Contents

1. Outline

(1) Outline of class field theory
(2) Abelian L-functions (Hecke, Tate)
(3) Non-abelian L-functions (Weil-Deligne group, local L- and ϵ- factors)
(4) Local Langlands program for GL_n
(5) Global: automorphic forms and representations

2. Outline of Class Field Theory

This is the study of abelian extensions of local/global fields. We first fix some notation. If K is a field, $\overline{K}$ will be a separable closure of K. Also, $\Gamma_K = \mathrm{Gal}(\overline{K}/K) = \varprojlim_{L/K finite} \mathrm{Gal}(L/K)$ (inverse limit). Then this is a profinite group with the Krull topology. Galois theory gives us two bijections:

(1) Closed subgroups of Γ_K with subfields $K \subset L \subset \overline{K}$
(2) Open subgroups correspond to finite extensions L/K

We'll also write $K^{\mathrm{ab}} \subset \overline{K}$ for the maximal abelian subextension. Then $\mathrm{Gal}(K^{\mathrm{ab}}/K) = \Gamma_K^{\mathrm{ab}} = \Gamma_K/\overline{[\Gamma_K, \Gamma_K]}$. Note that $\overline{K}$ is unique up to a (non-unique) isomorphism. So Γ_K is well-defined up to conjugation, and Γ_K^{ab} is well-defined.

2.1. **Local Fields.** At the moment, let F be a non-archimedean local field (finite extension of $\mathbb{Q}_p$ or Laurent series in one variable over a finite field). This has a normalized valuation $v = v_F : F^\times \to \mathbb{Z}$. Then we get a valuation ring $\mathcal{O} = \mathcal{O}_F$, also the ring of integers. Pick a uniformizer $\pi = \pi_F$ ($v(\pi) = 1$). Then let $k = k_F$ be the residue field, isomorphic to $\mathbb{F}_q$, where $q = p^r$.

Within $\overline{F} \subset F^{\mathrm{ab}} \subset F^{\mathrm{ur}}$, the maximal unramified extension of F. Within Γ_F we have $I_F = \mathrm{Gal}(\overline{F}/F^{\mathrm{nr}})$, and within that P_F the wild inertia group (maximal proper subgroup of I_F).

Now to describe the first layer, which tells us about unramified extensions. Then we have an isomorphism

$$\Gamma_F/I_F \xrightarrow{\sim} \mathrm{Gal}(\overline{k}/k) \xrightarrow{\sim} \hat{\mathbb{Z}}$$

given by reduction modulo π. Inside the Galois group we have $\varphi_q : x \mapsto x^q$, the arithmetic Frobenius map. It results in infinite confusion to send $1 \mapsto \varphi_q$, so it's standard to take $\mathrm{Frob}_q := \varphi_q^{-1}$, called the geometric Frobenius. We fix the isomorphism $\mathrm{Gal}(\overline{k}/k) \xrightarrow{\sim} \hat{\mathbb{Z}}$ so that $\mathrm{Frob}_q \mapsto 1$.

Now to describe the second layer (the group I_F/P_F), which tells us about tamely ramified extensions. Fix $\pi_n \in \overline{F}$ with $\pi_n^n = \pi$. Define

$$t(n) : I_F = \mathrm{Gal}(\overline{F}/F^{\mathrm{ur}}) \to \mu_n(\overline{k}),$$

for $(n, p) = 1$, sending

$$\gamma \mapsto \gamma(\pi_n)/\pi_n \pmod{\pi}.$$

Now this is the tame mod n character, and doesn't actually depend on all the choices we made and is a homomorphism $I_F \to \mu_n(\overline{k})$. Assembling all of these together, we obtain a map

$$I_F \to \varprojlim_{(n,p)=1} \mu_n(\overline{k}) = \prod_{l \neq p} \varprojlim_{m \geq 1} \mu_{l^m}(\overline{k}) = \mathbb{Z}_l(1)(\overline{k}),$$

and this last one is called the Tate module of $\overline{k}^\times$. Each of these is isomorphic to $\mathbb{Z}_l$ but not canonically so. The kernel $\hat{t}$ of this map is $\prod_{l\neq p} t_l = P_F$. Equivalently, the maximal tamely ramified extension of F is

$$\bigcup_{(n,p)=1} F^{\mathrm{nr}}(\sqrt{n}\pi),$$

which is also just Kummer theory.

