Last time.

1) L/K cyclic $\Rightarrow h(G(L/K), C_L) = [L:K]$

so $|H^2(G(L/K), C_L)| \geq [L:K]$

2) For general L/K, $r_{L/K}: \mathbb{A}_K / N\mathbb{A}_L^\times \longrightarrow G(L/K)^{ab}$

is surjective.

Today: Start analyzing $\ker(r_{L/K})$, via an inequality opposite to (1). (Part 2 of our outline)

Thm. Let L/K be a finite Galois extension.

Then a) $|C_K/NC_L|$ divides $[L:K]$

b) $H^1(G, C_L) = 0$

c) $|H^2(G, C_L)|$ is finite & divides $[L:K]$.

Today. Prove this where L/K is cyclic of prime order.

Two ways (a) in this $<$ (Tate/Milton) detailed Kummer theory / analytic properties of L-functions

Lemma. If L/K cyclic, (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c), & equality holds

Pf. We know $[L:K] = \dfrac{|H^2(C_L)|}{|H^1(C_L)|}$. Lemma is immediate from this & calc of coh. of cyclic groups.

$C_L^{G(L/K)} = C_K$

$1 \to L^\times \to \mathbb{A}_L^\times \to C_L \to 1$

<u>Lemma</u>. Suffices to prove theorem only for cyclic case (even of prime power order).

<u>Pf</u>. Assume known for all L/K cyclic of prime order. Now let L/K be any Gal ext. w/ $G(L/K)$ a p-group of order p^r. We show (a), (b), (c) by induction on r. $G(L/K)$ has a norm s/g so wma.

$$L \supset \overbrace{M \supset K}^{\text{Galois}}$$

Using inf-res

$$0 \to \underbrace{H^1(G(M/K), C_M = C_L^{G(L/M)})}_{0 \text{ by ind (b)}} \to H^1(L/K, C_L)$$

$$\to \underbrace{H^1(L/M, C_L)}_{\substack{0 \text{ by ind} \\ (r=1)}}$$

Look farther in H-S (b/c H^1's vanish, get)

$$0 \to H^2(M/K, C_M) \to H^2(L/K, C_L)$$

$$\to H^2(L/M, C_L)$$

$$\overset{\text{ind}}{\Rightarrow} |H^2(L/K, C_L)| \,\Big|\, |H^2(M/K, C_M)|\,|H^2(L/M, C_L)|$$

$$\Big| \, [M:K][L:M] = [L:K].$$

Likewise,

$$C_M/NC_L \longrightarrow C_L/NC_L \longrightarrow C_K/NC_M \quad \text{gives}$$

(a) by induction.

General case: L/K fin. Galois. STP a, b, c for p-primary parts in each case.

Fix p, let $G_p := G(L/L_p)$ be a p-Sylow subgroup of $G(L/K)$.

<u>Recall</u>. $\mathrm{res}: H^r(G(L/K), C_L)_{p^\infty} \longrightarrow H^r(G(L/L_p), C_L)$

is i

(b) & (c) in general now follows from p-groups.

(a) If $x \in \mathbb{A}_K/K^\times N(\mathbb{A}_L^\times)^{[p^\infty]}$, then $x = \widehat{\underset{L_p}{\alpha}}$,

$N(L/L_p)$ back Even though $\alpha \in L_p^\times$

So $x \in (C_K/N_L)[p^\infty]$ <u>and</u> $x^{[L_p:K]} = 1$, so $x = 1$. i.e.

$$C_K/NC_L[p^\infty] \longleftrightarrow C_{L_p}/N_{L/L_p}C_L.$$

<u>Rmk</u>. Next week we prove (a) in general using L-functions (or see Tate for a algebraic proof when L/K cyclic $r_{\mathbb{Z}}(K^\wedge)$ - Lemma.

Part 3. $r_{L/K} : C_K / N C_L \longrightarrow G(L/K)^{ab}$ and

$\bar{\phi}$ then by 1 & 2 is in fact an iso.

need not be for given higher

focus.

Main Thm.

$$0 \to Br(K) \to \bigoplus Br(K_v) \xrightarrow{\sum_v inv_{K_v}} \mathbb{Q}/\mathbb{Z} \to 0.$$

Recall $Br(L/K) = H^2(G(L/K), L^\times)$. So by H90,

$$0 \to Br(L/K) \to Br(K) \to Br(L).$$

is exact.

Prop. For all Galois L/K,

$$res : Br(L/K) \to \bigoplus_{v \text{ place}} B(L_w, \ldots).$$

Pf. Take L/K finite. We have shown (modulo L-functions) $H^1(G(L/K), C_L) = 0$. So LES on $G(L/K)$ coh. assoc'd to $1 \to L^\times \to \mathbb{A}_L^\times \to C_L \to 1$ gives

$$\begin{array}{ccccc} 0 \to & H^2(G, L^\times) & \to & H^2(G, \mathbb{A}_L^\times) & \to \cdots \\ & \| & & \| & \\ & Br(L/K) & & \bigoplus_v H^2(L_w/K_v, L_w^\times) = \bigoplus Br(L_w/K_v) & \end{array}$$

