Algebraic Geometry (Math 6130)

Utah/Fall 2016.

4. Products. We begin our study of the category of quasi-projective varieties over k by showing that products of quasi-projective varieties exist (but **not** with the product topology!). This allows us to formulate categorical analogues of *Hausdorff* and *compact* that are satisfied by quasi-projective and projective varieties, respectively. We conclude from this that projective varieties are complete, i.e. that they do not arise as open subsets of any other varieties.

Definition 4.1. A **product** of objects X, Y of a category $\mathcal{C}$ is a triple:

$$(Z, \pi_1, \pi_2)$$

consisting of an object Z, and morphisms $\pi_1 : Z \to X$ and $\pi_2 : Z \to Y$ that is *universal* in this sense that:

(*) For all triples (W, p_1, p_2), there is a unique $f : W \to Z$ such that:

$$\pi_1 \circ f = p_1 \text{ and } \pi_2 \circ f = p_2$$

Remark. A product, if it exists, is uniquely determined up to a unique isomorphism by the universal property.

Examples. (i) The Cartesian product of a pair of sets.

(ii) The set-theoretic product of topological spaces with the product topology in the category of topological spaces.

(iii) Products also exist in the category of differentiable manifolds, incorporating the sheaf of differentiable functions.

Proposition 4.1. Products of any pair of affine varieties exist within the category of affine varieties over k.

Proof. This is deceptively simple. Via the contravariant equivalence of categories between $\mathcal{A}_k$ and the category of k-algebra domains, a product in $\mathcal{A}_k$ would correspond to a *coproduct* of coordinate rings. But the tensor product:

$$k[X] \otimes_k k[Y]$$

of two finitely generated k-algebra domains is the coproduct in the category of k-algebras, and it is a domain if k is algebraically closed.

Example. The tensor product of polynomial rings satisfies:

$$k[x_1, ..., x_m] \otimes_k k[y_1, ..., y_n] \cong k[x_1, ..., x_m, y_1, ..., y_n]$$

from which it follows that k^{m+n} with the projections $\pi_1 : k^{m+n} \to k^m$ and $\pi_2 : k^{m+n} \to k^n$ is the product of k^m and k^n in the category of affine varieties. We will write $k^{m+n} = k^m \times k^n$ for short.

Moreover, if $X \subset k^m$ and $Y \subset k^n$ are embedded affine varieties, with $X = V(\langle f_1, ..., f_a\rangle)$ and $Y = V(\langle g_1, ..., g_b\rangle)$, then:

$$X \times Y = V(\langle f_1, ..., f_a, g_1, ..., g_b\rangle) \subset k^{m+n} = k^m \times k^n$$

which is, in particular, the Cartesian product of X and Y as a set.

Remark. The Zariski topology on $k^m \times k^n$ is **not** the product topology. For example, the only closed sets in the product topology on $k \times k$ are finite unions of horizontal and vertical lines and points, while the Zariski topology on k^2 contains the curves $V(f(x, y))$. For example, the diagonal $\Delta = V(x - y) \subset k \times k$ is not closed in the product topology.

The product on affine varieties is compatible with basic open sets. If $U_f = \text{mspec}(A[f^{-1}]) \subset \text{mspec}(A)$ and $V_g = \text{mspec}(B[g^{-1}]) \subset \text{mspec}(B)$ are basic open subsets of affine varieties, then:

$$U_f \times V_g = W_{fg} = \text{mspec}(A \otimes_k B[(fg)^{-1}]) \subset \text{mspec}(A \otimes_k B)$$

is the corresponding basic open subset of the product. The fact that affine varieties form a basis for the Zariski topology allows us to find products in the following larger category than $\mathcal{QP}_k$:

Definition 4.2. An object $(X, \mathcal{O}_X)$ of $\mathcal{N}_k$ is a **prevariety** if X is locally affine, i.e. each $x \in X$ is contained in an open neighborhood $x \in U_x$ that is an affine variety (with the induced sheaf).

