Review. k - alg closed field

We considered rings $A = k[x_1, \dots, x_n]/P$, P a prime ideal (f.g. k-algebra domains).

Schemes. Allow any comm. rings w/ 1.

- Nullstellensatz
-

m-spec. We discussed the correspondence (functor)

$$k\text{-algebra domains} \longleftrightarrow \mathcal{N}_k = \left\{ (X, \mathcal{O}_X) \;\middle|\; \begin{array}{l} X \text{ noetherian top. sp.} \\ \mathcal{O}_X \text{ sheaf of } k\text{-algebras} \end{array} \right\}$$

① $A \xrightarrow{\text{geometrification}} \operatorname{mspec}(A) := \begin{cases} X = \{\text{max ideals of } A\} \\ \mathcal{O}_X(U) = \{ \phi \in k(A) \mid \phi \text{ is defined at all } x \in U, \ \phi = \frac{f}{g} \mid g \neq 0 \} \end{cases}$

② $(A \xrightarrow{f} B) \longrightarrow \operatorname{mspec}(A \longrightarrow B) := \begin{cases} \Phi : \operatorname{mspec} B \to \operatorname{mspec} A \text{ (continuous)} \\ \quad m_y \mapsto f^{-1}(m_y) \\ \Phi^* : \mathcal{O}_{\operatorname{mspec}(A)}(U) \to \mathcal{O}_{\operatorname{mspec} Y}(f^{-1}(U)) \end{cases}$

Thm. Let $\operatorname{Aff}_k :=$ full subcategory of $\mathcal{N}_k$ whose objects are $\operatorname{mspec}(A)$. Then

$\operatorname{mspec} : \{A\} \longrightarrow \operatorname{Aff}_k$ is an equivalence of categories.

<u>pf.</u> $\mathcal{O}_X(X) = A$, when $X = \mathrm{mspec}(A)$. ∎

<u>An analogy.</u>

$$\begin{tikzcd}
\text{balls in } \mathbb{R}^n \arrow[d, leftrightsquigarrow] & \text{manifolds} \arrow[d, leftrightsquigarrow] & \text{locally Euclidean topological space} \arrow[d, leftrightsquigarrow] & \text{topological spaces} \\
\mathrm{Aff}_k & \mathrm{Var}_k & \mathrm{Prevar}_k & \mathcal{N}_k \arrow[u, rightsquigarrow]
\end{tikzcd}$$

$\mathrm{Var}_k \cup (X, \mathcal{O}_X)$ separated

Prevar_k = locally affine $(X, \mathcal{O}_X)$

<u>Next.</u> $R_\bullet = k[x_0, \dots, x_n] / P_\bullet$ ← think of as $\mathbb{N}$ graded

$\left(\text{cone: } A = k[x_0, \dots, x_n] / P \right)$

$\mathrm{mproj}\ R_\bullet \subset \mathbb{P}^n$

$$= \begin{cases} X = \text{homog. max ideals} \\ \text{sheaf of regular functions} =: \mathcal{O}_X \end{cases}$$

$$\left(k(R_\bullet)_0 = \left\{ \frac{f_d}{g_d} \right\} \right)$$

Then define $\mathcal{O}_X(U)$ the same way.

However, $\mathrm{mproj} : \{R_\bullet\} \longrightarrow \mathcal{N}_k$ is far from being an equivalence of categories to the <u>image</u>!

Example. $\operatorname{mproj}(R_\bullet) \underset{\text{in } \mathcal{N}_k}{\simeq} \operatorname{mproj}(R_{d\bullet})$ $\left(\begin{array}{l}\text{via Veronese,} \\ \mathbb{P}^n \hookrightarrow \mathbb{P}^{\binom{n+d}{d}} \\ \cup \\ X \hookrightarrow X\end{array}\right)$

$$\begin{tikzcd} X \arrow[dr, bend left] \arrow[d] & \subset & \mathbb{P}^n \arrow[d, dashed] \\ {} & & \mathbb{P}^r \end{tikzcd}$$

$R_\bullet = k[x_0, \dots, x_n]/P_\bullet$

$T_\bullet = k[x_0, \dots, x_r]/P_\bullet \cap k[x_0, \dots, x_n]$

if this exists, $\operatorname{mproj}(R_\bullet) \simeq \operatorname{mproj}(T_\bullet) \simeq X$.

