Yield functions/failure criteria for isotropic materials

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A new yield function formalism is developed for isotropic materials. A spectrum of forms are inclusive to the general form that has two yield parameters. One parameter is that of a scale type involving uniform expansion or contraction of the yield function. The other is a non-dimensional shape parameter that controls the shape of the function in stress space. Thus the entire range of different behavioural types is accessed by varying a single parameter, the shape parameter. At one extreme of the behaviour is the usual Mises form involving distortional control. At the other extreme is a highly dilatant type of behaviour strongly influenced by mean normal stress. An intermediate range of behaviour is identified as being fracture controlled. The overall forms apply to the initial yielding of materials as well as to strength type behaviour where appropriate.

1. Introduction

The generation, interpretation and use of yield functions is one of the most basic aspects of the mechanics of materials. This sub-field almost goes back to the origins of the overall field of mechanics. The compelling reason for this is that the safe and efficient utilization of materials, whether natural or artificial, gives high priority to understanding the limits of performance. Based upon issues of longevity and importance, yield function theory and strength characterization have all the attributes of a classical endeavour. However, in contrast to some classical fields which are considered to be completely formulated and thoroughly understood, the circumstance with yield functions is not what could be called entirely satisfactory. To probe this situation, first the highlights of some historical developments will be given.

The first major form was given by Coulomb (1773) and later elaborated by Tresca (1868) stating that the isotropic material yields or fails when the maximum shear stress reaches a critical value,

$$\tau_{\max} \leqslant k. \tag{1.1}$$

The notation used here is that the equality implies yielding and the inequality implies elastic behaviour.

Von Mises (1913) proposed the form

$$\tfrac{1}{2} s_{ij} s_{ij} \leqslant k^2, \tag{1.2}$$

where

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\sigma_{kk}, \tag{1.3}$$

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with s_{ij} the deviatoric stress tensor and rectangular cartesian tensor notation is followed. The form (1.2) initially was viewed as an approximation to the Tresca form, (1.1), obtained by eliminating mean normal stress through (1.3). Over the years and with countless observations, this view has shifted to the point where now the Mises form is considered to be the more fundamental.

Although both forms (1.1) and (1.2) have proven to be fairly close to each other and satisfactory for ductile metals, they are not satisfactory for other materials which show a pronounced and sometimes strong dependence upon mean normal stress. It is likely that the majority of materials are not well described by yielding behaviour in accordance with (1.1) and/or (1.2). Actually this situation was recognized quite early and another form with a mean normal stress effect was proposed, the Coulomb–Mohr form.

In the Coulomb (1773)–Mohr (1914) form, the failure surface is visualized to be acted upon by both normal and shear stress such that the yield condition would be given by

$$\tau \leqslant c - \mu\sigma, \tag{1.4}$$

where σ is the normal stress and c and μ are the associated material parameters, determined from tests on the material. When combined with the Tresca form and put into a three-dimensional format viewed in principal stress space, there is formed a six sided pyramid, which has the vertex in the direction of the positive principal stresses. This two-parameter yield function, although widely noted, has not proven to be generally realistic in modelling data.

Usefull accounts of yield functions and strength criteria have been given by Hill (1950) and Paul (1968). All the difficulties and shortcomings are described in those references and need not be repeated here, except for one or two observations. As noted by Paul (1968), the Coulomb–Mohr form becomes somewhat more realistic when subjected to a 'tension cutoff'. That is, the vertex of the Coulomb–Mohr pyramid is eliminated by some form of truncation. Paul himself recommended a three-parameter generalized pyramid, possibly with tension cutoffs. Wronski & Pick (1977), in considering polymers, recommended the three-parameter pyramidal criterion of Paul, but in so doing chose to ignore behaviour in the biaxial tension quadrant and triaxial tension octant. Drucker & Prager (1952) proposed a conical generalization in principal stress space of the Mises form. Other yield functions will be mentioned in later sections.

It is fair to say, in summing up, that no general yield function has been accepted for application across broad classes of material types that has the clarity and power that the Mises form brings to the specific application of ductile metals.

2. Objectives and requirements

A main objective is to generalize the Mises criterion to include the effect of mean normal stress. Although this has been approached many times in the past, a new formalism will be developed here. The starting point is to inquire as to what is the simplest possible generalization of the Mises form to include the mean normal stress effect. This in turn is approached by first asking what is the simplest possible dependence upon a hydrostatic stress state by itself. The logical answer would seem to be that of a conditional dependence upon hydrostatic stress such that there is yielding in the corresponding tensile hydrostatic state, but not in the compressive hydrostatic state for full density materials. This line of inquiry will be used to open the developments in the next section.

In more specific terms, the present work is aimed to develop a unified theory of isotropic material yielding that has distortional (Mises) yielding at one extreme of behaviour and has a dilatational-related yielding at the other extreme. Furthermore, it is desired that fracture type behaviour be reconciled as an explicit special case.

Whether or not a homogeneous type of yield theory can include fracture is somewhat controversial. The present theme asserts not only the possibility, but the necessity of including fracture-like characteristics in the spectrum of material response types which a comprehensive yield theory must encompass. However, the term fracture must be carefully and quantitatively defined in the present context. The usual statement that 'brittle' materials imply fracture is too vague to be helpful. It is presumed that any distribution of cracks causing fracture is macroscopically homogeneous and isotropic.

The term fracture, as used here, has a broader meaning than that implied by linear elastic fracture mechanics as purely an instability. In the present context of yield theory, the range where fracture occurs could take the form of an instability, but it also could mean the onset of an irreversible damage state in the sense of plasticity. Thus, as used here, fracture behaviour would be best termed as generalized fracture, or plastic fracture, but for simplicity the shorter term will be used, with the implied qualification. Some other considerations on fracture are given in the last section.

It is necessary to have an explicit criterion for fracture, so that it can be identified, if it exists, as part of the yield theory. The following fracture criterion will be used, perhaps others are possible. Start with a state of uniaxial tension having

$$\sigma_{11} \neq 0, \quad \text{other } \sigma_{ij} = 0.$$

Next, add a small increment of transverse normal stress, $\sigma = \sigma_{22} = \sigma_{33}$, and require that this small superimposed increment have no effect on the tensile yield stress, σ_{11}^{T}, if a state of fracture exists. This requirement is stated by

$$\left.\frac{\mathrm{d}\sigma_{11}^{\mathrm{T}}}{\mathrm{d}\sigma}\right|_{\sigma=0} = 0 \text{ at fracture,} \tag{2.1}$$

where $\sigma = \sigma_{22} = \sigma_{33}$.

This requirement is consistent with linear fracture mechanics for non-interacting cracks. It is really just a mathematical statement of the widely recognized practice that fracture depends only upon the largest tensile component of stress so long as the other normal components are not too large, and there are no shear components.

As an additional requirement on uniaxial tension at fracture, take the associated irreversible plastic flow to be given entirely by strain in the stress direction, thus,

$$\dot{\varepsilon}_{22}^{\mathrm{p}} = \dot{\varepsilon}_{33}^{\mathrm{p}} = 0, \quad \dot{\varepsilon}_{11}^{\mathrm{p}} \neq 0, \tag{2.2}$$

where $\dot{\varepsilon}_{ij}^{\mathrm{p}}$ is the plastic component of the increment of small strain, as prescribed by the inviscid theory of incremental plasticity. Consistent with this theory, the associated flow rule will be employed, i.e.

$$\dot{\varepsilon}_{ij}^{\mathrm{p}} = \lambda \frac{\partial f}{\partial \sigma_{ij}}, \tag{2.3}$$

where the function $f(\sigma_{ij})$ is the yield function, λ is a positive factor of proportionality, and the small strain has been decomposed as the sum of elastic and plastic parts.

