Last time (Kummer theory)

K field $\rightsquigarrow H^1(G_K, \overline{K}^\times) = 0$ (separable closure)

$\rightsquigarrow$ (Kummer theory) $\forall n$ prime to char(K),

$$K^\times/(K^\times)^n \xrightarrow{\sim} H^1(G_K, \mu_n)$$

LCFT & GCFT are $\approx H^2(G_K, \overline{K}^\times)$ for K local or global

is the cohomological Brauer group of K, which has an interpretation as $\left\{\begin{matrix}\text{central simple} \\ \text{algebras over } K\end{matrix}\right\} \Big/ A_1 \sim A_2$ if $A_1 \simeq M_{n_1}(D)$ & $A_2 \simeq M_{n_2}(D)$ for div. algebra D/K.

e.g. if $K = \mathbb{R}$, $H^2(G_{\mathbb{R}}, \mathbb{C}^\times) = \dfrac{(\mathbb{C}^\times)^{c=1}}{(1+c)\,\mathbb{C}^\times} = \mathbb{R}^\times/\mathbb{R}^\times_{>0} \simeq \pm 1.$

and $Br(\mathbb{R}) = \{\mathbb{R}, \mathbb{H}\}$.

Notation/Lemma. If L/K fin. Galois, then

$$0 \to \underset{\ker \mathrm{res}}{\underset{\parallel}{Br(L/K)}} \longrightarrow Br(K) \xrightarrow{\mathrm{res}} Br(L)$$

Lemma. $Br(L/K) = H^2(G(L/K), L^\times)$.

Pf. By Hilbert 90.

F - a field. $(n, \text{char } F) = 1$.

Generalized Kummer theory.

$$\underset{\text{ab grp}}{F^\times} \longrightarrow \underset{\substack{\text{ext. alg.}\\ \text{of } F^\times\\ (\text{in } \underline{Ab})}}{\Lambda^\bullet F^\times} \xrightarrow{\text{Kummer}} \overbrace{\bigoplus_{m\geq 0} H^m(G_F, \mu_n^{\otimes m})}^{\text{mysterious for } m>1}$$

$$\begin{array}{ccc} \Lambda^m F^\times & \longrightarrow & H^m(G_F, \mu_n^{\otimes m}) \\ \cup & & \\ a_1 \wedge \cdots \wedge a_m & \longmapsto & \delta(a_1) \cup \cdots \cup \delta(a_n) \end{array}$$

Def. Let $K_*^M(F)$ (Milnor K-theory of F) be the quotient by two-sided ideal gen'd by all elements $a \wedge (1-a)$ for $a \in F \setminus \{0,1\}$.

Lemma. Kummer factors thru Milnor K-theory.

$$K_m^M(F) / n K_m^M(F) \longrightarrow H^m(G_F, \mu_n^{\otimes m}).$$

To prove the lemma, must check $\delta(a) \cup \delta(1-a)$ in $H^2(G_F, \mu_n^{\otimes 2})$ for all $a \in F \setminus \{0,1\}$. To see this, let $D =$ s/g of $H^2(G_F, \mu_n^{\otimes 2})$ gen'd by all elements of the form $\mathrm{cor}_{E/F}(\underbrace{(a, 1-a)_E}_{\delta(a)\cup\delta(1-a)})$ for all fin. ext's E/F & $a \in E \setminus \{0,1\}$.

$$H^2(G_F, -) \xleftarrow{\mathrm{cor}} H^2(G_E, \mu_n^{\otimes 2})$$

Claim. D is n-divisible (but mult. by n is 0 on $H^2(G_F, \mu_n^{\otimes 2})$, so $D = 0$).

<u>Pf.</u> Consider E/F, $a \in E \setminus \{0,1\}$. Factor

$$x^n - a = \prod \underbrace{f_i(x)}_{\substack{\text{monic irred.} \\ \text{in } E[x]}} .$$

Let α_i be a root of f_i, and $E_i = E(\alpha_i)$. Then

$$1 - a = \prod f_i(1) = \prod_i N_{E_i/E}(1-\alpha_i).$$

And $$(a, 1-a)_E = \Big(a, \prod_i N_{E_i/E}(1-\alpha_i)\Big)_E$$

$$= \sum_i (a, N_{E_i/E}(1-\alpha_i))_E . \qquad (A)$$

Since

$$\begin{array}{ccc} E^\times & \xrightarrow{\delta} & H^1(G_E, \mu_n) \\ {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle \mathrm{cor}} \\ F^\times & \xrightarrow{\delta} & H^1(G_F, \mu_n) \end{array}$$