Problem. Given a variety X, what are the embeddings of $X \subset \mathbb{P}^n$ (i.e. $\{R_\bullet \mid \operatorname{mproj}(R_\bullet) = X\}$)

Equivalently, what is the equivalence rel. on $R_\bullet$

induced by
- iso of $\operatorname{mproj}(R_\bullet)$
- birational iso of $\operatorname{mproj}(R_\bullet)$

$$\begin{array}{ccccc} & & \text{Proj}_k & \subset & \text{Prop}_k \leftrightsquigarrow \text{cpt manifolds} \\ & & \cap & & \\ \text{Aff}_k & \subset & \text{QProj} & \subset & \text{Var}_k \\ & & \cup & & \\ & & (U, \mathcal{O}_U), & & \\ & & U \subset X & & \end{array}$$

Then, properties of varieties.

$X \leftrightsquigarrow k(X)$ field of rational functions

<u>Global Props.</u>

$\dim(X) :=$
1. Krull dim (longest chain of subsets)
   $\|$ (Krull's thm)
2. trans. deg. of $k(X)/k$
   $\| \rightarrow$ if $X = \text{mproj}(R_\bullet)$
3. degree of Hilbert poly
   $H_{R_\bullet}(d) = \dim_k R_d \gg 0$

<u>Rmk.</u> Constant term in $H_{R_\bullet}(d)$ is also an invariant of $X = \text{mproj}(R_\bullet)$, ~~an invariant~~

$$\chi(X, \mathcal{O}_X) = \sum_{i=0}^{n} (-1)^i \dim(H^i(X, \mathcal{O}_X)).$$

This will give us def. of genus:

$X = C$ nonsing. curve gives $\chi(X, \mathcal{O}_X) = 1 - g$,

which defines "arithmetic genus".

$\chi(E, \mathcal{O}_E)$, E an elliptic curve, is 0.

$\chi(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 1$.

<u>Local Properties</u>

Nature of $\mathcal{O}_{X,x}$

- **Normality**: $\mathcal{O}_{X,x} \subset k(X)$ integrally closed
- **Non-singularity**: $\mathcal{O}_{X,x} \subset k(X)$ regular, i.e.

$\dim_k m_x/m_x^2 = \dim X \Rightarrow \mathcal{O}_{X,x}$ is a UFD.

Preview: if X arbitrary, $\exists !$ finite $\overset{\text{birational}}{\vee}$ map

$f: \tilde{X} \longrightarrow X$ affine, $\tilde{X}$ proj. if X is.

$\uparrow$ normal

In contrast, if X a variety, char. 0, $\exists$ desingularization $f: \bar{X} \xrightarrow[\text{surjective}]{\text{birational,}} X$. f is a series of blow-ups of X at smooth centers.

Divisors

(Effective if $n_i \geq 0$)

Weil: $\sum n_i Z_i$, Z_i codim 1 closed subvars.

- Linear equivalence: via $\operatorname{div}(\phi)$ (defined when X is normal).

$$\operatorname{Div}(X) = \{\sum n_i Z_i\}/\sim \quad \left(\text{on normal vars}\right)$$

Cartier: $\{(U_\alpha, \phi_\alpha) \mid \phi_\alpha/\phi_\beta \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)\} / \sim$

Linear equiv. via $\{(X, \phi)\}$ (principal Cartier divisors)

$\mathrm{Pic}(X) = \{(U_\alpha, \phi_\alpha)\}/\sim$

HW: Compute $\mathrm{Pic}(X)$ for the

1. nodal cubic: $\overline{y^2 = x^2(x-1)} \subset \mathbb{P}^2$

   $k^* = \mathrm{Pic}^0(X)$

2. cuspidal: $\overline{y^2 = x^3} \subset \mathbb{P}^2$

   $k^* = \mathrm{Pic}^0(X)$

$\exists$ SES:

$$\underbrace{\mathrm{Pic}^0(X)}_{\text{group!}} \longrightarrow \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \longrightarrow$$

Ω_X - sheaf of differentials.