Last time. M fin $\in \mathcal{M}_{G_K} \rightsquigarrow H^i(K, M) \times H^{2-i}(K, M^*) \to \mathbb{Q}/\mathbb{Z}$

$$x\ ,\ y \longrightarrow \mathrm{inv}_K(x \cup y)$$

is a perfect duality. The other coh. result for local fields: Euler characteristic formula (Tate):

$$M, K \text{ as above} \quad \frac{\# H^0(K,M)\, \# H^2(K,M)}{\# H^1(K,M)} = (\# k_K)^{-v_K(\# M)}$$

normalized val'n.

So now to compute $|H^*(K,M)|$:

1) $M^{G_K} = \# H^0$

2) $(M^*)^{G_K} = \# H^2$

3) Use EC formula to get $\# H^1$.

Th'm (LCFT) For each fin. ext. $K/\mathbb{Q}_p$, there is a cts inj. w/ dense image

$$r_K : K^\times \longrightarrow G_K^{ab} = G(K^{ab}/K)$$

characterized by commutativity of ($\forall n \in \mathbb{Z}_{>0}$), i.e. $\forall a \in K^\times$ and $\forall \chi : G_K^{ab} \to \mathbb{Z}/n\mathbb{Z}$,

$$\boxed{\begin{array}{l}\mathrm{inv}_K(\chi \cup \delta a) \\ = \chi(r_K(a))\end{array}}\,.$$

$$\begin{tikzcd}
 & \mathrm{Hom}(K^\times, \mathbb{Z}/n\mathbb{Z}) \times K^\times/(K^\times)^n \arrow[r, "\text{eval}"] \arrow[d, "\delta"] & \mathbb{Z}/n\mathbb{Z} \arrow[d, hook] \\
\mathrm{Hom}(G_K^{ab}, \mathbb{Z}/n\mathbb{Z}) = H^1(K, \mathbb{Z}/n\mathbb{Z}) \arrow[ur, "r_K^\wedge"] & \times H^1(K, \mu_n) \arrow[r, "\langle , \rangle_K"] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

It satisfies: ① $\forall\, L/K$ fin.,

$$\begin{tikzcd} K^\times \arrow[r, "r_K"] & G_K^{ab} \\ L^\times \arrow[u, "N_{L/K}"] \arrow[r, "r_L"'] & G_L^{ab} \arrow[u] \end{tikzcd}$$

② $\forall\, L/K$ fin. Galois, r_K induces iso

$$K^\times / N_{L/K} L^\times \xrightarrow{\sim} G(L/K)^{ab}$$

③ $\forall\, L/K$ fin.,

$$\begin{tikzcd} K^\times \arrow[r, "r_K"] \arrow[d, phantom, "\cap" description] & G_K^{ab} \arrow[d, "Ver"] \\ L^\times \arrow[r, "r_L"'] & G_L^{ab} \end{tikzcd}$$

④ If $a \in K^\times$, $\chi \in Hom(G_K, \mathbb{Q}/\mathbb{Z})$, $\chi(r_K(a)) = inv_K(a \cup d\chi)$

$$\begin{tikzcd} Hom(G_K, \mathbb{Q}/\mathbb{Z}) \arrow[d, "d"] \\ H^2(G_K, \mathbb{Z}) \end{tikzcd}$$

"Existence Thm"

⑤

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_K^\times \arrow[r] \arrow[d] & K^\times \arrow[r, "v_K"] \arrow[d, "r_K"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I_{K^{ab}/K} \arrow[r] & G_K^{ab} \arrow[r] & G(K^{ur}/K) \arrow[r] \arrow[d, "\wr"] & 0 \\ & & & G_k & \end{tikzcd}$$

$$1 \mapsto Frob_k$$

⑥ The open s/g of fin. index in $K^\times$ are precisely the subgroups of the form $N_{L/K} L^\times$, L/K fin. ab.

This bijection satisfies all properties from first lecture on LCFT.

