<u>Recall</u>. aim is Ribet's Thm: For i odd, $3 \le i \le p-2$,

if $p \mid \zeta(1-i)$, then $C(\mathbb{F}^i) \neq 0$.

$\zeta \in \mathbb{Q}$

C = p-part of $Cl(\mathbb{Q}(\zeta_p))$

suffices to find a $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}_p})$ s.t.

1) $\bar{\rho}$ is a non-split extension $0 \to \overline{\mathbb{F}_p}(\mathbb{F}^i) \to \bar{\rho} \to \mathbb{F}_p \to 0$

2) $\bar{\rho}$ is unr. outside p

3) $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is split ($\simeq \mathbb{F}^i \oplus \mathbb{1}$)

For $\Gamma = SL_2(\mathbb{Z})$, Serre's Course in Arithmetic.

Shimura Intro to Arith. Theory of Aut Functions.

Miyake Modular Forms

<u>Hecke Operators</u>

Let $x \in M_2(\mathbb{Z})^+$. Decompose $\Gamma x \Gamma = \coprod_i \Gamma x_i$

and define

$$f|_{[\Gamma x \Gamma]} = \det(x)^{\frac{k}{2}-1} \sum f|_{[x_i]_k}$$

$f \in M_k$

where $f|_{[\sigma]_k} = \det \sigma^{k/2} \cdot j(\sigma, z)^{-k} f(\sigma z)$

<u>Lemma</u> 1) $f|_{[\Gamma x \Gamma]}$ is ind. of choice of reps x_i.

2) $f|_{[\Gamma x \Gamma]} \in M_k(\Gamma)$ and $\in S_k$ if f is.

<u>Pf</u>. 1) if replace x_i by γx_i for $\gamma \in \Gamma$,

then $f|_{[\gamma x_i]_k} = \det(x_i)^{k/2} j(\gamma x_i, z)^{-k} f(\gamma x_i z)$ $\quad = j(\gamma, x_i z)^k f(x_i z) = f|_{[x_i]_k}$.

$$\overset{\shortparallel}{(j(\gamma, x_i z) \bullet j(x_i, z))^{-k}}$$

2) Let $\gamma \in \Gamma$ and consider

$j(\sigma\tau, z) = j(\sigma, \tau z)\, j(\tau, z)$

$$f|_{[\Gamma x \Gamma]}|_{[\gamma]_k} = \sum_i f|_{[x_i]_k}|_{[\gamma]_k} = \sum_i f|_{[x_i \gamma]_k} = f|_{[\Gamma x \Gamma]}$$

But as x_i go thru coset reps, so does $\{x_i \gamma\}$

Typical e.g., $T_p = [\Gamma \begin{pmatrix} 1 & \\ & p \end{pmatrix} \Gamma]$

$$\overset{\shortparallel}{\coprod_{i=0}^{p-1} \Gamma \begin{pmatrix} 1 & i \\ & p \end{pmatrix} \amalg \Gamma \begin{pmatrix} p & \\ & 1 \end{pmatrix}}.$$

$$(T_p f)(z) = p^{k-1}\left(\sum_{i=0}^{p-1} p^{-k} f\left(\frac{z+i}{p}\right) + f(pz)\right)$$

(e.g. use this to see effect on Fourier expansions)

<u>Fact</u>. $\exists$ a mult. on the free $\mathbb{Z}$-module $\mathbb{Z}[[\Gamma x \Gamma]]_x = \mathbb{T}^{\text{abstract}}$ making this into an associative <u>and commutative</u> algebra.

<u>Fact</u>. $\exists$ a basis of $S_k(\Gamma)$ and using $G_{\mathbb{Q}}$ also of $M_k(\Gamma)$, consisting of Hecke eigenforms

eigen $\forall$ elts of $\mathbb{T}^{\text{abstract}}$

<u>Careful in higher levels</u>.

<u>Th'm (Deligne)</u>. Let $f \in S_k(\Gamma)$ be a <u>Hecke eigenform</u>. Write $T_p f = a_p f$ $\left(\text{so } L(f,s) \overset{\text{Hecke}}{=} \prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}\right)$.
$\mathbb{Q} \subset \mathbb{C}$

Then ($\forall\ \overline{\mathbb{Q}} \overset{i}{\hookrightarrow} \overline{\mathbb{Q}}_\ell$), $\exists\ \rho_{f,\ell} : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_\ell)$ s.t.

i) $\rho_{f,\ell}$ is unram'd away from ℓ (b/c $\Gamma = SL_2\mathbb{Z}$) and $\forall p \neq \ell$,

$$\text{charpoly}(\rho_{f,\ell}(\mathrm{Fr}_p)) = X^2 - a_p X + p^{k-1}$$

(independent of ℓ!)

requires D's work on Weil conjectures

ii) all a_p live in fin. ext. E_f of $\mathbb{Q}$,

so can replace "ℓs" w/ $\lambda | \ell$ a place of E_f.

($k=2$, Eichler + Shimura)

(roughly, find $\rho_{f,\ell}$ inside $H^1_{\text{ét}}(\text{mod curve}, \mathrm{Sym}^{k-2}\mathbb{Q}_\ell^2)$)

"Cor." f as in Thm. $\forall \lambda$, $\rho_{f,\lambda}$ is absolutely irreducible.

