Group cohomology

Γ - (abstract) group

$$\Gamma\text{-modules} \longrightarrow \underline{Ab}$$
$$M \longrightarrow M^{\Gamma} = \{ m \in M \mid \gamma m = m \ \forall \gamma \in \Gamma \}$$

This is left-exact, giving us its right derived functor (& $H^0(\Gamma, M) = M^{\Gamma}$).

Alternative approaches: ① $H^*(B\Gamma,$ local system $B\Gamma$ associated to M (i.e. a rep of $\pi_1(B\Gamma)) = \Gamma$

② Explicit cochain definitions

$$(H^i = Z^i / B^i, \quad Z^i(\Gamma, M) = \{ f : \Gamma^{i+1} \to M \mid df = 0 \}$$
$$B^i = \{ df \mid f : \Gamma^i \to M \}.$$

Let G be a profinite group, i.e.

① $G = \varprojlim$ (fin. groups)

② compact, hausdorff w/ basis of nbhds at 1 given by open normal subgroups

Our category of coefficients.

Let $\mathcal{M}_G$ = category of discrete G-mods i.e. G-mods M s.t. $\forall m \in M$, $\mathrm{stab}_G(m)$ is open in G.

(equivalently, $M = \bigcup_{\substack{\text{open} \\ \text{sg } U \\ \text{of } G}} M^U$)

Ex. K field, $G = \mathrm{Gal}(\overline{K}/K)$ (separable closure), then $\overline{K}$ is a discrete G-mod. If $\alpha \in \overline{K}$, $\mathrm{Stab}(\alpha) = \mathrm{Gal}(\overline{K}/K(\alpha))$, which is open in G since $K(\alpha)/K$ is finite.

<u>Basic properties of $\mathcal{M}_G$</u>

- $\mathcal{M}_G$ is abelian — G-modules are $\mathbb{Z}[G]$-modules, in the trad. sense.
- $\mathcal{M}_G$ has enough $\boxed{\text{injectives}}$ ($\mathrm{Hom}(\cdot, I)$ is exact), i.e. $\forall M \in \mathcal{M}_G$, $\exists$ infinite $I \in \mathcal{M}_G$ and $M \hookrightarrow I$,

Rmk: (Discretization) The forgetful functor $\mathcal{M}_G \to \mathbb{Z}[G]$-modules has a right adjoint
(forgetful by forgetting the topological structure)

$$\mathcal{M}_G \longrightarrow \mathbb{Z}[G]\text{-mods}$$

$$\mathcal{M}_G \longleftarrow \mathbb{Z}[G]\text{-mod}$$

$$\bigcup_{\substack{U \text{ open} \\ \subset G}} N^U = N^{disc} \longleftarrow N .$$

enforces the discrete condition

$$\mathrm{Hom}_{\mathbb{Z}[G]}(M, N) = \mathrm{Hom}_{\mathcal{M}_G}(M, N^{disc}),$$ i.e. (adjoint condition)

$M \to N$ factors

$$\begin{tikzcd} M \arrow[r] \arrow[dr, dashed] & N \arrow[d, hook] \\ & N^{disc} \end{tikzcd}$$

<u>Pf</u>.: (enough injectives)

<u>Overview</u>. <u>Ab</u> has enough injectives

$\rightsquigarrow$ $\mathbb{Z}[G]$-mods have enough injefives

$\rightsquigarrow$. If $M \in \mathcal{M}_G$ and $M \hookrightarrow I$ w/ I injective in $\mathbb{Z}[G]$-modules, then $M \hookrightarrow I^{disc}$ & I^{disc} is injective in $\mathcal{M}_G$.

(i.e. $\mathrm{Hom}_{\mathcal{M}_G}(\cdot, I^{disc}) = \mathrm{Hom}_{\mathbb{Z}[G]}(\cdot, I)$ is exact b/c $\mathcal{M}_G \to \mathbb{Z}[G]$ + forgetful + $\mathrm{Hom}(\cdot, I)$ are exact)

<u>Claim</u>. Ab has enough injectives.

Let $M \in$ <u>Ab</u>. For $m \in M$, $\exists$ $\langle m \rangle \hookrightarrow I_m$ for some injective I_m in <u>Ab</u> (if m is torsion, take $I_m = \mathbb{Q}/\mathbb{Z}$, else take $\mathbb{Z}$).

By exactness of $\mathrm{Hom}(\cdot, I_m)$, this map extends.

$$\begin{array}{ccc} \langle m \rangle & \hookrightarrow & I_m \\ \cap & \nearrow & \\ M & & \end{array}$$

Then $M \xrightarrow{\prod_{m \in M} f_m} \prod I_m$ embeds M into an injective. (Exercise: product of injectives is injective).