<u>Pf</u>. (0) Construction of r_K.

$$\operatorname{Hom}_{cts}(\underbrace{G_K}_{cpt}, \underbrace{\mathbb{Q}/\mathbb{Z}}_{disc}) = \varinjlim_n \operatorname{Hom}(G_K, \mathbb{Z}/n\mathbb{Z})$$

all have fin. image

Pontryagin dual of G_K^{ab}

$$= \varinjlim_n H^1(K, \mathbb{Z}/n)$$

$$\xrightarrow[\langle , \rangle_K]{} \varinjlim_n \operatorname{Ho}(H^1(K, \mu_n), \mathbb{Q}/\mathbb{Z})$$

$$\underset{\text{Kummer}}{\simeq} \varinjlim_n \operatorname{Hom}(K^\times/(K^\times)^n, \mathbb{Q}/\mathbb{Z}).$$

$\|$ $K^\times = \text{cpt} \times \mathbb{Z}$, then argue as before.

$$\operatorname{Hom}_{cts}(K^\times, \mathbb{Q}/\mathbb{Z})$$

not pontryagin dual of $K^\times$.

Now apply $\operatorname{Hom}_{cts}(\cdot, \mathbb{Q}/\mathbb{Z})$ to left & right extremes and get

$$G_K^{ab} \simeq \operatorname{Hom}_{cts}(\operatorname{Hom}_{cts}(\overset{\mathcal{O}_K \times \langle \varpi \rangle}{\overset{\shortparallel}{K}}, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$$

$$\simeq \mathcal{O}_K^\times \times \widehat{\langle \varpi \rangle} = \widehat{K}^\times = \text{profinite comp. of } K^\times.$$

$$\widehat{\langle \varpi \rangle} = \operatorname{Hom}(\operatorname{Hom}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) = \widehat{\mathbb{Z}}.$$

① L/K

$$\begin{tikzcd} K^\times \arrow[r, "r_K"] & G_K^{ab} \\ L^\times \arrow[u] \arrow[r, "r_L"'] & G_L^{ab} \arrow[u] \end{tikzcd}$$

Let $b \in L^\times$. Must check $r_K(N(b)) = r_L(b)$. Suffices to check $\chi(r_K(N(b))) = \chi(r_L(b))$

$\forall \chi : G_K^{ab} \to \mathbb{Q}/\mathbb{Z}$. <u>Compute:</u>

$$\chi(r_L(b)) \underset{\substack{\text{char.}\\ \text{diagram}}}{=} \mathrm{inv}_L(\chi \cup \delta b) = \mathrm{inv}_L(\mathrm{res}(\chi) \cup \delta b)$$

$$= \mathrm{inv}_K(\mathrm{cor}(\mathrm{res}(\chi) \cup \delta b))$$

$$= \mathrm{inv}_K(\chi \cup \mathrm{cor}(\delta b)).$$

$$\underset{\substack{\text{Kummer}\\ \text{theory}}}{=} \mathrm{inv}_K(\chi \cup \delta(N(b)))$$

$$\underset{\substack{\text{char.}\\ \text{diagram.}}}{=} \chi(r_K(N(b)))$$

② L/K fin. Galois (1) gives us

$$\begin{tikzcd} L^\times \arrow[r, "N"] \arrow[d, "r_L"'] & K^\times \arrow[r] \arrow[d, "r_K"] & K^\times/NL^\times \arrow[r] \arrow[d, "r_{L/K}"] & 0 \\ G_L^{ab} \arrow[r] & G_K^{ab} \arrow[r] & G(L/K)^{ab} \arrow[r] & 0 \end{tikzcd}$$

a) $r_{L/K}$ is surjective. Given $\bar{g} \in G(L/K)^{ab}$, $g \in G_K^{ab}$ with $g \mapsto \bar{g}$,

find sequence $(\alpha_i)_i \in K^\times$ s.t.

