X variety.

A sheaf of ab gps $\mathcal{F}$ on X is a sheaf of $\mathcal{O}_X$-modules if $\mathcal{O}_X(U) \times \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$.

Example. the sheaves $\tilde{M}$ on an affine variety

$$X \text{ affine} \longleftrightarrow k[X] =: A \quad \text{coordinate ring}$$

$$\tilde{M} \text{ sheaf of } \mathcal{O}_X\text{-modules} \longleftrightarrow M \text{ an } A\text{-module}$$

Def. If $\mathcal{F}$ is a sheaf of ab groups and $x \in X$ is a point, then

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U)$$

is the stalk at x.

Ex. $\mathcal{O}_X = \mathcal{F}$, $\mathcal{F}_x = \mathcal{O}_{X,x} = \{$ germs of rational functions defined at x $\}$

$k[X]_S = \varinjlim \mathcal{O}_X(U) \subset k(X)$ $= k[X]_S$, $S = k[X] \setminus m_x$

" $\tilde{M}$ is the sheaf of $\mathcal{O}_X$-modules whose stalks

$$\tilde{M}_x = M_S \quad (S = k[X] \setminus m_x)"$$

Honestly. $\quad \mathcal{U} \subset X = \text{mspec } A$

$$\tilde{\mathcal{M}}(\mathcal{U}) \subset \coprod_{x \in \mathcal{U}} \tilde{\mathcal{M}}_x$$

is defined by the property that $\forall x \in \mathcal{U}$, $\exists$ nbhd $x \in \mathcal{U}_f \subset \mathcal{U}$ s.t. each $\phi_f \in \mathcal{M}_f$

$$\{ \phi_x \in A_S \}$$

$S = \{1, f, f^2, \dots\}$

s.t. $\phi_x \sim \phi_f$ for all $x \in \mathcal{U}_f$.

$\phi_x \in \tilde{\mathcal{M}}_x$

$\Rightarrow \phi_x = \frac{m}{g} \rightsquigarrow \phi_y$ for $y \in \mathcal{U}_g$

(some $m \in M$, $\bar{g} \in \mathcal{O}_X(\mathcal{U}_g)$)

E.g. $\quad M = \mathcal{O}_X = A$

$$s \in \tilde{A}(\mathcal{U}) \subset \coprod \mathcal{O}_{X,x}$$

$$= \mathcal{O}_X(\mathcal{U})$$

$$A_{f_i} \longrightarrow A_{m_x}$$

$\forall x \in \mathcal{U}_f$

$\mathcal{U} = \bigcup \mathcal{U}_{f_i}$

$$\rho_{\mathcal{U}_{f_i}}(s) \in \mathcal{O}_X(\mathcal{U}_{f_i}) = A_{f_i}$$

$\phi_x \in \mathcal{O}_{X,x} \subset k(X)$ "germ of a rational function"

$\Rightarrow \phi_x \in k(X) \Rightarrow \dim(\phi_x) = \mathcal{U} \ni x$

and $\phi_x \in \mathcal{O}_X(\mathcal{U})$

$$\phi_x = \frac{f}{g} \Rightarrow \phi_x \in \mathcal{O}_X(\mathcal{U}_g)$$

Prop. There is a uniquely determined sheaf $\widetilde{M}$ of $\mathcal{O}_X$-modules s.t.

(1) $\widetilde{M}(U_g) = M_g \quad \forall g \in A$

(2) $\widetilde{M} \subset \coprod_{x \in X} M_{m_x}$

Rmk. $\forall Z \subset X$ (irr. closed subset) $\quad \widetilde{M}_Z := \lim_{U \cap Z \neq \emptyset} \widetilde{M}(U)$

$Z \longleftrightarrow P \subset A \qquad \widetilde{M}_Z \simeq A_P$

Pf. Nullstellensatz

$$\widetilde{M}(U) = \{ s_g \in \widetilde{M}(U_g) \mid U_g \subset U \}$$

$$\rho_{U_g, V}(s_g) = \rho_{U_h, V}(s_h)$$

$$U = \bigcup U_g$$

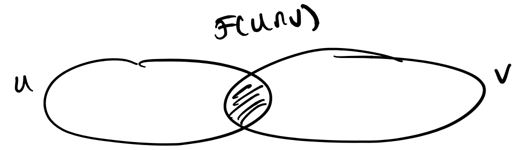
Def. A sheaf $\mathcal{F}$ on a variety X is quasi-coherent if $\exists\, X = \bigcup U_i$ s.t.

$\mathcal{F}|_{U_i} \simeq \widetilde{M}_i$ for some modules M_i over the ring $A_i = k[U_i]$.

<u>Example</u>. $\mathcal{F}$ restricts to $\tilde{A}_i$ over U_i.

$\mathcal{F} = \mathcal{O}_X$ does this, but so do line bundles.

generates $f \in A_U$ generates $A_V \ni g$



$\rho_{U\cap V}(f)$ $\rho_{U \cap V}(g)$

generate all the stalks on the intersection

$\mathcal{F}_x$ $x \in U \cap V$.

$\mathcal{F}$ locally free of rank 1,

$X = \bigcup U_i$ s.t. $\mathcal{F}|_{U_i}$ are free of rank 1.

$\Rightarrow$ system of invertible fns

$\phi_{ij} \in \mathcal{O}_X(U_i \cap U_j)$

(passing from $\mathcal{F}|_{U_i}$ to $\mathcal{F}|_{U_j}$)

ϕ_{ij} transition function satisfying

$$\phi_{jk} \circ \phi_{ij} = \phi_{ik} \in \mathcal{O}_X^*(U_i \cap U_j \cap U_k)$$

Choose. $(U_i, 1)$ $\rightsquigarrow$ Cartier divisor (U_j, ϕ_{ij})

(always doable, but not usually effective!)

$\hat{\mathcal{O}}_X(U_i)$

try to find: $(U_i, f_i \in \mathcal{O}_X(U_i))$

satisfying $f_i/f_j = \phi_{ij}$.

These correspond to global sections of the sheaf $\mathcal{F}$.