

# **Link Analysis (Part II)**

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# OUTLINE

- Link Analysis (Part II)
  - Solving the flow model:  $Mv = v$ 
    - Power Iteration and Gaussian Elimination
    - Dead-ends and spider traps (teleporting and taxation)
    - PageRank for Search Engines (MapReduce)
  - Topic-specific: (augment for general-popularity)
  - Trust-Rank: (fighting against link spams)
  - Hubs-and-Authorities: (multiple importances)

# **RECALL: THE FLOW MODEL**

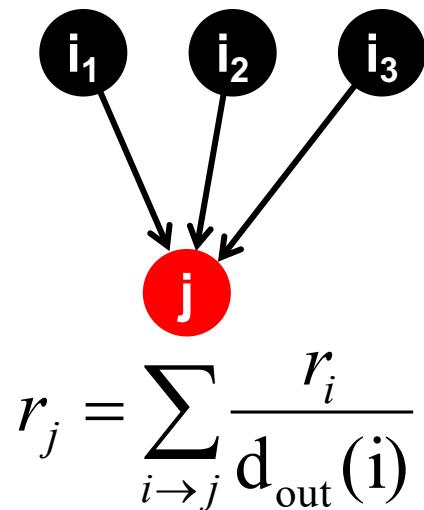
# Recall: Random Walk Interpretation

## ■ Imagine a random web surfer:

- At any time  $t$ , surfer is on some page  $i$
- At time  $t+1$ , the surfer follows an out-link from  $i$  uniformly at random
- Ends up on some page  $j$  linked from  $i$
- Process repeats indefinitely

## ■ Let:

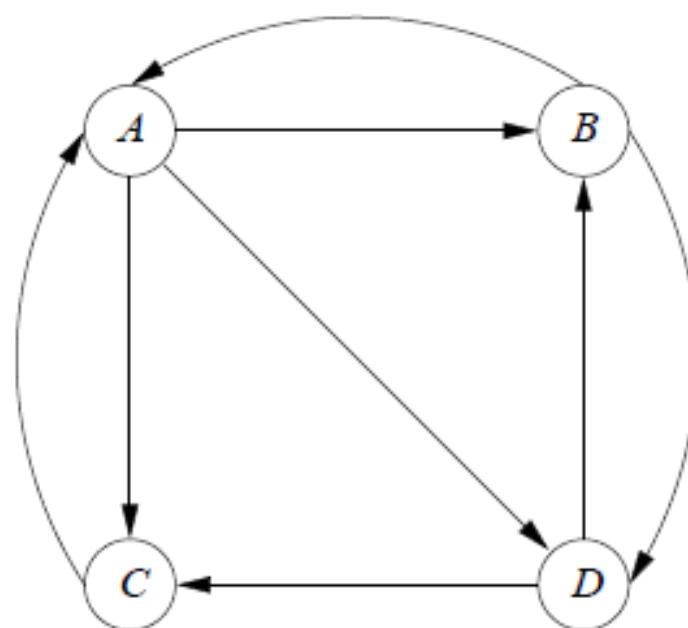
- $p(t)$  ... vector whose  $i^{\text{th}}$  coordinate is the prob. that the surfer is at page  $i$  at time  $t$
- So,  $p(t)$  is a probability distribution over pages
  - -> rank vector



# Recall: Transition Matrix

- $M[i,j] = \text{prob. of going from node } j \text{ to node } i$ 
  - If  $j$  has  $k$  outgoing edges, prob. for each edge =  $1/k$

$$M = \begin{matrix} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} \end{matrix}$$



# Prob. of Locations of Surfer

- Represented as a **column vector**,  $v$
- Initially, surfer can be at any page with equal probability

$$v^0 = \begin{matrix} A \\ B \\ C \\ D \end{matrix} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

# Limiting Distribution

- ◆  $v^1 = Mv^0$
- ◆  $v^2 = Mv^1 (= M^2 v^0)$
- ◆ ...
- ◆  $v^i = Mv^{i-1} (= M^i v^0)$
- ◆ ...
- ◆  $v = Mv$

(from some step  $k$  on, **v** does not change any more)

# Compute Next Distribution

$$\blacklozenge v^1 = Mv^0$$

$$\blacktriangleright v_i^1 = \sum_{j=1}^n M_{ij} v_j^0$$

$$\blacklozenge \text{E.g., } v_0^1 = 0 * \frac{1}{4} + \frac{1}{2} * \frac{1}{4} + 1 * \frac{1}{4} + 0 * \frac{1}{4} = \frac{3}{8}$$

$\blacktriangleright$  i.e., prob. at A is 3/8 (or 9/24) after step 1

$$\begin{array}{cccc} & \text{A} & \text{B} & \text{C} & \text{D} \\ \text{A} & 0 & 1/2 & 1 & 0 \\ \text{B} & 1/3 & 0 & 0 & 1/2 \\ \text{C} & 1/3 & 0 & 0 & 1/2 \\ \text{D} & 1/3 & 1/2 & 0 & 0 \end{array} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$v^0 = \begin{bmatrix} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \quad v^1 = \begin{bmatrix} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{bmatrix} \begin{bmatrix} 9/24 \\ 5/24 \\ 5/24 \\ 5/24 \end{bmatrix}$$

# **SOLVING $MV = V$**

# Two Approaches for Solving $Mv = v$

- Power Iteration
  - Repeatedly solving  $Mv^{k+1} = v^k$
- Finding the Eigenvectors
  - Solving  $Mv = \lambda v \rightarrow (M - I)v = 0$
  - Gaussian Elimination

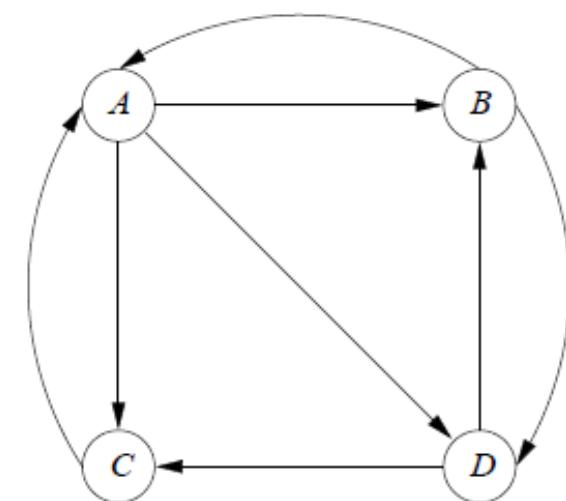
# Solving $Mv = v$ by Power Iteration

- Note:  $\sum_{i=1}^n v_i^k = 1$ , after every step k
- Usually stop when little change btw iterations
- In practice, 50-75 iterations for Web

$$\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \quad \begin{bmatrix} 9/24 \\ 5/24 \\ 5/24 \\ 5/24 \end{bmatrix}$$

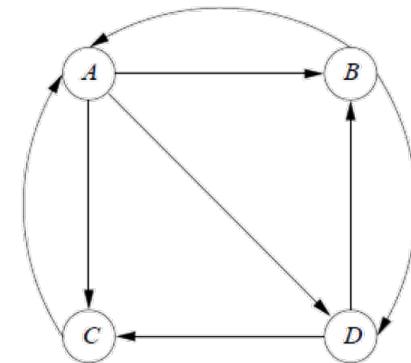
NOTE:  $x$  is an eigenvector with the corresponding eigenvalue  $\lambda$  if:  
 $Ax = \lambda x$

$$\begin{bmatrix} 11/32 \\ 7/32 \\ 7/32 \\ 7/32 \end{bmatrix}, \dots, \begin{bmatrix} 3/9 \\ 2/9 \\ 2/9 \\ 2/9 \end{bmatrix}$$



# Solving $Mv = v$ by Finding Eigenvector

$$M = \begin{bmatrix} A & B & C & D \\ A & 0 & 1/2 & 1 & 0 \\ B & 1/3 & 0 & 0 & 1/2 \\ C & 1/3 & 0 & 0 & 1/2 \\ D & 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$



- $Mv = \lambda v$ 
  - $\lambda$ : eigenvalue,  $v$ : eigenvector
- $M$  is a (left) stochastic matrix
  - Each column adds up to 1
- Largest eigenvalue of a stochastic matrix is 1
  - Corresponding eigenvector is principal eigenvector
  - Proof in the next few slides

# Characteristic Polynomial

Calculating the Eigenvalues

- $Mv = \lambda v$
- $(M - \lambda I)v = 0$
- $v$  is not a null vector , so

–  $\det(M - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 1/2 & 1 & 0 \\ 1/3 & -\lambda & 0 & 1/2 \\ 1/3 & 0 & -\lambda & 1/2 \\ 1/3 & 1/2 & 0 & -\lambda \end{vmatrix} = 0$$

$$M = \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

I is the identity matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} |A| &= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= aei + bfg + cdh - ceg - bdi - afh. \end{aligned}$$

Compute the determinant using cofactors in this column

# Characteristic Polynomial

$$\begin{vmatrix} \frac{1}{3} & -\lambda & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & -\lambda \end{vmatrix} - \lambda \begin{vmatrix} -\lambda & \frac{1}{2} & 0 \\ \frac{1}{3} & -\lambda & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & -\lambda \end{vmatrix}$$

$$= \lambda \begin{vmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & -\lambda \end{vmatrix} - 1/2 \begin{vmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{vmatrix} - \lambda(-1/2) \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{vmatrix} - \lambda \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{3} & -\lambda \end{vmatrix}$$

$$= \lambda \left( -\frac{1}{3} \lambda - \frac{1}{6} \right) - \lambda \left[ -\frac{1}{2} \left( -\frac{1}{2} \lambda - \frac{1}{6} \right) - \lambda \left( \lambda^2 - \frac{1}{6} \right) \right]$$

$$= \lambda^4 - \frac{3}{4} \lambda^2 - \frac{\lambda}{4} = \lambda(\lambda-1)(\lambda^2 + \lambda + \frac{1}{4}) = \lambda(\lambda-1)(\lambda + \frac{1}{2})^2$$

