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A new strategy for fatigue analysis in presence of general multiaxial time varying loadings

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Abstract

The object of this paper is to propose an energy based fatigue approach which handles multidimensional time varying loading histories.

Our fundamental thought is to assume that the energy dissipated at small scales governs fatigue at failure. The basis of our model is to consider a plastic behavior at the mesoscopic scale with a dependence of the yield function not only on the deviatoric part of the stress but also on the hydrostatic part. A kinematic hardening under the assumption of associative plasticity is also considered. We also follow the Dang Van paradigm at macro scale. The structure is elastic at the macroscopic scale. At each material points, there is a stochastic distribution of weak points which will undergo strong plastic yielding, which contribute to energy dissipation without affecting the overall macroscopic stress.

Instead of using the number of cycles, we use the concept of loading history. To accommodate real life loading history more accurately, mean stress effect is taken into account in mesoscopic yield function and non-linear damage accumulation law are also considered in our model. Fatigue will then be determined from the plastic shakedown cycle and from a phenomenological fatigue law linking lifetime and accumulated mesoscopic plastic dissipation.

Keywords: Fatigue; Energy; High cycle; Plasticity; Mean stress

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Nomenclature

S_{max}	maximum deviatoric stress during the loading cycles
σ_{-1}	fatigue limit for fully reversed condition
b	back stress
$\dot{w}$	energy dissipation rate at a certain scale
$\dot{W}$	energy dissipation rate at all scales
W	dissipated energy
W_{cyc}	dissipated energy per cycle
N	current number of cycles
N_F	number of cycles to failure
$\dot{\varepsilon}_p$	rate of effective plastic strain
W_F	dissipated energy to failure per unit volume
E	Young's modulus
$k = 500 \sim 800MPa$	hardening parameter
$\beta = 1 \sim 50$	weakening scales distribution exponent
$\gamma = 0 \sim 50$	material parameter from Chaboche law(Wohler curve exponent)
$\alpha = 1 - a\left\langle \dfrac{\frac{1}{2}\sigma_{vm}(t) - \sigma_{-1}\left(1 - 3c\sigma_{H,max}(t)\right)}{\sigma_u - \sigma_{vm}(t)} \right\rangle$	characterizes non-linearity of damage accumulation(c is constant)
a	material parameter from Chaboche law
M_0	material parameter in Chaboche law
σ_y	macroscopic yield stress(normal or shear)
$\lambda = 0 \sim 5$	hydrostatic pressure sensitivity
$\underline{\underline{S}} = dev\dot{\underline{\underline{\Sigma}}}$	deviatoric part of the stress tensor
Σ_H	macroscopic hydrostatic pressure
$A_{II} = \tau_{oct,a} = \sqrt{\frac{1}{3}J_{2,a}}$	the amplitude of octahedral shear stress
$S_{max} = \sigma_{VM} = \sqrt{6J_{2,a}}$	Von Mises stress
s_{-1}	tensile fatigue limit for $R = -1$
$\langle\ \rangle$	Macaulay bracket symbol.$\langle\ \rangle$ is defined as $\langle m \rangle = 0$ if $m \leqslant 0$

1 Weakening scales and yield function

1.1. The concept of weakening scales

We follow the Dang Van paradigm. The structure is elastic at the macroscopic scale. At each material points, there is a stochastic distribution of weak points which will undergo strong plastic yielding, without contributing to the overall macroscopic stress. From a microscopic point of view, there is a distribution of weakening scales, namely $s \in [1, \infty)$. Let S_{max} be the macroscopic stress intensity at present time. Let σ_y be the yield limit before weakening. Then we imagine that for a given scale s:

• either $1 \leqslant s \leqslant \sigma_y/S_{max}$, then $S_{max} \leqslant \sigma_y/s$, the material stays in the elastic regime and there is no energy dissipation at this scale.

• or $\sigma_y/S_{max} \leqslant s \leqslant \infty$, then $S_{max} \geqslant \sigma_y/s$, the material is in the plastic regime and there is dissipated energy at scale s, contributing to the fatigue limit, which evolve through kinematic hardening.

In more details, at each scale s of a plastic evolution process there is a weakened yield limit σ_y/s, zero initial plastic strain $\underline{\underline{\varepsilon}}_p$ and zero initial backstress $\underline{\underline{b}}$ at initial time t_0.

