$f: k[Y] \longrightarrow k[X]$ is injective $\iff$ $\Phi = \operatorname{mspec}(f): X \to Y$ has a dense image.

__Pf.__ $h \in \ker f \iff U_h \cap \operatorname{im}\Phi = \emptyset$, so $(\operatorname{im}\Phi \subseteq V(h))$

__Corollary.__ If $\Phi: X \to Y$ has a dense image, $\Phi^*: k(Y) \hookrightarrow k(X)$.

Conversely, $k(Y) \hookrightarrow k(X) \rightsquigarrow$ rational map $\Phi: X \to Y$, with dense image.

$$\begin{tikzcd} k[y_1, \dots, y_n] \arrow[d, two heads, squiggly] & & \\ f: k[Y] \arrow[r, hook] & k(Y) \arrow[r, hook] & k(X) \end{tikzcd}$$

$$\phi_i = f(\bar{y}_i)$$

$$\Phi = (\phi_1, \dots, \phi_m): X \dashrightarrow Y.$$

__Def.__ X, Y varieties then $\Phi: X \to Y$ is __dominant__ if $\operatorname{im}\Phi$ is dense.

__Obs.__ Φ is dominant $\iff$ $\forall$ affines $U \subset X$, $V \subset Y$ s.t.

$\Phi|_U : U \to V$ (i.e. is regular),

$(\Phi|_U)^* : k[V] \hookrightarrow k[U]$.

__Def.__ The field of rational functions of a variety of X is $k(X) := \varprojlim_{U \subset X} \mathcal{O}_X(U)$ (inverse limit).

__Rmk.__ $k(X) = k(V)$ for any open affine $V \subset X$.

Roughly, dominant rational maps $\longleftrightarrow$ field inclusions for fixed X, Y.

f surjective.

$f : k[Y] \to k[X]$ is surjective $\Leftrightarrow$ $\exists$ irred. closed subset

$$\begin{array}{c} Z \subset Y \\ \cap \\ k^n \end{array} \quad \text{such that}$$

f induces an isomorphism $f|_X : X \xrightarrow{\sim} Z$.

This is because $k[X] \simeq k[Y]/\ker f$. We can

let $Z = \ker f$. Then take mspec's.

Note: We can globalize this def, but must explain why irr. closed subset can be a variety (what's the topology? what's the sheaf of functions?)

f integral.

Def. f integral if $k[X]$ is f.g. as a module over $k[Y]$.

Rmk. If f is not injective, factor through the kernel:

$$k[Y] \to k[Y]/\ker f \hookrightarrow k[X].$$

$$Y \supset Z \xleftarrow{\text{dominant}} X$$

Example. (Noether Normalization)

Given X, $k[X] = k[x_1, \dots, x_n]/p$.

$k(X)$ has tr. deg. m over k.

Then $\exists$ m linear comb's $y_j = \sum a_{ij} x_i$

s.t. $k[y_1, \dots, y_m] \hookrightarrow k[X]$ is an integral map.

This corresponds to the diagram

$$\begin{tikzcd} k^n \arrow[r, "\text{projection}"] & k^m \\ \cup & \\ X \arrow[uur, "\text{finite}"'] & \end{tikzcd}$$

Def. $\operatorname{mspec}(f)$ is <u>finite</u> if $f : k[Y] \longrightarrow k[X]$ is integral.

Thm. A finite map $\Phi : X \longrightarrow Y$ satisfies

(i) $|\Phi^{-1}(y)| < \infty \quad \forall y \in Y$.

(ii) Φ maps closed sets to closed sets

In particular, if $f : k[Y] \longrightarrow k[X]$ is integral + injective, then $\Phi : X \longrightarrow Y$ is surjective, closed, finite fibers.

Pf. (i) by CRT. (ii) by Cohen-Seidberg.

(i) $\Phi : X \longrightarrow Y \qquad\qquad f : k[Y] \longrightarrow k[X]$

$\quad\;\; \cup \qquad\quad \in \qquad\qquad\qquad \cup$

$\quad \Phi^{-1}(y) \quad\;\; y \qquad\qquad\qquad m_y$

$$\Phi^{-1}(y) = \{ m_a \in k[X] \mid m_a \supseteq \langle f(m_y) \rangle \}$$

Consider $f : \underset{\substack{\cong \\ k}}{k[Y]/m_y} \longrightarrow \underbrace{k[X]/\langle f(m_y)\rangle}_{} =: A$

$\uparrow$ integral $\Longleftrightarrow$ f.d. vector space.

Claim. # max ideals in $A \leq \dim_k(A) < \infty$.

Example. $A = k[x]/(x-r_1)\cdots(x-r_n) \cong k^n$ has n maximal ideals $(x - r_i)$.

$A = k[x]/x^n$ has one, i.e. $\langle x \rangle$.

(iii) $\Phi : X \xrightarrow[\text{dominant}]{\text{finite}} Y$; $\overline{f(Z)} = V(f^{-1}(I)) \subset Y$.

$Z = V(I)$ Next time.

$I \subset k[X]$