The requirement (2.2) for fracture is taken from the discontinuity in displacement u_1 across an activated opening crack due to tensile stress. Only u_1 has a displacement discontinuity and this is taken to give the plastic component of strain as the crack grows.

In uniaxial tension, the form (2.1) will be used to identify the possible existence of fracture behaviour in the larger picture of the yield theory. If (2.1) is satisfied, the simultaneous satisfaction of (2.2) will be used as a consistency check.

Up until this point, both of the terms yielding and failure (or strength) have been used. Henceforth, only the term 'yield' will be used, with the understanding that it refers either to the initial yield function or to failure criteria in cases where the material undergoes macro-failure.

Although the present developments are given in small strain and associated stress forms, there is no difficulty in extending the results to full generality using Cauchy stress or other forms.

3. Yield/failure theory

The three standard-form invariants of a second-order tensor, stress in this case, are

$$\sigma_{ii},\ \sigma_{ij}\sigma_{ij}, \quad \text{and} \quad |\sigma_{ij}|.$$

Take a polynomial expansion through second-degree terms. Then the most general form for the corresponding isotropic material yield function is

$$\varDelta\sigma_{ii} + \beta\sigma_{ii}^2 + \gamma\sigma_{ij}\sigma_{ij} \leqslant 1, \tag{3.1}$$

where $\varDelta$, β and γ are material parameters. Many forms of the general type of (3.1) have been investigated with specific yield data sets. Bresler & Pister (1957) have used forms of (3.1) for the case of unreinforced concrete, as well as Hu & Pai (1963) along with Pai and many colleagues for polymers, and Raghava *et al.* (1973) and Asp *et al.* (1997) also for polymers. However, that possible course is not followed here, rather (3.1) is used to develop a different formulation.

Take a hydrostatic stress

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{3\mathrm{D}}. \tag{3.2}$$

As a physical hypothesis require that the yield function in (3.1) possesses a limit under tensile hydrostatic stress (3.2) but give no limit under compressive hydrostatic stress within the range of the polynomial expansion. With (3.2), (3.1) becomes

$$3\varDelta\sigma_{3\mathrm{D}} + 3(3\beta+\gamma)\sigma_{3\mathrm{D}}^2 \leqslant 1. \tag{3.3}$$

For no compressive root, (3.3) requires that

$$\beta = -\tfrac{1}{3}\gamma, \tag{3.4}$$

leaving

$$3\varDelta\sigma_{3\mathrm{D}} \leqslant 1, \tag{3.5}$$

which possesses a tensile root for $\varDelta$ non-negative. With (3.4), (3.1) becomes

$$\varDelta\sigma_{ii} + \gamma(-\tfrac{1}{3}\sigma_{ii}^2 + \sigma_{ij}\sigma_{ij}) \leqslant 1. \tag{3.6}$$

This form is a special case of (3.1) and has been studied with specific material data sets, as mentioned in connection with (3.1). Apparently the first time (3.6) was considered was by Schleicher (1926) with comments by Mises (1926). Stassi (1967) extensively studied form (3.6) both for different parameter values and different stress states.

Rewrite (3.6) with three parameters, ζ, α and k, but only two combinations of which can be independent. Then,

$$\zeta k\sigma_{ii} + \tfrac{1}{2}(1+\alpha)[-\tfrac{1}{3}\sigma_{ii}^2 + \sigma_{ij}\sigma_{ij}] \leqslant k^2. \tag{3.7}$$

For $\zeta = \alpha = 0$ equation (3.7) reduces to the Mises form in (1.2) and (1.3), where the term in brackets is just $s_{ij}s_{ij}$. Thus (3.7) is a generalization of the Mises form to include dilatational effects. The generalization is an expansion about the Mises case with ζ and α being the additional material parameters. In this expansion, it is important not only to include the ζ term in (3.7) but also the α term, as will be seen in the following derivation and resulting limiting cases.

Now, with k taken as an assigned parameter consistent with the above expansion, then there remains one independent condition between the two remaining parameters, ζ and α. There is freedom to invoke an additional condition relating ζ to α. Take this condition such that in the uniaxial stress case, the quadratic formula for the roots can be evaluated explicitly. This procedure now will be given.

In uniaxial stress

$$\sigma_{11} \neq 0, \quad \text{other } \sigma_{ij} = 0.$$

The yield form of (3.7) then becomes

$$\hat{\sigma}_{11}^2 + \frac{3\zeta}{1+\alpha}\hat{\sigma}_{11} - \frac{3}{1+\alpha} = 0, \tag{3.8}$$

where

$$\hat{\sigma}_{ij} = \frac{\sigma_{ij}}{k}. \tag{3.9}$$

The quadratic formula gives the solution of (3.8) as

$$\hat{\sigma}_{11} = \frac{3\zeta}{2(1+\alpha)}\left(-1 \pm \sqrt{1 + \frac{4(1+\alpha)}{3\zeta^2}}\right). \tag{3.10}$$

To evaluate this explicitly, take the terms inside the radical as comprising a perfect square. This is satisfied by

$$\zeta = \frac{\alpha}{\sqrt{3}}, \tag{3.11}$$

and the roots from (3.10) are then

$$\hat{\sigma}_{11} = \frac{\sqrt{3}\alpha}{2(1+\alpha)}\left(-1 \pm \left(1 + \frac{2}{\alpha}\right)\right). \tag{3.12}$$

Finally, the tensile and compressive roots in (3.12) are

$$\sigma_{11}^{\mathrm{T}} = \frac{\sqrt{3}k}{(1+\alpha)} \quad \text{and} \quad \sigma_{11}^{\mathrm{C}} = -\sqrt{3}k. \tag{3.13}$$

With (3.11), the yield function (3.7) takes the final form

$$\frac{\alpha k\sigma_{ii}}{\sqrt{3}} + \tfrac{1}{2}(1+\alpha)(-\tfrac{1}{3}\sigma_{ii}^2 + \sigma_{ij}\sigma_{ij}) \leqslant k^2, \tag{3.14}$$

where both parameters k and α are non-negative. From the roots, (3.13), the two material parameters are given by the yield values in uniaxial stress as

$$\alpha = \frac{|\sigma_{11}^{\mathrm{C}}|}{\sigma_{11}^{\mathrm{T}}} - 1 \quad \text{and} \quad k = \frac{|\sigma_{11}^{\mathrm{C}}|}{\sqrt{3}}, \tag{3.15}$$

where $\sigma_{11}^{\mathrm{T}} \leqslant |\sigma_{11}^{\mathrm{C}}|$.

It is important to note that the two material parameters have special characteristics. Changing parameter k uniformly expands or contracts the yield function, (3.14), thus it is called the scale parameter using terminology similar to that of a Weibull distribution. In contrast, the non-dimensional parameter α changes the shape of the yield function, thus it is called the shape parameter. It follows that the spectrum of behavioural types for different materials is displayed by varying the single parameter, α. This characteristic has great utility in examining different forms of behaviour, as is first done next.