1.2. Distribution of weakening scales

We assume the weakening scales have a probability distribution function of power law:

$$P(s) = Cs^{-\beta},$$

where β is a material constant and C is hardening constant. The choice of a power law has two reasons: on one hand, this type of distribution corresponds to a scale invariant process, on the other hand it leads in cyclic loading to a prediction of a number of cycles to life limit as a power law function of the stress intensity. More general laws can also be proposed.

The integrated probability ranging from macroscopic to microscopic stress is unity. From this we can conclude:

$$\int_1^\infty P(s)ds = \left[\frac{Cs^{1-\beta}}{1-\beta}\right]_1^\infty = 0 - \frac{C}{1-\beta} = 1.$$

Then we know $C = \beta - 1$, so the distribution is given by:

$$P(s) = Cs^{-\beta} = (\beta - 1)s^{-\beta}$$

The probability of weakening scales is shown in Figure 1.

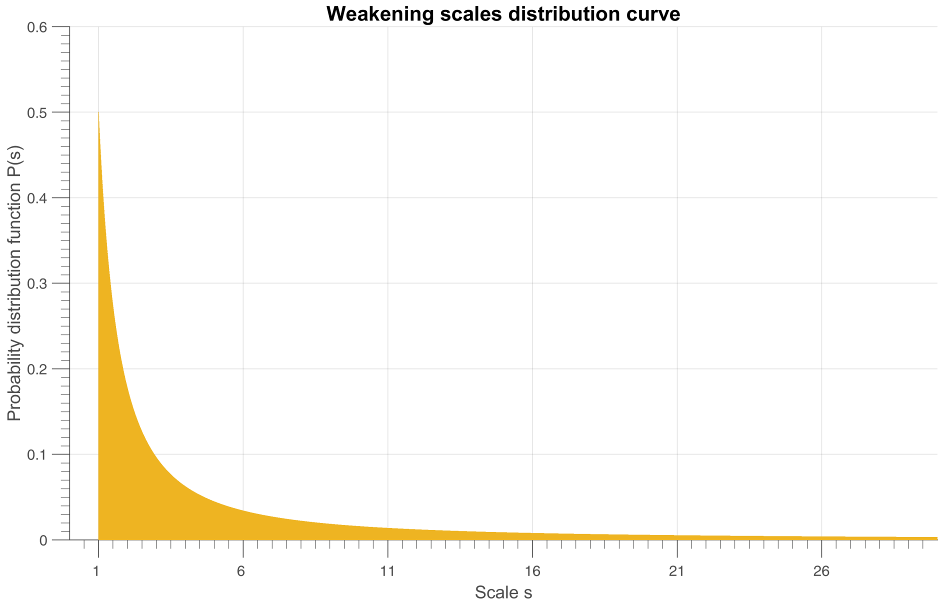


Figure 1: Weakening scales s probability distribution curve when $\beta = 1.5$

1.3. *Yield function with mean stress effect*

Positive mean stress clearly reduces the fatigue life of the material. In design evaluation of multiaxial fatigue with mean stress, a simplified, conservative relation between mean stress and equivalent alternating stress is necessary. We can improve the model by modifying the yield function σ_y and the localization tensor.

The idea is to consider as in Maitournam and Krebs[1] that the yield limit σ_y can be reduced in presence of positive mean stress. The mesoscopic yield function can therefore be written as:

$$f(s) = \|\underline{\underline{S}}(s) - \underline{\underline{b}}(s)\| + (\lambda \Sigma_H - \sigma_y)/s \leqslant 0 \tag{1}$$

with $\underline{\underline{S}}$ denoting the deviatoric part of the stress tensor at microscale, and $\underline{\underline{b}}(s)$ the corresponding backstress at the same scale. The material remain in elastic regime when $f < 0$ and in plastic regime when $f = 0$.

