Q. What are the morphisms $\Phi : k^n \to k^m$?

Get a pullback $\Phi^* : k[x_1, \dots, x_m] \to k[x_1, \dots, x_n]$

Then, $\Phi^*(x_i) \in k[x_1, \dots, x_n]$, and

$$\Phi = (\Phi^*(x_i), \dots)$$

Generalize $\Phi : \underset{N_k}{\underset{\cap}{X}} \to k^n$. $\Phi^*(x_i)$ still polynomial.

Ex. $X \subset k^n$

$$\begin{array}{c} X \subset k^n \\ \updownarrow \\ k[x_1, \dots, x_n] \twoheadrightarrow k[X]. \end{array}$$

mspec : Rings $\longrightarrow$ Affine Varieties

onto $\rightsquigarrow$ closed embeddings

injective $\rightsquigarrow$ dominating maps

integral $\rightsquigarrow$ finite maps

Graded rings

$R_\bullet$ finitely generated graded k-algebra domains

$R_0 = k$, $R_1 = \langle x_0, \dots, x_n \rangle$.

R_1 generates $R_\bullet$ as a graded ring.

Def. $R_+ = \bigoplus_{d>0} R_d$

Def. $X := \mathrm{mproj}(R_\bullet) = \left\{ \begin{matrix} \text{maximal } \underline{\text{prime}} \\ \text{homogeneous ideals} \end{matrix} \; m_x \subsetneq R_+ \right\}$

Zariski topology : $Z \subset X$ is $\underline{\text{closed}} \iff Z = \{ m_x \mid I \subset m_x \}$ for some homogeneous $I \subset R$.

$\underline{\text{Sheaf of regular functions}}$

$$K(X) = \left\{ \frac{F}{G} \;\middle|\; F, G \in R_d,\ G \neq 0 \right\} \subset k(R_\bullet)$$

$$\mathcal{O}_{X,x} = \left\{ \frac{F}{G} \;\middle|\; G \not\subset m_x \right\} \subset K(X)$$

$$\mathcal{O}_X(U) := \bigcap_{x \in U} \mathcal{O}_{X,x}$$

$\underline{Q}$: How is an elem of $\mathcal{O}_X(U)$ a continuous function $f : U \to k$.

$\underline{\text{Recall}}$. $\mathrm{mspec}(A)$, $\mathcal{O}_{X,x} = A_{m_x}$. For

$$\phi \in \mathcal{O}_{X,x}, \quad \phi(x) := \overline{\Phi} \in A_{m_x}/m_x A_{m_x} = k.$$

What is a maximal ideal in $R_\bullet$?

Choose a basis $\langle x_0, \dots, x_n \rangle = R_1$

$$0 \to P \longrightarrow S \twoheadrightarrow R \longrightarrow 0$$

$$y_0^{i_0} \cdots x_n^{i_n} \longrightarrow x_0^{i_1} \cdots x_n^{i_n}$$

poly mult ring mult

We know what the max ideals are in S:

$$\langle \underbrace{l_1, \dots, l_n}_{\text{lin. indep. elements on } S.} \rangle$$

If we consider $\langle l_1, \dots, l_n \rangle \subset R_\bullet$, then two possibilities

- they give a max ideal $(P \subset \langle l_1, \dots, l_n \rangle)$
- this ideal satisfies $\sqrt{I} = R_+$.

$$\Rightarrow x_i^N \in I \text{, some } N \ \forall i.$$

Then $R_d \subseteq I$ for $d \geq (n+1)(N-1)+1$

This gives us

$$\overset{F,G}{R_\bullet} / \underset{= \langle l_1, \dots, l_n \rangle}{m_x} \cong k[X]_\bullet \to k \text{ in every degree}$$

So for $\phi = \frac{F}{G}$, then $\bar{\Phi} = \frac{\bar{F}}{\bar{G}} \in k[x]$. So

$$\phi(m_x) := \frac{\bar{F}}{\bar{G}}$$

Take $X = \text{mproj}(R_\bullet)$. Given $G \in R_d$.

<u>Define</u>. $U_G ::$ basic open set $X \setminus V(G)$

<u>Prop</u>. $\mathcal{O}_X(U_G) = (R_\bullet[G^{-1}])_0 = \{ \frac{F}{G^n} \mid \deg F = n \deg G \}$

<u>Special case</u>: $G = 1$, $V(G) = \emptyset$; $U_G = X$, $\mathcal{O}_X(X) = R_0 = k$.

Pf of Special Case:

Given $\phi \in k(X)$, define

$$I_\phi = \text{ideal of denominators}$$

$$= \{ g \in R_\bullet \mid g \cdot \phi \in R_\bullet \}$$

$$= \langle G \in R_d \mid G \cdot \phi \in R_d \rangle$$

Assume $\phi \in \mathcal{O}_X(X) \Rightarrow V(I_\phi) = \emptyset \overset{\text{proj Null}}{\Rightarrow} x_i^N \in I_\phi \ \forall i.$

$$\Rightarrow R_d \subset I_\phi \text{ for } d \gg 0.$$

$$\overset{\text{cleverness}}{\Rightarrow} \phi \cdot R_d \subseteq R_d$$

Let $G \in R_d$. Then $G \cdot \phi \in R_d \Rightarrow G \cdot \phi^2 \in R_d$

$$\Rightarrow \cdots \Rightarrow G \cdot \phi^n \in R_d.$$

So $R_\bullet \subset R_\bullet + \phi R_\bullet \subset \cdots \subset G^{-1} R_\bullet$

$\Rightarrow$ eventually, there is a relation

$$\phi^n = f_{n-1}\phi^{n-1} + \cdots + f_0$$

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deg 0 $\quad f \in R.$

$$\Rightarrow \phi^n = c_{n-1}\phi^{n-1} + \cdots + c_0 \Rightarrow \phi = c.$$