Remark 2.1. This $t(n)$ extends to a map $\Gamma_F \to \mu_n$, given by the same formula, although it is not a homomorphism. Explicitly,

$$t(n)(\gamma\delta) = \gamma\delta(\pi_n)/\pi_n = \gamma(\pi_n)/\pi_n\gamma(\frac{\delta(\pi_n)}{\pi_n}) = t(n)(\gamma)\gamma(t(n)\delta) = t(n)(\delta),$$

where this last equality holds if $\gamma \in I_F$, but not in general. This means that $t(n)$ is a 1-cocycle.

2.2. **Local Class Field Theory.** We'll now just state local class field theory. The first part is the following theorem.

Theorem 2.2. *There is a unique family of continuous homomorphisms* $\mathrm{Art}_F : F^\times \to \Gamma_F^{\mathrm{ab}}$ *with dense image, characterized by*

(1) $F^\times \to \mathrm{Art}_F\Gamma_F^{\mathrm{ab}}$ *surjects onto* $\Gamma_F\mathcal{I}_F$ *"uniformizers map to geometric Frobenius" (see photo)*
(2) Often called the base-change property: (see other photo)

The second part is the existence theorem.

Theorem 2.3. *The inverse of the Artin map* Art_F^{-1} *induces a bijection between open subgroups of* Γ_F^{ab} *(finite abelian extensions of* F*) and open subgroups of* $F^\times$ *of finite index.*

The final part is that if $F = \mathbb{Q}_p$, then for $x = p^n y \in \mathbb{Q}_p^\times$, $y \in \mathbb{Z}_p^\times$. Then

$$F^{\mathrm{ab}} = \mathbb{Q}_p(\mu_\infty) = \bigcup \mathbb{Q}_p(\mu_n) = \mathbb{Q}_p^{\mathrm{nr}}(\mu_\infty).$$

Then

$$\mathrm{Art}_{\mathbb{Q}}(x)\big|_{\mathbb{Q}_p^{\mathrm{nr}}} = \mathrm{Frob}_p^n,$$

and

$$\mathrm{Art}_{\mathbb{Q}}(x)\big|_{\mathbb{Q}_p^{\mathrm{nr}}} = (\zeta_{p^n} \mapsto \zeta_{p^n}^{y \pmod{p^n}}).$$

On the level of finite extensions, you can rephrase the first theorem as: for E/F a finite Galois extension, we get $\mathrm{Art}_{E/F} : F^\times/N_{E/F}(E^\times) \xrightarrow{\sim} \mathrm{Gal}(E/F)^{\mathrm{ab}}$.

Finally, we note that Art_F induces an isomorphism

$$\mathcal{O}_F \simeq \mathrm{im}\ (I_F \to \Gamma_F^{\mathrm{ab}}),$$

which sends

$$(1 + \pi\mathcal{O}_F)^\times \xrightarrow{\sim} \mathrm{im}\ (P_F \to \Gamma_F^{\mathrm{ab}}).$$

Also, this is funtorial, namely if $F \simeq F'$, and extend it to an isomorphism $\overline{F} \simeq \overline{F'}$, which induces an isomorphism $\Gamma_F \simeq \Gamma_{F'}$ up to conjugacy, calling the abelianization of this isomorphism α_*^{ab}, then

$$\alpha_*^{\mathrm{ab}} \circ \mathrm{Art}_F = \mathrm{Art}_{F'} \circ \alpha_*^{\mathrm{ab}}.$$

2.3. **Weil group of F.** There is also a topological group W_F (not profinite). Here we take F to be a non-archimedean local field. It's related to the Galois group by the following. As an abstract group,

$$W_F = \{\gamma \in \Gamma_F \mid \exists n \in \mathbb{Z},\ \gamma\big|_{F^{\mathrm{nr}}} = \mathrm{Frob}_q^n\}.$$

This contains I_F. To topologize it, we dictate that I_F is an open subgroup with profinite topology. So W_F is a fibred product of topological groups.

(see photo)

We now give some motivation for this group. Then Art_F induces an isomorphism of topological groups

$$\mathrm{Art}_F^W : F^\times \xrightarrow{\sim} W_F^{\mathrm{ab}}.$$

Then note that

$$\mathrm{Art}_F : \mathcal{O}_F^\times \xrightarrow{\sim} \mathrm{inertialsubgroupof}\, \Gamma_F^{\mathrm{ab}}$$
$$F^\times/\mathcal{O}_F^\times \simeq \mathbb{Z} \to \mathrm{Gal}(F^{\mathrm{nr}}/F).$$

Now a comment on the proofs. There are two main proofs.