1.4. *Local plastic model*

First we should describe the mesoscopic stress state. The model considers a plastic behavior at the mesoscopic scale. The mesoscopic stress evolution equations are thus:

$$\dot{\underline{\underline{S}}}(s, M, t) = dev\dot{\underline{\underline{\Sigma}}}(M, t) - \frac{E}{1+\nu}\dot{\underline{\underline{\varepsilon}}}^p(s, M, t), \tag{2}$$

which defines a Taylor-Lin scale transition model with unit localization tensor[2].

$$\dot{\underline{\underline{b}}}(s, M, t) = \frac{kE}{E - k}\dot{\underline{\underline{\varepsilon}}}^p(s, M, t), \tag{3}$$

which is our kinematic hardening model.

$$\dot{\underline{\underline{\varepsilon}}}^p(s,M,t) = \gamma \frac{\partial f(s,M,t)}{\partial \underline{\underline{S}}}, \tag{4}$$

which is the associated plastic flow rule assuming $\gamma = 0$ when $f < 0$ and $\gamma \geqslant 0$ when $f = 0$.

Here E denotes the Young's modulus and k the hardening parameter. The local dissipated energy rate per volume at weakening scales s is given by the local entropy dissipation:

$$\dot{w}(s,M,t) = (\underline{\underline{S}} - \underline{\underline{b}})(s,M,t) : \dot{\underline{\underline{\varepsilon}}}^p(s,M,t). \tag{5}$$

2 Construction of an energy based fatigue approach

In a preliminary step, we will consider a simple macroscopic loading history $\underline{\underline{\Sigma}}(M,t)$ which is uniaxial and time periodic of deviatoric amplitude S_{max}, and a Von Mises flow rule to see if we get a prediction of local failure for a number of cycles N_F varying as $\Sigma^{-\beta}$.

In uniaxial cyclic loading, there will be 3 kinds of loading patterns, as is shown in Figure 2:

1. Elastic regime, in phase 2 and 4,where $\dot{\underline{\underline{\varepsilon}}}^p(s,M,t) = 0$, and $|\underline{\underline{S}} - \underline{\underline{b}}| < (\sigma_y - \lambda\Sigma_H)/s$.

2. Plastic regime according to plastic flow rule, with increasing plastic deformation, in phase 5 and 1, where $\dot{\underline{\underline{\varepsilon}}}^p(s,M,t) = \gamma \dfrac{\underline{\underline{S}}(s) - \underline{\underline{b}}(s)}{\|\underline{\underline{S}}(s) - \underline{\underline{b}}(s)\|} > 0$ with $\gamma = dev\dot{\Sigma}\left(\dfrac{kE}{E-k} + \dfrac{E}{1+\nu}\right)^{-1}$, with $\underline{\underline{S}} - \underline{\underline{b}} = (\sigma_y - \lambda\Sigma_H)/s$ and $\dot{\underline{\underline{S}}} - \dot{\underline{\underline{b}}} = 0$.

3. Plastic regime in the other direction, in phase 3, there is $\dot{\underline{\underline{\varepsilon}}}^p(s,M,t) < 0$, then $\underline{\underline{S}} - \underline{\underline{b}} = -(\sigma_y - \lambda\Sigma_H)/s$ and $\dot{\underline{\underline{S}}} - \dot{\underline{\underline{b}}} = 0$

In phase 1, a direct analysis yields the energy dissipation at scale s:

$$dW = (S - b)d\varepsilon^p = \frac{(E-k)(1+\nu)}{E(E+k\nu)} \frac{(\sigma_y - \lambda\Sigma_H)}{s} \left(S_{max} - \frac{(\sigma_y - \lambda\Sigma_H)}{s}\right) \tag{6}$$

A similar analysis yields

$$dW(phase1) = dW(phase5) = \frac{1}{2} dW(phase3).$$

We can then calculate the local dissipated energy W at point M during one cycle by cumulating the input of all sub-scales which result in plastic regime with their probabilities [3].

$$\begin{aligned} W_{cyc} &= 4 \int_{(\sigma_y - \lambda\Sigma_H)/S_{max}}^{\infty} dW(s,M,t)P(s)ds \\ &= 4 \int_{(\sigma_y - \lambda\Sigma_H)/S_{max}}^{\infty} \frac{(E-k)(1+\nu)}{E(E+k\nu)} \frac{(\sigma_y - \lambda\Sigma_H)}{s} \left(S_{max} - \frac{(\sigma_y - \lambda\Sigma_H)}{s}\right)(\beta - 1)\, s^{-\beta} ds \\ &= \frac{4(E-k)(1+\nu)(\beta-1)}{E(E+k\nu)\beta(\beta+1)} \frac{S_{max}^{\beta+1}}{(\sigma_y - \lambda\Sigma_H)^{\beta-1}}. \end{aligned} \tag{7}$$

