Recall. GCFT: the local reciprocity maps assemble to iso's

$\forall L/K$ fin Galois, $\mathrm{rec}_K : C_K / N_{L/K} C_L \xrightarrow{\sim} G(L/K)^{ab}$

$$\begin{tikzcd} C_K/N_{L/K}C_L \arrow[r, "\sim"] & G(L/K)^{ab} \\ K_v^\times \arrow[u] \arrow[r, "\mathrm{rec}_{K_v}"] & G(L_w/K_v)^{ab} \arrow[u] \end{tikzcd}$$

(2) Existence: norm s/g of C_K = open fin. index s/g's.

Application. K - # field, m integer. $\alpha \in (K^\times)^m$

$\Rightarrow \alpha \in (K_v^\times)^m$ $\forall$ places v of K.

Yes, this has a converse for $K = \mathbb{Q}$, by unique factorization.

e.g. (HW) $K = \mathbb{Q}(\sqrt{7})$, $m = 8$. $16 \in (K_v^\times)^8$ $\forall v$, but $16 \notin (K^\times)^8$.

Simpler version. Thm. Set $m = 2^t m'$ with m' odd. Let S be a fin. set of places of K, and assume $\alpha \in K^\times$ lies in $(K_v^\times)^m$ $\forall v \notin S$. Then

1) if $K(\mu_{2^t})/K$ is cyclic, then $\alpha \in (K^\times)^m$.

2) in general, $\alpha \in (K^\times)^{m/2}$. (eg. (1) holds if $t \leq 2$, i.e. $8 \nmid m$.)

Pf. (assuming GCFT) Suffices to prove (STP) for $m = p^r$ a prime power [if $(m,n) = 1$ and $\alpha \in (K^\times)^m \cap (K^\times)^n$, then writing $1 = mx + ny$ yields $\alpha = \alpha^{mx}\alpha^{ny} \in (K^\times)^{mn}$].

(*) Basic case: Assume $\mu_m \subset K$. Then always $\alpha \in (K^\times)^m$.

<u>Pf</u>. Consider $L = K(\sqrt[m]{\alpha})$. Goal $L = K$. Apply rec_K:

$$\mathbb{A}_K^\times / K^\times N_{L/K}\mathbb{A}_L^\times \xrightarrow{\sim} G(L/K)$$

$\forall v \notin S$, $N(\mathbb{A}_L^\times) \supset K_v^\times$ b/c $\sqrt[m]{\alpha} \in K_v^\times$.

Since $K^\times \subset K_S^\times$ is dense, $\mathbb{A}_K^\times = K^\times \cdot N_{L/K}\mathbb{A}_L^\times$.

[$x = \underbrace{x^S}_{\substack{\text{away}\\ \text{from}\\ S}} \cdot \underset{K_S^\times}{\underset{\cap}{x_S}} \longrightarrow$ $\forall$ open U, $x_S \in K^\times \cdot U$.

Take U small enough $\subset N(L_S^\times)$].

By rec, $G(L/K) = \{1\}$, so $\sqrt[m]{\alpha} \in K$.

<u>General case</u>. ($\zeta_m \notin K$). Form

$$\begin{array}{c} K' = K(\zeta_m) \\ | \\ K \end{array}$$

By (*), $\alpha = \beta^m$ for some $\beta \in K'$. Factor

$X^m - \alpha = \prod f_j(X)$ into irreducibles in $K[X]$. Each f_j has some root $\beta_j = \beta \cdot \zeta_m^{r_j}$. Now $\forall v \notin S$, $\alpha = \gamma_v^m$ for some $\gamma_v \in K_v^\times$. So γ_v a root of $f_i(X)$.

v splits completely, i.e.

$$\left[\begin{array}{c} K(\beta_i) \\ | \\ K \end{array}\right] \quad — K'$$

(i) Consider the subcase where K'/K has p-power order.

Then $\alpha \in (K^\times)^m$: if K'/K has p-power order & is cyclic. Then the intermediate fields are totally ordered and there is a minimal one among the $K(\beta_i)$.

WMA $K(\beta_1) \subset K(\beta_i)$ $\forall i$.

Then $\forall v \notin S$, v splits in some $K(\beta_{i=i(v)})$, hence splits in $K(\beta_1)$. By the argument in ⊛, it follows that $K(\beta_1) = K$. (b/c almost all places in $K(\beta_1)/K$). Thus $\beta_1 \in K^\times$ & $\alpha \in (K^\times)^m$.

