in pf. that for L/K cyclic, $|H^2(G(L/K), L^\times)| \le [L:K]$,

compute

$$h(G(L/K), \underbrace{L^\times}) = h(\mathcal{O}_L^\times) \cdot h(\mathbb{Z})$$

$\rightsquigarrow$ triv. mod (easy to compute)

$$0 \to \mathcal{O}_L^\times \to L^\times \to \mathbb{Z} \to 0$$

Last time. $Br(K) \xrightarrow[\text{inv}_K]{\sim} \mathbb{Q}/\mathbb{Z}$ & we deduced

$\Big\{$ M torsion $\in \mathcal{M}_{G_K}$, $H^r(G, M) = 0 \ \forall r \ge 3$

M finite $\in \mathcal{M}_{G_K}$, $\Rightarrow |H^r(G_K, M)| < \infty \quad \forall r$

Local Duality (Tate)

Def. (Tate dual of finite $M \in \mathcal{M}_{G_K}$) Set $M^* = \mathrm{Hom}(M, \mu_\infty)$ as abelian groups

with natural G_K-action.

(If $X, Y \in \mathcal{M}_{G_K}$, $\mathrm{Hom}(X, Y)$ is a G_K-module

via $(\sigma \cdot \varphi)(x) = \sigma\, \varphi(\sigma^{-1}(x))$.)

$g(x \otimes y) := gx \otimes gy$

Then $(M^*)^* \simeq M$. And have G_K-equivariant $M \otimes M^* \to \mu_\infty$.
$m, \varphi \to \varphi(m)$

$\rightsquigarrow$ Local duality pairing:

$$H^i(G_K, M) \times H^{2-i}(G_K, M^*) \xrightarrow{\cup} H^2(G_K, M \otimes M^*)$$

$$\downarrow \text{eval}$$

$$H^2(G_K, \mu_\infty) \xrightarrow[\sim]{\text{inv}_K} \mathbb{Q}/\mathbb{Z}.$$

$\bar{K}^\times$ $\hookrightarrow$ by Hilbert 90

Write $\langle\ ,\ \rangle_K : H^i(K, M) \times H^{2-i}(K, M^*) \to \mathbb{Q}/\mathbb{Z}$.

Thm. For any finite $M \in \mathcal{M}_{G_K}$, $\langle , \rangle_K$ is a perfect duality, i.e. it induces iso's

$$H^i(K, M) \xrightarrow{\sim} \mathrm{Hom}(H^{2-i}(K, M^*), \mathbb{Q}/\mathbb{Z})$$

(and vice versa)

Notation. $D^\vee = \mathrm{Hom}(D, \mathbb{Q}/\mathbb{Z})$ for ab gp D.

Rmk. 1) Plugging in $i=1$, $M = \mathbb{Z}/n$, (so $M^* = \mu_n$), we'll get reciprocity map of LCFT

2) H^0's are easy (invariants). So now H^2's are easy.

Pf. $\boxed{\text{Step 1}}$

Write $\theta_K : H^0(K, M^*) \longrightarrow H^2(K, M)^\vee$. WTS θ_K is an isomorphism.

$$\begin{array}{ccc} H^0(K, M^*) & & H^2(K, M)^\vee \\ \| & & \\ \mathrm{Hom}_{G_K}(M, \mu_\infty) & & \\ \varphi & \longrightarrow & \mathrm{inv}_K \circ \varphi \end{array}$$

Strategy. reduce to case $M = \mu_n$, then use Brauer case.

1st reduction. For L/K a finite extension,

$$\begin{array}{ccc} \mathrm{Hom}_{G_K}(M, \mu_\infty) & \xrightarrow{\theta_K} & H^2(K, M)^\vee \\ \cap & & \downarrow \mathrm{cor}^\vee \\ \mathrm{Hom}_{G_L}(M, \mu_\infty) & \xrightarrow{\theta_L} & H^2(L, M)^\vee \end{array}$$

why. $\mathrm{cor}^\vee(\Theta_K(\varphi))$ is

$$\begin{tikzcd}
H^2(L,M) \arrow[r, "\mathrm{cor}"] \arrow[d, equal, "\Theta_L(\varphi) \text{ is}"'] \arrow[drr, phantom, "\text{functoriality of cor}"] & H^2(K,M) \arrow[r, "\varphi"] & H^2(K,\mu_\infty) \arrow[r, "\mathrm{inv}_K"] \arrow[dr, phantom, "\text{Brauer compatibility}"] & \mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\
H^2(L,M) \arrow[rr] & & H^2(L,\mu_\infty) \arrow[u, "\mathrm{cor}"] \arrow[r, "\mathrm{inv}_L"'] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

Thus $\mathrm{cor}^\vee(\Theta_K(\varphi)) = \Theta_L(\varphi)$.

end of 1st reduction.