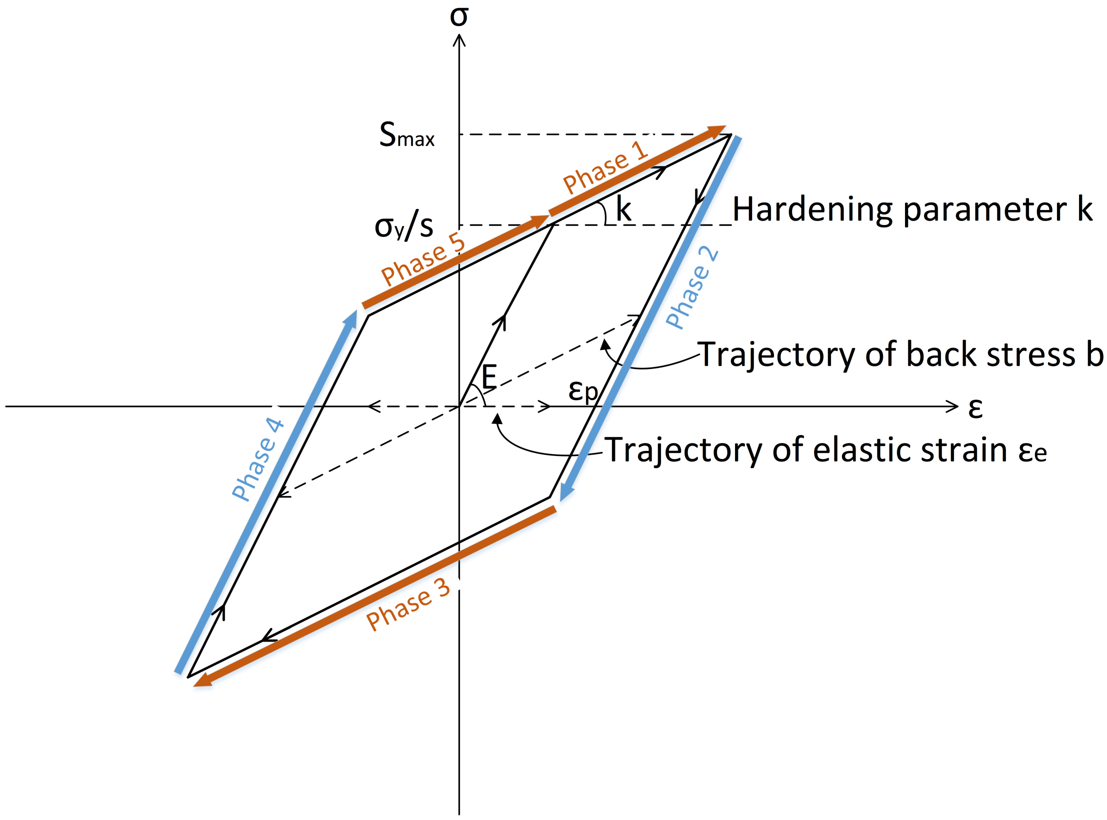


Figure 2: Uniaxial load with plastic dissipation

If the dissipated energy accumulates linearly until a failure value W_F, we can get directly the time to failure from Eq.(8):

$$T_{fail} = N_F t_{cyc} = \frac{W_F}{W_{cyc}} t_{cyc} = C(S_{max})^{-\beta-1}. \tag{8}$$

From Eq.(7), we then obtain that the model predicts as expected a power law dependence in function of S_{max}. However, experiments shows that the damage or the energy accumulation of a material evolves non-linearly in time. We should introduce below a method to handle such a nonlinearity.

