

$\mathcal{F} \in \text{Sh}_X$, sheaves on X . We then took an injective res

$$0 \rightarrow \mathcal{F} \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

We can look at

$$0 \rightarrow 0 \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

and apply global sections:

$$0 \rightarrow 0 \rightarrow \Gamma(I^0) \rightarrow \Gamma(I^1) \rightarrow \dots$$

Then we defined $H^p(X, \mathcal{F}) = H^p(\Gamma(I^\bullet))$. Some remarks about spectral sequences.

We want to form a category containing our abelian category A and all corresponding resolutions. and objects are isomorphic to their resolutions.

We can try category of complexes $C^*(A)$:

$$\longrightarrow c^{i-1} \rightarrow c^i \xrightarrow{d} c^{i+1} \rightarrow \dots c^i \in \text{Obj}(A),$$

$d^2=0$, where $* = \emptyset, b, +, -$; where b means bounded complexes, $+$ is bounded below, $-$ is bounded above, and \emptyset means no restrictions. The morphisms are families of morphisms with a commutation restriction.

In this category, we already have a desired property. Consider, for example,

$$\cdots \rightarrow \cdots \rightarrow 0 \xrightarrow{\deg^0} A \rightarrow 0 \rightarrow \cdots$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$0 \rightarrow 0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$$

This morphism is injective (?), and we can embed A into $C^*(A)$.

Let B be a category and S a family of morphisms of B . Consider all functors $F: B \rightarrow C$ such that $F(S)$ consists of only of isomorphisms.

Question. Does there exist $\mathcal{Q}, S^{-1}B$ such that

$$\begin{array}{ccc} & B & \\ \mathcal{Q} \searrow & \downarrow & \\ S^{-1}B & & \swarrow F \\ & C & \end{array}$$

Objects of $S^{-1}B$ are objects of B , and morphisms are complicated, but it does always exist.

In $C^*(A)$, we can define $H^p(C^\bullet) = \ker d^p / \text{im } d^{p-1}$. If $f: C^\bullet \rightarrow D^\bullet$ is a morphism of complexes, we get $H^p(f^\bullet): H^p(C^\bullet) \rightarrow H^p(D^\bullet)$.

Def. f^\bullet is a quasi-isomorphism if all $H^p(f^\bullet)$ are isomorphisms.

If we take ε^\bullet to be the morphism from $\cdots 0 \rightarrow A \rightarrow 0 \cdots$ to $\cdots I^0 \rightarrow I^1 \cdots$

is a quasi-isomorphism.

Def. The derived category $D^*(A)$ is the localization of $C^*(A)$ with $S = \{\text{all quasi-isomorphisms}\}$.

Rmk. $D^*(A)$ is not necessarily abelian, and morphisms are mysterious.

Morphisms

Morphisms and compositions are paths and killing the S allows certain edges to be invertible.

Ex. Look at A -modules. $\mathcal{B} \subset \mathcal{A}$ is a subcategory.

$S = \{\text{morphisms in } A \text{ with } \ker \text{ in } \mathcal{B}, \text{ im in } \mathcal{B}\}$.

To talk about this, we need to talk about the

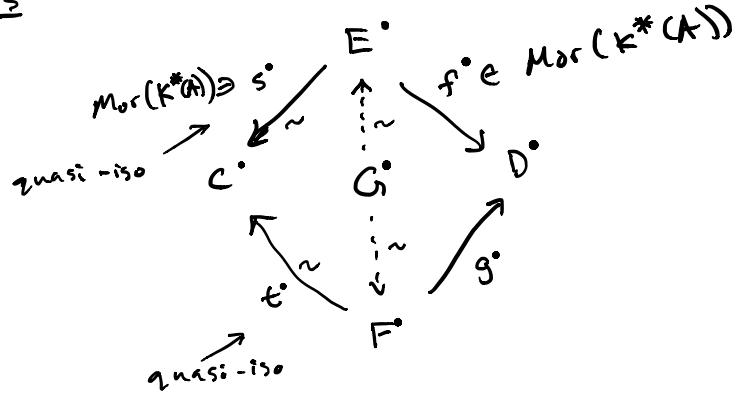
$K^*(A)$, the homotopic category of complexes.

Def. Take two complexes C^\cdot, D^\cdot and $f, g: C^\cdot \rightarrow D^\cdot$,
then f°, g° are homotopic if $\exists h_i: C^i \rightarrow D^{i+1}$
if $dh^i + h^i d = f^\circ - g^\circ$. Being homotopic is an
equivalence relation on maps.

Rmk. This kills the abelian-ness. Also, $H^p(f^\circ) = H^p(g^\circ)$,
if $f^\circ \sim g^\circ$.

