Yield Functions, Damage States, and Intrinsic Strength

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Abstract: A recently derived isotropic yield/failure criterion is here shown to give a measure of the state of the damage relative to an intrinsic (ideal) yield strength. The formulation of the intrinsic strength itself is obtained through a spectral analysis. A criterion for the ductile-brittle transition is found in the process of deriving the effect of pressure on the new yield function formalism.

Key Words: Yield functions, Mises criterion, intrinsic strength, ductile-brittle transition

1. INTRODUCTION

Considering the reality that the continuum mechanics–type analysis of materials is very highly developed after nearly 250 years of activity, it is disappointing that the intimately related field of material failure characterization is in such an unrefined and ambiguous state. It is not that this vitally important area of materials yielding and failure has been neglected, rather it is more likely a reflection of the difficulties and barriers that obstruct the path to understanding. While the subfield of the plastic behavior of ductile metals is well developed, this is the only particular material type for which that can be said. The specific evaluation and application of yield functions has become an unstructured even fragmentary operation with different forms for different types of materials, there being no apparent connection between them (see Paul [1] for an overview). However, in recent work, Christensen [2] has developed a theory of materials yielding and failure that has the generality to include most major isotropic materials types. This new formalism for the failure and yielding of isotropic materials will be further developed here.

The present work integrates this new view of yield functions with measures of damage, as well as with new concepts of ideal or intrinsic yield strength and concepts of transitions. Damage is herein modeled through the yield function. This is a completely different approach from the one involving the direct effect of damage on elastic properties. Both aspects are legitimate points of view and commonly employed. Lemaitre and Chaboche [3], Krajcinovic [4] give a reasonable view of the relationship of damage metrics and general continuum mechanics. The present work is concerned with initial yield functions for isotropic materials; thus, all related aspects of damage to be treated herein are themselves isotropic. Materials with isotropic or nearly isotropic damage are commonplace, as is the complementary class.

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In the next section, the new yield function formalism will be recalled and interpreted. Following that, the concept of intrinsic yield strength will be introduced, and this will be given a specific form by closely examining what was called the fracture transition in previous work. The exact nature of transitions in the present context will receive quite detailed treatment, ultimately resulting in a criterion for the ductile-brittle transition. Along the way, examples of damage states will be given after establishing a suitable definition for damage. Finally, it should be said that although this new formalism may possibly apply to all isotropic materials, this is in fact exploratory work in an immensely difficult and complicated field.

2. YIELD FUNCTION PRELIMINARIES

The recently derived yield function formalism, [2], resulted in the form for initial yield functions and/or failure criteria as

$$\frac{\alpha k}{\sqrt{3}}\sigma_{ii} + \frac{(1+\alpha)}{2}s_{ij}\,s_{ij} \leq k^2, \tag{1}$$

where equality implies the actual yielding or failure and s_{ij} is the deviatoric stress

$$s_{ij} = \sigma_{ij} - \frac{\delta_{ij}}{3}\sigma_{kk}\,, \tag{2}$$

with rectangular Cartesian tensor notation being used. This two-parameter form has α and k given by the uniaxial stresses at yield as

$$\alpha = \frac{|\sigma_{11}^C|}{\sigma_{11}^T} - 1$$

and

$$k = \frac{|\sigma_{11}^C|}{\sqrt{3}}, \tag{3}$$

where the theory is only applicable when $\sigma_{11}^T \leq |\sigma_{11}^C|$. From (1) it is seen that the yield theory is a generalization of the Mises criterion to include the effect of mean normal stress. From relations (3), the uniaxial yield stresses are given by

$$\begin{aligned} \sigma_{11}^T &= \frac{\sqrt{3}k}{(1+\alpha)} \\ \sigma_{11}^C &= -\sqrt{3}k, \end{aligned} \tag{4}$$

while the yield in simple shear is given by

$$\sigma_{12}^{Y} = \frac{k}{\sqrt{1+\alpha}}. \tag{5}$$

The parameters α and k were descriptively labeled as the shape and scale parameters because they admit the geometric interpretation suggested by these terms. The range of different materials-type behaviors is revealed by varying only the nondimensional parameter, α. These two descriptive terms, shape parameter and scale parameter, will not be further used here. In their place, some more closely materials-related terminology will be given that are more appropriate to the present purposes. Note also that the yield function (1) will be referred to in both the singular and the plural sense; the difference will be clear from the context.

The most important result of this new yield function formalism is that it admits fracture behavior as a special case. Specifically, at $\alpha = 1$ the behavior was characterized as a fracture transition because the yield function (1) was shown to possess fracturelike characteristics for materials with a distribution of randomly oriented, well-separated cracks. For values of $\alpha < 1$, a ductile-type behavior was found, whereas for $\alpha > 1$, a dilatant behavior increasingly related to mean normal stress was found to be controlling. This fracture transition at $\alpha = 1$ will play a major role in the developments to follow here. For present purposes, it suffices to recall that for $\alpha \ll 1$ the deviatoric stresses are controlling and for $\alpha \gg 1$ the dilatational stresses are controlling, whereas at the transition at $\alpha = 1$ the two physical effects are in balance, all this in a state of uniaxial tension at yield.