Example. Quasi-projective varieties are locally affine. If $U \subset X$ is an open subset of a projective variety, then U is covered by basic open sets, which are affine by Exercise 3.4.

We have the following Corollary of Proposition 4.1.

Corollary 4.1. Products of prevarieties exist and are prevarieties.

Proof. (Sketch) First note that a prevariety is covered by finitely many affine varieties, since a Noetherian topology is quasi-compact; every open cover has a finite subcover. Let X, Y be prevarieties with affine open covers $X = \bigcup U_i$ and $Y = \bigcup V_j$. Then we construct the product object in $\mathcal{N}_k$ as follows:

(i) As a set, it is the Cartesian product $X \times Y$.

(ii) Open sets in the topology on $X \times Y$ are defined locally; a subset $Z \subset X \times Y$ is open if and only if $Z \cap (U_i \times V_j)$ is open for each i, j, where $U_i \times V_j$ is the product affine variety with the corresponding topology. This definition is consistent because each $U_{i_1} \cap U_{i_2}$ may be covered by basic affine open subsets that match the topologies of $U_{i_1} \times V_j$ and $U_{i_2} \times V_j$ along their intersection.

(iii) The sheaf $\mathcal{O}_{X\times Y}$ is likewise defined locally, with $\mathcal{O}_{X\times Y}(U_i \times V_j)$ in particular being $\Gamma(U_i, \mathcal{O}_{U_i}) \otimes_k \Gamma(V_j, \mathcal{O}_{V_j})$. The basic open affines in U_i and V_j respectively show that the local definition is consistent. Sheaf property (ii) is employed to define $\mathcal{O}_{X\times Y}(W)$ for all open W.

The subcategory of prevarieties will remain nameless since it is too inclusive. It is analogous to the category of "premanifolds" which would be manifolds without the Hausdorff property.

Bad Example. The affine line with the doubled origin is a prevariety. Consider two copies of k, glued along the identity map on $k^* \subset k$. I.e.

$$X = k \cup 0' \text{ as a set}$$

with the Noetherian topology defined so that the two inclusions:

$$i : k \hookrightarrow X;\ i_1(0) = 0 \text{ and } i' : k \hookrightarrow X;\ i'(0) = 0'$$

are both continuous. In this (cofinite) topology, every open set that contains 0 also contains $0'$ and vice versa. We define the sheaf of regular functions on X locally, so that i^* and $(i')^*$ are both the identity on sections. With this structure, X is a prevariety.

Let $\mathcal{C}$ be a category of topological spaces, and let X be an object of $\mathcal{C}$ with the property that $X \times X$ exists in $\mathcal{C}$. The identity map defines a triple $(X, \mathrm{id}, \mathrm{id})$, hence a morphism:

$$\delta : X \to X \times X \text{ with } \pi_1 \circ \delta = \mathrm{id} = \pi_2 \circ \delta$$

Definition 4.3. In this setting X is **separated** (as an object of $\mathcal{C}$) if $\Delta := \delta(X)$ is a closed subset of the topological space $X \times X$.

Example. In the category $\mathcal{T}$ of all topological spaces, products have the product topology and X is separated if and only if X is Hausdorff: Every pair of points $x_1, x_2 \in X$ may be separated by open subsets $x_1 \in U_1, x_2 \in U_2$ such that $U_1 \cap U_2 = \emptyset$. As we remarked upon earlier, a Noetherian topology fails to separate points by open subsets. **But** by decorating the spaces with sheaves of regular functions, we have produced products that do not have the product topology, and indeed:

Proposition 4.2. Affine varieties are separated in $\mathcal{A}_k$.

