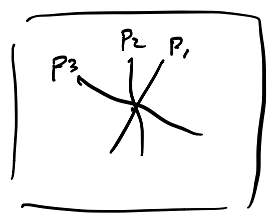
$\mathbb{P}^n_k$ is the set of lines thru the origin.
(one-dim quotients of k^{n+1})

$I \subset S$ homog.

$\rightsquigarrow V(I) \subset k^{n+1}$ is a cone thru the origin
$\rightsquigarrow V_h(I) :=$ set of rulings (lines thru origin) $\subset \mathbb{P}^n_k$ in $V(I)$

Conversely, given $X \subset \mathbb{P}^n_k$, define

$I(X) := I(C(X))$ where $C(X)$ is the union of rulings indexed by $X \subset \mathbb{P}^n_k$.

$X = \{p_1, \dots, p_m\} \subset \mathbb{P}^1_k$, then 

Convention: $I(\{\}) = I(\{0\}) = \langle x_0, \dots, x_n \rangle$

<u>Last Time</u>: $I(C(X))$ is homogeneous if k is infinite.

<u>Nullstellensatz for $\mathbb{P}^n_k$</u>

Given a homogeneous ideal $I \subsetneq \langle x_0, \dots, x_n \rangle \subset S$
then $I(V_h(I)) = \sqrt{I}$ for homogeneous ideals I.

<u>Correspondences</u>

$X \longleftrightarrow I \subset S \longleftrightarrow S/I = R.$ graded quotients algebraic sets $V_n(I)$ $\longleftrightarrow$ radical homogeneous ideals $\longleftrightarrow$ reduced graded quotient ring

varieties $\longleftrightarrow$ prime homog ideals $\longleftrightarrow$ domain graded quotients

points $\longleftrightarrow$ $I = \langle \ell_1, \dots, \ell_n \rangle$ $\longleftrightarrow$ $(S \to R \to 0)$

$$R = \bigoplus_{d \geq 0} (R_d = k)$$

Koszul $\to \bigoplus^n S(-1) \longrightarrow S \longrightarrow R \longrightarrow 0$

$(1, 0, \dots, 0) \mapsto \ell_1$

think about $k[x_0, \dots, x_n] / \langle x_1, \dots, x_n \rangle$

X proj. variety

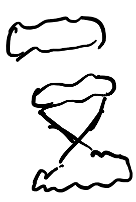
Def. (a) The homogeneous coordinate ring of $X \subset \mathbb{P}^n_k$ is $S/I(X) = k[X]$ = homog. coord. ring of the cone $C(X)$.

(b) the field of ~~fractions~~ rational functions on X is (not $k(C(X))$) is $k(X) = \{ \frac{F}{G} \mid F, G \in k[X]_d,\ G \neq 0 \}$ $\subset k(C(X))$.

Prop. $k(C(X)) \cong k(X)(x)$, $X \neq \{\}$

X

C(X)



Pf. Consider the integer graded ring

graded by difference in degrees

$$k(X) = R_0 \subset R = \left\{ \frac{F}{G} \;\middle|\; F, G \text{ homog}, G \neq 0 \right\} \subset k(C(X)).$$

Rmk. $R \simeq k(X)[x, x^{-1}]$, choose $x \in k[X]$, $x \neq 0$. .

Then $k(R) = k(C(X))$.

Def. (a) $\dim X$ = trans. deg. of $k(X)$ over k.

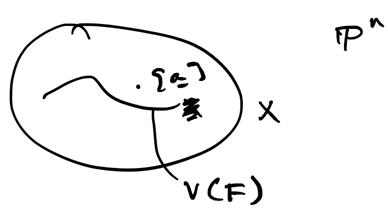
(b) X is non-singular at $[\underline{a}] \in X$ if

$$\operatorname{rank}\left(\frac{\partial F_i}{\partial x_j}\right)([\underline{a}]) = n - \dim X.$$

$\iff$ $[\underline{a}] \in C(X)$ is nonsingular

$\mathcal{I}(X) = \langle F_1, \dots, F_m \rangle$.

Ex. $I = \langle F \rangle$



F homog. of deg. d $\iff$

$$\sum x_i \frac{\partial F}{\partial x_i} = dF.$$

Want to know $\frac{\partial F}{\partial x_i}([\underline{a}]) = 0 \ \forall i$

$\iff$ $\underline{a} \in X$ is singular.

$\implies$ $\underline{a} \in X$.

Quadrics.

$$F \in S_2 \iff F = \sum_{i,j} \gamma_{ij} x_i x_j .$$

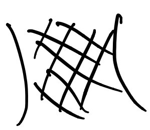
Let $\gamma_F = (\gamma_{ij})$. $V(F) \subset \mathbb{P}^n_k$ is a quadric

$$a = (a_0 : \cdots : a_n) \in V(F)$$
$$\iff (a_0 \cdots a_n)\, \Gamma_F \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = 0.$$

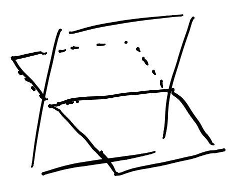
Exercise $V(F)$ non-singular $\iff \Gamma_F$ invertible.

In general, kernel of $\Gamma_F \subset k^{n+1}$ gives the ~~same~~ singular locus of $V(F) = Q$.

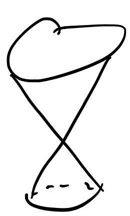
$\mathbb{P}^3_k$ non singular quadric



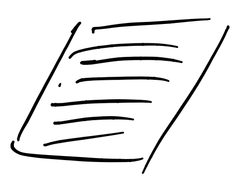
singular along a line



singular @ pt.



singular along a plane



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