3. SPECTRAL CHARACTERIZATION AND INTRINSIC STRENGTH

The transition behavior briefly described in the previous section occurs at $\alpha = 1$. This is in effect the transition between Mises behavior at one extreme, $\alpha = 0$, and material degradation or disintegration at the other extreme, $\alpha \to \infty$. At $\alpha = 1$ fracturelike behavior occurs. The transition at $\alpha = 1$ for (1) was found using nondimensional forms [2, 5]. Now, the transition behavior at $\alpha = 1$ found previously will be complemented by requiring that the dimensional aspects of the yield function (1) also possess a coordinated transition at $\alpha = 1$. By the term dimensional aspect is meant the behavior expressed in terms of units of stress. However, there is no sensible basis for requiring any single component of stress to have a transition at $\alpha = 1$, nor would there be for any combined state of stress components. The rational basis for imposing a coordinated dimensional transition is on the parameter k, which carries the dimensionality of the yield function. It is parameter k that will be endowed with a transitional property in the manner to be developed now.

Transitions are commonplace in materials science. Crystalline melting point is taken as a first-order transition involving a discontinuity in properties. Second-order transitions involve a change in the first derivative of properties as a function of the transition variable, often pressure or temperature. Atomic physics and polymer science when viewed on the molecular scale typically involve relaxation processes that inherently admit a transition-type behavior. The same general type of behavior is taken here. The parameter α is taken as the transition variable, and a general transition type behavior will be imposed on k at $\alpha = 1$.

The damage processes that would make parameter α very large would be expected to make parameter k very small at large α. The transition of k at $\alpha = 1$ will become the main feature of the reduction of k as α increases; this transition is and must be consistent with the transition already found. The entire process will be developed while still retaining the basic two-parameter form of the yield function, but one of the two parameters will require a new basis.

Spectra are widely used in chemistry and physics to characterize behavior. Spectra are ideal for use here to consider the behavior of parameter k as parameter α is varied, and to impose the transition on k at $\alpha = 1$. Probably, it would be possible to define a transition within the present context without using spectra, but the most useful and fundamental method certainly involves spectra. Spectra are usually specified in the time and frequency domain, but $k = k(\alpha)$ will be taken with certain monotone properties that enable parameter α to comprise the spectrum variable.

The spectral characterization to be developed now is essentially a mathematical procedure. After this process is completed, the physical significance of the transition at $\alpha = 1$ will be revealed.

Take parameter k such that $k = k(\alpha)$ and

$$\begin{aligned} k &= K \quad \text{at} \quad \alpha = 0 \\ k &\to 0 \quad \text{as} \quad \alpha \to \infty \end{aligned} \tag{6}$$

and take $k(\alpha)$ as being a monotone decreasing function of α. The second condition in (6) will be shown to correspond to complete material disintegration. Now take $k(\alpha)$ to be expressed by its spectrum, $H(\tau)$, through the standard form

$$k(\alpha) = \int_0^\infty H(\tau)\, e^{-\frac{\alpha}{\tau}} d(\log \tau) + k_\infty, \tag{7}$$

where the integration over τ is from $0 \leq \tau \leq \infty$. In relation (7), log is base 10, and this will be used throughout this work. Natural log could be used as well, with no change in results. Consistent with (6), it will be necessary to take $k_\infty = 0$, which in physical terms means this asymptotic value is taken to be negligible. See Ferry [6] for a standardized treatment of spectra in one aspect of materials science, namely, that of polymer science.

Let a new variable, ζ, be introduced as

$$\frac{\alpha}{\tau} = \zeta. \tag{8}$$

Note that the exponential weighting function in (7) has the characteristic that

$$\left| \frac{de^{-\zeta}}{d(\log \zeta)} \right| = \max \quad \text{at} \quad \zeta = 1. \tag{9}$$

Thus, $e^{-\zeta}$ has a type of mathematical transition at $\zeta = 1$.

In physical terms, a transition in $k(\alpha)$ will be taken at $\alpha = 1$. For a mathematical transition of $e^{-\zeta}$ at $\zeta = 1$ and for a physical transition of $k(\alpha)$ at $\alpha = 1$, there must be imposed some restrictions on the spectrum $H(\tau)$ because of relation (8) between α, τ, and ζ. For example, if $H(\tau)$ were a delta function at $\tau = 1$, then the proper behavior involving the transition of $k(\alpha)$ at $\alpha = 1$ would occur. Even if $H(\tau)$ just has a maximum at $\tau = 1$ and reasonable properties of decay around that value, then the proper transition at $\alpha = 1$ is still observed. This latter situation will be the case for the classes of functions to be considered for $k(\alpha)$ and $H(\tau)$. Require the maximum in $H(\tau)$ to occur at $\tau = 1$, subject to verification of the transition behavior in $k(\alpha)$ at $\alpha = 1$.