Proof. In case $X = k^n$, the diagonal $\Delta \subset k^n \times k^n$ is:

$$\Delta = V(\langle x_i - y_i \rangle) \subset\ k^{2n} = \mathrm{mspec}(k[x_1, ..., x_n, y_1, ..., y_n])$$

and in case $X \subset k^n$ is an affine variety, then $\Delta_X \subset X \times X$ is the intersection of $\Delta \subset k^n \times k^n$ with $X \times X$, which is closed in $X \times X$.

Definition 4.4. A **variety** is a separated object in the category of prevarieties. The full subcategory of varieties is denoted by $Var_k \subset \mathcal{N}_k$.

Bad Example Revisited. The affine line with the doubled origin is not a variety. Indeed,

$$X \times X = k^2 \cup 0' \cup 0'' \cup 0'''$$

is the affine plane with a quadruple origin (corresponding to the various ordered pairs of origins in X), and the diagonal $\delta \subset X \times X$ only picks up two of the four origins, hence is not closed. In a sense, non-separatedness of a prevariety always boils down to there being more than one limit to a germ of a smooth curve, generalizing this example. This is the *valuative criterion for separatedness*.

We next want to prove that the product of projective varieties is a projective variety, from which we will conclude that quasi-projective varieties are objects of $\mathcal{V}ar_k$. The first attempt to see this is too naive:

Observation. The projective space $\mathbb{P}_k^{m+n}$ is **not** the product $\mathbb{P}_k^m \times \mathbb{P}_k^n$.

This follows from the following startling proposition:

Theorem 4.1. There are **no** non-constant morphisms:

$$\Phi : \mathbb{P}_k^n \to \mathbb{P}_k^m \text{ when } n > m$$

Proof. Assume that the image of Φ intersects the set $U_0 \subset \mathbb{P}_k^m$ (changing coordinates if necessary), and consider the rational map:

$$\Phi : \mathbb{P}_k^n - - > \mathbb{P}_k^m; \quad \Phi(x) = (1 : \phi_1(x) : \cdots : \phi_n(x))$$

We may clear denominators by mutiplying each ϕ_i by a fixed $F_0 \in S_d$:

$$(1 : \phi_1(x) : \cdots : \phi_n(x)) = (F_0(x) : \cdots : F_n(x))$$

as maps to $\mathbb{P}_k^m$, and **because** S is a UFD, we may do this in such a way that the F_i have no common factors. We now claim that:

(a) Φ extends to $U = \mathbb{P}_k^n - V(\langle F_0, ..., F_m \rangle)$ as a morphism to $\mathbb{P}_k^m$, but it extends no further than that (see Exercise 3.6), and:

(b) An intersection $V(F_0) \cap \cdots \cap V(F_m) \subset \mathbb{P}_k^n$ is non-empty if $m < n$.

We see that Φ may be extended to U by dividing through by F_i whenever some $F_i(x) \neq 0$ to obtain the restricted morphism defined at $x \in U$. On the other hand, if Φ may be extended to $x \in \mathbb{P}_k^n$, then $x \in \Phi^{-1}(U_i)$ for some i, and then:

$$\Phi(x) = (\psi_0(x) : \cdots : 1 : \cdots : \psi_n(x)) \text{ for rational functions } \psi_i$$

But then $\psi_j(x) = F_j(x)/F_i(x)$, and it follows from the fact that the F_i have no common factors in the polynomial ring (and UFD) S that $\psi(x)$ is defined at x if and only if $F_i(x) \neq 0$. This gives (a).

For (b), we consider the intersection of the **cones** $V(F_i) \subset k^{n+1}$ whose rulings are $V(F_i) \subset \mathbb{P}^n_k$. By Krull's Theorem (see Exercise 3.2.) each irreducible component of an intersection $V(F) \cap X \subset k^{n+1}$ with an affine variety of Krull dimension m has Krull dimension $m-1$ or m (the latter occuring if $X \subset V(F)$). In particular, if $X \subset k^{n+1}$ is itself a **cone**, then the intersection $V(F) \cap X$ contains the origin, hence it is non-empty. Proceeding by induction, we see that every irreducible component of $V(F_0) \cap \cdots \cap V(F_m) \subset k^{n+1}$ has Krull dimension at least $n-m$, hence, if $n > m$, it contains points **other than** the origin, and thus the projective intersection is non-empty. $\square$

Remarks. (i) It follows that $\mathbb{P}^{m+n}_k$ is not the product of $\mathbb{P}^m_k$ and $\mathbb{P}^n_k$, since a product comes equipped with (surjective) projection maps.

