Recall. $K/\mathbb{Q}_p$ fin.

$$\begin{array}{ll}
\overline{K} & \\
\Big| & \Big\}\ \text{wild inertia, pro-}p. \\
K^t & = \text{max tame ext. } (e\text{'s prime to } p) \\
\Big| & \\
K^{ur} & = K(\zeta_n \mid (n,p)=1) \\
\Big| & \qquad \mathrm{Gal} \xrightarrow{\sim} \mathrm{Gal}(\overline{k}/k), \quad \overline{k} = k_{K^{ur}} \\
K & \qquad\qquad\qquad\qquad \text{"} \langle \mathrm{Fr}_p \rangle
\end{array}$$

Prop. $K^t = K^{ur}(\varpi_K^{1/n}\ \forall\ (n,p)=1)$. $\mathrm{Gal}(K^t/K^{ur}) \xrightarrow{\sim} \varprojlim_{(n,p)=1} \mu_n$

μ_n " $\{n\text{-th roots of unity}\}$

$$\sigma \xrightarrow[\text{(Kummer theory)}]{} \left(\frac{\sigma \varpi^{1/n}}{\varpi^{1/n}}\right)_n$$

Moreover, let $\varphi \in \mathrm{Gal}(K^t/K)$ lift $\mathrm{Fr}_K \in G(K^{ur}/K)$. Then $\forall\ \tau \in G(K^t/K^{ur})$, $\varphi\tau\varphi^{-1} = \tau^q$ for $q = |k_K|$, i.e.

$$G(K^t/K) \simeq \prod_{\ell \neq p} \mathbb{Z}_\ell \rtimes \hat{\mathbb{Z}}$$

where $1 \in \hat{\mathbb{Z}}$ acts on $\prod \mathbb{Z}_\ell$ by mult. by q.

Pf. The identification $G(K^t/K^{ur}) \xrightarrow{\sim} \varprojlim_{(n,p)=1} \mu_n$ reduces

$$G(K(\zeta_n, \varpi^{1/n})/K(\zeta_n)) \xrightarrow{\sim} \mu_n.$$

$$\sigma \longrightarrow \frac{\sigma \varpi^{1/n}}{\varpi^{1/n}}.$$

This is a hom: $\sigma\tau \longrightarrow \dfrac{\sigma\tau(\varpi^{1/n})}{\varpi^{1/n}} = \dfrac{\sigma(\tau(\varpi^{1/n}))}{\tau(\varpi^{1/n})} \dfrac{\tau(\varpi^{1/n})}{\varpi^{1/n}}$.

But τ fixes $\varpi^{1/n}$. The map is $\therefore$ indep. of choice of n-th root.

To compute $\varphi\tau\varphi^{-1}$ for any $\tau \in \mathrm{Gal}(K^t/K^{ur})$, compute

$\varphi\tau\varphi^{-1}(\varpi^{1/n})$ $\forall n$. $\varphi^{-1}\varpi^{1/n} = \zeta \cdot \varpi^{1/n}$ for some ζ,
then $\varphi\tau(\zeta\varpi^{1/n}) = \varphi\left(\zeta\varpi^{1/n} \cdot \dfrac{\tau(\varpi^{1/n})}{\varpi^{1/n}}\right)$

$$= \varpi^{1/n} \cdot \left(\frac{\tau(\varpi^{1/n})}{\varpi^{1/n}}\right)^{\varphi}.$$

<u>Claim</u> Remains to show $K^t = K^{ur}(\ldots)$. If L/K, M/K tamely ~~unr~~ramified and $e_{L/K} \mid e_{M/K}$, then LM/M is unramified.

<u>Pf.</u>

$$I_{LM/K} \hookrightarrow I_{L/K} \times I_{M/K}$$

all cyclic since tame. $(G_0/G_1 \hookrightarrow k^\times)$

$I_{LM/K}$ is cyclic of order $e_{M/K}$, hence LM/M unramified. ∎

Apply this. We know $K^{ur}(\varpi^{1/n}, (n,p)=1) \subset K^t$. Let L/K be any tame ext. Set $M = K(\mu_{e_{L/K}}, \varpi^{1/e_{L/K}})$, so $e_{L/K} = e_{M/K}$. Conclude that LM/M is unram, hence $L \subset M^{ur} \subset K^t$. ∎

<u>Rmk.</u>

$$\begin{array}{c} L \\ | \\ M \\ | \\ K \end{array}$$

$G(L/K) \longrightarrow G(M/K)$, the restriction, does not induce a restriction on the lower ram. groups. Only after ~~renormalizing~~ reindexing.