$r_K(a_i) \xrightarrow[i \to \infty]{} g$. Look at $(a_i \bmod NL^\times)_i$ in $\underbrace{K^\times/NL^\times}_{\text{compact}}$

Then $\exists$ a lim pt $\bar{\alpha} \in K^\times/NL^\times$ & $r_{L/K}(\bar{\alpha}) = \bar{g}$. ✓

b) $r_{L/K}$ is injective. If $\alpha \in K^\times$ w/ image $1 \in G(L/K)^{ab}$.

then $r_K(\alpha)$ is $\underrightarrow{\lim}\, r_L(b_i)$ for $b_i \in L^\times$, and if we

show $K^\times \cap \overline{N_{L/K} L^\times} = N_{L/K} L^\times$, then we get $\alpha \in NL^\times$.

<u>Show</u>. If $N_{L/K} b_i \xrightarrow[\hat{K}^\times]{} \alpha \Rightarrow \alpha \in N(L^\times)$.

<u>Pf</u>. Reduce to $\alpha \in \mathcal{O}_K^\times$ (by looking at valuation):

$$v_K(N(b_i)) = \underbrace{[L:K]}_{e_{L/K}} v_L(b_i) = f_{L/K} v_L(b_i)$$

So $v_K(\alpha) \in f_{L/K} \cdot \mathbb{Z}$. Choose $c \in L^\times$ s.t. $v_L(c) = \dfrac{v_K(\alpha)}{f_{L/K}} \in \mathbb{Z}$.

Then $N(b_i/c) \longrightarrow \dfrac{\alpha}{N(c)} \in \mathcal{O}_K^\times$. So we may assume $\alpha \in \mathcal{O}_K^\times$.

$\rightsquigarrow$ wma $b_i \in \mathcal{O}_L^\times$, $N(b_i) \longrightarrow \alpha$ & since $\mathcal{O}_L^\times$ is compact we extract a lim pt. $\{b_i\}$, say b, and then $N(b) = \alpha$.

③ $K^\times \subset L^\times \longleftrightarrow v: G_K^{ab} \to G_L^{ab}$. <u>HW</u>.

④ Recall $\underset{\substack{\uparrow \\ \mathrm{Hom}(G_K, \mathbb{Q}/\mathbb{Z})}}{\chi}(r_K(\underset{\substack{\uparrow \\ \hat{K}^\times}}{a})) = \mathrm{inv}_K(\chi \cup \delta a) = -\mathrm{inv}_K(\delta a \cup \chi)$

cup bndry $= -\mathrm{inv}_K(\delta(a \cup \chi))$

compatibilities! $= -\mathrm{inv}_K(a \cup \delta\chi)$

(5) First check commutativity of

$$\begin{tikzcd} K^\times \arrow[r, "v_K"] \arrow[d, "r_K"'] & \mathbb{Z} \arrow[d] & 1 \arrow[d, mapsto] \\ G_K^{ab} \arrow[r] & G_k^{ab} & \mathrm{Frob}_K^{-1} \end{tikzcd}$$

This normalization depends on char. diagram & in calc of $\mathrm{Br}(K)$, choice of $H^1(G_k, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$.

Let $a \in K^\times$, let $\chi \in H^1(G(K^{ur}/K), \mathbb{Q}/\mathbb{Z})$. Must show $\chi(r_K(a)|_{K^{ur}}) = \mathrm{Fr}_K^{-v_K(a)}$.

