$\Phi : X \longrightarrow \mathbb{P}^n$

$\Phi(x) = (\phi_0(x) : \cdots : \phi_{n+1}(x))$; $\phi_i \in k(X)$

- somehow extends to a morphism

As it is written, $\Phi : U \to \mathbb{P}^n$, e.g. for $U = \bigcap \operatorname{dom}\phi_i \setminus V(\{\phi_i\})$.

<u>Varieties</u>.

$\mathcal{A}_k$ affine varieties

$\cap$

$\mathcal{P}_k \subset \mathcal{QP}_k$ quasi-proj. variety

proj. varieties

<u>Def</u>. Let $\mathcal{C}$ be a category. X, Y have a product if:

there is a triple (Z, π_1, π_2) s.t. any other triple (W, p_1, p_2) admits a unique morphism

$$\begin{tikzcd} & W \arrow[dl, "p_1"'] \arrow[d, dashed, "f"] \arrow[dr, "p_2"] & \\ X & Z \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

<u>Ex</u>. • Category of sets. $f(w) = (p_1(w), p_2(w))$.

• Category of topological spaces (Cartesian product w/ product topology).

<u>Prop</u>. Products always exist in $\mathcal{A}_k$.

<u>Pf</u>. Have equivalence (contravariant)

$$\mathcal{A}_k \xrightarrow{\Gamma} \begin{array}{l} k\text{-algebra} \\ \text{domains} \end{array}$$

In k-algebra domains, $A, B \longrightarrow A \otimes_k B$ is a coproduct (CA result. If A, B domains, $k = \bar{k}$, $A \otimes_k B$ is a domain). Then mspec $(A \otimes_k B)$ is the product of mspec A, mspec B.

__Ex.__ Want $k^m \times k^n = k^{m+n}$.

Just check that if $A = k[x_1, \ldots, x_m]$, $B = k[y_1, \ldots, y_n]$, $A \otimes_k B = k[x_1, \ldots, x_m, y_1, \ldots, y_n]$.

__Surprise!__ $X \times Y$ __is__ the Cartesian product, but the topology is not the product topology.

__Ex.__ $k^2 = k^1 \times k^1$

$k^1 \times k^1$ w/ product top: closed sets are unions of lines and points (horizontal + vertical lines)

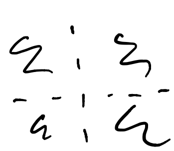
k^2 w/ Zariski top: diagonal is closed.

__Obs.__ The product is compatible w/ basic open sets, i.e.

$$\left.\begin{array}{l} U_f \subset X \\ V_g \subset Y \end{array}\right\} \Rightarrow \underset{\substack{\| \\ W_{fg}}}{U_f \times V_g} \subset X \times Y, \text{ and}$$

W_{fg} is basic open.

__Ex.__ $k^* \times k^* \subset k \times k$



Def. An object $(X, \mathcal{O}_X)$ of $\mathcal{N}_k$ is a prevariety if it is locally affine, i.e.

$$\forall x \in X, \quad \exists \quad x \in \underset{\substack{\text{open,} \\ \text{affine variety.}}}{U_x} \subset X$$

Rmk. $X = \bigcup_{x \in X} U_x = \bigcup_{\substack{i \in I, \\ |I| < \infty}} U_{x_i}$.

Prop. The full subcategory of prevarieties has products.

Sketch. $X, Y \to X \times Y$ is cartesian as a set.

Look at $X = \bigcup U_i$, $Y = \bigcup V_j$ and topologize $X \times Y$

by using affine products $U_i \times V_j$ to define the topology & sheaf $\mathcal{O}_X$ locally. Things glue by uniqueness.

Ugly example. The affine line w/ doubled origin

$X = k \cup 0'$. Topologized s.t.

$$i : k \to X \; ; \; 0 \to 0$$
$$i' : k \to X \; ; \; 0 \to 0'$$

are continuous. Sheaf of rational functions defined in obvious way. Want to rule this out — not let it be Hausdorff.

Categorical Alt. to Hausdorff

Def. $(X, \mathcal{O}_X)$ is separated if $\Delta \subset X \times X$ is closed, i.e. (X, id, id). By univ. property,

$$\begin{tikzcd}
& X \arrow[dl, "id"'] \arrow[d, dashed] \arrow[dr, "id"] & \\
X & X \times X \arrow[l, "\pi_2"] \arrow[r, "\pi_1"'] & X
\end{tikzcd}$$

Then $\Delta := \delta(X)$.

The doubled origin isn't separated, since

① diagonal only contains $(0,0)$, $(0',0')$.

② To be closed, must contain $(0',0)$, $(0,0')$.

On the other hand, affine varieties are separated.

$$\Delta \subset \underbrace{k^n \times k^n}_{\text{mSpec}(k[x_1, \dots, x_n, y_1, \dots, y_n])}$$

$$\Delta = V(\langle x_1 - y_1, \dots, x_n - y_n \rangle).$$

Def. A variety is a separated pre-variety.

Quest. Is the product of quasi-projective varieties quasi-projective?

Refined quest. Is the product of proj. varieties proj.?

Yes, but. $\mathbb{P}^n_k \times \mathbb{P}^m_k \not\cong \mathbb{P}^{n+m}_k$