Galois theory of local fieds

Take $K/\mathbb{Q}_p$ finite. Recall this is unramified if $e_{L/K}=1$, i.e. $\mathcal{O}_K \longrightarrow \mathcal{O}_L$ is unramified (étale). If is totally ramified if $f_{L/K}=1$, where $f_{L/K}=[k_L:k_K]$.

Prop: ① $e_{L/K}\cdot f_{L/K} = [L:K]$.

② $\mathcal{O}_L = \mathcal{O}_K[\beta]$, for some β

③ If L/K is unram., can take $\beta = \overset{\text{Teichmüller lift}}{\omega}(\bar{\beta})$ where $k_L = k_K(\bar{\beta})$. If L/K is totally ramified, $\beta = \varpi_L$

Rmk. ② is false for r.o.i. of number fields.

Pf. If $\varpi_K \in \mathcal{O}_K$ is a uniformizer, then any $\alpha \in K$ is uniquely expressable as $\sum_{i\geq i_0} \omega(\underbrace{\alpha_i}_{\in k_K})\, \varpi_K^i$.

Then $\alpha \equiv \alpha_0 \pmod{\varpi}$, so $\alpha - \omega(\alpha_0) \in (\varpi)$, etc. More generally, fix a set of reps $\tilde{\varsigma}_i$ of residue field & fixe γ_i of each possible val'n $v_K(\gamma_i)$. Then

$$\alpha = \sum_{j\geq i_0} \varsigma_{f(j)}\, \gamma_j$$

Let ϖ_L uniformizer of L and let $k_L = k_K(\bar{\beta})$ (ϖ_K uniformizer of K). Then any $\alpha \in L$ is

$$\sum_{i\geq i_0} \sum_{t=0}^{e_{L/K}-1} \sum_{s=0}^{f_{L/K}-1} \omega(\bar{\beta}^s \cdot \underset{\in k_K}{\alpha_{i,s,t}})\, \varpi_K^i \varpi_L^t$$

$$= \sum_{s,t} w(\bar{\beta})^s \varpi_L^t \sum_{i \geq i_0} w(\alpha_{i,s,t}) \cdot \varpi_K^i .$$

Conclude. $\{ w(\bar{\beta})^s w_L^t \}_{\substack{s = 0, \dots, f-1 \\ t = 0, \dots, e-1}}$ are a K-basis of L, even an $\mathcal{O}_K$-basis.

In particular, $[L:K] = ef$. (3) follows as well, since e, f specialize.

For (2), let f be a monic lifting to $\mathcal{O}_K$ of the min poly of $\bar{\beta}$. Certainly, $f(w(\bar{\beta})) \equiv 0 \pmod{\varpi_L}$.

Case 1: If $v_L(f(w(\bar{\beta}))) = 1$, then $f(w(\bar{\beta}))$ is a uniformizer, i.e. take $\varpi_L = f(w(\bar{\beta}))$.

Case 2. If $v_L(f(w(\bar{\beta}))) > 1$, then set $\beta = w(\bar{\beta}) + \varpi_L$ for some uniformizer. Then $v_L(f(\beta)) = 1$.

Take $w(\bar{\beta}) + \varpi_L$ in place of a $w(\bar{\beta})$ and for w_L take $f(w(\bar{\beta}) + \varpi_L) = f(w(\bar{\beta})) + \varpi_L f'(w(\bar{\beta})) + \varpi_L^3 \cdots$

Then $f(\beta) \rightsquigarrow \{ \beta^s f(\beta)^t \}$ are an $\mathcal{O}_K$-basis.

eg: $\mathbb{Q}_p(\zeta_{p^n}) = L$, $\mathbb{Q}_p = K$.

$$\mathcal{O}_L = \mathbb{Z}_p[1 - \zeta_{p^n}] = \mathbb{Z}_p[\zeta_{p^n}].$$

Structure of unramified ext'ns

Lemma: L/K algebraic (not nec. fin.) Then $\exists!$ intermediate extension $\begin{matrix} L \\ | \\ M \\ | \\ K \end{matrix}$ s.t. M/K is unram., L/M is tot. ram., and if $L/N/K$ and N/K unram., $N \subset M$. If $L \supset N \supset K$ w/ L/N totally ramified, $N \supset M$.

Pf. Set $M = K(\omega(\alpha) \mid \alpha \in k_L)$. Then $k_M = k_L$, so L/M is totally ramified.

Claim. M/K unr. $M = \bigcup_{k_L \supset k' \supset k_K} K(\omega(k'))$, so s.t.p. $K(\omega(k'))/K$ is unr. Let $k' = k_K(\bar{\beta})$. Let f be min. (monic) poly of $\omega(\bar{\beta})$ over $\mathcal{O}_K$. Hensel $\Rightarrow$ $f \bmod \pi_K$ is still ——, so $[K(\omega(k')):K] = \deg f = \deg f \bmod \varpi_K = [k' : k_K]$. So $K(\omega(k'))/K$ is unr.

(i) If N/K unr., then $N = K(\omega(k_N))$ by last lemma. $K(\omega(k_L)) = M$.

(ii) If L/N tot. ram., $k_L = k_N$, so $N \supset K(\omega(k_L)) = M$.

