Recall. S - fin. set of places $\supset \{v | \infty\}$

$$G_{K,S}\left[\begin{array}{l} K(S) = \text{max unram'd outside } S \text{ ext of } K \text{ in } \overline{K} \\ \quad | \\ K \end{array}\right.$$

M: finite G_S-mod

Assume $S \supset \{v : v(\# M) \neq 0\}$. Then get 9-term Poitou-Tate sequence.

$P_S^i(M) = \bigoplus_{v \in S} H^i(G_{K_v}, M)$

except for $i=0$, v real use H^0/norm.

$$\hookrightarrow H^i(G_S, M) \xrightarrow{\beta^i} P_S^i(M) \xrightarrow{\gamma^i} H^{2-i}(G_S, M^*)^\vee$$

β^i = local restriction

γ^i = defined via local duality

$$\gamma^i(\underset{\substack{\Uparrow \\ H^i(K_v, M)}}{(x_v)_{v \in S}}) = \Big[\phi \to \sum_{v \in S} \underset{\text{local duality pairing.}}{\langle \phi|_{K_v}, x_v \rangle_v}\Big]$$

Construction of the P-T sequence.

For GCFT, we studied $G(L/K)$-coh LES assoc'd to

$$1 \to L^\times \to \mathbb{A}_L^\times \to C_L^\times \to 1$$

For global duality, do an "unr. outside S" version:

$$\begin{array}{ccc} K^\times & \mathbb{A}_K^\times & C_K \\ \vdots & \vdots & \vdots \\ \mathcal{O}_K[1/S]^\times & K_S^\times & \mathbb{A}_K^\times / K^\times \prod_{v \notin S} \mathcal{O}_v^\times =: C_S(K) \end{array}$$

(use $C_S(K)$ instead of $K_S^\times / \mathcal{O}_K[1/S]^\times$ since it satisfies Galois descent)

For any L/K fin, $S = \{\text{places of } L \text{ above } S\}$.

<u>Lemma</u>. $K - L - M$ w/ M/L Galois.

$$\Rightarrow \begin{cases} \left(\mathcal{O}_M[1/S]^\times\right)^{G(M/L)} = \mathcal{O}_L[1/S]^\times \\ \left(M_S^\times\right)^{G(M/L)} = L_S^\times \end{cases}$$

and if M/L is unr. outside S (i.e. $M \subset L(S)$)

then $C_S(M)^{G(M/L)} = C_S(L)$.

As in GCFT, pass to limit

$$E_S = \varinjlim_{\substack{K \subset L \subset K(S) \\ \text{fin}}} \mathcal{O}_L[1/S]^\times \qquad J_S = \varinjlim_{K \subset L \subset K(S)} L_S^\times \qquad C_S = \varinjlim_{K \subset L \subset K(S)} C_S(L)$$

analogue of $E^\times$

analogue of $\varinjlim C_L$

<u>Lemma</u>. (1) For L/K fin., have exact sequence

$$0 \to \mathcal{O}_L[1/S]^\times \to L_S^\times \to C_S(L) \to Cl_S(L) \to 0$$

$Cl_S(L)$: S-class group of L

② $0 \to E_S \to J_S \to C_S \to 0$ is exact.

Pf. ① $Cl_S(L) = \mathbb{A}_L^\times / L_\infty L^\times \prod_{v \notin S} \mathcal{O}_v^\times L_S^\times$

② Take $\varinjlim_{K \subset L \subset K(S)}$ of the sequences in (1).

$$\rightsquigarrow \quad 0 \to E_S \to J_S \to C_S \to \boxed{\varinjlim Cl_S(L)} \to 0$$

this lim is 0 b/c every ideal class in L becomes trivial in some unr. outside S ext. (e.g. but unnecessarily strong: Hilbert class field)

Now let M be a G_S-mod as in stmt of P-T duality th'm. The P-T sequence is (after some difficult $\boxed{\text{identifications}}$) the sequence obtained by applying $\mathrm{Ext}^i_{G_S}(M, \cdot)$ to the SES of G_S-mods $1 \to E_S \to J_S \to C_S \to 1$.