Begin the process by inverting (7) to solve for $H(\tau)$ as determined by $k(\alpha)$. Rewrite (7) with $k_\infty = 0$ as

$$k(\alpha) = \beta \int_0^\infty \frac{H(\tau)}{\tau} e^{-\frac{\alpha}{\tau}} d\tau, \tag{10}$$

where

$$\beta = \log e. \tag{11}$$

Let

$$\tau = \frac{1}{\lambda}, \tag{12}$$

then (10) becomes

$$k(\alpha) = \beta \int_0^\infty \frac{H\left(\frac{1}{\lambda}\right)}{\lambda} e^{-\alpha\lambda} d\lambda. \tag{13}$$

Expression (13) has the standard form of the LaPlace transform when one lets $\alpha \to s$ and $\lambda \to t$. Formally then, the inversion formula for the LaPlace transform can be employed to obtain $H\left(\frac{1}{\lambda}\right)$ as

$$\frac{\beta H\left(\frac{1}{\lambda}\right)}{\lambda} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda\alpha} k(\alpha)\, d\alpha, \tag{14}$$

where γ is to the right of all singularities of $k(\alpha)$ in the complex (α) plane. Relation (14) can and will be explicitly evaluated using residue theory.

The class of functions to be considered for $k(\alpha)$ are power law forms. Such power law forms are commonly utilized and widely successful in technical applications. Specifically, take

$$k(\alpha) = \frac{K}{(1+\alpha)^m}, \quad m > 0 \tag{15}$$

where K is a constant (parameter), independent of α. Form (15) satisfies conditions (6). Solve for exponent m in (15) such that the spectrum $H(\tau)$ has the maximum at $\tau = 1$, as already

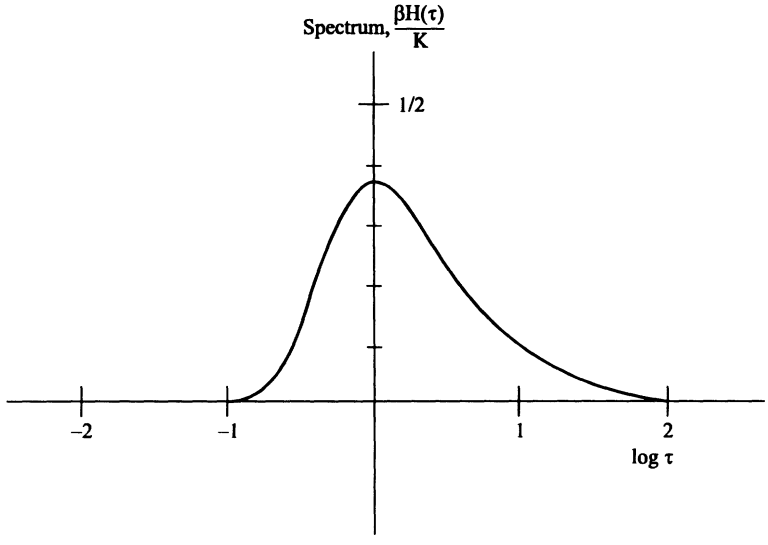

Fig. 1. Spectrum of $k(\alpha)$, equation (18).

discussed. Begin by considering the particular case $m = 1$. For $m = 1$, the spectrum from (14) can be shown to be

$$\frac{H\left(\frac{1}{\lambda}\right)}{\lambda} = \frac{Ke^{-\lambda}}{\beta} \tag{16}$$

with (15) as

$$k(\alpha) = \frac{K}{1+\alpha}. \tag{17}$$

With (12), spectrum (16) is

$$H(\tau) = \frac{K}{\beta\tau}e^{-\frac{1}{\tau}}. \tag{18}$$

It is now easy to verify that

$$H(\tau) = \frac{K}{\beta\tau}e^{-\frac{1}{\tau}} = \max \quad \text{at} \quad \tau = 1. \tag{19}$$

This spectrum is shown in Figure 1, where it is seen that the spectrum is rather narrow, not spread over many decades, thus the transition will be sharp. Any value of m in (15) other than $m = 1$ will give a maximum value of $H(\tau)$ at some value of τ other than $\tau = 1$. For example, $m = 1/2$ in (15) gives the maximum in $H(\tau)$ as occurring at $\tau = 2$. In future work, the $m = 1/2$ value will be shown to govern the corresponding two-dimensional case.

