4. K a number field. Show $\exists L/K$ finite such that every prime p of K becomes principal in L.

$$p = \prod_{i=1}^{r} \mathfrak{B}_i^{e_i}$$

For a cyclotomic extension, (abelian, in general), all e_i are equal. Can $e_i > h_L$?

$\overset{\shortparallel}{\varphi(p^n)}$

For $I \subset K$, I^{h_K} is principal.

$$I^{h_K} = a\mathcal{O}_K \qquad \text{Let } L = K(\sqrt[h_K]{a}).$$

$$(\sqrt[h_K]{a})^{h_K}\mathcal{O}_L = a\mathcal{O}_L = I^{h_K}\mathcal{O}_L$$

$$I^{h_K}\mathcal{O}_L = (\sqrt[h_K]{a})^{h_K} = a\mathcal{O}_L$$

$$I^{h_K}\mathcal{O}_L = (I\mathcal{O}_L)^{h_K} = a\mathcal{O}_L = (\sqrt[h_K]{a}\,\mathcal{O}_L)^{h_K}$$

$$\overset{\shortparallel}{a}\mathcal{O}_L = (\sqrt[h_K]{a}\,\mathcal{O}_L)^{h_K}$$

$\alpha := \sqrt[h_K]{a}$

<u>Claim</u>. $I\mathcal{O}_L = (\alpha)$

<u>Pf</u>. have $I^{h_K}\mathcal{O}_L = (\alpha)^{h_K}$, ~~so $\alpha^{h_K} \in I^{h_K}\mathcal{O}_L$~~

$$(\alpha^{-1}I)^{h_K}\mathcal{O}_L = \mathcal{O}_L$$

$$((\alpha)^{h_K})^{-1}(I\mathcal{O}_L)^{h_K} = \mathcal{O}_L$$

Assume otherwise. $A = (\alpha)$, $I := I\mathcal{O}_L$

$IA^{-1} \neq \mathcal{O}_L$ $\qquad (IA^{-1})$

$I^{h_K} A^{-h_K} = \mathcal{O}_L \Rightarrow h_K \mid h_L$.

$1 \in IA^{-1}$ since $1 \in \sum ia$ $\quad i \in I^{h_K}$, $a \in A^{h_K}$

$\mathcal{O}_L \subset IA^{-1}$.

$(IA^{-1})^{h_K} = \mathcal{O}_L$

Take $i \in I$, $a \in A^{-1}$. Then

$$ia \cdot 1 \cdots 1 \in \mathcal{O}_L.$$

Take a representative $I_1, \dots, I_{h_K}$ from each ideal class.

$$I_1 = a I_1'$$

$$I_1^{h_K} = a^{h_K} (I_1')^{h_K} = b\,\mathcal{O}_K . \quad I\mathcal{O}_L = (\beta)$$

adjoin $\beta := \sqrt[h_K]{b}$, move to $\mathcal{O}_L$.

$$\beta^{h_K} = a^{h_K} (I_1')^{h_K}$$

$$(\beta / a)^{h_K} = (I_1')^{h_K}$$

So $I_1' = \left(\frac{\beta}{a}\right)$. If I_1' integral, a must be a unit.

<u>writeup</u>. Take an ideal $I \subset \mathcal{O}_K$. Then

$I^{h_K} = a\mathcal{O}_K$. Let $\alpha = \sqrt[h_K]{a}$, $L = K(\alpha)$. Then

$I^{h_K}\mathcal{O}_L = a\mathcal{O}_L = \alpha^{h_K}\mathcal{O}_L$. Let $A = (\alpha)$. So

$I^{h_K}A^{-h_K} = \mathcal{O}_L$. Then $1 \in I^{h_K}A^{-h_K} \subset IA^{-1} \Rightarrow \mathcal{O}_L \subset IA^{-1}$.

But $(IA^{-1})^{h_K} = \mathcal{O}_L$, so $IA^{-1} \cdot 1 \cdots 1 \subset \mathcal{O}_L$.

If $J \in \bar{I} \in Cl(K)$, $I = Ju$, $u \in K^\times$. Then

$J^{h_K}u^{h_K} = a\mathcal{O}_L \Rightarrow J\mathcal{O}_L = \alpha u^{-1}\mathcal{O}_L$, still principal,

don't need any new elements.

1. S finite set of places

$\alpha_v \in K_v$ given for $v \in S$.

<u>Claim</u>. If $\varepsilon > 0$, $\exists\, \alpha \in K$, $|\alpha - \alpha_v|_v < \varepsilon$ for all

$v \in S$ and $|\alpha|_v \leq 1$ for all $v \notin S$.

$|\alpha|_v = p_v^{-v_{p_v}(\alpha)}$. $\alpha \equiv \alpha_v \pmod{p_v^N}$ for $N \gg 0$.

Take $T := \{ v \mid v \notin S, |\alpha|_v > 1 \}$, so need

$\alpha \to \alpha\beta$ where

$\beta \equiv 1 \pmod{p_v^N}$, $v \in S$

$\beta \equiv 0 \pmod{p_v^{v_{p_v}(\alpha)}}$, $v \in T$

$|\alpha|_v \leq 1 \Leftrightarrow v_{p_v}(\alpha) \geq 0$.

$v_{p_v}(\alpha\beta) = v_{p_v}(\alpha) + v_{p_v}(\beta) \geq 0$.

