Let $X \subset k^n$ be a variety. We want to

- topologize X
- define the sheaf of regular functions on X

Let $k[X] \subset k(X)$ be as before.

Def. $Z \subset X$ is closed if Z is an algebraic subset, i.e. $Z = V(I) = \{x \in X \mid f(x) = 0,\ \forall f \in k[X]\}$

Rmk.

closed sets $Z \subset X$ $\longleftrightarrow$ radical ideals in $k[X]$

varieties $Y \subset X$ $\longleftrightarrow$ prime ideals in $k[X]$

points $x \in X$ $\longleftrightarrow$ max ideals in $k[X]$.

Prop. this def of closed sets defines a topology on X.

$$V(I_1 \cdots I_m) = V(I_1 \cap \cdots \cap I_m)$$

Question. Why isn't $Z_1 \cup \cdots \cup Z_m = V(I_1 \cap \cdots \cap I_m)$

Def. A topology is Noetherian if every proper chain of closed sets: $Z_1 \supset \cdots$.

Def. A closed set $Z \subset X$ in a Noetherian topology is irreducible if $Z = Z_1 \cup Z_2 \Rightarrow Z = Z_i$.

Prop. Zariski is <u>Noetherian</u> and the <u>varieties</u> in X are the irreducible closed sets.

Pf. $Z_1 \supset \cdots$

$\updownarrow$

$I_1 \subset \cdots$

Corr. Every closed set is a finite union of varieties.

Pf. ~~Look at the set of all closed sets~~

Note that this automatically gives us primary decomposition.

Cor. Every open set $U \subset X$ is <u>dense</u>!

The Zariski topology is spectacularly non-Hausdorff.

Def. For any noetherian topology on an <u>irreducible</u> space X. Then the Krull dim. is the supremum of the lengths of chains of irreducible subsets.

$$X \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_n \supsetneq \phi$$

Thm. $\underset{\text{topological.}}{\underset{\downarrow}{\text{Krull dim}}} = \underset{\text{algebraic}}{\underset{\downarrow}{\text{tr. deg. } K(X)}}.$