Since $\mathrm{cor}(\mathrm{res}(x) \cup y) = x \cup \mathrm{cor}(y)$, (A) equals

$$\sum \mathrm{cor}_{E_i/E}\big((a, 1-\alpha_i)_{E_i}\big)$$

$$= \sum \mathrm{cor}_{E_i/E}\big((\alpha_i^n, 1-\alpha_i)_{E_i}\big)$$

$$= n \sum \mathrm{cor}(\alpha_i, 1-\alpha_i)_{E_i} .$$

Thm (Bloch-Kato, proven by Voevodsky). Kummer is an iso $\forall$ fields F, $\forall n$ prime to char F, $\forall m \geq 0$.

Local Classfield Theory.

Main technical point: compute $Br(K)$, $K/\mathbb{Q}_p$ finite.

Th'm. There is a canonical isomorphism $Br(K) \xrightarrow[\sim]{inv_K} \mathbb{Q}/\mathbb{Z}$.
satisfying the compatibilities for L/K,

$$\begin{tikzcd} Br(K) \arrow[r, "res"] \arrow[d, "inv_K"', "\sim"] & Br(L) \arrow[d, "inv_L"] \\ \mathbb{Q}/\mathbb{Z} \arrow[r, "\text{mult by}", "{[L:K]}"'] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

and

$$\begin{tikzcd} Br(K) & Br(L) \arrow[l, "cor"'] \arrow[d, "inv_L"] \\ \mathbb{Q}/\mathbb{Z} \arrow[from=u, "inv_K"'] & \mathbb{Q}/\mathbb{Z} \arrow[l, "id"] \end{tikzcd}$$

The first compatibility gives, plus Hilbert 90,

$$\underset{\ker(res)}{\overset{\shortparallel}{H^2(G(L/K), L^\times)}} \xrightarrow[\sim]{inv_K} \tfrac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}.$$

Pf overview: construction of inv_K

$$\begin{tikzcd}
H^2(G_K, \bar{K}^\times) \arrow[rrr, dashed, "\mathrm{inv}_K"] & & & \mathbb{Q}/\mathbb{Z} & \ni \underbrace{\varphi(\mathrm{Frob}_K)}_{\text{Frob or } -\mathrm{Frob}} \\
H^2(G(K^{ur}/K), K^{ur\times}) \arrow[u, "\mathrm{inf}", "\sim"'] \arrow[r, "\mathrm{ord}", "\sim"'] & H^2(G(K^{ur}/K), \mathbb{Z}) & & H^1(G(K^{ur}/K), \mathbb{Q}/\mathbb{Z}) \arrow[ll, "\delta"', "\sim"] \arrow[u, "\mathrm{eval}"', "\sim"] & \ni \bar{\varphi} \arrow[u]
\end{tikzcd}$$

normalized valuation

comes from

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

Pf. from easiest to hardest

1) eval is an iso ✓

2) δ is an iso: the LES gives you a sandwich

$$\cancel{H^1(\_, \mathbb{Q})} \to H^1(G(K^{ur}/K), \mathbb{Q}/\mathbb{Z}) \to H^2(G(K^{ur}/K), \mathbb{Z}) \to \cancel{H^2(\_, \mathbb{Q})}$$

shows up in HW

$\forall$ finite, hence $\forall$ pro-finite groups, $H^i(G, \mathbb{Q}) = 0$ $\forall i > 0$

b/c $|G| : \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}$ (and cor $\circ$ res $= |G|$).

3) ord $\quad 0 \to \mathcal{O}_{K^{ur}}^\times \to K^{ur\times} \xrightarrow{\mathrm{ord}} \mathbb{Z} \to 0$

we need to show for $r = 2, 3$, $H^r(G(K^{ur}/K), \mathcal{O}_{K^{ur}}^\times) = 0$.

$$\text{“} = \text{”} \varinjlim_{\text{fin } K^{ur} \supset L \supset K} (\quad).$$

Suffices to prove that $H^r(G(L/K), \mathcal{O}_L^\times) = 0$ for all finite. unram L/K.

Any such L/K is cyclic, so STP for $r=1,2$ (by periodicity of $H^i(G(L/K), -)$.

For $r=1$, $L^\times = \mathcal{O}_L^\times \times \langle p \rangle \Rightarrow H^r(G(L/K), \mathcal{O}_L^\times)$ is a direct summand of $H^r(G(L/K), L^\times)$. Then this vanishes by Hilbert 90.

