1. Grunwald-Wang Counterexamples

1.1. For $\mathbb{Q}$. We show that 16 is an 8-th power in $\mathbb{Q}_v$ for $v \neq 2$. To start, we have

$$X^8 - 16 = (X^2-2)(X^2+2)(X^2-2X+2)(X^2+2X+2).$$

The roots of $X^2 \pm 2X + 2$ are $1 \pm \sqrt{-1}, -1 \pm \sqrt{-1}$, respectively. Thus, the splitting field of $X^8 - 16$ is $K := \mathbb{Q}(\sqrt{2}, \sqrt{-1})$. Hence for p odd, we need to establish that one of $\sqrt{\pm 2}, \sqrt{-1}$ are in K. We do so by looking for integral solutions to $X^8 - 16$, in order to apply Hensel's lemma. If $2, -1$ are not squares in $\mathbb{F}_p$, then multiplicativity of the Legendre symbol shows -2 is a square. If $2, -2$ aren't, then -4 is, and since p is odd, this means -1 is. If $-1, -2$ aren't, then 2 is. This means a modulo p solution to $X^8 - 16$ is guaranteed, and since none of these solutions are zero, by Hensel's lemma there is a root to $X^8 - 16$ is $\mathbb{Z}_p$, for odd p.

1.2. For $\mathbb{Q}[\sqrt{7}]$. For p an odd prime, the same use of Hensel's lemma above shows that $\mathbb{Q}_p[\sqrt{7}] = \mathbb{Q}_p[\sqrt{2}, \sqrt{-1}][\sqrt{7}]$. For $p = 2$, $\mathbb{Q}_2[\sqrt{7}] = \mathbb{Q}_2[\sqrt{2}]$. Let $\alpha = \sqrt{7}$. Then

$$\begin{aligned} &\alpha^2 - 2\alpha + 4 + 2\alpha - 4 - 8 = -1 \\ \Rightarrow\ &(\alpha - 2)^2 + 2(\alpha - 2) - 7 = 0 \end{aligned}$$

Then $\alpha - 2 = -1 \pm 4\sqrt{2}$, so $(\alpha - 1)/4 = \pm\sqrt{2}$. Thus $X^8 - 16$ has a root in $\mathbb{Q}_2(\sqrt{7})$.

1.3. Relation to Grunwald-Wang.

2. Norms are Local Norms

2.1. Finite, Cyclic Extensions. Let L/K be finite cyclic extensions of number fields. We will show $a \in N_{L/K}L$ if and only if it is in $N_{L_w/K_v}L_w$ for all places w of L. This is true since we have the following maps

$$\begin{aligned} K^\times/N_{L/K}L^\times \cong H^2(G(L/K), L^\times) &\hookrightarrow H^2(G(L/K), \mathbb{A}_L^\times) \\ &\cong \bigoplus_v H^2(G(L_w/K_v), L_w^\times) \cong \bigoplus_v K_v^\times/N_{L_w/K_v}L_w^\times. \end{aligned}$$

2.2. Counterexample for Non-cyclic Extensions. Let $L = \mathbb{Q}(\sqrt{13}, \sqrt{17})$. We show that 25 is not a global norm but it is everywhere a local norm. Let $\alpha = a + b\sqrt{13} + c\sqrt{17} + d\sqrt{17 \cdot 13}$. Let $x = a + b\sqrt{13}, y = c + d\sqrt{13}$. Suppose $25 = N(\alpha)$. Then

$$\begin{aligned} 25 &= N_{\mathbb{Q}(\sqrt{13}/\mathbb{Q}}(N_{L/\mathbb{Q}(\sqrt{13})}\alpha) \\ &= N_{\mathbb{Q}(\sqrt{13})/\mathbb{Q}}(x^2 - 17y^2). \end{aligned}$$

3. Hilbert Class Field

3.1. Hilbert Class Field. Let $U = K^\times \prod_{v|\infty} K_v^\times \prod_{v\nmid\infty} \mathcal{O}_{K_v}^\times$. We claim this is the subgroup of C_K corresponding to the Hilbert class field. By the existence theorem, there is an H/K such that $N_{H/K}C_H = U$. Then by Artin reciprocity,

$$C_K/U \xrightarrow{\sim} \mathrm{Gal}(H/K).$$

For v an infinite place, looking at the local reciprocity map gives $K_v^\times/K_v^\times \xrightarrow{\sim} \mathrm{Gal}(H_w/K_v)$, so v is split completely. For v a finite place,

$$\mathcal{O}_{K_v}^\times/\mathcal{O}_{K_v}^\times K^\times \subset K_v^\times/\mathcal{O}_{K_v}^\times K^\times$$

is trivial, and so the local reciprocity map sends it to the identity element of $\mathrm{Gal}(H/K)$. Thus, v is unramified.

For the narrow Hilbert class field, for $v|\infty$ we take $(K_v^\times)^2$. Then the $\mathrm{Gal}(H_w/K_v)$, for $w|v$, have order two or one depending on if K_v is real or complex.