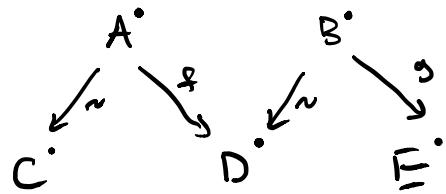
Take $\varphi: C^\cdot \rightarrow D^\cdot$ a morphism in $D^*(A)$. What
does this look like?

roots

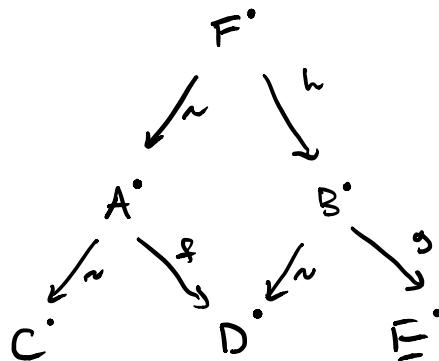


If the above diagram commutes, we say that the roots (E^*, s^*, f^*) , (F^*, t^*, g^*) are equivalent (if G^* exists).

composition



In homotopic category, $\exists F^*$ s.t.



s.t. $FADB$ commutes. \rightsquigarrow indicates

quasi-iso's, which are closed under composition.

Question. How can this be useful?

Thm. $D^*(A)$ is a triangulated category,

which means in this category there are distinguished triangles.

$\exists T: D^*(A) \hookrightarrow T(X^*)^i = X^{i+1}$ such that
 $d_{T(X^*)}^i = -d_{X^*}^{i+1}$

Def. A distinguished triangle is a

$$x^* \rightarrow y^* \rightarrow z^* \rightarrow T(x^*),$$

which is represented as

$$\begin{array}{ccc} & z^* & \\ \swarrow & & \uparrow \\ x^* & \longrightarrow & y^* \end{array}$$

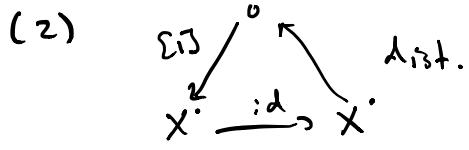
Def. Morphism of triangles is a family of maps

$$\begin{array}{ccccccc} x^* & \longrightarrow & y^* & \longrightarrow & z^* & \longrightarrow & T(x^*) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ x'^* & \longrightarrow & x''^* & \longrightarrow & y''^* & \longrightarrow & T(x'^*) \end{array}$$

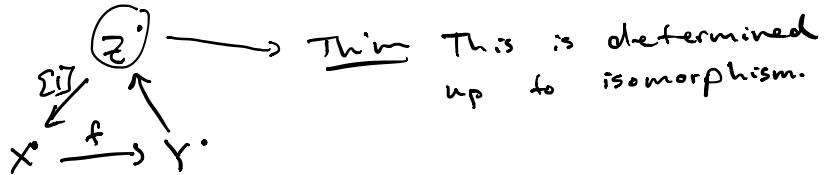
Claim. This is an additive category.

Axioms of D.T.

- (1) Any triangle iso. to a dist. triangle is distinguished.



(3) For any $f: X^{\cdot} \rightarrow Y^{\cdot}$, \exists d.t.



(4)

$$X^{\cdot} \xrightarrow{\Sigma\mathbb{I}} Y^{\cdot} \xrightarrow{s^{\cdot}} Z^{\cdot} \text{ is d.t.} \Leftrightarrow Y \xrightarrow{T(f)} Z \xrightarrow{T(s)} T(X^{\cdot}) \text{ is d.t.}$$

(5)

$$\begin{array}{ccccccc} X & \rightarrow & Y & \rightarrow & Z & \rightarrow & T(X) \\ f \downarrow & & \downarrow g & & \downarrow & & \downarrow T(f) \\ X' & \rightarrow & Y' & \rightarrow & Z' & \rightarrow & T(X') \end{array}$$

The above diagram commutes

(6) octahedral axiom.

Def. $F: \mathcal{C} \rightarrow \mathcal{A}$

$$\begin{array}{c} \text{abelian} \\ \downarrow \\ \mathcal{A} \end{array}$$

↑
triangulated

F is a coho functor if for every d.t.

$$\begin{array}{ccc} & \Sigma\mathbb{I}^0 & \\ \Sigma\mathbb{I} & \swarrow & \nearrow \\ X & \xrightarrow{\quad} & Y \end{array}$$

we have $F(X) \rightarrow F(Y) \rightarrow F(Z)$ is exact.

Claim. $H^0(-): D^*(A) \rightarrow A$ is a coh. functor.

\Rightarrow long exact seq. of coh. groups of a dist. triangle.