2nd reduction. $\mathrm{cor}^\vee$ is injective, and thus
[Θ_L is an iso $\Rightarrow$ Θ_K is an iso].

Pf of "thus". Suppose Θ_L is an iso.

Θ_L is $G(L/K)$ equivariant.

$$\begin{tikzcd}
\mathrm{Hom}_{G_K}(M,\mu_\infty) = \mathrm{Hom}_{G_L}(M,\mu_\infty)^{G(L/K)} \arrow[rr, "\hat{\Theta}_L"] \arrow[dr, "\Theta_K"'] \arrow[rr, phantom, shift right=6, "\text{by last step}"] & & {[H^2(L,M)^\vee]^{G(L/K)}} \\
& H^2(K,M)^\vee \arrow[ur, "\mathrm{cor}^\vee"'] &
\end{tikzcd}$$

Formally $\mathrm{cor}^\vee$ injective $\Rightarrow$ Θ_K is an iso.

To show $\mathrm{cor}^\vee$ is inj on H^2, we show $\mathrm{cor}: H^2(L,M) \longrightarrow H^2(K,M)$ is surj.

Recall cor can be expressed as

$$\begin{tikzcd} H^2(L,M) \arrow[r, "\mathrm{cor}"] & H^2(K,M) \\ H^2(K, \mathrm{Ind}_{G_L}^{G_K} M) \arrow[u, equal, "\sim"] \arrow[ur, "tr"'] & \end{tikzcd}$$

where $tr : \mathrm{Ind}\, M \to M$

$$f \longrightarrow \sum_{g \in G_K/G_L} g^{-1} \cdot f(g).$$

cor is surjective since the LES assoc'd to

$$0 \to \ker(tr) \to \mathrm{Ind}(M) \xrightarrow{tr} M \to 0.$$

gives $H^2(L,M) \xrightarrow{\mathrm{cor}} H^2(K,M) \longrightarrow \underbrace{\cancel{H^3(K, \ker(tr))}}_{\text{zero by last time.}}$

⊛ By these two reductions, suffices to prove Θ_K is an iso for $M = \mu_n$ (choose L/K fin. s.t. $M|_{G_L} \simeq \bigoplus \mu_{n_i}$) (take L trivializing M and all μ_{n_i}, so this reads $M|_L \simeq \bigoplus \mathbb{Z}/n_i$
$\simeq \bigoplus \mu_{n_i}$

Now Θ_K :

$$\begin{tikzcd} \mathrm{Hom}_{G_K}(\overset{=M}{\mu_n}, \mu_\infty) \arrow[r] \arrow[d, equal] & H^2(K,\mu_n)^\vee \\ \mathbb{Z}/n\mathbb{Z} \arrow[r, "\sim"] & \mathrm{Hom}(\tfrac{1}{n}\mathbb{Z}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \arrow[u, "\mathrm{inv}_K"'] \end{tikzcd}$$

$$r \longrightarrow \left(\tfrac{1}{n} \to \tfrac{r}{n}\right)$$

<u>Conclusion</u>. $\forall M, K$, $i = 0, 2$, $\langle\ ,\ \rangle_K$ is a perfect duality.

Deduce $i=1$ case from these cases. Recall we have an embedding $M \hookrightarrow \mathrm{Ind}_1^{G_K} M = \bigcup \{ f : G(L/K) \to M \}$
for L/K fin. Galois

Choose a finite ~~Galois~~ G_K submodule $N \subset \mathrm{Ind}_1^{G_K} M$ s.t. the map $H^j(G_K, M) \longrightarrow H^j(G_K, N)$ is the ZERO map. $\forall j$. (use finiteness of $H^*(G_K, M)$)

$$\begin{tikzcd} M \arrow[r, "\text{factor}"] \arrow[dr] & N \arrow[d] \\ & \mathrm{Ind}\, M \end{tikzcd}$$

Then look at LES's assoc'd to

$$0 \longrightarrow M \longrightarrow N \longrightarrow Q \longrightarrow 0$$

$$\begin{tikzcd} H^0(K,N) \arrow[r] \arrow[d, "\cong"] & H^0(K,Q) \arrow[r] \arrow[d, "\cong"] & H^1(K,M) \arrow[r] \arrow[d] & 0 \\ H^2(K,N^*) \arrow[r] & H^2(Q^*)^\vee \arrow[r] & H^1(K,M^*)^\vee \end{tikzcd}$$

(since $H^1(M) \to H^1(N)$ is zero)

duality maps

$$0 \longrightarrow Q^* \longrightarrow N^* \longrightarrow M^*$$

These squares commute (up to sign) by functoriality properties of $\cup$ (& def of inv_K).

<u>Diagram-chase</u> $\Rightarrow$ $H^1(K,M) \longrightarrow H^1(K,M^*)^\vee$ is injective.

By symmetry ($M \longleftrightarrow M^*$), get

$H^1(K, M^*) \longrightarrow H^1(K,M)^\vee$ is injective

and therefore both maps are iso's as inj. homs b/w fin. groups of the same size.

