Last time.

L/K fin. ext. $\quad c \in C_K/NC_L$.

Then $\delta(\{\text{places of } K \mid \varpi_v \in c\}) = \dfrac{1}{|C_K/NC_L|}$

Non-abelian version: Čebotarev density th'm

Th'm. Let L/K be a Galois ext.
Let $C \subset G(L/K)$ be a conjugacy class,
then $\delta(v : Fr_v = C) = \dfrac{|C|}{|C_K/NC_L|}$.

$$\begin{array}{ccl} L & w & \rightsquigarrow Fr_w \in G(L/K) \\ | & | & \\ K & v & \rightsquigarrow Fr_v \end{array}$$

(unr. in L/K)

Rmk. For L/K abelian, via Artin reciprocity, this reduces to the previous statement.

e.g. $L/K = \mathbb{Q}(\zeta_m)/\mathbb{Q}$. $\quad \underset{C_a}{C} : \zeta_m \mapsto \zeta_m^a$ for $(a,m)=1$.

Then $\delta(p : Fr_p \zeta_m = \zeta_m^p = \zeta_m^a) = \dfrac{1}{|\mathrm{Gal}|} = \dfrac{1}{|\varphi(m)|}$.

$p \equiv a \pmod m$

Pf of Čeb. Choose $\sigma \in G(L/K)$. Set $M = L^{\langle\sigma\rangle}$

$$\begin{array}{c} L \\ |\ \langle\sigma\rangle \\ M \\ | \\ K \end{array}$$

Let $X = \{\text{places } v \text{ of } K, \text{ unr in } L,\ Fr_w \in C\ \forall w|v\}$ (C: conj. class of σ)

$Y = \{\text{places } w \text{ of } L, \text{ unr}/K, \text{ s.t. } Fr_w = \sigma\}$

$Z = \{\text{places } u \text{ of } M, \text{ unr in } L/M, \text{ split over } K, \text{ s.t. } Fr_u = \sigma\}$

So, $L_X(s) = \sum_{v \in X} a_v^{-s}$

$= \sum_{w \in Y} a_{v = w|_K}^{-s} \Big/ \frac{[M:K]}{|C|}$

$= \sum_{u \in Z} q_u^{-s} \cdot \frac{|C|}{[M:K]}$

$\delta(Z)$ can be computed by abelian case.

So $\delta(X) = \frac{|C|}{[M:K]} \cdot \delta(Z)$

Given $v \in X$,

$\#\{w | v : \mathrm{Fr}_w = \sigma\}$

$\shortparallel$

$$\frac{|Z_{G(L/K)}(\sigma)|}{|\text{decomp. grp. of } w/v|}$$

$|\text{decomp. grp. of } w/v| = |\langle \sigma \rangle|$

$$= \frac{|G(L/K)|}{|C|} \cdot \frac{1}{|\langle \sigma \rangle|}$$

$$= \frac{[M:K]}{|C|}$$

$\overset{(\text{rec})}{=} \frac{|C|}{[M:K]} \delta(u \text{ of } M \text{ unr in } L \text{ s.t. } \sigma_u = \mathrm{rec}^{-1}_{L/M}(\mathrm{Fr}_u) \text{ in } C_M/NC_L)$

$$= \frac{|C|}{[M:K]} \cdot \frac{1}{|C_M/NC_L|} = \frac{|C|}{[L:K]}.$$

__Ab rec.__ given $C \subset G(L/K)$, ab., which v have $\mathrm{Fr}_v = C$?

__Nonab rec.__ given nonab., " " " " " ?

(original Langlands: connection w/ automorphic forms).

Along w/ Čeb, consequences of GCFT most sig. for non-ab. theory are __arithmetic duality__ theorems.

Global Duality.

Recall Local Duality. if $K/\mathbb{Q}_p$ is fin. and M is a fin. G_K-module, then $\forall i$, there is a perfect pairing

$$H^i(G_K, M) \times H^{2-i}(G_K, \underbrace{M^*}_{\mathrm{Hom}(M,\mu_\infty)}) \xrightarrow{\mathrm{inv}_K(\cdot \cup \cdot)} \mathbb{Q}/\mathbb{Z}$$

and $H^i = 0$ for $i \geq 3$, and $|H^i| < \infty$.

Recall. LCFT proven by choosing $\begin{matrix} M = \mathbb{Z}/n \\ M^* = \mu_n \end{matrix}$.

Recall. K # field, $H^i(G_K, M)$ can be infinite when M is finite (e.g. $\mathbb{Q}^\times/(\mathbb{Q}^\times)^n = H^1(G_\mathbb{Q}, \mu_n)$)

First pass at global duality corrects this by working w/ $G(K(S)/S)$ where

$$\begin{cases} S \text{ is a fin. set of primes } (\supset v|\infty) \\ K(S) = \text{max ext. of } K \text{ inside } \overline{K} \text{ that is unram. outside } S \end{cases}$$

Why this is a meaningful finiteness thm:

compare $\#\{\text{ext's of } \mathbb{Q} \text{ w/ degree} \le d\}$ - infinite

$\#\{$ ext's of K w/ deg. $\le d$ & unr. outside a fixed fin. set S $\}$ - finite, Hermite's theorem

Formulation of GD. (not most general)

Fix S a fin. set of primes ($\supset v|\infty$). Let M be a fin. $G_{K,S}$-mod (i.e. a G_K-mod where the action is unr. outside S) s.t. $\# M$ is invertible in $\mathcal{O}_K[1/S]$.

