Chapter 11

Matrix Inverse and Condition

Chapter Objectives

- Understanding the meaning of matrix and vector norms and how they are computed.
- Knowing how to use norms to compute the matrix condition number.
- Understanding how the magnitude of the condition number can be used to estimate the precision of solutions of linear algebraic equations.

Matrix Inverse

- Recall that if a matrix $[A]$ is square, there is another matrix $[A]^{-1}$, called the inverse of $[A]$, for which $[A][A]^{-1}=[A]^{-1}[A]=[I]$

Vector and Matrix Norms

- A *norm* is a real-valued function that provides a measure of the size or "length" of multi-component mathematical entities such as vectors and matrices.
- Vector norms and matrix norms may be computed in different ways.

Vector Norms

- For a vector $\{X\}$ of size *n*, the *p*-norm is:

$$\|X\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

- Important examples of vector *p*-norms include:

$$p = 1 : \text{sum of the absolute values} \quad \|X\|_1 = \sum_{i=1}^{n} |x_i|$$

$$p = 2 : \text{Euclidian norm (length)} \quad \|X\|_2 = \|X\|_e = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$$p = \infty : \text{maximum} - \text{magnitude} \quad \|X\|_\infty = \max_{1 \le i \le n} |x_i|$$

Consider the vector B=[1 -2 0]

$||B||_1$ =np.linalg.norm (B,1)= 3

$||B||_2$=np.linalg.norm(B,2)=np.linalg.norm(B)=2.236

$||B||_\infty$ =np.linalg.norm (B,np.inf)= 2

Matrix Norms

- Common matrix norms for a matrix [A] include:

column-sum norm $\quad \|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$

Frobenius norm $\quad \|A\|_f = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2}$

row-sum norm $\quad \|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$

spectral norm (2 norm) $\quad \|A\|_2 = (\mu_{\max})^{1/2}$

- Note that μ_{max} is the largest eigenvalue of A^T or A.

Consider A=[1 0 1

0 1 1

1 2 4].

$||A||_1$ =np.linalg.norm(A,1)= 6,

$||A||_{\infty}$=np.linalg.norm (A,np.inf)= 7

$||A||_f$ =np.linalg.norm (A,'fro')= $\sqrt{25}$= 5

$||A||_2$ =np.linalg.norm (A,2)=np.linalg.norm (A)= 4.89 (largest eigenvalue)

Matrix Condition Number

- The *matrix condition number* Cond[A] is obtained by calculating Cond[A]=$\|A\| \cdot \|A^{-1}\|$

  Using A above find cond(A). Also calculate norm(A)*norm(inv(A))

- It can be shown that for $AX=b$:

$$\frac{\|\Delta X\|}{\|X\|} \le \mathrm{Cond}[A] \frac{\|\Delta A\|}{\|A\|}$$

- The relative error of the norm of the computed solution can be as large as the relative error of the norm of the coefficients of [A] multiplied by the condition number.

Summary: Python Commands

- Python has built-in functions to compute both norms and condition numbers:
  - **`norm(`*`X`*`,`*`p`*`)`**

    Compute the *p* norm of vector *X*, where *p* can be any number, `inf`, or `'fro'` (for the Euclidean norm)
  - **`norm(`*`A`*`,`*`p`*`)`**

    Compute a norm of matrix *A*, where *p* can be 1, 2, `inf`, or `'fro'` (for the Frobenius norm)
  - **`cond(`*`X`*`,`*`p`*`)` or `cond(`*`A`*`,`*`p`*`)`**

    Calculate the condition number of vector *X* or matrix *A* using the norm specified by *p*.

n = np.linalg.norm(A,p) returns a different kind of norm, depending on the value of p.

If p is...	Then norm returns...
1	The 1-norm, or largest column sum of A, max(sum(abs(A))).
2	The largest singular value (same as norm(A)).
inf	The infinity norm, or largest row sum of A, max(sum(abs(A'))).
'fro'	The Frobenius-norm of matrix A, sqrt(sum(diag(A'*A)))

Example

Solve by hand the equations

$$x + 2y = 1$$
$$3x - y = 2$$

And the slightly perturbed equation

$$x + 2y = 1$$
$$3.001x - y = 2$$

The determinant of A is -7

cond(A)=1.46