It now can be shown that $k(\alpha)$ in (17) has a transition at $\alpha = 1$ as specified by

$$\left| \frac{dk}{d(\log \alpha)} \right| = \max \quad \text{at} \quad \alpha = 1. \tag{20}$$

The use of spectra in this derivation leads as a natural consequence to the use of logarithmic scales and hence to the result shown above.

The status now is that parameter k has been eliminated in favor of parameter K through (17). The yield function form (1) with the change of parameter form in (17) becomes

$$\frac{\alpha K}{\sqrt{3}} \sigma_{kk} + \frac{(1+\alpha)^2}{2} s_{ij}\, s_{ij} \leq \frac{K^2}{(1+\alpha)}, \tag{21}$$

where α is unchanged as

$$\alpha = \frac{|\sigma_{11}^C|}{\sigma_{11}^T} - 1, \tag{22}$$

but the new parameter K is given by

$$K = \frac{(\sigma_{11}^C)^2}{\sqrt{3}\sigma_{11}^T}. \tag{23}$$

Relations (21) through (23) comprise the yield function formalism to be used from this point on in this work. It is an alternative form of (1) through (3) and completely compatible with it. The yield stresses in uniaxial and shear states are given by

$$\begin{aligned} \sigma_{11}^C &= \frac{-\sqrt{3}K}{(1+\alpha)} \\ \sigma_{12}^Y &= \frac{K}{(1+\alpha)^{3/2}} \\ \sigma_{11}^T &= \frac{\sqrt{3}K}{(1+\alpha)^2}. \end{aligned} \tag{24}$$

At this point, it is important to circumscribe the physical meaning of the parameters α and K in this alternative formalism for yield. At $\alpha = 0$, relations (24) and (21) show the behavior to be that of purely Mises type, governed only by dislocation movement. This is taken to be the ideal condition. At $\alpha \to \infty$, the material in the stress states in (24) is not able to sustain load, and disintegration has occurred. The ideal state at $\alpha = 0$ then has parameter K as solely determining the yield strength. Accordingly, parameter K is termed as the ideal or intrinsic yield strength parameter, sometimes simply called the intrinsic strength, for brevity. The intrinsic strength is that which would occur if there were no damage or microstructure disturbance. As such, it represents atomic scale characteristics and, specifically, the nature and resistance of the atomic bonding state to dislocation motion. Parameter α easily could be viewed as a damage measure; however, that terminology would be too restrictive, as discussed further in the next section. In a more general sense, α will be referred to as the

Table 1. Materials classification

Type	$\frac{\sigma_{11}^T}{\lvert\sigma_{11}^C\rvert}$	α	$\log\alpha$	$\frac{\sigma_{11}^T}{\sqrt{3}K}$
Ideal, defect free state	1	0	$-\infty$	1
Some ductile metals	0.95	0.05	-1.3	0.90
Some glassy polymers	$\frac{2}{3}$	$\frac{1}{2}$	-0.3	$\frac{4}{9}$
Transition	$\frac{1}{2}$	1	0	$\frac{1}{4}$
Some cast irons	$\frac{1}{3}$	2	0.3	$\frac{1}{9}$
Some ceramics	$\frac{1}{10}$	9	0.95	$\frac{1}{100}$
Some geological materials	$\frac{1}{100}$	99	2.0	$\frac{1}{10,000}$
Collapse to null state, material disintegration	0	∞	∞	0

microstructure parameter since it represents any and all microstructure changes on any scale that cause deviations from the ideal state. Even if the microstructure disturbance actually is of macroscale size, it still must be small compared with the scale of the macro averaging to represent an effective continuum behavior. The two-parameter yield form (21) then is determined by the intrinsic yield strength K and the microstructure parameter α.

The further physical significance of these new results now will be seen by taking the difference between the uniaxial yield stresses in compression and tension. From (24),

$$\frac{\lvert\sigma_{11}^C\rvert - \sigma_{11}^T}{\sqrt{3}K} = \frac{\alpha}{(1+\alpha)^2}. \tag{25}$$

Relation (25) is shown in Figure 2. It is seen that the present formulation maximizes at $\alpha = 1$ the difference between $\lvert\sigma_{11}^C\rvert$ and σ_{11}^T normalized by the intrinsic strength. Any other value of m in (15) besides $m = 1$ would not provide this maximum at $\alpha = 1$, which was previously found to characterize the fracture transition. This maximal property of $\lvert\sigma_{11}^C\rvert - \sigma_{11}^T$ at $\alpha = 1$ coordinates with the physical notion that an idealized condition of fracture in the tensile stress state is particularly limiting relative to the failure level in the corresponding compressive stress state. Another compatible physical interpretation of the behavior of the yield function (21) at $\alpha = 1$ will be given in the section on the ductile-brittle transition.

By way of illustration, the yield function (21) for biaxial stress states is shown in Figure 3. Even though the stresses are normalized by the intrinsic strength, K, which would vary by material type, the general trend is still evident. Generally speaking, the larger the value of the microstructure parameter, α, the more constricted is the yield function until the yield function collapses to a point at $\alpha \to \infty$ in Figure 3.

