Last time: finished for L/K fin. ab.,

$$r_K : C_K/NC_L \simeq G(L/K)$$

and showed

$$0 \to Br(K) \to \bigoplus_v Br(K_v) \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0$$

is exact except (not yet) in the middle – but saw it is complete.

To finish this, need a lemma from group coho.:

Lemma. L/K cyclic & injective $\chi : G(L/K) \to \mathbb{Q}/\mathbb{Z}$.

$\rightsquigarrow \delta\chi \in H^2(G(L/K), \mathbb{Z})$. Then

$\cup \delta\chi$ is an iso $\underbrace{K^\times/N(L^\times)}_{H^0(G, L^\times)} \xrightarrow{\sim} H^2(G(L/K), L^\times)$

Lemma. G fin. cyclic group & a γ generator of $H^2(G, \mathbb{Z})$.

Then $\cup\gamma : H^0(M)/N(M) \xrightarrow{\sim} H^2(G, M)$.

Pf. Have two sequences of G-modules that are split as $\mathbb{Z}$-mod:

$G = \langle \sigma \rangle$

$$0 \to I_G \to \mathbb{Z}[G] \xrightarrow{\text{avg}} \mathbb{Z} \to 0$$

$$1 \longmapsto N$$

$$0 \to \mathbb{Z} \to \mathbb{Z}[G] \xrightarrow{\sigma - 1} I_G \to 0.$$

B/c split over $\mathbb{Z}$, still get SES after $\otimes_{\mathbb{Z}} M$ for any G-mod M.

$$\underset{\forall M}{\rightsquigarrow} \quad \underbrace{H^0(M) \xrightarrow{\delta} H^1(G, M \otimes I_G)}_{\substack{\text{avg}\,(\mathbb{Z}[G] \otimes M) \\ H^0(G,M)/N(M)}} \longrightarrow H^2(G,M)$$

Want $H^0(M)/N(M) \xrightarrow{\cup \delta} H^2(G,M)$ is an iso.

We can take $\gamma = \delta\delta(\gamma_0)$ for γ_0 a gen. of $\underset{\mathbb{Z}/|G|\mathbb{Z}}{H^0(G,\mathbb{Z})/N(\mathbb{Z})}$. The map $m \longrightarrow m \cup \gamma = m \cup \delta\delta\gamma_0$
$= \delta\delta(m \cup \gamma_0)$.

Since $\delta\delta$ is an iso $\forall M$, reduced to checking that $m \longrightarrow m \cup \gamma_0$ is an automorphism of $H^0(G,M)/NM$.

So $\cup\gamma_0$ is mult. by an elem of $\mathbb{Z}$ coprime to $|G|$. ∎

<u>Next</u>) 1) Finish proving exactness of Brauer seq.

2) Prove existence th'm.

3) L-fn arg to get 2^{nd} equality + Čebotarev density th'm.

4) Global duality.

Th'm. The sequence

$$0 \to Br(K) \to \bigoplus_v Br(K_v) \xrightarrow{\sum inv_v} \mathbb{Q}/\mathbb{Z} \to 0$$

e.g. $H = \left(\frac{-1,-1}{\mathbb{Q}}\right)$

$\mathrel{\rotatebox{90}{$\in$}}$ $Br(\mathbb{Q})[2]$ s.t. $inv_v(res_v(H)) = \begin{cases} 0 & \neq 2,\infty \\ \frac{1}{2}, & v = 2, \infty \end{cases}$

② For all fin. Galois L/K, the sequence

$$0 \to Br(L/K) \to \bigoplus Br(L_w/K_v) \xrightarrow{\sum inv_v} \frac{1}{n_0}\mathbb{Z}/\mathbb{Z} \to 0$$

where $n_0 = \mathrm{lcm}(\{[L_w : K_v]\})$, n_0 need not $= [L:K]$.

e.g. $L = \mathbb{Q}(\sqrt{13}, \sqrt{17})$ $\quad n_0 = 2$, $[L:K] = 4$.

$\quad |$

$K = \mathbb{Q}$

Pf. L/K fin. Galois. Take LES assoc'd to

$$1 \to L^\times \to \mathbb{A}_L^\times \to C_L \to 1$$

surj in general: let H^2 be the image.

$$\rightsquigarrow \quad 0 \to H^2(G, L^\times) \longrightarrow H^2(G, \mathbb{A}_L^\times) \longrightarrow H^2(G, C_L)$$

$H^2(G, L^\times) = Br(L/K)$

$H^2(G, \mathbb{A}_L^\times) = \bigoplus_v Br(L_w/K_v)$

$\downarrow \sum inv_v$. surj. by def'n and local theory

$\frac{1}{n_0}\mathbb{Z}/\mathbb{Z}$

Since top row is exact & $\to \to \to$ is a complex,

we get well-defined

$$\sum \mathrm{inv}_v : \overline{H^2} \longrightarrow \frac{1}{n_0}\mathbb{Z}/\mathbb{Z}.$$

Recall for general L/K, we have $|H^2(G, C_L)| \leq [L:K]$

For L/K cyclic, $=$ holds.

