Problem 1a. Let $I = I(X), J = I(Y)$. Then we claim $X \cup Y = V(IJ)$. Clearly, $V(IJ) \supset X \cup Y$, since for $h := \sum_{f_i \in I(X), g_i \in I(Y)} f_i g_i$, if $x \in X$, $h(x) = \sum f_i(x) g_i(x) = \sum 0 \cdot g_i(x)$. Similarly, for $y \in Y$, $h(y) = 0$.

On the other hand, these two sets differ only if there is a point $p \notin X \cup Y$ such that for any h as above, $h(p) = 0$. In particular, this means that for any $f, g \in I, J$, $f(p)g(p) = 0$. Hence $p \in V(I) = X$, a contradiction.

If we take $\mathbb{C}[X]$, then consider the family of ideals $\langle x - n \rangle$, $n \in \mathbb{Z}$. These are maximal ideals, hence prime, so their vanishing sets are varieties. The union of these varieties is $\mathbb{Z}$, but this is not an algebraic set, since any polynomial vanishing on all of $\mathbb{Z}$ must be the zero polynomial. But that would vanish on all of $\mathbb{C}$, not just $\mathbb{Z}$.

Problem 1b. Here, we claim that $X \cap Y = V(I + J)$. Clearly, $X \cap Y \subset V(I + J)$, since one way for a sum $f + g$ ($f, g \in I, J$) to vanish is for each term to vanish. On the other hand, if we have $p \notin X \cap Y$ but $p \in V(I + J)$, then $f(p) = 0$ for every $f \in I$, contradicting the fact that $X = V(I)$.

If we consider $k[X]$, and we take the prime ideals $\langle x \rangle, \langle 1 - x \rangle$, then their sum is the unit ideal, which is not prime since it isn't proper.

Problem 1c. Using the Lasker-Noether theorem, we can conclude that for any algebraic set $X = V(I)$, we can write I as an intersection of primary ideals $P_1, ..., P_n$. If we let $r(J)$ be the radical of J, using the fact that I is a radical ideal and $r(J \cap K) = r(J) \cap r(K)$, we find that $I = r(I) = r(P_1) \cap \cdots \cap r(P_n)$. Since the radical of a primary ideal is prime, this proves what we wanted.

Problem 2a. Without loss of generality, let p, q, r have distinct x coordinates. Let $f = (x - p_1)(x - q_1)(x - r_1)$ and let $g = y - q(x)$, where $q(x)$ is a quadratic passing through the points p, q, r. Then it's clear $V(\langle f, g \rangle) = S$.

Problem 2b. The question amounts to whether $\langle f, g \rangle$ is a geometric ideal. If it is, then by the Nullstellensatz, $\langle f, g \rangle = I(V(\langle f, g \rangle))$. Since $I(S)$ is geometric, $I(S) = I(V(I(S)))$. Since $S = V(\langle f, g \rangle)$ is algebraic, then $S = V(I(S))$, hence $V(\langle f, g \rangle) = V(I(S))$. Taking I of both sides would give us $\langle f, g \rangle = I(S)$. So the failure must be that $\langle f, g \rangle$ is not a geometric ideal.

To see that this happens, take $k = \mathbb{C}$. Let the three points be $(0,0), (1,0), (2,i)$. Then $f = x(x-1)(x-i), g = y - \frac{i}{2}x(x-1)$. We claim that a linear combination of f, g must be at least quadratic in x, if it is non-zero. Note that a linear combination of these is always cubic or larger in x, unless the f-coefficient is 0 or if we balance out the powers.

If the f-coefficient is zero, then any non-zero polynomial (that is a linear combination) will be at least quadratic in x. If we balance out the powers, with a non-zero f-coefficient, we need to be more careful. Let $h = af + bg$. By factoring, one can see that $h = x(x-1)(ax - ai - \frac{bi}{2}) + by$. Then $a \neq 0$, so the degree of x in the third factor is at least 0, so h is still at least quadratic.

However, if we take $h(x, y) = 2iy^2 + xy$, it vanishes at the three points above, but it can't be a linear combination of f, g, since it's not quadratic in x.

Problem 2c. Without loss of generality, we can assume all y-coordinates are 0. Let $P_i = \langle x - x_i, y \rangle$. Then if $p, q, r = (x_1, 0), (x_2, 0), (x_3, 0)$, then $S = V(\prod_{i=1}^{3} P_i)$. Then $I(S) = I(V(P_1 P_2 P_3)) = \sqrt{P_1 P_2 P_3} = \sqrt{P_1}\sqrt{P_2}\sqrt{P_3} = P_1 P_2 P_3$.

Problem 2d.

Problem 3a. Let $q = ax^2 + bxy + cy^2$. Otherwise, q would factor into two linearly independent factors. Now assume q has a singular point (x_0, y_0) different than the origin. Then q_x, q_y vanish at that point, which induces several requirements.

- $b \neq 0$, otherwise q is degenerate.
- $c \neq 0$, otherwise q is degenerate.

Hence $y_0 = -2ax_0/b = -bx_0/2c$, hence $b^2 = 4ac$. Then $\frac{1}{2b}q_x q_y = \frac{1}{2b}(2ax + by)(2cy + bx) = \frac{1}{2b}(2abx^2 + (b^2 + 4ac)xy + 2bcy^2)$. Notice that $b^2 = 4ac$ and $4ac/b = b$. Hence $(b^2 + 4ac)/2b = 8ac/2b = b$. Hence $q = \frac{1}{2b}q_x q_y$, as desired.

On the other hand, if $q = fg$, where f, g are homogenous linear polynomials, then $q_x = f_x g + g_x f$ and $q_y = f_y g + f g_y$. Let $f = ax + by + c, g = dx + ey + f$. Then the unique solvability of the equations $q_x = q_y = 0$ requires that $ae - bd \neq 0$, or that $a/d \neq b/e$, which requires f, g to be linearly independent.

An example of a singular conic that doesn't factor in more than 2 variables is $x^2 - yz$.

Problem 3b. Let $h(x, y) = y^d - f(x)$, and let h_x, h_y be the partials. Then there is a simultaneous solution to $h = 0, h_x = 0, h_y = 0$ if and only if f, f_x share a root, which is true if and only if f has a double root.

Problem 4a.