$Q := V(F) \iff F(a_0,\dots,a_n) = 0 \iff \underline{a}$ is an isotropic vector for Γ

$\underline{a} \in \mathrm{Sing}(Q) \iff \underline{a} \in \ker \Gamma$

Gramm-Schmidt $\Rightarrow \exists$ basis $v_0,\dots,v_n$ for k^{n+1} s.t.

$$\Gamma \sim \begin{pmatrix} \underbrace{1 \ \ddots \ 1}_{m} & \\ & 0 \ \ddots \ 0 \end{pmatrix}, \quad F \sim x_0^2 + \dots + x_m^2 .$$

So $\ker \Gamma = \langle v_{m+1}, \dots, v_n \rangle$. Hence <u>all</u> non-singular quadrics are equivalent under change of basis.

Consider the space of all quadrics $\mathbb{P}_k^{\binom{n+2}{2}-1}$.

Basis is the symmetric tensors: $e_0^{\otimes 2}, \dots, e_n^{\otimes 2}$, $e_i \otimes e_j + e_j \otimes e_i$

<u>Rmk</u>. The <u>veronese</u> map $v_{n,2} : \mathbb{P}_k^n \longrightarrow \mathbb{P}_k^{\binom{n+2}{2}-1}$

$$\sum a_i e_i \longrightarrow \left(\sum a_i e_i\right) \otimes \left(\sum a_i e_i\right)$$

In coordinates,

$$(a_0 : \dots : a_n) \longrightarrow (a_0^2 : \dots : \underset{i<j}{a_i a_j} : \dots a_n^2)$$

<u>Observation</u>. The image of $v_{n,2}$ is an intersection of quadrics in $\mathbb{P}_k^{\binom{n+1}{2}-1}$

$\underline{\text{Ex}}$ $(n=1)$: $\quad \nu_{1,2}: \mathbb{P}^1_k \longrightarrow \mathbb{P}^2_k$

$$(a_0 : a_1) \longrightarrow (\underset{y_0}{a_0^2} : \underset{y_1}{a_0 a_1} : \underset{y_2}{a_1^2}).$$

Image is $\overbrace{V(y_1^2 - y_0 y_2)}^{\text{the non-singular}}$ conic in $\mathbb{P}^2_k$.

Conversely, if $b_1^2 - b_0 b_2 = 0$, $\frac{b_1}{b_0} = \frac{b_2}{b_1}$.

WLOG, let $b_0 = 1$, $b_1^2 = b_2$. Then let $a_0 = b_0$, $a_1 = b_1$.

$(n=2)$ $\quad \nu_{2,2}: \mathbb{P}^2_k \longrightarrow \mathbb{P}^5_k$

$$(a_0 : a_1 : a_2) \longrightarrow (a_0^2 : a_0 a_1 : a_0 a_2 : a_1^2 : a_1 a_2 : a_2^2)$$

$\nu_{2,2}(\mathbb{P}^2_k) =$ rank one locus of

$$A = \begin{bmatrix} x_{00} & x_{01} & x_{02} \\ x_{01} & x_{11} & x_{12} \\ x_{02} & x_{12} & x_{22} \end{bmatrix}, \text{ i.e.}$$

the intersection of all quadrics given by 2×2 minors.

This is also the locus of "quadrics" in $\mathbb{P}^2_k$ that are singular along a line

$\underline{\text{Observation}}$. There are only 5 linearly indep. 2×2 matrices.

$\mathbb{P}_k^2 \subset \mathbb{P}_k^5$

dim = 2 dim = 5

$X \subset \mathbb{P}_k^n$ has codim m
and $I(X)$ gen. by m poly's, then
X is a complete intersection

(iii) $V(\det(A))$ is a cubic hypersurface
in $\mathbb{P}_k^5$, which parameterizes lines.

And. $\mathrm{Sing}(V(\det A)) = V_{2,2}(\mathbb{P}_k^2)$.

Generalize (d-uple embeddings)

Fix V a vector space of dim $n+1$.

$$\nu_{n,d} : \mathbb{P}(V) \longrightarrow \mathbb{P}(\mathrm{Sym}^d V)$$
$$\lambda \underline{v} \longrightarrow (\lambda \underline{v})^{\otimes d} = \lambda^d (v^{\otimes d})$$
$$(a_0 : \cdots : a_n) \longrightarrow (a_0^d : a_0^{d-1} a_1 : \cdots)$$
$$\prod a_i^{d_i} \text{ s.t. } \sum d_i = d.$$

Segre embedding V, W vector spaces

$$S(V, W) = \mathbb{P}(V) \times \mathbb{P}(W) \longrightarrow$$
$$(\lambda \underline{v}, \mu \underline{w}) \longrightarrow \lambda\mu\, \underline{v} \otimes \underline{w}.$$

For instance

$$s_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

$$(a_0:a_1),(b_0:b_1) \longrightarrow (\overset{x}{a_0b_0} : \overset{y}{a_0b_1} : \overset{z}{a_1b_0} : \overset{w}{a_1b_1})$$

$\text{Image} = V(xw - yz)$

<u>Grassmanian</u>.

$G(m,V) \subset \mathbb{P}(\Lambda^m V)$ $\quad$ $(\dim V = n > m)$

$=$ {decomposable tensors},

i.e. elements are $\sum_{\sigma \in S_n} (-1)^{\mathrm{sign}\,\sigma}\, \underline{v}_{\sigma(1)} \otimes \cdots \otimes \underline{v}_{\sigma(n)}$. $\quad$ $\underline{v}_1, \ldots, \underline{v}_m \in V$ (lin. indep.)

There is a natural bijection

$$\left\{ \begin{array}{c} m\text{-dimensional} \\ \text{subspaces } U \subset V \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{points of} \\ G(m,V) \end{array} \right\}$$

$m=1$ $\quad G(1,V) = \mathbb{P}(V)$

$m=n-1$ $\quad G(n-1,V) = \mathbb{P}(V)$

Grassmanian is an intersection of quadrics, called Plücker quadrics

There is a map

$$V \times \cdots \times V \dashrightarrow G(m,V)$$

$$(\underline{v}_1, \ldots, \underline{v}_m)$$

$\bigcup$

$\mathcal{U} =$ {indep. vectors}

send vectors to their span.

<u>key obs.</u> $GL(V)$ acts freely on $\mathcal{U}$ with <u>quotient</u> $G(m,V)$.

$\dim = mn$

$$\begin{tikzcd} & \arrow[d] \\ GL(V) \arrow[r] & \mathcal{U} & \arrow[l] \text{fibration} \\ \arrow[u] \text{dim}=m^2 & \arrow[d] \\ & Gr(m,V) \rightsquigarrow \text{dim} = mn - m^2. \end{tikzcd}$$

Ex. $Gr(2,4) \subset \mathbb{P}^5$, w/ dim 4, so it is a quadric.

$$\begin{bmatrix} 0 & x_{01} & x_{02} & x_{03} \\ -x_{01} & 0 & x_{12} & x_{13} \\ -x_{02} & -x_{12} & 0 & \\ -x_{03} & & & 0 \end{bmatrix}$$

$Gr(2,4) = V(\text{Pfaffian}(\quad))$.