(a) Cohomological, see Artin-Tate, Cassels-Frohlich. For E/F finite and Galois, we want to show that

$$\mathrm{Art}\, E/F : F^\times/N_{E/F}E^\times \xrightarrow{\sim} \mathrm{Gal}(E/F)^{\mathrm{ab}}.$$

It proceeds by noting that the left-hand side has a group-cohomological interpretation, i.e. as $\hat{H}^0(G, E^\times)$ where the hat refers to Tate cohomology. Then we have that for $G = \mathrm{Gal}(E/F)$, $G^{\mathrm{ab}} = H_1(G,\mathbb{Z}) = \hat{H}^{-2}(G,\mathbb{Z})$. The main step is then to show that $H^2(G, E^\times) = \hat{H}^2(G, E^\times) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z} = H^2(\Gamma_F, \overline{F}^\times)$, the Brauer group $\mathrm{Br}(F)$ of F. Then define $\mathrm{Art}_{E/F}^{-1}$ to be the product with geneartor of $\hat{H}^2(G, E^\times)$, which is a map $\hat{H}^{-2}(G,\mathbb{Z}) \to \hat{H}^0(G, E^\times)$. It is then rather formal that this is an isomorphism, and a bonus is that it generalizes to duality theorems, where one looks at $H^*(G, M)$ for some arbitrary G-module M. One issue however is that it is not very explicit, and very much tied to abelian extensions.

(b) The other approach is with formal groups. Recall that $\mathbb{Q}_p^{\mathrm{ab}} = \mathbb{Q}_p^{\mathrm{nr}}(\mu_{p^\infty}) = \mathbb{Q}_p^{\mathrm{nr}}(\mathrm{torsionin}\hat{\mathbb{G}}_m)$, where $\hat{\mathbb{G}}_m$ is roughly $(1+\mathfrak{p}_{\overline{\mathbb{Q}_p}})^\times \supset \mu_{p^\infty}$. This generalizes to any $F/\mathbb{Q}_p$, $F^{\mathrm{ab}} = F^{\mathrm{nr}}(\mathrm{torsionin}\mathcal{G}_\pi)$, where $\mathcal{G}_\pi$ is the "Lubin-Tate formal group." Reference for this is Iwasawa, and a paper on LCFT by T. Yoshida. The advantage of this is that it is explicit and gives both Artin map and existence theorem. It has a natural generalization to non-abelian extensions. The downside is that it doesn't give the duality theorems.

(c) Finally, there is also Neukirch's method. Take E/F abelian and finite. Neukirch's idea is that for $g \in \mathrm{Gal}(E/F)$, there is only one possibility for $\mathrm{Art}_{E/F}^{-1}(g) \in F^\times/N_{E/F}E^\times$, due to the following lemma. There is only one possibility because if we look at $\langle g\rangle = \mathrm{Gal}(E/F') \subset \mathrm{Gal}(E/F)$ cyclic,

Lemma 2.4. *There exists a finite K/F' such that $K \cap E = F'$, so $\mathrm{Gal}(KE/K) \simeq \langle g\rangle$, and KE/K is unramified, and we have the diagram in the photo. So we have to have $g'\big|_E = g = \mathrm{Frob}_{KE/K}^a$, and $\mathrm{Art}_{KE/K}^{-1}(g') = \pi_K^a \pmod{N_{KE/K}(KE^\times)}$. Hence $\mathrm{Art}_{E/F}^{-1}(g) = N_{K/F}(\pi_K^a) \pmod{(N_{E/F}(E^\times))}$.*

The problem is to show that this doesn't depend on choices and is a homomorphism. But so far we haven't used any number theory, and it's these last two steps that involves the number theory.

For completeness, we should also describe the archimedean setting. If F is the complex numbers, then $W_{\mathbb{C}} = \mathbb{C}^\times$, and $\mathrm{Art}_{\mathbb{C}}^W$ is the identity map on $\mathbb{C}$. If F is the real numbers, then $\mathrm{Art}_F : \mathbb{R}^\times \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$. Then $W_{\mathbb{R}} = \langle \mathbb{C}^\times, \sigma \mid \sigma^2 = -1 \in \mathbb{C}^\times, \sigma z \sigma^{-1} = \overline{z} \forall z \in \mathbb{C}^\times \rangle$. Thus we have the s.e.s.

$$1 \to \mathbb{C}^\times \to W_{\mathbb{R}} \to \Gamma_{\mathbb{R}} \to 1$$

where we send $z \mapsto 0, \sigma \mapsto 1$. Then $\left(\mathrm{Art}_{\mathbb{R}}^W\right)^{-1} : W_{\mathbb{R}}^{\mathrm{ab}} \xrightarrow{\sim} \mathbb{R}^\times$ by taking $z \mapsto z\overline{z}, \sigma \mapsto -1$. While these look ad-hoc, they are not.

