Arizona Winter School: Perfectoid spaces

- Deadline: Nov. 11

<u>Existence Th'm</u>. For every open s/g. $\overset{\text{finite index}}{\vee}$ $U \subset C_K$, there is a finite abelian L/K s.t. $U = N_{L/K} C_L$.

<u>Pf</u>. We perform a few reductions that allow reducing to the special case: p prime, $\mu_p \subset K$, and C_K/U is killed by p, using Kummer theory.

<u>Reductions</u>.

<u>Lemma 1</u>: Let $U = N_{L/K} C_L$ be a norm s/g, and let $V \supset U$ be any s/g. Then $V =$ a norm s/g.

<u>Pf</u>.

$$\begin{tikzcd}
C_K/U \arrow[r, "\sim", "r_{L/K}"'] \arrow[d, two heads] & \mathrm{Gal}(L/K) \arrow[d, two heads, "\wr"] & \\
C_K/V \arrow[r, dashed, "\sim", "r_{M/K}"'] & G(M/K) & \overset{\text{for some}}{\hookrightarrow M \supset K}
\end{tikzcd}$$

But $\ker(r_{M/K}) = N_{M/K} C_M$.

<u>Lemma 2</u>. Let $U \subset C_K$ be open s/g of finite index. If $\exists$ a fin. ext. K'/K s.t. $N^{-1}_{K'/K}(U) \subset C_{K'}$ is a norm s/g, then $U \subset C_K$ is a norm s/g.

<u>Pf</u>. By assumption, $\exists$ abelian extension L/K' s.t.

$N^{-1}_{K'/K} U = N_{L/K'} C_L$ (L/K need not be abelian).

Sub-Thm (Norm limitation thm) Let L/K be a finite ~~abe~~ ext. and let $M = L \cap K^{ab}$ be the maximal abelian ext. Then $N_{L/K} C_L = N_{M/K} C_M$.

Then $U \supset N_{K'/K}(N^{-1}_{K'/K} U) = N_{L/K} C_L = N_{L \cap K^{ab}/K} C_{L \cap K^{ab}}$

Pf of norm limitation. $r_K : C_K \to G_K^{ab}$ is surjective (in contrast to local CFT).

Taking $\varprojlim_L r_{L/K}$, get $r_K(C_K)$ has dense image.

But r_K factors

$$\mathbb{A}_K^\times / K^\times \cdot (K_\infty^\times)^\circ$$

} compact since C_K^1 is compact.

connected comp of 1 (e.g. $\mathbb{R}^\times_{>0}$ or $\mathbb{C}^\times$)

Thus, $r_K(C_K)$ is compact, hence closed, so all of $G^{ab}(K)$

Return to L/K finite

$$\begin{tikzcd}
G^{ab}(L) \arrow[r] & G^{ab}(K) \arrow[r, two heads] & G(\overbrace{L \cap K^{ab}}^{=:M}/K) \\
C_L \arrow[u, two heads, "r_L"] \arrow[r, "N_{L/K}"'] & C_K \arrow[u, two heads] \arrow[ur, "r_{M/K}"'] &
\end{tikzcd}$$

$G(M/K) \simeq G_K^{ab} / \operatorname{im} G_L^{ab}.$

$$\underset{r_L}{\overset{=}{\text{surj. of}}}\; G_K^{ab} / r_K N_{L/K} C_L \simeq C_K / N_{L/K} C_L .$$

$N_{L/K} C_L \subseteq N_{M/K} C_M$ + above iso $\Rightarrow$ equality holds.

<u>Corollary</u>. L/K fin. Galois induces

$$r_{L/K} : C_K / N C_L \xrightarrow{\sim} G(L/K)^{ab}$$

<u>Lemma 3</u>. It suffices to prove: if $\mu_p \subset K$, $U \subset C_K$ open, fin. index, C_K/U killed by p, then U is norm s/g.

<u>Pf</u>. Take any K, U as in stmt. of existence. Induct on $|C_K/U|$. Choose $p \mid |C_K/U|$. By Lemma 2, STP for $N^{-1}_{K(\mu_p)/K} U \subset C_K(\mu_p)$. Note that p still divides this index.

Now choose $U \subset U_1 \subset C_K$. By assumed case, $U_1 = N_{K'/K} C_{K'}$ for some K'/K. Then

$$\begin{array}{ccc} C_{K'} / N^{-1}_{K'/K} U & \xrightarrow{N_{K'/K}} & C_K/U \\ & \searrow^{\sim}_{N_{K'/K}} & \cup \\ & & U_1/U . \end{array}$$

$\Rightarrow [C_{K'} : N^{-1}U] = [U_1 : U] = \dfrac{[C_K/U]}{p}$.