# Characteristic Polynomial

- $Mv = \lambda v$
- $(M - \lambda I)v = 0$
- $\det(M - \lambda I) = 0$

$$M = \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1/2 & 1 & 0 \\ 1/3 & -\lambda & 0 & 1/2 \\ 1/3 & 0 & -\lambda & 1/2 \\ 1/3 & 1/2 & 0 & -\lambda \end{vmatrix} = 0$$

$$\lambda(\lambda-1)(\lambda + \frac{1}{2})^2 = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = \lambda_4 = -\frac{1}{2}$$

# Solving $Mv = v$ by Finding Eigenvector

- $Mv = \lambda v$
- $(M - \lambda I)v = 0$
- $\det(M - \lambda I) = 0$

$$M = \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

$$\lambda(\lambda-1)(\lambda+\frac{1}{2})^2=0 \Rightarrow \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = \lambda_4 = -\frac{1}{2}$$

- Problem becomes solving  $(M - I)v = 0$
- Use Gaussian elimination

# Solving $Mv = v$ by Gaussian Elimination

$$\bullet (M - I)v = 0$$

$$\begin{bmatrix} -1 & 1/2 & 1 & 0 \\ 1/3 & -1 & 0 & 1/2 \\ 1/3 & 0 & -1 & 1/2 \\ 1/3 & 1/2 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -1 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & -1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & -\frac{1}{2} & -1 & 0 \\ \frac{1}{3} & -1 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & -1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & -\frac{5}{6} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \\ 0 & \frac{2}{3} & \frac{1}{3} & -1 \end{bmatrix}$$

# Solving $Mv = v$ by Gaussian Elimination

$$\bullet (M - I)v = 0$$

$$\begin{bmatrix} -1 & 1/2 & 1 & 0 \\ 1/3 & -1 & 0 & 1/2 \\ 1/3 & 0 & -1 & 1/2 \\ 1/3 & 1/2 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & -\frac{5}{6} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \\ 0 & \frac{2}{3} & \frac{1}{3} & -1 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & -\frac{6}{15} & -\frac{3}{5} \\ 0 & \frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \\ 0 & \frac{2}{3} & \frac{1}{3} & -1 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & -\frac{6}{15} & -\frac{3}{5} \\ 0 & 0 & -\frac{3}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & -\frac{3}{5} \end{bmatrix}$$

# Solving $Mv = v$ by Gaussian Elimination

$$\bullet (M - I)v = 0$$

$$\begin{bmatrix} -1 & 1/2 & 1 & 0 \\ 1/3 & -1 & 0 & 1/2 \\ 1/3 & 0 & -1 & 1/2 \\ 1/3 & 1/2 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & -\frac{6}{15} & -\frac{3}{5} \\ 0 & 0 & -\frac{3}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & -\frac{3}{5} \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & -\frac{6}{15} & -\frac{3}{5} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\xrightarrow{\quad} \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Solving $Mv = v$ by Gaussian Elimination

$$\bullet (M - I)v = 0$$

$$\begin{bmatrix} -1 & 1/2 & 1 & 0 \\ 1/3 & -1 & 0 & 1/2 \\ 1/3 & 0 & -1 & 1/2 \\ 1/3 & 1/2 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[ \begin{array}{cccc} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \left. \begin{array}{l} v_1 - 3/2 v_4 = 0 \\ v_2 - v_4 = 0 \\ v_3 - v_4 = 0 \\ v_1 + v_2 + v_3 + v_4 = 1 \end{array} \right\} \quad \begin{array}{l} v_1 = 1/3 \\ v_2 = v_3 = v_4 = 2/9 \end{array}$$

# Comparison of Solutions

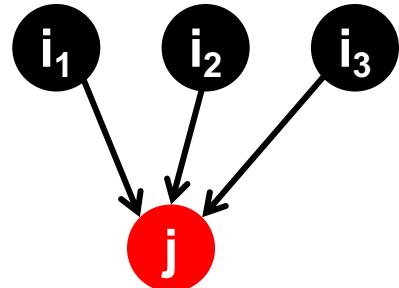
- $Mv = v$ 
  - Solution 1: power iteration
  - Solution 2: Gaussian elimination
- Power iteration
  - Complexity:  $O(kn^2)$ , where  $k$  = # of iterations
- Gaussian elimination
  - Complexity:  $O(n^3)$
  - $n^2 + (n-1)^2 + \dots + 2^2$

# **SPIDER TRAPS AND DEAD-ENDS**

# The Stationary Distribution

- **Where is the surfer at time  $t+1$ ?**

- Follows a link uniformly at random



$$p(t + 1) = M \cdot p(t)$$

- Suppose the random walk reaches a state  
then is **stationary distribution** of a random walk
- **Our original rank vector** satisfies
  - So, is a stationary distribution for  
the random walk

# Existence and Uniqueness

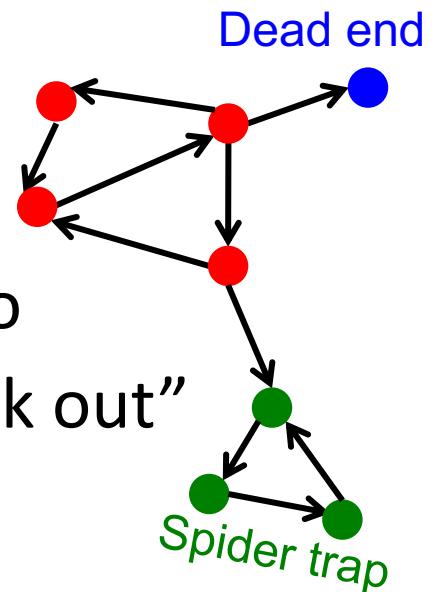
- A central result from the theory of random walks (a.k.a. Markov processes):

For graphs that satisfy certain conditions,  
the stationary distribution is unique and  
eventually will be reached no matter what the  
initial probability distribution at time  $t = 0$

# PageRank: Problems

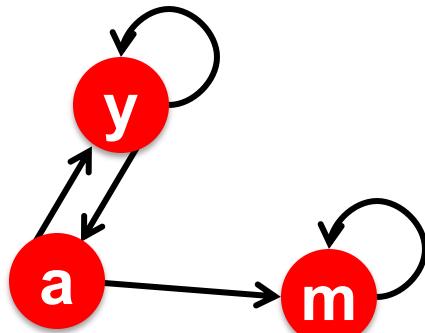
## Two problems:

- (1) Some pages are **dead ends** (have no out-links)
  - Random walk has “nowhere” to go to
  - Such pages cause importance to “leak out”
- (2) **Spider traps:**  
(all out-links are within the group)
  - Random walked gets “stuck” in a trap
  - And eventually spider traps absorb all importance



# Problem: Spider Traps

All outlinks are within a group



m is a spider trap

	y	a	m
y	1/2	1/2	0
a	1/2	0	0
m	0	1/2	1

$$\mathbf{r}^{k+1} = \mathbf{M}\mathbf{r}^k$$

$$r_y = r_y/2 + r_a/2$$

$$r_a = r_y/2$$

$$r_m = r_a/2 + r_m$$

- Example:

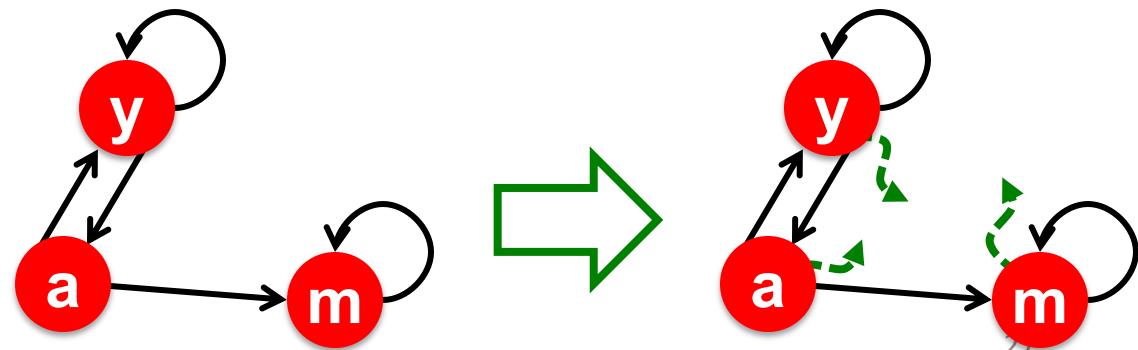
$$\begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{matrix} 1/3 & 2/6 & 3/12 & 5/24 & 0 \\ 1/3 & 1/6 & 2/12 & 3/24 & \dots & 0 \\ 1/3 & 3/6 & 7/12 & 16/24 & & 1 \end{matrix}$$

Iteration 0, 1, 2, ...