(ii) Part (b) generalizes to show that projective subvarieties of $\mathbb{P}^n_k$ always intersect when they should, by a dimension count.

We look instead to the Segre embedding (Example 2.4).

Proposition 4.3. The Segre embedding, defined set-theoretically by:

$$s_{m,n} : \mathbb{P}^m_k \times \mathbb{P}^n_k \to \mathbb{P}^{(m+1)(n+1)-1}_k$$

$$((a_0 : ... : a_m), (b_0 : ... : b_n)) \mapsto (... : a_i b_j : ...)$$

exhibits the product of projective spaces as a projective variety.

Proof. The map $s_{m,n}$ is injective and its image is the variety:

$$X = V(\langle x_{ij}x_{kl} - x_{il}x_{jk} \rangle)$$

which endows the Cartesian product with the Noetherian topology and sheaf of regular functions making it a projective variety. The two maps $\pi_1 : X \to \mathbb{P}^m_k$ and $\pi_2 : X \to \mathbb{P}^n_k$ are the projection maps:

$$\pi_1(... : a_i b_j : ...) = (a_0 b_0 : ... : a_m b_0) \text{ and}$$

$$\pi_2(... : a_i b_j : ...) = (a_0 b_0 : ... : a_0 b_n)$$

which extend to all of X, defining morphisms that set-theoretically agree with the Cartesian projection maps. It only remains to prove that this triple is universal (in the category of prevarieties!).

But suppose (W, p_1, p_2) is a triple in the cateogry of prevarieties. Then because X is the Cartesian product, we have the desired map $\Phi : W \to X$ uniquely defined on the level of sets. The only question is whether it is a morphism of locally affine varieties. This may be checked locally.

Suppose $U \subset W$ is an open affine variety. Then each of the pair of morphisms p_i is achieved with rational functions:

$$p_1 : U \to \mathbb{P}_k^m;\ p_1(x) = (\phi_0(x) : \cdots \phi_m(x)) \text{ and}$$

$$p_2 : U \to \mathbb{P}_k^n;\ p_1(x) = (\psi_0(x) : \cdots \psi_n(x));\ \phi_i, \psi_j \in k(U)$$

from which it follows that:

$$f : U \to X \text{ is defined by } f(x) = (... : \phi_i(x)\psi_j(x) : ...)$$

and it is a morphism of varieties, as desired. □

This extends in a straightforward manner to projective varieties. If $Y = V(\langle F_1, ..., F_a\rangle) \subset \mathbb{P}_k^m$ and $Z = V(\langle G_1, ..., G_b\rangle) \subset \mathbb{P}_k^n$ are defined by homogeneous prime ideals, then:

$$Y \times Z = V(\langle F_i(x_{00}, ..., x_{m0}), G_j(x_{00}, ..., x_{0n})\rangle) \subset X \subset \mathbb{P}_k^{(m+1)(n+1)-1}$$

is the projective variety product.

We may also see this directly from the graded rings. If $Y = \text{mproj}(R_\bullet)$ and $Z = \text{mproj}(Q_\bullet)$, then $\bigoplus R_d \otimes Q_d$ is a graded domain, and:

$$Y \times Z = \text{mproj}\left(\bigoplus_{d=0}^{\infty} R_d \otimes_k Q_d\right)$$

Finally, if $U \subset Y$ and $V \subset Z$ are open subsets, then:

$$U \times V = Y \times Z - \pi_2^{-1}(Z - V) - \pi_1^{-1}(Y - U) \subset Y \times Z$$

is open, with the induced structure of a quasi-projective variety.