Table 1 shows typical values of yield stress ratios and thereby α for a variety of materials. Some of these are taken from the previous work; all are qualified as applying only to some materials within each materials type since there can be a wide range of behaviors. The

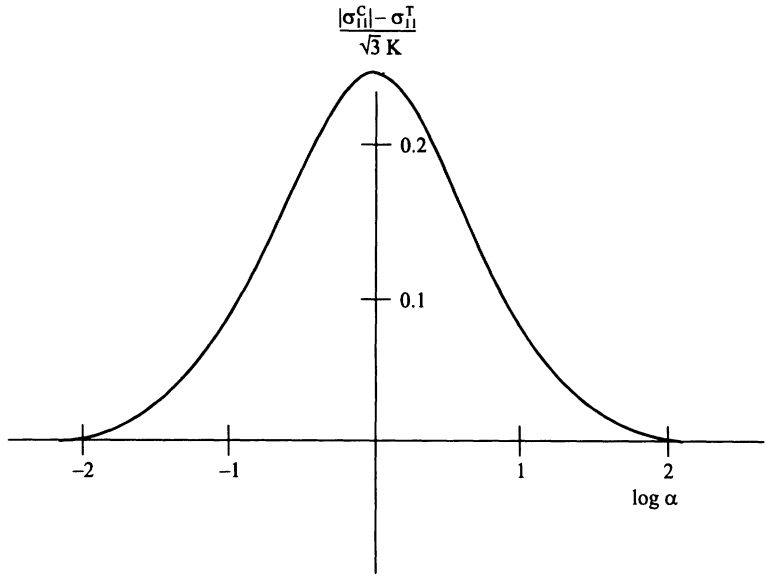

Fig. 2. Fracture behavior maximum at $\alpha = 1$, equation (25).

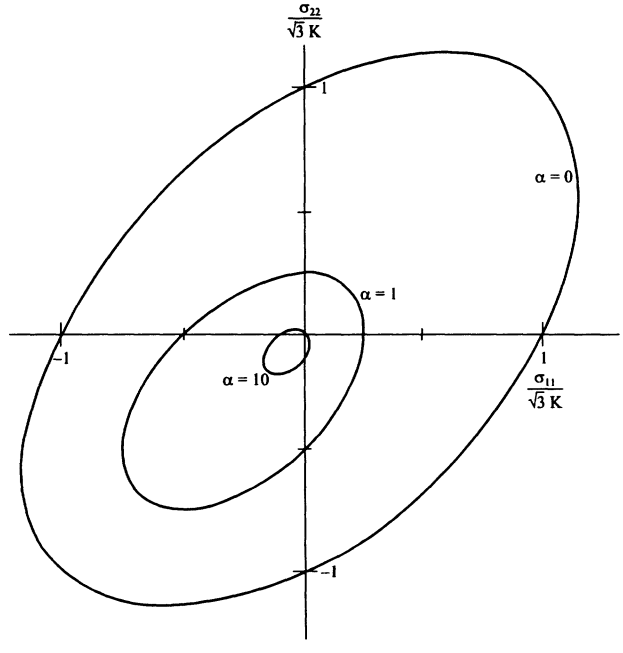

Fig. 3. Biaxial normal stress yielding.

microstructure parameter, α, provides a measure of the deviation from the ideal state at $\alpha = 0$. Even for ductile metals with α near zero, it may be that the material behavior is far from ideal in an asymptotic sense, when viewed through the value of $\log \alpha$. Certainly the value of $\log \alpha$ in Table 1 gives a meaningful sense of the degree of microstructure disruption from the ideal state. At first consideration, the transition at $\alpha = 1$ has an unbalanced appearance relative to the range from $\alpha = 0$ to ∞. However, when properly viewed on the logarithmic scale with a range from $-\infty$ to ∞, the transition at $\log \alpha = 0$ has an attractive symmetry surrounding it. Furthermore, the major materials groups appear to cluster somewhat symmetrically on either side of $\log \alpha = 0$ at ambient temperature and pressure. The last column in Table 1 will be discussed in the next section.

The advantage of the yield formalism given by (21) through (23) is that from measurements on any isotropic material in any possible state of damage, the ideal yield strength is deduced directly from measurements on the actual damaged material. The intrinsic yield strength parameter, K, from (23) affords this perspective according to the present theoretical formulation. This is the main result of the present work, and its implications will be considered next.