(*) If $n_0 = [L:K]$, then above $\underline{\text{inv}}$ $: H^2 \longrightarrow \frac{1}{n_0}\mathbb{Z}/\mathbb{Z}$

$\twoheadrightarrow \frac{1}{n_0}\mathbb{Z}/\mathbb{Z}$, imply

$$\overline{H^2} = H^2(G, C_L)$$
$$\downarrow$$
$$\frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}.$$

Then get ES

$$0 \to \mathrm{Br}(L/K) \to \bigoplus \mathrm{Br}(L^v/K^v) \to \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z} \to 0$$

This condition (*) holds when

L/K is cyclic, so (2) of Thm true for L/K cyclic.

Use this to prove (1) of Thm (& later to do

(2) in general).

Let $\mathbb{Q}_\infty/\mathbb{Q}$ be the $\hat{\mathbb{Z}}$ extension inside $\mathbb{Q}^{ab}/\hat{\mathbb{Z}}^\times$.

Consider $\Omega = \mathbb{Q}_\infty K$. $G(\Omega/K)$ is $\simeq$ to $\hat{\mathbb{Z}}$ (open

s/g of $\hat{\mathbb{Z}} = G(\mathbb{Q}_\infty/\mathbb{Q})$) $\forall n$, have cyclic

$$\begin{array}{c}\Omega\\ |\\ \Omega_n\\ |\\ k\end{array}$$

Take $\varinjlim_n$ of (② for Ω_n/k):

<u>Exact</u>. $0 \to Br(\Omega/k) \to \bigoplus_v Br(\Omega_n/k_v) \xrightarrow{\sum inv_v} \mathbb{Q}/\mathbb{Z} \to 0$

Our proof of rec. law showed Ω_k is big enough to split all Brauer classes, i.e.

$$Br(\Omega/k) \hookrightarrow Br(k) \text{ is surjective,}$$

hence an iso. (ditto locally). So this exact sequence $\xrightarrow[inf]{\sim}$ the (exact) seq

$$0 \to Br(k) \to \bigoplus_v Br(k_v) \xrightarrow{\sum inv_v} \mathbb{Q}/\mathbb{Z} \to 0$$

(i.e. ① of Thm is true)

<u>Pf of ② in general</u>. Let L/k be fin. Galois.

Take

$$\begin{tikzcd}
0 \arrow[r] & Br(k) \arrow[r] \arrow[d, "res"'] & \bigoplus_v Br(k_v) \arrow[r] \arrow[d, "\oplus res"] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, "{[L:k] = \sum_{w|v}[L_w:k_v]}"] & 0 \\
0 \arrow[r] & Br(L) \arrow[r] & \bigoplus Br(L_w) \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

for each v

This last map is since

$$\begin{tikzcd}
Br(K_v) \arrow[r] & inv_v(x_v) \\
x_v \arrow[u, phantom, "\in" sloped] \arrow[d] & \sum\limits_{w|v} [L_w : K_v]\, inv_v(x_v) \\
(res_{w/v}(x_v))_{w|v} \arrow[r] & \sum\limits_{w|v} inv(res_{w/v}\, x_w) \arrow[u, phantom, "\shortparallel"]
\end{tikzcd}$$

Now snake lemma gives ② in general.

$$\Downarrow$$

$$0 \to Br(L/K) \to \bigoplus_v Br(L_w/K_v) \to \underbrace{\tfrac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}}_{} \to coker$$

image of this is $\frac{1}{n_0}\mathbb{Z}/\mathbb{Z}$.

Rmk. this shows that when $n_0 \neq [L:K]$, the map $H^2(G, \mathbb{A}_L^\times) \to H^2(G, C_L)$ is not surjective.

(in particular, $H^3(G, L^\times) \neq 0$ in these cases).

However, $H^3(G_K, \bar{K}^\times) = 0$, which is important for arithmetic duality.

Let $X = \operatorname{Spec} \mathcal{O}_K \hookleftarrow \operatorname{Spec} K$. Div. sequence:

$$0 \to \mathbb{G}_{m,X} \to j_* \mathbb{G}_{m, \operatorname{Spec} K} \to \bigoplus_{v \text{ finite prime}} \mathbb{Z}|_v \to 0$$

Onto existence thm.

Thm. For every open s/g $U \subset C_K$ of finite index, there is a unique fin. ab. ext. L/K s.t. $U = N_{L/K} C_L$. (L is the "class field" associated to U, e.g.

$$U = \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times \times K_\infty^\times \rightsquigarrow L \text{ is Hilbert class field}$$

(maximal abelian unramified ext of K & split at ∞))

This Galois corresp. $L \longleftrightarrow N(C_L)$ satisfies:

- $L_1 \supset L_2 \iff N C_{L_1} \subset N C_{L_2}$
- $N(C_{L_1}) \cap N(C_{L_2}) = N(C_{L_1 L_2})$
- $N(C_{L_1 \cap L_2}) = N(C_{L_1}) N(C_{L_2})$

(recall: follows formally from rec. law)

Def. Call $U \subset C_K$ a <u>norm subgroup</u> if $U = N_{L/K} C_L$ for L/K fin. abelian. Eventually prove norm limitation thm in global setting.

Except for $K = \mathbb{Q}$, quad. imag., explicit description of K^{ab} not known.