mspec: k-algebras $\longrightarrow \mathcal{N}_k$

$$\mathrm{mspec}(A) = \{\text{max ideals in } A\}$$

$$\mathrm{mspec}(f: A \to B) = \underline{\Phi}: \mathrm{mspec}(B) \longrightarrow \mathrm{mspec}(A)$$

$$\underline{\Phi}(m_y) = f^{-1}(m_x)$$

mproj: graded k-algebras $R_\bullet$ $\left(\begin{array}{c} R_0 = k \\ \text{generated by } R_1 \end{array} \right) \longrightarrow \begin{array}{c} \mathcal{P}_k \\ \cap \\ \mathcal{N}_k \end{array}$

$$\mathrm{mproj}(R_\bullet) = \{\text{max homog. prime ideals } m_x \supsetneq R_+\}$$

$$R_\bullet \xrightarrow{p_x} k[x]_\bullet$$

$$\phi = \frac{F}{G} \longrightarrow \phi(x) = \frac{p_x(F)}{p_x(G)}$$

$$\mathrm{mproj}(f_\bullet: R_\bullet \to Q_\bullet) \rightsquigarrow \underline{\Phi}(m_y) = f^{-1}(m_y).$$

<u>Problem</u>. $f^{-1}(m_y)$ may not be a maximal prime. Instead, it may satisfy $\sqrt{f_\bullet^{-1}(m_y)} = R_+$.

<u>Consider</u>. $f: k[x] \longrightarrow S$

$x \longrightarrow x_0 \in S_1$

$\mathrm{mproj}(S) = \mathbb{P}^n_k$

$\mathrm{mproj}(k[x]) = pt.$

$\mathrm{mproj}(f): \mathbb{P}^n_k \longrightarrow pt.$

$m_y = \langle H \rangle \subset S, \quad H \subset S_1$

$H \subset \langle x_0, \dots, x_n \rangle$

(i) $x_0 \notin H \Rightarrow f^{-1}(\langle H \rangle) = 0 = m_{k[x]}$

(ii) $x_0 \in H \Rightarrow f^{-1}(\langle H \rangle) = \langle x \rangle = k[x]_+$

__Exception__. if $f: R_\bullet \to Q_\bullet$ is surjective, then

$\mathrm{mproj}(f): \mathrm{mproj}(Q_\bullet) \to \mathrm{mproj}(R_\bullet)$

__Ex__. $S \to R_\bullet = k[X]$, $X \subset \mathbb{P}^n_k$.

__Deep question__. Which $R_\bullet$ have $X = \mathrm{mproj}(R_\bullet)$ for a given X?

__Ex (HW).__ $\mathrm{mproj}(R_\bullet) \simeq \mathrm{mproj}(R_{d\bullet})$, $R_{d\bullet} = \bigoplus_{k=0}^{\infty} R_{dk}$ in P_k.

__Definition__. $QP_k \subset N_k$ consists of the objects $(X, \mathcal{O}_X)$ that are isomorphic $(U, \mathcal{O}_U)$ for some open subset of a projective variety.

__Ex__. Show that the qp variety $\mathbb{P}^2_k \setminus (0:0:1)$ is neither projective or quasi-affine.

$$\begin{array}{ccc} P_k & \subset & QP_k \\ & & \cup \\ A_k & \subset & QA_k \end{array}$$

__Reality check__.

How to think about a morphism to $\mathbb{P}^n_k$?

Let X be a quasi-projective variety

$$\Phi: X \to \mathbb{P}^n_k, \qquad \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n}) = k$$

$$\begin{array}{ccc} \cup & & \cup \\ U := \Phi^{-1}(U_0) & \to & U_0 = \{(1, x_1, \dots, x_n) \mid \cdots\} \end{array}$$

Assume $\Phi^{-1}(U_0) \neq \phi$, then $\Phi|_U = (\phi_1, \dots, \phi_n)$. Then $\phi_i \in \mathcal{O}_X(U) \subset K(X)$. $\Phi|_U : U \longrightarrow \mathbb{P}^n_k$

" $(1 : \phi_1 : \dots : \phi_n)$

<u>And</u>. Φ is an <u>extension</u> of $\Phi|_U$ to all of X.

<u>Def.</u> A rational map $\Phi : X \dashrightarrow \mathbb{P}^n_k$ is given by $\Phi = (\Phi_0 : \dots : \Phi_n)$. Apparent domain is

$$\bigcap \operatorname{dom} \Phi_i \setminus V(\Phi_0, \dots, \Phi_n).$$

Actual domain may be larger.

<u>Rmk.</u> $(\phi_0 : \dots : \phi_d)$ for any ϕ.

<u>Examples.</u>

<u>projection</u>: $\pi : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^1_k$

$$\pi(x_0 : x_1 : x_2) = \left(1 : \frac{x_1}{x_0}\right).$$

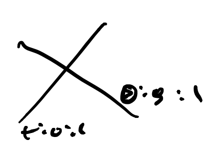
This is defined where $x_0 \neq 0$. But also equals $\left(\frac{x_0}{x_1} : 1\right)$, which is defined at $x_1 \neq 0$.

Write $\pi(x_0 : x_1 : x_2) = (x_0 : x_1)$. But x_0 is not a function on $\mathbb{P}^2_k$.

$\pi(t : 0 : 1) = (1 : 0)$

$\pi(0 : s : 1) = (0 : 1)$

let $t, s \to 0$??



Let $Y = V(x_0x_2 - x_1^2) \subset \mathbb{P}^2$. If $\pi : Y \dashrightarrow \mathbb{P}^1_k$ by

$$(x_0 : x_1 : x_2) \longrightarrow \left(1 : \frac{x_1}{x_0}\right)$$
$$= \left(\frac{x_0}{x_1} : 1\right)$$
$$= \left(\frac{x_1}{x_2} : 1\right)$$

$\to$ since $\dfrac{x_0}{x_1} = \dfrac{x_1}{x_2}$ in $k(Y)$.

<u>Moral</u>. There can be morphisms $\Phi : \underset{\mathbb{P}^m_k}{\overset{\cup}{X}} \longrightarrow \mathbb{P}^n_k$ that <u>do</u> <u>not</u> extend.

<u>Def</u>. Let X, Y be quasi-projective varieties. Then any morphism $\Phi : \underset{X}{\overset{\cap}{U}} \longrightarrow Y$ ($\to$ non-empty, open in X) is deemed a rational map $\Phi : X \dashrightarrow Y$.