(ii) general case, w/ p odd.

$$\begin{tikzcd}
K(\mu_{m=p^r}) = K' \arrow[d, dash] \arrow[d, dash, bend left=60, "\begin{array}{l}\text{cyclic of } p\text{-power}\\ \text{order } (p \text{ odd}),\\ \text{so (i) shows}\\ \alpha \in (K(\mu_p)^\times)^m\end{array}"] \\
K(\mu_p) \arrow[d, dash] \arrow[d, dash, bend right=60, "\begin{array}{l}\text{prime to}\\ p \text{ degree}\end{array}"'] \\
K
\end{tikzcd}$$

So $\alpha = \beta^m$, $\beta \in K(\mu_p)^\times$

Apply $N_{K(\mu_p)/K}$:

$$\alpha^{[K(\mu_p):K]} = N(\beta)^m.$$

Since $[K(\mu_p):K]$ is prime to p, deduce that $\alpha \in (K^\times)^m$. (Write $1 = [K(\mu_p):K]x + py \ldots$).

Finally, $p=2$. K'/K is 2-power. If it is cyclic, we win (by (i).). So assume <u>not</u> cyclic.

Nevertheless,

$$\begin{tikzcd} K' \arrow[d, dash] \arrow[d, bend left, dash, "\text{cyclic}"] \\ K(\sqrt{-1}) \arrow[d, dash] \\ K. \end{tikzcd}$$

(i) $\Rightarrow$ $\alpha = \beta^m$ for $\beta \in K(\sqrt{-1})$. Taking N,

$$\alpha^2 = N(\beta)^m.$$

Recall $m = 2^t m'$, and $t \geq 2$ in this case, so

$$\alpha = \pm N(\beta)^{m/2}.$$

<u>Claim</u>. Have $+$:

<u>Proof</u>. $\dfrac{\alpha}{N(\beta)^{2^{t-1}}} = -1$, ~~Suppose~~ Since α a square in K (by the case $m=2$ and $2^{t-1} \geq 2$.

then $-1 \in (K^\times)^2$, contradicting $K(\sqrt{-1}) \neq K$.

<u>Another e.g.</u> $16 \in \mathbb{Q}_v^{8}$ $\forall\, v \neq 2$. But $16 \notin (\mathbb{Q}^\times)^8$.

<u>Cor</u>. Very useful. K - # field. S - fin. set of places (WMA contains ~~∞~~-places. Suppose we are given finite-order $\chi_v : G_{K_v}^{ab} \longrightarrow \mathbb{Q}/\mathbb{Z}$ $\forall\, v \in S$. Then

$$\begin{matrix} G_{K_v}^{ab} \\ \wr| \\ K_v^\times \end{matrix}$$

$\exists\, \chi : G_K^{ab} \longrightarrow \mathbb{Q}/\mathbb{Z}$ s.t. $\chi|_{G_{K_v}} = \chi_v$ for all $v \in S$.

Rmk. If $v \nmid \infty$, any cts $G_{K_v} \to S^1$ has finite order. @ $v | \infty$, have chars like $z/|z| : \mathbb{C}^\times \to S^1$.

Pf of Cor. If STP Lemma. $\forall m$, $\exists$ open s/g U of $\prod_{v \notin S} \mathcal{O}_v^\times$ s.t. $\mathcal{O}_K[1/S]^\times \cap U \subset (\mathcal{O}_K[1/S]^\times)^m$.

why suffices. Want to build a $\chi : \mathbb{A}_K^\times \to \mathbb{Q}/\mathbb{Z}$ corresponding to desired $G_K^{ab} \to \mathbb{Q}/\mathbb{Z}$ under rec

$$\underbrace{K^\times}_{\substack{\text{declare} \\ \chi = 1 \text{ here}}} \cdot \Big(\underbrace{\prod_{v \in S} K_v^\times}_{\substack{\text{declare} \\ \chi = \prod_{v \in S} \chi_v \\ \text{here}}} \cdot \underbrace{U}_{\substack{\text{declare} \\ \chi = 1 \text{ here,} \\ \text{where } U \\ \text{is the output} \\ \text{of last lemma} \\ \text{for } m = \text{lcm(orders of the } \chi_v) \\ v \in S.}} \Big) \subset \mathbb{A}_K^\times .$$

For $\chi|_{K^\times \cdot (\prod_{v \in S} K_v^\times \cdot U)}$ to be well-defined, need

on $K^\times \cap (\prod_{v \in S} K_v^\times \cdot U)$, $\prod_{v \in S} \chi_v \times \prod_{v \notin S} 1$ should equal 1. But $K^\times \cap (\prod K_v^\times \cdot U) \subset \mathcal{O}_K[1/S]^\times \cap U \subset (\mathcal{O}_K[1/S]^\times)^m$, and all χ_v are trivial on m-th powers.