Th'm. ① $\mathrm{Ext}^i_{G_S}(M, E_S) \simeq H^i(G_S, M^*)$ $\forall i \geq 0$

② $\mathrm{Ext}^i_{G_S}(M, J_S) \simeq P^i_S(M^*)$ for $i \geq 1$

③ $\mathrm{Ext}^i_{G_S}(M, C_S) \simeq H^{2-i}(G_S, M)^\vee$ for $i \geq 1$.

Construction of P-T sequence from the th'm:

$$\mathrm{Ext}^1_{G_S}(M, E_S) \to \mathrm{Ext}^1_{G_S}(M, J_S) \to \mathrm{Ext}^1(M, C_S) \to$$

$$\to \mathrm{Ext}^2(M, E_S) \to \cdots$$

identifies to

$$H^1(G_S, M^*) \longrightarrow P^1_S(M^*) \longrightarrow H^1(G_S, M)^\vee \to$$

$$\to H^2(G_S, M^*) \to P^2_S(M^*) \longrightarrow H^0(G_S, M)^\vee \to 0.$$

To get full 9-terms, take 6-term seq swapping M & M^* and apply $(\ )^\vee$, and then splice together the 2 6-term sequences.

<u>Rmk on proof</u>. ③ is the hardest step. Here is the map:

$$\begin{tikzcd}
H^{2-i}(G_S, M) \arrow[d, "\wr"] & \times\ \mathrm{Ext}^i_{G_S}(M, C_S) \arrow[r, "\text{via}", "\text{ext}"'] & \mathrm{Ext}^2_{G_S}(\mathbb{Z}, C_S) \arrow[d, "\wr"] & \\
\mathrm{Ext}^{2-i}_{G_S}(\mathbb{Z}, M) & & H^2(G_S, C_S) \arrow[r, "\inf"] & H^2(G_K, C \\
& & & \varinjlim C_L \arrow[from=u]
\end{tikzcd}$$

Frequently in applications, one is interested not in the whole $H^1(G_S, M)$ but in a s/g satisfying local conditions.

Def'n. M finite G_K-mod. A Selmer system is a collection of s/g's $L_v \subset H^1(G_{K_v}, M)$ $\forall$ places v s.t. for a.e. v, $L_v = H^1(G_{K_v}/I_{K_v}, M^{I_{K_v}}) \underset{inf}{\subset} H^1(G_{K_v}, M)$

Def. Let $\mathcal{L} = \{L_v\}$ be a Selmer system. The Selmer grp of $\mathcal{L}$ is $H^1_{\mathcal{L}}(K, M) = \ker\left(H^1(G_K, M) \longrightarrow \prod_{\text{all } v} \frac{H^1(G_{K_v}, M)}{L_v}\right)$

(i.e. $x \in H^1(K, M) \mid x| \in L_v \ \forall v$)

Equivalently,

Lemma. If S is a fin. set of places s.t. $G_K \circlearrowright M$ factors through $G_S \circlearrowright M$, $S \supset \{\infty\}$, $S \supset \{v \mid L_v \text{ unr. coh.}\}$. Then

$$0 \to H^1_{\mathcal{L}}(K, M) \to H^1(G_S, M) \to \bigoplus_{v \in S} \frac{H^1(K_v, M)}{L_v}$$

Pf. By def'n, $0 \to H^1_{\mathcal{L}}(K, M) \to H^1(K, M) \to \bigoplus_{v \in S} \frac{H^1(K, M)}{L_v} \bigoplus_{v \notin S} \mathrm{Hom}(I_{K_v}, M)$

So classes in $H^1_{\mathcal{L}}(K, M)$ are trivial on $\left\langle \begin{array}{l} \text{closed s/g} \\ \text{of } G_K \text{ gen'd} \\ \text{by } I_{K_v} \text{ for} \\ v \notin S \end{array} \right\rangle = G(\bar{K}/K(S))$

