G profinite group

$\mathcal{M}_G$ = discrete G-module (= M s.t. $m \in M$ has <u>open</u> stab)

- ab. category w/ enough injectives
- left-exact $\mathcal{M}_G \longrightarrow Ab \overset{\text{derived functors}}{\rightsquigarrow} H^i(G,M)$

  $m \longrightarrow m^G$
- important construction: if $H \leq G$ is closed s/g,

$$M \in \mathcal{M}_H \rightsquigarrow \mathrm{Ind}_H^G(M) \overset{\{f: G \to M \mid f \text{ is loc. const}, \ f(hg) = h f(g) \ \forall h \in H\}}{=} \dots \in \mathcal{M}_G$$

(Ind_H^G is exact, adjoint to forgetful)

<u>Constructions</u> relating $H^i(G,M)$ for varying G.

<u>Last time</u> 1 (Shapiro) $H^i(H, \underset{\mathcal{M}_H}{\underset{\cap}{M}}) \xrightarrow{\sim} H^i(G, \mathrm{Ind}_H^G M)$

$$i=0: \quad M^H \longrightarrow (\mathrm{Ind}_H^G M)^G$$
$$\quad\ \ m \longrightarrow (g \longrightarrow m)$$

2 <u>Restriction</u>. $H \leq G$ closed

s/g $\rightsquigarrow$ natural transformations (of δ-functors)

$$H^i(G,M) \longrightarrow H^i(H,M).$$

extending $(i=0)$ $M^G \subset M^H$

since $(H^i(G,\cdot))_{i\geq 0}$ is a universal δ-functor $\mathcal{M}_G \to Ab$, & $(H^i(H,\cdot))_{i \geq 0}$ is a δ-functor $\mathcal{M}_G \to \mathcal{M}_H \to Ab$, we get ! res $H^i(G,M) \longrightarrow H^i(H,M)$. (Topologically, pullback is pullback under $BH \to BG$).

③ Corestriction/transfer/norm. For $H \leq G$ open s/g ($\Rightarrow$ fin. index), have $H^0(H, \cdot) \longrightarrow H^0(G, \cdot)$ cuz G compact

$$m \longmapsto \sum_{g \in \bar{g} \in G/H} g \cdot m$$

(well-defined since $m \in M^H$).

Claim. $(H^i(H, \cdot))_{i \geq 0} : \mathcal{M}_G \to Ab$ is a universal δ-functor.

Granted claim, we get transformations on $\mathcal{M}_G$

$$\mathrm{cor} : H^i(H, M) \longrightarrow H^i(G, M)$$

To check the claim; as before, suffices to check each $H^i(H, -)$, iso., on $\mathcal{M}_G$ is effaceable. Let $M \in \mathcal{M}_G$, $\exists\, M \underset{\text{in } Ab}{\hookrightarrow} I$ w/ I injective abelian grp.

$$\rightsquigarrow \quad M \hookrightarrow \mathrm{Ind}_1^G(I) \qquad \underline{\text{Claim}}\ H^i(H, \mathrm{Ind}_1^G I) = 0\ \forall i > 0.$$

$$m \longrightarrow (g \to gm)$$

This follows from decomposing

$$\mathrm{Ind}_1^G(I)\big|_H \xrightarrow{\sim} \bigoplus_{s \in G/H} \mathrm{Ind}_1^H(I)$$

$$\varphi \longrightarrow (h \to \varphi(sh))_{s \in G/H}.$$

and noting that each $\mathrm{Ind}_1^H(I)$ is an injective H-module (recall Ind preserves injectives).

Lemma. H open s/g of G.

$$\begin{tikzcd} H^i(G,M) \arrow[r, "res"] \arrow[rr, bend right, "\text{mult. by } [G:H]"'] & H^i(H,M) \arrow[r, "cor"] & H^i(G,M) \end{tikzcd}$$

(on $\underline{H^0}$: $m \longmapsto m \longmapsto \sum gm = [G:H]m$

Cor. If G is a finite group, then $\forall$ G-modules M, $|G|$ annihilates $H^i(G,M)$ $\forall i$.

Pf. Use lemma w/ $H = \{e\}$.

$$\begin{tikzcd} H^i(G,M) \arrow[r] \arrow[rr, bend right, "|G|"'] & H^i(\{e\},M) \arrow[r] & H^i(G,M) \end{tikzcd}$$

Other descriptions of res/cor.

