Th'm TFAE

① C_K^1 is compact

② $Cl(\mathcal{O}_K)$ is finite AND Dirichlet unit th'm.

③ $\forall$ finite sets S of places $\ni$ inf., $cl(\underbrace{\mathcal{O}_K[1/S]}_{\text{invert primes in } S})$ finite and $\mathrm{rk}\, \mathcal{O}_K[1/S]^\times = |S|-1$

Th'm. (Dirichet unit th'm) $\log : \mathcal{O}_K^\times \longrightarrow \Delta_\infty = \ker(\mathbb{R}^{\text{r+s}} \xrightarrow{\text{sum}} \mathbb{R})$

$x \longmapsto (\log |x|_v)_{v|\infty}$ (r+s of these)

$\log(\mathcal{O}_K^\times)$ is a lattice in the r+s−1 dim. space.

Pf of ① ⟺ ③ For any finite $S \supset \{v|\infty\}$, have sequence

$\{(x_v) \mid \prod_{v\in S} |x_v|_v = 1\}$

$$1 \longrightarrow \frac{K_S^{1} \times \prod_{v\notin S} \mathcal{O}_v^\times}{\mathcal{O}_K[1/S]^\times} \to \mathbb{A}_K^1 / K^\times \longrightarrow Cl_S(K) \longrightarrow 1$$

$$(x_v)_v \longrightarrow \prod_{v\notin S} \mathfrak{p}_v^{v(x_v)}$$

where $K_S = \prod_{v\in S} K_v$. To analyze kernel, use

$$\frac{\prod_{v \text{ real}} \pm 1 \times \prod_{v \text{ cmplx}} S^1 \times \prod_{v \text{ fin}} \mathcal{O}_v^\times}{\mu_\infty(K)} \to \frac{K_S^1 \times \prod_{v\notin S} \mathcal{O}_v^\times}{\mathcal{O}_K[1/S]^\times} \xrightarrow{\log} \frac{\Delta_S = \ker(\mathbb{R}^S \xrightarrow{\text{sum}} \mathbb{R})}{\log \mathcal{O}_K[1/S]^\times}$$

$$(x_v)_{v\in S} \times (y_v)_{v\notin S} \longrightarrow (\log |x_v|_v)_{v\in S}.$$

Let $S = \{v \mid \infty\}$, so $\log$ is surjective.

Assume ①. Then $\dfrac{K_S^1 \times \prod_{v \notin S} \mathcal{O}_v^\times}{\mathcal{O}_K[1/S]^\times}$ is sandwiched

b/w two compact groups. By fin. of $Cl_S(K)$,

$\mathbb{A}_K^1 / K^\times$ is also compact.

$\boxed{2 \Rightarrow 1}$ If C_K^1 is compact, then $Cl(K) = \text{cpct}/\text{open sg}$

= finite. Since $K_S^1 \times \prod_{v \notin S} \mathcal{O}_v^\times / \mathcal{O}_K[1/S]^\times$ open, also closed, hence compact,

so its quotient $\dfrac{\Delta_\infty}{\log \mathcal{O}_K^\times}$ is compact.

To complete ① $\Leftrightarrow$ ② $\Leftrightarrow$ 3, check that $S = \{v \mid \infty\}$

case of unit thm $\Rightarrow$ general S-unit thm.

Take

$$\mathcal{O}_K^\times \to \mathcal{O}_K[1/S]^\times \to \bigoplus_{v \in S \setminus \{v \mid \infty\}} \mathbb{Z} \to$$

$$x \longmapsto (v(x))$$

To show image has full rank, consider $\forall v \in S \setminus \{v \mid \infty\}$.

$$\mathfrak{p}_v^{\overset{\text{class } \#}{h_K}} = \text{principal ideal } (a)$$

$$\Rightarrow \text{val}(a) = (0, \dots, h_K, \dots, 0)$$

Ideles & field extensions.

For L/K finite ext., have
$$\begin{array}{ccc} \mathbb{A}_K^\times & \hookrightarrow & \mathbb{A}_L^\times \\ \cup & & \cup \\ K^\times & \hookrightarrow & L^\times \end{array}$$
extending

via $(x_v)_v \longmapsto \prod_{v \text{ of } K} \prod_{w \mid v} (x_v)_w$.

