Th'm Proj varieties are proper.

Cor. Given any morphism $\Phi : X \longrightarrow Y$, for X proj, Y separated, has a closed image.

Pf. Factor Φ through the graph

$$\begin{tikzcd} & X \times Y \arrow[r, "(\Phi, id)"] \arrow[d, "\pi_2"] & Y \times Y \\ X \arrow[ur, "\gamma_\Phi = (id, \Phi)"] \arrow[r, "\Phi"] & Y & \end{tikzcd}$$

Let $\Gamma_\Phi = \operatorname{im} \gamma_\Phi = \{ (x, \Phi(x)) \}$

$$\| $$

$$(\Phi, id)^{-1}(\Delta) = \{ (x,y) \mid \Phi(x) = y \}$$

Γ_Φ is closed, since Y is separated. $\Phi(X) = \pi_2(\Gamma)$ is closed since X is proper.

Application: Hypersurfaces

$$V(F) = X_F \subset \mathbb{P}^n_k \quad , \quad F \in k[x_0, \dots, x_n]_d$$

Have multiplication: $\mu : k[x_0, \dots, x_n]_e \times k[x_0, \dots, x_n]_{d-e} \longrightarrow k[x_0, \dots, x_n]_d$

$$\rightsquigarrow \tilde{\mu} : P(k[x_0, \dots, x_n]_e) \times P(k[x_0, \dots, x_n]_{d-e}) \longrightarrow P(k[x_0, \dots, x_n]_d)$$

For X_F in the images, F factors. Let $U_{var} \subset P(k[x_0, \dots, x_n]_d)$ be varieties, i.e. (X_F s.t. F is irreducible). So the set of irreducible varieties is closed.

$\underline{\text{Ex.}}$ Look at $\mu : \mathbb{P}(k[x_0,\dots,x_n]_1)^n \longrightarrow \mathbb{P}(k[x_0,\dots,x_n]_d) = \mathbb{P}^N$

$\qquad\qquad\qquad\qquad \overset{\shortparallel}{\mathbb{P}^n} \qquad\qquad\qquad\qquad F = \prod H_i$

$(H_1,\dots,H_d) \longrightarrow \prod H_i$

$\mu : (\mathbb{P}^n)^d \longrightarrow (\mathbb{P}^n)^d/\Sigma_d = \mathrm{Sym}^d(\mathbb{P}^n) \overset{\text{(closed)}}{\subset} \mathbb{P}^N$

$\underline{\text{Consider}}$ the universal hypersurface of degree d.

$\mathbb{P}(k[x_0,\dots,x_n]_d) = \mathbb{P}^{\binom{n+d}{d}-1}$

$$X = V\Big(\underbrace{\sum_{|I|=d} y_I x^I}_{\text{bihomog. of bidegree } (d,1)}\Big) \subset \underbrace{\mathbb{P}^n}_{S = k[x_0,\dots,x_n]} \times \underbrace{\mathbb{P}^{\binom{n+d}{d}-1}}_{R = k[y_I]}$$

$Z := V\Big(\dfrac{\partial F}{\partial x_i}\Big)_i \subset X$ is the locus of singularity of fibers of the map $\pi|_X$.

$(d-1,1)$ — try Euler

$\pi(Z) \subset \mathbb{P}^{\binom{n+d}{d}-1}$ is closed ($\underline{\text{Not}}$ everything, e.g. $x_0^d + \dots + x_n^d = 0$ is non-singular.

Morphisms.

Def. A regular map of varieties is a rational map $\Phi : X \to Y$ that is everywhere defined.

Rmk. Every regular map is locally a map of affine varieties.

$$\Phi : X \longrightarrow Y$$
$$\quad\ \ \ \rotatebox{90}{$\in$} \qquad\ \ \rotatebox{90}{$\in$}$$
$$\quad\ \ x \qquad\ \ y$$

$$\operatorname{dom}\Phi \cap \Phi^{-1}(U) \longrightarrow U \leftarrow \text{affine}$$
$$\cup$$
$$\text{affine} \to W$$

i.e. $\exists$ affine nbhds $x \in W$, $\Phi(x) \in U$ s.t.

$$\Phi|_W : W \longrightarrow U$$

is a map of affines.

$$\Phi : X \longrightarrow Y \quad \longleftrightarrow \quad \Phi^* : k[Y] \longrightarrow k[X]$$

- k-algebra homomorphism.

(a) closed embeddings
$\Phi : X \simeq Z \hookrightarrow Y$
$\quad \hookrightarrow$ closed, irred.

(b) dense image

(c) finite maps

Property. (i) Fibers are finite
(ii) $\Phi(Z)$ is closed if $Z \subset X$ is closed

(a) • surjective

(b) • injective

(c) • integral. Via Φ^*, $k[X]$ is a finite $k[Y]$-module.

Examples.

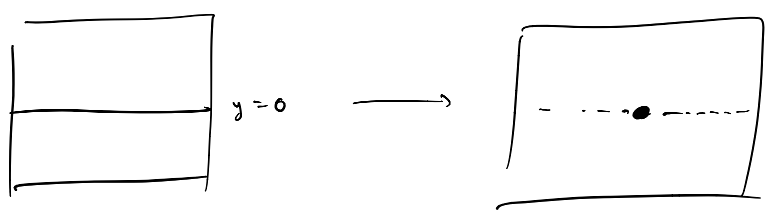
(a) $X \subset k^n \quad \leftrightsquigarrow \quad \Phi^* : k[x_1, \ldots, x_n] \twoheadrightarrow k[X]$.

(b) Projection: $\pi : X \times Y \longrightarrow Y \quad \leftrightsquigarrow \quad \pi^* : k[Y] \hookrightarrow k[X] \otimes k[Y]$

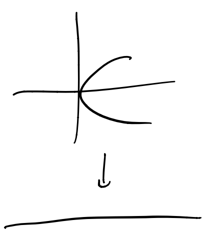
Inclusion of $U_f \hookrightarrow X$ a basic open $\qquad i^* : k[X] \hookrightarrow k[X][f^{-1}]$.

$\uparrow$ not a fin. gen. module.

Blow down: $\sigma : k^2 \longrightarrow k^2$, $\sigma(x,y) = (xy, y)$ $\qquad \sigma^* : k[x,y] = k[xy, y] \subset k[x', y']$.



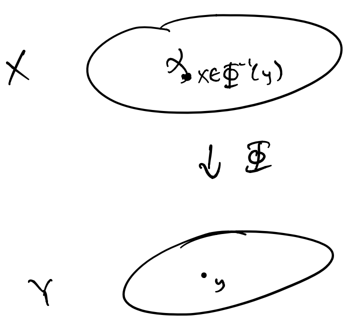
(c)



$P = (x = y^2) \subset k^2$

$\pi : P \longrightarrow k^1 \qquad \pi^* : k[x] \longrightarrow k[x,y]/\langle x - y^2 \rangle$

- Upper-semi-continuity of the dimension of fibers:



dimensions of components of fibers can only jump down.

(set of pts. where dim is $\geq$ is Zariski closed)

Contrast Diff geo.