$X$ variety

$X$ is normal along $Z \subset X$ if $\mathcal{O}_{X,Z}$ is integrally closed (in $k(X)$).

<u>Seen</u>. If $\mathcal{O}_{X,x}$ int closed for some $x \in Z$, then $\mathcal{O}_{X,Z}$ is. (Localization & integral closure commute).

<u>Def</u>. $X$ is <u>normal</u> if $X$ is normal at every $x$.

<u>Prop</u>. Let $X$ be a variety. The set $\{x \in X \mid X \text{ normal at } x\}$ is open and non-empty.

<u>Pf</u>. $r: \hat{X} \longrightarrow X$, the normalization is birational so $\hat{X}, X$ share a common open.

$$\left(\begin{array}{ccc} r: \tilde{X} & \longrightarrow & X \\ & & \cup \\ r^{-1}(U) & \xrightarrow{\sim} & U \end{array}\right)$$

So $U$ is normal $\iff$ $X$ is normal at each $x \in U$.

Suppose $X$ is normal at $x \in X$, choose an affine <u>nbhd</u> $x \in V \subset X$.

Then
$$\begin{array}{ccc} k[V] & \subset & \overline{k[V]} \\ \cap & & \cap \\ \mathcal{O}_{X,x} & & \\ \cap & & \\ k(X) & & \end{array}$$

Let $\phi_1, \ldots, \phi_n$ generate $\overline{k[V]}$ as a $k[V]$-module. Then

$\phi_i \in \mathcal{O}_{X,x}$ so $\phi_i = f_i/g_i$, $g_i(x) \neq 0$. Let $g = \prod_{i=1}^{n} g_i$. Then

$$\overline{k[V]_g} = \left(\overline{k[V]}\right)_g = k[V]_g.$$

So $\overset{x}{\underset{\cap}{V_g}} \subset V$ is <u>normal</u>.

**Thm.** $X = m\mathrm{Proj}(R_\bullet)$, $\left[ R_0 = k,\ R \text{ gen by } \langle x_0, \dots, x_n \rangle = R_1 \right]$

then $\tilde{X}$ is also proj.

**Pf.** Idea: Morally, $\tilde{X} = m\mathrm{Proj}(\overline{R_\bullet})$.

Claim. $\overline{R_\bullet}$ is graded.

Pf. $R_\bullet \overset{\text{as a graded ring}}{\subset} k(X)[x] \subset k(X)(x) = k(C(X))$ provides the grading

$x \in R_1 \setminus \{0\}$

$$Q_\bullet := \overline{R_\bullet} \subset k(X)[x] = \overline{k(X)[x]} \subset k(C(X))$$

$R_\bullet \subset Q_\bullet$ as a graded ring, but

$Q_\bullet$ may not be generated in degree 1.

Let $y_0, \dots, y_m$ generate $Q_\bullet$, $y_i \in Q_{d_i}$;

$\deg(y_i) = d_i$.

Ex. $Q_\bullet$ freely gen. by $\deg(y) = 2$, $\deg(z) = 3$,

$\deg(w) = 5$. Then $Q_\bullet = k[y, z, w]$

<u>Last time</u>: There is a value $D$ such that

$$Q_{D^\bullet} = Q_0 \oplus Q_D \oplus Q_{2D} \oplus \cdots$$

<u>is</u> generated by $Q_D$.

Let $d = \operatorname{lcm}\{d_i\}$; $d = d_i e_i$, some $e_i$. Then

$D = d \cdot (m+1)$ will do.

$Q_{kD}$ as a $k$-vector space is spanned by monomials in $y_0, \ldots, y_m$ of weighted degree $k \cdot d \cdot (m+1)$.

Thus our task is to see that such monomials are generated by $Q_D = Q_{d(m+1)}$.

Given such monomials $y_0^{s_0} \cdots y_m^{s_m}$ $(k > 1)$

$$\begin{aligned} \sum s_i d_i &= kd(m+1) \\ &\geq 2(d(m+1)) \\ &= \sum d_i e_i + d(m+1) \end{aligned}$$

then some $s_1 \geq e_i$, WLOG $i=1$. Pull out $\underbrace{y_{i_1}^{e_{i_1}}}_{\deg d}$.

Repeatable $m+1$ times, so we get

$$\underbrace{(y_0^{e_{i_0}} \cdots y_m^{e_{i_m}})}_{\in\, Q_D} \underbrace{(\text{monomial})}_{\in\, Q_{(k-1)D}}$$

Ex. $d=30$, $m+1=3$.

$$Q_0 \oplus Q_{90} \oplus Q_{180} \oplus \cdots$$

Easy $Q = k[x,y]$
$\uparrow\ \uparrow$
$1\ \ 2$

$$Q_0 \oplus Q_4 \oplus Q_8 \oplus \cdots$$

$\langle x^4, x^2y, y^2\rangle$ $\quad k[f,g,h]/\langle g^2-fh\rangle$

$f \quad g \quad h \qquad g^2=fh$

Idea. Instead of $R_\bullet \subset Q_\bullet$

$$\underbrace{R_{D\bullet} \subset Q_{D\bullet}}_{\text{finite}} = \overline{(R_{D\bullet})}$$

$$\mathrm{mProj}(R_\bullet) \simeq \mathrm{mProj}(R_{D\bullet}) \xleftarrow[\text{(normalization)}]{\text{finite}} \mathrm{mProj}(Q_{D\bullet}).$$

$R_D = \langle x_0, \dots, x_N\rangle$

$\cap$

$Q_D = \langle x_0, \dots, x_N, \underbrace{x_{N+1}, \dots, x_M}_{\text{integral over } R_D}\rangle$

$$(0:\cdots:0:x_M) \in \mathrm{mProj}(Q_{D\bullet}) \subset \mathbb{P}^M$$
$$\downarrow \text{projection}$$
$$\mathrm{mProj}(R_{D\bullet}) \subset \mathbb{P}^N$$

## Non-singular varieties

Let $X$ be a variety

$X$ is nonsingular along $Z \subset X$ if

$m_Z \subset \mathcal{O}_{X,Z}$ is a regular local ring, i.e.

$$\dim_{\mathcal{O}/m_Z} (m_Z/m_Z^2) = \dim \mathcal{O}_{X,Z}$$ ← chain of prime ideals.

↑ as a vector space

Review.

1. Regular local rings localize to regular local rings. (e.g. if $\mathcal{O}_{X,x}$ is regular, then $\mathcal{O}_{X,Z}$ is)
2. RLR are integrally closed.

If $V \subset k^n$ is an affine variety, $I(V) = \langle f_1, \dots, f_m \rangle$

then for all $\underline{a} = (a_1, \dots, a_n) \in V$,

$$\dim (m_x/m_x^2) = \dim \ker \left| \frac{\partial f_i}{\partial x_j}(a) \right|.$$

This shows that

$\dim (m_a/m_a^2)$ is an upper-semi cont

function $e : X \to \mathbb{Z}$.

Idea. Let $I(a)$ = ideal of $a \in V$.
$\subset k[V]$.

$m_a \in \mathcal{O}_{V,a} = k[V]_{I_a}$

Then $I(a)/I(a)^2 \xrightarrow{\sim} m_a/m_a^2$.

Show directly that $I(a) = \langle x_i - a_i \rangle$

$$I^2(a) = \left\langle \frac{\partial f_i}{\partial x_j}(a)(x_j - a_j) \right\rangle.$$