Ex. $K = \mathbb{Q}_p$ Aim: describe the maximal abelian extension $\mathbb{Q}_p^{ab}$. Answer: $\mathbb{Q}_p^{ab} = \mathbb{Q}_p(\mu_\infty)$

Pf. $\mathbb{Q}_p^\times = \langle p \rangle \times \mathbb{Z}_p^\times = \langle p \rangle \times \mu_{p-1} \times (1 + p\mathbb{Z}_p)$

$\Rightarrow$ cofinal in the collec of open subgroups of finite index in $\mathbb{Q}_p$ are $\underset{U_{n,m} :=}{\langle p^m \rangle \times (1 + p^n \mathbb{Z}_p)}$ ($\forall$ open U, U contains one of these). By existence thm, $U_{n,m} = N_{L_{n,m}/\mathbb{Q}_p}(L_{n,m}^\times)$ for some fin. ab. ext. $L_{n,m}/\mathbb{Q}_p$ so suffices to show $\forall n, m$, $\exists$ cyclotomic ext., $\mathbb{Q}_p(\mu_r) = L$ s.t. $N_{L/\mathbb{Q}_p} L^\times \subset U_{n,m}$.

(then by last time, $L_{n,m} \subset L \subset \mathbb{Q}_p(\mu_\infty)$).

First, if $\mathbb{Q}_{p,m}$ is the unramified ext. of deg m, ($\subset \mathbb{Q}_p(\zeta_N : (N,p) = 1)$) then $N \mathbb{Q}_{p,m}^\times \subset \langle p^m \rangle \times \mathbb{Z}_p^\times$. We'll check that $N \mathbb{Q}_p(\zeta_{p^n}) = \langle p \rangle \times (1 + p^n \mathbb{Z}_p)$, and then conclude

$$N(\{\mathbb{Q}_{p,m} \cdot \mathbb{Q}_p(\zeta_{p^n})\}^\times) = N \mathbb{Q}_{p,m}^\times \cap N \mathbb{Q}_p(\zeta_{p^n})^\times \subset \langle p^m \rangle \times (1 + p^n \mathbb{Z}_p).$$

$$\begin{tikzcd} p\mathbb{Z}_p \arrow[r, "\exp", "\sim"'] \arrow[d, "(p-1)p^{n-1}"', "\sim"] & 1 + p\mathbb{Z}_p \arrow[d, "(\ldots)^{(p-1)p^{n-1}}"] \\ p^n\mathbb{Z}_p \arrow[r, "\sim", "\exp"'] & 1 + p^n\mathbb{Z}_p \end{tikzcd}$$

To compute $N \mathbb{Q}_p(\zeta_{p^n})^\times$, observe that

$$N_{\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p}(1 + p\mathbb{Z}_p) = (1 + p\mathbb{Z}_p)^{(p-1)p^{n-1}} \underset{\text{check}}{=} 1 + p^n \mathbb{Z}_p$$

$$\Rightarrow \langle p \rangle \times (1 + p^n \mathbb{Z}_p) \subset N \mathbb{Q}_p(\zeta_{p^n})^\times$$

But the indices on both sides are equal (index in $\mathbb{Q}_p^\times$). ∎

Recall last time, defined for a unif. $\varpi \in K$, a totally ramified ext. $K_\varpi = (K^{ab})^{r_K(\varpi)}$. In this case, $\varpi = p$, check $\mathbb{Q}_{p,\varpi} = \mathbb{Q}_p(\mu_{p^\infty})$ (reason $p = N(1-\zeta_{p^n})$) (composite of all L s.t. $r_K(\varpi)|_L = \mathrm{id}$, i.e. s.t. $\varpi \in N(L^\times)$.

Global classfield theory.

