Thm

We were working on showing (a) for $L \subset K(\zeta_m)$, Reduced this to checking:

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad G(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wr\!\!\mid\ \text{LCFT} \end{array}$$

Lemma. $r_{\mathbb{Q}_p} = s_{\mathbb{Q}_p}$ where $s_{\mathbb{Q}_p} : \mathbb{Q}_p^\times \longrightarrow G(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \times G(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$

$$p \longrightarrow (\mathrm{Frob}_p, 1)$$

$$\text{cyclotomic char } \kappa: \quad \mathbb{Z}_p^\times \xleftarrow{\sim} 1 \times G(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$$

Pf. of Lemma: In Milne, use Lubin-Tate (purely local).

The lemma follows from

Sub-Lemma. $\forall$ $\boxed{\text{cyclotomic}}$ (contained in) $L/\mathbb{Q}_p$, $s_{\mathbb{Q}_p}(N_{L/\mathbb{Q}_p}L^\times)\big|_L = 1$ & consequently induces an iso $\mathbb{Q}_p^\times/NL^\times \xrightarrow{\sim} G(L/\mathbb{Q}_p)$

Since: $s_{\mathbb{Q}_p}(p) = r_{\mathbb{Q}_p}(p)$ b/c they agree on $\mathbb{Q}_p^{ur}$ ($= \mathrm{Frob}_p$) and $r_{\mathbb{Q}_p}(p)\big|_{\mathbb{Q}_p(\mu_{p^r})} = 1$ since $N(1-\zeta_p^r) = p$, so $p \in N\mathbb{Q}_p(\mu_{p^r})^\times$. Fix $r \geq 1$ & $u \in \mathbb{Z}_p^\times$. Must show $r_{\mathbb{Q}_p}(u)\big|_{\mathbb{Q}_p(\mu_{p^r})} = s_{\mathbb{Q}_p}(u)\big|_{\mathbb{Q}_p(\mu_{p^r})}$

For any $\alpha \in \mathrm{Hom}(G(\mathbb{Q}_p^{ur}/\mathbb{Q}_p), G(\mathbb{Q}_p(\mu_{p^r})/\mathbb{Q}_p))$, let

M_α = fixed field of $\mathrm{graph}(\alpha) \hookrightarrow G(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \times G(\mathbb{Q}_p(\mu_{p^r})/\mathbb{Q}_p)$

Then $pu \in NM_\alpha^\times \iff s_{\mathbb{Q}_p}(pu)|_{M_\alpha} = 1 \iff a(\mathrm{Frob}_p) = s_{\mathbb{Q}_p}(u)|_{\mathbb{Q}_p(\mu_{p^\infty})}$

$$\Updownarrow$$

$$r_{\mathbb{Q}_p}(pu)|_{M_\alpha} = 1 \overset{\text{defn of } M_\alpha}{\iff} (\mathrm{Frob}_p) = r_{\mathbb{Q}_p}(u)$$

$$\|$$

$$(\mathrm{Frob}_p, r_{\mathbb{Q}_p}(u))$$

Now choose $\alpha \in a_i \mathbb{Q}$

Proof of sublemma. Whole sublemma of this follows from $s_{\mathbb{Q}_p}(NL^\times) = 1$. Clearly is surjective, and $LCFT \Rightarrow [\mathbb{Q}_p^\times : NL^\times] = [L : \mathbb{Q}_p]$.

To check, ~~...~~ $s_{\mathbb{Q}}(NL^\times)|_{L} = 1$ and continued to swear, claiming the ~~construction~~/set-up

$$\begin{array}{ccc} \mathbb{A}_{\mathbb{Q}_n} & \overset{\text{diff}}{\dashrightarrow} & G(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \\ \downarrow & \swarrow_{\cong} & \\ \mathbb{A}_f^{\mathbb{Q}}/\mathbb{Q}^\times \mathbb{R}_{>0}^\times & & \end{array}$$

Check: $\boxed{s_{\mathbb{Q}}(N/\mathbb{A}_L^\times) = 1}$ for a $\underline{\text{m}}$...

Let $L \subset \mathbb{Q}(\zeta_n)$ ~~...~~ $\exists$ open sg in

$u \to \mathbb{A}_L^\times$. $\exists$ open sg. $\{x \mid x \in \hat{\mathbb{Z}}, x \equiv 1 \text{ mod } n\}$.

(Choose a finite set S of places (not dividing m.C s.t. $\Rightarrow$ $c(^{ur}/L_v)$.

$\mathbb{A}_L^\times = L^\times \cdot L_S^\times U$. So $\alpha(L\,\mathbb{Q}^\times)$, and

$s_{L/\mathbb{Q}}(Nu) = 1$ (since $Nu = 1$ ~~and~~ $\pm$ d mental.

$j\alpha: \alpha(G_{\mathbb{Q}_p}/\mathbb{Q}_p)$ and K explain.

$= G(\mathbb{Q}_p/\mathbb{Q}$.

So it STP for ever real $v \in S$,

$s_{v/\mathbb{Q}_w}$ (NFT following).

So $N_{L_v/\mathbb{Q}_p} \underset{\substack{ur \\ loc \\ adel}}{=} (\mathbb{Z}_p^\times, p^{[L_v:\ ]})$

but $\boxed{s_{L/\mathbb{Q}}(\mathbb{Z}_p^\times) = 1}$ for $p \nmid m$. B.

Such $s_{L/\mathbb{Q}}(p^{[L_v:\mathbb{Q}_p]})$ and $s_{L/\mathbb{Q}}$ be

a similar.