Corollary 4.2. Quasi-projective varieties are varieties.

Proof. We start with projective space. The diagonal in $\mathbb{P}_k^n \times \mathbb{P}_k^n$ is:

$$\Delta = \{((a_0 : ... : a_n), (b_0 : ... : b_n))\ |a_i b_j = a_j b_i \text{ for all } i, j\}$$

which is closed, defined by linear polynomials: $\Delta = V(\langle x_{ij} - x_{ji}\rangle) \subset X$ and it follows that Δ is closed in $U \times U$ for any quasi-projective variety.

Example. The image of the Segre embedding s_{11} is the quadric:

$$\mathbb{P}_k^1 \times \mathbb{P}_k^1 = Q = V(x_{00}x_{11} - x_{01}x_{10}) \subset \mathbb{P}^3$$

and the diagonal is the intersection of the quadric with a hyperplane:

$$\Delta = V(x_{01} - x_{10}) \subset Q$$

which (substuting x_{10} for x_{01}) is the conic $C = V(x_{00}x_{11} - x_{10}^2) \subset \mathbb{P}_k^2$.

Corollary 4.3. The locus of agreement: $\{x \in U\ |\Phi_1(x) = \Phi_2(x)\} \subset U$ of a pair of morphisms $\Phi_1, \Phi_2 : U \to V$ of varieties is closed.

Proof. The two morphisms determine a morphism to the product:

$$f = (\Phi_1, \Phi_2) : U \to V \times V$$

and the locus of agreement is $f^{-1}(\Delta)$, which is closed since the variety V is separated and f is continuous. $\square$

Corollary 4.4. Let $\Phi : X - - > Y$ be a rational map of projective varieties, and let $U \subset X$ be an open subset on which $\Phi|_U$ is a morphism. Then an extension of Φ to a morphism defined on a larger open subset $U \subset V \subset X$ is unique, if it exists.

Next, we turn to a categorical version of compactness. Let $\mathcal{C}$ be a category of topological spaces and continuous maps in which products always exist, and let X be a separated object of $\mathcal{C}$.

Definition 4.5. X is **proper** if for all objects Y of $\mathcal{C}$, the projection:

$$\pi_2 : X \times Y \to Y \text{ is a closed map}$$

(i.e. π_2 maps closed sets to closed sets).

Remark. In a category $\mathcal{C}$ of topological spaces in which products have the product toplogy, compact spaces are proper. There are categories in which the reverse implication fails, but they are pretty pathological.

Example. The affine line is not proper. Consider the projection:

$$\pi_2 : k \times k \to k$$

The hyperbola $V(xy - 1)$, which is closed, projects to k^*, which is not.

All quasi-projective varieties that are not projective fail to be proper:

Proposition 4.4. Let $U \subset V$ be a nonempty open subset, properly contained in a variety V. Then U is not a proper object of $\mathcal{V}ar_k$.

Proof. The closed diagonal $\Delta \subset V \times V$ intersects $U \times V$ in a closed subset which projects to $U \subset V$, which is not closed.

In this sense proper varieties are *complete*.

Theorem 4.2.(Grothendieck) Projective varieties are proper varieties.

Proof.. Let $X \subset \mathbb{P}^n_k$ be a (closed) embedded projective variety. If $\mathbb{P}^n_k$ is proper, then any closed subset $Z \subset X \times Y$ is closed in $\mathbb{P}^n_k \times Y$, so it projects to a closed subset of Y and we conclude that X is proper. Thus it suffices to prove $\mathbb{P}^n_k$ itself is proper.

