Čech resolution

X topological space

$\mathcal{U} = \{U_i\}$ an open cover of X

Define $U_{i_0,\dots,i_n} = \bigcap_{k=0}^{n} U_{i_k}$. Then if we have a sheaf $\mathcal{F}$, can restrict $\mathcal{F}|_{U_{i_0,\dots,i_n}}$.

<u>Def.</u> If $\mathcal{G}$ a sheaf on $U \subset X$, and $j: U \hookrightarrow X$, then $j_*(\mathcal{G})(V) = \mathcal{G}(V \cap U)$ for $V \subset X$ open. This is called the <u>direct image</u> of $\mathcal{G}$.

This allows us to consider $\mathcal{F}|_{U_{i_0,\dots,i_n}}$ as a sheaf via the inclusions $j_{i_0,\dots,i_n}: U_{i_0,\dots,i_n} \hookrightarrow X$.

<u>Def.</u> $C^n(\mathcal{U},\mathcal{F}) = \prod_{i_0 < \dots < i_n} j_{i_0,\dots,i_n *}(\mathcal{F}|_{U_{i_0,\dots,i_n}})$, and let

$d: C^n \to C^{n+1}$ by $(ds)_{i_0,\dots,i_{n+1}}(V) = \sum_{k=0}^{n+1} (-1)^k s_{i_0,\dots,\hat{i_k},\dots,i_{n+1}}\Big|_{U_k \cap V}$

<u>Example</u>. Suppose $\mathcal{U} = \{U, V\}$. Then if $i: U \hookrightarrow X$, $j: V \hookrightarrow X$, and $k: U \cap V \hookrightarrow X$

$$C^0(\mathcal{U},\mathcal{F}) = i_*(\mathcal{F}|_U) \oplus j_*(\mathcal{F}|_V)$$

$$C^1(\mathcal{U},\mathcal{F}) = k_*(\mathcal{F}|_{U \cap V})$$

Take W, open in X. Let $s_u \in \mathcal{F}(U \cap W)$, and $s_v \in \mathcal{F}(V \cap W)$.
Then $d(s_u, s_v)(W) = \underset{\underset{\mathcal{F}(U \cap V \cap W)}{\cap}}{s_u|_{W \cap V}} - \underset{\ni}{s_v|_{U \cap W}}$

So we have

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varepsilon} C^0(\mathcal{U}, F) \xrightarrow{d} C^1(\mathcal{U}, \mathcal{F}) \longrightarrow 0$$

where $\varepsilon(u) = (u|_u, u|_v) \in C^0(\mathcal{U}, V)$. This says that to study $\mathcal{F}$, we can study it on $U, V, U \cap V$. This is the analogue of the Kunneth formula from AT. ∎

<u>Def</u>. Čech resolution of a sheaf $\mathcal{F}$ is

$$0 \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \longrightarrow C^1(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots$$

When X is an algebraic variety, $\mathcal{U}$ is finite so this resolution also is.

<u>Th'm</u>. $\mathcal{F}$ sheaf. The following is exact.

$$0 \longrightarrow F \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \longrightarrow C^1(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots$$

<u>Def</u>. $H^p(\Gamma(C^\bullet(\mathcal{U}, \mathcal{F}))) =: \check{H}^p(\mathcal{U}, \mathcal{F})$ is the Čech cohomology.

<u>Example</u>. $\mathbb{P}^1 = \underset{\underset{\mathbb{C}}{\cong}}{(\mathbb{P}^1 \setminus \{\infty\})} \cup \underset{\underset{\mathbb{C}}{\cong}}{(\mathbb{P}^1 \setminus \{0\})}$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow C^0(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) \longrightarrow C^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) \longrightarrow 0$$

<u>Remark</u>. Note that $\check{H}^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}) = \underbrace{H^0(X, \mathcal{F})}_{\text{Grothendieck cohomology}}$.

Also note that if $\mathcal{F}$ is <u>flasque</u>, $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0 = H^p(X, \mathcal{F})$ for $p > 0$.

In general, Čech and Grothendieck cohomology not the same. But, take a Čech resolution and injective

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[dr] & C^0(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d, dashed] & \cdots \arrow[d, dashed] & \\
& & I_0 \arrow[r] & I_1 \arrow[r] & \cdots
\end{tikzcd}$$

by general nonsense. So we have a map $\check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$.

<u>Th'm</u>. Let $\mathcal{F}$ be a sheaf on X such that $H^p(U_{i_0, \ldots, i_n}, \mathcal{F}) = 0$ $\forall p > 0$, $i_0, \ldots, i_n \in I$. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$ is an isomorphism.

<u>Remark</u>. If $\mathcal{U}$ is a finite affine cover of X an alg. variety, then $\mathcal{U}_{i_0, \ldots, i_n}$ is also affine by Hausdorff property.

So if $\mathcal{F}$ is a q.c. $\mathcal{O}_X$-module, $\mathcal{F}|_{U_{i_0,\dots,i_n}}$ is quasi-coherent.

So Serre $\Rightarrow$ $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$.