We know $\chi(r_K(a)) = -\mathrm{inv}_K(\underset{H^q(K^{ur}/K, K^{ur\times})}{a \cup} \underset{H^2(K^{ur}/K, \mathbb{Z})}{\delta\chi})$

<u>Recall</u>.

$$\begin{tikzcd} H^2(K^{ur}/K, K^{ur\times}) \arrow[r, "\mathrm{inv}_K", "\sim"'] \arrow[d] & \mathbb{Q}/\mathbb{Z} \\ H^2(K^{ur}/K, \mathbb{Z}) & H^1(K^{ur}/K, \mathbb{Q}/\mathbb{Z}) \arrow[l, "\delta"'] \arrow[u, "ev"'] \end{tikzcd}$$

Hence $= -ev \circ \delta^{-1} \circ v_K(a \cup \delta\chi)$

$= -ev(\chi^{v_K(a)}) = \chi(\mathrm{Frob}_K^{-v_K(a)})$

So we get a diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}_K^\times \arrow[r] \arrow[d, dashed] & K^\times \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & I_{K^{ab}/K} \arrow[r] & G_K^{ab} \arrow[r] & G_k^{ab} \arrow[r] & 1
\end{tikzcd}$$

<u>Claim</u>. $r_K|_{\mathcal{O}_K}$ is an iso. know r_K is inj.

- $r_K|_{\hat{\mathcal{O}}_K}$ surjects onto any finite quotient of $I_{K^{ab}/K}$, hence $r_K(\mathcal{O}_K^\times)$ is dense in $I_{K^{ab}/K}$, hence $r_K(\mathcal{O}_K^\times)$ is compact, hence closed, dense subset of $I_{K^{ab}/K}$, so $=$.

<u>Checking density</u>.

Let L/K fin. ab.

$$\begin{array}{l} L \\ \mid \Big\} I_{L/K} \\ M \\ \mid \\ K \end{array}$$

Fix $h \in I_{L/K}$. We know $\exists a \in K^\times$ s.t. $r_{L/K}(a) = h$. Also, $r_K(a)|_M = h|_M = 1!$

$r_K(a) :=$

$r_K(a)|_M \;(5.i)\,\|\; \mathrm{Fr}_K^{\,v_K(a)}$

$\Rightarrow f_{L/K} \mid v_K(a)$

$\rightsquigarrow \exists b \in L^\times$ s.t. $\dfrac{v_K(a)}{f_{L/K}} = v_L(b)$.

Hence $u = a \cdot N(b)^{-1} \in \mathcal{O}_K^\times$ and $r_K(u) = h$.

(6) We have $\mathcal{O}_K^\times \xrightarrow[\text{top iso}]{\sim} I_{K^{ab}/K} = \varprojlim_{L/K \text{ fin}} I_{L/K}$

$$\xleftarrow{\sim} \varprojlim_{L/K \text{ fin}} \mathcal{O}_K^\times / \mathcal{O}_K^\times \cap N(L^\times).$$

$\Rightarrow$ the usual profin. top. on $\mathcal{O}_K^\times$ agrees w/ topology induced by norm s/g's.

Now let $U \subset K^\times$ be any open fin. index s/g.

Want $U \overset{?}{=}$ norm.

$\underbrace{U \cap \mathcal{O}_K^\times \text{ is open fin. idx } \subset \mathcal{O}_K^\times}$ and $v_K(U) \supset m\mathbb{Z}$ for some $m \geq 1$.

$\exists$ L/K s.t. $N(L^\times) \cap \mathcal{O}_K^\times \subset U \cap \mathcal{O}_K^\times$. Replacing L by $L \cdot$ unr. ext., get $v_K(N(L^\times)) \subset v_K(U)$.

Hence $N(L^\times) \subset U \subset K^\times$.

<u>Claim</u>. Having $U \supset N(L^\times)$ implies U is itself a norm s/g.
(uses rec: given in 1st LCFT lecture)

$$\begin{array}{ccc} r_K : K^\times/NL^\times & \xrightarrow{\sim} & G(L/K) \\ \downarrow & & \downarrow \ \exists\, M \subset L \\ r_K : K/U & \xrightarrow{\sim} & G(-/K) \end{array}$$

$$\Rightarrow U^\times = \ker(r_K|_M) = N_{M/K}(M^\times)$$

So all open fin. ind $\in K^\times$ = norms of fin. ab. ext's.

■