4. DAMAGE STATES AND OTHER CONDITIONS

The microstructure parameter at $\alpha = 0$ is the only defect-free, damage-free perfect arrangement of microstructure, with yield governed by the Mises condition. That is, there are no special features of microstructure to degrade performance below that at $\alpha = 0$. In the present context, the term *damage* could be used to mean any nonideal state caused by any aspect of microstructure variation. Such deviations from the ideal could be due to any effect on any scale from atomic to completely macroscopic. Thus, one could interpret any state with $\alpha \neq 0$ as that of damage, but that would be a rather arbitrary and unduly restrictive view that would not allow differentiation between major materials types. For example, glassy polymers have a range of α from about 0.6 to 0.8 or even greater. The damage state could be viewed as measured by those values of α relative to the highest possible values within the material type. Clearly, the best way to specify a material type is by taking only those materials with the same intrinsic strength, K. In the glassy polymer example, that would be an α of about $\alpha = 0.8$. That polymers as a general material type cannot approach the ideal case of $\alpha = 0$ is due to the long-chain nature of the polymeric constitution that cannot permit the ideal dislocation mechanism of yielding, which is necessary for the $\alpha = 0$ ideal case. The term *damage* will be used both in the sense of deviations from the best value of α in the materials class having the same intrinsic strength, K, and in the sense of deviations from the ideal state. The difference will be clear from the context. Thus, the microstructure parameter α also becomes the damage parameter when material type is fixed. In the present context, the term *damage* has far broader implications than just effects due to idealized cracks.

Figure 4 shows the uniaxial stress yield values as a function of α, equations (24). The shear stress at yield normalized by K would fall between the two forms in Figure 4. The forms in Figure 4 can be viewed as the effect of damage, as α increases. These results also suggest that the approach to the ideal defect-free state is an asymptotic situation and not easily realized as a state of perfect behavior. The last column in Table 1 gives the value of the uniaxial tensile

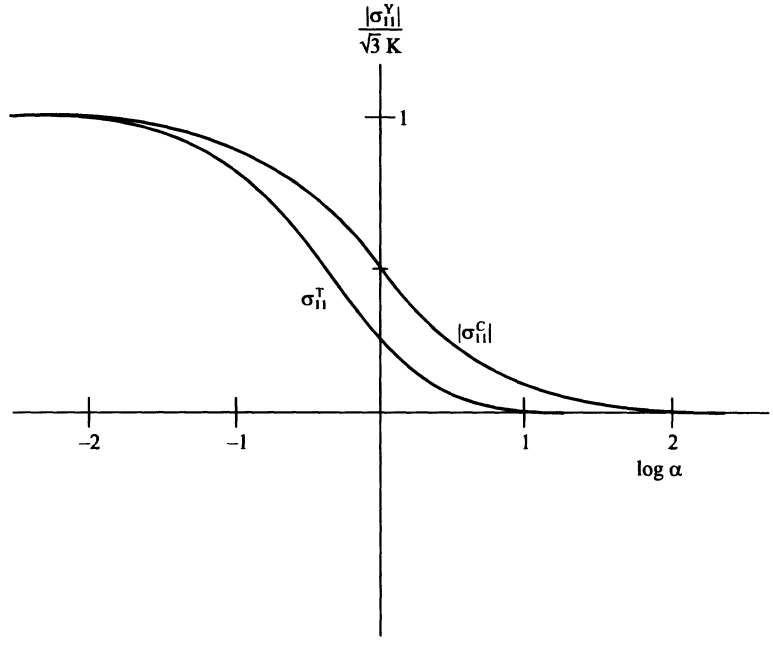

Fig. 4. Uniaxial stress yielding.

yield from (24) as ratioed with the ideal value. As such, this is a direct measure of the state of damage in the material. For example, for cast iron, the yield strength in tension is about 1/9 of the value that would occur in the ideal state. Similar to the uniaxial results, the biaxial stress states shown in Figure 3 can be looked on as the effect of damage as α increases.

The special case of equal biaxial stress states is revealing when compared with uniaxial and hydrostatic cases. The three cases are shown below for both tensile and compressive yielding at $\alpha = 1$.

State	$\frac{\sigma^T}{\sqrt{3}K}$	$\frac{\sigma^C}{\sqrt{3}K}$
Uniaxial	$\frac{1}{4}$	$-\frac{1}{2}$
Equi-biaxial	0.18	−0.68
Equi-triaxial	$\frac{1}{6}$	$-\infty$

Comparing the uniaxial, biaxial, and triaxial states at $\alpha = 1$, it is seen that the triaxial state is the most damage degraded in tension, but the uniaxial state is the most damage degraded in compression.