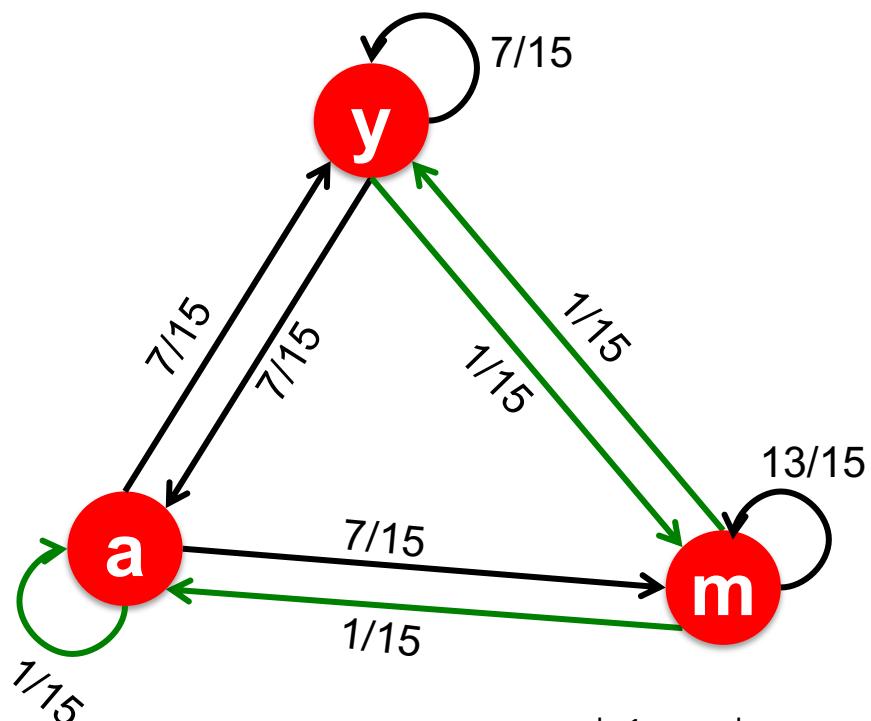
All the PageRank score gets “trapped” in node m

# Solution: Teleports!

- The Google solution for spider traps: **At each time step, the random surfer has two options**
  - With prob.  $\beta$  follow a link at random
  - With prob.  $1-\beta$  jump to some random page
  - Common values for  $\beta$  are in the range 0.8 to 0.9
- Surfer will teleport out of spider trap within a few time steps



# Random Teleports ( $\beta = 0.8$ )



$$M \quad [1/N]_{NxN}$$

1/2	1/2	0
1/2	0	0
0	1/2	1

+

1/3	1/3	1/3
1/3	1/3	1/3
1/3	1/3	1/3

$$A$$

y	7/15	7/15	1/15
a	7/15	1/15	1/15
m	1/15	7/15	13/15

$r_y$	1/3	0.33	0.24	0.26	...	7/33
$r_a$	1/3	0.20	0.20	0.18	...	5/33
$r_m$	1/3	0.46	0.52	0.56	...	21/33

# Matrix Formulation

- Suppose there are  $N$  pages
  - Consider a page  $j$ , with set of outlinks  $O(j)$
  - We have  $M_{ij} = 1/|O(j)|$  when  $j \rightarrow i$  and  $M_{ij} = 0$  otherwise
  - The random teleport is equivalent to
    - adding a **teleport link** from  $j$  to every other page with probability  $(1-\beta)/N$
    - reducing the probability of following each outlink from  $1/|O(j)|$  to  $\beta/|O(j)|$
    - Equivalent: tax each page a fraction  $(1-\beta)$  of its score and redistribute evenly

# Taxation

- Deal with spider traps and dead ends
  - By allowing surfer to jump to a random page
  - This jumping is also called “teleporting”

$$v' = \beta M v + (1 - \beta) e/n$$

Tax: 20%

Distribute it evenly

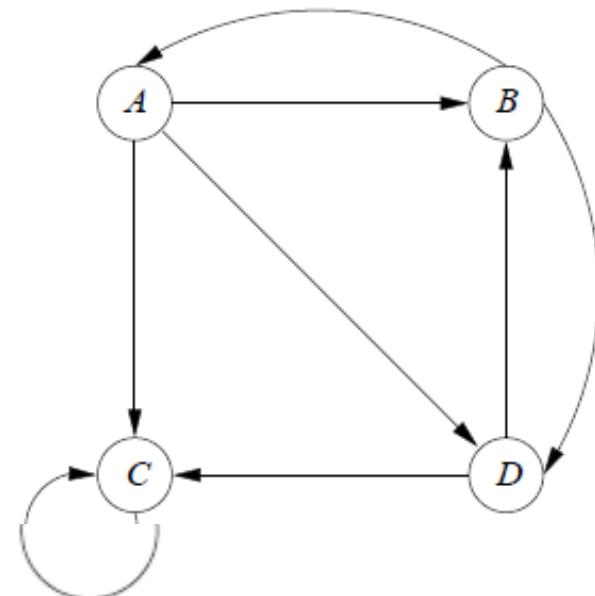
- $\beta$ , damping factor, e.g., set to 80%
- $e$ , an n-dimensional vector with all 1's
- n, # of nodes in the graph

- This in effect makes graph “strongly-connected”

# Taxation on Spider Trap

- Suppose  $\beta = .8$

$$M = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 1 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} \quad v^0 = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$



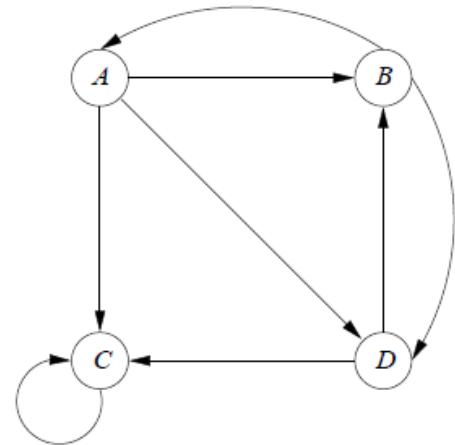
$$v^1 = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \left( .8 \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \right) + .2 \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 1/10 \\ 1/6 \\ 11/30 \\ 1/6 \end{bmatrix} + \begin{bmatrix} 1/20 \\ 1/20 \\ 1/20 \\ 1/20 \end{bmatrix} = \begin{bmatrix} 3/20 \\ 13/60 \\ 25/60 \\ 13/60 \end{bmatrix}$$

$C = 0.8 * 1/4 = 1/5; A, D = 1/6; \text{total} = 11/30$

(A and D:  $0.8 * 1/4 * (1/3 + 1/2) = 1/6$ ; C:  $0.8 * 1/4 = 1/5$ )

- Suppose  $\beta = .8$

$$M = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 1 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} \quad v' = .8Mv + .2 \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$



$$\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}, \begin{bmatrix} 9/60 \\ 13/60 \\ 25/60 \\ 13/60 \end{bmatrix}, \begin{bmatrix} 41/300 \\ 53/300 \\ 153/300 \\ 53/300 \end{bmatrix}, \begin{bmatrix} 543/4500 \\ 707/4500 \\ 2543/4500 \\ 707/4500 \end{bmatrix}, \dots, \begin{bmatrix} 15/148 \\ 19/148 \\ 95/148 \\ 19/148 \end{bmatrix}$$

A	C	D	.8(1/3 A)	.8(1/2 D)	.8C	.2 * 1/4	Next C
1/4	1/4	1/4	1/15	1/10	1/5	1/20	25/60 (.4)
9/60	25/60	13/60	3/75	13/150	1/3	1/20	153/300 (.5)
...	...	...	...	...	...	...	...
15/148	95/148	19/148	4/148	19/370	76/148	1/20	95/148 (.64)

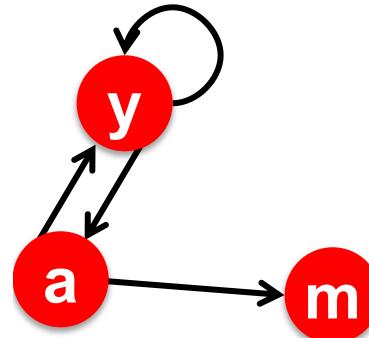
decreasing    decreasing    increasing    fixed    increasing

# Problem: Dead Ends

- **Power Iteration:**

Pages with no outlinks are  
“dead ends” for the  
random surfer:

Nowhere to go on next step



	y	a	m
y	1/2	1/2	0
a	1/2	0	0
m	0	1/2	0

$$r_y = r_y/2 + r_a/2$$

$$r_a = r_y/2$$

$$r_m = r_a/2$$

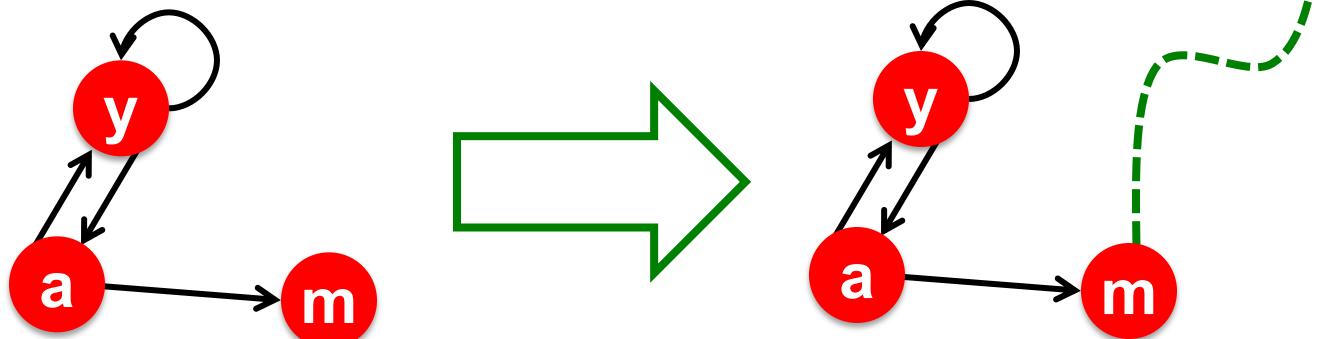
- **Example:**  $r^{k+1} = Ar^k$

$$\begin{pmatrix} r_y \\ r_a \\ r_m \end{pmatrix} = \begin{pmatrix} 1/3 & 2/6 & 3/12 & 5/24 & 0 \\ 1/3 & 1/6 & 2/12 & 3/24 & \dots & 0 \\ 1/3 & 1/6 & 1/12 & 2/24 & & 0 \end{pmatrix}$$