K - # field. $\forall$ places v of K, have (LCFT)

$r_{K_v} : K_v^\times \longrightarrow \mathrm{Gal}(K_v^{ab}/K_v)$ ($v|\infty$: $\mathbb{R}^\times / N_{\mathbb{C}/\mathbb{R}} \mathbb{C}^\times \simeq \mathrm{Gal}(\mathbb{C}/\mathbb{R})$

$\mathbb{C}^\times / N\mathbb{C}^\times \simeq G(\mathbb{C}/\mathbb{C})$)

Define. (Artin Map) $r_K : \mathbb{A}_K^\times \xrightarrow{\varprojlim r_{L/K}} \mathrm{Gal}(K^{ab}/K) \to \mathrm{Gal}(L/K)$

$(x_v)_v \longrightarrow \prod_v r_{K_v}(x_v)$

Precisely. Let L/K be a fin. ab. ext. $\forall v$ of K and $w|v$ of L, the decomposition groups (as w varies $|v$),

$G(L_w/K_v) \hookrightarrow G(L/K)$ are conjugate

but L/K ab. $\Rightarrow$ the decomp. group is indep. of w.

For a.e. v, L/K is unr at v, so $r_{L_w/K_v}(\mathcal{O}_v^\times) = 1$, and so the prod $\prod_v r_{L_w/K_v}(x_v)$ is actually a fin. product, and since L/K is abelian, "$\prod$" does not depend on order of the terms.

Since local maps vanish on norms, we get

$$r_{L/K} : \mathbb{A}_K^\times / N_{L/K}\mathbb{A}_L^\times \longrightarrow G(L/K).$$

__Th'm__ ① $\forall\, L/K$ fin. ab., this map induces

$$r_{L/K} : \mathbb{A}_K^\times / K^\times N_{L/K}\mathbb{A}_L^\times \xrightarrow{\sim} \mathrm{Gal}(L/K).$$

$$\overset{\shortparallel}{\left(C_K / N_{L/K} C_L \right)}$$

② (Existence) The norm subgroups $N_{L/K}(C_L) \subset C_K$ for fin. ab. ext's. L/K are precisely the open subgroups of finite index of C_K.

①' get same thing as LCFT (order reversing inclusions based on norm)

__e.g.__ $U = K^\times \cdot K_\infty^\times \cdot \prod_{v \nmid \infty} \mathcal{O}_v^\times$, so

$$\mathbb{A}_K^\times / U \xrightarrow{\sim} Cl(K)$$

By LCFT, this says H/K is unr. at all finite places, & is the maximal such ab. ext.

__Existence__. $\exists$ fin. ab. ext H/K s.t. $\boxed{N C_H = U/K^\times} \subset C_K$ and $r_{H/K} : Cl(K) \xrightarrow{\sim} \mathrm{Gal}(H/K)$. ($H$ is the Hilbert class field).

(misleadingly simple)

$\mathbb{Z}$ has no interesting units.

$$\mathbb{A}_{\mathbb{Q}}^{\times} \xleftarrow{\sim} \mathbb{Q}^{\times} \times \mathbb{R}_{>0}^{\times} \times \prod_p \mathbb{Z}_p^{\times}.$$

__Th'm.__ (Kronecker-Weber Th'm) The maximal ab. ext. of $\mathbb{Q}$ is $\mathbb{Q}(\mu_\infty)$.

__Rmk.__ For general K, no glimpse of an explicit description of K^{ab}. (only exceptions: $\mathbb{Q}$, $\mathbb{Q}(\sqrt{-d})$ $d>0$)

__Pf.__ From $\mathbb{A}_{\mathbb{Q}}^{\times} = \mathbb{Q}^{\times} \times \mathbb{R}_{>0}^{\times} \times \prod_p \mathbb{Z}_p^{\times}$, suffices to check commutation:

$$\begin{tikzcd}
C_{\mathbb{Q}}/\mathbb{R}_{>0}^{\times} \arrow[r, "r_{\mathbb{Q}}", "\sim"'] \arrow[d, "\text{projection}"', "\sim"] & \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \arrow[d, "\mathrm{res}"] \\
\prod_p \mathbb{Z}_p = \widehat{\mathbb{Z}} & \mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \arrow[l, "\sim"', "\kappa^{-1}"]
\end{tikzcd}$$

Suffices to show $\forall m$,
$$\begin{tikzcd}
\widehat{\mathbb{Z}}^{\times} \arrow[r, "r_{\mathbb{Q}}"] \arrow[d] & \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \arrow[d] \\
(\mathbb{Z}/m\mathbb{Z})^{\times} & \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \arrow[l, "\kappa^{-1}"]
\end{tikzcd}$$

$(\mathbb{Z}/m\mathbb{Z})^{\times}$ is generated by primes $p : (p,m)=1$.