3 Nonlinearity of damage accumulation

3.1. Energy approach with Chaboche law

The Chaboche law[4] is essentially a damage incremental law for cyclic loading of stress intensity σ with a deviatoric part A_{II} and hydrostatic part Σ_H, defining the damage increase by:

$$\delta D = \left(1 - (1-D)^{\gamma+1}\right)^{\alpha} \left(\frac{A_{II}}{M(\sigma_H)(1-D)}\right)^{\gamma} \delta N \tag{9}$$

With $A^*_{II} = A_{II}/(1-D)$ evolving with damage D. And the mean stress effect is in both exponential α and denominator $M(\sigma_H)$.

$$\alpha = 1 - a\left\langle \frac{\frac{1}{2}\sigma_{vm}(t) - \sigma_{-1}\left(1 - 3c\sigma_{H,max}(t)\right)}{\sigma_u - \sigma_{vm}(t)} \right\rangle,$$
$$M(\sigma_H) = M_0\left(1 - 3c\sigma_{H,max}(t)\right).$$

Eq.(9) writes equivalently as Eq.(10):

$$\delta[1-(1-D)^{\gamma+1}]^{1-\alpha} = (1-\alpha)(\gamma+1)\left(\frac{A_{II}}{M(\Sigma_H)}\right)^{\gamma}\delta N = \frac{1}{N_F(\sigma)}\delta N. \tag{10}$$

Here $N_F(\sigma)$ denotes the number of cycles at intensity σ to failure as obtained by integration of Eq.(10) from $D=0$ to $D=1$. In our model, in case of a simple uniaxial cyclic loading, we propose to replace $\frac{1}{N_F(\sigma)}$ which is the relative unit increment of energy by $\frac{W^*_{cyc}}{W_F}$.

$$W^*_{cyc} = n(D)W_{cyc}$$

Assume the number of defects, meaning the number of possible occurrence of weak scales to be:

$$n(D) = n_0\frac{1}{(1-D)^{\gamma}} \quad , \tag{11}$$

with n_0 the initial number of defects at local scale, which increases with damage increment.

Thus the dissipated energy in one cycle evolves with damage D:

$$W^*_{cyc} = \frac{n_0}{(1-D)^{\gamma}}\frac{4(E-k)(1+\nu)(\beta-1)}{E(E+k\nu)\beta(\beta+1)}\frac{S_{max}^{\beta+1}}{(\sigma_y-\lambda\Sigma_H)^{\beta-1}}. \tag{12}$$

The nonlinear damage incremental law using energy dissipation:

$$\begin{aligned}
\delta D &= \left(1-(1-D)^{\gamma+1}\right)^{\alpha}\delta W \\
&= \left(1-(1-D)^{\gamma+1}\right)^{\alpha}\frac{W^*_{cyc}}{W_F}\delta N \\
&= \left(1-(1-D)^{\gamma+1}\right)^{\alpha}\frac{n_0}{(1-D)^{\gamma}}\frac{4(E-k)(1+\nu)(\beta-1)}{E(E+k\nu)\beta(\beta+1)}\frac{S_{max}^{\beta+1}}{(\sigma_y-\lambda\Sigma_H)^{\beta-1}}\frac{\delta N}{W_F}.
\end{aligned} \tag{13}$$

We compare Eq.(9) and Eq.(13), there is

$$\beta + 1 = \gamma.$$

Similar to Eq.(10), we do the integration and get:

$$\begin{aligned}
\delta[1-(1-D)^{\gamma+1}]^{1-\alpha} &= (1-\alpha)(\gamma+1)\frac{4n_0(E-k)(1+\nu)(\beta-1)}{E(E+k\nu)\beta(\beta+1)}\frac{S_{max}^{\beta+1}}{(\sigma_y-\lambda\Sigma_H)^{\beta-1}}\frac{\delta N}{W_F} \\
&= n_0(1-\alpha)(\gamma+1)\frac{W_{cyc}\delta N}{W_F}.
\end{aligned} \tag{14}$$

3.2. Generalized damage accumulation

Formula (10) is a general accumulation law which can be applied for any cyclic loading sequence provided that we can identify the multiscale value of the dissipated energy per cycle.

