

Last time.

$$\left\{ \begin{array}{l} \text{spanning rational} \\ \text{maps } \Phi: X \dashrightarrow \mathbb{P}^V \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{linear series} \\ \mathbb{P} \subset |\mathcal{O}| \text{ for} \\ \text{some effective} \\ \text{Cartier divisor } D \text{ on } X \end{array} \right\}$$

$$\Phi = (\phi_0: \dots: \phi_n) \xleftarrow[\phi_0, \dots, \phi_n]{\text{choose basis}} \mathbb{P} \subset |\mathcal{O}|$$

$\mathbb{A}^1_{\mathbb{A}^1/k^*}$

Thm (Hilbert scheme)  $X$  non-singular, projective  
 The locus of effective divisors is a countable union of projective "varieties" (schemes of finite type /  $k$ ).

Moreover, the equivalence classes of effective divisors are also a countable union of projective varieties (schemes of finite type /  $k$ ).

$$\text{And, } \text{Eff}(X) = \bigcup_{NS(X)} \text{Eff}(X) \longrightarrow \text{CDiv}(X) = \bigcup_{NS(X)} \mathbb{P}^{\text{Pic}(X)}$$

$\searrow \quad \nearrow$   
 $\bigcup \mathbb{Z}s$

Remark.  $\text{CDiv}(X)$  is a group.  
 $\text{Pic}(X)$

Neron-Severi  
 $= NS$

$$0 \longrightarrow \text{Pic}^0(X) \longrightarrow \text{Pic}(X) \longrightarrow NS(X) \longrightarrow 0$$

$$\Phi: \text{Eff}_g(X) \xrightarrow[\text{maps of Proj. vars.}]{\text{regular}} \text{Pic}^g(X)$$

$$\begin{array}{ccc} \mathcal{O} & & \\ \downarrow & \longrightarrow & [D] \end{array}$$

$$\Phi^{-1}([D]) = |\mathcal{O}| \quad (\text{proj. space})$$

Ex.  $X = \mathbb{P}^n$ ,  $\text{Eff}_d(\mathbb{P}^n) = \mathbb{P}(k[x_0, \dots, x_n]_d)$

$d \geq 0$ .  $\sum d_i z_i \longleftarrow F = \prod F_i^{d_i}$

$$Z_i = V(F_i)$$

$$\text{Eff}_d(\mathbb{P}^n) \longrightarrow \text{Pic}^d(\mathbb{P}^n) = \text{pt.}$$

$$F \sim F' \text{ via } \phi = F'/F.$$

Ex.  $X = \mathbb{C}$ , non-sing. curve.

$$\text{Eff}(\mathbb{C}) = \bigsqcup_{d \geq 0} \text{Eff}_d(\mathbb{C}).$$

$$\frac{\text{Eff}_d(\mathbb{C})}{\sim} \subset \mathbb{C}^d / \sum_d = \text{Sym}^d(\mathbb{C})$$

is a projective variety of dimension  $d$   
(also non-singular!)

$$\Phi_d: \text{Sym}^d(\mathbb{C}) \longrightarrow \text{Pic}^d(\mathbb{C})$$

$\hookrightarrow$  ab. var. of dimension  $g$ .

$d = 1, \dots, g-1$ ; the map is "generically" injective,

i.e. the image has  $\dim = d$ .

Thm (Riemann).  $\Phi_g$  is surjective.

$E$  - elliptic curve

$$\begin{aligned} \Phi_1: E &\xrightarrow{\sim} \text{Pic}^1(E) = (\text{Pic}^0(E)) \\ p &\longrightarrow [p] \longmapsto [p - p_0] \end{aligned}$$

$$\Phi_2: \text{Sym}^2(C) \longrightarrow \text{Pic}^2(C)$$

$$p' = [K_C] \longrightarrow [K_C].$$

Coherent sheaves.

Recall. Affine varieties  $X \longleftrightarrow k[X]$ ,  $k$ -algebras

Proj- varieties  $X \longleftrightarrow R$ , graded  $k$ -algebra

$$R_0 = k$$

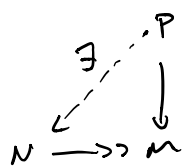
$R_1$  fin. dim  
generates  $R$ .

Generalize  $M$  module over  $k[X]$  ( $X$  affine)  
to a sheaf of  $\mathcal{O}_X$ -modules over  $\mathcal{O}_X$ .

Main Ex's:

$$\textcircled{1} \quad 0 \longrightarrow I \longrightarrow k[X] \longrightarrow k[X]/I \longrightarrow 0$$

$\textcircled{2}$  Projective modules



$P$  inject into free modules.  
Over a local ring, projective = free.

③ Injective

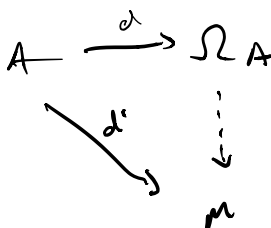
④ Module of differentials (~~if  $M$  is~~)

Def.  $M$  module over  $k[X]$ , then a  $k$ -linear map  $d: A \rightarrow M$

is a differential

if the usual properties hold.

Def.  $d: A \rightarrow \Omega_A$  is the universal module of differentials



$\Omega_A = (\text{free module on } d\alpha) / (\text{the usual relations that } d \text{ satisfies})$ .

Ex. If  $A = k[x_1, \dots, x_n]$ , then  $\Omega_A = \text{free module on } dx_i$

$\Rightarrow$  If  $A = k[x_1, \dots, x_n] / \langle f_1, \dots, f_n \rangle$

then  $\Omega_A = \langle dx_i \rangle / \langle df_j \rangle$  i.e. f.g., maybe not free.

Ex. If  $\text{mspec}(A)$  is non-singular, then  $\Omega_A$  is locally free (projective!)

Rmk. If  $S \subset A$  is a mult. set, then

$$(\Omega_A)_S = \{ \frac{m}{f} \mid f \in S \}$$

Goal. To find the sheaf of differentials  $\Omega_X$  over  $X$ .  
canonical

Def. A sheaf of  $\mathcal{O}_X$ -modules consists of:

(i)  $\mathcal{F}$ , a sheaf of ab. groups;

$\mathcal{F}(U)$  ab. grp.

$\uparrow_{U \subset X.}$

$$\mathcal{F}(U) \xrightarrow{res} \mathcal{F}(V)$$

$V \subset U.$

(ii) Each  $\mathcal{F}(U)$  has a consistent structure of an  $\mathcal{O}_X(U)$ -module.

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{F}(U) & \xrightarrow{\text{mult}} & \mathcal{F}(U) \\ \downarrow \text{res} & & \downarrow \text{res} \\ \mathcal{O}_X(V) \times \mathcal{F}(V) & \xrightarrow{\text{mult}} & \mathcal{F}(V) \end{array}$$

Moral.  $\mathcal{F}$  is q.e. if  $\mathcal{F}$  is "locally  $m$ "  
 $\uparrow$   
 over  $k[u]$   
 $\uparrow$  affine.

coherent if q.c. & "locally" f.g.