<u>Recall</u>. K number field, v - valuation of K

$$\begin{array}{c} L \\ | \\ K \end{array}$$ Galois, possibly ∞. Let w extend v to L.

$$G(L/K)_w = \{ \sigma \in G(L/K) \mid w \circ \sigma = w \}.$$

<u>Prop</u>. $G(L/K)$ acts trasitively on $\{ w | v \} \ni w, w'$

Last time, did finite case. General case!

$$L = \bigcup_{L \supset M/K \text{ fin. Galois}} M$$

For each M, $X_M := \{ \sigma \in G(L/K) \mid w \circ \sigma|_M = w'|_M \}$

M/K fin. $\Rightarrow X_M \neq \emptyset$ by finite case. Also $X_M \subset G(L/K)$ is closed ($\sigma \notin X_M \Rightarrow \sigma G(L/M)$ disjoint from M, open s/g in $G(L/K)$ by def of topology)

<u>Claim</u>. $\bigcap_M X_M \neq \emptyset$.

or, filtered inverse lim at non-empty compact sets is non-empty.

<u>Lem</u>. (Local Galois $\hookrightarrow$ global Galois)

1. The decomposition groups G_w are <u>closed</u> in G. Likewise for analogues of higher ramification groups.
2. $\forall \sigma \in G$, $\sigma^{-1} G_w \sigma = G_{w \circ \sigma}$, ditto above.
3. $G(L_w/K_v) \xrightarrow{\text{res}} G(L/K)_w < G$.

Convention. If L/K finite, L_w is the completion. For L/K infinite, $L_w = \bigcup$ (completions of finite ext's)

Pf. ① If $\sigma \in \overline{G_w} \hookrightarrow G$, then $\forall$ fin ext M/K,

$\sigma \cdot G(L/M) \cap G_w \neq \emptyset$. Take σ_M in that set.

$\Rightarrow w \circ \sigma|_M = w \circ \sigma_M|_M = w|_M$.

True $\forall M \Rightarrow w \circ \sigma = w$.

② $G_{w \circ \sigma} \xrightarrow{\sim} G_w$

$\tau : w \circ \sigma \circ \tau = w \circ \sigma \to \sigma \tau \sigma^{-1}$

$\Rightarrow w \circ (\sigma \tau \sigma^{-1}) = w$

③ Note $G_w = \{\sigma \in G \mid \sigma$ is cts w.r.t. $w\}$

$\subseteq$ obvious

$\supseteq$ if $\sigma \in$ RHS, then $\forall x$, $|x|_w < 1 \Leftrightarrow |\sigma x|_w < 1$.

We saw this implies $w \circ \sigma = w$ (they are equiv & extend common v)

Look at res $G(L_w/K_v) \xrightarrow{\text{res}} G(L/K)$.

res is inj. b/c $L \subset L_w$ is dense.

Claim. the image is precisely $\sigma \in G$ that are cont wrt w. (and hence extend to L_w)

Bookkeeping: Normalized absolute values.

Motivation. For # field K, we will choose representatives $|\cdot|_v$ for each place v s.t.

Product Formula. $\forall \alpha \in K^\times$, $\prod_v |\alpha|_v = 1$

Def. If v is a finite place, normalize ~~the~~ v (as val'n) s.t. $v: K^\times \to \mathbb{Z}$ and set $\underset{K_v}{|\alpha|_v} = (\# k_v)^{-v(\alpha)}$. If v is an ∞ place, it corresponds to $\{\tau : K \to \mathbb{C}\}/\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and let

$$|\alpha|_v = \begin{cases} |\tau\alpha|_\infty & \text{if } \tau(K) \subset \mathbb{R}, \\ \underbrace{|\tau\alpha|^2_\infty}_{\text{norm on } \mathbb{C}} & \text{if } \tau(K) \not\subset \mathbb{R} \end{cases}$$

usual $\mathbb{R}$-abs

$$v \text{ fin.} \longleftrightarrow \{\tau : K \hookrightarrow \overline{\mathbb{Q}_p}\}/\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$$