Consider the limiting cases of physical behaviour. It is convenient to write the yield function, (3.14), in the most compact form using deviatoric stress,

$$\frac{\alpha k \sigma_{ii}}{\sqrt{3}} + \tfrac{1}{2}(1+\alpha) s_{ij} s_{ij} \leqslant k^2. \tag{3.16}$$

At $\alpha = 0$ the Mises form is the limiting case,

$$\tfrac{1}{2} s_{ij} s_{ij} \leqslant k^2, \quad \alpha = 0. \tag{3.17}$$

The other limiting case from (3.16) is at $\alpha \to \infty$, with k remaining bounded,

$$\tfrac{1}{2} s_{ij} s_{ij} \leqslant -\left(\frac{k}{\sqrt{3}}\right) \sigma_{ii}, \quad \alpha \to \infty. \tag{3.18}$$

Thus both limiting cases reduce to simple one-parameter forms. Considering the limit, (3.18), first note that $\sigma_{ii} > 0$ is not possible. Furthermore, for $\sigma_{ii} = 0$ then (3.18) gives $s_{ij} = 0$ so this case too is null. There can be no non-trivial values for s_{ij} unless $\sigma_{ii} < 0$, that is, the mean normal stress must be compressive. The mean normal stress has a profound influence upon yielding in this limiting case. For all states in between $\alpha = 0$ and $\alpha \to \infty$, the mean normal stress has an influence, with the degree of the physical effect increasing as α increases. For very large, but not infinite α, the resulting behaviour is like that of the Coulomb–Mohr form, this is easy to show using the binomial expansion. But aside from that special case, the present yield function, (3.14) or (3.16), is fundamentally, and in a physical sense, enormously different from the Coulomb–Mohr form.

The two limiting cases are revealing. At $\alpha = 0$ the Mises form effectively models yielding due to the shear stress driven flow of dislocations, applicable to ductile metals. At $\alpha \to \infty$ the yield function, (3.18), is that for a cohesionless solid. For materials with a small degree of cohesion, then the general yield function, (3.14), would have α large but not infinite. The damage conditions that drive α to be very large would also be expected to render k as very small. Thus the mathematical limiting cases span the entire range of extreme physical effects. It is also noted that the $\alpha \to \infty$ limiting case, (3.18), would be less transparent from the more primitive and somewhat standard form in (3.1) or (3.6), the key here is the shape parameter.

As a last matter of relevance for the general theory, it is important to examine the yield function (3.14) in order to ascertain whether or not a fracture type behaviour is somehow embedded as a special case in the general theory. The criterion for fracture is that of (2.1) as discussed in the previous section. The problem is posed as follows. Is there some value of the shape parameter, α, at which the fracture criterion (2.1) is satisfied? First it is necessary to form the derivative of the type shown in (2.1). The yield form of (3.14) gives

$$\frac{\alpha}{\sqrt{3}}(\hat{\sigma}_{11} + 2\hat{\sigma}) + \tfrac{1}{3}(1+\alpha)(\hat{\sigma}_{11} - \hat{\sigma})^2 = 1, \tag{3.19}$$

where σ is as in (2.1) and where the notation in (3.9) is used. Take the derivative of (3.19) with respect to $\hat{\sigma}$ and then set $\hat{\sigma} = 0$ to obtain

$$\left.\frac{\mathrm{d}\hat{\sigma}_{11}}{\mathrm{d}\hat{\sigma}}\right|_{\hat{\sigma}=0} = \frac{-\sqrt{3}\alpha + (1+\alpha)\hat{\sigma}_{11}}{\tfrac{1}{2}(\sqrt{3}\alpha) + (1+\alpha)\hat{\sigma}_{11}}. \tag{3.20}$$

Using the roots shown in (3.13) gives (3.20) as

$$\left.\frac{\mathrm{d}\sigma_{11}^{\mathrm{T}}}{\mathrm{d}\sigma}\right|_{\sigma=0} = \frac{1-\alpha}{1+\tfrac{1}{2}\alpha} \quad \text{and} \quad \left.\frac{\mathrm{d}\sigma_{11}^{\mathrm{C}}}{\mathrm{d}\sigma}\right|_{\sigma=0} = \frac{1+2\alpha}{1+\tfrac{1}{2}\alpha}. \tag{3.21}$$

At $\alpha = 0$ both derivatives give the value of one which simply means that the yield is independent of superimposed hydrostatic pressure, as must be true in the Mises limit.

The fracture criterion, (2.1), is satisfied by (3.21) at

$$\alpha = 1, \quad \text{fracture}, \tag{3.22}$$

thus fracture, or generalized fracture type behaviour, has been identified as existing at $\alpha = 1$ for the yield function, (3.14). Although $\sigma_{22} = \sigma_{33} = \sigma$ was used in fracture criterion (2.1), it can be shown that the same result can be established more generally for any ratio of σ_{22} to σ_{33}. In the next section it will be shown that the second part of the fracture criterion (2.2), also is satisfied at the same value of α. Much of the rest of the work here is occupied with this fracture case since this special case is so interesting and important.

4. General and special cases

Several different aspects and consequences of the yield function developed in the last section will now be given.

The inviscid plasticity flow rule (2.3) can be written in dilatational and deviatoric components as

$$\dot{\varepsilon}^{\mathrm{p}}_{kk} = 3\lambda\frac{\partial f}{\partial \sigma_{kk}} \quad \text{and} \quad \dot{e}^{\mathrm{p}}_{ij} = \lambda\frac{\partial f}{\partial s_{ij}}. \tag{4.1}$$

The elastic components of strain are given by

$$2\mu_{\mathrm{m}}\dot{e}^{\mathrm{e}}_{ij} = \dot{s}_{ij} \quad \text{and} \quad 3k_{\mathrm{m}}\dot{\varepsilon}^{\mathrm{p}}_{kk} = \dot{\sigma}_{kk}, \tag{4.2}$$

where

$$\varepsilon_{ij} = \varepsilon^{\mathrm{e}}_{ij} + \varepsilon^{\mathrm{p}}_{ij}, \tag{4.3}$$

and where μ_{m} and k_{m} are the elastic moduli.

Using the yield function (3.14) and relations (4.1)–(4.3) the parameter λ and the elastic and plastic components of strain can be eliminated in the elastic perfectly-plastic case to obtain

$$\dot{e}_{ij} = \frac{\dot{s}_{ij}}{2\mu_{\mathrm{m}}} + \left(\frac{(1+\alpha)s_{kl}\dot{e}_{kl} + \dfrac{\alpha k}{2\sqrt{3}\mu_{\mathrm{m}}}\dot{\sigma}_{kk}}{k - \dfrac{\alpha}{\sqrt{3}}\sigma_{kk}} \right) \frac{s_{ij}}{k}, \tag{4.4}$$

$$\dot{\varepsilon}_{kk} = \frac{\dot{\sigma}_{kk}}{3k_{\mathrm{m}}} + \frac{\sqrt{3}\alpha}{2(1+\alpha)} \left(\frac{(1+\alpha)s_{kl}\dot{e}_{kl} + \dfrac{\alpha k}{2\sqrt{3}\mu_{\mathrm{m}}}\dot{\sigma}_{kk}}{k - \dfrac{\alpha}{\sqrt{3}}\sigma_{kk}} \right). \tag{4.5}$$

These completely coupled equations govern the material constitutive response. In the case when the shape parameter $\alpha = 0$ then (4.4) and (4.5) become

$$\dot{e}_{ij} = \frac{\dot{s}_{ij}}{2\mu_{\mathrm{m}}} + \frac{(s_{kl}\dot{e}_{kl})}{k^2}s_{ij} \quad \text{and} \quad \dot{\varepsilon}_{kk} = \frac{\dot{\sigma}_{kk}}{3k_{\mathrm{m}}}, \tag{4.6}$$

which are the well-known Prandtl–Reuss equations (Prager & Hodge 1951).