But the notion of cycle itself may be hard to identify for general loadings. The idea is then to replace the relative increment of dissipated energy per cycle by the relative increment of dissipated energy per unit time, yielding:

$$\begin{aligned}\delta D &= \left(1-(1-D)^{\gamma+1}\right)^{\alpha} \frac{\dot{W}_{total}\delta t}{W_F} \\ &= \left(1-(1-D)^{\gamma+1}\right)^{\alpha} \frac{n(D)\dot{W}\delta t}{W_F} \\ &= \left(1-(1-D)^{\gamma+1}\right)^{\alpha} \frac{n_0\dot{W}\delta t}{(1-D)^{\gamma} W_F}\end{aligned} \tag{15}$$

We put all the D on the left side to get:

$$\delta[1-(1-D)^{\gamma+1}]^{1-\alpha} = n_0(1-\alpha)(\gamma+1)\frac{\dot{W}\delta t}{W_F}. \tag{16}$$

In a general loading case, $\dot{W}$ is the microscopic energy dissipation rate of unit defect. By integrating Eq.(5) over all microscales, we get:

$$\dot{W}(M,t) = \int_{s=1}^{\infty} \dot{w}(s,M,t)P(s)ds = \int_{s=1}^{\infty} \left(\underline{\underline{S}} - \underline{\underline{b}}\right)(s,M,t) : \underline{\underline{\dot{\varepsilon}}}^p(s,M,t)P(s)ds. \tag{17}$$

The evolution of $\underline{\underline{S}}$, $\underline{\underline{b}}$ and $\underline{\underline{\dot{\varepsilon}}}^p$ are given in section 1.4. Equation (16) and (17) are therefore our proposed damage law.

This is a nonlinear law with a constant α, there will be no sequence effect. In other words, when applying two successive cycles of different intensities, the failure will occur at the same number of cycles whatever the order of the loading(high then low versus low then high). In numerical implementation of complex loading case, to take into account the load sequence effect, α changes at every time step.

$$\alpha = 1 - \left(1 - \frac{1}{s_{min}}\right)^{a}. \tag{18}$$

Where s_{min} is the minimum scale that causes energy loss, meaning the deviatoric stress is greater than local yield stress:

$$S_{max} > \frac{(\sigma_y - \lambda\Sigma_H)}{s_{min}},$$

$$s_{min} = \frac{(\sigma_y - \lambda\Sigma_H)}{S_{max}}.$$

which will change over time in loading history, we can now use numerical method to verify our analysis in 1-D cyclic loading.

4 Loop on time and scales

4.1. Integration rules for $\dot{W}$ and δD

Our first approach takes one cycle as unit time. We compute analytically the energy dissipation at each scale during this cycle. The method is valid for simple loading history and which includes the integration on all weakening scales. The damage D is accumulated after each cycle.

However, there are certain limitations of this method. Firstly we need a load history decomposition in cycles. Secondly in real life the perfect close loop cycle is hardly applicable.

Thus we propose in Eq.(16) a more general method which can be integrated by a step by step strategy. We compute numerically the dissipation at different scales using an implicit Euler time integration of the constitutive laws of section 1.4. After which we make a numerical integration on different scales. Then we can update the damage and go to next time step.

Instead of doing the scale integration directly which can be difficult for complex loading, the Gaussian Quadrature rule with Legendre points is used to give the value of local dissipated energy rate.

To use the Gaussian quadrature rule the limit range of integral must be from −1 to 1, while the total dissipated energy is expressed by integrating all the weakening scale s ranging from 1 to infinity with their occurrence probabilities:

$$\dot{W} = \int_1^\infty \dot{w}(s)(\beta - 1)(s)^{-\beta} ds.$$

To change the limit range of integral from $[1, \infty]$ to $[1, 0]$ we take as new integration variable $u(s) = s^{-p}$ with $p = \beta - 1$, yielding $u(1) = 1$ and $u(\infty) = 0$ with

$$du = -ps^{-p-1} ds$$

that is

$$du = (-\beta + 1)s^{-\beta} ds = (-\beta + 1)s^{-\beta} ds.$$

Therefore the dissipated energy summed on all scales is:

$$\begin{aligned}
\dot{W} &= \int_1^\infty \dot{w}(s)(\beta - 1)(s)^{-\beta} ds \\
&= \int_1^0 \dot{w}\left(u^{\frac{1}{1-\beta}}\right)(\beta - 1)\frac{1}{-\beta + 1} du \\
&= \int_0^1 \dot{w}\left(u^{\frac{1}{1-\beta}}\right)(\beta - 1)\frac{1}{\beta - 1} du \\
&= \int_0^1 \dot{w}\left(u^{\frac{1}{1-\beta}}\right) du \\
&= \frac{1}{2}\int_{-1}^1 \dot{w}\left[\left(\frac{x+1}{2}\right)^{\frac{1}{1-\beta}}\right] dx
\end{aligned} \tag{19}$$

given $u = \dfrac{x+1}{2}$.