To global setting.

Back to K = general field w/ non-triv abs. values $|\cdot|$. $K_{|\cdot|}$ = completion.

Prop. Let $L = K[\alpha]$ be a fin. sep. ext. Let $f(x) \in K[X]$ be the min poly. Then $\exists$ bijection

$$\left\{ \begin{array}{l} \text{extensions of} \\ |\cdot| \text{ to } L \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{irreducible} \\ \text{factors of} \\ f(x) \in K_{|\cdot|}[x] \end{array} \right\}.$$

Pf. If $|\cdot|_L$ is an extension, form
$L_{|\cdot|_L} := L \cdot K_{|\cdot|}$ [LHS is complete + contains L]

$$\begin{array}{ccc} L & \hookrightarrow & L_{|\cdot|_L} \\ \cup & & \cup \\ K & \hookrightarrow & K_{|\cdot|} \end{array}.$$

Let $g(x)$ be the min. poly of $\alpha \in L_{|\cdot|_L}$ over $K_{|\cdot|}$, $g(\alpha)=0$.
so $g(X) \mid f(x)$ in $K_{|\cdot|}[X]$.

Conversely, if $g(X) \mid f(X)$ in $K_{|\cdot|}[X]$, form $K_{|\cdot|}[X]/g(X)$. By local theory, $|\cdot|$ extends uniquely to $K_{|\cdot|}[X]/g(X)$ and gives

$$\begin{array}{c} L \longrightarrow K_{|\cdot|}[X]/g(X) \\ \alpha \longrightarrow X \end{array}$$

by restriction, an abs. value on L.

Prop. setup as last Prop. $L \otimes_K K_{|\cdot|} \simeq \prod_{\substack{\text{extensions} \\ |\cdot|_i \text{ of } |\cdot| \\ \text{to } L}} L_{|\cdot|_i}$.

e.g. $\mathbb{Q}(i) \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \begin{cases} \mathbb{Q}(i)_{(p)} & p \equiv 3 \\ \mathbb{Q}_p \oplus \mathbb{Q}_p & \end{cases}$

$p \equiv 1 \ (4)$

<u>pf</u>. $L = K[\alpha]$ w/ f min. poly. of α/K.

$$L \otimes_K K_{|\cdot|} \simeq K[x]/f(x) \otimes_K K_{|\cdot|}$$

$$\simeq K_{|\cdot|}[x]/f(x) \simeq \prod_{\substack{\text{irr. factors} \\ g \text{ of } f \in K_{|\cdot|}[x]}} K_{|\cdot|}[x]/g(x)$$

$$\simeq \prod_{\substack{\text{ext} \\ |\cdot|_i \text{ of } |\cdot|}} L_{|\cdot|_i} . \blacksquare$$

<u>Cor</u>. $\forall \beta \in L$, $N_{L/K}(\beta) = \prod_{\substack{\text{extensions} \\ |\cdot|_i \text{ of } |\cdot|}} N_{L_{|\cdot|_i}/K_{|\cdot|_i}}(\beta)$.

Same w/ traces.

<u>pf</u>. $N_{L/K}(\beta) = \det(\beta : L \to L)$
$= \det(\beta : L \otimes_K K_{|\cdot|} \to L \otimes_K K_{|\cdot|})$.

$$\overset{\text{Prop}}{=} \prod_i \det(\beta : L_{|\cdot|_i} \to L_{|\cdot|_i}) . \blacksquare$$

<u>Cor</u>. L/K fin ext of # fields. Then $\forall$ non-triv valuations on K, $[L:K] = \sum_{\substack{\text{ext's} \\ |\cdot|_i \text{ of} \\ |\cdot|}} e_{L_{|\cdot|_i}/K_{|\cdot|_i}} f_{L_{|\cdot|_i}/K_{|\cdot|_i}}$.

<u>Better</u> <u>proposition</u>: L/K fin. ext. of # fields.
For any non-archimedean abs. value $|\cdot|$ on K,

the iso ✩ induces $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K_{|\cdot|}} \xrightarrow{\sim} \prod_{|\cdot|_i \text{ ext. } |\cdot|} \mathcal{O}_{L_{|\cdot|_i}}$

<u>Pf.</u> Under the map of the field version

$\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K_{|\cdot|}}$ lands in $\prod \mathcal{O}_{L_{|\cdot|_i}}$ and spans

$\prod L_{|\cdot|_i}$ over $K_{|\cdot|}$ $\Rightarrow$ $\operatorname{im}(\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K_{|\cdot|}}) \supset m \cdot \prod \mathcal{O}_{L_{|\cdot|_i}}$

for some integer m. By weak approximation,

$\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K_{|\cdot|}} \twoheadrightarrow \prod \mathcal{O}_{L_{|\cdot|_i}} / m \cdot (\quad)$.

