Čebotarev.

Cor. of existence thm. For any number field, r_K induces an iso.

$$\mathbb{A}_K \Big/ \overline{K^\times (K_\infty^\times)^\circ} \xrightarrow{\sim} G_K^{ab}$$

Pf. Last time (using rec) showed surj. To identify $\ker(r_K) = \bigcap_{\substack{\text{fin. ab.}\\ L/K}} \ker(r_{L/K})$. Note this $\bigcap$ contains $\overline{K^\times (K_\infty^\times)^\circ}$ and we have to check that $\overline{K^\times (K_\infty^\times)^\circ}$ is equal to the $\bigcap$ of open s/g's containing it.

$$\overset{\text{(Existence thm)}}{=} \bigcap_{\substack{U \text{ open}\\ \text{s/g of } \mathbb{A}_K^\times}} K^\times U$$

To see this, note that

$$\mathbb{A}_K^\times \Big/ \overline{K^\times (K_\infty^\times)^\circ} \xleftarrow{\sim} \mathbb{A}_K^{\infty,\times} \Big/ \overline{K_+^\times}$$

away from ∞

$$K_+^\times = \{ x \in K^\times \mid \iota(x) > 0 \ \forall \iota : K \hookrightarrow \mathbb{R} \}$$

Since $\mathbb{A}_K^{\infty,\times}$ is locally profinite, $\overline{K_+^\times} = \bigcap$ open s/g containing it, so same holds for $\overline{K^\times (K_\infty^\times)^\circ} \subset \mathbb{A}_K^\times$.

Rmk. once K has interesting units ($K \neq \mathbb{Q}$, quad. imag.), this quotient is hard to understand explicitly.

Abelian L-functions.

Goals: - prove 2nd ineq. $|C_K/NC_L| \leq [L:K]$

- Čebotarev density thm.

Recall $\zeta(s) = \sum_{n \geq 1} n^{-s} \underset{Re(s)>1}{=} \prod_p (1-p^{-s})^{-1}$

abs & uniform cpt in $Re(s) > 1$ so hol.

$\zeta(s)$ admits meromorphic con't w/ simple pole at $s=1$.

Let $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ be a character. Define

$$L(\chi, s) = \sum_{n \geq 1} \chi(n) n^{-s}, \quad \chi(n) = 0 \text{ if } (n,m) > 1.$$

$$= \prod_p (1 - \chi(p) p^{-s})^{-1}$$

If χ is $\neq 1$, $L(\chi, s)$ admits analytic continuation. and $L(\chi, 1) \neq 0$.

comes from. Dedekind zeta function of $K = \mathbb{Q}(\zeta_m)$,

$$\zeta_K(s) = \prod_{\substack{\text{integral} \\ \text{prime ideals}}} (1 - N(\mathfrak{p})^{-s})^{-1}$$

mer cont. w/ simple pole at 1.

Note. $\zeta_K(s) = \prod_{\chi} L(\chi, s)$

Consequence. $\chi \neq 1 \Rightarrow L(\chi, 1) \neq 0$ (both sides have simple pole at $s=1$, and $\zeta(s)$ simple pole @ 1, $L(\chi \neq 1, s)$ hol @ 1.

Cor of $L(\chi,1) \neq 0$: $\forall (a,m)=1$, there are ∞'ly many primes $\equiv a \pmod m$.

Def'n. K - # field. X: any set of finite places of K.

Dirichlet density:

(if it exists) $\delta(X) = \lim_{s \to 1^+} \frac{-L_X(s)}{\log(s-1)}$ | The upper density $\overline{\delta}(X)$ is $\limsup_{s \to 1^+}$

$L_X(s) = \sum_{v \in X} q_v^{-s}$

$q_v = \# k_v$

Basic properties.

- if $\delta(X)$ exists, then $\delta(X) = \overline{\delta}(X)$.
- $X \subset Y \Rightarrow \delta(X) \leq \delta(Y)$ and $\overline{\delta}$
- $|X| < \infty \Rightarrow \delta(X)$ exists $\delta = 0$

- $X \cap Y = \phi \Rightarrow \bar{\delta}(X \cup Y) \leq \bar{\delta}(X) + \bar{\delta}(Y)$, and

<u>More interesting</u>: $\delta(\{v \text{ s.t. } q_v \text{ is not prime}\}) = 0$

<u>Pf</u>. $L_X(s) = \sum_{v \in X} q_v^{-s} \leq [K:\mathbb{Q}] \underbrace{\sum_p p^{-2s}}_{\text{holomorphic at } Re(s) > 1/2,}$ (measures split primes)

so $\delta(X) = 0$.

Fix L/K finite. We know C_K/NC_L is finite, so all its chars are finite order. Fix a char $\chi : C_K/NC_L \to \mathbb{C}$ (analogue of Dir. char)

Let S be a fin set of primes containing ∞ & ramified primes for χ. Define

$$L^S(\chi, s) = \prod_{v \notin S} (1 - \chi(\varpi_v)\, q_v^{-s})^{-1}$$

<u>Thm</u>. $L^S(\chi, s)$ has meromorphic con't to $Re(s) > 1 - \varepsilon$. (some ε, e.g. $\frac{1}{[K:\mathbb{Q}]}$). If $\chi \neq 1$, then it is analytic in this domain and if $\chi = 1$, it has a simple pole at $s=1$.

<u>Consequences</u>. ① $\delta(\{\text{all places of } K\}) = 1$.