"pf" — Thm (Faltings) One further knows that $\rho_{f,\lambda}|_{G_{\mathbb{Q}_\ell}}$, $\lambda | \ell$ is Hodge-Tate (even de Rham). Suppose $\rho_{f,\lambda}$ is reducible, so

$$\sim \begin{pmatrix} \chi_1 & * \\ & \chi_2 \end{pmatrix} \text{ for } \chi_1, \chi_2 : G_{\mathbb{Q}} \to \mathbb{Q}_\ell^\times$$

Fact: $\rho_{f,\lambda}|_{G_{\mathbb{Q}_\ell}}$ Hodge-Tate $\underset{\text{(Thm of Tate)}}{\Longrightarrow}$ $\chi_i|_{G_{\mathbb{Q}_\ell}}$ is "locally algebraic"

(i.e. $\chi_i : G_{\mathbb{Q}_\ell}^{ab} \longrightarrow \overline{\mathbb{Q}_\ell}^\times$

$$\begin{tikzcd} G_{\mathbb{Q}_\ell}^{ab} \arrow[r] & \overline{\mathbb{Q}_\ell}^\times \\ I_{\mathbb{Q}_\ell} \arrow[u, "\text{"ab"}"] \arrow[r, "\sim", "\text{LCFT}"'] & \mathbb{Z}_\ell^\times \supset U \arrow[u] \end{tikzcd}$$

$\exists$ fin. index s/g

is $z \to z^n$ for some $n \in \mathbb{Z}$.

$\Rightarrow \chi_i|_{\text{fin. index s/g of } I_{\mathbb{Q}_\ell}} = x^{n_i}$ for some $n_i \in \mathbb{Z}$.

$\Rightarrow \chi_i x^{-n_i}$ is finite.

(b/c factors $\underbrace{\mathbb{A}_{\mathbb{Q}}^\times / \mathbb{Q}^\times \prod_{p \nmid \ell} \mathbb{Z}_p^\times \times U_\ell \cdot \mathbb{R}_{>0}^\times}_{\text{finite index}}$

<u>Consequence</u>. $\exists\ \psi : \mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times} \to \mathbb{C}^{\times}$ s.t.

$$\forall\ p \neq \ell \quad \underset{\substack{\uparrow \\ (1,\dots,p,\dots,1,\dots)}}{\psi_i(p)} = \chi_i(\mathrm{Fr}_p) \in \overline{\mathbb{Q}} \subset \mathbb{C}$$

$$\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell .$$

(i.e. the Gal rep'n assoc'd to ψ_i is χ_i).

$$|\cdot|_{\mathbb{A}_{\mathbb{Q}}} \longleftrightarrow \{\kappa_\ell\}_\ell$$

$$\Rightarrow L(f,s) = L(\psi_1, s)\, L(\psi_2, s)$$

$$\Rightarrow L(f \otimes \psi^{-1}, s) = L(1,s)\, L(\psi_2, s).$$

$\therefore$ a contradiction, since we can form $L(f \otimes \psi^{-1}, s)$ on time,

$L(1,s)$ = pole at $s=1$

$L(\psi_2/\psi_1, s)$ = non-van @ $s=1$,

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this is Gal rep'n for cuspidal form. <u>Much easier</u>. Gal rep for Eisenstein series. Recall

$$T_p G_k = (1+p^{k-1})\, G_k$$

$$L(G_k, s) = \prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}$$

a Gal rep'n for G_k should have

$$\mathrm{tr}(\rho_{G_k,\ell}(\mathrm{Fr}_p)) = a_p \equiv 1 + p^{k-1} \quad \& \quad \det(\rho_{G_k,\ell}(\mathrm{Fr}_p)) = p^{k-1}.$$

Set $\forall \ell$,

$$\rho_{G_k,\ell} : G_{\mathbb{Q},\ell} \to GL_2(\mathbb{Q}_\ell)$$

equal to $\begin{pmatrix} \kappa^{k-1} & 0 \\ 0 & 1 \end{pmatrix}$

<u>Strategy for Ribet's Th'm</u>. $p \mid \zeta(1-j)$

$q = e^{2\pi i z}$

Note $G_k(z) = \sum \frac{1}{(mz+n)^k} \overset{q\text{-exp}}{=} \sum_{n \geq 0} a_n q^n$

$$= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$$

$\sigma_r(n) = \sum_{d \mid n} d^r$

<u>Renormalize</u>. $E_k^*(z) = \frac{\zeta(1-k)}{2} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n$.

$\parallel$ re-scaling of G_k

<u>Idea</u>: When $p \mid \zeta(1-k)$, $E_k^*(q) \equiv \sum_{n \geq 1} \sigma_{k-1}(n) q^n$ <u>mod p</u>.

i.e. E_k^* mod p is a cusp form. One shows:

$\exists$ cuspidal <u>Hecke eigenform</u> (requires work) $f(\tau) = \sum_{n \geq 1} \overset{\overset{\mathbb{Z}}{\cup}}{a_n} q^n$

s.t. $f(q) \bmod p = E_k^*(q) \bmod p$ (= between cusp form & Eisenstein series).

This implies

$$\mathrm{char}(\rho_{f,\ell}(\mathrm{Fr}_p)) \overset{\bmod p}{\equiv} \mathrm{char}(\rho_{E_k^*,\ell}(\mathrm{Fr}_p))$$

i.e. $$\underbrace{(\rho_{f,i} \bmod p)}_{\text{irr.}}{}^{ss} \cong \underbrace{(\rho_{E_k^*,i} \bmod p)}_{\text{is } 1 \oplus \chi^{k-1}}$$