Why ⊛: $H^2(G, \mathbb{A}_L) = \varinjlim_{\substack{S \text{ finite} \\ \text{set of} \\ \text{places of} \\ K.}} \mathbb{A}_L^{\times S}$

$$= \prod_{v \in S} \prod_{w|v} L_w^\times \times \prod_{v \notin S} \prod_{w|v} \mathcal{O}_w^\times .$$

$$= \varinjlim_S H^2(G, \mathbb{A}_L^{\times S}) = \varinjlim_S \Big(\prod H^2(G, \prod_{w|v} L_w^\times) \times \prod H^2(G, \prod_{w|v} \mathcal{O}_w^\times) \Big)$$

(see last time)

$$\underset{\text{Shapiro}}{=} \varinjlim_S \prod_{v \in S} H^2(G(L_w/K_v), L_w^\times) \times \underbrace{\prod_{v \notin S} H^2(G(L_w/K_v), \mathcal{O}_w^\times)}_{\substack{\text{for } L/K \text{ unr @ } v, \\ \text{these groups vanish} \\ \text{(proven during LCFT)}}}$$

$\Rightarrow H^2(G, \mathbb{A}_L^\times)$

$= \varinjlim_S \bigoplus_{v \in S} Br(K_v)$

$= \bigoplus_v Br(K_v)$

(e.g. a div. alg / K that splits over every K_v is just K itself).

Thm (finish next time) a) $\forall L/K$ fin. abelian,

$r_{L/K}(K^\times) = 1.$

b) $\forall L/K$ Galois and all $\alpha \in Br(L/K)$,

$\sum_v inv_v(\alpha) = 0$, i.e. the sequence in Main Th'm on $Br(K)$ is a complex.

Preliminary reductions. Lemma: • If (a) holds for L/K, then it holds $\forall K'/K$ w/ $L \supset K' \supset K$.

• If (a) for L/K, then (a) holds $\forall LK'/K'$ for any K'/K.

Pf.

$$\begin{tikzcd} \mathbb{A}_K^\times \arrow[r, "r_{L/K}"] \arrow[d, equal] & G(L/K) \arrow[d, "(res)"] \\ \mathbb{A}_K^\times \arrow[r, "r_{K'/K}"] & G(K'/K) \end{tikzcd}$$

gives 1st point.

Set $L' = K'.L$. Have

$$\begin{tikzcd} \mathbb{A}_{K'}^\times \arrow[r, "r_{L'/K'}"] \arrow[d, "N_{K'/K}"] & G(L'/K') \arrow[d, hook] \\ \mathbb{A}_K^\times \arrow[r, "r_{L/K}"] & G(L/K) \end{tikzcd}$$

$\leftarrow$ injective

How to exploit these reductions.

Prop. Let $L/K = \mathbb{Q}(\zeta_m)/\mathbb{Q}$. Then

$r_{\mathbb{Q}(\zeta_m)/\mathbb{Q}} : \mathbb{A}_{\mathbb{Q}}^\times \to G(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ satisfies

$r_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(\mathbb{Q}^\times) = 1$. Consequently (by last lemma), for any K, if $L \subset K(\zeta_m)$, then $r_{L/K}(K^\times) = 1$.

$\Rightarrow$ [(a) for $\mathbb{Q}(\zeta_m)/\mathbb{Q}$ $\Rightarrow$ (a) for $K(\zeta_m)/K$ $\Rightarrow$ for any L/K w $L \subset K(\zeta_m)$]

<u>Note</u>. For general K, not all ext's are cyclotomic ($\subset K(\mu_\infty)$).

<u>Pf of prop</u>. Need $r_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(\mathbb{Q}^\times) = 1$, STP for $m = p^r$. Then STP

$$r(-1) = 1$$
$$r(\ell) = 1, \quad \ell \neq p$$
$$r(p) = 1.$$

This reduces to a local calc identifying $r_{\mathbb{Q}_p}$ to the map $s_{\mathbb{Q}_p} : \mathbb{Q}_p^* \longrightarrow G(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$

$$= G(\mathbb{Q}_p^{un}/\mathbb{Q}_p) \times G(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$$

geometric

$$p \longrightarrow (\mathrm{Frob}_p, 1)$$

$$\mathbb{Z}_p^\times \xleftarrow[\text{cycl. char}]{\sim} 1 \times G(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$$

<u>cycl char</u>. $G(K(\zeta_m)/K) \xrightarrow{\kappa} (\mathbb{Z}/m\mathbb{Z})^\times$ given by $\sigma(\zeta) = \zeta^{\kappa(\sigma)}$ $\forall \zeta \in \mu_m$, $\sigma \in \mathrm{Gal}$

(Did not prove this in LCFT).

<u>Given that $r_{\mathbb{Q}_p} = s_{\mathbb{Q}_p}$,</u> calculate:

1) $r(-1) = r_\infty(-1) r_p(-1)$ $\quad$ (b/c $-1 \in \mathbb{Z}_\ell^\times$ $\forall \ell$)

$= -1 \cdot -1 = 1$

2) $\ell \neq p$ $\quad r(\ell) = r_\ell(\ell) \cdot r_p(\ell) = \underset{\substack{\downarrow \\ \text{Geometric} \\ \text{frob}}}{\ell^{-1}} \cdot \ell \quad -1$

3) $r(p) = r_p(p) = 1$ on $\mathbb{Q}_p(\mu_{p^\infty})$.

Next time, prove the given. Rmk. Can be proven purely locally (using Lubin-Tate theory). But we'll give a global proof.