## Construction of $\mathrm{inv}_K$.

Write $H^2(L/K) := H^2(\mathrm{Gal}(L/K), L^\times)$.

Remains to show inf is an isomorphism.

Let $L/K$ be fin. Galois. We had

$$\begin{tikzcd}
0 \arrow[r] & \ker \arrow[r] \arrow[d, dashed] & H^2(K^{ur}/K) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{inf}"] & H^2(L^{ur}/L) \arrow[d, "\mathrm{inf}"] \\
0 \arrow[r] & H^2(L/K) \arrow[r] & H^2(\overline{K}/K) \arrow[r, "\mathrm{res}"] & H^2(\overline{L}/L)
\end{tikzcd}$$

$K^{ur}L = L^{ur}$

We've seen $\ker \simeq \mathbb{Z}/[L:K]$, since res $\leftrightarrow$ mult. by $[L:K]$ on $\mathbb{Q}/\mathbb{Z}$. We'll show $|H^2(L/K)| \leq [L:K]$ so $\ker \to H^2(L/K)$ must be an iso, & conclude that

$$H^2(K^{ur}/K) \xrightarrow{\mathrm{inf}} H^2(\overline{K}/K) = \varinjlim_{L/K} H^2(L/K)$$

is an iso since inf is injective & surj. follows from surj of $\ker \to H^2(L/K)$ $\forall$ finite $L/K$.

① Let $G = \mathrm{Gal}(L/K)$. <u>Claim.</u> $\exists$ $G$-stable open s/g $V \subset \mathcal{O}_L^\times$ s.t. $H^r(G,V) = 0$ $\forall r \geq 1$.

<u>Pf.</u> In finite index, exp is an $G$-equivariant iso $\mathcal{O}_L \supset \square \xrightarrow{\exp} \square \subset \mathcal{O}_L^\times$

so we can replace $\mathcal{O}_L^\times$ by $\mathcal{O}_L$. The Normal Basis Theorem shows $L \simeq K[G]$ as $G$-modules. Take a normal basis $\{\tau(x)\}_{\tau \in G}$, rescaling by powers of $p$,

... all $\tau(x) \in \mathcal{O}_L$.

Take $V = \sum_\tau \mathcal{O}_K \tau(x) \simeq \mathcal{O}_K[G] = \mathrm{Ind}_1^G(\mathcal{O}_K)$ which has 0 coh.

(2) Now treat case of $L/K$ cyclic (not necessarily ur). Take $V \subset \mathcal{O}_L^\times$ from (1). Since $\mathcal{O}_L^\times/V$ finite & $H^r(G,V)=0$ for $r \geq 1$, then $h(\mathcal{O}_L^\times)$ makes sense

& $h(\mathcal{O}_L^\times) = \underbrace{h(\mathcal{O}_L^\times/V)}_{1 \text{ since } \mathcal{O}_L^\times/V \text{ finite}} \cdot h(V) = 1.$

$L^\times = \mathcal{O}_L^\times \times \langle \varpi_L \rangle$

$$\frac{h(L^\times)}{h(\langle\varpi_L\rangle)} \quad \& \quad |H^2(G, L^\times)| = \left|\frac{H^2(G,\langle\varpi_L\rangle)}{H^1(G,\langle\varpi_L\rangle)}\right| \leq [L:K].$$

$$H^1(G, \langle\varpi_L\rangle) = \frac{\ker(N)}{\mathrm{im}(\sigma - 1)} = 0$$

$$H^2(G, \langle\varpi_L\rangle) = \frac{\ker(\sigma-1)}{\mathrm{im}\, N} \text{ thus order } \leq [L:K].$$

(3) Reduce general case to cyclic case, using solvability of $\mathrm{Gal}(L/K)$ (ramification filtration?).

For general fin. Gal. $L/K$, either $L/K$ cyclic or $\exists\ \underbrace{K \subset K'}_{\text{cyclic}} \subset L$, and then

by Hilbert 90 + H-S spec sequence.

$$0 \longrightarrow H^2(K'/K) \longrightarrow H^2(L/K) \longrightarrow H^2(L/K')$$

so $|H^2(L/K)| \leq |H^2(K'/K)|\,|H^2(L/K')| \leq [K':K][L:K'] = [L:K]$.

## Using $Br(K)$ to understand Gal. coh. of local fields (Tate Local Duality).

First key consequences:

**Cor.** $H^2(G_K, \mu_n) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$ (use Kummer theory $1 \to \mu_n \to \bar{K}^\times \to \bar{K}^\times \to 1$ to identify $H^2(G_K, \mu_n) \xrightarrow{\sim} Br(K)[n] \xrightarrow{\text{inv}} \mathbb{Z}/n\mathbb{Z}$).

**Cor.** ① $Br(K^{ur}) = 0$.

$I$ is inertia group

② $H^2(I_K, \mu_n) = 0$

③ If $M$ is a torsion $I_K$-module, then $H^r(I_K, M) = 0 \quad \forall r \geq 2$.

