Étale morphisms

$$AG \xleftrightarrow{GAGA} DG$$

étale morphism $\longleftrightarrow$ local iso. of manifolds

Given a morphism $Y \to X$ of schemes (say varieties). Etale if flat + unramified. If $y \mapsto x$, get

$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y,y}$ is ~~fai~~.

Flat $\longrightarrow$ faithfully flat

unram $\longrightarrow$
$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r] \arrow[d] & \mathcal{O}_{Y,y} \arrow[d] \\ k(x) \arrow[r] & k(y) \end{tikzcd}$$
finite separable

$k(?) = \mathcal{O}_?/m_?$

This square is Cartesian, i.e.

$$\mathcal{O}_{Y,y} \otimes_{\mathcal{O}_{X,x}} k(x) = \mathcal{O}_{Y,y} / m_x \mathcal{O}_{Y,y}$$

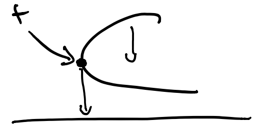
Rmk (flatness).

- For f locally finite, flat $\Rightarrow$ open.
- $\{y \in Y \mid \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y} \text{ is flat}\}$ as open.
- faithfully + finite type $\Rightarrow$ strict epi. This is part of Descent Theory

$$\begin{tikzcd} Y \arrow[d, "f"'] & Y' \arrow[l] \arrow[d, "f'"] \\ X & X' \arrow[l] \end{tikzcd}$$
faithfully flat + quasi-cpt.

If you have this Cartesian diagram, lots of f' properties descend.

Rmk (unram).



TFAE

(a) f is unramified

(b) $\forall x \in X$, $Y_x \longrightarrow k(x)$
$= \operatorname{Spec} k(x)$

is unramified.

$$\mathcal{O}_{Y_y, y} \simeq \mathcal{O}_{Y,y} / m_x \mathcal{O}_{Y,y}$$

(c) all geometric fibers are unramified

$$Y \times_{k(x)} k \longrightarrow k$$

$$k \longrightarrow X$$

$$k \supseteq k(x)$$

(d) $\forall x$, Y_x has an open covering by spectra of finite separable $k(x)$-algebra.

Def. An k-algebra A is separable $\iff$ the following equivalent conditions hold

(1) $A \otimes_k \bar{k}$ has Jacobson ideal 0

(2) $A \otimes_k \bar{k} \simeq \prod_{\text{finite}} \bar{k}$

(3) $A = \prod_{\text{finite}} k_i/k$ separable

(d) $\iff$ $Y_x = \coprod \operatorname{Spec} k_i/k$

Ex. open immersions, compositions, étale twisted by anything is étale.

Examples

1. k a field, $p(t) \in k[t]$ monic.

$$k[t]/(p) \longleftarrow k$$

Etale $\iff$ p is separable.

$$A[t]/(p) \longleftarrow A$$

unramified

$\Updownarrow$

$\overline{p} \ni 1$

Chapter 3, Section 10, Th'm 3.3, Mumford.

$$\begin{tikzcd} X \arrow[r, "\text{funr}"] & S \\ Y \arrow[u, "\text{étale}"] \arrow[ur, "\text{étale}"'] & \end{tikzcd}$$

The different

$$\delta_{Y/X} = \mathrm{ann}_{\mathcal{O}_Y} \Omega^1_{Y/X}$$