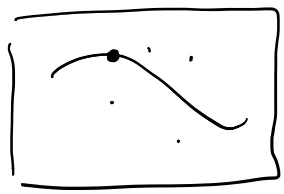
$X \subset K^n \longleftrightarrow k[x_1, \dots, x_n]/_p$

Zariski topology on X

Closed sets = alg. sets

Irreducible closed sets = varieties

Ex:



k^2

$\emptyset \subset pt \subset C \subset k^2$

Rmk. The Zariski topology is generated by the basis $U_f := X \setminus V(f)$, for $f \in k[X]$.

Rmk. Not every open set is a basis element. Take U_x, U_y.

$$k^2 \supset U_x \cup U_y = k^2 - \{0\},$$ no such function vanishes only at origin (for k algebraically closed).

Sheaf of regular functions.

(Piggy back off of $k(X)$)

Define $\mathcal{O}_{X,x} := \{ \frac{f}{g} \mid g(x) \neq 0 \} = k[X]_{m_x}$

where $m_x = \{ h \mid h(x) = 0 \}$, is a local ring, with max ideal $m_x \cdot \mathcal{O}_{X,x} = \{ \frac{f}{g} \mid {f(x)=0 \atop g(x) \neq 0} \}$.

If $V = V(P)$ is a variety,

$$\mathcal{O}_{X,V} := k[X]_P = \{ \frac{f}{g} \mid g \notin P \},$$

for P prime, is also a local ring.

__Ex__. $\phi \in \mathcal{O}_{X,V}$ iff $\exists\, x \in V$, $\phi(x) \neq 0$.

Given $\phi \in k(X)$ define $\mathrm{dom}(\phi) := \{ x \in X \mid \phi \in \mathcal{O}_{X,x} \}$
ϕ is "regular" on its domain.

__Prop__. dom ϕ is Zariski open. ∮

__Pf__. Let $I_\phi = \{ g \mid g \cdot \phi \in k[X] \}$. This is an ideal. If $\phi = \frac{f_1}{g_1} = \frac{f_2}{g_2} = \frac{f_1+f_2}{g_1+g_2}$. ●

__Def__. The presheaf of regular functions on X is :

$$\mathcal{O}_X : (\text{open sets in } X) \longrightarrow \left(\begin{matrix} k\text{-algebra} \\ \text{domains} \end{matrix} \right)$$

$$U \longrightarrow \bigcap_{x \in U} \mathcal{O}_{X,x}$$

$$= \{ \phi \in k(X) \mid U \subset \mathrm{dom}\, \phi \}$$

$$\subset k(X), \supset k[X].$$

$\mathcal{O}_X(U \subset V)$ = restriction of functions $\rho_{VU} : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$.

__Def__. An element $s \in \mathcal{O}_X(U)$ is called a section of the presheaf over U.

(i) Sections that are locally zero are globally zero. If $s \in \mathcal{O}_X(U)$; $U = \bigcup U_i$; $\rho(U, U_i)(s) = 0$, then $s = 0$.

(ii) Sections that can be pieced together form sections.

i.e. if $U = \bigcup U_i$, $s_i \in \mathcal{O}_X(U_i)$,

if $\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j)$

$\forall i, j$, then $\exists s \in \mathcal{O}_X(U)$ s.t. $\rho_{U, U_i}(s) = s_i$.

Let $\mathcal{C}$ be the category of pairs $(X, \mathcal{O}_X)$,

- X noetherian topology
- $\mathcal{O}_X$ sheaf of functions $U \to k$.

Morphisms

$$\Phi : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

Need

- $\Phi : X \longrightarrow Y$ continuous
- pullsback sections of $\mathcal{O}_Y$, i.e.
  $\exists\ \Phi^* : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\Phi^{-1}(U))$.

So far

Identified $(X, \mathcal{O}_X)$ as an object of the category $\mathcal{C}$ where X is an affine variety.

However, we don't need to specify an embedding.

In fact,

$k[X] \overset{\text{Mspec}}{\rightsquigarrow}$ finitely gen k-alg domain

- points of $X \longleftrightarrow$ maximal ideals in $m_x \subseteq k[X]$
- closed sets $\longleftrightarrow Z(I) = \{x \mid m_x \supseteq I\}$
- $k(X)$ = field of functions of $k[X]$
- $\mathcal{O}_{X,x} = k[X]_{m_x}$

$= (X, \mathcal{O}_X)$.

Def An object $(X, \mathcal{O}_X)$ of $\mathcal{C}$ is an affine variety if $(X, \mathcal{O}_X) \simeq \text{mspec}(A)$ for some fin. gen. k-algebra domain.

Rmk. If $(X, \mathcal{O}_X)$ is in $\mathcal{C}$, $(U, \mathcal{O}_X)$ also in $\mathcal{C}$, where $U \subset X$ is open,

$$\mathcal{O}_X|_U(V) := \mathcal{O}_X(V)$$

Rmk. If $(X, \mathcal{O}_X) \simeq \text{mspec}(A)$ is affine, $\boxed{(U, \mathcal{O}_X|_U)}$ may or may not be affine.

However, $U = U_f$ are, since it's just $\text{mspec}(A[f^{-1}])$.

quasi-affine.