Now consider the case of simple shear deformation. For shear stress σ_{12} the yield function (3.14) is

$$s_{12}^2 = \sigma_{12}^2 = \frac{k^2}{1+\alpha}, \tag{4.7}$$

and (4.4) is satisfied. Relation (4.5) becomes

$$\dot{\varepsilon}_{kk} = \tfrac{1}{2}(\sqrt{3}\alpha)\frac{s_{12}\dot{e}_{12}}{k}, \tag{4.8}$$

from which for ε_{12} positive

$$\varepsilon_{kk} = \tfrac{1}{2}\sqrt{3}\frac{\alpha}{\sqrt{1+\alpha}}\varepsilon_{12} + \text{const.} \tag{4.9}$$

This relation quantifies the volume change ε_{kk} caused by the shear strain ε_{12}. That shear deformation causes a positive volume change is an effect known as dilatancy. For the shape parameter α being quite large, the amount of volume change relative to the size of the shear strain becomes correspondingly large.

The other case to be considered here will use the flow rule, (2.3), directly to show physical effects rather than going through the more complex forms, (4.4) and (4.5). The flow rule, (2.3), combined with the yield function, (3.14), gives the components of plastic strain increment as

$$\left.\begin{aligned}
\frac{\dot{\varepsilon}_{11}^{\mathrm{p}}}{\lambda} &= \frac{\alpha k}{\sqrt{3}} + \tfrac{1}{3}(1+\alpha)(2\sigma_{11} - \sigma_{22} - \sigma_{33}), & \frac{\dot{\varepsilon}_{12}^{\mathrm{p}}}{\lambda} &= 2(1+\alpha)\sigma_{12},\\
\frac{\dot{\varepsilon}_{22}^{\mathrm{p}}}{\lambda} &= \frac{\alpha k}{\sqrt{3}} + \tfrac{1}{3}(1+\alpha)(-\sigma_{11} + 2\sigma_{22} - \sigma_{33}), & \frac{\dot{\varepsilon}_{23}^{\mathrm{p}}}{\lambda} &= 2(1+\alpha)\sigma_{23},\\
\frac{\dot{\varepsilon}_{33}^{\mathrm{p}}}{\lambda} &= \frac{\alpha k}{\sqrt{3}} + \tfrac{1}{3}(1+\alpha)(-\sigma_{11} - \sigma_{22} + 2\sigma_{33}), & \frac{\dot{\varepsilon}_{31}^{\mathrm{p}}}{\lambda} &= 2(1+\alpha)\sigma_{31}.
\end{aligned}\right\} \tag{4.10}$$

Note from (4.10) that

$$\dot{\varepsilon}_{kk}^{\mathrm{p}} = \sqrt{3}\lambda\alpha k, \tag{4.11}$$

and because λ, α and k are all positive,

$$\dot{\varepsilon}^{\mathrm{p}}_{kk} \geqslant 0. \tag{4.12}$$

The plastic volume change is positive and the materials are dilational under all conditions of stress except that of hydrostatic compression.

For uniaxial stress with $\sigma_{11} \neq 0$ then from (4.10)

$$\begin{aligned} \frac{\dot{\varepsilon}^{\mathrm{p}}_{11}}{\lambda} &= \frac{\alpha k}{\sqrt{3}} + \tfrac{2}{3}(1+\alpha)\sigma_{11}, \\ \frac{\dot{\varepsilon}^{\mathrm{p}}_{22}}{\lambda} &= \frac{\dot{\varepsilon}^{\mathrm{p}}_{33}}{\lambda} = \frac{\alpha k}{\sqrt{3}} - \tfrac{1}{3}(1+\alpha)\sigma_{11}. \end{aligned} \tag{4.13}$$

Use the yield roots, (3.13), in uniaxial tension and compression, then for tension,

$$\begin{aligned} \frac{\dot{\varepsilon}^{\mathrm{p}}_{11}}{\lambda k} &= \frac{(2+\alpha)}{\sqrt{3}}, \\ \frac{\dot{\varepsilon}^{\mathrm{p}}_{22}}{\lambda k} &= \frac{\dot{\varepsilon}^{\mathrm{p}}_{33}}{\lambda k} = \frac{(-1+\alpha)}{\sqrt{3}}, \end{aligned} \tag{4.14}$$

and for compression,

$$\begin{aligned} \frac{\dot{\varepsilon}^{\mathrm{p}}_{11}}{\lambda k} &= \frac{-(2+\alpha)}{\sqrt{3}}, \\ \frac{\dot{\varepsilon}^{\mathrm{p}}_{22}}{\lambda k} &= \frac{\dot{\varepsilon}^{\mathrm{p}}_{33}}{\lambda k} = \frac{(1+2\alpha)}{\sqrt{3}}. \end{aligned} \tag{4.15}$$

From (4.14) and (4.15) it is seen in both tension and compression that the plastic volume change is given by (4.11) and for large α the swelling effect is nearly spherical in the case of uniaxial tension, i.e.

$$\dot{\varepsilon}^{\mathrm{p}}_{11} \cong \dot{\varepsilon}^{\mathrm{p}}_{22} = \dot{\varepsilon}^{\mathrm{p}}_{33}. \tag{4.16}$$

This result indicates or suggests spherical void nucleation for materials with large α, which later will be shown to include ceramics.

The main thing to observe from this case of uniaxial tension is that the second half of the fracture criterion, (2.2), is satisfied by (4.14) at $\alpha = 1$, corroborating the result in (3.21) and (3.22).

The effect of hydrostatic stress upon shear yield stress is usually considered to be of importance. From (3.14) the yield stress in shear is

$$\sigma^{\mathrm{Y}}_{12} = \frac{k}{\sqrt{1+\alpha}}. \tag{4.17}$$

Now letting hydrostatic stress be $\sigma_{\mathrm{3D}} = \sigma_{11} = \sigma_{22} = \sigma_{33}$ then the yield function, (3.14), can be used to show

$$\left.\frac{\mathrm{d}\sigma^{\mathrm{Y}}_{12}}{\mathrm{d}\sigma_{\mathrm{3D}}}\right|_{\sigma_{\mathrm{3D}}=0} = \frac{-\sqrt{3}\alpha}{2\sqrt{1+\alpha}}. \tag{4.18}$$

Thus σ^{Y}_{12} increases with pressure, and the effect is stronger with increasing values of the shape parameter, α.