Here the PageRank “leaks” out since the matrix is not column stochastic (not all columns total to 1)

# Solution 1 for Dead Ends: Always Teleport!

- **Teleports: Follow random teleport links with probability 1.0 from dead-ends**
  - Adjust matrix accordingly



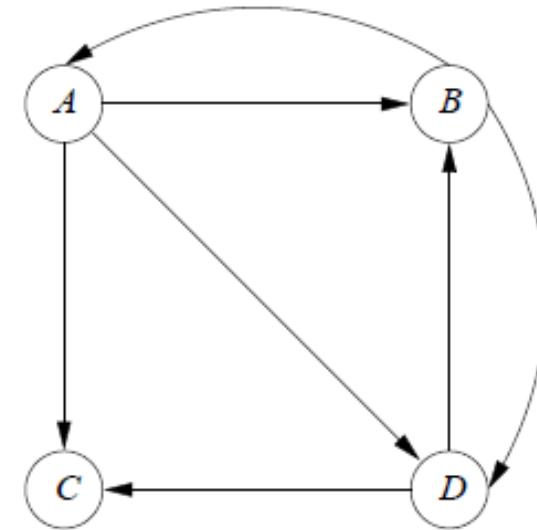
	y	a	m
y	$\frac{1}{2}$	$\frac{1}{2}$	0
a	$\frac{1}{2}$	0	0
m	0	$\frac{1}{2}$	0

	y	a	m
y	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$
a	$\frac{1}{2}$	0	$\frac{1}{3}$
m	0	$\frac{1}{2}$	$\frac{1}{3}$

# Solution 2 for Dead Ends: Taxation

- Suppose  $\beta = .8$

$$M = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

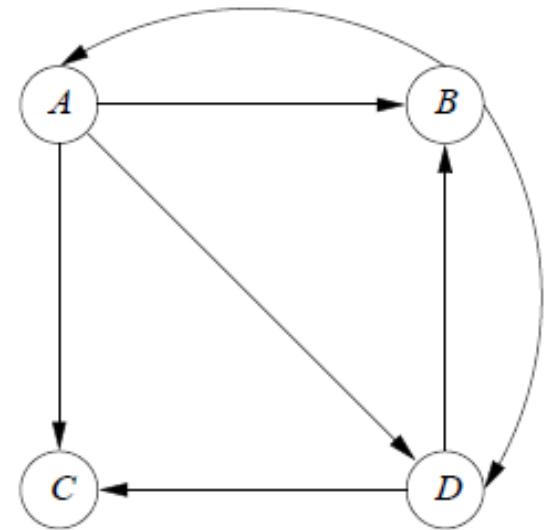


$$v' = .8 \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} v + .2 \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} (.8v) + .2 \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

# Taxation on Dead End

- Suppose  $\beta = .8$

$$M = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} \quad v^0 = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$



$$v^1 = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \left( .8 \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \right) + .2 \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 1/10 \\ 1/6 \\ 1/6 \\ 1/6 \end{bmatrix} + \begin{bmatrix} 1/20 \\ 1/20 \\ 1/20 \\ 1/20 \end{bmatrix} = \begin{bmatrix} 3/20 \\ 13/60 \\ 13/60 \\ 13/60 \end{bmatrix}$$

Total: 48/60

Lost:  $8/10 * 1/4 + 2/10 * 1 - 2/10 * 1 = 1/5 = 12/60$

sink     
  tax     
  teleport (gain)

# Taxation on Dead End

- Suppose  $\beta = .8$

$$M = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} \quad v^0 = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}, \quad v^1 = \begin{bmatrix} 3/20 \\ 13/60 \\ 13/60 \\ 13/60 \end{bmatrix}$$

$$v^2 = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \left( .8 \begin{bmatrix} 3/20 \\ 13/60 \\ 13/60 \\ 13/60 \end{bmatrix} \right) + .2 \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 13/150 \\ 19/150 \\ 19/150 \\ 19/150 \end{bmatrix} + \begin{bmatrix} 1/20 \\ 1/20 \\ 1/20 \\ 1/20 \end{bmatrix} = \begin{bmatrix} 41/300 \\ 53/300 \\ 53/300 \\ 53/300 \end{bmatrix}$$

Total:  $2/3$

$$\text{Lost } 8/10 * 13/60 + 2/10 * 48/60 - 2/10 = 13/75 + 12/25 - 1/5 = 2/15$$

$$\text{Total loss so far: } 1/5 + 2/15 = 1/3$$

# After 100 Iterations

$$\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}, \begin{bmatrix} 3/20 \\ 13/60 \\ 13/60 \\ 13/60 \end{bmatrix}, \begin{bmatrix} 41/300 \\ 53/300 \\ 53/300 \\ 53/300 \end{bmatrix}, \dots, \begin{bmatrix} .1 \\ .13 \\ .13 \\ .13 \end{bmatrix}$$

Remaining: 1       $4/5$        $2/3$       .49

- Net loss = column 2 + column 3 – column 4

At the end of iteration #	Loss due to dead end	Loss due to taxation (20% tax)	Gain due to teleporting	Net loss	Total net loss	Remaining ranks
1	$8/10 * 1/4$	$2/10 * 1$	$2/10 * 1$	$1/5 (.2)$	$1/5 (.2)$	$4/5$
2	$8/10 * 13/60$	$2/10 * 4/5$	$2/10 * 1$	$2/15 (.13)$	$1/5+2/15 (.33)$	$2/3$
...	...	...	...	...	...	...

Net loss will be zero when loss and gain balance out

## Without taxation

At the end of iteration #	Lost Prob.	Total loss	Remaining
1	1/4 (.25)	1/4 (.25)	3/4
2	5/24 (.21)	1/4 + 5/24 = 11/24 (.45)	1-11/24 = 13/24
3	7/48 (.15)	11/24+7/48 = 29/48 (.6)	1-29/48 = 19/48
...	...	...	...

## With taxation

At the end of iteration #	Loss due to dead end	Loss due to taxation (20% tax)	Gain due to teleporting	Net loss	Total net loss	Remaining ranks
1	8/10 * 1/4	2/10 * 1	2/10 * 1	1/5 (.2)	1/5 (.2)	4/5
2	8/10 * 13/60	2/10 * 4/5	2/10 * 1	2/15 (.13)	1/5+2/15 (.3)	2/3
...	...	...	...	...	...	...

Net loss will be zero when loss and gain balance out

# Why Do Teleports Solve the Problem?

**Why are dead-ends and spider traps a problem and why do teleports solve the problem?**

- **Spider-traps**
  - PageRank scores are **not** what we want
  - **Solution:** Never get stuck in a spider trap by teleporting out of it in a finite number of steps
- **Dead-ends**
  - The matrix is **not column stochastic** so our initial assumptions are not met
  - **Solution:** Make matrix column stochastic by always teleporting when there is nowhere else to go

# Alternative for Dealing with dead-ends

- **Prune and propagate**
  - Preprocess the graph to eliminate dead-ends
  - This may create new dead-ends
  - Might require multiple passes
  - Compute page rank on reduced graph
  - Approximate values for dead-ends by propagating values from reduced graph

# How is Page Rank Information Used in a Search Engine?

- Each search engine has a **secret formula** that **decides the order in which to show pages** to a user in response to a search query
  - Google: thought to use over 250 different properties of pages
- **Search engine decides on a linear order of search results**
- To be considered for ranking, **page has to have at least one of the query search terms**
  - Unless all search terms are present, little chance of being in top ten results
- **Among qualified pages, score computed for each**
  - **PageRank** is an important component of the score
  - **Other components:** presence/absence of search terms in headers, links to the page

# PageRank Iteration using MapReduce

- If  $n$  is small enough that each Map task can store the full vector  $v$  and  $v'$  in main memory
  - With additional steps to multiply each component of  $Mv$  by constant  $\beta$  and to add  $(1-\beta)/n$
- Given the size of the Web,  $v$  is much too large to fit in main memory
- Break  $M$  into vertical stripes (chp. 2.3.1) and break  $v$  into corresponding horizontal stripes to execute the MapReduce

