Steps in pf of CFT (reciprocity)

Recall for fin Galois L/K, have

$$\begin{tikzcd} r_{L/K} : \mathbb{A}_K^\times \arrow[r] & G(L/K)^{ab} \\ r_{L_w/K_v} : K_v^\times \arrow[r] \arrow[u, hook] & G(L_w/K_v)^{ab} \arrow[u] \end{tikzcd}$$

(for all places v of K)

LCFT $\Rightarrow$ $r_{L/K}$ factors $\mathbb{A}_K^\times / N_{L/K}(\mathbb{A}_L^\times) \to G(L/K)^{ab}$.

We must show

① $r_{L/K}$ is surjective

② $|\mathbb{A}_K^\times / K^\times N(\mathbb{A}_L^\times)| \leq [L:K]$

③ $K^\times \subset \ker(r_{L/K})$

Combining ①, ②, ③ gives that $r_{L/K} : C_K/N(C_L) \xrightarrow{\sim} G(L/K)$ at least when L/K abelian, and not hard to $\xrightarrow{\sim} G(L/K)^{ab}$ in general.

(*) Everything follows from understanding of $H^*(G(L/K), C_L)$. (e.g. L/K cyclic $\Rightarrow$ $H^2(G(L/K), C_L)$
$= C_K/NC_L$

① Show L/K cyclic $\Rightarrow$
$|C_K/NC_L| \geq [L:K]$
$\Downarrow$
surj. of $r_{L/K}$ $\Leftarrow$ "not too many primes split"

$\boxed{\text{recall } \; e_L^{G(L/K)} = C_K.}$

② Opposite ineq + "Hilbert 90 for C_L"

↳ we'll use info from global L-functions (analytic)

③ hardest part — calculate ~~the way~~ $Br(K)$ along the way.

Onto details of ①. Let L/K be cyclic (until specified otherwise). $G(L/K) =: G$,

so $h(G, M) = \dfrac{\# H^2}{\# H^1}$ for G-modules M s.t. both orders are finite. If M finite, $h(G,M) = 1$. Multiplicative in SES's.

Th'm. L/K cyclic $\Rightarrow h(G, C_L) = [L:K]$.

Cor. $|C_K / N C_L| \geq [L:K]$

Pf. Let S be a finite set of places K

- ∞ places
- places ram'd in L/K
- generators of $Cl(\mathcal{O}_L)$.

$K_S := \prod_{v \in S} K_v$

$L_S := \prod_{w | v \in S} L_w$

So $L_S^\times \twoheadrightarrow \mathbb{A}_L^\times / L^\times L_\infty \prod_{w \nmid \infty} \mathcal{O}_w^\times$ is surjective, and

$C_L = \mathbb{A}_L^\times / L^\times = L_S^\times \cdot \prod_{w \notin S} \mathcal{O}_w^\times / \mathcal{O}_L[1/S]^\times$, so reduce

to computing $h(L_S^\times)$, $h(\prod_{w \notin S} \mathcal{O}_w^\times)$, $h(\mathcal{O}_L[1/S]^\times)$.

$h(L_S^\times) = h(G, L_S^\times) = \prod_{v \in S} h(G(L/K), \prod_{w | v} L_w^\times)$

(Shapiro) $= \prod_{v \in S} h(G(L_{w_0}/K_v), L_{w_0}^\times)$

b/c $\prod_{w|v} L_w^\times = \mathrm{Ind}_{G(L_{w_0}/K_v)}^{G(L/K)} (L_{w_0}^\times)$

$$= \prod_{v \in S} \frac{\# H^2}{\# H^1} = \prod_{v \in S} [L_w : K_v].$$

Hilbert 90
Local Brauer

Then $h(\prod_{w \notin S} \mathcal{O}_w^\times) = \prod_{v \notin S} h(G(L/K), \prod_{w|v} \mathcal{O}_w^\times)$

$$\overset{(\text{Shapiro})}{=} \prod_{v \notin S} h(G(L_w/K_v), \mathcal{O}_w^\times)$$

Since the H^1's are trivial as $\mathcal{O}_w^\times$ is a summand.

Actually, H^2 is also trivial as norm is surj. on units for unram. ext's. So

$$h(\prod_{w \notin S} \mathcal{O}_w^\times) = 1.$$

Claim. $h(\mathcal{O}_L[1/S]^\times) = h\left(\ker\left(\bigoplus_{v \in S} \mathrm{Ind}_{G(L_w/K)}^{G(L/K)} \mathbb{Z} \xrightarrow{f \mapsto f(1)} \mathbb{Z}\right)\right)$

Granted claim, $h(\mathcal{O}_L[1/S]^\times) = \dfrac{\prod_{v \in S} h(\mathrm{Ind}_{G(L_w/K_v)}^{G(L/K)} \mathbb{Z})}{h(\mathbb{Z})} =$

$$H^1(\langle s \rangle, \mathbb{Z}) = \frac{\ker N}{\mathrm{im}(s-1)} = 0.$$

$$H^2 = \frac{\ker(s-1)}{\mathrm{im}\, N} = \mathbb{Z}/\text{order}$$

Hence $= \dfrac{\prod_{v \in S} [L_w : K_v]}{[L:K]}$. Combining all this,

$$h(C_L) = [L:K].$$

<u>Remains to prove claim</u>.

