Zariski topology on X & sheaf of rings of continuous functions $\phi : U \to k$ (w.r.t. Zariski topology):

$$\mathcal{O}_X(U) = \{ \phi \mid U \subseteq \mathrm{dom}(\phi) \} \subseteq k(X)$$

$\downarrow$

sections of the sheaf over $U \subset X$

<u>Def</u>. $\mathcal{N}_k$ is the category of Noetherian topological spaces X with a choice $\mathcal{C}_X$ of a non-zero sheaf of continuous functions $f : U \to k$.

<u>Def</u>. A morphism in $\mathcal{N}_k$ consists of $\Phi : X \to Y$ continuous which induces $\Phi^* : \mathcal{C}_Y(U) \to \mathcal{C}_X(\Phi^{-1}(U))$.

<u>Example</u>. Fix $X \neq \{pt\}$ a Noetherian topological space.

(i) $(X, \mathbb{K})$, where $\mathbb{K}(U) = k$, $\forall U$.

Rmk. The identity map $X \to X$ gives a morphism $id : (X, \mathcal{C}_X) \to (X, \mathbb{K})$, which shows constant functions are always legal. In some sense, $(X, \mathbb{K})$ is a final object.

(ii) $(X, \mathcal{F})$, where $\mathcal{F}$ is the sheaf of all con't functions. The identity $(X, \mathcal{F}) \to (X, \mathcal{C}_X)$ is a morphism.

<u>Def</u>. An object $(X, \mathcal{C}_X)$ in $\mathcal{N}_k$ is an affine variety (over k) if $(X, \mathcal{C}_X) \simeq (X, \mathcal{O}_X)$, $X \subset k^n$, i.e. the <u>category of affine varieties</u> $\mathcal{A}_k$ is the full subcategory whose objects are <u>affine varieties</u>.

Rmk. Given an object $(X, \mathcal{O}_X)$ in $\mathcal{N}_k$ and $U \subset X$, then we get a new object $(U, \mathcal{O}_X|_U)$. Rigged

given an induced topology

so that $i : U \hookrightarrow X_\varepsilon$ is a morphism.

Def. An object $(Y, \mathcal{O}_Y) \in \mathcal{N}_k$ is a quasi-affine variety over k if $(Y, \mathcal{O}_Y) \simeq (U, \mathcal{O}_X|_U)$ for some $U \subset X \subset k^n$.
QA_k is category of these.

Example. $k^\times \subset k$ has cofinite topology. Sheaf is
$\mathcal{O}_{k^\times}(U) := \mathcal{O}_k(U)$, $U \subset k^\times$.

$$\Gamma(k^\times, \mathcal{O}_{k^\times}) := \mathcal{O}_{k^\times}(k^\times) = k[x, x^{-1}].$$

Consider $X = V(xy-1) \subset k^2$. $(X, \mathcal{O}_X)$ is affine, and

$\Gamma(X, \mathcal{O}_X) = k[x,y]/(xy-1) \cong k[x, x^{-1}]$. Also,

$$k^\times \longleftrightarrow X \quad \text{by}$$

$$x \longrightarrow (x, x^{-1})$$

$$x \longleftarrow (x, y)$$

Th'm. The global sections functor:

$$\Gamma : (X, \mathcal{O}_X) \longrightarrow \Gamma(X, \mathcal{O}_X) := \mathcal{O}_X(X) = \text{k-algebra of global sections of } X$$

has a pullback

$$((X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)) \longrightarrow (\underset{\text{global pullback}}{\Phi^*} : \underbrace{\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)}_{\text{k-algebra homo's}})$$

which is an equivalence of categories A_k & finitely generated k-algebra domains.

Ex. $k^\times \subset k$, $k[x, x^{-1}] \supset k[x]$.

Pf. Construct the inverse.

$$\text{mspec}: (k\text{-algebra domains}) \longrightarrow \mathcal{A}_k \subset \mathcal{N}_k.$$

$\text{mspec}(A) = \{\text{max ideals of } A\}$, topologized by

$Z \subset \text{mspec}(A)$ is closed $\iff Z = Z(I) = \{m \mid \begin{smallmatrix} m \text{ maximal} \\ m \supset I \end{smallmatrix}\}$, some I

w/ sheaf: $\mathcal{O}_{X,x} = \{\phi \in k(X) \mid \phi = \frac{f}{g},\ g \notin m_x\} = A_{m_x}$

$$\mathcal{O}_X(U) = \bigcap_{x \in X} \mathcal{O}_{X,x}. \quad \boxed{\text{Rmk. } \mathcal{O}_X(U_h) = A[h^{-1}].}$$

Q. How is $\phi \in \mathcal{O}_X(U)$ a function $U \to k$?

Evaluate ϕ at m_x by

$$\phi \in A_{m_x}, \text{ consider } \bar{\phi} \in A_{m_x}/m_x A_{m_x} = k.$$

By the remark,

$$A \xrightarrow{\text{mspec}} (X, \mathcal{O}_X) \longrightarrow \mathcal{O}_X(X) = A$$

$$(X, \mathcal{O}_X) \longrightarrow k[X] \longrightarrow (X, \mathcal{O}_X).$$

Q. How does mspec connect $f: A \to B$

to $f^\#: \underset{\text{mspec}(B)}{\overset{\shortparallel}{(Y, \mathcal{O}_Y)}} \longrightarrow \underset{\text{mspec}(A)}{\overset{\shortparallel}{(X, \mathcal{O}_X)}}$?

We define it by $f^\#(m_y) := f^{-1}(m_y) \subset A$. Is

$f^\#$ continuous?

Claim. $(f^{\#})^{-1}(U_h) = U_{f(h)}$.

$\quad\quad\quad \overset{\shortparallel}{\{m_y \mid h \notin m_y\}}$

Does it pull back regulars to regulars? Need a map

$$(f^{\#})^{*} : \mathcal{O}_X(U_h) \longrightarrow \mathcal{O}_Y\left[(f^{\#})^{-1}(U_h)\right]$$

$$f : \quad \overset{\shortparallel}{A[h^{-1}]} \longrightarrow B[f(h)^{-1}]$$

Ex. If $n \geq 2$, $k^n \setminus \{0\} \subset k^n$ is not affine variety.

Check. $\mathcal{O}_{k^n}(k^n - \{0\}) = k[x_1, \dots, x_n]$.