$\mathcal{A}$ abelian category

$D(\mathcal{A})$ derived category — additive

- triangulated

Def. A functor $D(A) \xrightarrow{F} D(B)$ is <u>exact</u> if it is additive and sends distinguished triangles to distinguished triangles.

$$\begin{tikzcd}[column sep=small] & Z^\cdot \arrow[dl, "{[1]}"'] & \\ X^\cdot \arrow[rr] & & Y^\cdot \arrow[ul] \end{tikzcd} \longrightarrow \begin{tikzcd}[column sep=small] & F(Z^\cdot) \arrow[dl] & \\ F(X^\cdot) \arrow[rr] & & F(Y^\cdot) \arrow[ul] \end{tikzcd}$$

Def. $F: D(A) \longrightarrow B$ is a cohomological functor if

$$\begin{tikzcd}[column sep=small] & Z^\cdot \arrow[dl, "{[1]}"'] & \\ X^\cdot \arrow[rr] & & Y^\cdot \arrow[ul] \end{tikzcd} \xrightarrow{\quad F \quad} F(X^\cdot) \longrightarrow F(Y^\cdot) \longrightarrow F(Z^\cdot)$$

is exact.

$$\Rightarrow \quad F(X^\cdot) \to F(Y^\cdot) \to F(Z^\cdot) \to F(T(X^\cdot)) \longrightarrow F(T(Y^\cdot)) \longrightarrow F(T(Z^\cdot)) \to \cdots$$

is exact.

Def. $H^p: X^\cdot \longrightarrow H^p(X^\cdot)$

Claim. H^0 is a cohomological functor. $H^p = H^0 \circ T^p \Rightarrow$ long exact sequence

Take k to be a field. Vect_k vector spaces over k. The functor $V \longrightarrow \mathrm{Hom}(V, k)$ is exact. Vect_k^{fd} finite-dim. We have two functors

$$V \longrightarrow V$$

$$V \longrightarrow (V^*)^*. \qquad V^{**} \simeq V.$$

R is a commutative ring with 1. $\mathcal{M}(R)$ modules over R.

<u>Def.</u> $V^* = \mathrm{Hom}(V, R)$.

Then $V \to V^*$ is a contravariant functor $\mathcal{M}(R) \circlearrowleft$ which is left exact. Note that it can happen that $V \neq 0$, $V^* = 0$, and $V \to (V^*)^*$ is not an iso. of functors. Of course, recall we can fill in a left exact sequence with Ext.

Consider $\mathcal{M}_{fg}(R)$. Then $(R^p)^{**} \simeq R^p$. Also, $R^* = \mathrm{Hom}(R, R)$, so $R^* \simeq R$. Assume now that R is noetherian. Then given V, we can construct a left-projective resolution.

Note that if P is projective, $P \oplus Q \simeq R^p$.

$$\begin{tikzcd} P \arrow[d] \oplus Q \arrow[d] \simeq R^p \arrow[d, "\simeq"] \\ P^{**} \oplus Q^{**} & (R^p)^{**} \end{tikzcd}$$

So $P^{**} \simeq P$. Assume R also has finite homological dim.

<u>Ex.</u> Polynomial rings have finite homological dimension (szygy theorem).

<u>Def.</u> $D^b(R) = D^b(\mathcal{M}_{fg}(R))$. $\quad P^\cdot \xrightarrow{\sim} V^\cdot$

$\mathbb{D} : D^b(R) \circlearrowleft$. $\quad \mathbb{D}(V^\cdot) = \mathbb{D}(P^\cdot) = (P^\cdot)^{**}$

$\underline{\mathbb{D} \circ \mathbb{D} \cong \mathrm{id}}$

This a replacement of the fact that $V \simeq V^{**}$.

Let V be fin. gen. module, $D(V)$ be the cplx

$$\cdots \quad 0 \longrightarrow V \longrightarrow 0 \quad \cdots$$

Then $H^p(D(D(V))) = \mathrm{Ext}^p_R(V, R)$. If all Ext are 0, then $0 \longrightarrow D(V)$ is a quasi-isomorphism. So if $V \neq 0$, some Ext is non-zero.

$\mathrm{Ext}^p_R(\mathrm{Ext}^q_R(V, R), V) \Rightarrow \mathrm{Gr}^{p+q} V$ in the sense of spectral sequence.

<u>Derived Functors</u>

$A \xrightarrow{F} B$ is left exact. Construct derived categories $RF: D^b(A) \longrightarrow D^b(B)$. Let $V \in \mathrm{Obj}\, A$. Then $D(V) \in D^b(A)$.

$$(R^pF)(V) = H^p(RF(DV))$$