We'll show (i) $\mathcal{O}_L[1/S]^\times \otimes \mathbb{Q} \simeq \ker\left(\bigoplus_{v \in S} \mathrm{Ind}_{G(L_w/K_v)}^{G(L/K)} \mathbb{Q} \to \mathbb{Q}\right)$

as $G(L/K)$-mods,

(ii) Two G-mods that have $\underset{\sim}{M, N}$ lattices in common.

(a) ~~weet~~ w/ G-action have the same Herbrand quotient.

[(ii): up to rescaling, $M \subset N$. <u>lattices</u> $\Rightarrow$ N/M's finite so $h(N/M) = 1$. and so $h(M) = h(N)$.

For (i), we'll check the iso after $\underset{\mathbb{Q}}{\otimes} \mathbb{R}$ & then invoke

Lemma. Let G be a finite gp & $E \subset E'$ be infinite fields. Let M, N be $E[G]$-mods f.d. $/E$. Then

$$M \underset{E[G]}{\cong} N \Rightarrow M \otimes_E E' \underset{E'[G]}{\cong} N \otimes_E E'.$$

(Apply this with $E = \mathbb{Q}$, $E' = \mathbb{R}$.)

Pf. (of Lemma). $\mathrm{Hom}_{E[G]}(M, N \otimes_E E')$

$$\cong \mathrm{Hom}_{E'[G]}(M \otimes E', N \otimes E').$$

Let $\alpha_1, \dots, \alpha_m$ be an E-basis to.

By assumption, $\exists\, x_i \in E$ s.t. $\sum x_i \alpha_i$ is an iso,

hence $\det(\sum x_i \alpha_i) \neq 0$. So poly $\det(y_1, \dots, y_m)$

is not identically.

Since E is finite, there are $y_i \in E$ s.t.

$$\det\left(\sum X_i \alpha_i\right)\Big|_{X_i = y_i} \neq 0.$$

To finish pf, need:

$\mathcal{O}_L[1/S]^\times \otimes \mathbb{R} \cong \ker \Gamma$ then check ∂ ~

$\iota_{\log}$ al $\simeq$ as $G(L/K)$-

$\bigoplus_{w|v\in S} \mathbb{R} \xrightarrow{\text{sum}} \mathbb{R}$ clearly

$(\log|x|_w)_w$

$G(L/K)$ action by permuting funct'r over S.

Define iso by

$x \otimes 1 \longmapsto$

$$\begin{tikzcd}
\mathcal{O}_L[1/S]^\times \otimes \mathbb{R} \arrow[r, "\text{define this via}", "\text{other arrows}"'] \arrow[dr, "(\text{S-unit thm})"'] & \ker\left(\bigoplus_{v\in S} \mathrm{Ind}_{G(L_w/K_v)}^{G(L/K)} \mathbb{R} \to \mathbb{R}\right) \\
& \ker\left(\bigoplus_{w|v\in S} \mathbb{R} \xrightarrow{\text{sum}} \mathbb{R}\right) \arrow[u, "(\text{as } G(L/K)\text{-mod})"']
\end{tikzcd}$$

☒

We've shown $h(G(L/K), C_L) = [L:K]$ $\forall$ cyclic L/K.

Application. Surjectivity of $r_{L/K}$ in general.

Prop. If L/K fin. ~~If~~ all but finitely many places of K split completely in L, then $K = L$.

Pf. Only deal w/ Galois

First assume L/K cyclic. Let S be a finite set of places containing

- ∞ places
- ram'd in L/K
- the places not split completely (fin. by assumption).

Then the cor. above gives

$$[L:K] \le \left| \mathbb{A}_K^\times / K^\times N \mathbb{A}_L^\times \right|$$

$$= \left| \mathbb{A}_K^\times / K^\times N(L_S^\times) \right|$$

$$= \left| \mathbb{A}_K^\times / K^\times N(L_S^\times)\, \mathbb{A}_K^{\times S} \right|$$

$$= K_S^\times / K^\times N(L_S^\times)$$

since $K^\times \hookrightarrow K_S^\times$ is dense by weak approx. $= 1$,
and $N(L_S^\times)$ is open

So $[L:K] \le 1$. Now take L/K Galois. If $L \neq K$, then $\exists\ \underbrace{L \supseteq M \supset K}_{\text{cyclic}}$. If all but finitely many places of K split in L, then ditto for M to L. General L/K: exercise (replace L by Galois closure).

<u>Rmk</u>. Later, we'll prove a much stronger version of this prop (Čebotarev density thm.

<u>Cor</u>. L/K fin. Galois. Let S be a finite set of places of K. Then $\mathrm{Gal}(L/K)$ is generated by the decomposition groups $G(L_w/K_v)$ for $w|v \notin S$. Consequently, $r_{L/K}: \mathbb{A}_K^\times \longrightarrow G(L/K)^{ab}$ is surjective.

<u>Pf</u>. Let H = s/g of $G(L/K)$ gen'd by these decomp groups. Any unramified prime $\notin S$ of K splits completely in L^H/K.

$$\begin{array}{c} L \\ | \\ L^H \\ | \\ K \end{array}$$

So $L^H = K$, and $H = G(L/K)$.

$$\mathbb{A}_K^\times \longrightarrow G(L/K)^{ab}$$

$v \notin S$ & not ram'd in L/K: $\varpi_v \longrightarrow \mathrm{Frob}_v$, which generates $G(L_w/K_v)$

so $r_{L/K}$ is surjective.