In these terms,

$$|\alpha|_v = |\tau\alpha|_p^{[K_v:\mathbb{Q}_p]}.$$

unique ext of normalized abs val on $\mathbb{Q}_p$

Why these agree: $|\alpha|_v = (p^{-f_v})^{e_v \cdot v_p(\alpha)}$

$$= |\tau\alpha|_p^{[K_v:\mathbb{Q}_p]}$$

Lemma. If L/K f.n., v place of K, $\alpha \in L$, then

① If $w|v$, then $\underset{\text{(NORMALIZED)}}{|\alpha|_w} = |N_{L_w/K_v}\, \alpha|_v$

② $|N_{L/K}(\alpha)|_v = \prod_{w|v} |\alpha|_w$

③ (*) $\forall \alpha \in K$, $\prod_v |\alpha|_v = 1$.

Pf. Why $1+2 \Rightarrow 3$. Saw 3 was true for $\mathbb{Q}$. For general K,

$$\prod_{v \text{ of } K} |\alpha|_v = \prod_p \prod_{w|p} |\alpha|_v$$

$$= \prod_p |N_{K/\mathbb{Q}}(\alpha)|_p = 1$$

by $\mathbb{Q}$ case.

Ex. $K = \mathbb{Q}(i)$ $\alpha = 3$. $\exists!\ v|3$, $f_v = 2$.

$$|\alpha|_v = 9^{-1}$$

For $v|\infty$, $\underset{\text{(not normalized)}}{|\alpha|_v} = |\tau\alpha|_\infty^2 = 9$.

Pf of 1. Write $|\cdot|_v$ for the ! extension of $|\cdot|_v : K_v \longrightarrow \mathbb{R}$. Characterized by $|\alpha|_v$ ($\alpha \in L_v$)

$= |N_{L_w/K_v}(\alpha)|_v^{1/[L_w:K_v]}$

Now $|\alpha|_w = \# k_w^{-w(\alpha)} = \# k_v^{-f_{w/v} \cdot e_{w/v} \cdot v(\alpha)}$

$= |\alpha|_v^{[L_w : K_v]}$

<u>Pf of 2.</u> $|N_{L/K}(\alpha)|_v = \prod_{w|v} |N_{L_w/K_v}(\alpha)|_v$

from $L \otimes K_v \simeq \bigoplus_{w|v} L_w$

$= \prod_{w|v} |\alpha|_w.$ ∎

⊛ Local-global packaging of info on automorphic side

<u>Adeles</u> of K.

<u>Def.</u> $\mathbb{A}_K$, the ring of adeles of K, is the restricted direct product

$$\prod_{v\,\{\mathcal{O}_{K_v}\}}{}' K_v$$

w.r.t. the $\mathcal{O}_{K_v}$, i.e.

$$\left\{ (x_v) \in \prod_v K_v \;\middle|\; \begin{array}{l}\text{almost all } x_v \\ \text{belong to } \mathcal{O}_{K_v}\end{array} \right\}.$$

<u>Def.</u> <u>ideles</u> are the group of units of the adeles, i.e.

$$\prod_{v\,\{\mathcal{O}_{K_v}^\times\}}{}' K_v^\times$$

Global content comes from embeddings

$$\begin{array}{ccc} K & \hookrightarrow & \mathbb{A}_K \\ K^\times & \hookrightarrow & \mathbb{A}_K^\times \\ x & \longrightarrow & (x, \dots x, \dots) \end{array}$$

Adeles + embedding is what is important.

Moreover. $\mathbb{A}_K^\times$ is a locally cpt. Hausdorff group w/ natural topology on res. dir. prod.

Explicitly. a neighborhood basis of ~~(subgroups)~~ 1 is given by a product of open sets at finitely many places and the ring of integers everywhere else. (one must be infinite place).

Ex. $K = \mathbb{Q}$, $U = \mathbb{R}^\times_{>0} \times \prod_{\substack{\text{primes} \\ p \neq 5}} \mathbb{Z}_p^\times \times (1 + 125\,\mathbb{Z}_5)$

Here, $S = \{5, \infty\}$

Rmk. $\mathbb{A}_K^\times$ is not topologized via subspace topology from $\mathbb{A}_K$.