The fully expanded, component form of the yield function (21) should be recorded. This form is

$$\begin{aligned} & \frac{\alpha K}{\sqrt{3}}\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right) \\ + \quad & (1+\alpha)^2\left[\frac{\left(\sigma_{11}-\sigma_{22}\right)^2+\left(\sigma_{22}-\sigma_{33}\right)^2+\left(\sigma_{33}-\sigma_{11}\right)^2}{6}\right. \\ + \quad & \left.\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2\right)\right] \leq \frac{K^2}{(1+\alpha)}, \end{aligned} \tag{26}$$

where (22) and (23) give parameters α and K. In contrast, the most compact form of (21) is given by

$$\frac{1}{2}s_{ij}\,s_{ij} \leq \eta\,K^2, \tag{27}$$

where

$$\eta = \frac{1-\dfrac{\alpha\,(1+\alpha)}{\sqrt{3}K}\sigma_{ii}}{(1+\alpha)^3}. \tag{28}$$

In (27) and (28), η is a nondimensional scaling factor on K^2, determined by mean normal stress. A nontrivial solution of (27) is possible only if

$$\sigma_{ii} < \frac{\sqrt{3}K}{\alpha\,(1+\alpha)}. \tag{29}$$

An important restriction emerges from the forms in (27) and (28). The yield function (27) can never match or exceed the intrinsic strength, no matter how high the pressure through σ_{ii}. The intrinsic strength is given by

$$\frac{1}{2}s_{ij}\,s_{ij} \leq K^2. \tag{30}$$

Comparing (27) and (30) requires $\eta < 1$; thus,

$$\frac{K}{(1+\alpha)^3} - \frac{\alpha\sigma_{ii}}{\sqrt{3}\,(1+\alpha)^2} < K. \tag{31}$$

Relation (31) then gives

$$\sigma_{ii} > -3\sqrt{3}K\left[1+\frac{\alpha^2}{3\,(1+\alpha)}\right]. \tag{32}$$

For small and large α, (32) becomes

$$\begin{aligned} \sigma_{ii} &> -3\sqrt{3}K \quad \text{at} \quad \alpha = 0 \\ \sigma_{ii} &> -3\sqrt{3}\alpha K \quad \text{at} \quad \alpha \gg 4. \end{aligned} \tag{33}$$

Thus (32) provides limits on $\sigma_{ii} < 0$ in specific cases. This does not contradict the possibility of unlimited $\sigma_{kk} < 0$ when $s_{ij}\, s_{ij} = 0$.

5. DUCTILE-BRITTLE TRANSITION

Suppose for a given state of stress, an additional pressure is imposed on the material. It is the purpose now to show that the effect of the additional pressure is equivalent to a change in the effective value of the microstructure parameter, α at yield. Take the additional pressure as specified by Δp, then the yield criterion, (21), becomes

$$\frac{\alpha K}{\sqrt{3}}(\sigma_{kk} - 3\Delta p) + \frac{(1+\alpha)^2}{2} s_{ij}\, s_{ij} = \frac{K^2}{1+\alpha}. \tag{34}$$

On the other hand, relative to the given state of stress, take a change in α by amount $\Delta\alpha$, then (21) becomes

$$\frac{(\alpha+\Delta\alpha)K}{\sqrt{3}}\sigma_{kk} + \frac{(1+\alpha+\Delta\alpha)^3}{2} s_{ij}\, s_{ij} = \frac{K^2}{1+\alpha+\Delta\alpha}. \tag{35}$$

Using small values of Δp and $\Delta\alpha$, with expansions, then (34) and (35), respectively, become

$$\frac{\alpha K}{\sqrt{3}}\sigma_{kk} + \frac{(1+\alpha)^2}{2} s_{ij}\, s_{ij} - \frac{K^2}{(1+\alpha)} = \sqrt{3}\alpha K \Delta p \tag{36}$$

and

$$\frac{\alpha K}{\sqrt{3}}\sigma_{kk} + \frac{(1+\alpha)^2}{2} s_{ij}\, s_{ij} - \frac{K^2}{(1+\alpha)} = -\left[\frac{K\sigma_{kk}}{\sqrt{3}} + (1+\alpha)\, s_{ij}\, s_{ij} + \frac{K^2}{(1+\alpha)^2}\right]\Delta\alpha\,. \tag{37}$$

Equating the right-hand sides of (36) and (37) and going to the limit of infinitesimal $\Delta\alpha$ and Δp gives

$$\frac{d\alpha}{dp} = \frac{-\sqrt{3}\alpha K}{\dfrac{K}{\sqrt{3}}\sigma_{kk} + (1+\alpha)\, s_{ij}\, s_{ij} + \dfrac{K^2}{(1+\alpha)^2}}. \tag{38}$$

The derivative $d\alpha/dp$ is the result sought. It shows that the effect of the pressure is equivalent to a change in α, and specifies the size of the physical effect.