For a field iso., $\sigma : L \simeq \sigma L$, get

$$\sigma : \mathbb{A}_L^\times \xrightarrow{\sim} \mathbb{A}_{\sigma L}^\times$$

$$a \longrightarrow (\sigma a)_{\sigma w} = \sigma(\alpha_w).$$

i.e. w is a place of L. σ induces

$$L_w \xrightarrow{\sim} (\sigma L)_{\sigma w = w \circ \sigma^{-1}}.$$

Then $x \in L_w$, $|x|_w = |\sigma x|_{\sigma w}$. In particular, if L/k is Galois, $G(L/k)$ acts on $\mathbb{A}_L^\times$.

$|x|_w = |\sigma x|_{\sigma w}$ (so σ extends to completions)

$$\sigma((x_w)_w) = (\sigma(x_{\sigma^{-1}w}))_w.$$

<u>Lemma</u> L/k Galois $\Rightarrow (\mathbb{A}_L^\times)^{G(L/k)} = \mathbb{A}_k^\times$.

<u>Pf.</u> $\supseteq$ is clear. Let $(x_w)_w \in (\mathbb{A}_L^\times)^{G(L/k)}$

<u>First</u>: $\forall w$, $x_w \in K_v^\times \hookrightarrow L_w^\times$. For $\sigma \in G_w$,

invariance $\Rightarrow \sigma(x_w) = x_w$ (use formula defining the action).

This $\forall \sigma \in G_w \Rightarrow x_w \in K_v^\times = (L_w^\times)^{G(L_w/K_v)}$

Now conclude by transitivity of $G(L/k) \circlearrowright \{w | v\}$ that x_w in $K_v^\times$ is independent of $w|v$.

Alternatively, think of this as follows:

$$\mathbb{A}_K \otimes_K L \xrightarrow{\sim} \mathbb{A}_L^\times \quad \text{induced by}$$

$$K_v \otimes_K L \simeq \bigoplus_{w|v} L_w$$

$G(L/K)$-equivariant (for the action just defined on RHS and on $\sigma(\alpha \otimes x) = \alpha \otimes \sigma x$)

and $\mathcal{O}_{K_v} \otimes_{\mathcal{O}_K} \mathcal{O}_L \xrightarrow{\sim} \bigoplus_{w|v} \mathcal{O}_{L_w}$ on LHS.

Check of equivariance:

$$\begin{tikzcd} L \arrow[r, "\sigma"] \arrow[d, "\iota_{\sigma^{-1}w}"'] & L \arrow[d, "\iota_w"] \\ L_{\sigma^{-1}w} \arrow[r, "\sigma"] & L_w \end{tikzcd}$$

$K_v \otimes L \xrightarrow{\sim} \bigoplus_{w|v} L_w \qquad \iota_w : L \hookrightarrow L_w$

$$x \otimes y \longrightarrow (x \cdot \iota_w(y))_w$$

$$G(L/K) \ni \sigma \downarrow$$

$$x \otimes \sigma y \longrightarrow (x \cdot \iota_w(\sigma y))_w$$
$$= (x \cdot \sigma(\iota_{\sigma^{-1}w}(y)))_w$$
$$= (\sigma(x\, \iota_{\sigma^{-1}w}(y)))_w$$
$$\underset{\text{def.}}{=} \sigma((x \cdot \iota_w(y))_w)$$

So have $\mathbb{A}_K^\times \hookrightarrow \mathbb{A}_L^\times$ (and $C_K \longrightarrow C_L$).

In fact, $C_K \hookrightarrow C_L$ $\quad (L^\times \cap \mathbb{A}_K^\times = K^\times)$.

In other direction,

Def. $N_{L/K} : \mathbb{A}_L^\times \longrightarrow \mathbb{A}_K^\times$ induced by

$$\bigoplus_{w|v} L_w^\times \longrightarrow K_v^\times$$

by $\quad (x_w)_w \longmapsto \prod_{w|v} N_{L_w/K_v}(x_w)$.

$N_{L/K}$ extends $N_{L/K} : L^\times \longrightarrow K^\times$ by

$$N_{L/K}(\underset{\in L^\times}{\alpha}) = \prod_{w|v} N_{L_w/K_v}(\alpha)$$

so it induces $N_{L/K} : C_L \longrightarrow C_K$

(alt., $N_{L/K}(\underset{\in \mathbb{A}_L^\times}{\alpha}) = \det(\alpha \circlearrowright \underbrace{\mathbb{A}_K \otimes L = \mathbb{A}_L}_{\text{as } \mathbb{A}_K\text{-module.}})$

Rmk. we'll show $C_K = C_L^{G(L/K)}$: this follows from Hilbert 90 & LES on $G(L/K)$-cohom. assoc'd to $1 \to L^\times \to \mathbb{A}_L^\times \to C_L^\times \to 1$.

