Def. $f : k[Y] \longrightarrow k[X]$ is *integral* if $k[X]$ is fin. gen. as a $k[Y]$-module.

$$\Phi : X \longrightarrow \overline{\Phi(X)} = Z \subset Y \iff k[Y] \longrightarrow \overset{k[Z]}{\overset{\shortparallel}{k[Y]/\ker f}} \longrightarrow k[X]$$

In practice, usually assume Φ is dominant.

Example. $k[X] = k[x_1, \ldots, x_n]/I(X)$. If X has dim $= m$, then $\exists$ integral $f : k[y_1, \ldots, y_m] \subset k[X]$ where $y_i = \sum a_{ij} x_j$ and

$$\underbrace{X \subset k^n \xrightarrow{\text{projection}} k^m}_{\text{finite}}$$

Thm. If $\Phi : X \to Y$ is a finite map of affine varieties then

(a) Φ is finite-to-one ($|\Phi^{-1}(y)| \leq \#$ of generators of $k[X]$ as a $k[Y]$-module

(expect: if Φ is finite & dominant

$[k(X) : k(Y)]$ - to - one)

(b) Φ is a closed map

Pf. Assume $Y = \overline{\Phi(X)}$ and Φ is dominant. Let

(a) $f = \Phi^* : k[Y] \longrightarrow k[X]$.

$Y \leftrightarrow \operatorname{mspec}(k[Y])$
$= \{ m_y \subset k[Y] \}$

Then. $\Phi^{-1}(y) = \Phi^{-1}(m_y) = \{ m_x \subset k[X] \mid m_x \supset f(m_y) \}$

$\Updownarrow$

$f^{-1}(m_x) = m_y$

$k = k[Y]/m_y \subset A = k[X]/\langle f(m_y) \rangle$

algebraic, so A is a finite-dim k-alg of dim $\leq$ # generators of $k[X]$.

$\overset{(Ex)}{\Rightarrow}$ A has $\leq \dim(A)$ max ideals.

<u>Ex.</u> $\Phi : k^* \longrightarrow k$ is <u>not</u> surjective

$$f : k[x] \hookrightarrow k[x, x^{-1}]$$
$$(x) \longrightarrow (x) = k[x, x^{-1}]$$

(b) Suffices to <u>prove</u> $\Phi : X \longrightarrow Y$ is surjective if Φ is finite & dominant. (Why? can restrict Φ to irreducible closed subset)

Given $m_Y \subset k[Y]$, WTS $\exists\, m_X \subset k[X]$ s.t. $m_X \cap k[Y] = m_Y$. Localize at m_Y in $k[X]$. By correspondence, suffices to find max ideal $m \subset k[X]$ s.t.

$$m \cap k[Y]_{m_y} = m_y k[Y].$$

Choose a max'l ideal $m \subset k[X]_S$ and consider the subring $B := k[Y]_S / m \cap k[Y]_S \subset k[X]_S$. Need to show B is a field.

We <u>know</u> $k[X]_S$ is a finitely generated $k[Y]_S$-module and $k[X]_S/m$ is a fin. gen. $k[Y]_S / m \cap k[Y]_S = B$-module.

Choose $\phi \in B - \{0\}$. If ϕ were not invertible, we'd have

$B \subsetneq \phi^{-1}B \subsetneq \phi^{-2}B \subsetneq \cdots \quad \subsetneq k[X]_s/m \ni \phi^{-1}$. Contradicts Noetherian. ∎

<u>Rmk.</u> Same idea proves <u>Null</u>.

k <u>arbitrary</u> <u>field</u> (infinite)

Let $m \subset k[x_1, \dots, x_n]$ be a maximal ideal.

$$K = k[x_1, \dots, x_n]/m$$

Consider K/k

(a) Algebraic (tr. deg. 0) (if $k = \bar{k} \Rightarrow m = \langle x_i - a_i \rangle$)

(b) tr. deg. $d > 0$

In that case, apply Noether norm. $(d > 0)$

$$\underbrace{k[y_1, \dots, y_d] \subset K}_{\text{integral}}$$

<u>Then</u> let $f \in k[y_1, \dots, y_d]$ be <u>any</u> <u>poly</u> of pos deg, and consider

$$k[y_1, \dots, y_d] \subsetneq f^{-1}k[y_1, \dots, y_d] \subsetneq \cdots \subsetneq K$$

Eventually $f^{-n}k[y_1, \dots, y_d] = f^{-n-1}k[y_1, \dots, y_d]$

$\Rightarrow f^{-n-1} = g \cdot f^{-n} \Rightarrow g = f^{-1}$, contradiction.

<u>Noether.</u> $k[X] = k[x_1, \dots, x_n]/p$, $\dim X = m$.

$\overset{?}{\Rightarrow}$ $\exists\, y_i = \sum a_{ij} x_j$ s.t.

$$\underbrace{k[y_1,\dots,y_m] \subset k[X]}_{\text{integral}}.$$

Pf. By induction on $n-m$. $n-m=0$, ✓.

$$f(\bar{x}_1,\dots,\bar{x}_n)=0, \quad \bar{x}_i \in k[X].$$

If $f = c\bar{x}_n^d + (\text{lower deg terms})$.

Then $k[X]$ gen by $1, \bar{x}_n, \dots, \bar{x}_n^{d-1}$

as a module over $k[x_1,\dots,x_{n-1}]/p \cap k[x_1,\dots,x_{n-1}]$.

Problem. What if f doesn't have this form?

Try $y_1 = x_1 + a_1 x_n, \ \dots \ y_{n-1} = x_{n-1} + a_{n-1} x_n$.

Then make $a_1,\dots,a_{n-1}$ large enough.