From last time,

- products of affine varieties exist as affine varieties
- products of prevarieties exist as prevarieties

Def. A prevariety is separated if $\Delta \subset Y \times Y$ is closed.

Claim. Affine varieties are separated $(\Delta = V(\langle x_i - y_i \rangle) \subset k^n \times k^n)$

Def. $Var_k \subset \mathcal{N}_k$ is the category of varieties.

$\mathcal{A}_k \subset Var_k$ is full sub-category of affine varieties.

$$\begin{array}{ccc} \mathcal{A}_k & \subset & Var_k \\ & & \cap \\ QP_k & \subset & Prev_k \end{array}$$

Last time. $\mathbb{P}^n_k \times \mathbb{P}^m_k \neq \mathbb{P}^{n+m}_k$.

Th'm. There are no morphisms $\Phi : \mathbb{P}^n_k \longrightarrow \mathbb{P}^m_k$, $m < n$.

Pf. Given $\Phi : \mathbb{P}^n_k \dashrightarrow \mathbb{P}^m_k$. Suppose $U_0 \cap \operatorname{im}\Phi \neq \phi$

so $\Phi(x) = (1 : \varphi_1(x) : \cdots : \varphi_m(x)) = (F_0 : F_1 : \cdots : F_m)$ by clearing denominators w/ $LCM(F_0, \ldots, F_m)$.

Claim. Φ extends exactly to $\mathbb{P}^n_k \setminus \underset{\overset{\shortparallel}{U_0}}{V(F_0)} \cap \cdots \cap \underset{\overset{\shortparallel}{U_m}}{V(F_m)}$. When $m < n$, $V(F_0) \cap \cdots \cap V(F_m) \neq \phi$.

Pf. If $x \in U$, then $F_i(x) \neq 0$ for some i, and

$\Phi = \left(\frac{F_0}{F_i} : \cdots : \frac{F_m}{F_i}\right)$, so Φ extends to x. Conversely,

if Φ extends to x, suppose $\Phi(x) \in U_i$. Then

$\Phi = (\psi_0 : \cdots : 1 : \cdots : \psi_m)$ and $\psi_i = \frac{F_0 \cdots F}{F \cdots F}$. Then $F_i(x) \neq 0$.

Now use Krull. Consider $V(F_0) \cap \cdots \cap V(F_m) \subset K^{n+1}$, which is a cone containing the origin.

By induction and Krull, ~~the~~ each irreducible component of $V(F_0) \cap \cdots \cap V(F_m)$ has $\dim \geq n-m \geq 0$ (by assumption). <u>And</u> there is one such component, consists of <u>more</u> than just $\{0\}$. Then $V(F_0) \cap \cdots \cap V(F_m) \neq \emptyset$ as a subset of $\mathbb{P}^n_k$. ■

So, what is $\mathbb{P}^m_k \times \mathbb{P}^n_k$?

<u>Answer</u>. Segre embedding $s_{m,n} : \mathbb{P}^m_k \times \mathbb{P}^n_k \hookrightarrow \mathbb{P}_k^{(n+1)(m+1)-1}$

$$((a_0 : \cdots : a_m), (b_0 : \cdots : b_n)) \longrightarrow (\cdots : \underset{x_{ij}}{\underset{\shortparallel}{a_i b_j}} : \cdots)$$

Let $X \subset \mathbb{P}_k^{(m+1)(n+1)-1}$ be $X = V(\langle x_{ij} x_{k\ell} - x_{i\ell} x_{jk} \rangle)$

$= \operatorname{im}(s_{m,n})$.

<u>Claim</u>. X is the product of $\mathbb{P}^m_k, \mathbb{P}^n_k$ in Var_k.

(i) exhibit the projections

(ii) Prove the universal property in Prevar_k

(iii) Check separatedness.

(i) $\pi_1(\cdots : a_i b_j : \cdots) = (a_0 b_0 : \cdots : a_m b_0)$,

$\pi_2(\quad " \quad) = (b_0 a_0 : \cdots : b_n a_0)$

Note that X is the Cartesian product, so we have:

$$\begin{tikzcd} & W \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[d, dashed, "f\ \text{of sets}"] & \\ \mathbb{P}^m & X \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathbb{P}^n \end{tikzcd}$$

prevariety.

Let $w \in W$, $w \in U \subset W$, then $p_1(w) = (\varphi_0 : \cdots : \varphi_m)$
$\uparrow$ affine $\qquad p_2(w) = (\psi_0 : \cdots : \psi_n)$

Then $f(w) = (\cdots : \varphi_i \psi_j : \cdots)$. ∎

So $\mathbb{P}^m_k \times \mathbb{P}^n_k = X \hookrightarrow$ in P_k. $\rightsquigarrow$ Product of quasi-projectives is quasi-projective.
$\llcorner$ in Var_k

<u>Corollary</u>. $\mathbb{P}^n_k$ is separated.

$$\Delta \subset X = \mathbb{P}^m_k \times \mathbb{P}^n_k$$

$$\psi$$

$$((a_0 : \cdots : a_m), (b_0 : \cdots : b_n)) \in \Delta \Longleftrightarrow \quad a_i b_j = a_j b_i \quad \forall i, j.$$

So $\Delta = V(\langle x_{ij} - x_{ji} \rangle)$.

<u>Ex</u>. $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$

$= (a_0 b_0 : a_0 b_1 : a_1 b_0 : a_1 b_1)$

$= V(x_{00} x_{11} - x_{01} x_{10}) = Q$

diag $= Q \cap V(x_{01} - x_{10})$.

$= V(x_{00} x_{11} - x_{01}^2) \subset \mathbb{P}^2_k$

$\llcorner$ conic $\cong \mathbb{P}^1_k$.

<u>Def</u>. Suppose $\mathcal{C}$ is the category of topological spaces in which products ~~objects~~ exist, and X is a separated object of $\mathcal{C}$. Then X is <u>proper</u> if X is

"universally closed," i.e. $p_2 : X \times Y \longrightarrow Y$ are <u>closed</u> maps for any $y \in Y$.

<u>Rmk</u>. X Compact in top. spaces $\Rightarrow$ X proper.

(almost)

<u>Ex</u>. k is not proper.

$V(xy-1) = Z \subset k \times k$.

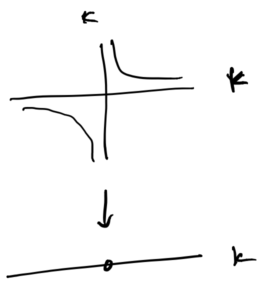

image is $k^\times$, which is not closed.