Def. K^{ur} be the maximal unramified ext of K inside (a choice of alg. closure) $\bar{K}$.

Let $I_K = \mathrm{Gal}(\bar{K}/K^{ur})$, the <u>inertia</u> <u>group</u> of K.
If L/K is fin. Galois, let $K \subset M \subset L$, and set $I_{L/K} = \mathrm{Gal}(L/M) \trianglelefteq \mathrm{Gal}(L/K)$.

M/K Galois b/c it is <u>the</u> maximal unram ext.

$$1 \longrightarrow I_K \longrightarrow \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{Gal}(K^{ur}/K) \longrightarrow 1.$$

$\exists$ natural hom $\mathrm{Gal}(K^{ur}/K) \longrightarrow \mathrm{Gal}(k_{K^{ur}}/k_K)$

<u>Lemma</u>. This map is an <u>iso</u>.

<u>Pf</u>. <u>Injective</u>. If $\sigma \to 1$, then

$$\sigma(\underset{\overset{\uparrow}{\bar{k}}}{\omega(\alpha)}) = \omega(\sigma(\alpha)) = \omega(\alpha).$$

<u>Surj</u>. $\forall$ fin. ur. L/K,

$$\begin{array}{ccc} G(K^{ur}/K) & \hookrightarrow & G(\bar{k}/k) \\ \downarrow & & \downarrow \\ G(L/K) & \hookrightarrow & G(k_L/k_K) \end{array}$$

iso. since same size

$\hookrightarrow |G(L/K)| = |G(k_L/k_K)|$.

Take inverse limit to get isomorphism.

$$1 \longrightarrow I_K \longrightarrow G(\bar{K}/K) \longrightarrow G(K^{ur}/K) \underset{(x \mapsto x^{\# k})}{\xrightarrow{\sim}} G(\bar{k}/k) \simeq \hat{\mathbb{Z}}$$

$\mathrm{Frob}_K \longleftarrow 1$

<u>To understand I_K</u>

<u>Def</u>. L/K fin. Galois, the <u>lower ramification subgp</u>

$$G(L/K)_i := \{\sigma \in G(L/K) \mid \underset{\forall \alpha}{\sigma\alpha \equiv \alpha} \bmod \varpi_L^{i+1}\}.$$

e.g. $i=0$, $I_K = G(L/K)_0$.

<u>Lemma</u> (eases computation). For $i \geq 1$, $G(L/K)_i = \{\sigma \mid \sigma\varpi_L \equiv \varpi_L \ (\varpi_L^{i+1})\}$

(don't need to check all α)

<u>Pf</u>. Saw. $\mathcal{O}_L = \mathcal{O}_{L \cap K^{ur}}[\varpi_L]$.

<u>Prop</u>. $G(L/K)$ has a separated decreasing filtration by normal subgroups $G \supset G_0 \supset G_1 \supset \cdots$, and the successive quotients satisfy

$$G/G_0 \xrightarrow{\sim} G(k_L/k_K)$$

$$G_0/G_1 \hookrightarrow k_L^\times$$

and for all $i \geq 1$, $G_i/G_{i+1} \hookrightarrow \dfrac{1+(\varpi_L)^i}{1+(\varpi_L)^{i+1}} \hookrightarrow k_L$. (as an additive gp.)

<u>Cor</u>. $G(L/K)$ is solvable ($G(\bar{K}/K)$ is pro-solvable). $G(L/K)_1$ is the unique p-Sylow subgp of $G(L/K)$ ("wild ramification group").

<u>Pf</u>. $G_i \triangleleft G$ follows from definition since G preserves v_L.

$$G_0/G_1 \hookrightarrow k_L^\times \qquad \sigma \mapsto \frac{\sigma\varpi_L}{\varpi_L} \pmod{\varpi_L}$$

$$G_i/G_{i+1} \longrightarrow \frac{1+m_L^i}{1+m_L^{i+1}} \qquad \sigma \longrightarrow \frac{\sigma\varpi_L}{\varpi_L} \bmod 1+m_L^{i+1}.$$

Note that $\frac{1+m_L^i}{1+m_L^{i+1}} \rightsquigarrow \kappa_L$ by $1+\bar{\varpi}_L^i x \longrightarrow x$. This iso depends on choosing a uniformizer. ●

Def. L/K is tamely ram if $(e_{L/K}, p) = 1$,

i.e. G_1 is trivial.

Note. A composite of tame extensions is tame.

$L_1, L_2/K$ tame. Have

$$G(L_1L_2/K) \hookrightarrow G(L_1/K) \times G(L_2/K)$$

$$\cup$$

$$I_{L_1L_2/K} \hookrightarrow I_{L_1/K} \times I_{L_2/K}$$

Take p-Syl. s/g is $G(L_1L_2/K)_1 \hookrightarrow G(L_1/K)_1 \times G(L_2/K)_1$.

$\rightsquigarrow$ Makes sense to talk about **the** maximal tame extension K^t of K in $\bar{K}$.

Prop.

$$\begin{array}{l} \bar{K} \\ | \\ K^t = K^{ur}(\varpi_K^{1/n} \mid (n,p)=1). \\ | \\ K^{ur} = K(\omega(\bar{k})) = K(\zeta_n \mid (n,p)=1) \\ | \\ K \end{array}$$

Compute Galois group next time.