X variety.

$x \in X \rightsquigarrow \mathcal{O}_{X,x}$ local ring of germs of regular functions

$Z \subset X \rightsquigarrow \mathcal{O}_{X,Z}$ " " of germs of regular functions along Z.

irreducible, closed

$$\dim \mathcal{O}_{X,Z} = \dim X - \dim Z$$
$$= \dim k(X) - \dim k(Z).$$

<u>Def.</u> X is <u>normal</u> if $\mathcal{O}_{X,Z} \subset k[X]$ are integrally closed $\forall Z \subset X$.

<u>Def.</u> Let A be a domain, $A \subset K$. $x \in K$ <u>integral</u> over A if x satisfies monic polynomial w/ coeff. in A.

The <u>integral closure</u> of A in K is the set of all $x \in K$ integral over A. A is <u>integrally closed</u> if it equals its integral closure.

<u>Rmk.</u> ϕ integral over $A \iff A[\phi]$ is fin. gen.

<u>Pf.</u> $(\Rightarrow)$ $\phi^n + a_{n-1}\phi^{n-1} + \cdots + a_0 = 0 \Rightarrow \phi^{k \geq n} = \sum_{i<n} a_i \phi^i$.

$(\Leftarrow)$ $A[\phi] = Ay_1 + \cdots + Ay_n$

$y_i y_j = \sum a_{ij} y_j$

<u>Cor.</u> The integral closure is a ring.

<u>Rmk.</u> $\overline{A_S} = (\overline{A})_S$.

Cor. Let X be an affine variety. Then X is normal $\iff k[X] \subset k(X)$ is <u>integrally closed</u>.

Pf. $k[X]$ int. closed $\Rightarrow k[X]_S$ is integrally closed at localizations, i.e. at prime ideals, which gives that $\mathcal{O}_{X,Z}$ are integrally closed.

($\Rightarrow$) suffices to know that $k[X]_m$ are integrally closed? ∎

Ex. A UFD is always integrally closed.

Pf. Rational root th'm, plus lemma:

$(a,bc)=1$ and $(a,c)=1$, $a \mid b$.

Ex 1. $X = k^n$ is <u>normal</u>. $k[X] = k[x_1, \dots, x_n]$ is a UFD.

Ex 2. $X = V(y^2 - x^3)$ is not <u>normal</u>.

$$k[X] = k[x,y]/\langle y^2 - x^3 \rangle \simeq k[t^2, t^3] \subset k[t] = k(X).$$

$$x \longleftarrow t^2$$
$$y \longleftarrow t^3$$

But $t \in k(t)$ is integral over $k[t^3, t^2]$.

Ex 3. $X = V(y^2 - x^2(x+1))$

$k[x,y]/\langle y^2 - x^2(x+1) \rangle$ is not integrally closed.

Fact. If X is affine, $\dim = 1$, normal, then $k[X]$ is a <u>Dedekind domain</u>.

Along the same lines:

X any normal variety,

$Z \subset X$ has codim 1.

Then $\mathcal{O}_{X,Z}$ is a DVR.

$$\begin{array}{c}\mathcal{O}_{X,Z}\\ \cup \\ m_Z = \langle \pi \rangle.\end{array}$$

Thm. Given X a variety, then $\exists\ r : \tilde{X} \to X$ regular, finite, dominant, birational, s.t. $\tilde{X}$ is normal, and $\tilde{X}$ is the <u>normalization</u>. r is universal:

$$\begin{tikzcd} Y \arrow[r, dashed, "(!)"] \arrow[dr, "\text{dominant}"'] & \tilde{X} \arrow[d] \\ & X \end{tikzcd}$$

Moreover if X is affine then $\tilde{X}$ is affine; similarly for projective.

<u>Pf (affine case).</u> X affine. $k[X] \subset \overline{k[X]} \in k(X)$

<u>Q</u>: Is $\overline{k[X]}$ a fin. gen. k-algebra?

<u>Q2</u>: $k[X] \subset \overline{k[X]}$ is <u>integral</u> (so $\overline{k[X]}$ is a f.g. module).

More generally: Suppose $A \subset K(A) \subset L$ separable

Suppose $A = \bar{A}$. Then the integral closure of A in L is a f.g. module over A.