So the dissipated energy rate integrated over all scales takes the form of Eq.(20):

$$\dot{W} = \frac{1}{2}\int_{-1}^{1} \dot{w}\left[\left(\frac{x+1}{2}\right)^{\frac{1}{1-\beta}}, t\right] dx \approx \frac{1}{2}\sum_{i} \omega_i \dot{w}\left[\left(\frac{x_i+1}{2}\right)^{\frac{1}{1-\beta}}, t\right], \tag{20}$$

where ω_i and x_i are respectively the weights and nodes of the Gauss Legendre integration rule used for the numerical integration. The use of Gaussian Quadrature rule changes the integrand s from infinity to finite fixed values without affecting the integration results. In this work, we used 64 points[5].

After changing the integration limit, $\left(\frac{x+1}{2}\right)^{\frac{1}{1-\beta}}$ represents the weakening scale s.

Damage accumulation is deduced from Eq.(16):

$$g_{n+1} = g_n + n_0(1-\alpha)(\gamma+1)\frac{W}{W_F} \tag{21}$$

with $g_n = \left[1-(1-D_n)^{\gamma+1}\right]^{1-\alpha}$.

We upgrade the damage step by step following Eq.(21). When D reaches one, the material fails.

4.2. Regime determination under multiple scales

The material could be both in elastic and plastic regime at different scales. To be more elaborate, we reuse the fundamental equations in different regimes. At scale s, we have a dissipation rate given by:

$$\dot{w}(s) = \left(\underline{\underline{S}} - \underline{\underline{b}}\right) : \underline{\underline{\dot{\varepsilon}}}^p,$$

which differs between plastic and elastic regime.

Elastic regime:

There we have $\underline{\underline{\dot{\varepsilon}}}^p = 0$, $\underline{\underline{\dot{b}}} = 0$ and $\underline{\underline{\dot{S}}} = dev\underline{\underline{\dot{\Sigma}}}$, so

$$\underline{\underline{\dot{S}}} - \underline{\underline{\dot{b}}} = dev\underline{\underline{\dot{\Sigma}}},$$

yielding

$$\left(\underline{\underline{S}} - \underline{\underline{b}}\right)(t+dt) = \left(\underline{\underline{S}} - \underline{\underline{b}}\right)(t) + dev\underline{\underline{\dot{\Sigma}}}dt := \left(\underline{\underline{S}} - \underline{\underline{b}}\right)_{trial}(t+dt). \tag{22}$$

We are in elastic regime at scale s as long as we satisfy

$$\left(\underline{\underline{S}} - \underline{\underline{b}}\right)(t+dt) \leqslant (\sigma_y - \lambda\Sigma_H)/s.$$

Plastic regime:

When we leave elastic regime at scale s, we have:

$$
\begin{cases}
\dot{\underline{\underline{\varepsilon}}}^p = \gamma \dfrac{\underline{\underline{S}} - \underline{\underline{b}}}{\left\| \underline{\underline{S}} - \underline{\underline{b}} \right\|}, \gamma > 0, & \text{plastic flow}, \quad (23)\\
\left\| \underline{\underline{S}} - \underline{\underline{b}} \right\| = \left(\sigma_y - \lambda \Sigma_H \right) / s, & \text{yield limit}, \quad (24)\\
\left(\underline{\underline{S}} - \underline{\underline{b}} \right) : \left(\dot{\underline{\underline{S}}} - \dot{\underline{\underline{b}}} \right) = 0, & \text{yield limit time invariance}, \quad (25)\\
\dot{\underline{\underline{b}}} = \dfrac{kE}{E-k} \dot{\underline{\underline{\varepsilon}}}^p, & \text{kinematic hardening rule}, \quad (26)\\
\dot{\underline{\underline{S}}} = dev \dot{\underline{\underline{\Sigma}}} - \dfrac{E}{1+\nu} \dot{\underline{\underline{\varepsilon}}}^p, & \text{localisation rule}. \quad (27)
\end{cases}
$$

