Phaseful Hadamard Type

1 Introduction

Hadamard type is used to represent all quantum states with superposition yet without entanglement, i.e.,

$$|\phi_n\rangle = \prod_{i=0}^{n} \left(\alpha_i \left(|0\rangle + \omega_N^{k_i} |1\rangle \right) \right).$$

$$|\phi_n\rangle = \prod_{i=0}^{n} \left(\frac{1}{\sqrt{2^n}} \left(|0\rangle + \omega_N^{k_i} |1\rangle \right) \right).$$

Since amplitudes and kets are known for the given shape of states, only phases needs to be recorded. `(nat<seq>, nat)` is sufficient as a representation of a `Had`-typed quantum state. The type is purely *phaseful*.

2 Phase Kickback

We may define **phase kickback** as a special semantics for the phaseful Hadamard type. Given a common phase kickback setting,

$$U_f \left(\frac{1}{\sqrt{2}} |\phi_n\rangle \otimes (|0\rangle - |1\rangle) \right) = \frac{1}{\sqrt{2}} |\phi_n\rangle \otimes (-1)^{f(x)} (|0\rangle - |1\rangle)$$

where $f(x,y) = (x, y + f(x))$. In our language, this operation is a special case that interacts an `En` state with a *single-qubit* `Had` state. The general strategy would be coverting the `Had` state to `En`, taking the cartesian product with the existing `En` state and applying f over the entangled state as an entity. This results in a convoluted representation of phase kickback:

$$\sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |0 + f(x)\rangle - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |1 + f(x)\rangle \tag{1}$$

$$= \left[\begin{array}{l} f(x) = 0 \Rightarrow \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |0\rangle - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |1\rangle \\[2ex] f(x) = 1 \Rightarrow \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |1\rangle - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |0\rangle \end{array} \right. \tag{2}$$

$$= \left[\begin{array}{l} f(x) = 0 \Rightarrow \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |0\rangle - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |1\rangle \\[2ex] f(x) = 1 \Rightarrow - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |1\rangle + \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |0\rangle \end{array} \right. \tag{3}$$

$$\equiv \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |0\rangle - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |1\rangle \tag{4}$$

This is undesireble because

1. it requires the equality modulo permutation to prove (4), and

2. proofs by cases over the codomain of $f(x)$ are needed to bridge the relation between kets and phases.

3   A Phaseful Approach

An alternative approach is to axoimatize the phase kickback phenonmenon using the fact that Had type itself is phaseful: if an oracle operator application involves a Had range/locus, its effect over kets should be treated phasefully. Given effect $f(x, y) = (x, y + f(x))$, this is translated using phase calculus only.

$$\sum_{x \in \phi_n} \frac{1}{\sqrt{2}} \omega_2^{f(x)} |x\rangle |0\rangle + \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} \omega_2^{1-f(x)} |x\rangle |1\rangle$$

Generalizing to arbitrary Had type, we have

$$U_f \left(\sum_{x \in \phi_n} \frac{1}{\sqrt{2}} \omega_N^0 |x\rangle |0\rangle + \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} \omega_N^{k_x} |x\rangle |1\rangle \right) \tag{5}$$

$$= \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} \omega_N^0 |x\rangle |0 + f(x)\rangle + \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} \omega_N^{k_x} |x\rangle |1 + f(x)\rangle \tag{6}$$

$$= \sum_{x \in \phi_n} \frac{1}{\sqrt{2^N}} \begin{bmatrix} f(x) = 0 \Rightarrow \omega_N^0 |x\rangle |0\rangle + \omega_N^{k_x} |x\rangle |1\rangle \\ f(x) = 1 \Rightarrow \omega_N^0 |x\rangle |1\rangle + \omega_N^{k_x} |x\rangle |0\rangle \end{bmatrix} \tag{7}$$

$$= \sum_{x \in \phi_n} \frac{1}{\sqrt{2^N}} \begin{bmatrix} f(x) = 0 \Rightarrow \omega_N^0 |x\rangle |0\rangle + \omega_N^{k_x} |x\rangle |1\rangle \\ f(x) = 1 \Rightarrow \omega_N^{k_x} |x\rangle |0\rangle + \omega_N^0 |x\rangle |1\rangle \end{bmatrix} \tag{8}$$

$$= \sum_{x \in \phi_n} \frac{1}{\sqrt{2^N}} \begin{bmatrix} f(x) = 0 \Rightarrow \omega_N^{k_x(0+0)} |x\rangle |0\rangle + \omega_N^{k_x(1-0)} |x\rangle |1\rangle \\ f(x) = 1 \Rightarrow \omega_N^{k_x(0+1)} |x\rangle |0\rangle + \omega_N^{k_x(1-1)} |x\rangle |1\rangle \end{bmatrix} \tag{9}$$

$$\equiv \sum_{x \in \phi_n} \frac{1}{\sqrt{2^N}} \omega_N^{k_x(0+f(x))} |x\rangle |0\rangle + \sum_{x \in \phi_n} \frac{1}{\sqrt{2^N}} \omega_N^{k_x(1-f(x))} |x\rangle |1\rangle \tag{10}$$

$$= \sum_{x \in \phi_n} \frac{1}{\sqrt{2^N}} \omega_N^{k_x \cdot f(x)} |x\rangle |0\rangle + \sum_{x \in \phi_n} \frac{1}{\sqrt{2^N}} \omega_N^{k_x(1-f(x))} |x\rangle |1\rangle \tag{11}$$

3.1   Effects on Different Phase Degrees

- Zeroth degree phase corresponds to $k = 0$. For $|+\rangle$ where $k = 0$, and the oracle operator is equivalent to identity.
- First degree phase

3.2   Entanglement is a Must

Had type requires the $|0\rangle$ term to have the phase shift 1. It is tempting to add a global phase shift to cancel this to keep Had an En locus seperate. However, this is incorrect because the shift added is *local* and dependent of x:

$$\sum_{x \in \phi_n} \omega_N^{k(0+f(x))} \frac{1}{\sqrt{2}} \left(|x\rangle |0\rangle + \omega_N^{k(1-2f(x))} |x\rangle |1\rangle \right).$$

Therefore, the oracle still entangles En and Had as others do, although the effect is purely phaseful.

4   Generalization: n-qubit Had States

TODO