Now return to the case of uniaxial stress, σ_{11}. The yield function, (3.16), in non-dimensional form is

$$\frac{\alpha\hat{\sigma}_{kk}}{\sqrt{3}} + \tfrac{1}{2}(1+\alpha)\hat{s}_{ij}\hat{s}_{ij} = 1. \tag{4.19}$$

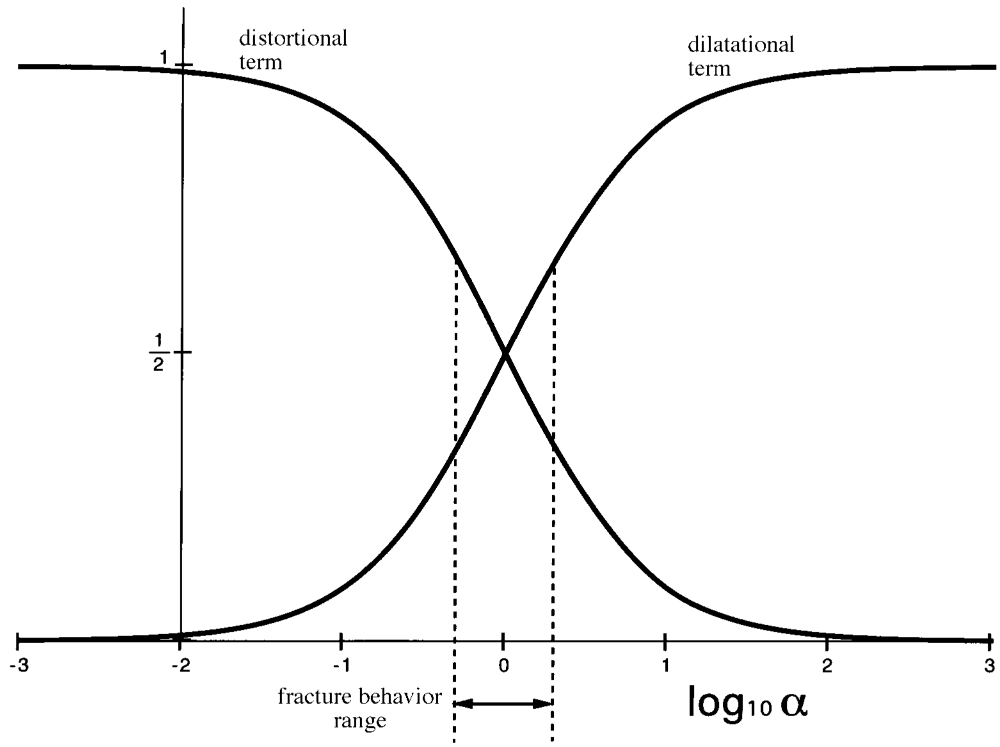

Figure 1. Uniaxial tensile yielding, equation (4.21).

For the tensile root, (3.13),

$$\hat{\sigma}_{11}^{\mathrm{T}} = \frac{\sqrt{3}}{1+\alpha}, \tag{4.20}$$

then these combine to give (4.19) as

$$\underbrace{\frac{\alpha}{1+\alpha}}_{\substack{\text{dilational}\\ \text{term}}} + \underbrace{\frac{1}{1+\alpha}}_{\substack{\text{distortional}\\ \text{term}}} = 1, \tag{4.21}$$

The dilatational term is that due to σ_{kk} and the distortional or deviatoric term is that due to $s_{ij}s_{ij}$ in (4.19). Relation (4.21) is plotted in figure 1. It is seen that at $\alpha = 1$ the dilatational and distortional terms are of equal weight in the yield criterion in uniaxial tension. Shape parameter $\alpha = 1$ is the case of fracture as already discussed. The fracture case at $\alpha = 1$ has dilatational and distortional effects in balance, with neither predominating. As $\alpha > 1$ the dilatational effects begin to dominate in this case and as $\alpha < 1$ the distortional effects begin to dominate. The fracture range near $\alpha = 1$ is especially important, not just because it contains a large class of materials but because it also is the threshold of transition between the other two extremes, the fracture transition.

As a practical matter there would be a range of fracture centred around $\alpha = 1$. With some degree of 'reasonable arbitrariness', that range is taken such that either term in (4.21) differs from the other by no more than a factor of two. This gives the reasonable range of fracture as

$$\tfrac{1}{2} \leqslant \alpha \leqslant 2, \tag{4.22}$$

and as shown in figure 1. The corresponding range of uniaxial tensile to compressive yield stress is, from (3.13),

$$\tfrac{1}{3} \leqslant (\sigma_{11}^{\mathrm{T}}/|\sigma_{11}^{\mathrm{C}}|) \leqslant \tfrac{2}{3}, \tag{4.23}$$

while at the explicit value for fracture $\alpha = 1$

$$(\sigma_{11}^{\mathrm{T}}/|\sigma_{11}^{\mathrm{C}}|) = \tfrac{1}{2}, \quad \text{fracture, } \alpha = 1. \tag{4.24}$$

Consider different materials that come within, or near, the range of fracture according to (4.23). From Paul (1968) cast iron in many cases has

$$(\sigma_{11}^{\mathrm{T}}/|\sigma_{11}^{\mathrm{C}}|) = \tfrac{1}{3} \quad \text{cast iron.}$$

For some but not all epoxies, data fall near

$$(\sigma_{11}^{\mathrm{T}}/|\sigma_{11}^{\mathrm{C}}|) = 0.75 \quad \text{some epoxies.}$$

From the *Handbook for applied engineering science*, for polystyrene

$$(\sigma_{11}^{\mathrm{T}}/|\sigma_{11}^{\mathrm{C}}|) = 0.61 \quad \text{polystyrene.}$$

All of these examples fall near the range of fracture behaviour (4.23). Polycarbonate has values

$$(\sigma_{11}^{\mathrm{T}}/|\sigma_{11}^{\mathrm{C}}|) = (9.0/11.0) = 0.82 \quad \text{polycarbonate,}$$

which certainly is beyond the fracture range, and toward ductile behaviour.

Now consider ceramics. From the *Handbook for materials science* in two separate formulations of alumina

$$\left.\begin{aligned} (\sigma_{11}^{\mathrm{T}}/|\sigma_{11}^{\mathrm{C}}|) &= (220/2239) = 0.10, \\ (\sigma_{11}^{\mathrm{T}}/|\sigma_{11}^{\mathrm{C}}|) &= (206/2549) = 0.081, \end{aligned}\right\} \quad \text{alumina.}$$

Some other ceramics have values both above and below these values. The important thing to note is that ceramics as a class of materials do not fall within the range of fracture behaviour in the present theory. For the alumina examples shown above, the shape parameter is about $\alpha = 10$ which is far beyond the fracture range (4.22). Ceramics are often thought of as being fracture controlled, but in tensile yield they are dominated by dilatational effects. The present theory provides a coherent yield function to be used in the application of ceramics.

As a last item in this section it is useful to graphically show the range of different behaviours for different classes of materials. Biaxial stress states are suitable for this purpose. Take $\sigma_{11} \neq 0$ and $\sigma_{22} \neq 0$ with all other stress components vanishing. Figure 2 shows three classes of behaviour from (3.14). The $\alpha = 0$ case is ductile metals and nothing more need be said of that case since it is so well understood. The $\alpha = 1$ fracture type behaviour would be typical of some glassy polymers and cast iron. It must be cautioned that it is too easy to generalize, and within any one class such as glassy polymers there is likely to be a wide range of behaviours that make inclusive classification quite risky. Similarly figure 2 shows $\alpha = 10$ as being typical of some ceramics. The limiting case $\alpha \rightarrow \infty$ is little different from that for $\alpha = 10$ in figure 2 except that it goes through the origin. Note that as α increases the tensile range properties become very limiting. The plots in figure 2 are non-dimensionalized so that the yield surface for one material type may be very magnified or contracted relative to another type. Finally, note that the three yield surfaces shown in figure 2 are not merely the same shape translated with respect to each other.

