Sketch of weak Mordell-Weil th'm

Th'm. K - # field, A/K ab. variety, $n \in \mathbb{Z}_{\geq 1}$, then $A(K)/n(A(K))$ is finite.

Kummer seq. assoc'd to $0 \to A[n] \to A \xrightarrow{[n]} A \to 0$

$$\begin{array}{lcccl} \rightsquigarrow & 0 \longrightarrow & A(K)/nA(K) & \hookrightarrow & H^1(G_K, A[n]) \longrightarrow \cdots \\ & & \downarrow & & \text{Ш} \\ \forall v, & 0 \longrightarrow & A(K_v)/nA(K_v) & \xhookrightarrow{\delta_v} & H^1(G_{K_v}, A[n]) \longrightarrow \cdots \\ & & y & \longmapsto & \sigma\tilde{y} - \tilde{y} \mid [n]\tilde{y} = y. \end{array}$$

Note. $A(K)/n(A(K)) \subset \{x \in H^1(G_K, A[n]) \mid \forall v, \ x|_v \in \operatorname{im} \delta_v\}$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \parallel$ (notation)

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \mathrm{Sel}^n(A/K)$

Claim. $\mathrm{Sel}^{(n)}(A/K)$ is finite.

More precisely. Let S be a finite set of places, ~~the~~ s.t. $S \supset \{v \mid n, v \mid \infty, v : A \text{ has bad reduction}\}$

Then $\mathrm{Sel}^{(n)}(A/K) \subset \underbrace{H^1(G_S, A[n])}_{\text{we've seen this is finite.}}$

Pf. Need to show $\phi \in \mathrm{Sel}^{(n)}(A/K) \Rightarrow \phi|_{I_v} = 0 \ \forall v \notin S$.

By assumption, $\phi|_{G_{k_v}}(\sigma) = \sigma\tilde{y} - \tilde{y}$ for some $\tilde{y} \in A(\mathcal{O}_v')$

$\mathcal{O}_v'$ roi in some ext of K_v.

Reduce to k_v' = res field of $\mathcal{O}_v'$.

$$\tilde{y} \longrightarrow (\bar{y} \in A(k'))$$

reduced to $A(k_v')$

Thus $\sigma \in I_{k_v} \Rightarrow \sigma\bar{y} - \bar{y} = 0$, i.e. $\bar{\phi}|_{I_{k_v}}$ is 0.

But for $v \nmid n$, $A(\mathcal{O}_v')[n] \longrightarrow A(k_v')[n]$ is injective,

b/c kernel is a pro-v-group.

$\Rightarrow \phi|_{I_{k_v}} = 0$

<u>Thm</u>. (global Euler char formula) Let M be a fin. G_S-mod. $S \supset \{v | \infty; v \mid v(\#M) \neq 0\}$

Then $$\frac{\# H^1(G_S, M)}{\# H^0(G, M)\, H^2(G, M)} \cdot \prod_{v \mid \infty} H^0(K_v, M) = \# M^{[k:\mathbb{Q}]}.$$

$*$

<u>Cor</u>. As above, but now let $\mathcal{L} = \{L_v\}$ be a Sel. system for M (recall $\mathcal{L}^\perp = \{L_v^\perp\} \subset H^1(K_v, M^*)$).

Then $$\frac{\# H^1_{\mathcal{L}}(K, M)}{\# H^1_{\mathcal{L}^\perp}(K, M^*)} = \frac{\# H^0(K, M)}{H^0(K^*, M^*)} \cdot \prod_{v \in S} \frac{\# L_v}{\# H^0(K, M)}.$$

Rmk. if v unramified and $L_v = \underbrace{H^1(G_K/I_K, M)}_{(M/(Fr_v - 1)M)}$, then

$\# L_v = \# H^0(K_v, M)$.

Pf. (check)

$$0 \to M^{Fr_v = 1} \to M \xrightarrow{Fr_v - 1} M/(Fr_v - 1)M$$

Galois representations.

A/K – abelian variety.

