(1) (Cohomology of pro-cyclic groups) Let $G = \widehat{\mathbb{Z}}$, and let F be a topological generator of G. Show that for any torsion G-module, $H^1(G, M) \cong M/(F-1)M$. Show that $H^i(G, M) = 0$ for $i \geq 2$.

Solution. We have

$$H^1(G, M) = \varinjlim_{n \in \mathbb{N}} H^1(\widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}}, M^{n\widehat{\mathbb{Z}}})$$

We can now use the description of H^1 as the crossed homomorphisms modulo the principal crossed homomorphisms. A crossed homomorphism $\varphi : \widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}} \to G$ is determined by its value on a generator $\sigma_n \in \widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}}$ (which we can take as the projection of the generator F). Moreover, its value on a general element can be described as

$$\begin{aligned} \varphi(a\sigma_n) &= \varphi(\sigma_n + (a-1)\sigma_n) \\ &= \sigma_n \cdot \varphi((a-1)\sigma_n) + \varphi(\sigma_n) \\ &\vdots \\ &= (\sigma_n^{a-1} + \cdots + \sigma_n + \mathrm{id}_{\mathbb{Z}/n\mathbb{Z}})\varphi(\sigma_n). \end{aligned}$$

Since $n\sigma_n = \mathrm{id}_{\widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}}}$, the other requirement for an $m \in M^{n\widehat{\mathbb{Z}}}$ to be the image of σ_n is that m should be killed by the action of $N_n := \mathrm{id}_{\widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}}} + \sigma_n + \cdots + \sigma_n^{n-1}$. If we let ${}_N A$ denote the elements of A annhilated by N, then we have that the crossed homomorphisms are isomorphic to ${}_{N_n} M^{n\widehat{\mathbb{Z}}}$. On the other hand, it is clear that the principal crossed homomorphisms is isomorphic to ${}_{N_n}(\sigma_n - 1)M^{n\widehat{\mathbb{Z}}}$. This gives us

$$H^1(\widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}}, M^{n\widehat{\mathbb{Z}}}) \cong {}_{N_n} M^{n\widehat{\mathbb{Z}}}/(\sigma_n - 1)M^{n\widehat{\mathbb{Z}}}.$$

The direct limit commutes with quotient, and since M is torsion, $M = \varinjlim {}_{N_n} M^{n\widehat{\mathbb{Z}}}$. Finally, we should use some fact about the isomorphism

$$\widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}.$$

(2) Let G be a profinite group, M a finite G-module. Consider extensions

$$1 \to M \to E \xrightarrow{\pi} G \to 1,$$

where M is a normal subgroup of E. The G-action on E is given by lifting $g \in G$ to E and allowing it to act on M by conjugation. Let $\mathrm{Ext}(G, M)$ be the set of equivalence classes of E where $E \simeq E'$ if there is an isomorphism $E \to E'$ indicuing the identity on M and G.

a. Show that there is a natural isomorphism $\mathrm{Ext}(G, M) \cong H^2(G, M)$.

Solution. Given $\varphi \in Z^2(G, M)$, construct $E \in \mathrm{Ext}(G, M)$ by letting $E = M \times G$ as a set, and give it a group law $(m, g) \cdot (m', g') = (m + g(m') + \varphi(g, g'), gg')$. For an identity element to exist, we must have $g' = \mathrm{id}_G$ and $g(m') + \varphi(g, \mathrm{id}_G) = 0$. One can take $m' = 0$, as we claim that there is a φ' in the same cohomology class of φ such that $\varphi'(g, \mathrm{id}_G) = 0$. From the condition

$$g\varphi(g', g'') + \varphi(g, g'g'') = \varphi(g, g') + \varphi(gg', g''),$$

by setting $g' = g'' = \mathrm{id}_G$, we have

$$g\varphi(\mathrm{id}_G, \mathrm{id}_G) = \varphi(g, \mathrm{id}_G).$$

Let $\phi(g) = \varphi(\mathrm{id}_G, \mathrm{id}_G)$. Then $d^1\phi$ is a coboundary satisfying

$$d^1\phi(g, g') = g\phi(g') - \phi(gg') + \phi(g).$$

Notice that for $\varphi' := \varphi - d^1\phi$, we have

$$\varphi'(g, \mathrm{id}_G) = \varphi(g, \mathrm{id}_G) - d^1\phi(g, \mathrm{id}_G) = g\varphi(\mathrm{id}_G, \mathrm{id}_G) - g\varphi(\mathrm{id}_G, \mathrm{id}_G) - \phi(g) + \varphi(g) = 0.$$

From this, we also see that $(0, \mathrm{id}_G) \in E$ is an identity element.

Finally, by writing out an equation for associativity of the product of $(m, g), (m', g'), (m'', g'')$, we can see that we need to have

$$g\varphi(g', g'') + \varphi(g, g'g'') = \varphi(g, g') + \varphi(gg', g'').$$

This is exactly the condition that $d\varphi(g, g', g'') = 0$.