Suffices to check

$$\begin{tikzcd}
(p,\dots,1,\dots) = p(1,\dots,\tfrac{1}{p},1,\dots) \arrow[r, "r_{\mathbb{Q}}"] \arrow[d, "p\text{th component}"] & \mathrm{Frob}_p^{-1} \arrow[dl, bend left, "\kappa^{-1}"] \\
p &
\end{tikzcd}$$

So res is an iso, i.e. $\mathbb{Q}^{ab} = \mathbb{Q}(\mu_\infty)$.

Rmk. We get more info about LCFT from this calc:

$$r_{\mathbb{Q}_p}(u) \in \mathrm{Gal}(\mathbb{Q}_p(\mu_\infty)/\mathbb{Q}_p^{ur}).$$
$$\text{with } u \in \mathbb{Z}_p^\times$$

This calculation $\boxed{r_{\mathbb{Q}_p}(u)(\zeta_{p^n}) = \zeta_{p^n}^{u^{-1}}}$ this is very hard to prove locally.

Intro to group cohomology.

- for us, $H^*(G, M)$ is purely algebraic.

A top. def. of G (abstract group, finite group,) i.e.

$H^*(G, R) := H^*(BG, R)$ where BG is a $K(G,1)$, i.e. a space s.t. $\pi_q(BG) = \begin{cases} G & \text{if } q=1 \\ 0, & \text{else} \end{cases}$ (htpy equiv CW) | BG is the base of a universal G-torsor

(e.g. $G = \mathbb{Z}$, then $K(\mathbb{Z},1) = S^1$).

(e.g. $G = \mathbb{Z}/(2)$. $\forall n$, have $\mathbb{Z}/2$-covers S^n covering $\mathbb{RP}^n$.

Have compatibly

$$\begin{array}{ccccc} S^n & \hookrightarrow & S^{n+1} & \overset{\lim}{\rightsquigarrow} & S^\infty \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{RP}^n & \hookrightarrow & \mathbb{RP}^{n+1} & & \mathbb{RP}^\infty \end{array}$$

and now S^∞ is contractible.

Conclusion. $B(\mathbb{Z}/2) \simeq \mathbb{RP}^\infty$, i.e. $H^*(\mathbb{Z}/2, R) = H^*(\mathbb{RP}^\infty, R)$.

$\mathbb{RP}^\infty$ has cellular structure w/ one k-cell in each dimension. Attaching maps are $S^{k-1} \to \mathbb{RP}^{k-1}$.

$\rightsquigarrow$ cellular chain complex is

$$\to \cdots \xrightarrow{0} R \xrightarrow{2} R \xrightarrow{0} R \to \cdots$$

$\rightsquigarrow$ cochain complex

$$\xleftarrow{2} R \xleftarrow{0} R$$

$$\Rightarrow H^*(\mathbb{RP}^\infty, R) = \begin{cases} R & * = 0 \\ R[2] & * = \text{odd} \\ R/2R & * = \text{even} \end{cases}$$

e.g. if $R = \mathbb{Z}/2$, $\underset{\text{(rings)}}{\cong} \mathbb{Z}/2[x]$

② $R = \mathbb{Z}/m$, m odd, $H^*(\mathbb{RP}^\infty, \mathbb{Z}/m) = 0$ for $* > 0$.

<u>Why?</u>. $H^1(G, R) = H^1(BG, R) = \mathrm{Hom}(H_1(BG, R), R)$

$$= \mathrm{Hom}(\pi_1(BG)^{ab}, R)$$

$$= \mathrm{Hom}(G^{ab}, R)$$

we want to study $\mathrm{Gal}(\bar{F}/K)^{ab}$

$$\Updownarrow$$

$\mathrm{Hom}(\mathrm{Gal}(\bar{F}/K), \mathbb{Z}/m)\ \forall m.$