Example.

If $Z \subset \mathbb{P}^n$ and $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ is a projection with Λ s.t. $\Lambda \cap Z = \emptyset$,

(Def. $\Lambda = V(x_0, \dots, x_m)$)

then $\pi : Z \to \overline{\pi(Z)}$ is finite and dominant.

Fibers of a morphism.

Proposition. Suppose Y is an affine variety, and $Z \subset Y$ is a closed subvariety of codimension $c = \dim Y - \dim Z$. Then $\exists\, f_1, \dots, f_c \in k[Y]$ s.t. Z is a component of $V(f_1, \dots, f_c)$. Conversely, if X is a variety, and $g_1, \dots, g_c \in \Gamma(X, \mathcal{O}_X)$, then every component of $V(g_1, \dots, g_c) \subset X$ has codim $\leq c$.

Pf (Krull). Krull says in an affine variety Y, every component of $V(f)$ has codimension one.

$$Z \subset Y \Rightarrow \begin{cases} Z = Y, & \text{done.} \\ Z \subsetneq Y, & \text{and choose } f \in k[Y] \text{ s.t.} \end{cases}$$

$f(Z) = 0$. Then $\underset{\dim\, n-c.}{Z} \subset \underset{\dim = n-1}{W} = \text{component of } V(f) \subsetneq \underset{\dim = n}{Y}.$

Other direction. Every component $V(g_1) \subset X$ has

codim $= 0, 0+1, \dots$.

Cor. If $\phi: X \to Y$ is a dominant regular map of varieties and $W \subset Y$ is a closed subvariety of codim c, and $Z \subset \Phi^{-1}(W)$ is a component, (that dominates W)

$$X \supset \Phi^{-1}(W) \supset Z$$

then codim of Z in $X \leq c$.

Pf. Assume Y is affine. Choose $U \subset Y$ affine s.t. $U \cap W \neq \phi$.

$$\begin{array}{ccccc} X & \supset V = \Phi^{-1}(U) & \supset & Z \cap V \\ \downarrow & \downarrow & & \downarrow \\ Y & \supset U & & \supset W \cap U \end{array}$$

By the prop, $W \stackrel{cT}{\text{is}}$ a component of $V(f_1, \dots, f_c)$ and then $V = \Phi^{-1}(W) \subset V(\Phi^*(f_1), \dots,)$

$$\Phi^*(f_i) = g_i \in \Gamma(X, \mathcal{O}_X)$$

Let $Z \subset V$ be a component that dominates W, and contains $Z \subset Z' =$ component of $V(\Phi^*(f_1), \dots, \Phi^*(f_c))$.

Then $W = \overline{\Phi(Z)} = \overline{\phi(Z')} \subset V(f_1, \dots, f_c)$,

$$\Rightarrow \underbrace{Z \overset{=}{\subset} Z' \subset V}_{\text{components}} \quad . \blacksquare$$

$$\Phi: X \to Y$$

Thm. Over an open subset of Y, the fibers have the expected dimension $\dim(\cancel{X}) - \dim Y =: r$

In fact, there is an open subset $U \subset Y$ s.t.

(i) $U \subset \Phi(X)$

(ii) For every $W \subset Y$ intersecting U, and $Z \subset \Phi^{-1}(U) = V$ intersecting $\Phi^{-1}(U)$,

$$\dim(V) = \dim(W) + r.$$

Pf. Assume Y is affine, and X.

$$X = \bigcup V_i$$

$$\Phi : V_i \longrightarrow Y \rightsquigarrow U_i .$$

then $U = \bigcap U_i$. How to prove it for $\Phi : X \longrightarrow Y$ a map of affines.

Idea. $\Phi^* = f \circ k[Y] \hookrightarrow k[X]$.

Consider. $A := k[X] \otimes_{k[Y]} k(Y) \subset k(X)$ is a domain fin. gen. as a $k(Y)$-algebra.

Apply Noether normalization to A:

$$k(Y)\overbrace{[x_1, \ldots, x_r] \subset A}^{\text{integral}}$$