

Phaseful Hadamard Type

1 Introduction

Hadamard type is used to represent all quantum states with superposition yet without entanglement, i.e.,

$$|\phi_n\rangle = \prod_{i=0}^n \left(\alpha_i \left(|0\rangle + \omega_N^{k_i} |1\rangle \right) \right).$$

$$|\phi_n\rangle = \prod_{i=0}^n \left(\frac{1}{\sqrt{2^n}} \left(|0\rangle + \omega_N^{k_i} |1\rangle \right) \right).$$

Since amplitudes and kets are known for the given shape of states, only phases needs to be recorded. (nat<seq>, nat) is sufficient as a representation of a Had-typed quantum state. The type is purely *phaseful*.

2 Phase Kickback

We may define **phase kickback** as a special semantics for the phaseful Hadamard type. Given a common phase kickback setting,

$$U_f \left(\frac{1}{\sqrt{2}} |\phi_n\rangle \otimes (|0\rangle - |1\rangle) \right) = \frac{1}{\sqrt{2}} |\phi_n\rangle \otimes (-1)^{f(x)} (|0\rangle - |1\rangle)$$

where $f(x, y) = (x, y + f(x))$. In our language, this operation is a special case that interacts an En state with a *single-qubit* Had state. The general strategy would be coverting the Had state to En, taking the cartesian product with the existing En state and applying f over the entangled state as an entity. This results in a convoluted representation of phase kickback:

$$\sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |0 + f(x)\rangle - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |1 + f(x)\rangle \quad (1)$$

$$= \begin{cases} f(x) = 0 \Rightarrow \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |0\rangle - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |1\rangle \\ f(x) = 1 \Rightarrow \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |1\rangle - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} |x\rangle |0\rangle \end{cases} \quad (2)$$

$$= \begin{cases} f(x) = 0 \Rightarrow \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |0\rangle - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |1\rangle \\ f(x) = 1 \Rightarrow - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |1\rangle + \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |0\rangle \end{cases} \quad (3)$$

$$\equiv \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |0\rangle - \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} (-1)^{f(x)} |x\rangle |1\rangle \quad (4)$$

This is undesirable because

1. it requires the equality modulo permutation to prove (4), and
2. proofs by cases over the codomain of $f(x)$ are needed to bridge the relation between kets and phases.

3 A Phaseful Approach

An alternative approach is to axiomatize the phase kickback phenomenon using the fact that Had type itself is phaseful: if an oracle operator application involves a Had range/locus, its effect over kets should be treated phasefully. Given effect $f(x, y) = (x, y + f(x))$, this is translated using phase calculus only.

$$\sum_{x \in \phi_n} \frac{1}{\sqrt{2}} \omega_2^{f(x)} |x\rangle |0\rangle + \sum_{x \in \phi_n} \frac{1}{\sqrt{2}} \omega_2^{1-f(x)} |x\rangle |1\rangle$$

Generalizing to arbitrary Had type, we have

$$U_f \left(\left(\sum_{x \in \phi_n} \alpha_x \omega_N^{k_x} |x\rangle \right) \otimes \frac{1}{\sqrt{2}} (\omega_N^0 |0\rangle + \omega_N^k |1\rangle) \right) \quad (5)$$

$$= \sum_{x \in \phi_n} \alpha_x \omega_N^{k_x} |x\rangle |0 + f(x)\rangle + \sum_{x \in \phi_n} \alpha_x \omega_N^{k_x + k} |x\rangle |1 + f(x)\rangle \quad (6)$$

$$= \sum_{x \in \phi_n} \frac{1}{\sqrt{2^N}} \left[f(x) = 0 \Rightarrow \omega_N^{k_x} |x\rangle |0\rangle + \omega_N^{k_x + k} |x\rangle |1\rangle \right] \quad (7)$$

$$\equiv \sum_{x \in \phi_n} \frac{1}{\sqrt{2^N}} \left[f(x) = 0 \Rightarrow \omega_N^{k_x} |x\rangle |0\rangle + \omega_N^{k_x + k} |x\rangle |1\rangle \right] \quad (8)$$

$$= \sum_{x \in \phi_n} \frac{1}{\sqrt{2^N}} \omega_N^{k_x} \left[f(x) = 0 \Rightarrow \omega_N^0 |x\rangle |0\rangle + \omega_N^k |x\rangle |1\rangle \right] \quad (9)$$

$$\equiv \sum_{x \in \phi_n} \frac{1}{\sqrt{2^N}} \omega_N^{k_x} \omega_N^{k(0+f(x))} |x\rangle |0\rangle + \sum_{x \in \phi_n} \frac{1}{\sqrt{2^N}} \omega_N^{k_x} \omega_N^{k(1-f(x))} |x\rangle |1\rangle \quad (10)$$

$$= \sum_{x \in \phi_n} \frac{1}{\sqrt{2^N}} \omega_N^{k_x + k \cdot f(x)} |x\rangle |0\rangle + \sum_{x \in \phi_n} \frac{1}{\sqrt{2^N}} \omega_N^{k_x + k(1-f(x))} |x\rangle |1\rangle \quad (11)$$

3.1 Effects on Different Phase Degrees

- Zeroth degree phase corresponds to $k = 0$. For $|+\rangle$ where $k = 0$, and the oracle operator is equivalent to identity.
- 1st degree phases is exactly what's here.

3.2 Entanglement is a Must

Had type requires the $|0\rangle$ term to have the phase shift 1. It is tempting to add a global phase shift to cancel this to keep Had an En locus separate. However, this is incorrect because the shift added is *local* and dependent of x :

$$\sum_{x \in \phi_n} \omega_N^{k(0+f(x))} \frac{1}{\sqrt{2}} \left(|x\rangle |0\rangle + \omega_N^{k(1-2f(x))} |x\rangle |1\rangle \right).$$

Therefore, the oracle still entangles En and Had as others do, although the effect is purely phaseful.

4 Generalization: n -qubit Had States

TODO