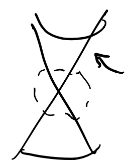
$x \in X$ non-sing if m_x/m_x^2 has dim $= \dim X$.

Think about. $\mathcal{O}_{X,x}$ is a UFD.

- Every $\phi \in \mathcal{O}_{X,x}$ factors uniquely
- Every minimal prime ideal is principal prime ideal.
  height 1
- Something about covering every ~~closed~~ subvariety.

Nonexample.

$X = V(y^2 - xz)$ $\quad$ Z = ruling of the cone (e.g. $(0,0,t)$)



Check: Z is not principal in $\mathcal{O}_{X,0}$

For all $f, g \in \mathcal{O}_{X,x}$, the ideal $(f):(g) = \{h \in \mathcal{O}_{X,x} \mid h \cdot g \in (f)\}$

$$0 \longrightarrow ((f):(g)) \longrightarrow \mathcal{O}_{X,x} \xrightarrow{g} \mathcal{O}_{X,x}/f \cdot \mathcal{O}_{X,x}$$

Suppose $\hat{\mathcal{O}}_{X,x}$ is a UFD, then

$\hat{h}$

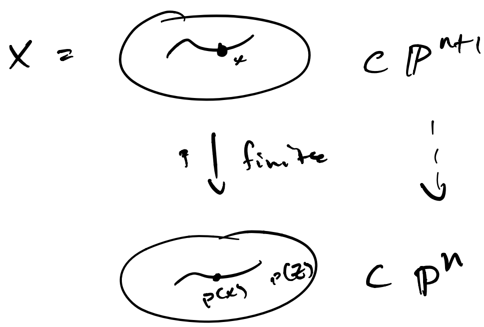
$$0 \longrightarrow (\text{principal}) \to \hat{\mathcal{O}}_{X,x} \longrightarrow \hat{\mathcal{O}}_{X,x}/f \cdot \hat{\mathcal{O}}_{X,x}$$

Rmk. Completion is faithfully flat.

Alternately,

$$\begin{array}{ccc} X & \dashleftarrow\dashrightarrow & H \subset \mathbb{P}^{n+1} \\ \cup & & \cup \\ x \in U & \cong & U \ni x \end{array}$$

Assume X is a hypersurface



(Severys' arg)

<u>X Normal</u>: $\mathcal{O}_{X,x}$ is integrally closed.

$\mathcal{O}_{X,Z}$ " " "

In particular, $\mathcal{O}_{X,Z}$ is a DVR if $Z \subset X$ has codim 1.

<u>X non-singular</u>. $\mathcal{O}_{X,x}$ is a UFD

$$\Big(\Rightarrow \forall x \in \underset{\substack{\uparrow \\ \text{codim } 1}}{Z} \subset X, \ \exists \text{ eq'n } Z = V(f) \text{ for } Z \Big)$$

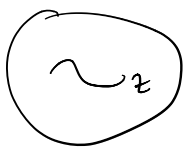
Consequence.

X normal, $\Phi : X \dashrightarrow \mathbb{P}^n$

$$\phi = (\phi_0 : \cdots : \phi_n) \qquad \phi_i \in k(X)$$

Choose $Z \subset X$ codim 1 and regard

$$\phi_i \in k(X) = k(\underset{\text{DVR}}{\mathcal{O}_{X,Z}})$$



$\mathcal{O}_{X,Z}$ DVR $\quad (\varpi) = m_{X,Z}$.

Then $\quad \phi_i = u_i \varpi^{d_i}$. Multiply through by $\varpi^{-\min_i d_i}$. Then

$$\phi = (\cdots : \phi_i : \cdots) = (\cdots : u_{i'} : \cdots) \quad \text{for some } i'.$$

Then all coordinates are in $\mathcal{O}_{X,Z}$, so we have extended ϕ to an open subset of Z.

E.g. $\dim C = 1$ (normal $\Leftrightarrow$ non-singular)

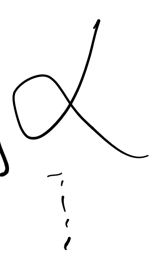
$\Phi : C \longrightarrow \mathbb{P}^n \qquad$ i.e. choose $\phi \in k(C)$

$$\phi : C \dashrightarrow \mathbb{P}^1$$
$$x \longmapsto (1 : \phi(x))$$

extends to

$$\Phi : C \longrightarrow \mathbb{P}^1$$

Not-normal counter example:

$k[t(t^2-1), t^2-1]$  $\subset k^2$

"origin wants to go to two different points.

U $\quad$ V

$k[t]$ $\longrightarrow$ $k^1 \subset \mathbb{P}^1$

even with pt at infinity, can't extend this map"

<u>Divisors</u>. X normal, $\phi \in k(X)$. $Z \subset k(X)$
$\uparrow$ codim 1 subvariety

$\mathrm{mult}_Z\, \phi := d$ s.t. $\phi = u \cdot \varpi^d \in \mathcal{O}_{X,Z}$. Think of it as order of pole/zero.

<u>Def</u>. $$\operatorname{div} \phi = \sum_{\substack{Z \subset X \\ \text{codim } 1}} \mathrm{mult}_Z\, \phi \cdot [Z],$$

Note that this sum is finite.

<u>Ex</u>. $X = C$, non-singular, projective curve

$$\operatorname{div} \phi = \sum_{p \in C} \mathrm{mult}_p\, \phi \cdot [p]$$

<u>Def</u>. X normal variety. $Z(X) :=$ free ab gp on $Z \subset X$.

$$\left\{ \sum_{\text{finite}} n_Z [Z] \right\} \quad \left(\begin{array}{c} \text{Weil} \\ \text{divisors} \end{array} \right)$$

$Z(X)$
$\cup$
$\mathrm{Prin}(X) :=$ { divisors of the form $\operatorname{div}\phi$ }

Rmk. $k(X)^* \simeq \mathrm{Prin}(X)$.

Def. Divisor class group $:= Z(X)/\mathrm{Prin}(X)$.