Set

$$P_S^i(M) = \begin{cases} \bigoplus_{v\in S} H^i(G_{K_v}, M) & \text{for } i \ge 1 \\ \bigoplus_{v \in S\setminus\{v|\infty\}} H^0(G_{K_v}, M) \oplus \bigoplus_{v|\infty} H_0(G_{K_v}, M)/N_{\bar{K}/K_v}(M) & \boxed{i=0} \end{cases}$$

Then local duality gives perfect pairing

$$P_S^i(M) \times P_S^{2-i}(M^*) \xrightarrow{\Sigma\langle\,,\,\rangle} \mathbb{Q}/\mathbb{Z}$$

Then there is an exact sequence, natural in M, (Poitou-Tate)

$$0 \to H^0(G_S, M) \xrightarrow{\beta^0} P_S^0(M) \xrightarrow{\gamma^0} H^2(G_S, M^*)^\vee$$

$$\to H^1(G_S, M) \xrightarrow{\beta^1} P_S^1(M) \xrightarrow{\gamma^1} H^1(G_S, M^*)^\vee$$

$$\to H^2(G_S, M) \xrightarrow{\beta^2} P_S^2(M) \xrightarrow{\gamma^2} H^0(G_S, M^*)^\vee \to 0$$

and all $|H^i| < \infty$ and for $i \geq 3$

$$H^i(G_S, M) \xrightarrow{\sim} \bigoplus_{v \text{ real}} H^i(G_{K_v}, M)$$

(or $P_S^i(M)$)

(in particular, if K is imaginary, then $H^i(G_S, M) = 0$ for $i \geq 3$).

<u>What are the maps?</u>

- β^i are just restriction ($G_{K_v} \to G_S$, $v \in S$ are injections usually. if S is "large enough")

proven in early 2000's via Gal rep assoc'd to aut. forms

• γ^i are local duality

$$P_S^i(M) \xrightarrow{\gamma^i} H^{2-i}(G_S, M^*)^\vee$$

$$(\phi_v)_{v\in S} \longmapsto \Big[\psi \longrightarrow \sum_{v\in S} \langle \phi_v \,|\, \psi|_{G_{k_v}} \rangle \Big]$$

local duality pairing

A couple concrete ~~eg~~ eg's of this exactness:

<u>Cor.</u> The images of $H^1(G_S, M) \longrightarrow P_S^1(M)$ and $H^1(G_S, M^*) \longrightarrow P_S^1(M^*)$ are <u>exact annihilators</u> under $P_S^1(M) \times P_S^1(M^*) \to \mathbb{Q}/\mathbb{Z}$.

(This reformulates exactness of $P_S^1(M)$ given def'ns of β^i, γ^i)

<u>Cor.</u> Set $Ш_S^i(M) = \ker(H^i(G_S, M) \to P_S^i(M))$. Then, via boundary map, there is a perfect pairing

$$Ш_S^1(M) \times Ш_S^2(M^*) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

<u>e.g.</u> $M = \mu_n$. $Ш_S^1(\mu_n) = \ker\left(\mathcal{O}_K[1/S]^\times / \mathcal{O}_K[1/S]^{\times n} \to \bigoplus_{v\in S} K_v^* / K_v^{*n} \right)$

(sub-e.g. $S = \{v \mid \infty\}$, this gives the Grunwald-Wang obstruction grp).

<u>Pf. of Ш cor</u>: Exactness at $H^2(G_S, M^\times)^\vee \Rightarrow$

$$\operatorname{coker}(\gamma^0) \xrightarrow{\sim} \operatorname{Ker}(\beta^1) = Ш^1_S(M)$$

$$\frac{H^2(G_S, M^\times)^\vee}{\gamma^0(P^0_S(M))} \simeq \begin{pmatrix} f \in H^2(G_S, M^\times) : \\ \langle f, P^0_S(M) \rangle = 0 \end{pmatrix}^\vee$$

$$\underset{\substack{\text{local}\\ \text{dual}}}{=} \begin{pmatrix} f \in H^2(G_S, M^*) : \\ f|_S = 0 \end{pmatrix}^\vee = Ш^2_S(M^\times)^\vee.$$

<u>Cor.</u> $\forall i$, $H^i(G_S, M)$ is finite (M as in Th'm).

<u>Pf.</u> For $i \geq 3$, this is part of the construction of the sequence:

$i = 0$ obvious

Exactness of the sequence reduces fin. of H^2 to finiteness of H^1. Finiteness of H^1 follows from Hermite.

- if M trivial $= \operatorname{Hom}_{cts}(G_S, M)$, any such factors thru an ext. of deg $\leq$ exponent of M & unr. outside $S \rightsquigarrow$ fin'ly many possibilities.

To reduce to the case of trivial M, use inf-res w/ $K - L - K(S)$ s.t. G_L acts triv on M.

$$\begin{tikzcd}
 & H^1(G(K(S)/K), M) \arrow[r, "\text{res}"] & \overbrace{H^1(G(K(S)/L), \underbrace{M}_{\text{triv}})}^{\text{fin by Hermite or abel. case}} \\
0 \arrow[r] & H^1(G(L/K), M) \arrow[u, "\text{inf}"] &
\end{tikzcd}$$

$H^1(G(L/K), M)$] finite b/c $|G(L/K)| < \infty$.