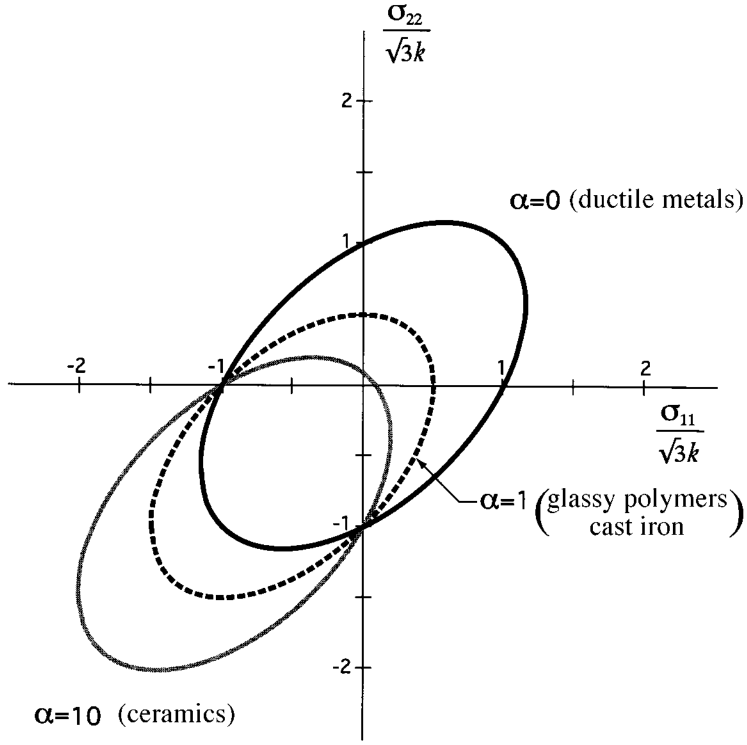

Figure 2. Bi-axial stress yielding, equation (3.14).

5. Two-dimensional yield/failure criterion

It will be useful to have a two-dimensional isotropic yield function for use in fracture evaluations. This is not a specialization of the three-dimensional yield form to plane-stress or plane-strain conditions. Rather, this is the independent development of a true two-dimensional form as though a third dimension does not exist.

Following exactly the same steps as used in the three-dimensional derivation resulting in (3.14), gives the following two-dimensional governing forms. The yield function is

$$\alpha_{2D}k_{2D}\sigma_{ii} + (1+\alpha_{2D})(-\tfrac{1}{2}\sigma_{ii}^2 + \sigma_{ij}\sigma_{ij}) \leqslant 2k_{2D}^2, \quad i,j = 1,2, \tag{5.1}$$

or

$$\alpha_{2D}k_{2D}\sigma_{ii} + (1+\alpha_{2D})s_{ij}s_{ij} \leqslant 2k_{2D}^2, \quad i,j = 1,2, \tag{5.2}$$

where the two-dimensional deviatoric stress is

$$s_{ij} = \sigma_{ij} - \tfrac{1}{2}\delta_{ij}\sigma_{kk}, \quad i,j,k = 1,2. \tag{5.3}$$

Parameters $\alpha_{2D} \geqslant 0$ and $k_{2D} \geqslant 0$ are the shape and scale parameters.

In uniaxial stress, σ_{11}, the yield values given by (5.1) are

$$\sigma_{11}^{T} = \frac{2k_{2D}}{1+\alpha_{2D}}, \tag{5.4}$$

and

$$\sigma_{11}^{C} = -2k_{2D}.$$

From these, the two material parameters are determined by

$$\alpha_{2\mathrm{D}} = (|\sigma_{11}^{\mathrm{C}}|/\sigma_{11}^{\mathrm{T}}) - 1 \quad \text{and} \quad k_{2\mathrm{D}} = \tfrac{1}{2}|\sigma_{11}^{\mathrm{C}}|. \tag{5.5}$$

The possibility of fracture type behaviour is controlled by a criterion of the type (2.1), but in the two-dimensional context. Express (5.1) explicitly in terms of σ_{11} and σ_{22}. Take the derivative with respect to σ_{22} and then set $\sigma_{22} = 0$ and finally evaluate at the roots given in (5.4). The result is

$$\left.\frac{\mathrm{d}\sigma_{11}^{\mathrm{T}}}{\mathrm{d}\sigma_{22}}\right|_{\sigma_{22}=0} = \frac{2-\alpha_{2\mathrm{D}}}{2+\alpha_{2\mathrm{D}}} \quad \text{and} \quad \left.\frac{\mathrm{d}\sigma_{11}^{\mathrm{C}}}{\mathrm{d}\sigma_{22}}\right|_{\sigma_{22}=0} = \frac{2+3\alpha_{2\mathrm{D}}}{2+\alpha_{2\mathrm{D}}}. \tag{5.6}$$

From a criterion of the type of (2.1) it is seen from (5.6) that fracture behaviour exists at

$$\alpha_{2\mathrm{D}} = 2, \quad \text{fracture.} \tag{5.7}$$

At this condition of fracture, the yield values (5.4) give

$$(\sigma_{11}^{\mathrm{T}}/|\sigma_{11}^{\mathrm{C}}|) = \tfrac{1}{3}, \quad \text{two-dimensional fracture } \alpha_{2\mathrm{D}} = 2. \tag{5.8}$$

A form similar to (5.1) can be extracted from recent work on the failure of fibre composite materials (Christensen 1997).

6. Two-dimensional fracture

Fracture mechanics is most highly developed in the two-dimensional case, where the crack is through the thickness to render the problem as two dimensional. It is useful here to recall some general aspects of fracture mechanics, as they relate to the present yield theory. Then a specific evaluation will be given comparing the present yield-theory prediction with that of fracture mechanics.

The elastic crack tip stress field is given by Rice (1968) and many others as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{Bmatrix} = \frac{K_{\mathrm{I}}}{(2\pi r)^{1/2}} \cos\tfrac{1}{2}\theta \begin{Bmatrix} 1 - \sin\frac{1}{2}\theta \sin\frac{3}{2}\theta \\ \sin\frac{1}{2}\theta\cos\frac{3}{2}\theta \\ 1 + \sin\frac{1}{2}\theta\sin\frac{3}{2}\theta \end{Bmatrix}, \tag{6.1}$$

where polar coordinates are used and K_{I} is the mode I stress intensity factor. These stresses can be expressed in terms of dilatational and deviatoric components as

$$\sigma_{kk} = \frac{\sqrt{2}K_{\mathrm{I}}}{\sqrt{\pi r}}\cos\tfrac{1}{2}\theta \quad \text{and} \quad \sqrt{s_{ij}s_{ij}} = \frac{K_{\mathrm{I}}}{2\sqrt{\pi r}}\sin\theta, \quad i,j,k = 1,2. \tag{6.2}$$

In connection with the results in figure 1 it was seen that at fracture the yield theory has positive dilatational effects and distortional effects in balance, and it is seen here that fracture mechanics has the same ingredients in effect with an elastic crack-tip stress field.

Another general feature of fracture mechanics is the elegant, and fundamental, path-independent J integral given by Rice (1968). This is stated by

$$J = \int_{\Gamma}\left(W\,\mathrm{d}x_2 - \sigma_{ij}n_j\frac{\partial u_i}{\partial x_1}\,\mathrm{d}s\right), \quad i,j = 1,2, \tag{6.3}$$

which gives the energy release rate in terms of the strain energy W, and $\boldsymbol{n}$ is the unit normal vector to the path of integration. Relation (6.3) can be decomposed into deviatoric and dilatational terms as

$$J = J_{\mathrm{DEV}} + J_{\mathrm{DIL}},$$

where

$$\begin{aligned} J_{\mathrm{DEV}} &= \int_{\Gamma} [\mu_{\mathrm{m}} e_{ij} e_{ij} \,\mathrm{d}x_2 - s_{ij} n_j u_{i,1} \,\mathrm{d}s] \quad \text{and} \\ J_{\mathrm{DIL}} &= \int_{\Gamma} (\tfrac{1}{2} K_{\mathrm{m}} \varepsilon_{kk}^2 \,\mathrm{d}x_2 - \tfrac{1}{2} \sigma_{kk} n_i u_{i,1} \,\mathrm{d}s), \quad i,j,k = 1,2, \end{aligned} \tag{6.4}$$

where μ_{m} is the shear modulus and K_{m} is a two-dimensional bulk modulus. Again, it is seen that dilatational and deviatoric effects play a balanced role in fracture mechanics, as they do in the present yield theory at the fracture condition.

