Quasi-coherent sheaves

(Follows Hartshorne qc sheaves)

$A = k[X]$

M module over A

Given $S \subset A$ a multiplicative set, have

$$M_S = \left\{ \frac{m}{f} \;\middle|\; \begin{matrix} m \in A \\ f \in S \end{matrix} \right\} \Big/ \sim$$

__Consider.__ $\varphi : M \longrightarrow \coprod_{p \text{ prime}} M_p$

__Lemma.__

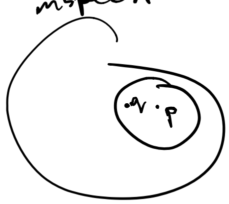
(1) φ is injective

(2) φ surjects onto the set of "coherent" elements, i.e.

$$\left(\frac{m_p}{f_p}\right) \quad \cancel{\text{s.t. } \forall \text{ prime } P \subset A}$$

$$\text{if } \forall q \not\ni f_p,$$

mspec A



$$\frac{m_p}{f_p} \sim \frac{m_q}{f_q} \quad (\text{in } M_q)$$

__Pf.__ (1) Suppose $\varphi(m) = \varphi(n)$ i.e. $m \sim n$ in M_p $\forall p$

$$\Rightarrow \exists h \text{ s.t. } h(m-n) = 0.$$

$$\Rightarrow \operatorname{ann}(m-n) \not\subset \mathbb{P}, \ \forall P \subset A$$

$\Rightarrow \operatorname{ann}(m-n) = A \Rightarrow m = n.$

(2) Suppose $\left(\ldots, \frac{m_p}{f_p}, \ldots\right)$ is a set of coherent elements of M_p's. Then $\exists$ finite set $\left\{\frac{m_i}{f_i}\right\}$, $\frac{m_i}{f_i} \in M_{f_i}$ s.t. $\forall P$, $\exists f_i \notin P$ and

$\frac{m_i}{f_i} \sim \frac{m_p}{f_p}$.

$$U_i := \operatorname{mspec}(A) \setminus V(f_i)$$

$$\frac{m_i}{f_i} \sim \frac{m_j}{f_j} \text{ on } U_i \cap U_j = \operatorname{mspec}(A_{f_i f_j})$$

Therefore $\exists n_{ij}$ s.t. $(f_i f_j)^{n_{ij}}(m_i f_j - m_j f_i) = 0$

Let $n = \max\{n_{ij}\}$. Then

$$f_i^n f_j^n (m_i f_j - m_j f_i) = 0$$

$$f_j^{n+1}(f_i^n m_i) - f_i^{n+1}(f_i^n m_j) = 0$$

$$\begin{array}{cccc} \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ f_j & m_i & f_i & m_j \end{array}$$

$$\Rightarrow m_i f_j = m_j f_i$$

Since $X = \bigcup U_i$, it follows from Null that

$\langle f_i \rangle = A$. Write $1 = \sum f_i g_i$.

Let $m = \sum g_i m_i$.

Claim. $m = \frac{m_j}{f_j}$ $\forall j$.

$$m f_j = \sum g_i m_i f_j = \sum g_i m_j f_i$$
$$= m_j \sum g_i f_i = m_j.$$

Define. $\tilde{M}$ sheaf of $\mathcal{O}_X$-modules, $X = \operatorname{mspec}(A)$.

$$\tilde{M}(U) = \left\{ \text{coherent sets } \left(\ldots, \frac{m_p}{f_p}, \ldots\right) \text{ for } p \text{ s.t. } V(p) \cap U \neq \phi \right\}$$

Thm. $\tilde{M}(X) = M$ (by (2))

$\tilde{M}(U_f) = M_f$

Also, $(\tilde{M})_p = \varinjlim_{f \notin p} M_f = M_p$.

stalk at $V(p)$

Def. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is quasi-coherent if the sections $\mathcal{F}(U) \subset \coprod_{\substack{Z \cap U \\ Z \neq \emptyset}} \mathcal{O}_{X,Z}$ are coherent sets.

$\mathcal{F}(U \cap \underbrace{U_i}_{= \text{mspec}(A_i)})$ are coherent systems for some module M over A_i.

$$\mathcal{F} \rightsquigarrow \overbrace{M_i}^{= \mathcal{F}(U_i)} \quad \text{on } U_i = \text{mspec}(A_i)$$

affine open cover $X = \bigcup U_i$

$\uparrow$ module over A_i

$$\mathcal{F}|_{U_i} = \tilde{M}_i$$

Ex. $\mathcal{F}$ is locally free if $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module $\forall x \in X$.

Rmk. The rank of a locally free qc sheaf of $\mathcal{O}_X$-modules is well-defined, ind. of x.

If M_p is free, then M_f is free for some $f \notin P$. $A_p^r \simeq M_p$ $\leftarrow$ module / local ring $\supset \{\frac{m_i}{f}\}$

$\Rightarrow A_f^r \simeq M_f^r$

<u>Def.</u> coherent if finite rank.

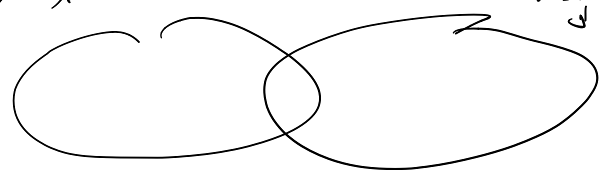
In that case, then we may cover

$$X = \bigcup u_i \quad u_i = \operatorname{mspec}(A_i)$$

$$\mathcal{F}(u_i) \simeq A_i^r.$$

$(e_1, \dots, e_r) = A_i^r$ $\qquad\qquad$ $A_j^r = (f_1, \dots, f_r)$

u_i $\qquad\qquad$ u_j



$$f_i = \sum \phi_{ij} e_i$$

$$\phi_{ij} \in \mathcal{O}_X^*(u_i \cap u_j).$$

$\Rightarrow \phi_{ij}$ transition matrix.