$\mathcal{F} \in Sh_X$, sheaves on X. We then took an injective res

$$0 \longrightarrow \mathcal{F} \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

We can look at

$$0 \longrightarrow 0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

and apply global sections:

$$0 \longrightarrow 0 \longrightarrow \Gamma(I^0) \longrightarrow \Gamma(I^1) \longrightarrow \cdots$$

Then we defined $H^p(X, \mathcal{F}) = H^p(\Gamma(I^\bullet))$. Some remarks about spectral sequences.

We want to form a category containing our abelian category $\mathcal{A}$ and all corresponding resolutions, and objects are isomorphic to their resolutions.

We can try <u>category of complexes</u> $C^*(\mathcal{A})$:

$$\longrightarrow C^{i-1} \longrightarrow C^i \xrightarrow{d} C^{i+1} \longrightarrow \cdots \quad C^i \in \mathrm{Obj}(\mathcal{A}),$$

$d^2 = 0$, where $* = \phi, b, +, -$; where b means bounded complexes, $+$ is bounded below, $-$ is bounded belows and ϕ means no restrictions. The morphisms are families of morphisms with a commutation restriction.

In this category, we already have a desired property. Consider, for example,

$$\begin{tikzcd}
\cdots \arrow[r] & \cdots \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \overset{\deg 0}{A} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
& 0 \arrow[r] & 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots
\end{tikzcd}$$

This morphism is injective (?), and we can embed A into $C^*(A)$.

Let B be a category and S a family of morphisms of B. Consider all functors $F: B \to C$ such that $F(S)$ consists of only of isomorphisms.

<u>Question.</u> Does there exist Q, $S^{-1}B$ such that

$$\begin{tikzcd}
& B \arrow[dl, "Q"'] \arrow[ddr] & \\
S^{-1}B \arrow[drr, bend left, "F"'] & & \\
& & C
\end{tikzcd}$$

Objects of $S^{-1}B$ are objects of B, and morphisms are complicated, but it does always exist.

In $C^*(A)$, we can define $H^p(C^\cdot) = \ker d^p / \operatorname{im} d^{p-1}$.

If $f: C^\cdot \to D^\cdot$ is a morphism of complexes, we get $H^p(f^\cdot): H^p(C^\cdot) \to H^p(D^\cdot)$.

<u>Def</u>. $f^\cdot$ is a <u>quasi-isomorphism</u> if all $H^p(f^\cdot)$ are isomorphisms.

If we take $\varepsilon^\cdot$ to be the morphism from

$$\cdots\ 0 \to A \to 0\ \cdots$$

$$\cdots\ I^0 \to I^1\ \cdots$$

is a <u>quasi-isomorphism</u>.

<u>Def</u>. The <u>derived</u> <u>category</u> $D^*(A)$ is the localization of $C^*(A)$ with $S = \{$all quasi-isomorphisms$\}$.

<u>Rmk</u>. $D^*(A)$ is not necessarily abelian, and morphisms are mysterious.

<u>Morphisms</u>

Morphisms and compositions are paths and killing the S allows certain edges to be invertible.

<u>Ex</u>. Look at A-modules. $\mathcal{B} \subset \mathcal{A}$ is a subcategory.

$S = \left\{ \begin{array}{l} \text{morphisms in } A \text{ with} \\ \ker \text{ in } \mathcal{B}, \text{ im in } \mathcal{B} \end{array} \right\}$.

To talk about this, we need to talk about the $K^*(A)$, the homotopic category of complexes.

<u>Def</u>. Take two complices $C^\cdot, D^\cdot$ and $f, g : C^\cdot \longrightarrow D^\cdot$, then $f^\bullet, g^\bullet$ are <u>homotopic</u> if $\exists\, h_i : C^i \longrightarrow D^{i-1}$ if $dh^\cdot + h^\cdot d = f^\bullet - g^\bullet$. Being homotopic is an equivalence relation on maps.

<u>Rmk</u>. This kills the abelian-ness. Also, $H^p(f^\bullet) = H^p(G^\bullet)$, if $f^\bullet \sim g^\bullet$.

