Phaseful Hadamard Type

1 Introduction

Hadamard type is used to represent all quantum states with superposition yet without entanglement, i.e.,

$$|\phi_n\rangle = \prod_{i=0}^{n} \left(\alpha_i \left(|0\rangle + \omega_N^{k_i} |1\rangle \right) \right).$$

$$|\phi_n\rangle = \prod_{i=0}^{n} \left(\frac{1}{\sqrt{2^n}} \left(|0\rangle + \omega_N^{k_i} |1\rangle \right) \right).$$

Since amplitudes and kets are known for the given shape of states, only phases needs to be recorded. `(nat<seq>, nat)` is sufficient as a representation of a `Had`-typed quantum state. The type is purely *phaseful*.

2 Phase Kickback

We may define **phase kickback** as a special semantics for the phaseful Hadamard type. Given a common phase kickback setting,

$$U_f \left(\frac{1}{\sqrt{2}} |\phi_n\rangle \otimes (|0\rangle - |1\rangle) \right) = \frac{1}{\sqrt{2}} |\phi_n\rangle \otimes (-1)^{f(x)} (|0\rangle - |1\rangle)$$

where $f(x, y) = (x, y + f(x))$. In our language, this operation is a special case that interacts an `En` state with a *single-qubit* `Had` state. The general strategy would be coverting the `Had` state to `En`, taking the cartesian product with the existing `En` state and applying f over the entangled state as an entity. This results in a convoluted representation of phase kickback:

$$\begin{aligned}
&\sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |0 + f(x)\rangle - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |1 + f(x)\rangle && (1)\\
&= \begin{bmatrix} f(x) = 0 \Rightarrow \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |0\rangle - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |1\rangle \\ f(x) = 1 \Rightarrow \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |1\rangle - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |0\rangle \end{bmatrix} nn && (2)\\
&= \begin{bmatrix} f(x) = 0 \Rightarrow \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |0\rangle - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |1\rangle \\ f(x) = 1 \Rightarrow -\sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |1\rangle + \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |0\rangle \end{bmatrix} && (3)\\
&\equiv \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |0\rangle - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |1\rangle && (4)
\end{aligned}$$

This is undesireble because

1. it requires the equality modulo permutation to prove (4), and
2. proofs by cases over the codomain of $f(x)$ are needed to bridge the relation between kets and phases.

3 A Phaseful Approach

An alternative approach is to axoimatize the phase kickback phenonmenon using the fact that `Had` type itself is phaseful: if an oracle operator application involves a `Had` range/locus, its effect over kets should be treated phasefully. Given effect $f(x, y) = (x, y + f(x))$, this is translated using phase calculus only.

$$\sum_{x \in \phi_n} \frac{1}{\sqrt{2}} \omega_2^{f(x)} |x\rangle |0\rangle + \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} \omega_2^{1-f(x)} |x\rangle |1\rangle$$

Generalizing to arbitrary `Had` type, we have

$$\sum_{x \in \phi_n} \frac{1}{\sqrt{2}} \omega_N^{k(0+f(x))} |x\rangle |0\rangle + \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} \omega_N^{k(1-f(x))} |x\rangle |1\rangle .$$

For $|+\rangle$, take $k = 0$, and the oracle operator is equivalent to identity.

`Had` type requires the $|0\rangle$ term to have the phase shift 1. It is tempting to add a global phase shift to cancel this to keep `Had` an `En` locus seperate. However, this is incorrect because the shift added is *local* and dependent of x:

$$\sum_{x \in \phi_n} \omega_N^{k(0+f(x))} \frac{1}{\sqrt{2}} \left(|x\rangle |0\rangle + \omega_N^{k(1-2f(x))} |x\rangle |1\rangle \right) .$$

Therefore, the oracle still entangles `En` and `Had` as others do, although the effect is purely phaseful.

4 Generalization: n-qubit `Had` States

TODO