<u>Pf</u>. Take $\chi = 1$, $S = \{\infty\}$. Then $L^{\infty}(1, s) = \prod_{v \nmid \infty} (1 - q_v^{-s})^{-1}$

$$= \frac{1}{s-1} \underbrace{f(s)}_{\substack{\text{hol. and} \\ \text{non-vanishing} \\ \text{in nbhd of } 1.}}$$

$\Rightarrow \log(s-1) + \log f(s)$

$= \sum -\log(1 - q_v^{-s})$

$= \sum_v q_v^{-s} + \sum_v \sum_{n \geq 2} q_v^{-ns}/n$. Divide by $-\log(s-1)$

$\longrightarrow \quad 1 + \frac{\text{hol}}{\log(s-1)} = \frac{\sum_v q_v^{-s}}{-\log(s-1)}$. $\lim_{s \to 1^+}$ to get the claim. $\blacksquare$

② Let L/K be fin. ext. Then $\overline{\delta}(v \text{ of } K \mid \varpi_v \in K^\times N_{L/K} \mathbb{A}_L^\times)$

$$\geq \frac{1}{[L:K]}$$

(So, many primes are norms — i.e. split).

<u>Pf</u>. This $\overline{\delta}(\quad) \geq \overline{\delta}(v \mid q_v \text{ is prime and } \varpi_v \in N\mathbb{A}_L^\times)$

$\geq \overline{\delta}(v \mid \exists w \mid v \text{ unramified w/ } q_w \text{ prime and } \varpi_v \in N\mathbb{A}_L^\times)$

$= \overline{\delta}(v \mid \exists w \mid v \text{ unram w/ } q_w \text{ prime})$

Now $\displaystyle\sum_{\substack{v \mid \exists w \mid v \\ \text{unr w/ } q_w \\ \text{prime}}} q_v^{-s} \geq \frac{1}{[L:K]} \underbrace{\sum_{\substack{w \text{ of } L \\ \text{unr. } \cancel{/K} \\ \text{w/ } q_w \text{ prime}}} q_w^{-s}}_{\substack{\text{by (1), this set of} \\ \text{primes has } \delta=1.}}$

Th'm. L/K fin. ext. (1) 2nd ineq. $|C_K/NC_L| \leq [L:K]$.

(2) If $\chi : C_K/NC_L \to \mathbb{C}^\times$ is $\neq 1$, then $L^S(\chi,1) \neq 0$.

(3) Abelian Čebotarev. Let $c \in C_K/NC_L$. Then $\delta(v \text{ of } K \mid \varpi_v \in c) = \dfrac{1}{|C_K/NC_L|}$

Pf. Let $S = \{v \mid \infty\} \cup \{v \text{ ram'd in } L/K\} \subset$ places of K.

Let $m_\chi = \operatorname{ord}_{s=1} L^S(\chi, s)$ so our th'm says

$$m_\chi = \begin{cases} 0 & \text{if } \chi \neq 1 \\ 1 & \text{, else} \end{cases}$$

Write $L^S(\chi,s) = (s-1)^{m_\chi} \cdot \underbrace{g_\chi(s)}_{\substack{\text{hol \& non-vanishing} \\ \text{at nbhd of } 1}}$

Let $c \in C_K/NC_L$, and consider

$$-\sum_\chi \chi(c)^{-1} \log \underset{\shortparallel}{L^S(\chi,s)} = \sum_\chi \chi(c)^{-1} \sum_v \log(1-\chi(\varpi_v)q_v^{-s})$$

$\prod (1-\chi(\varpi_v) q_v^{-s})^{-1}$

$$\underset{\substack{\text{(log, def}\\ \text{of } m_\chi)}}{=} \sum_\chi \chi(c)^{-1} m_\chi (-\log(s-1)) + \left(\begin{matrix}\text{hol in nbhd}\\ \text{of } 1\end{matrix}\right)$$

$$\underset{\text{(Taylor log)}}{=} -\sum_\chi \chi(c) \sum_v \sum_{n\geq 1} \frac{\chi(\varpi_v)^n q_v^{-ns}}{n}$$

$$= -\sum_v \sum_\chi \chi(\varpi_v c^{-1}) q_v^{-s} + \left(\text{hol in } \mathrm{Re}(s) > \tfrac{1}{2}\right)$$

$n \geq 2$ terms

$$\underset{(\sum_\chi)}{=} -|C_K/NC_L| \sum_{v:\, \varpi_v \in c} q_v^{-s} + \left(\text{hol in } \mathrm{Re}(s) > \tfrac{1}{2}\right)$$

Combine.

$$-\sum_\chi \chi(c)^{-1} m_\chi = \frac{|C_K/NC_L| \sum_{v:\varpi_v \in c} q_v^{-s}}{-\log(s-1)} + \frac{\text{hol near } 1}{\log(s-1)}$$

Take $\lim_{s\to 1^+}$:

$$\boxed{-\sum_\chi \chi(c)^{-1} m_\chi = |C_K/NC_L| \cdot \delta(v : \varpi_v \in c)}.$$

Take $c = 1$:

$$1 - \sum_{\chi \neq 1} m_\chi = |C_K/NC_L|\, \delta(v : \varpi_v = 1)$$

RHS > 0, and all $m_\chi \in \mathbb{Z}_{\geq 0}$ ($\chi \neq 1$) $\Rightarrow$ all $m_\chi = 0$ for $\chi \neq 1$. (2)

So for any c, we get $1 = |C_K/NC_L| \cdot \delta(v : \varpi_v \in c)$

(3).

Going back to $c=1$, $\delta(v : \varpi_v = 1) \geq \frac{1}{[L:K]}$ and so $|C_K/NC_L| \leq [L:K]$. (1). this finishes all proofs of CFT.