(ii) For $\mathcal{M}_G$, we use the following

<u>Construction</u>. Let $H \leq G$ be a closed subgroup. Define for any discrete H-modules $M \in \mathcal{M}_H$

$\mathrm{Ind}_H^G(M) = \{ f : G \to M \mid f(hg) = h\cdot f(g)\}$
continuous $\quad \forall h \in H$, and s.t.
f is locally constant.

and let G act on $\mathrm{Ind}_H^G(M)$ by $(g\cdot f)(a) = f(ag)$.

<u>Sub-lemma</u>. ① $\mathrm{Ind}_H^G : \mathcal{M}_H \to \mathcal{M}_G$, i.e. $\mathrm{Ind}(M)$ is discrete when $M \in \mathcal{M}_H$.

② Ind_H^G is <u>exact</u>.

③ left adj $\left[\ \mathcal{M}_G \xrightarrow{\text{forget}} \mathcal{M}_H \right.$

$\left. \mathcal{M}_G \xleftarrow[\mathrm{Ind}_H^G]{} \mathcal{M}_H \ \right]$ right adj.

④ For a sequence of closed subgroups $G > H > K$, there is a natural iso. $\mathrm{Ind}_K^G \simeq \mathrm{Ind}_H^G \circ \mathrm{Ind}_K^H$.

⑤ If I is injective in $\mathcal{M}_H$, then $\mathrm{Ind}_H^G(I)$ is inj in $\mathcal{M}_G$.

<u>Pf of 1</u>. If $M \in \mathcal{M}_H$, need to show $\underline{\mathrm{Ind}(M) \in \mathcal{M}_G}$.
i.e. $\mathrm{Ind}(M) = \bigcup_{V \text{ open}} \mathrm{Ind}(M)^V$
Let $f : G \to M \in \mathrm{Ind}(M)$.
(locally const) $\forall g \in G$, $\exists\, U_g \ni g$ s.t. $f(U_g) = f(g)$
$\exists V_g,\ f(V_g) = f(1)$
$\Rightarrow \exists$ open s.g. V_g s.t. $gV_g \subset U_g$.
$\Rightarrow f(gV_g) = gf(1) = f(g)$
Since G is compact, the cover $\bigcup_{g\in G} gV_g = G$ refined to a fin. subcover $\bigcup_{i=1}^{r} g_i V_{g_i}$. Let

$V = \bigcap_{i=1}^{r} V_{g_i}$ (so V open s/g). Then $\forall v \in V$,

$v \cdot f = f$.

Pf of 2. The tricky part is right exactness, which follows from

Ind(M) $\simeq M \otimes \mathbb{Z}[G]$ for G finite, and inverse limits are exact

Fact. G profinite group $G > H > K$, then $G/K \overset{\pi}{\twoheadrightarrow} G/H$ admits a continuous section $s : G/H \longrightarrow G/K$ s.t. $s \circ \pi = id$.

map, not necessarily homo

Pf. First treat case $[H:K] < \infty$, so K is open in H. Then $\exists$ open normal s/g $U \triangleleft G$ s.t. $H \cap U \subset K$. Let $G = \coprod_{i=1}^{n} U g_i H$ and define $U g_i H/K \longrightarrow U g_i H/H$.

$$u g_i K \xleftarrow{s} u g_i h H$$

(check: s is well-defined & continuous)

The general case. $(T,s) \geq (T', s')$ if $T \subset T'$
$$\begin{tikzcd} G/T \arrow[r] & G/T' \arrow[r] & G/H \arrow[l, bend right, "s'"'] \arrow[ll, bend right=50, "s"'] \end{tikzcd}$$

Consider the poset of pairs (T, s) where $H \supset T \supset K$ and s is a section of $G/T \longrightarrow G/H$.

We'll apply Zorn's lemma:

- if $\{(T_i, s_i)\}$ is a totally ordered subset, define its max by taking $T_\infty = \bigcap T_i$ and appropriate s_∞ gluing s's.

So $\exists$ max element (T, s) of our poset.

Claim. $T = K$.

If not $T \supsetneq U \supset K$, and
$$\begin{tikzcd} G/K \arrow[r] & G/U \arrow[r] & G/T \arrow[r] \arrow[l, bend right, dashed] & G/H \arrow[l, bend right, dashed] \arrow[ll, bend left, dashed, "\text{compose}"] \end{tikzcd}$$

contradicting maximality.

Pf of 5. Use Frob reciprocity, 3.

Sub-lemma $\Rightarrow$ $\mathcal{M}_G$ has enough injectives.

If $M \in \mathcal{M}_G$, $\exists\, M \overset{i}{\hookrightarrow} I$ w/ I injective ab grp. in Ab

and then $M \hookrightarrow \mathrm{Ind}_{\{1\}}^{G} I$ } by 5 an injective
$m \longrightarrow [g \longmapsto i(gm)]$ object of $\mathcal{M}_G$.