(res) H-closed.

$$\begin{tikzcd} H^i(G,M) \arrow[r, "\text{induced by}"] \arrow[dr, "res"'] & H^i(G, \mathrm{Ind}_H^G M) \\ & H^i(H,M) \arrow[u, "\sim"', "\text{shapiro}"] \end{tikzcd}$$

(cor). H open

$$\begin{tikzcd} H^i(H,M) \arrow[r] \arrow[dr] & H^i(G,M) \\ & H^i(G, \mathrm{Ind}_H^G M) \arrow[u, "(tr)"'] \end{tikzcd}$$

$tr : \mathrm{Ind}_H^G M \longrightarrow M$ by

$$\varphi = \sum_{g \in G/H} g^{-1} \cdot \varphi(g)$$

One more. (Inflation) If $H \trianglelefteq G$ is closed & normal, get $H^i(G/H, m) \to H^i(G, m)$ by extending

$$M^{G/H} = \underbrace{M^G}_{\text{regard as } G\text{-module via } G \twoheadrightarrow G/H.}$$

If we can start by a G-mod M, then get

$$H^i(G/H, m^H) \xrightarrow{\text{inf}} H^i(G, M).$$

Hoschild-Serre Spectral Sequence

Point. Analyze $H^*(G, M)$ in terms of $H^i(H, m)$, $H^i(H, \cdot)$, and $H^i(G/H, \cdot)$.

Thm. There ($M \in \mathcal{M}_G$) is a spectral sequence

$$E_2^{p,q} := H^p(G/H, \underbrace{H^q(H, m)}_{\text{a } G/H\text{-module via } g \in G/H \to \text{mg}\ (m \to gm)}) \Longrightarrow H^{p+q}(G, M).$$

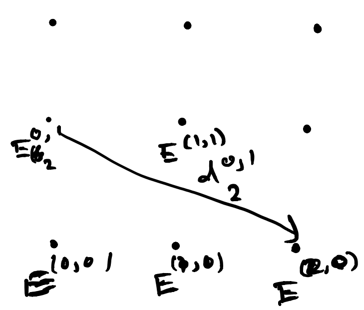
This extends to G/H-action on $H^q(H, m)$ using that $H^q(H, \cdot) : \mathcal{M}_G \to Ab$ is universal.

We checked this for H fin. index, general case reduces to this, by a tool we'll see next time).

Note! H acts trivially on $H^*(H,M)$.

Review of spectral sequences: Have abelian groups,

$E_2^{p,q}$ w/ diff's $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$



Take coh: $$E_3^{p,q} = \frac{\ker(d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1})}{\mathrm{im}(d_2^{p,q} : E_2^{p-2,q+1} \to E_2^{p,q})}.$$

~~Now take spec seq is~~. These come w/ diff'ls,

$$d_3^{p,q} : E_3^{p,q} \to E_3^{p+3,q-2}$$

$$\vdots$$

$$d_n^{p,q} : E_n^{p,q} \to E_n^{p+n,q-n+1}$$

Convergence of the spec. sequence to $H^{p+q}(G,M)$ ($E^{p+q} :=$)

means

1) $E_n^{p,q}$ stabilizes for $n \gg |G|$.?

Define $E_\infty^{p,q}$ as this stable value.

E^{p+q} is filtered $E^{p+q} = \mathrm{Fil}^0 E^{p+q} \supset \dots \supset \mathrm{Fil}^{p+q} E_\infty^{p+q}$

s.t. $\mathrm{gr}^p E^{p+q} \simeq E_\infty^{p,q}$.

$$\parallel$$

$$\mathrm{Fil}^p/\mathrm{Fil}^{p+1}$$

<u>Low degree example</u> (5-term exact sequence)

There is an exact seq:

$$E_\infty^{0,0} = E_2^{0,0}$$

$$E_\infty^{1,0} = E_2^{1,0}$$

$$E_\infty^{0,1} = E_3^{0,1} = \ker(E_2^{0,1} \to E_2^{2,0})$$

$$E_\infty^{2,0} = E_3^{2,0} = \mathrm{coker}(E_2^{0,1} \to E_2^{2,0})$$

The fil. on E^1 is

$$0 \to E_\infty^{1,0} \to E^1 \to E_\infty^{0,1} \to 0$$

i.e.