# Matrix-Vector Multiplication by MapReduce

- Matrix M of size n\*n, vector v of length n
- $M v = x$
- the vector x of length n, whose  $i_{th}$  element  $x_i$  is given by

$$x_i = \sum_{j=1}^n m_{ij} v_j$$

# Use of Combiners to Consolidate the Result Vector

- Partition the vector  $v$  into  $k$  stripes and matrix into  $k^2$  blocks
- E.g.,  $k=4$

$$\begin{array}{c|c} \mathbf{v}_1' \\ \hline \mathbf{v}_2' \\ \hline \mathbf{v}_3' \\ \hline \mathbf{v}_4' \end{array} = \begin{array}{|c|c|c|c|} \hline M_{11} & M_{12} & M_{13} & M_{14} \\ \hline M_{21} & M_{22} & M_{23} & M_{24} \\ \hline M_{31} & M_{32} & M_{33} & M_{34} \\ \hline M_{41} & M_{42} & M_{43} & M_{44} \\ \hline \end{array} \begin{array}{c|c} \mathbf{v}_1 \\ \hline \mathbf{v}_2 \\ \hline \mathbf{v}_3 \\ \hline \mathbf{v}_4 \end{array}$$

$$x_i = \sum_{j=1}^n m_{ij} v_j$$

# Use of Combiners to Consolidate the Result Vector (Cont'd)

- Use  $k^2$  Map tasks (e.g., 16)
- Each task gets **one square of the matrix M ( $M_{ij}$ )** and **one stripe of the vector v ( $v_j$ )**
  - each stripe of the vector is sent to  $k$  (e.g., 4) different Map tasks
  - $v_j$  is sent to the task handling  $M_{ij}$  for each of the  $k$  possible values of  $i$
- Advantage: keep both the  $j_{th}$  stripe of  $v$  and the  $i_{th}$  stripe of  $v'$  in main memory as we process  $M_{ij}$

$$\begin{array}{c|c} \mathbf{v}_1' \\ \hline \mathbf{v}_2' \\ \hline \mathbf{v}_3' \\ \hline \mathbf{v}_4' \end{array} = \begin{array}{|c|c|c|c|} \hline M_{11} & M_{12} & M_{13} & M_{14} \\ \hline M_{21} & M_{22} & M_{23} & M_{24} \\ \hline M_{31} & M_{32} & M_{33} & M_{34} \\ \hline M_{41} & M_{42} & M_{43} & M_{44} \\ \hline \end{array}$$

$$\begin{array}{c|c} \mathbf{v}_1 \\ \hline \mathbf{v}_2 \\ \hline \mathbf{v}_3 \\ \hline \mathbf{v}_4 \end{array}$$

$$x_i = \sum_{j=1}^n m_{ij} v_j$$

Note that all terms generated from  $M_{ij}$  and  $v_j$  contribute to  $v'_i$  and no other stripe of  $v'$

# PageRank Iteration Using MapReduce

- Assume rank vector  $v$  does not fit in main memory
- **Block stripe update method:** break  $M$  into vertical stripes and break  $r$  into corresponding horizontal stripes
- **Allows us to execute MapReduce efficiently**
- Process no more of  $v$  at any one Map task than can fit in main memory
- **Trying to use a combiner gets more complicated:** Section 5.2.3
  - Partition matrix  $M$  into  $k^2$  blocks, vector  $r$  in  $k$  stripes
  - Use  $k^2$  Map tasks
  - Each Map task gets: **one square of matrix:  $M_{ij}$ , one stripe of vector,  $r_j$**
  - **Each stripe  $r_j$  sent to k Map tasks**, transmitted over network  $k$  times
  - **But each  $M_{ij}$  transmitted only once**

# Some Problems with Page Rank

- **Measures generic popularity of a page**
  - Biased against topic-specific authorities
  - **Solution:** Topic-Specific PageRank
- **Susceptible to Link spam**
  - Artificial link topographies created in order to boost page rank
  - **Solution:** TrustRank
- **Uses a single measure of importance**
  - Other models of importance
  - **Solution:** Hubs-and-Authorities

# Topic-Specific PageRank

# Topic-Specific PageRank

- Instead of generic popularity, can we measure popularity **within a topic?**
- **Goal:** Evaluate Web pages not just according to their popularity, but **by how close they are to a particular topic**, e.g. “sports” or “history”
- **Allows search queries to be answered based on interests of the user**
  - **Example:** Query “Trojan” wants different pages depending on whether you are interested in sports, history, and computer security

# Topic-Specific PageRank

- Random walker has a small probability of teleporting at any step
- **Where a Teleport can go:**
  - **Standard PageRank:** Any page with equal probability
  - **Topic Specific PageRank:** A topic-specific set of “relevant” pages (**teleport set**)
- **Idea: Bias the random walk**
  - When walker teleports, she picks a page from a set  $S$
  - $S$  contains only pages that are relevant to the topic
    - E.g., Open Directory (DMOZ) pages for a given topic/query
  - **For each teleport set  $S$ , we get a different vector  $r_s$**

# Discover Topic PageRank Vectors $r_s$ for Topic Set S

- If the search engine can deduce the user's interests, then it can do a better job of returning relevant pages
  - A private PageRank vector that gives importance of each page to that user (not feasible!)
- Topic-Sensitive PageRank: creates one vector for some small number of topics
  - Bias PageRank to favor pages of that topic
- Create different PageRank vectors for different topics
  - The 16 top-level categories of the Open Directory (DMOZ): arts, business, sports,...
- If we can determine that the user is interested in one of these topics (e.g., by content of pages they recently viewed), use the PageRank vector for that topic when deciding on ranking of pages for query results

# Biased Random Walk

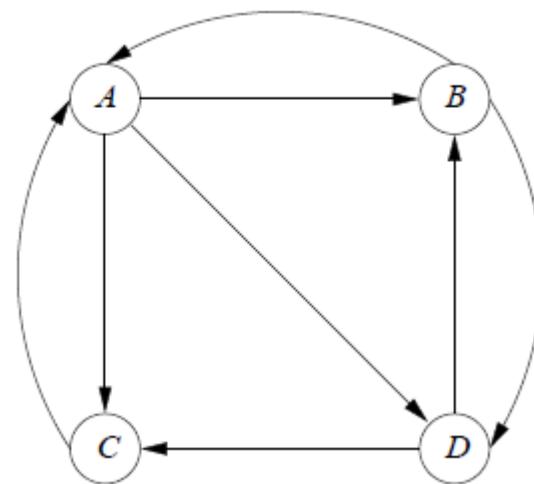
- Bias toward a set  $S$  of pages
  - $S$  called teleport set
  - $e_S$ : a vector with 1 for pages in  $S$ , 0 otherwise

$$\mathbf{v}' = \beta M \mathbf{v} + (1 - \beta) \mathbf{e}_S / |S|$$

# Example

- Teleport set  $S = \{B, D\}$ ,  $\beta = .8$

$$M = \begin{matrix} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} \end{matrix}$$

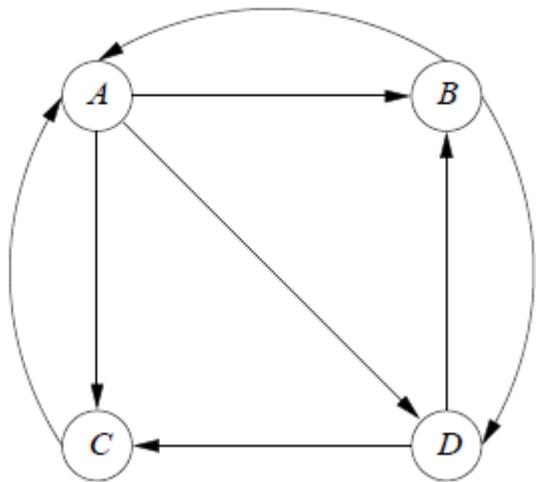


$$\mathbf{v}' = \beta M \mathbf{v} + (1 - \beta) \mathbf{e}_S / |S|$$

$$\mathbf{v}' = \begin{bmatrix} 0 & 2/5 & 4/5 & 0 \\ 4/15 & 0 & 0 & 2/5 \\ 4/15 & 0 & 0 & 2/5 \\ 4/15 & 2/5 & 0 & 0 \end{bmatrix} \mathbf{v} + \begin{bmatrix} 0 \\ 1/10 \\ 0 \\ 1/10 \end{bmatrix}$$

2/10 \* 1/2

# Example



$$\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}, \begin{bmatrix} 9/24 \\ 5/24 \\ 5/24 \\ 5/24 \end{bmatrix}, \begin{bmatrix} 15/48 \\ 11/48 \\ 11/48 \\ 11/48 \end{bmatrix}, \begin{bmatrix} 11/32 \\ 7/32 \\ 7/32 \\ 7/32 \end{bmatrix}, \dots, \begin{bmatrix} 3/9 \\ 2/9 \\ 2/9 \\ 2/9 \end{bmatrix}$$

Unbiased: A has the largest

$$\begin{bmatrix} 0/2 \\ 1/2 \\ 0/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 2/10 \\ 3/10 \\ 2/10 \\ 3/10 \end{bmatrix}, \begin{bmatrix} 42/150 \\ 41/150 \\ 26/150 \\ 41/150 \end{bmatrix}, \begin{bmatrix} 62/250 \\ 71/250 \\ 46/250 \\ 71/250 \end{bmatrix}, \dots, \begin{bmatrix} 54/210 \\ 59/210 \\ 38/210 \\ 59/210 \end{bmatrix}$$

Biased: B and D has largest PageRanks

# Integrating Topic-Sensitive PageRank Into a Search Engine

1. **Decide on topics** for which to create specialized PageRank vectors
2. **Pick a teleport set for each of these topics**
  - Use that set to **compute topic-sensitive PageRank for that topic**
3. **Determine the topic or set of topics most relevant to a particular search query**
  - More on this in next slide
4. **Use the PageRank vector for those topic(s) to order the responses to the search query**

# How to Select the Topic Set that is Most Relevant to the Search Query?

- Tricky
- Several methods proposed:
  1. User can pick from a topic menu
  2. Infer the topic(s) from the context of the query
    - Words that appear in Web pages recently searched by user or recent user queries
    - E.g., query is launched from a web page talking about a known topic
      - History of queries: e.g., “basketball” followed by “Kobe”
    - Classification of documents by topic: studied for decades
  3. Infer the topic(s) from information about the user
    - E.g., the user’s bookmarks, their interest on Facebook, etc.