2.4. **Relative Weil groups.** Suppose we took F non-archimedean, E/F Galois. Define

$$W_{E/F} = \{\gamma \in \mathrm{Gal}(E^{\mathrm{ab}}/F) \mid \gamma\big|_{F^{\mathrm{nr}}} = \mathrm{Frob}_q^n, n \in \mathbb{Z}\} = W_F/\overline{[W_E, W_E]},$$

topologized with quotient topology. Then $W_{\overline{F}/F} = W_F$ and $W_{F/F} \simeq F^\times$ by local class field theory. Now take E/F finite. Then we have

$$1 \to \mathrm{Gal}(E^{\mathrm{ab}}/E) \to \mathrm{Gal}(E^{\mathrm{ab}}/F) \to \mathrm{Gal}(E/F) \to 1.$$

$$1 \to W_E^{\mathrm{ab}} \simeq E^\times \to W_{E/F} \to \mathrm{Gal}(E/F) \to 1.$$

So $\varprojlim E, \varprojlim \mathrm{norm} E^\times = \{1\}$. So $\overline{F}^\times$ is not visible in $W_F = \varprojlim W_{E/F}$.

We get some equality in the short exact sequences if and only if element of

$$H^2(\mathrm{Gal}(E/F), E^\times) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

2.5. **Global Class Field Theory.** By K being a global field, we mean a number field or the function field of a smooth projective absolutely irreducible curve over a finite field. Let Σ_K be the set of places of K, and for a number field this can be split into $\Sigma_{K,\infty}$ the infinite places and Σ_K^∞ the finite places, which are the embeddings into $\mathbb{C}$ and the prime ideals of $\mathcal{O}_K$, respectively. For a function field, the places are parameterized by the closed points of C, which are also the orbits of $\overline{\mathbb{F}_q}$ under the action of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$. Now if we have $v \in \Sigma_K$ we can consider the inclusion $K \hookrightarrow K_v$. Sometimes it's convenient to write $v|\infty$ to indicate that $K_v = \mathbb{R}$ or $\mathbb{C}$. If $v \in \Sigma_K^\infty$ then write $\mathcal{O}_v$ for the valuation ring of the completion.

It's convenient to normalize absolute values associated to these valuations. There are two different ways to normalize over p-adic fields. Suppose v is non-archimedean, then we have for π_v a uniformizer, $K_v \supset \mathcal{O}_v \supset \pi_v \mathcal{O}_v$, and let $q_v = |\mathcal{O}_v/\pi_v\mathcal{O}_v|$. Then the normalized absolute value associated to v is $|x|_v = q_v^{-v(x)}$. When v is real, we take $|x|_v = |x|$. When v is complex, we take $|x|_v = x\overline{x}$. Ultimately, the reason we choose these valuations is that if $x \in K^\times$, then

$$\prod_{v \in \Sigma_K} |x|_v = 1.$$

2.6. **Adeles and Ideles.** We write $\mathbb{A}_K$ for the restricted tensor product over all places of K. The restricted means that for all but finitely many $v \in \Sigma_K^\infty$, an element $(x_v)_v$, $x_v \in \mathcal{O}_v$. This is so that $\mathbb{A}_K$ is locally compact, and $\mathbb{Q}$ is actually discrete inside $\mathbb{A}_K$. Alternative notations for this are

(1) $K_\infty \times \hat{K}$, where $K_\infty = \mathbb{A}_{K,\infty} = \prod_{v|\infty} K_v = K \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, or nothing if char $K > 0$. Also, $\hat{K} = \mathbb{A}_K^\infty = \sum'_{v|\infty} K_v = \bigcup_{S \subset \Sigma_K^\infty, |S| < \infty} \prod_S K_v \times \prod_{v \in \Sigma_K^\infty \ S} \mathcal{O}_v$. Within this, $\hat{\mathcal{O}_K} = \prod_{v \not| \infty} \mathcal{O}_v$, which is the completion of $\mathcal{O}_K$ is the number field case. More precisely,

$$\hat{\mathcal{O}_K} = \varprojlim_{\mathfrak{a}} \mathcal{O}_K/\mathfrak{a} = \varprojlim_N \mathcal{O}_K/N\mathcal{O}_K = \mathcal{O}_K \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}.$$

(2) $\mathbb{A}_{K,\infty} \times \mathbb{A}_K^\infty$.