$\boxed{\text{Claim. } \mathbb{A}_K^\times = GL_1(\mathbb{A}_K)}$

<u>Lemma.</u> $K^\times \hookrightarrow \mathbb{A}_K^\times$ is discrete ($\Rightarrow$ closed), so the quotient topology makes

$$\boxed{C_K := \mathbb{A}_K^\times / K^\times}$$ a locally cpct Hausdorff grp

idele class group

$GL_1(\mathbb{A}_K)/GL_1(K)$.

<u>Pf:</u> Let $U = \{\alpha \in \mathbb{A}_K^\times \mid |\alpha_v|_v = 1$ for all finite primes and $|\alpha_v - 1|_v < 1$ for all ∞ $v\}$

U is open and if $x \in K^\times \cap U \setminus \{1\}$, then

$$1 = \prod_v |x-1|_v < \prod_{v \nmid \infty} |x-1|_v = 1, \ \lightning.$$

$$\underset{\substack{\| \\ 1}}{\max(|x|_v, |1|)}$$

■

<u>Relation to classical class group.</u>

<u>Lemma.</u> $\mathbb{A}_K^\times / K^\times \prod_{v|\infty} K_v^\times \prod_{v \nmid \infty} \mathcal{O}_v^\times \xrightarrow{\sim} Cl(\mathcal{O}_K)$.

<u>Pf.</u> Define $\mathrm{div}: \mathbb{A}_K^\times \longrightarrow \mathrm{Div}(\mathcal{O}_K)$ = free ab grp on primes of $\mathcal{O}_K$

$$(\alpha_v) \longrightarrow \sum_{v \nmid \infty} P_v^{v(\alpha_v)}$$

$$\left(\text{or } \sum_v [v]\, v(\alpha_v)\right)$$

By <u>def'n.</u> this induces $\mathbb{A}_K^\times / K^\times \twoheadrightarrow Cl(\mathcal{O}_K)$.

The kernel is $\prod_{v|\infty} K_v^\times \prod_{v\nmid\infty} \mathcal{O}_v^\times$.

Rmk. Eventually, $H^1(G(L/K), C_L)$ will be a fundamental part of CFT.

Rmk. $\forall$ collections of open subgroups $U_v \subseteq \mathcal{O}_v^\times$, w. a.e. $U_v = \mathcal{O}_v^\times$, we can think of

$$\mathbb{A}_K^* / K^* \prod_{v \mid \infty} K_v^\times \prod_{v \nmid \infty} U_v .$$

Useful Lemma. For a la

Def. For any fin. set. of places S containing the archimedean places, let

$$\mathbb{A}_K^{\times S} = \prod_{v \in S} K_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times .$$

So $\mathbb{A}_K^\times = \bigcup_S (\mathbb{A}_K^\times)^S$.

Lemma. For S finite & suff. large,

$$\mathbb{A}_K^\times = K^\times \cdot \mathbb{A}_K^{\times, S}$$

Pf. Let S be large enough that primes in S generate $Cl(\mathcal{O}_K)$ [FINITE!] (and $S \supset \{v \mid \infty\}$).

Let $I_1, I_2, \dots, I_k$ be ideals of $\mathcal{O}_K$, supported on S, that represent all elements of $Cl(\mathcal{O}_K)$.

For $\alpha \in \mathbb{A}_K^\times$, $\mathrm{div}(\alpha) = I_j \cdot \mathrm{div}(a)$. Hence $\mathrm{div}(a^{-1}\alpha) = I_j$, and for $v \notin S$, $a^{-1}\alpha \in \mathcal{O}_v^\times$

$\Rightarrow \alpha \in K^\times \cdot \mathbb{A}_K^{\times, S}$.

b/c I_j is supp'd on S.

<u>Define</u>. $|\cdot| : \mathbb{A}_K^\times \to \mathbb{R}^\times$ by

$$(\alpha_v) \longmapsto \prod |\alpha_v|_v$$

Set $\mathbb{A}_K^1 := \ker |\cdot| \underset{\text{prod form}}{\supset} K^\times$. So can form

$$C_K^1 = \mathbb{A}_K^\times / K^\times$$

<u>Thm</u>. C_K^1 is compact. This statement is logically equivalent to [$Cl(\mathcal{O}_K)$ finite AND Dirichlet unit thm].