Dfn. Since $\mathcal{M}_G$ is an abstract category w/ enough injectives, and $M \longmapsto M^G$ is a left-exact functor $\mathcal{M}_G \to$ Ab, we can define $H^i(G,M) = i$-th right derived functor $M \to M^G$.

[i.e. choose a quasi-isomorphism $M \to I^\bullet$ where (M regarded as obj in deg 0) $I^\bullet$ is an injective res, and set

$$H^i(G,M) = H^i((I^\bullet)^G) \;]$$
$\hookrightarrow (I^0 \to I^1 \to \cdots) \to (0 \to (I^0)^G \to \cdots)$

Def. $M^\bullet \to N^\bullet$ is a quasi-iso if it induces iso's on cohomology.

The derived functor formalism gives:

- $\forall\, i \geq 0$, $H^i(G, \cdot)$ is a functor $\mathcal{M}_G \to$ Ab.
  For $i = 0$, $H^i(G,M) = M^G$.

- For every SES in $\mathcal{M}_G$ $0 \to A \to B \to C \to 0$, get a LES
  $$\cdots \to H^n(G,A) \to H^n(G,B) \to H^n(G,C) \xrightarrow{\delta} H^{n+1}(G,A) \to \cdots$$

& these LES are natural in the SES, i.e.

$$\begin{array}{ccccccccc} 0 \to A \to B \to C \to 0 \\ \downarrow \quad \downarrow \quad \downarrow \\ 0 \to A' \to B' \to C' \to 0 \end{array} \rightsquigarrow \begin{tikzcd} H^n(G,C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & H^{n+1}(G,A) \arrow[d] \\ H^n(G,C') \arrow[r] & H^{n+1}(G,A') \end{tikzcd}$$

Summarize these by saying $(H^n(G,\cdot), \delta_n)_n$ is a δ-functor.

<u>Def</u>. A δ-functor is <u>universal</u> if for any δ-functor (G_n, ε_n) and natural transf. $H^0 \to G^0$, $\exists!$ natural trans of $H^n \to G^n$ $\forall n \geq 0$ commuting w/ boundary maps δ, δ'.

<u>Lemma</u>. $(H^i(G, \cdot), \delta_i)$ is a universal δ-functor.

<u>Pf</u>. Grothendieck observed that a δ-functor (G^i, ε_i) w/ all G_i <u>effaceable</u> for $i>0$ is automatically universal.

G^i effaceable means $\forall M$, $\exists$ $M \overset{f}{\hookrightarrow} N$ s.t. $G^i(M) \xrightarrow{G^i(f)} G^i(N)$ is zero map.

For us, any $M \in \mathcal{M}_G$ embeds $M \hookrightarrow I$ into I injective, and for all $i>0$, $H^i(G,I) = 0$.

<u>Rmk</u>. To "compute" $H^i(G,M)$, can instead use an acyclic resolution

$$M \xrightarrow{q\text{-iso}} J^\bullet \text{ where } H^n(G, J^r) = 0 \ \forall r \text{ and } \forall n > 0.$$

Now we use universality to make key constructions.

<u>Lemma.</u> (Shapiro) $H < G$ closed s/g $M \in \mathcal{M}_H$. Then there is a natural iso.

$$H^i(H, \cdot) \xrightarrow{\sim} H^i(G, \mathrm{Ind}_H^G(\cdot))$$

<u>Pf.</u> Check that both sides are <u>universal</u> δ-functors, w/ a natural iso. in degree zero.

<u>On H^0:</u>

$$M^H \longrightarrow (\mathrm{Ind}_H^G(m))^G$$

$$m \longrightarrow (g \longmapsto m)$$

$\hookrightarrow$ H equivariant since $m \in M^H$.

We already know $(H^i(H, \cdot), \delta_i)$ is universal. We need

$$\left(\begin{array}{l} \mathcal{M}_H \longrightarrow Ab \\ m \longrightarrow H^i(G, \mathrm{Ind}_H^G M) \end{array} \right)_i \text{ is a univ } \delta\text{-functor}$$

It is a δ-functor b/c Ind is <u>exact</u> & $H^i(G, \cdot)$ is a δ-functor. It is effaceable, hence universal, b/c $\forall M$, let $M \hookrightarrow I$ an inj. embedding in $\mathcal{M}_H$.

$\rightsquigarrow \mathrm{Ind}_H^G M \hookrightarrow \mathrm{Ind}_H^G(I) \rightsquigarrow$ inj. in $\mathcal{M}_G$.

Then, $H^n(G, \mathrm{Ind}_H^G M)$

$$\downarrow$$

$H^n(G, \mathrm{Ind}\, I) = 0 \quad \forall n > 0$ gives effaceability.