Next, if Y is a variety and $Z \subset \mathbb{P}^n \times Y$ is closed but $\pi_2(Z) \subset Y$ is not, let $Y = \bigcup U_i$ be an open cover by affine varieties. Then for all U_i, the intersection $Z \cap (\mathbb{P}^n_k \times U_i)$ is closed, but for some U_i, the projection, $\pi_2(Z) \cap U_i$, is not closed. So we may assume Y is affine.

Finally, if $Y \subset k^m$ is a (closed) embedded affine variety and the projection $\pi_2 : \mathbb{P}^n_k \times k^m \to k^m$ is a closed map, it follows immediately that the projection $\pi_2 : \mathbb{P}^n_k \times Y \to Y$ is also a closed map. Thus it suffices to show that for all m and n,

$$\pi_2 : \mathbb{P}^n_k \times k^m \to k^m \text{ is a closed map}$$

We accomplish this by reinterpreting the topology on $\mathbb{P}^n_k \times k^m$ in terms of a graded algebra over the *ring* $k[y_1, ..., y_m]$. Namely, consider:

$$R_\bullet := S \otimes_k k[y_1, ..., y_m] = \bigoplus_{d=0}^{\infty} S_d \otimes_k k[y_1, ..., y_m]$$

generated in degree one as an algebra over $R_0 = k[y_1, ..., y_m]$.

An element $F \in R_d$ therefore has the form $F = \sum m_I(x) \otimes g_I(y)$ (summed over multiindices I with $|I| = d$) and as in the case of ordinary projective space, the locus of points $(a, b) \in \mathbb{P}^n_k \times k^m$ for which:

$$F(a, b) = F((a_0 : ... : a_n), (b_1, ..., b_m)) = 0$$

is well-defined. We obtain a topology on $\mathbb{P}^n_k \times k^m$ by letting

$$Z(I) = \{(a, b) \in \mathbb{P}^n_k \times k^m \mid F(a, b) = 0 \text{ for all } F \in I_d\}$$

be the closed sets, for homogeneous ideals $I \subset R_\bullet$.

Lemma 4.1. This topology agrees with the topology on $\mathbb{P}^n_k \times k^m$ as a product in the category $\mathcal{V}ar_k$.

Proof. In the latter topology, a subset $Z \subset \mathbb{P}^n_k \times k^m$ is closed if and only if $Z \cap (U_i \times k^m)$ is closed in $U_i \times k^m$ for each basic open set $U_i \subset \mathbb{P}^n_k$. Clearly a closed subset $Z(I)$ in the former topology is closed in the latter, since:

$$Z(I) \cap (U_i \times k^m) = \left\{(a, b) \mid \frac{F}{x_i^d}(a, b) = 0 \ \forall F \in I_d\right\}$$

In the reverse direction, suppose that $Z \subset \mathbb{P}^n_k \times k^m$ is closed in the latter topology, and take a point $(a, b) \in (U_i \times k^m) - Z$. Then some

$$f \in k\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}, y_1, ..., y_m\right]$$

vanishes along $Z \cap (U_i \times k^m)$ but not at (a, b). Then for some d, the product $F = x_i^d f$ is in R_d and also vanishes on $(U_i \times k^m) \cap Z$ and not at (a, b). Multiplying by one additional x_i if necessary, we obtain $x_i F \in R_{d+1}$ vanishing at **all** points of Z but not (a, b).

Thus every point $(a,b) \in \mathbb{P}^n_k \times k^m$ may be distinguished from Z in this way by some $F \in R_d$. It follows that $Z = Z(I)$ for:

$$I = I(Z) = \langle F \in R_d \mid F(a,b) = 0 \text{ for all } (a,b) \in Z \rangle$$

i.e. Z is closed in the former topology.

Lemma 4.2. Suppose $Z \subset \mathbb{P}^n_k \times k^m$ is closed and $I = I(Z) \subset R_\bullet$ is the homogeneous ideal of elements of $R_\bullet$ that vanish at all points of Z. Then $\pi_2(Z) = V(I_0) \subset k^m$. In particular, $\pi_2(Z) \subset k^m$ is closed.