STATEMENTS OF CFT

LOCAL CFT Let $K/\mathbb{Q}_p$ be a finite ext.

$\boxed{\text{Thm}}$ ① (reciprocity law) Let K^{ab} = max abel ext. of K inside $\overline{K}$. There is a unique hom

$$r_K := \mathrm{rec}_K : K^\times \longrightarrow G(K^{ab}/K) =: G_K^{ab}$$

such that

(a) $\forall\ \varpi$ of K uniformizers and all unram. ext's, $r_K(\varpi)\big|_L = \mathrm{Frob}_K\big|_L$, (so $r_K(\mathcal{O}_K^\times)\big|_L = \mathrm{id}_L$)

(b) $\forall$ fin. abel ext L/K, r_K induces

$$r_{L/K} : K^\times / N_{L/K}(L^\times) \xrightarrow{\sim} G(L/K).$$

①′ : $L \longrightarrow N_{L/K}(L^\times)$ is a bijection

$$\left\{\begin{matrix}\text{fin. ab}\\ L/K\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{norms of}\\ \text{fin ab } L/K\end{matrix}\right\} \left(= \left\{\begin{matrix}\text{norms of}\\ \text{all fin}\\ \text{ext.'s of } K\end{matrix}\right\}\right)$$

satisfying: (i) $L_1 \subseteq L_2 \iff N(L_2^\times) \subseteq N(L_1^\times)$

(ii) $N((L_1 L_2)^\times) = N(L_1^\times) \cap N(L_2^\times)$

(iii) $N((L_1 \cap L_2)^\times) = N(L_1^\times) \cdot N(L_2^\times)$

(iv) If a s/g $U \subset K^\times$ <u>contains</u> a normal sg then it equals a normal sg.

<u>Pf of 1 ⇒ 1'.</u> $L_1 \subset L_2 \Rightarrow NL_2 \subset NL_1$.

In particular, $N(L_1L_2)^\times \subset NL_1^\times \cap NL_2^\times$.

If $x \in NL_1^\times \cap NL_2^\times$, then $r_K(x)|_{L_i} = 1$, $i=1,2$ by ①. So $r_K(x)|_{L_1L_2} = 1$, and so $x \in N(L_1L_2)^\times$.

Now if $NL_2^\times \subset NL_1^\times$, $N(L_2^\times) \cap N(L_1^\times) = NL_2^\times$

$$\overset{\text{by (ii)}}{=} N((L_1L_2)^\times)$$

$$\Rightarrow [L_1L_2 : K] = [L_2 : K].$$

$$\Rightarrow L_1 \subset L_2.$$

For (iv), if $U \supset N(L^\times)$

$$\begin{tikzcd} r_K : K^\times/N(L^\times) \arrow[r, "\sim"] \arrow[d] & G(L/K) \arrow[d, two heads, "\text{By Galois theory.}"] \\ K^\times/U \arrow[r, "\sim"] & G(M/K) \end{tikzcd}$$

$\Rightarrow U = N_{M/K}(M^\times)$.

<u>Leftover question.</u> What are the norm subgroups?

② (Existence thm) The norm subgroups $\{N_{L/K}(L^\times)\}_{L/K \text{ fin ab.}}$ are precisely the open subgroups of finite index in $K^\times$.

(2 $\Rightarrow$ 2') r_K induces a top. iso.

$$r_K : \widehat{K^\times} \xrightarrow{\sim} G_K^{ab}$$

if $K^\times = \varpi^{\mathbb{Z}} \times \mathcal{O}_K^\times \Rightarrow \widehat{K^\times} = \varpi^{\hat{\mathbb{Z}}} \times \mathcal{O}_K^\times$.

By Galois theory, construct

$$K^{un} = (K^{ab})^{r_K(\mathcal{O}_K^\times)}$$

For any uniformizer ϖ, $K_\varpi = (K^{ab})^{r_K(\varpi)}$ - this is a totally ramified ext (which depends on ϖ).

__Next time__. $K = \mathbb{Q}_p$, $\varpi = p$, $K_\varpi = \mathbb{Q}_p(\mu_{p^\infty})$