④ If $M$ is a torsion $G_K$-mod, then $H^r(G_K, M) = 0 \quad \forall r \geq 3$.

⑤ If $M$ is a finite $G_K$-mod, then $H^r(G_K, M)$ is finite $\forall r$.

(one says the cohom. dimension $G_K$ is 2).

**Pf.** ① $Br(K^{ur}) = H^2(\bar{K}/K^{ur}) = H^2(I_K, \bar{K}^\times) = \operatorname{colim}_{\text{res maps}} H^2(G_L, \bar{K}^\times)$

$I_K = \bigcap G_L$, $K^{ur} \supset L \supset_{\text{fin}} K$

$= \operatorname{colim}_{L' \supset L,\ \text{mult by } [L':L]} \mathbb{Q}/\mathbb{Z}$

$= 0$.

$$\begin{array}{ccc} Br(L) & \xrightarrow{\text{res}} & Br(L') \\ \downarrow{\scriptstyle \text{inv}_L} & & \downarrow{\scriptstyle \text{inv}_{L'}} \\ \mathbb{Q}/\mathbb{Z} & \xrightarrow{[L':L]} & \mathbb{Q}/\mathbb{Z} \end{array}$$

for any el't of $Br(L)$, say of order $m$, take $L'$ = deg $m$ unr ext of $L$, then this el't vanishes in $Br(L) \to Br(L')$.

② $H^2(I_K, \mu_n) = 0 \ \forall n$ is ① + Kummer theory

$$\left( H^2(I_K, \mu_n) \hookrightarrow Br(K^{un})[n] = 0 \right)$$

③ suffices to prove $H^2(I_K, M) = 0 \ \forall$ <u>finite</u> modules $M$.

<u>Reduction</u>. $H^2(I_K, m) = 0 \ \forall$ fin. $M$, then $H^2(I_K, M) = 0$ since $\forall$ torsion $M$ since torsion $= \bigcup$ finite sub-mods.

If $H^2(I_K, M) = 0 \ \forall$ torsion $M$, then $H^r(I_K, M) = 0 \ \forall$ tor $M$ & all $r \geq 2$. Now use "dimension shifting"

Can embed $M \hookrightarrow I$ for $I$ an injective <u>torsion</u> ob. of $\mathcal{M}_{I_K}$.

by our construction of an inj. res.

Then we get $0 \to M \to I \to Q \to 0$ where $Q$ is also torsion. Then consider the LES.

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To show $H^2(I_K, M) = 0 \ \forall$ fin. $M$ : wma $M = M[\ell^\infty]$ for some prime $\ell$. Recall $res : H^2(I_K, M) \to H^2(Syl_\ell(I_K), M)$ is then injective. Inducting on $\# M$, wma $M$ is a simple $Syl_\ell(I_K)$-module, and in particular $M = M[\ell]$.

Now, an irred. rep. of a (pro-)$\ell$ group on a f.d. $\mathbb{F}_\ell$ vector space is trivial, so $M = \mathbb{Z}/\ell$ as $Syl_\ell(I_K)$-mod.
But $\mathbb{Z}/\ell \cong \mu_\ell$ as $Syl_\ell(I_K)$-modules since $[K^{un}(\mu_\ell) : K^{un}]$ is prime.

As $Syl_\ell(I_K) = \bigcap_{\substack{\text{certain} \\ L/K \text{ fin}}} I_L$, we get $\varinjlim H^2(Syl_\ell(I_K), \mu_\ell) = 0$.

and thus $H^2(I_K, M) = 0$, $\forall$ fin. $M$.

④ $H^r(G_K, M) = 0$ $\forall$ torsion $M$ & all $r \geq 3$.

Pf. $H^p(G(K^{ur}/K), H^q(I_K, M)) \Rightarrow H^{p+q}(G_K, M) = E_2^{p,q}$

So, only $E_2^{0,0}$, $E_2^{1,0}$, $E_2^{0,1}$, $E_2^{1,1}$ can be non-zero.

⑤ If $M$ is finite, then $H^r(G_K, M)$ is finite $\forall r$.

Choose $L/K$ fin. st. $M|_{G_L}$ is trivial, so $\bigoplus \mathbb{Z}/n_i\mathbb{Z}$,

and all $\mu_{n_i}$ are trivial $G_L$-module, so $M|_{G_L} \simeq \bigoplus \mathbb{Z}/n_i\mathbb{Z}$

$\simeq \bigoplus \mu_{n_i}$

Then H-S $\Rightarrow H^p(G(L/K), H^q(G_L, M)) \Rightarrow H^{p+q}(G_K, M)$.

$\forall q$, $H^q(G_L, M) = \bigoplus H^q(G_L, \mu_n)$ is finite

$$\begin{cases} q = 0 & \text{clear} \\ q = 1 & \text{Kummer} \\ q = 2 & Br(L)[n_i] \\ q > 2 & \text{vanishing} \end{cases}$$

$G(L/K)$ fin. $\Rightarrow$ all $E_2^{p,q}$ terms are finite $\Rightarrow H^n(G_K, M)$ is finite $\forall n$.