Proof. If $F \in I_0$, then by definition, $F(a,b) = 0$ for all $(a,b) \in Z$, but $F(a,b) = F(b)$ is independent of a, so $F(b) = 0$ for all $b \in \pi_2(Z)$. Thus:

$$\pi_2(Z) \subseteq Z(I_0)$$

The other direction is the interesting one. Suppose $b \in k^m - \pi_2(Z)$. We want to show that $b \in k^m - Z(I_0)$. That is, if $m_b \subset k[y_1, ..., y_m]$ is the maximal ideal associated to $b \in k^m$, we want to show that:

$$m_b + I_0 = k[y_1, ..., y_m]$$

The closed set $\pi_2^{-1}(\{b\}) = \mathbb{P}^n_k \times \{b\} \subset \mathbb{P}^n_k \times k^m$ is equal to $Z(R_\bullet \cdot m_b)$. By the projective Nullstellensatz, if closed sets $V(I), V(J) \subset \mathbb{P}^n_k$ defined by homogeneous polynomials have an empty intersection, then:

$$I_d + J_d = R_d \text{ for all } d >> 0$$

and the same holds here (Exercise). Since $\pi_2^{-1}(b) \cap Z(I) = \emptyset$, we have:

$$(*)\ R_d \cdot m_b + I_d = R_d \text{ for all } d >> 0$$

Now consider the $k[x_1, ..., x_m]$-module $M = R_d/I_d$. By $(*)$, we have: $m_b \cdot M = M$ and it follows from Nakayama's Lemma (Exercise) that:

$$(1 + f) \cdot M = 0 \text{ for some } f \in m_b$$

hence $(1+f)R_d \in I_d$, and then since $(1+f)x_i^d \in I_d$ for all d, we must have $(1+f) \in I_0$, as desired. □

Corollary 4.5. If X is a projective variety, then the image of X is closed under any morphism $\Phi : X \to Y$ of varieties.

Proof. Consider the graph of the map Φ:

$$\Gamma_\Phi = \{(x,y) \mid y = \Phi(x)\} \subset X \times Y$$

Then $\Gamma_\Phi = (\Phi, \mathrm{id})^{-1}(\Delta)$ for $(\Phi, \mathrm{id}) : X \times Y \to Y \times Y$ defined by $(\Phi, \mathrm{id})(x,y) = (\Phi(x), y)$. This is closed, since Y is separated, and:

$$\Phi(X) = \pi_2(\Gamma_\Phi)$$

which is closed since X is proper. □

We finish this section with a few applications to hypersurfaces.

Consider the multiplication maps for each $e < d$:

$$\mu : k[x_0, ..., x_n]_e \times k[x_0, ..., x_n]_{d-e} \to k[x_0, ..., x_n]_d$$

These are bilinear and non-degenerate, so they define:

$$\mu : P(k[x_0, ..., x_n]_e) \times P(k[x_0, ..., x_n]_{d-e}) \to P(k[x_0, ..., x_n]_d)$$

whose image is closed, by Corollary 4.5. Thus:

$$U_{\text{var}} = \{\text{irreducible } F \in k[x_0, ..., x_n]_d\} \subset \mathbb{P}^{\binom{n+d}{d}-1} = P(k[x_0, ..., x_n]_d)$$

is the Zariski open subset parametrizing hypersurface **varieties** in $\mathbb{P}^n$, as it is the complement of the union of the images of the μ maps.

At the other extreme, the most reducible polynomials are the image:

$$\rho : k[x_0, ..., x_n]_1 \times \cdots \times k[x_0, ..., x_n]_1 \to k[x_0, ..., x_n]_d$$

and the image of the resulting map on projective varieties:

$$\rho : \mathbb{P}^n \times \cdots \times \mathbb{P}^n \to \text{Sym}^d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{d}-1}$$

is the symmetric power of $\mathbb{P}^n$. By restricting, we obtain quasi-projective symmetric powers of quasi-projective varieties.