<u>Functoriality properties of $\langle\, ,\rangle_K$ are essential to its usefulness</u>

<u>a) Lemma</u>. For L/K finite $\quad x \in H^i(K,M)$, $y \in H^{2-i}(L, M^*)$ $\Rightarrow \langle x, \mathrm{cor}(y)\rangle_K = \langle \mathrm{res}(x), y\rangle_L$

<u>Pf</u>. $\langle x, \mathrm{cor}(y)\rangle_K = \mathrm{inv}_K(x \cup \mathrm{cor}(y))$

$\overset{\text{cup product identity}}{=} \mathrm{inv}_K(\mathrm{cor}(\mathrm{res}(x) \cup y))$

$\underset{\mathrm{inv}_K}{=} \mathrm{inv}_L(\mathrm{res}(x) \cup y) = \langle \mathrm{res}(x), y\rangle_L$.

<u>b) Lemma</u> $\quad x \in H^i(K,M)$, $y \in H^{2-i}(K, M^*)$ $\Rightarrow \langle \mathrm{res}(x), \mathrm{res}(y)\rangle_L = [L:K]\langle x,y\rangle_K$

<u>Pf</u>. $\mathrm{inv}_L(\mathrm{res}(x) \cup \mathrm{res}(y)) = \mathrm{inv}_L(\mathrm{res}(x \cup y))$

$\overset{\cup}{=} [L:K]\, \mathrm{inv}_K(x \cup y) \overset{\mathrm{inv}}{=} [L:K]\langle x,y\rangle_K$

c) <u>Background</u>. $\sigma: K_1 \xrightarrow{\sim} K_2$, then $\langle x,y\rangle_{K_1} = \langle \sigma x, \sigma y\rangle_{K_2}$

Given $\sigma: K_1 \longrightarrow K_2$. $\tilde\sigma$ induces

$$\begin{array}{ccc} \overline{K_1} & \overset{\tilde\sigma}{\dashrightarrow} & \overline{K_2} \\ \cup & & \cup \\ K_1 & & K_2 \end{array} \qquad \begin{array}{ccc} G(\overline{K_1}/K_1) & \longrightarrow & G(\overline{K_2}/K_2) \\ g & \longmapsto & \tilde\sigma g \tilde\sigma^{-1} \end{array}$$

If M is any G_{K_1}-mod, form G_{K_2}-mod M, by ${}^{\sigma}M = M$ as ab. gp. and use the iso above to transport the action. Then get iso's

$$H^j(G_{K_1}, M) \xrightarrow{\sim} H^j(G_{K_2}, {}^{\tilde\sigma}M)$$

& the iso only depends on $\tilde\sigma$, only σ.

This claim reduces to the case $\sigma = id : K \to K$, and think of $\tilde\sigma$ as identifying two choices of $\overline{K}$.

Phil. LCFT gives (almost an iso)

ditto $H^n(K, M)$

$$r_K : \underbrace{K^\times}_{\text{intrinsic}} \longrightarrow \underbrace{G_K^{ab}}_{\substack{\text{requires choice}\\ \text{of algebraic}\\ \text{closure, up}\\ \text{to inner auto's}}}$$

G_K^{ab} is indep. of this choice.

Can't expect a canonical iso

$$\left\{\begin{matrix}\text{intrinsic}\\ \text{data to } K\end{matrix}\right\} \longleftrightarrow G_K.$$

This is one reason local Langlands program

$$\underbrace{\left\{\begin{matrix}\text{rep'ns of}\\ GL_n(K)\end{matrix}\right\}}_{\text{intrinsic to } K} \cdots\cdots \left\{\begin{matrix}\text{rep'ns of}\\ G_K\end{matrix}\right\} / \cong$$

eliminates inner automorphism ambiguity in defining G_K.

An example for next time.

$$\begin{array}{ccc} H^1(G_K, \mathbb{Z}/n) & \xrightarrow{\sim} & H^1(K, \mu_n)^\vee \\ \| & & \| \\ \mathrm{Hom}(G_K, \mathbb{Z}/n) & & \mathrm{Hom}(K^\times/(K^\times)^n, \mathbb{Q}/\mathbb{Z}) \end{array}$$

Dualize again.

1) If G profinite abelian group,

$$\mathrm{Hom}_{cts}(\underbrace{\mathrm{Hom}_{cts}(G, \mathbb{Q}/\mathbb{Z})}, \mathbb{Q}/\mathbb{Z})$$

$$\sim\uparrow$$

$$G$$

topology: discrete.
(but this is really the compact-open topology on the Pontryagin dual.)

G profinite $\Rightarrow$ any cnts hom $G \to GL_n(\mathbb{C})$ has finite image.

2) different e.g.

$$\mathrm{Hom}(\mathrm{Hom}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) = \hat{\mathbb{Z}}$$