# TrustRank: Combatting Web Spam

# What is Web Spam?

- **Spamming:**
  - Deliberate action to boost a webpage's position in search results
  - Not commensurate with page's real value
- This is a very broad definition
  - **SEO** industry might disagree!
  - SEO = search engine optimization
- Approximately **10-15%** of web pages are spam

# Web Search

- **Early search engines:**
  - Crawl the Web
  - Index pages by words they contained
  - Respond to **search queries** (lists of words) with the pages containing those words
- **Early page ranking:**
  - Attempt to order pages matching a search query by “importance”
  - **First search engines considered:**
    - **(1) Number of times** query words appeared
    - **(2) Prominence of word position**, e.g. title, header

# First Spammers: Term Spam

- How do you make your page appear to be about movies?
- (1) Add the word movie 1,000 times to your page
  - Set text color to the background color, so only search engines would see it
- (2) Or, run the query “movie” on your target search engine
  - See what page came first in the listings
  - Copy it into your page, make it “invisible”
- These and similar techniques are called term spam

# Google's Solution to Term Spam

- Believe what people say about you, rather than what you say about yourself
  - Use words in the anchor text (words that appear underlined to represent the link) and its surrounding text
- PageRank as a tool to measure the “importance” of Web pages
  - Doesn't depend on content of pages

# Why It Works?

- Our hypothetical shirt-seller loses
  - Saying he is about movies doesn't help, because others don't say he is about movies
  - His page isn't very important, so it won't be ranked high for shirts or movies
- Example:
  - Shirt-seller creates 1,000 pages, each links to his with "movie" in the anchor text
  - These pages have no links in, so they get little PageRank
  - So the shirt-seller can't beat truly important movie pages, like IMDB

# Google vs. Spammers: Round 2!

- Once Google became the dominant search engine, spammers began to work out ways to fool Google
- Spam farms** were developed to concentrate PageRank on a single page

The screenshot shows the Freepublic.com homepage. At the top, there's a navigation bar with icons for back, forward, home, and search, along with the URL http://freepublic.com/. Below the URL is a search bar containing the placeholder "What you need, when you need it". To the right of the search bar are links for "February 2, 2010", "English", "Make this your homepage", and "Bookmark this page". The main content area features a large graphic of overlapping colored squares (red, yellow, green, blue) on the left. To its right is a "Related Searches" section with a list of links, each preceded by a green checkmark. Below this is a search bar with a magnifying glass icon and a red "Search" button. At the bottom of the page are two "Popular Categories" sections, each with a red header and a grid of links.

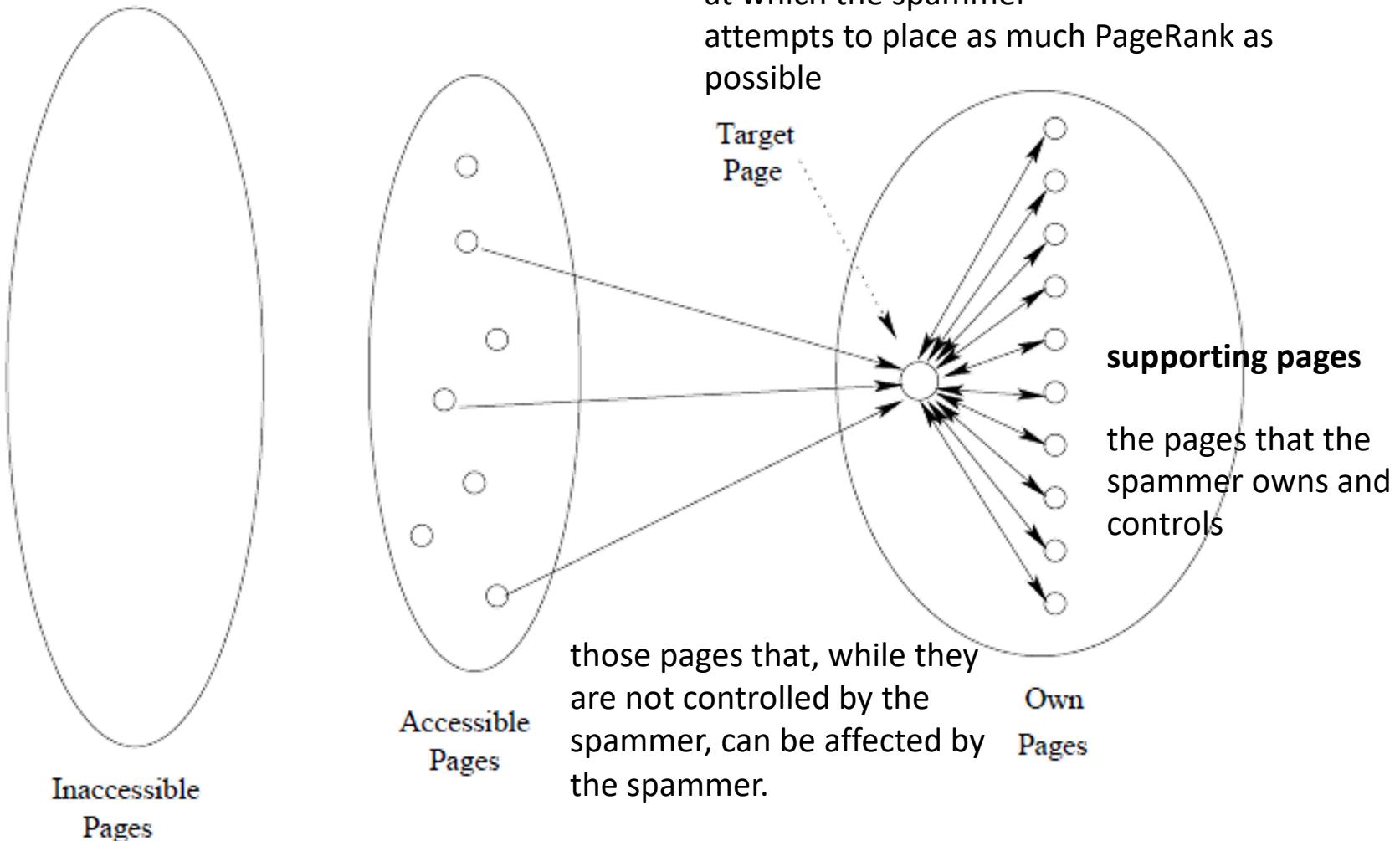
Popular Categories	
Travel <a href="#">Airline Tickets</a> <a href="#">Hotels</a> <a href="#">Car Rental</a> <a href="#">Flights</a> <a href="#">South Beach Hotels</a>	Finance <a href="#">Free Credit Report</a> <a href="#">Online Payment</a> <a href="#">Credit Card Application</a> <a href="#">Car Insurance</a> <a href="#">Health Insurance</a>
Business <a href="#">Foreclosures</a> <a href="#">Houses For Sale</a> <a href="#">Mortgage</a> <a href="#">People Search</a> <a href="#">Real Estate Training</a>	Employment <a href="#">Work From Home</a> <a href="#">Reorder Checks</a> <a href="#">Used Cars</a> <a href="#">Business Opportunities</a>

## Link spam:

- Creating link structures that boost PageRank of a particular page

# Spam Farm

at which the spammer attempts to place as much PageRank as possible



the pages that the spammer cannot affect. Most of the Web is in this part

# Spam Farm Analysis

- $x$ : contributed ranks to target page  $t$  from accessible pages
- $y$ : PageRank of target page
- $m$ : # of supporting pages
- $\beta$ : damping factor (so  $1 - \beta$  is tax percentage)
- $n$ : # of pages in the entire Web
- PageRank of each supporting page:

$$\beta y/m + (1 - \beta)/n$$

# Compute PageRank of Target Page



- Contributed by supporting pages
  - $\beta m(\beta y/m + (1 - \beta)/n)$
- Contributed by accessible pages
  - $x$
- Teleporting

- $(1 - \beta)/n$ , negligible;  $y = x + \beta m(\beta \frac{y}{m} + (1 - \beta)/n) = x + \beta^2 y + \frac{\beta(1-\beta)m}{n}$

$$y = \frac{x}{1-\beta^2} + \frac{\beta m}{(1+\beta)n} = \frac{x}{1-\beta^2} + c \frac{m}{n},$$

where  $c = \frac{\beta}{1+\beta}$       If  $\beta = .85$ , then  $1/(1-\beta^2) = 3.6$ ,  $c = .46$

Amplify external contribution by 360%

Plus additional PageRank: 46% of fraction of farm pages in the entire Web

# HITS: Hubs and Authorities

# HITS Ideas

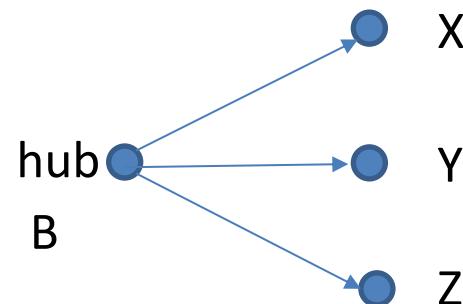
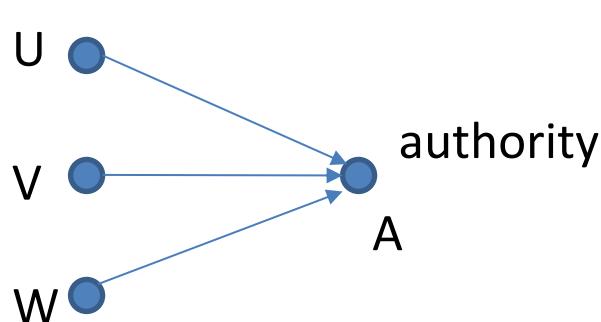
- A variant of topic-sensitive PageRank
  - Topic = a set of trustworthy pages
- Identify trustworthy pages
  1. Manually examine pages with high PageRanks
  2. Pick pages within controlled domains: edu, gov, mil
  3. Avoid borderline pages, e.g., blog sites, forums

