Last time.

$$\left\{ \begin{array}{l} \text{spanning rational} \\ \text{maps } \Phi : X \dashrightarrow \mathbb{P}^V \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{linear series} \\ \mathbb{P} \subset |D| \text{ for} \\ \text{some effective} \\ \text{Cartier divisor } D \text{ on } X \end{array} \right\}$$

$$\Phi = (\phi_0 : \cdots : \phi_n) \xleftarrow[\phi_0, \dots, \phi_n]{\text{choose basis}} \mathbb{P} \subset |D|$$

$$\mathbb{P} = V \setminus 0 / k^*$$

Thm (Hilbert scheme) X non-singular, projective. The locus of effective divisors is a countable union of projective "varieties" (schemes of finite type $/k$).

Moreover, the equivalence classes of effective divisors are also a countable union of projective varieties (schemes of finite type $/k$).

And,

$$\begin{tikzcd} \mathrm{Eff}(X) = \bigsqcup_{\delta \in NS(X)} \mathrm{Eff}_\delta(X) \arrow[rr] \arrow[dr] & & \mathrm{CDiv}(X) = \bigsqcup_{\delta \in NS(X)} \mathrm{Pic}^\delta(X) \\ & \bigsqcup Z_\delta \arrow[ur] & \end{tikzcd}$$

Neron-Severy $= NS$

Rmk. $\mathrm{CDiv}(X) = \mathrm{Pic}(X)$ is a group.

$$0 \longrightarrow \mathrm{Pic}^\circ(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow NS(X) \longrightarrow 0$$

$$\Phi: \mathrm{Eff}_\delta(X) \xrightarrow[\text{maps of proj. var's}]{\text{regular}} \mathrm{Pic}^\delta(X)$$

$$\begin{array}{ccc} \cup & & \\ D & \longrightarrow & [D] \end{array}$$

$$\Phi^{-1}([D]) = |D| \quad (\text{proj. space})$$

Ex. $X = \mathbb{P}^n$, $\mathrm{Eff}_d(\mathbb{P}^n) = \mathbb{P}(k[x_0, \dots, x_n]_d)$

$d \geq 0$.

$$\sum d_i Z_i \longleftarrow F = \prod F_i^{d_i}$$

$$Z_i = V(F_i)$$

$$\mathrm{Eff}_d(\mathbb{P}^n) \longrightarrow \mathrm{Pic}^d(\mathbb{P}^n) = pt.$$

$F \sim F'$ via $\varphi = F/F'$.

Ex. $X = C$, non-sing. curve.

$$\mathrm{Eff}(C) = \bigsqcup_{d \geq 0} \mathrm{Eff}_d(C).$$

$$\underset{\sum n_i p_i}{\overset{\mathrm{Eff}_d(C)}{\cup}} \in C^d/\Sigma_d = \mathrm{Sym}^d(C)$$ is a projective variety of dimension d (also non-singular!)

$$\Phi_d : \mathrm{Sym}^d(C) \longrightarrow \mathrm{Pic}^d(C)$$

$\hookrightarrow$ ab. var. of dimension g.

$d = 1, \dots, g-1$; the map is "generically" injective, i.e. the image has dim $= d$.

<u>Thm (Riemann).</u> Φ_g is surjective.

E - elliptic curve

$$\Phi_1 : E \xrightarrow{\sim} \mathrm{Pic}^1(E) = (\mathrm{Pic}^0(E))$$
$$p \longrightarrow [p] \ (\longmapsto [p - p_0])$$

$$\Phi_2 : \mathrm{Sym}^2(C) \longrightarrow \mathrm{Pic}^2(C)$$
$$\mathbb{P}^1 = |K_C| \longrightarrow [K_C].$$

<u>Coherent sheaves.</u>

<u>Recall.</u> Affine varieties $X \longleftrightarrow k[X]$, k-algebras

Proj. varieties $X \longleftarrow R_\bullet$ graded k-algebra

$R_0 = k$

R_1 fin. dim.

generates $R_\bullet$

Generalize M module over $k[X]$ (X affine)

to a sheaf of $\mathcal{O}_X$-modules over $\mathcal{O}_X$.

<u>Main Ex's:</u>

① $0 \to I \to k[X] \to k[X]/I \to 0$

② Projective modules

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d] \\ N \arrow[r, two heads] & M \end{tikzcd}$$

P inject into free modules.

Over a <u>local</u> <u>ring</u>, <u>projective</u> = <u>free</u>.

③ Injective

④ Module of differentials ~~(if M m~~

<u>Def.</u> M module over $k[X]$, then a k-linear map
$$\overset{\shortparallel}{A}.$$

$$d: A \longrightarrow M \quad \text{is a differential}$$

if the usual properties hold.

<u>Def.</u> $d: A \to \Omega_A$ is the universal module of differentials

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "d'"'] & \Omega_A \arrow[d, dashed] \\ & M \end{tikzcd}$$

$\Omega_A = (\text{free module on } da) \big/ \left(\begin{array}{c}\text{the usual} \\ \text{relations that} \\ d \text{ satisfies}\end{array}\right).$

<u>Ex</u>. If $A = k[x_1, \ldots, x_n]$, then Ω_A = free module on dx_i

$\Rightarrow$ If $A = k[x_1, \ldots, x_n] / \langle f_1, \ldots, f_n \rangle$

then $\Omega_A = \langle dx_i \rangle / \langle df_j \rangle$ i.e. f.g., maybe not free.

Ex. If mSpec(A) is non-singular, then Ω_A is locally free (projective!)

Rmk. If $S \subset A$ is a mult. set, then

$(\Omega_A)_S$

$M_S = \{ m/f \mid f \in S \}$

Goal. To find the sheaf of differentials $\underset{\text{canonical}}{\underbrace{\Omega_X}}$ over X.

Def. A sheaf of $\mathcal{O}_X$-modules consists of:

(i) $\mathcal{F}$, a sheaf of ab. groups:

$\mathcal{F}(U)$ ab. grp.
$\uparrow$ $U \subset X$. $\quad \mathcal{F}(U) \xrightarrow{res} \mathcal{F}(V)$
$V \subset U$.

(ii) Each $\mathcal{F}(U)$ has a consistent structure of an $\mathcal{O}_X(U)$-module.

$$\begin{array}{ccc}
\mathcal{O}_X(U) \times \mathcal{F}(U) & \xrightarrow{mult} & \mathcal{F}(U) \\
\downarrow res & & \downarrow res \\
\mathcal{O}_X(V) \times \mathcal{F}(V) & \xrightarrow{mult} & \mathcal{F}(V)
\end{array}$$

Moral. $\mathcal{F}$ is q.c. if $\mathcal{F}$ is "locally $\mathcal{M}$"

over $k[U]$

affine.

coherent if q.c. & "locally" f.g.