By induction $N_{K'/K}^{-1}U = N_{L/K'}C_L$ for some L/K'. Apply Lemma 2 to conclude U is a norm s/g.

<u>Key case: Prop.</u> Suppose K contains μ_p. Then every open s/g of fin. index $U \subset C_K$ s.t. $p \cdot C_K/U = 0$ is a norm s/g. (& so by Lemma 3, Existence Thm is true in general)

<u>Pf</u>. (explicit Kummer theory). Let S be a finite set of places s.t.

- $S \supset \{v \mid \infty\}$
- $S \supset \{v \mid p\}$
- $\mathbb{A}_K^\times = K_S^\times \cdot \prod_{v \notin S} \mathcal{O}_v^\times \cdot K^\times$

  (e.g. $S \supset$ gen's of class group)

Let $L = K((\mathcal{O}_K[1/S]^\times)^{1/p})/K$, corresponding to the s/g. $\mathcal{O}_K[1/S]^\times K^{\times p}/K^{\times p} \subset K^\times/K^{\times p} \xrightarrow{\sim} \mathrm{Hom}(G_K, \mu_p)$

$$a \longmapsto \left(\sigma \mapsto \frac{\sigma a^{1/p}}{a^{1/p}}\right)$$

Set $E = \prod_{v \in S} (K_v^\times)^p \cdot \prod_{v \notin S} \mathcal{O}_v^\times$

Claim. $E \cdot K^\times = (N_{L/K} \mathbb{A}_L^\times) \cdot K^\times$

First, claim $\Rightarrow$ Prop: Let $\mathcal{U} \subset C_K$ be as in the prop. and let

$$\begin{array}{ccc} \tilde{\mathcal{U}} & \subset & \mathbb{A}_K^\times \\ \downarrow & & \downarrow \\ \mathcal{U} & \subseteq & C_K \end{array}$$

be the preimage. $\tilde{\mathcal{U}}$ is open, so $\exists$ a fin. set T of places K s.t. $\tilde{\mathcal{U}} \supset \prod_{v \in T} \{1\} \times \prod_{v \notin T} \mathcal{O}_v^\times$.

But p kills $C_K/\mathcal{U} \Rightarrow \tilde{\mathcal{U}} \supset (\mathbb{A}_K^\times)^p$, so $\tilde{\mathcal{U}} \supset \prod_{v \in T} (K_v^\times)^p \times \prod_{v \notin T} \mathcal{O}_v^\times$. This is still true after enlarging T, so wma T satisfies the hyp's on S in the claim.

$$\Rightarrow \tilde{\mathcal{U}} \supset K^\times \cdot \prod_{v \in T} (K_v^\times)^p \times \prod_{v \notin T} \mathcal{O}_v^\times \overset{\text{claim}}{=} K^\times \cdot N(\mathbb{A}_L^\times),$$

where L is defined as in claim, w/ $T = S$.

$\Rightarrow \mathcal{U} \supset N C_L$, so $\mathcal{U}$ is a norm s/g.

Pf of claim. Note $E \subset N_{L/K} \mathbb{A}_L^\times$: for $v \notin S$, L/K is unramified at v, so local norms $N(\mathcal{O}_{L_w}^\times) = \mathcal{O}_{K_v}^\times$ are surj. on units.

For $v \in S$, $K_v^{\times p} \subset N L_w^\times$ b/c L/K is exponent p.

$\subset G(L_w/K_v)$

$\cong$

$K_v^\times / N L_w^\times$ = gp killed by p.