Finally, consider an evaluation of the present yield theory using fracture mechanics. Under uniaxial tension, the yield stress is given by (5.4). Under simple shearing stress the yield is given from (5.1),

$$\sigma_{12}^{\mathrm{Y}} = \frac{k_{2\mathrm{D}}}{\sqrt{1+\alpha_{2\mathrm{D}}}}. \tag{6.5}$$

From the ratio of (5.4) and (6.5) we obtain

$$\frac{\sigma_{11}^{\mathrm{T}}}{\sigma_{12}^{\mathrm{Y}}} = \frac{2}{\sqrt{1+\alpha_{2\mathrm{D}}}}, \tag{6.6}$$

and at fracture, from (5.7),

$$\frac{\sigma_{11}^{\mathrm{T}}}{\sigma_{12}^{\mathrm{Y}}} = \frac{2}{\sqrt{3}}. \tag{6.7}$$

Next evaluate this same ratio using fracture mechanics.

The energy release rate for fracture is

$$G = \frac{K_{\mathrm{I}}^2}{E} + \frac{K_{\mathrm{II}}^2}{E}, \tag{6.8}$$

where in the present context E is the two-dimensional uniaxial modulus and for modes I and II

$$K_{\mathrm{I}} = \sigma\sqrt{\pi a}, \quad K_{\mathrm{II}} = \tau\sqrt{\pi a}, \tag{6.9}$$

with a being the crack half length and σ and τ being the applied normal and shear stresses. Taking cracks with the worst orientation gives the energy release rates in the two cases as

$$G = \frac{\sigma_{11}^2 \pi a}{E} \quad \text{and} \quad G = \frac{\sigma_{12}^2 \pi a}{E}. \tag{6.10}$$

Taking the critical energy release rates to be the same in the two cases and eliminating G between these gives

$$(\sigma_{11}/\sigma_{12}) = 1. \tag{6.11}$$

Alternatively, consider the case where the cracks are taken to have a random orientation. First for application of stress σ_{11}, the stresses oriented with the crack are

$$\sigma_n = \sigma_{11}\cos^2\theta, \quad \tau_{nt} = -\sigma_{11}\sin\theta\cos\theta, \tag{6.12}$$

where θ is the orientation of the crack from direction x_1. Averaging over all possible crack orientations gives the energy release rate as

$$\left(\frac{E}{\pi a}\right) G = \frac{\sigma_{11}^2}{\pi} \int_0^{\pi} (\cos^4\theta + \sin^2\theta\cos^2\theta)\,\mathrm{d}\theta = \tfrac{1}{2}\sigma_{11}^2. \tag{6.13}$$

For applied stress field σ_{12}, the stresses oriented with the crack are

$$\sigma_n = 2\sigma_{12}\sin\theta\cos\theta, \quad \tau_{nt} = \sigma_{12}(\cos^2\theta - \sin^2\theta). \tag{6.14}$$

In performing the integrations to give all possible crack orientations, the integral involving σ_n can only involve positive values of σ_n for crack opening, thus

$$\left(\frac{E}{\pi a}\right) G = \frac{4\sigma_{12}^2}{\pi} \int_0^{(\pi/2)} \sin^2\theta\cos^2\theta\,\mathrm{d}\theta + \frac{\sigma_{12}^2}{\pi}\int_0^{\pi} (\cos^2\theta - \sin^2\theta)^2\,\mathrm{d}\theta = \tfrac{3}{4}\sigma_{12}^2. \tag{6.15}$$

Eliminating G between (6.13) and (6.15) gives in this case

$$(\sigma_{11}/\sigma_{12}) = \sqrt{\tfrac{3}{2}}. \tag{6.16}$$

Collecting the results from (6.6), (6.7), (6.11) and (6.16) gives

$$\left.\begin{aligned} \left.\frac{\sigma_{11}^{\mathrm{T}}}{\sigma_{12}^{\mathrm{Y}}}\right|_{\underbrace{\alpha_{2\mathrm{D}} = 0}_{\text{distortional}}} &= 2 \\ \left.\frac{\sigma_{11}^{\mathrm{T}}}{\sigma_{12}^{\mathrm{Y}}}\right|_{\underbrace{\alpha_{2\mathrm{D}} \to \infty}_{\text{dilatant}}} &= 0 \end{aligned}\right\} \quad \text{yield theory range,}$$

$$\left.\frac{\sigma_{11}^{\mathrm{T}}}{\sigma_{12}^{\mathrm{Y}}}\right|_{\alpha_{2\mathrm{D}}=2} = 1.155, \quad \text{yield theory at fracture,} \tag{6.17}$$

$$\left.\begin{aligned} \left.\frac{\sigma_{11}^{\mathrm{T}}}{\sigma_{12}^{\mathrm{Y}}}\right|_{\text{worst orientation}} &= 1 \\ \left.\frac{\sigma_{11}^{\mathrm{T}}}{\sigma_{12}^{\mathrm{Y}}}\right|_{\text{random orientation}} &= 1.225 \end{aligned}\right\} \quad \text{fracture mechanics.}$$

The yield theory at $\alpha_{2\mathrm{D}} = 2$ and classical fracture mechanics can be brought into coincidence if the critical energy release rate is taken to be stress state dependent. Even without this effect, the two theories seem to be in reasonable agreement in the overall scheme of material behaviour.

7. Three-dimensional fracture

It is interesting to compare the two-dimensional and three-dimensional forms of fracture that follow from the yield theory. In uniaxial stress, from (4.24) and (5.8), repeated here,

$$\frac{\sigma_{11}^{\mathrm{T}}}{|\sigma_{11}^{\mathrm{C}}|} = \begin{cases} \tfrac{1}{2}, & \text{three-dimensional fracture, } \alpha = 1, \\ \tfrac{1}{3}, & \text{two-dimensional fracture, } \alpha_{2\mathrm{D}} = 2. \end{cases} \tag{7.1}$$

It is seen that relative to the compressive yield values, the tensile yield values at fracture are more severely degraded in the two-dimensional case than in three dimensions. It is also interesting to note that Griffith (1924) predicted that at fracture, $(\sigma_{11}^{\mathrm{T}}/|\sigma_{11}^{\mathrm{C}}|) = \frac{1}{8}$, a value not in accord with common practice.

