle u_1, \ldots, u_n \rangle = m_x$.

<u>Usual form.</u> Take max ideal m of A, $\Phi: M \to N$ an A-module homomorphism. Φ surjective $\Leftrightarrow \overline{\Phi}: M/mM \to N/mN$ is.

Last time.

Said that

$$I(x)/I(x)^2 \xrightarrow{\sim} m_x/m_x^2,$$

$I(x) \subset k[U]$ is the ideal of $x \in U$.

Pf. Injectivity: $I(x) \subset m_x$

$I(x) \cap m_x^2 = I(x)^2$.

Surjectivity. Suppose $\frac{f}{g} \in m_x$; $f \in I(x)$, $s(x) \neq 0$.

Then $$\overset{m_x}{\overset{\cup}{\frac{f}{s}}} - \overset{I(x)}{\overset{\cup}{f\cdot(s(x))^{-1}}} = \frac{f(1-s\cdot s(x)^{-1})}{s} \in m_x^2.$$

So $$\frac{f}{s} = f\cdot s(x)^{-1} \in m_x/m_x^2.$$

Let $U = V(f_1,\dots,f_m) \subset k^n$. Then

$$\overset{\langle x_i - a_i\rangle}{\overset{\shortparallel}{I(a)}}/I(a)^2 = \left(\bigoplus k\cdot(x_i-a_i)\right.$$

$\sum b_i(x_i-a_i) \in I(a)^2$?

Claim. $\iff$ $\sum b_i(x_i-a_i) \in \langle \underbrace{\sum \frac{\partial f_j}{\partial x_i}(a)(x_i-a_i)}_{\text{First terms w/ Taylor poly of } f_j} \rangle$

$0 = f_j(a) = \sum \frac{\partial f_j(a)}{\partial x_i}(x_i-a_i) + (I(a))^2$.

been

First terms w/ Taylor poly of f_j.

Compare w/.

$$J(f_1, \dots, f_m)(x) = \begin{bmatrix} \frac{\partial f_i}{\partial x} \\ \frac{\partial f_1}{\partial \dots} \end{bmatrix}$$

Cor. The function

$$e : X \longrightarrow \mathbb{Z}$$

$$e(a) = \dim_k m_a / m_a^2$$

is upper-semicontinuous.

e.g. $X_r := \{ a \in X \mid e(a) \geq r \} \subset X$ is closed.

$$X_r \cap U = \{ a \in U, \ rk(J(a)) \leq n - r \}$$

$$= \left\{ a \in U \;\middle|\; \begin{array}{l} \text{all } n-r+1 \times n-r+1 \\ \text{minors of } J(x) \text{ vanish} \\ \text{at } a \end{array} \right\}$$

Observation. The minimum value of e is $r = \dim X$ (attained on an open subset of X).

Pf. Assume $X = U$ is affine.

Then consider $k(U)$; $td. = \dim(U) = n$

$\exists$ transcendentals $u_1, \ldots, u_n \in k(U)$

s.t. $k(U) / k(u_1, \ldots, u_n)$ is finite.

$\Rightarrow \exists \alpha$, $k(u_1, \ldots, u_n)(\alpha) = k(U)$.

$\Rightarrow U$ is <u>birational</u> to a hypersurface in k^{n+1}

defined by the poly. satisfied by α.

$$\alpha^d + \phi_1 \alpha^{d-1} + \cdots + \phi_d = 0$$

Clear denom's, so $f y^d + g_1 y^{d-1} + \cdots + g_d = 0$,

defining $X \subset k^{n+1}$ ← variables $u_1, \ldots, u_n, y$.

We know $U \dashrightarrow X$ s.t.

$$\begin{tikzcd} U \arrow[r, dashed] & X \\ \exists V \arrow[u, phantom, "\cup"] \arrow[ur, phantom, "\subset"] & \end{tikzcd}$$

<u>Now</u>. $e(\underline{a})$ for $a \in V$ can be computed on X.

For $X = V(h)$,

$$\dim m_a / m_a^2 = \begin{cases} n+1 & \text{if } \frac{\partial h}{\partial x_i}(a) = 0, \forall i \\ n, & \text{otherwise.} \end{cases}$$

If $\nabla h \equiv 0$, $h \mid \frac{\partial h}{\partial x_1} \Leftrightarrow \frac{\partial h}{\partial x_i} \equiv 0$.

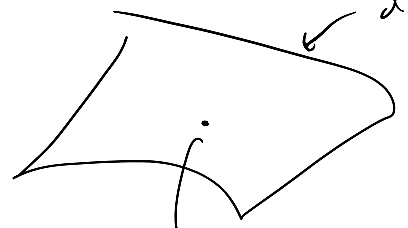
Then $h(x_1^p, \ldots, x_{n+1}^p) = 0$

$= p$-th power.

$X \subset \mathbb{P}^N$

$\Big\downarrow$ project "generic"ally

$V(H) \subset \mathbb{P}^{n+1}$



$\dim \mathfrak{m}_x/\mathfrak{m}_x^2 > \dim X$

nonsingular pts

$\dim \mathfrak{m}_y/\mathfrak{m}_y^2 = \dim X$

singularities? finite!

$$\hat{\tilde{X}} \xrightarrow{\text{desing?}} \tilde{X} \xrightarrow[\text{rational}]{r} X$$

nonsingular