$s_{L/\mathbb{Q}} = \mathbb{A}_{\mathbb{Q}}^\times = \mathbb{Q}^\times \times \mathbb{R}_{>0}^\times \times \hat{\mathbb{Z}}^\times \times p^{\mathbb{Z}}/\mathbb{Z} \twoheadrightarrow \hat{\mathbb{Z}}^\times$

$\hat{\mathbb{Z}}_p \to 1$

$\twoheadrightarrow (\mathbb{Z}/n\mathbb{Z})^\times \Leftrightarrow G_1(\mathbb{Q})$ says top

$s_{L/\mathbb{Q}}(1, \dots, p, 1, \dots) = \mathrm{Frob}_p.$

$(p, \dots)\ (p^{-1}, p^{-1}, \dots, 1, p^{-1}, \dots)$

$$\downarrow s_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}.$$

$$1 \cdot p^{-1} \in (\mathbb{Z}/n\mathbb{Z})^\times$$

$$\uparrow \kappa$$

$$\begin{pmatrix}\text{geom}\\ \text{frob}\end{pmatrix} = \mathrm{Frob}_p \in G(\mathbb{Q}(\zeta_n)/\mathbb{Q}).$$

This finishes proof of $r_{\mathbb{Q}_p} = s_{\mathbb{Q}_p}$ & thus of (a) in $L \subset K(\zeta_m)$ case.

<u>Completion of pf of thm</u>.

<u>Rmk</u>. We just proved Kronecker weber.

Now relate parts (a), (b).

used to bootstrap: <u>Lemma</u>. Let L/K abelian, (1): if (b) holds for L/K then (a) holds for L/K.

(2) If L/K is cyclic [(a) for $L/K \Rightarrow$ (b) for L/K]

<u>Claim</u>. For L/K abelian and any $\chi \in \mathrm{Hom}(G(L/K), \mathbb{Q}/\mathbb{Z})$, have comm. diag:

$$\begin{tikzcd}
K^\times \arrow[r] \arrow[d, "\cup\delta\chi"] & \mathbb{A}_K^\times \arrow[r, "r_{L/K}"] \arrow[d, "\cup\delta\chi"] & G(L/K) \arrow[d, "\chi"] \\
H^2(G(L/K), L^\times) \arrow[r] & H^2(G(L/K), \mathbb{A}_L^\times) \arrow[r, "-\Sigma \mathrm{inv}_v"] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

$$H^2(G(L/K), \mathbb{A}_L^\times) \cong \bigoplus Br(L_w/K_v)$$

Here, $\delta\chi \in H^2(G, \mathbb{Z})$ (so e.g. $H^0(G, L^\times) \xrightarrow{\cup\delta\chi} H^2(G, L^\times)$.

- LH square commutes by prop's of $\cup$
- RH by $\chi(r_{L_w/K_v}(a)) = -\mathrm{inv}(a \cup \delta\chi)$.

(ii) If (b) holds for L/K, then for $a \in K^\times$, $\chi(r_{L/K}(a)) = 0$ $\forall \chi : G(L/K) \to \mathbb{Q}/\mathbb{Z}$ hence $r_{L/K}(a) = 1$.

(2) if L/K is cyclic, then for any choice of injective χ, the map $\cup\delta\chi$ induces an iso $K^\times/NL^\times \xrightarrow{\sim} H^2(G(L/K), L^\times)$.

So in the cyclic case assume (a) holds. Then $\cup\delta\chi$ being an iso in LH square implies $H^2(G(L/K), L^\times) \xrightarrow[-\Sigma \mathrm{inv}_v]{} \mathbb{Q}/\mathbb{Z}$ is zero.

i.e. (b) holds. $\blacksquare$

<u>Conclusion</u>. <u>pf of Theorem</u>. We know (a) holds for all cyclotomic extensions. By the Lemma, (b) holds for the cyclic cyclotomic extensions.

<u>For the general case of (b)</u>: Let $\beta \in Br(K)$.

<u>Claim</u>. $\beta|_L = 0$ for some cyclic cyclotomic L/K.

Granted the claim, $\sum inv_v(\beta) = 0$ by (b) in cyclic cyclotomic case. So (b) holds in general.

By the Lemma, (a) then holds in general.

<u>Pf</u> (that $\beta|_L = 0$ for some cyclic cyclotomic L/K:

recall

$$\begin{tikzcd} Br(K) \arrow[r, "res"] \arrow[d, "inv_v"] & Br(L) \arrow[d, "inv_w"] \\ \mathbb{Q}/\mathbb{Z} \arrow[r, "{[L_w:K_v]}"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

, and

we have $Br(L) \hookrightarrow \bigoplus_w Br(L_w)$. So STP

$\beta|_{L_w} = 0 \ \forall w$, i.e. $[L_w:K_v] \cdot inv_v(\beta) = 0 \ \forall$ places.

Given $\mathcal{B} \in Br(k)$, $inv_v(\mathcal{B}) = 0$ $\forall v \notin S$, for S some finite set of places. So $\exists\, m \in \mathbb{Z}_{\geq 1}$ s.t. $m \cdot inv_v(\mathcal{B}) = 0$ $\forall$ places v. So it STP that given an integer m & a finite set of places S, $\exists\, L/k$ cyclic cyclotomic s.t. $m \mid [L_w : k_v]$ $\forall v \in S$.

By enlarging m, we can reduce to the case $k = \mathbb{Q}$: if $m' \mid [L_w : \mathbb{Q}_v]$, then

$$m' \mid [(Lk)_w : k_v][k_v : \mathbb{Q}],$$ these

all divide $[k : \mathbb{Q}]$. So, solve the problem for $m' = m[k : \mathbb{Q}]$ over $\mathbb{Q} \rightsquigarrow$ field L. Then use Lk/k to win for m over k. The pf for $k = \mathbb{Q}$ is elementary Gal. thry of $\mathbb{Q}(\mu_\infty)/\mathbb{Q}$.