We're after: p odd prime. $2 \leq j \leq p-3$ even.

Then $\underbrace{p \mid \zeta(1-j)}_{\text{same as } p \mid B_j} \Rightarrow Cl(\mathbb{Q}(\zeta_p))_p(\omega^{\overset{1-j}{\shortparallel}}_{p-j}) \neq 0$ "isotypic component"

$\circledast$ We've talked <u>existence</u> of <u>irreducible</u> ℓ-adic Galois rep'ns associated to cuspidal Hecke eigenforms.

(for $k \geq 2$, Deligne's thm. $k=1$ (Deligne-Serre))

$\circledast$ We gave eg's of Eisenstein series $\overset{CFT}{\rightsquigarrow}$ irreducible ℓ-adic Gal rep'ns

e.g. $$E_k^*(z) = \frac{\zeta(1-k)}{2} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n$$

$$T_p E_k^* = \underbrace{(1+p^{k-1})}_{tr(Fr_p \mid \mathbb{1}_\ell \oplus \kappa_\ell^{k-1})} E_k^*$$

$\mathbb{1}_\ell \oplus \kappa_\ell^{k-1}$: so this is the Gal rep assoc'd to E_k

This matching $tr(\rho(Fr_p)) \longleftrightarrow T_p$-e-value uniquely characterizes ρ^{ss} — use Čebotarev for $\rho: G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{Q}_\ell})$

Suppose $p \mid \zeta(1-k)_{(k=j)}$

<u>e.g.</u> $k=12$. $p = 691 \mid \zeta(1-12)$

The way the pf works, need a $p=691$ congruence between

1) E_{12}^*

2) Hecke eigenform $\in S_{12}(SL_2(\mathbb{Z}))$

$\exists$ a basis el't $\Delta \in S_{12}(\Gamma)$ ("Ramanujan Δ-function")

$$\Delta(z) = \sum_{n\geq 1} \tau(n) q^n \;;\; \tau(1)=1.$$

Ramanujan <u>conjectured</u>: $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$.

This $\equiv$ is equiv. to $\Delta \equiv E_{12}^* \bmod 691$.

| $E_k := \frac{2}{\zeta(1-k)} E_k^*$.

Δ is in fact given by: $\Delta = \frac{E_4^3 - E_6^2}{1728}$

| By an index card in

$$E_{12}^* = \frac{1}{65,520}(441 E_4^3 + 250 E_6^2)$$

Then $\Delta \equiv E_{12}^* \ (691)$

$\Updownarrow$

$1728 \cdot 441 \equiv 65,520 \ (691)$

$(\& \ 250 \equiv -441 \ (691))$

<u>Fact</u>. $M_*(\Gamma) = \mathbb{C}[E_4, E_6]$

Sketch of general arg.

$$p \mid \zeta(1-k) \Rightarrow \exists\ \bar{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{F}}_p)$$

s.t. 1) $\bar{\rho}$ unram'd outside p

2) $\bar{\rho}$ is a non-split ext.

$$0 \to \overline{\mathbb{F}}_p(\bar{\kappa}^{k-1}) \to \bar{\rho} \to \mathbb{1} \to 0 \quad (*)$$

3) $(*)|_{G_{\mathbb{Q}_p}}$ splits.

Step 1. $p \mid \zeta(1-k)$, so $E_k^* = \sum_{n \geq 1} \sigma_{k-1}(n) q^n \mod p$.

Then $\exists\ f' \in S_k(\Gamma)$ s.t.

$$f' \equiv E_k^* \mod p \quad (\text{i.e. } = \text{ of Fourier expansions})$$

Step 2. In ①, we can replace f' w/ a Hecke eigenform $f \in S_k(\Gamma)$

Step 3. From ②, we have $\rho_{f,\mathfrak{p}} : G_{\mathbb{Q}} \to GL_2(E_{f,\mathfrak{p}})$

$\mathfrak{p} \mid p$ lies above in field gen'd by Fourier coeff's

unram'd outside p and irreducible (absolutely)

and $(\bar{\rho}_{f,\mathfrak{p}})^{ss} \simeq \bar{\rho}_{E_k^*,p} = \mathbb{1} \oplus \bar{\kappa}^{k-1}$

Step ④. (Ribet's lattice lemma) For such an irr $\rho_{f,\mathfrak{p}}$,
$\exists\ \mathcal{O}_{f,\mathfrak{p}}$-lattice L stable under $G_{\mathbb{Q}}$, s.t.

$$\underbrace{L/\mathfrak{p}L}_{\text{our } \bar{\rho}} \simeq \begin{pmatrix} 1 & * \\ & \bar{\kappa}^{k-1} \end{pmatrix} \text{ w/ } * \text{ non-split.}$$

this $\bar{\rho}$ satisfies 1) and 2).

<u>Step 5</u>. (show $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is split). Use Thm (Wiles)

$$\underbrace{\rho_{f,\beta}}_{\text{char } 0} \Big|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \kappa^{k-1} & * \\ & 1 \end{pmatrix}$$

<u>Conclusion</u>. ④ + ⑤ $\Rightarrow$ $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ splits since it has stable lines on which $G_{\mathbb{Q}_p}$ acts, by

$\mathbb{1}$ & κ^{k-1}.

<u>Some details</u>.