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to E^2$$

is exact.

<u>Prop.</u> For group coho, the 5-term exact seq. resolves to

$$0 \to H^1(G/H, M^H) \xrightarrow{\text{inf}} H^1(G,M) \xrightarrow{\text{res}} H^1(H,M)^{G/H}$$

$$\xrightarrow[\text{"transgression"}]{} H^2(G/H, M^H) \xrightarrow{\text{inf}} H^2(G,M)$$

Called <u>inflation</u> <u>restriction</u> <u>sequence</u>.

<u>Ex.</u> Suppose $E_2^{p,q} = 0 \ \forall q \geq 1$, $H^n(G,M) \simeq H^n(G/H, M)$.

One way of deriving this spec. sequence, use: Grothendieck-Leray spectral sequence of a composite functor:

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

abelian cat's A, B, C. Assume $\mathcal{A}, \mathcal{B}$ have enough injectives. If $F(\text{injectives in } \mathcal{A}) = G$-acyclic, then there is an E_2 spec. sequence

$$R^p G \circ R^q F(\cdot) \Rightarrow R^{p+q}(G \circ F).$$

Apply this to the composite

$$\begin{tikzcd} \mathcal{M}_G \arrow[r, "(\ )^H"] \arrow[rr, bend right, "(\cdot)^G"'] & \mathcal{M}_{G/H} \arrow[r, "(\ )^{G/H}"] & Ab \end{tikzcd}$$

Another way to construct group coho. Recall can compute derived functors via acyclic resolutions. Standard resolution of a G-mod M: This will be a $M \xrightarrow{q\text{-iso}} X^\bullet(G, M)$ w/ each $X^n(G, M)$ acyclic.

$$X^n(G, M) := Map_{cts}(G^{n+1}, M)$$

$\circlearrowleft$ G-mod via $g \in G \rightsquigarrow (g \cdot \varphi)(g_0, \dots, g_n)$

$$= g \cdot \varphi(g^{-1} g_0, \dots, g^{-1} g_n).$$

Boundary op. $G^n \to M$

$$X^{n-1}(G, M) \xrightarrow{d} X^n(G, M)$$

$$\| $$

$$\sum_{i=0}^{n} (-1)^i d_i^* \quad \text{where } d_i : G^{n+1} \to G^n$$

$$(g_0, \dots, g_n) \to (g_0 \dots \hat{g_i} \dots g_n)$$

<u>Lemma</u>. $M \to X^0(G,M)$ makes $M \to X^\bullet(G,M)$ an
$m \mapsto (g \mapsto m)$

<u>acyclic</u> <u>resolution</u> of M.

<u>Pf</u>. Why acyclic G-mods:

check this is iso of G-mods & acyclicity follows from Shapiro.

$$X^n(G,M) \xrightarrow{\sim} \mathrm{Ind}_1^G(X^{n-1}(G,M)).$$

$$\varphi \longmapsto \Theta_\varphi : G \to X^{n-1}(G,M) \text{ given by}$$

$$\Theta_\varphi(g)(g_1,\dots,g_n) = g\cdot\varphi(g^{-1}, \dots, g^{-1}g_n).$$

$\blacksquare$

So, compute $H^i(G,M)$ as cohom. of the complex

$$\begin{array}{ccccccc} X^0(G,M)^G & \to & X^1(G,M)^G & \to & \cdots & \to & X^n(G,M)^G \\ & & & & & & \downarrow \wr \\ C^0(G,M) & \to & C^1(G,M) & \to & \cdots & \to & C^n(G,M). \end{array}$$

where $C^n(G,M) = \{\varphi : G^{n-1} \to M, \text{ loc. const}\}$.

The iso $C^{n-1}(G,M) \xrightarrow{\sim} X^{n-1}(G,M)^G$

$$\varphi \longmapsto \left[(g_1,\dots,g_n) \mapsto g_1\,\varphi(g_1^{-1}g_2,\dots,g_1^{-1}g_n)\right]$$

The induced boundary map is

$$df(g_1,\dots,g_n) = g_1 f(g_2,\dots,g_n) + \sum_{i=1}^{n-1}(-1)^i f(g_1,\dots,g_{i-1},g_ig_{i+1},\dots g_n) + (-1)^n f(g_1,\dots,g_{n-1}).$$