$\forall n$, have $A[n] \circlearrowleft G_K$. If $\dim A = g$, in a basis

this gives hom $G_K \longrightarrow GL_{2g}(\mathbb{Z}/n)$

(as an abstract grp., $A[n] \simeq (\mathbb{Z}/n)^{2g}$ ... think $/\mathbb{C}$, $A = \mathbb{C}^g/\text{lattice}$, lattice $\simeq \mathbb{Z}^{2g}$)

Compatible. $A[mn] \xrightarrow{[m]} A[n]$.

Gives

$$\begin{tikzcd} & GL_{2g}(\mathbb{Z}/mn) \arrow[d] \\ G_K \arrow[ur] \arrow[r] & GL_{2g}(\mathbb{Z}/n) \end{tikzcd}$$

Taking limit over n: $\varprojlim_n A[n]$, get

$$G_K \curvearrowright T(A) \text{ (Tate module)}$$

In a basis, this is $G_K \to GL_{2g}(\hat{\mathbb{Z}})$ cts for profinite top. By CRT,

$$TA = \prod_{\substack{\text{prime} \\ \ell}} \underbrace{T_\ell A}.$$

$$\rho_{A,\ell}: G_K \to GL_{2g}(\mathbb{Z}_\ell).$$

More generally, if X/K is any variety, have $H^*_{ét}\left(X_{\bar{K}}, \begin{smallmatrix} \mathbb{Q}_\ell \\ \mathbb{Z}_\ell \\ \mathbb{Z}/\ell^n \end{smallmatrix}\right) \circlearrowleft G_K$, so get cts reps $G_K \longrightarrow GL(H^i_{ét}(X_{\bar{K}}, \mathbb{Q}_\ell))$. For ab. var's,

$$(T_\ell A)^\vee \simeq H^1_{ét}(A_{\bar{K}}, \mathbb{Z}_\ell).$$

__Next time.__ Deformation theory of Galois rep'ns.

Basic aim is to understand ℓ-adic lifts of a given mod ℓ G_K-rep, i.e.

$$\begin{tikzcd} & GL_d(\mathbb{Z}_\ell) \arrow[d] \\ G_K \arrow[r] \arrow[ur, dashed, "?"] & GL_d(\mathbb{F}_\ell) \end{tikzcd}$$

Conj (now Thm). Let $E/\mathbb{Q}$ be an elliptic curve. Then E is modular.

Meaning: Most convenient & generalizable:

$\exists N$ (= conductor of elliptic curve) and an $f \in S_2(\Gamma_0(N))$ (weight 2 cuspidal form on level N),

i.e. a Hecke eigenform E s.t.

$$L(E,s) = L(f,s).$$

Here. f is a holomorphic function $\mathfrak{h}$ = UHP of $\mathbb{C}$.

s.t. ☆ $\forall \gamma \in \Gamma_0(N)$,

$$\Gamma_0(N) = \left\{ x \in SL_2\mathbb{Z} \mid x \equiv \begin{pmatrix} \cdot & \cdot \\ & \cdot \end{pmatrix} (\text{mod } N) \right\}$$

$$f(\gamma \cdot z) = (cz+d)^2 \cdot f(z).$$

$\circledast$ growth cond: at the cusps, f extends to a hol. func. (modular form) + vanishes at the cusps (def. of a cusp form).

L-fn. Since $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \in \Gamma_0(N)$, $f(z+1) = f(z)$, so we have Fourier exp:

$$f(z) = \sum_{n \geq 1} a_n e^{2\pi i n z}$$

↳ b/c cusp form.

when f is a Hecke e-form, $a_1 \neq 0$, so wma $a_1 = 1$.

$\rightsquigarrow L(f,s) = \sum_{n \geq 1} a_n \cdot n^{-s}$ (a priori a hol fn in $\mathrm{Re}(s) \gg 0$).

$\exists$ algebra $\mathbb{T}$ (Hecke alg) acting on $S_2(\Gamma_0(N))$.