On the other hand, given an extension E, fix a section $s : G \to E$. Write the group operation of E, G additively and multiplicatively, respectively. We have the action of G on M is $\sigma m = s(\sigma) + m - s(\sigma)$, or $\sigma m + s(\sigma) = s(\sigma) + m$. Give σ, σ', notice that $s(\sigma) + s(\sigma')$ and $s(\sigma\sigma')$ are sent to the same element $\sigma\sigma'$ by π. Hence they differ by an element $\varphi(\sigma, \sigma') \in M$, i.e.

$$s(\sigma) + s(\sigma') = \varphi(\sigma, \sigma') + s(\sigma\sigma').$$

Given $\sigma, \sigma', \sigma''$, by the associativity of $s(\sigma) + s(\sigma') + s(\sigma'')$, we can deduce (replacing sums of the form $s(a) + s(b)$ by $\varphi(a, b) + s(ab)$, and using the commutation rule $\sigma \cdot m + s(\sigma) = s(\sigma) + m$)

$$\varphi(\sigma, \sigma') + \varphi(\sigma\sigma', \sigma'') = \sigma\varphi(\sigma', \sigma'') + \varphi(\sigma, \sigma'\sigma'').$$

This is precisely the condition that $d\varphi(\sigma, \sigma', \sigma'') = 0$.

Moreover, for another section s', repeat the above process to obtain a $\varphi' : G^2 \to M$. Let $s'' = s' - s$. Note that $\pi \circ s'' = 0$, so any $s''(\sigma) \in M$. Then

$$\begin{aligned}
s'(\sigma) + s'(\sigma') &= \varphi'(\sigma, \sigma') + s''(\sigma\sigma') + s(\sigma\sigma') \\
&= \varphi'(\sigma, \sigma') + s''(\sigma\sigma') - \varphi(\sigma, \sigma') + s(\sigma) + s(\sigma') \\
&= \varphi'(\sigma, \sigma') + s''(\sigma\sigma') - \varphi(\sigma, \sigma') - s''(\sigma) + s'(\sigma) - s''(\sigma') + s'(\sigma') \\
&= \varphi'(\sigma, \sigma') + s''(\sigma\sigma') - \varphi(\sigma, \sigma') - s''(\sigma) - \sigma s''(\sigma') + s'(\sigma') + s'(\sigma') \\
&= \varphi'(\sigma, \sigma') - \varphi(\sigma, \sigma') + s''(\sigma\sigma') - s''(\sigma) - \sigma s''(\sigma') + s'(\sigma') + s'(\sigma').
\end{aligned}$$

Hence $\varphi' - \varphi = s''(\sigma) - s''(\sigma\sigma') + \sigma s''(\sigma') = d^1 s''(\sigma, \sigma'')$. Hence φ, φ' are in the same cohomology class.

b. Verify that the trivial element of $H^2(G, M)$ corresponds to the semi-direct product.

Solution. Identify M with $M \times \mathrm{id}_G$ and identify G with $0 \times G$. Then it's clear their intersection only consists of the identity element $(0, \mathrm{id}_G)$. Moreover, we have

$$(m, \mathrm{id}_G) \cdot (0, g) = (m + \mathrm{id}_G(0) + \varphi(\mathrm{id}_G, g), g) = (m, g).$$

Hence the trivial element of $H^2(G, M)$ gives the semi-direct product.

c. Suppose we have $E \leftrightarrow \phi_E$. If $f : H \to G$ is a profinite group homomorphism, show that f lifts to E if and only if $f^*(\phi_E) = 0$.

Solution. Since M is a G-module, and $f : H \to G$, we can define the action of H on M to be $h \cdot m := f(h) \cdot m$. Then of course $f : H \to G$ and $M \to M$ (sending M as a G-module to M as an H-module) forms a compatible pair of homomorphisms.

If f lifts to $\tilde{f} : H \to E$, then $\tilde{f}(h) + \tilde{f}(h') = \phi(h, h') + \tilde{f}(hh')$ for some ϕ. If we take a particular lift $\tilde{f} = s \circ f$, then

(3) Compute $H^2(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}), H^2(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$, and groups of order p^3.

Solution. Regarding $\mathbb{Z}/p\mathbb{Z}$ as a $\mathbb{Z}/p\mathbb{Z}$-module means we have a map $\mathbb{Z}/p\mathbb{Z} \xrightarrow{\varphi} \mathrm{Aut}(\mathbb{Z}/p\mathbb{Z})$. Since

$|\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z})|$ has order $p-1$, and since $p = |\mathrm{im}\varphi||\ker\varphi|$, φ must be the zero map, so that $\mathbb{Z}/p\mathbb{Z}$ is a trivial $\mathbb{Z}/p\mathbb{Z}$-module.

A homogenous 2-cochain $\varphi \in \widehat{Z}^2(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ is determined by its values on $(1,0)$ and $(0,1)$, which as we have computed earlier (at least in the case of inhomogenous cochains...)