Consider the *universal hypersurface* in $X = \mathbb{P}^n_k \times \mathbb{P}_k^{\binom{n+d}{d}-1}$ defined by the bihomogeneous polynomial equation:

$$F = \{\sum_I y_I x^I\} = 0$$

This defines a closed subset of X (Exercise). Moreover, the locus of *relative singular points:*

$$V\left(\frac{\partial F}{\partial x_i}\right) \subset V(F) \subset X$$

is closed, whose projection to $\mathbb{P}_k^{\binom{n+d}{d}-1}$ is therefore closed, and:

$$U_{\text{ns}} = \{\text{nonsingular varieties}\} \subset U_{\text{var}}$$

is also Zariski open. To see that it is always **nonempty**, consider:

The Fermat Hypersurfaces.

$$Z_d = V(x_0^d + \cdots + x_n^d) \subset \mathbb{P}^n_k$$

are non-singular hypersurfaces for all n and d.

Remark. A similar analysis can be undertaken for homogeneous ideals $I \subset S$ with a fixed Hilbert polynomial. The difficult part for non-principal ideals is constructing the projective **Hilbert** scheme that parametrizing these ideals. Once that is done, the openness of the loci of varieties and smooth varieties is straightforward.

Exercises 4.

1. Prove that compact spaces are proper in any category of topological spaces with products in which products have the product topology.

2.. Let $R_\bullet = S \otimes_k k[y_1, ..., y_m]$. Prove that closed subsets $Z(I)$ and $Z(J)$ in $\mathbb{P}^n_k \times k^m$ associated to homogeneous ideals $I, J \subset R_\bullet$ fail to intersect if and only if:

$$I_d + J_d = R_d \text{ for all } d >> 0$$

3. Prove the version of Nakayama's Lemma we used in Theorem 4.2. If M is a finitely generated module over $k[y_1, ..., y_m]$ and $I \cdot M = M$ for some ideal $I \subset k[y_1, ..., y_m]$, then show that:

$$(1 + f)M = 0 \text{ for some } f \in I$$

4. (a) Prove by passing to open covers that the closed sets in the product topology on $\mathbb{P}^n \times \mathbb{P}^m$ are all of the form $Z(I)$ where $I \subset k[x_0, ..., x_n, y_0, ..., y_m]$ are *bihomogeneous* ideals, i.e. I is generated by polynomials F satisfying:

$$F(\lambda x, \mu y) = \lambda^d \mu^e F(x, y)$$

(b)Reconcile this with the claim that:

$$\mathbb{P}^n \times \mathbb{P}^m = \mathrm{mproj}\left(\bigoplus_{d=0}^{\infty} S_d \otimes R_d\right)$$

where $S = k[x_0, ..., x_n]$ and $R = k[y_0, ..., y_m]$.

(b) Investigate the irreducible curves (bihomogeneous hypersurfaces) in the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3_k$. Which of them are intersections of Q with a second hypersurface $S \subset \mathbb{P}^3_k$?

5. Prove that the second symmetric product:

$$\mathrm{Sym}^2(\mathbb{P}^n_k)$$

is isomorphic to $\mathbb{P}^2_k$ when $n = 1$, but that otherwise it is singular. Hint: Think about the subvariety of the projective space of quadrics.

6. Prove that an intersection of projective varieties $X_1, X_2 \subset \mathbb{P}^n_k$ of dimensions $\dim(X_i) = d_1$ is nonempty provided that the codimensions of the X_i satisfy:

$$(n - d_1) + (n - d_2) \le n$$

Hint: Consider the cones $C(X_i) \subset k^{n+1}$ and identify their intersection with the intersection: $\Delta \cap (C(X_1) \times C(X_2)) \subset k^{2n+2}$. Now use the fact that Δ is cut out by the "right" number of equations.