So these classes actually belong to $H^1(G_S, M) \subset H^1(G_K, M)$

Cor. For any Selmer system, $H^1_{\mathcal{L}}(K,M)$ ← finite G_K-mod is finite

(b/c $\subset H^1(G_S, M)$ for some S)

Analogous P-T seq. on Selmer groups:

Def. Given Selmer system $\mathcal{L} = \{L_v\}$, the dual Sel. system $\mathcal{L}^\perp = \{L_v^\perp\}$ for M^* is the collection $L_v \subset H^1(K_v, M^*)$ given by $L_v^\perp = \underbrace{\mathrm{ann}}_{\text{under local duality.}}(L_v)$

Lemma. For a.e. v, $L_v^\perp = $ unr. coh.

Cor. Take S as in last lemma.

$G_S \curvearrowright M$

$S \supset \{v \mid L_v \text{ unr}\}$

$S \supset \infty$

then $\exists$ ES.,

$$0 \to H^0(G_S, M) \to P_S^0(M) \to H^2(G_S, M^*)^\vee$$

$$\hookrightarrow H^1_{\mathcal{L}}(K,M) \to \bigoplus_{v \in S} L_v \to H^1(G_S, M^*)^\vee$$

$$\hookrightarrow H^1_{\mathcal{L}^\perp}(K, M^*)^\vee \to 0.$$

e.g.'s. $Ш^1_S(M) = H^1_{\mathcal{L}}(K,M)$ where L_v = unr. coh. $\forall v \notin S$
$L_v = 0 \quad v \in S$

sub-e.g. $M = \mathbb{Z}/n$ (trivial)

$$Ш^1_S(M) = \ker\left(\mathrm{Hom}(G_{K_S}, \mathbb{Z}/n) \to \bigoplus_{v \in S} \mathrm{Hom}(G_{K_v}, \mathbb{Z}/n)\right)$$

$$\simeq \mathrm{Hom}\left(\mathbb{A}_K^\times / K^\times K_\infty^\times \prod_{v \notin S} \mathcal{O}_v^\times K_S^\times, \mathbb{Z}/n\right)$$

$$= \mathrm{Hom}(Cl_S(K), \mathbb{Z}/n)$$

- $M = \mu_n$, $Ш^1_S(\mu_n) \overset{\text{(Kummer)}}{=} \dfrac{\mathcal{O}_K[1/S]^\times \cap \prod_{v \in S} (K_v^\times)^n}{(\mathcal{O}_K[1/S]^\times)^n}$

(came up in pf of existence thm).

Kummer theory for an ab. variety. Let A/K be an ab. var. (e.g. elliptic curve).

A has good reduction outside a finite set S (eq'ns of A can be reduced mod p to smooth variety), i.e. have a scheme $\mathcal{A} \to \operatorname{Spec} \mathcal{O}_K[1/S]$

Let n be an integer, $(n, \operatorname{char} K) = 1$.
Then have exact seq. of G_K-mods
$0 \to A(\bar{K})[n] \to A(\bar{K}) \xrightarrow{[n]} A(\bar{K}) \to 0.$

(classic Kummer theory: $\mathbb{G}_m$ in place of A)

Take LES in G_K-coh:

$$0 \to A(K)/nA(K) \to H^1(G_K, A[n]) \to H^1(G_K, A)[n] \to 0$$

<u>Aim</u>. <u>Weak Mordell-Weil thm</u>. $A(K)/n(A_K)$ is a fin. group.

Also have for all v,

$$0 \to A(K_v)/nA(K_v) \xrightarrow{\delta_v} H^1(G_{K_v}, A[n])$$

$$\longrightarrow H^1(G_{K_v}, A)[n]$$

$\mathrm{Sel}^{(n)}(A/K) = \{ x \in H^1(G_K, A[n]) : \forall v, \mathrm{res}_v(x) \in \mathrm{im}\ \delta_v \}$

We'll show this is fin.

<u>Compare</u>. What is $\ker(H^1(K, A) \xrightarrow{\sim} \prod_v H^1(K_v, A))$.

/

Shafarevich-Tate group of a tape