<u>Step 1</u>. need to find $h \in M_k(\Gamma)$ w/ $a_0(h) = 1$ and all $a_n(h) \in \overline{\mathbb{Z}_p}$, s.t. $\underbrace{E_{12}^* - \frac{\zeta(1-k)}{2} h}_{\substack{\text{would give a non-zero} \\ \text{el't of } S_k(\Gamma) \text{ s.t.} \\ f' \equiv E_k^* \pmod p}} \not\equiv 0.$

We've done case $k = 12$. Assume $k \geq 16$ (all smaller k the th'm is trivial).

$\forall k \geq 16$, k is expressible in more than one way as $4a + 6b = k$. Take $h = E_4^a \cdot E_6^b$. At least one choice of (a,b) gives $E_k^* - \frac{\zeta(1-k)}{2} E_4^a \cdot E_6^b \not\equiv 0$: if no choice of (a,b) made this $\not\equiv 0$, would have linear dependence between $E_4^{a_1} E_6^{b_1}$ & $E_4^{a_2} E_6^{b_2}$ for

$(a_1, b_1) \neq (a_2, b_2)$.

More generally, the forms $\{E_4^a E_6^b\}_{4a+6b=k}$ are lin. indep. over $\mathbb{C}$. Why. Suppose

$$\sum_i c_i E_4^{a_i} E_6^{b_i} = 0$$

some $a_i, b_i = a_-, b_+$, w/ b_+ maximal.

$$\rightsquigarrow \sum_i c_i \frac{E_4^{a_i - a_-}}{E^{b_i - b_+}} = 0$$

Observe: $\forall i, j$, $4a_i + 6b_i = 4a_j + 6b_j$

$\Rightarrow 2(a_i - a_j) = 3(b_j - b_i)$

$$\| \qquad \sum c_i \left(\frac{E_4^3}{E_6^2}\right)^{d_i}$$

$\Rightarrow$ either all $c_i = 0$ or $\underbrace{\frac{E_4^3}{E_6^2} \in \mathbb{C}}_{\text{False.}}$ since it satisfies a poly $/\mathbb{C}$.

Step 2. Replacing f' by eigenform (cuspidal f).

is content of "Deligne - Serre lifting ~~theorem~~ lemma"

Step 3 ✓

(4) Lattice lemma: Let $E/\mathbb{Q}_p$ fin. $\mathcal{O} = \mathcal{O}_E$, G cpct gp. $\rho: G \to GL(V)$ (E-lin cts. rep), on 2-dim'l E-vect V. Assume ρ is irr but.

$\bar{\rho}^{ss} \simeq \chi_1 \oplus \chi_2$ for characters $\chi_i : G \to (\mathcal{O}/\varpi)^\times$.

Then $\exists$ a G-stable $\mathcal{O}$-lattice $L \subset V$ s.t.

$$L/\varpi L \simeq \begin{pmatrix} \chi_1 & * \\ & \chi_2 \end{pmatrix} \text{ w/ } * \text{ non-split.}$$

Pf. A matrix calculation: set $P = \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix}$,

$$\begin{pmatrix} 1 & \\ & \varpi \end{pmatrix} \begin{pmatrix} a & \varpi b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \\ & \varpi^{-1} \end{pmatrix} = \begin{pmatrix} a & b \\ \varpi c & d \end{pmatrix}$$

By assumption, a reduction of any G-stable lattice has the form (in some basis)

$$\begin{pmatrix} \chi_1 & * \\ & \chi_2 \end{pmatrix} \text{ or } \begin{pmatrix} \chi_1 & 0 \\ * & \chi_2 \end{pmatrix}.$$

By we can always find some lattice w/ red. of the form . Assume that EVERY G-stable $\mathcal{O}$-lattice w/ red'n of form

$$0 \to \chi_1 \to L/\varpi L \to \chi_2 \to 0 \quad \text{is}$$

split.

Fix a basis of L

Aim. Construct a seq. $M_i = \begin{pmatrix} 1 & t_i \\ & 1 \end{pmatrix}$ s.t.

$$M_i \rho(G) M_i^{-1} \subset \begin{pmatrix} * & \varpi^i \\ \varpi & * \end{pmatrix}. \text{ Then}$$

$$M = \lim M_i \text{ exists } \& \ M \rho(G) M^{-1} \subset \begin{pmatrix} * & 0 \\ * & * \end{pmatrix},$$

contradicting irreducibility of $\bar{\rho}$. Find M_i's by induction.

$\underline{i=1}$. our assumption that we have a given split $L/\varpi L$.

Assume known for step i, i.e. $\exists M_i$ s.t. $M_i \rho(G) M_i^{-1} \subset \begin{pmatrix} * & \varpi^i \\ \varpi & * \end{pmatrix}$.

relevant to thesis?

Then $P^i M_i \rho(G) M_i^{-1} P^{-i} \subset \begin{pmatrix} * & * \\ \varpi^{i+1} & * \end{pmatrix}$. Reducing mod ϖ, this has the form $\begin{pmatrix} \chi_1 & * \\ & \chi_2 \end{pmatrix}$. We have assumed in this case, $*$ always splits, so $\exists U = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}$ s.t.

$$U P^i M_i \rho(G) M_i^{-1} P^{-i} U^{-1} \subset \begin{pmatrix} * & \varpi \\ \varpi^{i+1} & * \end{pmatrix}$$

Set $M_{i+1} = P^{-i} U P^i M_i$