$$\mathrm{Mat}_n \supset X_{n-1} \supset X_{n-2} \supset \cdots \supset X_r \supset \cdots$$

$$\mathrm{Mat}_n \cong k^{n^2}$$

$$X_r = \{ A \mid \dim \mathrm{rk}(A) \leq r \}$$

$$(\dim \ker(A) \geq n-r)$$

<u>Last time:</u> $X_r \subset \mathrm{Sing}(X_{r+1})$

Every $\mathfrak{m}_x/\mathfrak{m}_x^2$ for $x \in X_r \subset X_{r+1}$ has largest possible $\dim = n^2$.

$$I_r = \{ (A, \Lambda) \mid \Lambda \leq \ker(A) \} \subset X_r \times G(n-r, n)$$

$$\begin{tikzcd} & I_r \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ X_r & & G(n-r,n) \end{tikzcd}$$

$$G(n-r,n) \ni \Lambda \qquad \pi^{-1}(\Lambda) \cong k^{rn}$$

<u>Rmk.</u> I_r is a smooth variety of $\dim = \dim(G(n-r,n)) + rn$

$$= (n-r)r + rn$$
$$= n^2 - (n-r)^2$$

$(\Lambda = \langle e_1, \dots, e_{n-r} \rangle;$

$$A = \begin{array}{c} \begin{array}{cc} \overbrace{\quad\;}^{n-r} & \overbrace{\quad\;}^{r} \end{array} \\ \left[\begin{array}{cc} 0 & * \end{array} \right] \end{array} \Big\} n$$

$$\Lambda = \langle e_1, \dots, e_{n-r} \rangle$$

Open set containing Λ:

$$\begin{array}{c} \overbrace{\qquad\qquad}^{n-r} \\ \left(\begin{array}{c} 1 \\ \ddots \\ 1 \\ * \end{array}\right) \end{array} \Big\}\, n-r$$

$\uparrow$ $r(n-r)$ coordinates.

(on $G(n-r,n)$)

near Ω

So we have

$$\begin{array}{ccc} \pi^{-1}(V) & \subset & I_r \\ \downarrow & & \downarrow \\ V & \subset & G(n-r,n) \end{array}$$

$$= \left\{ \left(\begin{array}{cccc} a_{11} & a_{12} & & * \\ & & a_{ij} & \end{array}\right)\left(\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{array}\right) = \underbrace{\left(\begin{array}{ccc} a_{11}+* & & \\ & \ddots & \\ & & a_{ij}+* \end{array}\right)}_{\text{affine linear equations}} = 0 \right\}$$

in $\left(\quad \middle| * \right) \times \left(\frac{\quad}{*} \right)$

Then we have $\simeq \Omega$

$$\begin{array}{ccccc} & (A, \ker(A)) & & & \\ A \nearrow & & \text{bijection } W \subset I_r & & \text{non-singular} \\ & X_r \setminus X_{r-1} \subset X_r & \swarrow & \searrow & G(n-r,n) \end{array}$$

Direct Pf that $X_r \setminus X_{r-1}$ is non-sing.

$X_n \setminus X_{n-1}$ is homog. for $GL(n,k) = Mat_n \setminus X_{n-1}$

(right mult)

Can get ~~to any~~ from one rank r to any other

Stab of $A \in X_r \setminus X_{r-1} \simeq GL(\ker(A))$.

Then $\dim = n^2 - \dim \mathrm{stab} = n^2 - (n-r)^2$

Consider a cone, I homog. ideal.

$V(I) \subset k^{n+1}$

E.g. $I = \langle F \rangle$

(i) $F = \sum a_i x_i \rightsquigarrow V(I) \simeq k^n$

is non-sing at 0.

$\deg\left(\frac{\partial F}{\partial x_i}\right) \geq 1$, homog. $\Rightarrow \nabla F(0) = 0$.

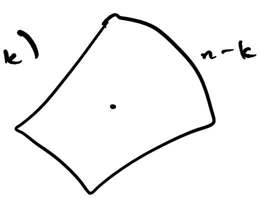
(ii) Arbitrary homog. ideal I,

$I = \langle L_1, \dots, L_k, F_i \rangle$

higher deg.

$I = \langle F_i \rangle$

$V(L_1, \dots, L_k)$



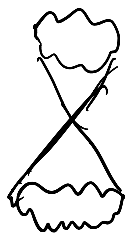
$V(I)$ sing. at 0, w/ $J(0) \equiv 0$

Away from 0, $V(I) = C(X)$ is birational to $X \times k$.

$X = V(I) \subset \mathbb{P}^n$.

$I \subset k[x_0, \dots, x_n]$.

Then $R_\bullet = k[x_0, \ldots, x_n]/I$ may be integrally closed.



(localizing)
$\Rightarrow$ $V(I)$ is normal (including at 0!)

<u>E.g.</u> Suppose $X \subset \mathbb{P}^n$ is normal. Does it follow that $C(X)$ is normal? Not necessarily.

$X \subset \mathbb{P}^n$

$R_\bullet \subset \overline{R_\bullet} \subset k(R_\bullet)$

$\rightsquigarrow$

$R_{d\bullet} \subseteq \overline{R_{d\bullet}}$ for an appropriate d.

Properties of non-singular points.

<u>Prop.</u> If $x \in X$ non-singular, then

$$\widehat{\mathcal{O}_{X,x}} \cong k[[x_1, \ldots, x_m]]$$

$(m = \dim X)$.

Let $u_1, \ldots, u_m$ gen $m \subset A$.

Get $k[[u_1, \ldots, u_m]] \longrightarrow \hat{A}$ (Look up Cohen or something)

When (A, m) is regular and

$\bar{x}_1, \dots, \bar{x}_n \in m/m^2$ is a basis,

then $k[[\cancel{x}_1, \dots, x_n]] \to \bar{A}$ is an iso.