The yield function (21) can be used to write the derivative (38) in several alternative forms. For example, (38) is equivalent to

$$\frac{d\alpha}{dp} = \frac{-\sqrt{3}\alpha K}{\dfrac{(1-\alpha)}{(1+\alpha)}\dfrac{K}{\sqrt{3}}\sigma_{kk} + \dfrac{3K^2}{(1+\alpha)^2}} \tag{39}$$

or

$$\frac{d\alpha}{dp} = \frac{-\sqrt{3}\alpha^2 K}{-\dfrac{(1-\alpha^2)}{2}s_{ij}\,s_{ij} + \dfrac{(1+2\alpha)}{(1+\alpha)^2}K^2}. \tag{40}$$

Note that from (38),

$$\left.\frac{d\alpha}{dp}\right|_{\alpha=0} = 0. \tag{41}$$

Thus, the Mises state remains a Mises state.

At $\alpha = 1$, (38) and the yield form (21) can be used to obtain an explicit result not dependent on individual σ_{kk} and s_{ij} terms, namely,

$$\left.\frac{d\alpha}{dp}\right|_{\alpha=1} = -\frac{4}{\sqrt{3}K}. \tag{42}$$

From (42), it is seen that an increase in pressure reduces the effective value of α, moving the behavior toward the Mises end of the scale. That is, moving the behavior toward the ductile case. Alternatively, decreasing the pressure moves α toward larger values, which is taken to be that of brittle behavior, in the sense that this is the opposite of ductility. This view will be amplified shortly.

Relation (42) gives $\frac{d\alpha}{dp} < 0$ at $\alpha = 1$, which is the expected behavior. Now establish the conditions under which $\frac{d\alpha}{dp}$ has this same behavior for all values of α. From (38), it is seen that

$$\frac{d\alpha}{dp} < 0 \quad \text{for} \quad \sigma_{kk} \geq 0. \tag{43}$$

Next consider the case when $\sigma_{kk} < 0$. From (39), for $\frac{d\alpha}{dp} < 0$

$$\frac{(1-\alpha)K}{\sqrt{3}}\sigma_{kk} + \frac{3K^2}{1+\alpha} > 0. \tag{44}$$

Relation (44) is satisfied by

$$\alpha \geq 1, \quad \sigma_{kk} < 0. \tag{45}$$

For the range of $\alpha < 1$, (44) gives

$$\sigma_{kk} > -\frac{3\sqrt{3}K}{1-\alpha^2}, \quad \alpha < 1. \tag{46}$$

Relation (46) is a restriction on the dilatational part of the stress state. Another restriction, (32), was found in the preceding section. Comparing (32) and (46) shows that (46) is always satisfied and superseded by (32). Thus, all conditions are met for $d\alpha/dp \leq 0$ for all values of α.

It should be observed that at the transition, $\alpha = 1$, the value of $d\alpha/dp$, (42), is independent of the stress state, as occurs with a material property. At all other values of α, the derivative $d\alpha/dp$ depends explicitly on the stress state as seen in (38) or (40). This means that the relative weights of the σ_{kk} and $s_{ij}\, s_{ij}$ terms in (38) do not matter at the transition $\alpha = 1$—the two effects are in balance. This is another (physical) signature property of the transitional behavior at $\alpha = 1$. This behavior would not have occurred if the yield form in (1) were used to find $d\alpha/dp$ rather than using (21) involving the intrinsic strength.

At the transition, $\alpha = 1$, (24) gives

$$\frac{\sigma_{11}^T}{|\sigma_{11}^C|} = \frac{1}{2}, \quad \text{Transition.} \tag{47}$$

This ratio of yield strengths in uniaxial stress is the criterion for the transition at $\alpha = 1$, according to the present theory. This transition at $\alpha = 1$ is the same as was detailed to be the fracture transition in the previous work. For

$$\frac{1}{2} < \frac{\sigma_{11}^T}{|\sigma_{11}^C|} \leq 1, \quad \text{Ductile Behavior,} \tag{48}$$

taken as ductile behavior since the corresponding values of $\alpha < 1$ place the yield function toward $\alpha = 0$, the Mises yield case. Complementary to (48), then, is

$$0 \leq \frac{\sigma_{11}^T}{|\sigma_{11}^C|} < \frac{1}{2}, \quad \text{Brittle Behavior.} \tag{49}$$

In this latter range of $\alpha > 1$, the yield form (21) effectively represents a failure criterion, with behaviors typified by cast iron and ceramics as seen in Table 1. Accordingly, this behavior is termed as that of a brittle type. Thus, the transition at $\alpha = 1$ is taken as that of the ductile-brittle transition, commonplace in materials science (Cottrell [7], Rice and Beltz [8], Needleman [9], Xu, Argon, and Ortiz [10]).

As seen in the preceding developments, the effective value of the microstructure parameter, α, depends on the pressure, which is imposed over and above that of the specified stress state. A dependence of α on pressure also implies a dependence on temperature, T, thus

$$\alpha = \alpha\,(T,p)\,.$$

The formal criterion for the ductile-brittle transition is then

$$\alpha\,(T,p) = 1. \tag{50}$$

Relation (50), along with (22) and (47), can be used to solve for temperature as a function of pressure as the locus of the transition.

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