Take $\varphi : C^\bullet \to D^\cdot$ a morphism in $D^*(\mathcal{A})$. What does this look like?

roofs

$$
\begin{tikzcd}
 & E^\bullet \arrow[dl, "\mathrm{Mor}(K^*(A)) \ni s^\bullet"', "\sim"] \arrow[dr, "f^\bullet \in \mathrm{Mor}(K^*(A))"] & \\
C^\bullet & G^\bullet \arrow[u, dashed, "\sim"'] \arrow[d, dashed, "\sim"'] & D^\bullet \\
 & F^\bullet \arrow[ul, "t^\bullet", "\sim"'] \arrow[ur, "g^\bullet"'] &
\end{tikzcd}
$$

quasi-iso

quasi-iso

If the above diagram commutes, we say that the roofs $(E^\bullet, s^\bullet, f^\bullet)$, $(F^\bullet, t^\bullet, g^\bullet)$ are equivalent(if $G^\bullet$ exists).

Composition

$$
\begin{tikzcd}
 & A^\bullet \arrow[dl, "\sim"'] \arrow[dr, "f"] & & B^\bullet \arrow[dl, "\sim"'] \arrow[dr, "g"] & \\
C^\bullet & & D^\bullet & & E^\bullet
\end{tikzcd}
$$

In homotopic category, $\exists\ F^\bullet$ s.t

$$
\begin{tikzcd}
 & & F^\bullet \arrow[dl, "\sim"'] \arrow[dr, "h"] & & \\
 & A^\bullet \arrow[dl, "\sim"'] \arrow[dr, "f"] & & B^\bullet \arrow[dl, "\sim"'] \arrow[dr, "g"] & \\
C^\bullet & & D^\bullet & & E^\bullet
\end{tikzcd}
$$

s.t. FADB commutes. $\xrightarrow{\sim}$ indicates quasi-iso's, which are closed under composition.

<u>Question</u>. How can this be useful?

<u>Th'm</u>. $D^*(A)$ is a <u>triangulated</u> <u>category</u>,

which means in this category there are distinguished triangles.

$\exists\ T: D^*(A) \circlearrowleft \quad T(X^\bullet)^i = X^{i+1}$ such that

$d^i_{T(X^\bullet)} = -d^{i+1}_{X^\bullet}$

<u>Def</u>. A distinguished triangle is a

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow T(X^\bullet),$$

which is represented as

$$\begin{tikzcd} & Z^\bullet \arrow[dl, "{[1]}"'] & \\ X^\bullet \arrow[rr] & & Y^\bullet \arrow[ul] \end{tikzcd}$$

<u>Def</u>. Morphism of triangles is a family of maps

$$\begin{tikzcd} X^\bullet \arrow[r] \arrow[d] & Y^\bullet \arrow[r] \arrow[d] & Z^\bullet \arrow[r] \arrow[d] & T(X^\bullet) \arrow[d] \\ X'^\bullet \arrow[r] & X'^\bullet \arrow[r] & Y'^\bullet \arrow[r] & T(X'^\bullet) \end{tikzcd}$$

<u>Claim</u>. This is an <u>additive</u> <u>category</u>.

<u>Axioms of D.T.</u>

(1) Any triangle iso. to a dist. triangle is distinguished.

(2)
$$\begin{tikzcd} & 0 \arrow[dl, "{[1]}"'] & \\ X^\bullet \arrow[rr, "id"] & & X^\bullet \arrow[ul] \end{tikzcd}$$
dist.

(3) For any $f: X^\bullet \to Y^\bullet$, $\exists$ d.t.
$$\begin{tikzcd} & Z^\bullet \arrow[dl, "{[1]}"'] & \\ X^\bullet \arrow[rr, "f"] & & Y^\bullet \arrow[ul] \end{tikzcd}$$
→ Thm This is determined up to isomorphism.

(4)
$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd}$$
is d.t. $\iff$
$$\begin{tikzcd} & T(X) \arrow[dl, "-T(f)"'] & \\ Y \arrow[rr, "g"'] & & Z \arrow[ul] \end{tikzcd}$$
is d.t.

(5)
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, dotted] & T(X) \arrow[d, "T(f)"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X') \end{tikzcd}$$

The above diagram commutes

(6) octahedral axiom.

Def. $F: \mathcal{C} \to A$

abelian

triangulated

F is a **coho** functor if for every d.t.
$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd},$$
we have $F(X) \to F(Y) \to F(Z)$ is exact.

Claim. $H^0(-): D^*(A) \to A$ is a coho. functor.

$\Rightarrow$ long exact seq. of coho. groups of a dist. triangle.