Th'm Given X a variety, there is a universal normalization $r : \tilde{X} \longrightarrow X$ such that

(i) $\hat{X}$ is normal (each $\mathcal{O}_{X,Z}$ is integrally closed)

(ii) r is finite & birational

(iii) $\Phi : Y \longrightarrow X$ is dominant, then

$$\Phi : Y \dashrightarrow \tilde{X} \longrightarrow X$$

unique (implies r is unique)

(iv) If X is affine/proj, so is $\tilde{X}$.

Pf. Start with $k[X] \subset \overline{k[X]} \subset k(X)$

Goal: To show $\overline{k[X]}$ is a fin. gen. $k[X]$-module.

Proposition. Suppose $k[X] \subset k(X) \subset L$ finite separable field ext.

Then the integral closure of $k[X]$ in L is a fin. gen. module over $k[X]$.

Pf (Step 1): Assume $k[X]$ is integrally closed in $k(X)$. Choose a basis $\alpha_1, \dots, \alpha_n$ for $L/k(X) = K$. There is a dual basis $\beta_1, \dots, \beta_n$ for the trace map

$$tr_{L/K}(\alpha, \beta) := tr_{L/K}(\alpha\beta).$$

separable $\Longleftrightarrow$ this is non-degenerate.

$\operatorname{tr}(\alpha_i \alpha^{-1}) = \operatorname{tr}(1) = n.$

Rmk 1. $k(A) = L$, due to following. Let $S = k[x] \setminus 0$. We saw

$$\overline{(k[x]_S)} = \overline{k(x)} = L$$

in L

$$\parallel$$

$$\left(\overline{k[x]}\right)_S = A_S.$$

in L

As a result of this remark, we can multiply $\alpha_1, \dots, \alpha_n$ by some $f \in k(x)$ to get $f\alpha_i \in A$.

So just assume $\alpha_1, \dots, \alpha_n \in A$.

Rmk 2. $\operatorname{tr}(a) \in k[x]$ for all $a \in A$.

a is integral over $k[x]$, so a satisfies a monic poly w/ coeff's in $k[x]$

$$a^m + b_1 a^{m-1} + \dots + b_n = 0.$$

All roots integral over $k[x] \Rightarrow$ the coeff's are, so they belong to $k(x)$.

Then $A \subset \sum \beta_i k[x]$, β_i dual basis. Why?

Given $a \in A$, write $a = \sum \gamma_j \beta_j$

$\gamma_j \in k$ (in $k[x]$?) ; β_j basis for L/k.

To check this, $\operatorname{tr}(a\alpha_i) = \gamma_i \in A$ by Remark 2.

ii) General case, $k[X]$ not necessarily integrally closed.

$$\underbrace{k[x_1,\dots,x_d] \subset k[X]}_{\text{finite}} \subset A \subset L$$

$\downarrow$ noether normal

Since $k[x_1,\dots,x_d]$ UFD, int. closed, so A ~~B~~ is a finite $k[x_1,\dots,x_d]$ ~~is~~ module.

Now have $\quad k[X] \subset \overline{k[X]} = k[\hat{X}] \subset k(X)$, and

$$r : \hat{X} \to X.$$

If $\quad Y \to X$ dom, then

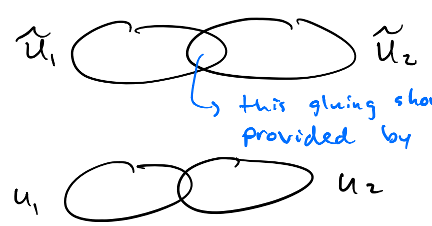
$$k[X] \hookrightarrow k[Y] \quad (\text{int. closed.})$$

$$\begin{tikzcd}
k[Y] & & & k(Y) \\
k[X] \arrow[u, hook] & \subset \overline{k[X]} \arrow[ul, dashed] & \subset & k(X) \arrow[u, hook]
\end{tikzcd}$$

__Brash__. By universality, every variety has a normalization

$$X = \bigcup U_i$$

$\tilde{U}_i \to U_i$ ↳ affines



↳ this gluing should be provided by universality.