The other matter to be considered here is the relationship of the present yield theory to possible energy forms for such criteria, particularly at the fracture condition. For an isotropic elastic material the strain energy is given by

$$W = \tfrac{1}{2}k_{\mathrm{m}}\varepsilon_{kk}^2 + \mu_{\mathrm{m}}e_{ij}e_{ij}. \tag{7.2}$$

In terms of stresses,

$$W = \frac{\sigma_{kk}^2}{18k_{\mathrm{m}}} + \frac{s_{ij}s_{ij}}{4\mu_{\mathrm{m}}}. \tag{7.3}$$

Putting this aside for a moment, the yield function (3.14) in states of dilatation and distortion gives

$$\underbrace{\sigma_{kk} = \frac{\sqrt{3}k}{\alpha}}_{\text{dilatation}} \quad \text{and} \quad \underbrace{\sigma_{12}^2 = \frac{k^2}{1+\alpha}}_{\text{distortion}} \tag{7.4}$$

From (7.3) and (7.4) at dilatational yield the energy is

$$W_{\mathrm{DIL}} = \frac{k^2}{6\alpha^2 k_{\mathrm{m}}}, \tag{7.5}$$

and at distortional or deviatoric yield, the energy is

$$W_{\mathrm{DEV}} = \frac{k^2}{2(1+\alpha)\mu_{\mathrm{m}}}. \tag{7.6}$$

The ratio of these is

$$\frac{W_{\mathrm{DEV}}}{W_{\mathrm{DIL}}} = \frac{2\alpha^2}{(1+\alpha)}\frac{(1+\upsilon_{\mathrm{m}})}{(1-2\upsilon_{\mathrm{m}})}, \tag{7.7}$$

where υ_{m} is Poisson's ratio. At fracture, $\alpha = 1$, (7.7) gives

$$\frac{W_{\mathrm{DEV}}}{W_{\mathrm{DIL}}} = \begin{cases} 1, & \text{at } \upsilon_{\mathrm{m}} = 0, \\ 2, & \text{at } \upsilon_{\mathrm{m}} = \frac{1}{5}, \\ \frac{5}{2}, & \text{at } \upsilon_{\mathrm{m}} = \frac{1}{4}, \\ 4, & \text{at } \upsilon_{\mathrm{m}} = \frac{1}{3}. \end{cases}$$

It is seen that for fracture under dilatation and under distortion separately, both cases have the same energy at fracture when $\upsilon_{\mathrm{m}} = 0$. For other values of υ_{m} the ratio changes. This shows the fundamental characteristic that the yield criterion at fracture involves both dilatation and distortion and at least partially there can be given an energy interpretation similar to the Mises case.

Next consider more general states than that of dilatation and distortion separately. Strain energy will be shown to provide a bound on yield according to the present theory. At fracture, $\alpha = 1$, the yield function (3.14) is

$$\frac{k\sigma_{kk}}{\sqrt{3}} + s_{ij}s_{ij} = k^2. \tag{7.8}$$

The strain energy, (7.3), can be written as

$$\frac{2}{9}\left(\frac{\mu_{\mathrm{m}}}{k_{\mathrm{m}}}\right)\sigma_{kk}^2 + s_{ij}s_{ij} = 4\mu_{\mathrm{m}}W. \tag{7.9}$$

At $\upsilon_{\mathrm{m}} = 0$, (7.9) becomes

$$\tfrac{1}{3}\sigma_{kk}^2 + s_{ij}s_{ij} = 4\mu_{\mathrm{m}}W. \tag{7.10}$$

The yield form (7.8), and the energy (7.10), have a similar structure. Now prove that the energy form (7.10) provides a bound of the type

$$\tfrac{1}{3}\sigma_{kk}^2 + s_{ij}s_{ij} \leqslant k^2, \tag{7.11}$$

where k^2 is that in (7.8). Proving (7.11) subject to (7.8) is the same as proving

$$\tfrac{1}{3}\sigma_{kk}^2 + s_{ij}s_{ij} \leqslant (k\sigma_{kk}/\sqrt{3}) + s_{ij}s_{ij}, \tag{7.12}$$

or

$$\sigma_{kk} \leqslant \sqrt{3}k. \tag{7.13}$$

From (7.8)

$$\sigma_{kk} = \sqrt{3}k - (\sqrt{3}/k)s_{ij}s_{ij}, \tag{7.14}$$

thus (7.13) is satisfied and (7.11) is proved. The consequence of this is that in (7.11) equality is obtained for either dilatational or distortional states by themselves, but the inequality applies more generally. Thus an energy criterion is a bound, that, if used, would be conservative for predicting fracture according to the present theory for $\upsilon_{\mathrm{m}} = 0$. Probably other energy type bounding forms could be given relative to the yield criterion, but it hardly seems worthwhile to pursue this line. The direct use of the yield criterion is far preferable.

8. Interpretation and conclusions

The approach here is at the next plateau of generality beyond that of Mises, namely that of additionally allowing yielding under hydrostatic tensile stress. Within this physical condition, the final result, (3.14) or (3.16), is the most general isotropic tensor polynomial form of second degree.

The form, (3.6), from which the explicit form, (3.14), comes, has been used many times in the past in fitting data for particular materials, used along with other trial functions. It is even clear that von Mises, himself, recognized that the form (3.6) would constitute a generalization of his criterion. Whether in the distant past or in more recent times, it does not appear that the form (3.6) has been proven by anyone to have the comprehensive and fundamental nature that is constructed here using the formalism of (3.14) or (3.16).

The two-parameter yield function, (3.16), has surprising generality and versatility. In particular, the limiting cases go from distortional control at one extreme to a dilatant type behaviour at the other extreme. The most noteworthy aspect of the new yield theory is that it includes classical fracture mechanics of non-interacting cracks as a special case, presuming a macroscopically homogeneous distribution of cracks. The fracture behaviour was found by examining the different types of response as a function of the non-dimensional shape parameter, α. At $\alpha = 1$ the fracture behaviour occurs and the corresponding ratio of uniaxial-tensile to compressive-yield stress is $(\sigma_{11}^{\mathrm{T}}/|\sigma_{11}^{\mathrm{C}}|) = \frac{1}{2}$. Therefore this value of this ratio is a distinguishing feature of fracture, according to the present theory. As the shape parameter increases beyond the value $\alpha = 1$, the yield function behaves in accordance with a state of increasing crack density or any other physical weakening. Shape parameter, α, in effect serves a dual role, including that of a simple and effective damage parameter. Within this scope of effects only the case at $\alpha = 0$, Mises behaviour, has no effective damage in its natural state. At $\alpha \to \infty$ the damage becomes completely debilitating, unless a compressive state of mean normal stress exists. The previously discussed dilatancy effect that exists for large values of α is directly due to the heavy state of material damage that is the root cause of large α.

The term fracture, as used here for behaviour at or near $\alpha = 1$, actually corresponds to fracture mechanics for non-interacting cracks. Beyond this range near $\alpha = 1$ up to $\alpha \to \infty$ has simply been called yield or failure. In some cases it could as well be called fracture in the sense implied by sudden failure. Thus the range near $\alpha = 1$ might best be termed precisely as that of classical fracture mechanics involving non-interacting cracks, while that beyond that range could be fracture of a more general type. But to avoid confusion, the latter range is simply termed as yield or failure here.

The yield and failure formalism given here is targeted to be applicable to full density or nearly full density materials. It is easy to see conceptually that these results could not be applicable to quite porous or extremely porous materials. Low density materials typically suffer compressive failure, including hydrostatic compressive failure, due to micro-buckling of the micro-material members. Effects such as these are outside the scope of the present theory. Although the present theory is restricted to isotropic materials, it certainly suggests guidelines for generalization to anisotropic conditions, such as orthotropy. Extensions to forms appropriate for large deformation, large rotation conditions are direct and obvious.

The forms derived here retain the simplicity of the Mises function. Whether used as initial yield functions or as failure criteria, these forms have an essential physical basis combined with a utility of application. With only two material parameters, it is not likely that each different material sample which might be examined could be modelled with comparably high accuracy, but it is likely that the results here can cover a broader range of material types with reasonable accuracy than can other forms, some with much greater complication. This assertion is founded upon the rigorous basis of the present derivation. Experiments on modern materials will be needed to establish the validity of this assertion or to disprove it.

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