# HITS Algorithm

- Hyperlink-induced topic search
- Each page has two scores
  - **Hub**: does it provide many pointers to the topic?
  - **Authority**: how important the page is about topic?
- Examples
  - Hub: department pages listing all courses
  - Authority: individual course pages

# Authority and Hub

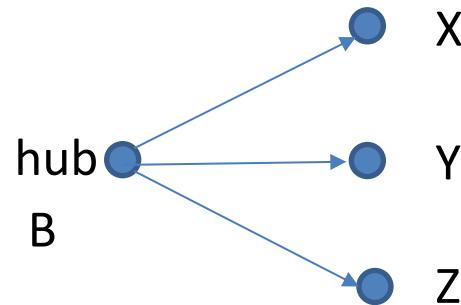
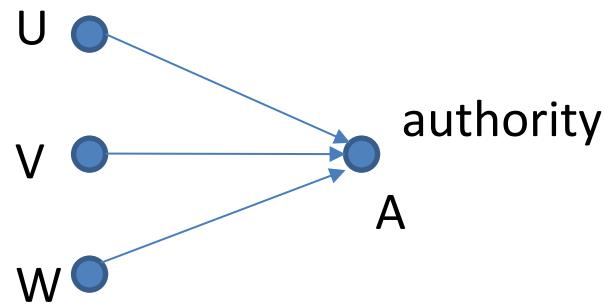
- A good authority page is one
  - pointed by many good hub pages



- A good hub page is one
  - pointing to many good authority pages

# Computing Authority and Hubbiness

- Authority of a page =
  - sum of hubbiness of pages pointing to it
  - $a(A) = h(U) + h(V) + h(W)$
- Hubbiness of a page =
  - sum of authority of pages it points to
  - $h(B) = a(X) + a(Y) + a(Z)$

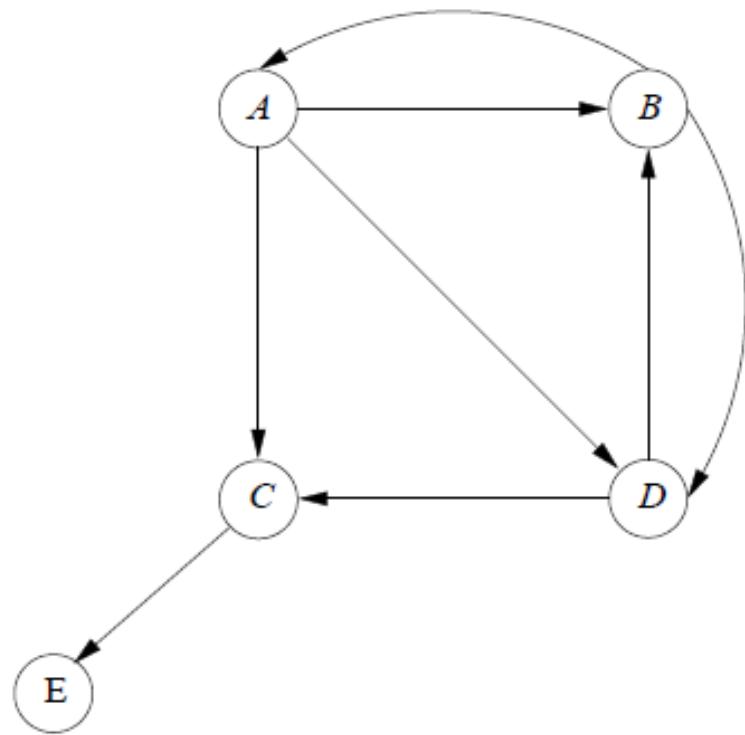


# Link Matrix

- $L[i,j] = 1$  if there is a link from node  $i$  to node  $j$
- $L^T$  is the un-normalized  $M$  in PageRank

$$L = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$



# Computing Authority and Hubbiness

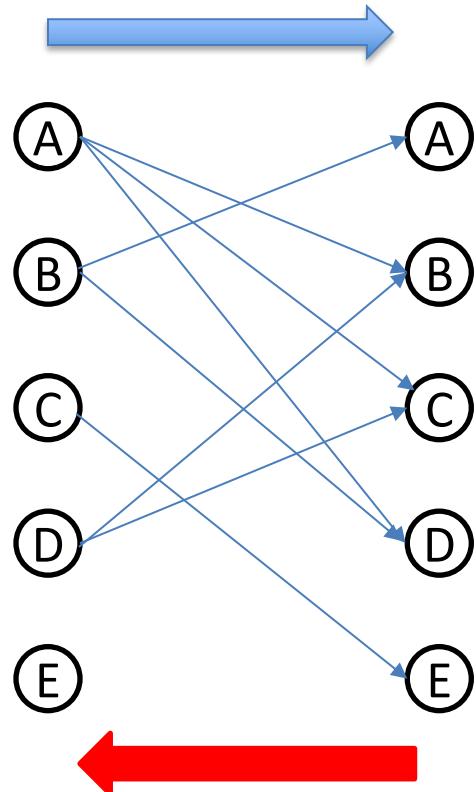
- $\mathbf{h} = \lambda L\mathbf{a}, \quad \mathbf{a} = \mu L^T \mathbf{h}$ 
  - $\lambda, \mu$ : scaling factors to normalize  $\mathbf{h}$  and  $\mathbf{a}$
- $\mathbf{h} = \lambda \mu LL^T \mathbf{h}, \quad \mathbf{a} = \lambda \mu L^T L \mathbf{a}$
- $LL^T$  and  $L^T L$  likely dense
  - Expensive to use direct recursion on  $\mathbf{h}$  and  $\mathbf{a}$
  - e.g., iteratively computing  $\mathbf{h} = \lambda \mu LL^T \mathbf{h}$
- Use mutual recursion instead

# Compute via Mutual Recursion

1. Start with  $\mathbf{h}$  = a vector of all 1's
2. Compute  $\mathbf{a} = L^T \mathbf{h}$ , then scale it so that largest component is 1
3. Compute  $\mathbf{h} = L\mathbf{a}$  and similarly scaled
4. Repeat steps 2 and 3 until  $\mathbf{h}$  and  $\mathbf{a}$  do not change much