The ideles, or $J_K = \mathbb{A}_K^\times = \prod'_v K_v^\times$, where almost every entry is in $\mathcal{O}_v^\times$.

The topology on the adeles is taken so that $K_\infty \times \hat{\mathcal{O}_K}$ to be open with the product topology. The topology on the ideles is given in the same way, by taking $K_\infty^\times \times \hat{\mathcal{O}_K}^\times$ to be open with the product topology. However, this topology on J_K is not the induced topology given by the inclusion. It is actually the one indued from taking $J_K \hookrightarrow \mathbb{A}_K \times \mathbb{A}_K$ taking $x \mapsto (x, x^{-1})$. This is similar in how we make GL_n an affine variety.

A basic fact is that $K^\times \subset J_K$ is a discrete subgroup, and the idele class group is $C_K = J_K/K^\times$. This comes with a continuous homomorphism

$$|\cdot|_{\mathbb{A}_K} : (x_v)_v \mapsto \prod_{v \in \Sigma_K} |x_v|_v,$$

which defines a map $|\cdot|_{\mathbb{A}} : C_K \to \mathbb{R}_{>0}^\times$, with *compact* kernel! This is the conjunction of two theorems: finiteness of the class number and Dirichlet's unit theorem.

Let's look at $K = \mathbb{Q}$ and $J_{\mathbb{Q}} = \mathbb{R}^\times \times \prod'_p \mathbb{Q}_p^\times$. Then there exists a unique $y \in \mathbb{Q}^\times$ such that $\mathrm{sgn}(y) = \mathrm{sgn}(x_\infty)$ and for all p, $v_p(y) = v_p(x_p)$. We can actually write $J_{\mathbb{Q}} = \mathbb{Q}^\times \times (\mathbb{R}_{>0}^\times \times \prod_p \mathbb{Z}_p^\times)$, which the product of a discrete and locally compact, so

$$C_{\mathbb{Q}} = \mathbb{R}_{>0}^\times \times \hat{\mathbb{Z}}^\times = \mathbb{R}_{>0}^\times \times \ker|\cdot|_{\mathbb{A}}.$$

Now $C_{\mathbb{Q}} \supset \mathbb{R}_{>0}^\times$ is the maximal connected subgroup and the intersection of all open subgroups containing 1. The other piece is totally disconnected as it's profinite. Also, $\pi_0(C_K) = C_K/C_K^0$, so $\pi_0(C_{\mathbb{Q}}) = \hat{\mathbb{Z}}^\times \simeq \mathrm{Gal}(\mathbb{Q}(\{\mu_n\}_n)/\mathbb{Q})$. So global class field theory is just a generalization of this to all number fields.

We elaborate on this remark. For L/K finite Galois, v a place of K, place $w|v$ of L (for infinite places this just means that $L \hookrightarrow L_w = \mathbb{R}$ or $\mathbb{C}$ extends $K \hookrightarrow K_v$. Then we have the decomposition group

$$\mathrm{Gal}(L_w/K_v) \subset \mathrm{Gal}(L/K).$$

If L/K is abelian, this depends only on v. Now we can define the global Artin map

$$\mathrm{Art}_{L/K} : J_K \to \mathrm{Gal}(L/K),$$

if L/K is abelian. Note that if $x_v \in \mathcal{O}_v^\times$, where L/K is unramified at $v \in \Sigma_K^\infty$, then $\mathrm{Art}_{L_w} K_v(x_v) = 1$. So we can define for an arbitrary idele,

$$\mathrm{Art}_{L/K}((x_v)_v) = \prod_v \mathrm{Art}_{L_w/K_v}(x_v).$$

Now if we pass to the limit over L/K, we get

$$\mathrm{Art}_K : J_K \to \Gamma_K^{\mathrm{ab}}.$$

As an exercise, check continuity of these maps.

Then we have the following

Theorem 2.5 (Artin Reciprocity)**.** $\mathrm{Art}_K(K^\times) = \{1\}$*, so defines a map* $\mathrm{Art}_K : C_K \to \Gamma_K^{\mathrm{ab}}$*.*

Something about something being profinite, so this can't be an isomorphism of topological groups. So we need to look at kernel and image.