Because $E \subset N \mathbb{A}_L^\times$, the claim is equivalent to

$$\left| \mathbb{A}_K^\times / K^\times E \right| = \left| \mathbb{A}_K^\times / K^\times \cdot N_{L/K} \mathbb{A}_L^\times \right| .$$

We'll compute each of these orders by hand & use that they agree.

$$\underline{\text{RHS}} \overset{\text{rec}}{=} [L:K] \overset{\text{Kummer}}{=} \left| \mathcal{O}_K[1/S]^\times K^{\times p} / K^{\times p} \right|$$

$$= \left| \mathcal{O}_K[1/S]^\times / \mathcal{O}_K[1/S]^{\times p} \right|$$

$$\overset{\substack{\text{S-unit thm} \\ + \mu_p \subset K}}{=} p^{|S|} .$$

$$\underline{\text{LHS}} = \left| \mathbb{A}_K^\times / K^\times E \right| \underset{\text{choice of } S}{=} \left| K^\times K_S^\times \prod_{v \notin S} \mathcal{O}_v^\times / K^\times E \right|$$

$$= \frac{[K_S^\times \prod_{v \notin S} \mathcal{O}_v^\times : E]}{[K^\times \cap (K_S^\times \cdot \prod_{v \notin S} \mathcal{O}_v^\times) : K^\times \cap E^\times]} .$$

(If $A \supset B$, C all s/g,

$$0 \to \frac{A \cap C}{B \cap C} \to \frac{A}{B} \to \frac{AC}{BC} \to 0$$

$$A = K_S^\times \prod_{v \notin S} \mathcal{O}_v^\times \;, \; B = E^\times \;, \; C = K^\times \quad)$$

numerator is local $= \prod_{v \in S} [K_v^\times : K_v^{\times p}]$

For $v \mid \infty$, $[K_v^\times : K_v^{\times p}] = 1$.

v fin. $[K_v^\times : K_v^{\times p}] = p \cdot \underline{[\mathcal{O}_v^\times : \mathcal{O}_v^{\times p}]}$

Note: $h(G = \mathbb{Z}/p, \mathcal{O}_v^\times) = \dfrac{[\mathcal{O}_v^\times : \mathcal{O}_v^{\times p}]}{|\mathcal{O}_v^\times[p]|}$

$$\left.\begin{matrix} p^{[K_v:\mathbb{Q}_p]} & \text{if } v \mid p \\ 1 & \text{otherwise} \end{matrix}\right\} = \frac{[\mathcal{O}_v : p\mathcal{O}_v]}{|\mathcal{O}_v[p]|} = h(G, \mathcal{O}_v)$$

// b/c $\mathcal{O}_v \simeq$ fin. idx s/g of $\mathcal{O}^\times$

$$\Rightarrow [\mathcal{O}_v^\times : \mathcal{O}_v^{\times p}] = \begin{cases} p, & v \nmid p \\ p \cdot p^{[K_v:\mathbb{Q}_p]}, & v \mid p. \end{cases}$$

Thus, $\displaystyle\prod_{v \in S} [K_v : K_v^{\times p}] \overset{p \text{ odd.}}{=} \prod_{v \in S} p^2 \prod_{v \mid p} p^{[K_v:\mathbb{Q}_p]}$

$= p^{2|S_{fin}| + [K:\mathbb{Q}]}$

$\left(\begin{matrix} K_v = \mathbb{C} \\ \forall v \mid \infty \end{matrix}\right)$ for p odd $= p^{2|S|}$

Same for when $p = 2$, though.

Denominator: $\left| \mathcal{O}_K[1/S]^\times / \mathcal{O}_K[1/S]^\times \cap (K_S^\times)^p \right|$

$$\underset{\text{(Lemma)}}{=} \left| \mathcal{O}_K[1/S]^\times / \mathcal{O}_K[1/S]^{\times p} \right| = p^{|S|}.$$

see our local-global for n-th powers discussion

Lemma. $K \supset \mu_n$. S fin. set of places $\supset v|\infty$, $v|n$, gen's of $Cl(K)$.

Then $K^\times \cap \left(K_S^{\times n} \prod_{v \notin S} \mathcal{O}_v^\times\right) \subset (K^\times)^n$.

Pf. Let $a \in$ LHS. Consider the abelian ($\mu_n \subset K$) ext. $L = K(a^{1/n})/K$. rec $\Rightarrow \mathbb{A}_K^\times / K^\times N(\mathbb{A}_L^\times) \xrightarrow{\sim} G(L/K)$

By assumption on S, $N\mathbb{A}_L^\times \supset \prod_{v \notin S} \mathcal{O}_v^\times \cdot K_S^\times$ so

$$\mathbb{A}_K^\times / K^\times N \mathbb{A}_L^\times \qquad \Rightarrow a^{1/n} \in K.$$

$$\Uparrow$$

$$\mathbb{A}_K^\times / K^\times \cdot K_S^\times \prod_{v \in S} \mathcal{O}_v^\times \underset{\substack{\uparrow \\ S \text{ generates } Cl}}{=} \{1\}$$