2. $\mathbb{A}_K = \prod_{v\ \{\mathcal{O}_{K_v}\}} K_v$, topologized by nbhd basis of 1 given by:

$$\prod_{\substack{v\in S\\ S \text{ finite}}} U_v \times \prod_{v\notin S} \mathcal{O}_{K_v}$$

$\mathbb{A}_K^\times = \prod_{v\ \{\mathcal{O}_{K_v}^\times\}} K_v^\times$ has nbhd basis of 1 given by:

$$\prod_{\substack{v\in S\\ S \text{ finite}}} U_v \times \prod_{v\notin S} \mathcal{O}_{K_v}^\times$$

(a) $a_n = (\alpha_v)_v$, $\alpha_v = p_n$ if $v = v_{p_n}$, else 1.

$a_n \notin \mathcal{O}_{K_{v_{p_n}}}^\times$ $\forall n$, so $a_n \notin \prod_v \mathcal{O}_{K_v}^\times$ $\forall n$.

(b) Same sequence, same reason, it doesn't converge in $\mathbb{A}_K^\times$.

$$U_S = \prod_{v\in S} 1 + p_v^{n_v}(\mathcal{O}_{K_v}) \prod_{v\notin S} \mathcal{O}_{K_v}$$

For $n > \max_v n_v$, $a_n \in U_S$.

(c) $\mathbb{A}_K^{\times S} = \prod_{v\in S} K_v^\times \times \prod_{v\notin S} \mathcal{O}_{K_v}^\times$ w/ product topology.

$U = \prod_{v\in S} U_{K_v} \times \prod_{v\notin S} \mathcal{O}_{K_v}^\times$ $\quad U_{K_v} \subset K_v^\times$ open.

$U \cap \mathbb{A}_K^{\times S} = U$, open in $\mathbb{A}_K^{\times S}$ ~~is open~~.

$$U \cap \mathbb{A}_K^{\times S'} = \prod_{v \in S \cap S'} U_{K_v} \times \underbrace{\prod_{v \in S \setminus S'} U_{K_v} \cap \mathcal{O}_K^{\times}}$$

open. π $m_{\mathcal{O}_{K_v}} \cap K_v^{\times}$

$$\prod_{v \in S'} K_v^{\times} \times \prod_{v \notin S'} \mathcal{O}_K^{\times} \quad \times \prod_{v \in S' \setminus S} \mathcal{O}_{K_v}^{\times} \times \prod_{v \notin S \cup S'} \mathcal{O}_{K_v}^{\times}$$

is open.

Now suppose $U \subset \bigcup_S \mathbb{A}_K^{\times S}$ is open, so

$U \cap \mathbb{A}_K^{\times S}$ open $\forall S$. This means

$U \cap \mathbb{A}_K^{\times S}$ of the form, if $U = \prod_v U_v$

$$U \cap \mathbb{A}_K^{\times S} = \prod_v W_v$$

Take $S = \{v_0\}$.

$$K_{v_0}^{\times} \times \prod_{v \neq v_0} \mathcal{O}_v^{\times}$$

$$U_{v_0} \cap K_{v_0}^{\times} = W_v$$

$K = \mathbb{Q}[\sqrt{-31}]$ $\quad N(I) \leq 3.5\ldots$

$\mathcal{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{-31}}{2}\right]$ $\quad \frac{2a+b+b\sqrt{-31}}{2}$

$\mathbb{Z}[\sqrt{-31}]$ $\quad c+d\sqrt{-31}$

$$\frac{2a+b-2c+b-2d\sqrt{-31}}{2}$$

$\left|\mathcal{O}_K/\mathbb{Z}[\sqrt{-31}]\right| = 2 \quad \frac{1+\sqrt{-31}}{2}, 0$

<u>mod 3</u>

$x^2+31 \equiv x^2+1$ is irreducible, so 3 stays prime.

<u>mod 2</u>.

$$(2) = \left(2, \frac{1+\sqrt{-31}}{2}\right)\left(2, \frac{1-\sqrt{-31}}{2}\right)$$

$$2 = 2 \cdot \frac{1+\sqrt{-31}}{2} + 2 \cdot \frac{1-\sqrt{-31}}{2}$$

Proves $\subset$. $4 \in (2)$, $1+\sqrt{-31} \in (2)$, and

$$\frac{(1+\sqrt{-31})(1-\sqrt{-31})}{4} = \frac{1-(-31)}{4} = 8.$$

$$\left(2, \frac{1+\sqrt{-31}}{2}\right)\left(2, \frac{1+\sqrt{-31}}{2}\right)$$

$$= \left(4, 1+\sqrt{-31}, \frac{1+\sqrt{-31}}{2}\right)$$

$$\left(4, \frac{1+\sqrt{-31}}{2}\right)\left(2, \frac{1-\sqrt{-31}}{2}\right)$$

$$\left(8, 1+\sqrt{-31}, 1-\sqrt{-31}, 8\right)$$

$= (2)$

$\mathfrak{p}_2^2 \bar{\mathfrak{p}}_2 = 1.$ $\qquad \mathfrak{p}_2^3 \overset{?}{=} 1$

$\mathfrak{p}_2 \bar{\mathfrak{p}}_2 = 1$ $\qquad \mathfrak{p}_2^3 \bar{\mathfrak{p}}_2 = \mathfrak{p}_2$

$\boxed{\mathfrak{p}_2^4 \bar{\mathfrak{p}}_2 = 1.}$ obtain this by squaring repeatedly.

$\mathfrak{p}_2^3 = 1$

$$d_{K(\sqrt{-1})/\mathbb{Q}} = d_{K/\mathbb{Q}}^2 \, N_{K/\mathbb{Q}}\left(d_{K(\sqrt{-1})/K}\right)$$

$$400 \cdot N_{K/\mathbb{Q}}$$