To get χ on $\mathbb{A}_K^\times$, take any ext. across the fin. index inclusion $K^\times \cdot \prod_{v \in S} K_v^\times \cdot U \hookrightarrow \mathbb{A}_K^\times$. Now have $\chi : \mathbb{A}_K^\times / K^\times \cdot K_\infty^\times \cdot U \longrightarrow \mathbb{Q}/\mathbb{Z}$. By existence thm, $K^\times K_\infty^\times U / K$ is a norm s/g of C_K,

i.e. $\chi : \mathbb{A}_K^\times / K^\times N_{L/K} \mathbb{A}_K^\times \longrightarrow \mathbb{Q}/\mathbb{Z}$ w/ desired local restrictions. $\Big| \wr$ rec

$$C_K \twoheadrightarrow G(L/K)$$

<u>Pf of Lemma</u>. We may assume $K \ni \sqrt{-1}$.

[Given m, $\exists d$ s.t. $(\mathcal{O}_K[1/S]^\times)^m \supset \mathcal{O}_K[1/S]^\times \cap (\mathcal{O}_{K(\sqrt{-1})}[1/S]^\times)^d$

Apply lemma to $K(\sqrt{-1})$ & d to get $\hat{U}$ s.t. $\hat{U} \cap \mathcal{O}_{K(\sqrt{-1})}[1/S]^\times \subset (\mathcal{O}_{K(\sqrt{-1})}[1/S]^\times)^d$. Take U (for K) to $\hat{U} \cap \mathbb{A}_K^\times$, so

$$\mathcal{O}_K[1/S]^\times \cap U = \mathcal{O}_K[1/S]^\times \cap \hat{U} \subset \mathcal{O}_K[1/S]^\times \cap \mathcal{O}_{K(\sqrt{-1})}[1/S]^{\times d}$$
$$\subset (\mathcal{O}_K[1/S]^\times)^m. \quad]$$

So wma $\sqrt{-1} \in K$ ✓. WMA $m = p^r$ (~ reduce as in prev. thm). $\Rightarrow K(\mu_m)/K$ is cyclic. Since $\mathcal{O}_K[1/S]^\times$ is f.g., we get a fin. # $a_1, \dots, a_s$ of rep's of $\mathcal{O}_K[1/S]^\times / (\mathcal{O}_K[1/S]^\times)^m$. If $a_i \notin (K^\times)^m$, then THM tells us $a_i \notin (K_{v_i}^\times)^m$ for some prime v_i.

Then take $U = \prod_{i=1}^{s} (\mathcal{O}_{v_i}^\times)^m \times \prod_{v \in S \cup \{v_1, \dots, v_s\}} \mathcal{O}_v^\times$.

For $i = 1, \dots, s$, $a_i \notin U \cap \mathcal{O}_K[1/S]^\times \supset (\mathcal{O}_K[1/S]^\times)^m$.

$$\cap$$

$$\mathcal{O}_K[1/S]^\times .$$

Thus $U \cap \mathcal{O}_K[1/S]^\times = (\mathcal{O}_K[1/S]^\times)^m$, and we win. ∎

<u>Sharp form.</u> (Grunwald-Wang thm) K - # field. S finite set of places

$$\ker\left(K^\times/(K^\times)^m \longrightarrow \prod_{v \notin S} K_v^\times/(K_v^\times)^m \right) \text{ is}$$

trivial unless "G-W" special case" in which it has order 2.

<u>Special case.</u> (K, S, m) satisfying

① $K(\mu_{2^{s_K+1}})/K$ is biquadratic where

$s_K = \max\ s$ s.t. $K \supset \mathbb{Q}(\mu_{2^s})^+$ ↪ totally real subfield

② $\operatorname{ord}_2(m) > s_K$

③ S contains all 2-adic places at which

$K_v(\mu_{2^{s_K+1}})/K_v$ is biquadratic.

Moreover, the non-trivial element is expressed by $(2+\zeta_{2^{8k}}+\zeta_{2^{8k}}^{-1})^{m/2}$.