$\underbrace{f \text{ is a Hecke eigenform}}$ if it is eigen for $\mathbb{T}$.

$\downarrow$

for such f, the L-fn has an Euler product

then $L(f,s) = \prod_p L_p(f,s)$. Explicitly,

$$L_p(f,s) = \begin{cases} (1 - \overset{\text{from Fourier expansion}}{a_p} p^{-s} + p^{(1-2s)})^{-1}, & p \nmid N \\ (1 - a_p p^{-s})^{-1}, & p \mid N \end{cases}$$

There is a "big" sub-alg $\mathbb{T}'$ of $\mathbb{T}$ s.t. $S_2(\Gamma_0(N))$ is diagonalizable for $\mathbb{T}'$.

~~Thm~~

<u>Th'm</u> (Hecke) If f is a Hecke eigenform, then $L(f,s)$ admits a meromorphic continuation (<u>entire</u>

if f is cuspidal $)_a$ to all of $\mathbb{C}$ & a functional eq'n w/ $s \longmapsto 2-s$ symmetry.

Def'n of $L(E,s)$ is via Euler product:

$$L(E,s) = \prod L_p(E,s), \text{ where}$$

$$L_p(E,s) = \det\left(1 - \mathrm{Fr}_p\, p^{-s} \Big|_{(T_\ell E)^{I_{\mathbb{Q}_p}}_{\otimes \mathbb{Q}_\ell}}\right)^{-1}.$$

for some (ANY!) $\ell \neq p$.
(Thm)

$$\text{Explicitly, } L_p(E,s) = \begin{cases} (1 - a_p(E)p^{-s} + p^{1-2s})^{-1}, & p \text{ good red'n} \\ 1, & p \text{ additive red'n (cuspidal)} \\ (1 \mp p^{-s})^{-1}, & p \text{ mult. red'n (nodal)} \end{cases}$$

where $a_p(E) = p+1 - \#E(\mathbb{F}_p)$.

$-$ in split nodal
$+$ in non-split nodal

[Weight monodromy conjecture]

Thm. $\forall\, E/\mathbb{Q}$, $\exists f$ $L(f,s) = L(E,s)$.

Cor: $L(E,s)$ has analytic continuation.

Note. $L(E,s)$ is a priori hol. in $\mathrm{Re}(s) > 3/2$.

Cor: Can formulate BSD conj:

$$\mathrm{ord}_{s=1} L(E,s) = \mathrm{rk}\, E(\mathbb{Q}).$$

Very roughly, Shimura-Taniyama

Want to show that $\rho_{E,\ell}$ is modular. Instead of just proving this,

$$\left\{ \text{all "nice" lifts } \begin{tikzcd} & GL_2(\mathbb{Z}_\ell) \arrow[d] \\ G_{\mathbb{Q}} \arrow[ur, "\rho"] \arrow[r, "\bar\rho_{E,\ell} \bmod \ell"'] & GL_2(\mathbb{F}_\ell) \end{tikzcd} \right\} \text{ are modular.}$$

$L(\rho, s) = L(\text{some mod. form}).$

To study these ρ's all at once, study all lifts of $\rho_{E,\ell}$ (mod ℓ) to appropriate coefficient rings.

all lifts of $\rho_{E,\ell}$ mod ℓ to all complete local noeth rings R $(R/m_R = \mathbb{F}_\ell)$

- form a geometric object $\operatorname{Spec} R^{\text{nice}}_{\rho_{E,\ell} \bmod \ell}$ "universal deformation space"

Strategy of pf: relate $R^{\text{nice}}_{\bar\rho}$ to a Hecke algebra $\mathbb{T}_{\bar\rho}$ — CLN $\mathbb{Z}_\ell$-alg whose $\overline{\mathbb{Q}}_\ell$-pts correspond to modular form f $\bar{\rho}_{f,\ell} \cong \bar{\rho}_{\ell}$.

<u>Conclude</u>: all "nice" lifts of $\bar{\rho}$ are modular...