Explicitly, $\forall\, (a_i) \in \prod \mathcal{O}_{L_{|\cdot|_i}}$, $\exists\, \alpha \in \mathcal{O}_L$ s.t.

$|\alpha - a_i|_i \le |m|_i \;\; \forall i$.

surjectivity follows.

One more description of ext's of valuations.

L/K global finite $\quad \left\{ \begin{matrix} \text{extensions of} \\ |\cdot| \text{ to } L \end{matrix} \right\} \xleftrightarrow{\sim} \left\{ \begin{matrix} K\text{-embeddings} \\ L \hookrightarrow \overline{K_{|\cdot|}} \end{matrix} \right\} \Big/ \operatorname{Gal}(\overline{K_{|\cdot|}}/K_{|\cdot|})$

$|\cdot|, K \rightsquigarrow K_{|\cdot|}$

$\downarrow$

alg. closure $\overline{K_{|\cdot|}}$ $\quad \left(\begin{matrix} \text{restriction of} \\ |\cdot| \text{ on } \overline{K_{|\cdot|}} \\ \text{to } L \underset{\tau}{\hookrightarrow} \overline{K_{|\cdot|}} \end{matrix} \right) \longleftarrow \tau$

For an ext. $|\cdot|_L \rightsquigarrow$ choose a $K_{|\cdot|}$ embedding $\quad$ $L_{|\cdot|} \hookrightarrow \overline{K_{|\cdot|}}$

$\begin{matrix} \cup \\ L \end{matrix} \dashrightarrow$

Rmk. Same holds for L/K alg & infinite if instead of completion $L_{|\cdot|_L}$, use $\bigcup_{\substack{L/L'/K \\ \text{fin}}} L'_{|\cdot|_L}$.

How local Galois fits into global Galois

L/K Galois ext (possibly infinite). Let $G = G(L/K)$.
field

Notation. For an abs. value $|\cdot|$ on K ($= |\cdot|_v$) write v for a corr. val in non-arch case, K_v = completion (previously $K_{|\cdot|_v}$).

Also write $K_v = K_{|\cdot|_v}$ in the arch. setting, even though "v" does not exist.

If w ($|\cdot|_w$) is in a class of abs. values extending $|\cdot|_v$, write $w|v$.

Def'n. A place (or prime) v of a # field K is an equivalence class of non-triv abs. values.

Let L/K be Galois. Let v be a place of K. $G = G(L/K)$ acts on $\left\{\begin{matrix}\text{ext of} \\ v \text{ to } L\end{matrix}\right\}$ by $w \longrightarrow w \circ \sigma$

Prop. This action is transitive.

Pf. Let w and w' be extensions. First, assume L/K finite. Assume $\{w \circ \sigma \mid \sigma \in G\}$ & $\{w' \circ \sigma \mid \sigma \in G\}$ are disjoint.

By weak approximation, $\exists x \in L$ s.t. $|\sigma x|_w < 1 \ \forall \sigma$ and $|\sigma x|_{w'} > 1$ for all $\sigma \in G$. But

$$|N_{L/K}(x)|_v = \begin{cases} \prod_\sigma |\sigma x|_w < 1 \\ \prod_\sigma |\sigma x|_w > 1 \end{cases}, \ \text{↯}.$$

$5 = (2+i)(2-i)$

$$\mathbb{Q}(i) \to \mathbb{C}, \quad \mathbb{Q}(i) \to \mathbb{C}$$

related by $G(\mathbb{Q}i/\mathbb{Q})$.

For L/K infinite, first recall ∞ Galois theory.

Digression. If L/K is any Gal ext., $\mathrm{Gal}(L/K)$ is topologized by making a basis of nbhds at $\sigma \in \mathrm{Gal}(L/K)$

$= \{\sigma\, G(L/M)$, for M/K finite$\}$.

($\leadsto$ compact Hausdorff group (profinite))

$$\mathrm{Gal}(L/K) \xrightarrow{\sim} \varprojlim_{\substack{\text{fin.} \\ \mathrm{Gal}(M/K)}} \underbrace{G(M/K)}_{\text{discrete}}.$$

Main thm. field extensions $\longleftrightarrow$ closed subgroups.

fin. extensions $\longleftrightarrow$ open subgroups

Galois extensions $\longleftrightarrow$ normal subgroups

See $\underline{\text{IV}}$ of Neukirch.