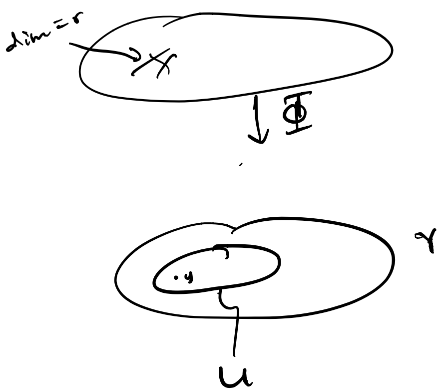
$\Phi : X \longrightarrow Y \qquad \dim(X) = \dim(Y) + r$

- Every fibre has all components with $\dim \geq r$, i.e. if $Z \subset \Phi^{-1}(y)$ is an irred. comp., $\dim Z \geq r$.

- "generically, the components have $\dim = r$"



Thm. Assume Φ is <u>dominant</u>. Then $\exists\ U \subset \Phi(X)$ s.t.
open

$\forall y \in U,\ \forall Z \subset \Phi^{-1}(y),\ \dim(Z) = \dim X - \dim Y$

<u>Def</u>. $\Phi : X \dashrightarrow Y$ is <u>birational</u> if Φ is dominant and $\Phi^* : k(Y) \longrightarrow k(X)$ is an isomorphism.

<u>Corollary</u>. If Φ is birational, then $\exists$ open $U \subset Y$, $V \subset \operatorname{dom} \Phi$ s.t. $\Phi|_V : V \longrightarrow U$ is an isomorphism.

<u>Pf</u>. $\Phi : \operatorname{dom}(\Phi) \longrightarrow Y$

By the theorem, $\exists\, U \subset Y$ over which the fibers of Φ are finite. WLOG, U is an affine. Choose $V \subset \Phi^{-1}(U)$ also affine.

Then $\Phi(X \setminus V)$ is a proper closed subset of Y. (every component has $\dim \leq \dim Y$.

Choose $f \in k[U]$ s.t. $U_f \cap \Phi(X \setminus V) = Z \subset U$

$\hookrightarrow$ closed, smaller dim

$\Rightarrow Z \subset V(f) \subset U$

$\Updownarrow$

$Z \cap U_f = \emptyset$.

Reduce the corollary to :

$\Phi : V_{\Phi^*(f)} \to U_f$

contrary to iso. open subsets. Note: $\left(V_{\Phi^*(f)} = \Phi^{-1}(U_f)\right)$.

$\boxtimes$

As a result, assume X & Y affine Given

$\Phi : X \to Y$

$\Phi^* : k[X] \to k[Y]$

$\Phi^{**} : k(Y) \xrightarrow{\sim} k(X)$

Problem. This ~~means~~ a basic ~~open~~ subset $U_g \subset Y$

i.e. $\Phi \; V_{\Phi^*(g)} \xrightarrow{\sim} U_g$.

Ideas. Choose generators for $k[X] = k[x_1, \ldots, x_n]/I$

Rmk. $x_i \in k(X) = k(Y) =$ fractions & $k[Y]$. Then

$\forall i, x_i = 0 = \frac{f_i}{g_i}, \quad f_i g_i \ldots$

Corr. Given a dominant map $\Phi : X \to Y$. Define

$e : X \to \mathbb{Z}^{\geq 0}$ s.t.

$$e(x) = \max\{\dim(Z) \mid Z \text{ is an irred. comp. of } \Phi^{-1}(\Phi(x))\}.$$

$$X_n := \{x \in X \mid e(x) \geq n\}$$

Ex. $k^2 \to k^2$ via

$(x, y) \to (x, xy)$

Then $X_0 = k^2$, $X_1 =$ y-axis, X_2, $X_3 = (\emptyset)$.

pf. Let $r = \dim X - \dim Y$. By easy thm, $e(x) \geq r$. $X = X_0 = \dots = X_r$. By hard thm, $X_{r+1} \neq X$. In fact, $X_{r+1} \subset \Phi^{-1}(Y \setminus U)$ for U as in the thm. Restrict Φ to the components of $\Phi^{-1}(Y \setminus U)$ and proceed by induction.

Thm (Chevalley). If G is a proper algebraic group over k, then G is abelian.

Def. (Irreducible) Alg. Group is a $G \in \mathrm{Var}_k$, and $\exists$.

$m : G \times G \to G$, $e \in G$,

$i : G \to G$.

<u>Pf.</u> Consider the map $c: G \times G \longrightarrow G$
$(h, g) \longrightarrow ghg^{-1}$.

Think of c as a family of maps

$$c_h : G \to G.$$
$$g \to ghg^{-1}$$

Then c_e is the constant map. Want to show $c_h(G) = h$ $\forall h$ (or even just constant). This follows from

<u>Lemma.</u> If X is proper, $\Phi : X \times Y \longrightarrow Z$ (thought of as a family $\Phi_y : X \longrightarrow Z$) satisfies

$$\Phi_{y_0}(X) = z_0$$

for some (y_0, z_0), then Φ_y are all constant.