# Bipartite View of Iterations

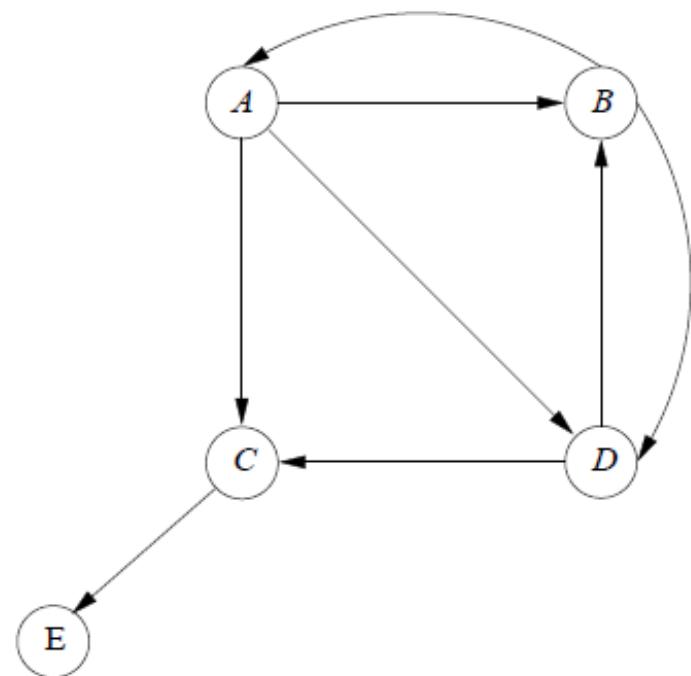
Compute authority from hub



Compute hub from authority

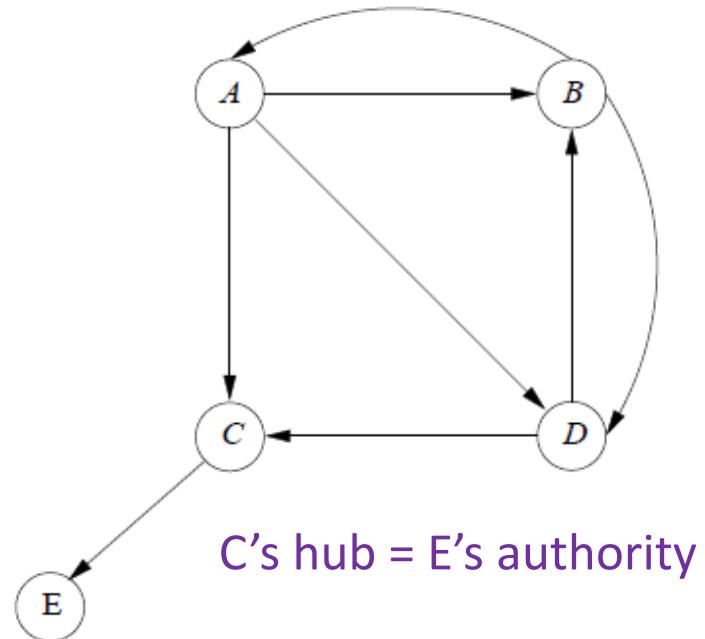
Hub

Authority



# Example

$$\mathbf{h}^0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

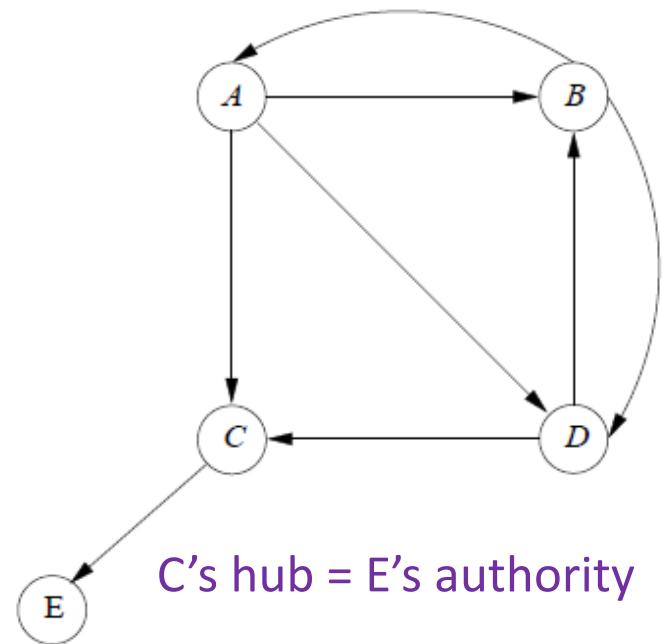


$$a^0 = L^T \mathbf{h}^0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 1 \\ 1 \\ 1/2 \end{bmatrix}$$

Can you predict the changes to hub scores next?

# Example

$$\mathbf{h}^0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{a}^0 = \begin{bmatrix} 1/2 \\ 1 \\ 1 \\ 1 \\ 1/2 \end{bmatrix}$$

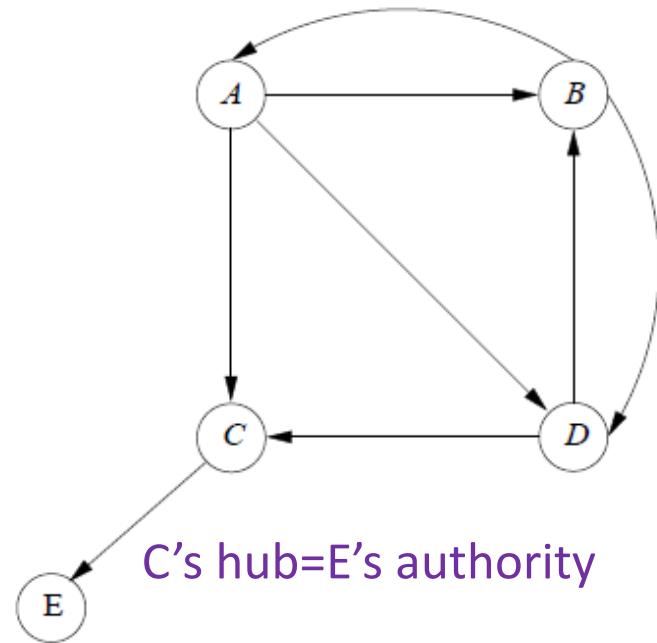


$$h^1 = L\mathbf{a}^0 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \\ 1 \\ 1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3/2 \\ 1/2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/6 \\ 2/3 \\ 0 \end{bmatrix}$$

Will E's authority decrease? Will E have  $> 0$  hub score?

# Example

$$h^0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad a^0 = \begin{bmatrix} 1/2 \\ 1 \\ 1 \\ 1 \\ 1/2 \end{bmatrix}$$

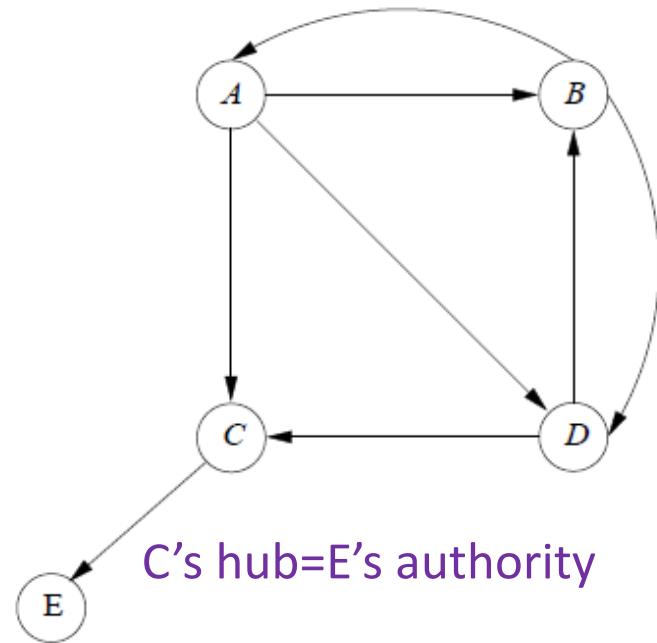


$$h^1 = \begin{bmatrix} 1 \\ 1/2 \\ 1/6 \\ 2/3 \\ 0 \end{bmatrix} \quad a^1 = \begin{bmatrix} 1/2 \\ 5/3 \\ 5/3 \\ 3/2 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 3/10 \\ 1 \\ 1 \\ 9/10 \\ 1/10 \end{bmatrix} \quad h^2 = \begin{bmatrix} 29/10 \\ 6/5 \\ 1/10 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 12/29 \\ 1/29 \\ 20/29 \\ 0 \end{bmatrix}$$

Decrease due to normalization

# Example

$$h^0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad a^0 = \begin{bmatrix} 1/2 \\ 1 \\ 1 \\ 1 \\ 1/2 \end{bmatrix}$$



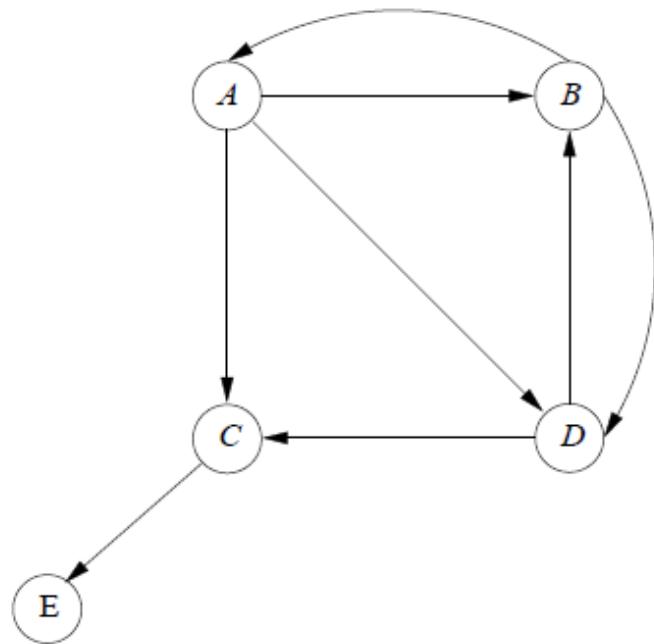
$$h^1 = \begin{bmatrix} 1 \\ 1/2 \\ 1/6 \\ 2/3 \\ 0 \end{bmatrix} \quad a^1 = \begin{bmatrix} 3/10 \\ 1 \\ 1 \\ 9/10 \\ 1/10 \end{bmatrix}$$

$$h^2 = \begin{bmatrix} 1 \\ 12/29 \\ 1/29 \\ 20/29 \\ 0 \end{bmatrix} \quad a^2 = \begin{bmatrix} 12/29 \\ 49/29 \\ 49/29 \\ 41/29 \\ 1/29 \end{bmatrix} = \begin{bmatrix} 12/49 \\ 1 \\ 1 \\ 41/49 \\ 1/49 \end{bmatrix}$$

Decrease due to normalization

# At the Limit

- A: greatest hub
- B, C: greatest authority



$$\begin{aligned}
 \mathbf{h}^0 &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} & \mathbf{h}^1 &= \begin{bmatrix} 1 \\ 1/2 \\ 1/6 \\ 2/3 \\ 0 \end{bmatrix} & \mathbf{h}^2 &= \begin{bmatrix} 1 \\ 12/29 \\ 1/29 \\ 20/29 \\ 0 \end{bmatrix}, \dots, \mathbf{h}^n &= \begin{bmatrix} 1 \\ .36 \\ 0 \\ .72 \\ 0 \end{bmatrix} \\
 \mathbf{a}^0 &= \begin{bmatrix} 1/2 \\ 1 \\ 1 \\ 1 \\ 1/2 \end{bmatrix} & \mathbf{a}^1 &= \begin{bmatrix} 3/10 \\ 1 \\ 1 \\ 9/10 \\ 1/10 \end{bmatrix} & \mathbf{a}^2 &= \begin{bmatrix} 12/49 \\ 1 \\ 1 \\ 41/49 \\ 1/49 \end{bmatrix}, \dots, \mathbf{a}^n &= \begin{bmatrix} .21 \\ 1 \\ 1 \\ .79 \\ 0 \end{bmatrix}
 \end{aligned}$$

# PageRank and HITS

- PageRank and HITS are two solutions to the same problem
- Rank the relative value of pages
- Use these values in search engine algorithms to determine the order of search results