For $r=2$, calc. of coh. of cyci groups, we need to show $N_{L/K}: \mathcal{O}_L^\times \to \mathcal{O}_K^\times$ is surjective (FOR L/K unr). Filter $\mathcal{O}_L^\times \supset 1+p\mathcal{O}_L \supset 1+p^2\mathcal{O}_L \supset \cdots$.

p is a uniformizer for K

w/ s/q's (i) $\mathcal{O}_L^\times / 1+p\mathcal{O}_L \xrightarrow{\sim} k_L^\times$ } are $G(L/K)$ equivariant.

(ii) $1+p^i\mathcal{O}_L / 1+p^{i+1}\mathcal{O}_L \xrightarrow{\sim} k_L$

And, (i) intertwines $N_{L/K}$ w/ $N_{k_L/k}$ and (ii) intertwines $N_{L/K}$ w/ $\mathrm{tr}_{k_L/k}$. $\underset{=}{}$ $G(k_L/k)$

<u>Sub-lemma.</u> $\forall r \geq 1$, $H^r(G(L/K), k_L^\times)$, $H^r(G(L/K), k_L)$ $=0$.

Granted sublemma, we show $N: \mathcal{O}_L^\times \twoheadrightarrow \mathcal{O}_K^\times$.

Let $u \in \mathcal{O}_K^\times$, $\exists v_0 \in \mathcal{O}_L^\times$ s.t. $u/N(v_0) \in 1+p\mathcal{O}_K$. Then $\exists v_1 \in 1+p\mathcal{O}_L$ s.t. $\frac{u}{N(v_0)N(v_1)} \in 1+p^2\mathcal{O}_K$ by Lemma + iso (ii)

The sequence $v_0, v_0v_1, \ldots$ converges in $\mathcal{O}_L^\times$ to some v & $N(v)=u$.

Notation. G : cyclic M : a G-module $\in \mathcal{M}_G$. Define the Herbrand quotient $h(G,M) = \dfrac{\# H^1(G,M)}{\# H^2(G,M)}$

whenever both the H^i's are both finite.

Lemma. 1) If M is finite, then $h(G,M) = \dfrac{\ker(N)/\mathrm{im}(\sigma-1)}{\ker(\sigma-1)/\mathrm{im}(N)}$

$$= \frac{\ker N \cdot \mathrm{im}\, N}{\ker(\sigma-1)\, \mathrm{im}(\sigma-1)} = 1.$$

2) If $0 \to M \to N \to P \to 0$ and if $h(G,\cdot)$ is defined for two/three, then defined for the third & satisfies

$$h(G,N) = h(G,M)\, h(G,P).$$

Pf. LES + periodic coh. $\rightsquigarrow$ ∎

Pf of sublemma. Know it for $r=1$, (NBT + H-90). Since k_L and $k_L^\times$ are fin., we get vanishing for $r=2$ as well. Hence $\forall r \geq 1$. ∎

To complete construction of inv_K, we have to show $\inf : H^2(G(K^{ur}/K), K^{ur\times}) \xrightarrow{\sim} H^2(G_K, \bar{K}^\times)$.

First check compatibilities w/ res/cor:

Prop. $\forall L/K$ fin. deg. $n = [L:K]$, have

$$\begin{tikzcd}
H^2(G(K^{ur}/K), K^{ur\times}) \arrow[r, "res"] \arrow[d, "eval \circ \delta^{-1} \circ ord = inv_K"'] & H^2(G(L^{ur}/L), L^{ur\times}) \arrow[d, "inv_L", "\sim"'] \\
\mathbb{Q}/\mathbb{Z} \arrow[r, "{[L:K]}"] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

<u>Pf</u>. Break into 3 comm. squares,

$$\begin{tikzcd}
H^2(G(K^{ur}/K), K^{ur\times}) \arrow[r, "ord_K"] \arrow[d, "res"] & H^2(G(K^{ur}/K), \mathbb{Z}) \arrow[d, "e_{L/K} \cdot res"] & H(-, \mathbb{Q}/\mathbb{Z}) \arrow[l, "\delta", "\sim"'] \arrow[r] \arrow[d, "e_{L/K} \cdot res"] & \mathbb{Q}/\mathbb{Z} \arrow[d, "e_{L/K} \cdot f_{L/K}"] \\
\text{ditto} \arrow[r, "ord_L"] & \text{ditto} \arrow[r] & \text{ditto} \arrow[r, "eval"'] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

Then get prop. by $e_{L/K} \cdot f_{L/K} = [L:K]$.

<u>Prop (ii)</u>. Likewise for L.

<u>Pf</u>. $cor \circ res = [L:K]$, & res is surjective,

so cor must correspond to identity.