Coherent Sheaves

$A = k[X]$, $\text{mspec}\, A =: X$

M f.g. module $\rightsquigarrow$ $\tilde{M}$ a coherent sheaf

$$\Gamma(\tilde{M}, U_f) = \tilde{M}(U_f) = M_f$$

$$\tilde{M}_z = M_{P = v(z)}$$

Let X arbitrary variety.

A sheaf of $\mathcal{O}_X$-modules is coherent if $\exists\, X = \bigcup U_i$ ← affines

s.t. $\mathcal{F}|_{U_i} = \tilde{M}_i$ where M_i are f.g. modules $/ A_i = k[U_i]$.

Projective Varieties.

$X = \text{mproj}\,(R_\bullet)$

$R_\bullet$ graded

$R_0 = k$

R_1 generates $R_\bullet$

$M_\bullet$ a f.g. graded module over $R_\bullet$

$$M_\bullet = \bigoplus_{d=n}^{\infty} M_d \;; \quad R_d \times M_e \longrightarrow M_{d+e}$$

Want $\tilde{M}_\bullet$, coherent sheaf on $X = \text{mproj}\,(R_\bullet)$

Def. Given $f \in R_d$,

$$(M_\bullet)_f = \left\{ \frac{m}{f^k} \;\middle|\; \deg(m) = dk \right\} / \sim$$

$$(M_\bullet)_S = \left\{ \frac{m}{f} \;\middle|\; \begin{array}{l} m, f \text{ homog of same degree} \\ f \in S \end{array} \right\} \Big/ \sim$$

e.g. $S = R_\bullet \setminus P_\bullet$, $(M_\bullet)_{P_\bullet} := (M_\bullet)_S$.

$$\tilde{M}_\bullet (U_f) := (M_\bullet)_f$$

$$(\tilde{M}_\bullet)_x := (M_\bullet)_{P_\bullet}$$

$\Gamma(\tilde{M}_i, U) =$ as in the affine case

This coherent because $\tilde{M}_\bullet|_{U_f} = \widetilde{(M_\bullet)_f}$

<u>Q</u>: $M_\bullet \rightsquigarrow \tilde{M}_\bullet \overset{(?)}{\rightsquigarrow} M_\bullet$

<u>affine case</u>: $M \rightsquigarrow \tilde{M} \rightsquigarrow M = \Gamma(X, \tilde{M})$.

$A \rightsquigarrow \mathcal{O}_X = \tilde{A} \rightsquigarrow \Gamma(\mathcal{O}_X) = k[X] = A$.

<u>projective</u>. $R_\bullet \rightsquigarrow \tilde{R}_\bullet = \mathcal{O}_X \rightsquigarrow \Gamma(X, \mathcal{O}_X) = k$

oops.

<u>Serre's Twisting sheaf</u>.

$$M_\bullet := R_\bullet(e) = \bigoplus_{d=-e}^{\infty} R_{d+e}$$

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graded module over $R_\bullet$

$R_\bullet = S_\bullet = k[x_0, \dots, x_n]$

$\widetilde{S_\bullet(e)} =: \mathcal{O}_X(e)$

coherent sheaf on $\mathbb{P}^n = \text{mproj}(S_\bullet)$

Look at $\mathcal{O}_X(e)\big|_{U_i} = \widetilde{\Gamma(U_i, \mathcal{O}_X(e))}$

$= \left\{ \frac{F}{x_i^d} \;\middle|\; \deg F = d+e \right\}$

as a module over $k[U_i]$ $\cong k\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$.

$\left\{ x_i^e \left(\frac{F}{x_i^{d+e}} \right) \right\} \overset{(!)}{\Rightarrow} \mathcal{O}_X(e)$ is a line bundle.

$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(e)) = (S_\bullet(e))_0 = S_e = k[x_0, \dots, x_n]_e.$

$s \in$

$\| \; \left\{ s_i = \frac{F}{x_i^d} \;\middle|\; \rho_{U_i \cap U_j}(s_i) = \rho_{U_i \cap U_j}(s_j) \right\}$

$$\frac{F_i}{x_i^d} = \frac{F_j}{x_j^d} \iff x_i^d, x_j^d \mid F_i, F_j$$

$$\Rightarrow G_i = G_j = G \in k[x_0, \dots, x_n]_e$$

<u>Rmk.</u> We can recover

$$S_\bullet = \bigoplus_{e \geq 0} \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(e))$$

$\uparrow$ is a graded ring

Try this for a graded module over $R_\bullet$.

$$M_\bullet(e) = \bigoplus M_{d+e}$$

<u>sheafify</u>: $\widetilde{M_\bullet(e)} = \tilde{M}_\bullet \otimes \mathcal{O}_X(e) = \widetilde{R_\bullet(e)}$

$$M_\bullet(e) = M_\bullet \otimes R_\bullet(e)$$

<u>global sections.</u> $\Gamma(X, \tilde{M}_\bullet \otimes \mathcal{O}_X(e))$

Serre, FAC.

$$\boxed{\begin{gathered}\bigoplus_{e=0}^{\infty} \Gamma(X, \tilde{M}_\bullet \otimes \mathcal{O}_X(e)) \text{ and } M_\bullet \\ \text{agree in all sufficiently high degrees.}\end{gathered}}$$

<u>Ex.</u> $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$

$(s : t) \longrightarrow (s^4 : s^3 t : s t^3 : t^4)$

image $X = \operatorname{mproj} k[s^4, s^3 t, s t^3, t^4]$

$$\left(\begin{tikzcd} & \mathbb{P}^4 \arrow[d, "\text{project.}"] \\ \mathbb{P}^1 \arrow[ur, "(s^4 : \cdots : t^4)"] \arrow[r, hook] & \mathbb{P}^3 \end{tikzcd} \right.$$

$$\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^1}$$

$$\mathcal{O}_X(1) \overset{(!)}{=} \mathcal{O}_{\mathbb{P}^1}(4)$$

$$R_\bullet \rightsquigarrow \mathcal{O}_X \overset{(?)}{\rightsquigarrow} \bigoplus \Gamma(X, \mathcal{O}_X(e)) \text{ vs } R_\bullet$$

$$\bigoplus \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4e))$$

$$k \oplus k[s,t]_4$$

$$R_1 = \langle s^4, s^3t, \dots, t^4 \rangle$$

$$R_2 = \langle s^8, s^7t, \dots \rangle$$

$$\overset{!!!}{=} k[s,t]_8$$