In all cases, we get(see annex 'Multi-dimensional analysis')

$$
\left(\underline{\underline{S}} - \underline{\underline{b}} \right)(s, t+dt) = \frac{\left(\underline{\underline{S}} - \underline{\underline{b}} \right)_{trial}(s, t+dt)}{1+\eta}, \tag{28}
$$

with

$$
\eta = max \left\{ \underbrace{0}_{elastic\ regime}, \quad \underbrace{\frac{\left\| \underline{\underline{S}} - \underline{\underline{b}} \right\|_{trial}}{\left(\sigma_y - \lambda \Sigma_H \right) / s} - 1}_{plastic\ regime\ when\ this\ number\ is\ positive} \right\},
$$

$$
\left(\underline{\underline{S}} - \underline{\underline{b}} \right)_{trial}(s, t+dt) = \left(\underline{\underline{S}} - \underline{\underline{b}} \right)(s, t) + dev \dot{\underline{\underline{\Sigma}}}(t) dt.
$$

That is to say, when the structure is in elastic regime at time t and scale s, we have $\left(\underline{\underline{S}} - \underline{\underline{b}} \right)(s,t) = \left(\underline{\underline{S}} - \underline{\underline{b}} \right)_{trial}(s,t)$. Otherwise, if the norm of $\left(\underline{\underline{S}} - \underline{\underline{b}} \right)_{trial}(s,t)$ is greater than the local yield limit $\left(\sigma_y - \lambda \Sigma_H \right)(1-D)^{\delta}/s$, $\left(\underline{\underline{S}} - \underline{\underline{b}} \right)(s,t)$ will be projected on the yield limit.

Knowing the distinction between elastic and plastic regime under multiple scales, we compute the general expression of the dissipated energy rate.

$$
\dot{w}(s) = \left(\underline{\underline{S}} - \underline{\underline{b}} \right) : \dot{\underline{\underline{\varepsilon}}}^p = \gamma \frac{\left(\sigma_y - \lambda \Sigma_H \right)}{s}. \tag{29}
$$

From Eq.(A.5) and Eq.(A.8) in annex, we get:

$$
E\gamma dt = \left\langle \left\| \underline{\underline{S}} - \underline{\underline{b}} \right\|_{trial} - \frac{\left(\sigma_y - \lambda \Sigma_H \right)}{s} \right\rangle / \left(\frac{1}{1+\nu} + \frac{k}{E-k} \right) = \left\langle \left\| \underline{\underline{S}} - \underline{\underline{b}} \right\|_{trial} - \frac{\left(\sigma_y - \lambda \Sigma_H \right)}{s} \right\rangle \frac{(E-k)(1+\nu)}{(E+k\nu)}, \tag{30}
$$

where $\langle\ \rangle$ is Macaulay bracket symbol defined as $\langle m \rangle = 0$ if $m \leqslant 0$, otherwise $\langle m \rangle = m$.

We replace γ deduced from Eq.(30) in Eq.(29) to give the expression of local energy dissipation rate at scale s:

$$
\dot{w}(s) dt = \frac{(E-k)(1+\nu)}{E(E+k\nu)} \left\langle \left\| \underline{\underline{S}} - \underline{\underline{b}} \right\|_{trial} - \frac{\left(\sigma_y - \lambda \Sigma_H \right)}{s} \right\rangle \frac{\left(\sigma_y - \lambda \Sigma_H \right)}{s}. \tag{31}
$$

With Eq.(20), the final expression of energy dissipation W during time step dt writes:

$$
\begin{aligned}
W &= \dot{W}dt \\
&= \frac{1}{2}\sum_i \omega_i \dot{w}\left[\left(\frac{x+1}{2}\right)^{\frac{1}{1-\beta}}\right]dt \\
&= \frac{(E-k)(1+\nu)}{2E(E+k\nu)}\sum_i \omega_i \left\langle \left\| \underline{\underline{S}} - \underline{\underline{b}} \right\|_{trial} - \frac{(\sigma_y - \lambda\Sigma_H)\, M(\sigma_H)}{\left(\frac{x_i+1}{2}\right)^{\frac{1}{1-\beta}}} \right\rangle \frac{(\sigma_y - \lambda\Sigma_H)\, M(\sigma_H)}{\left(\frac{x_i+1}{2}\right)^{\frac{1}{1-\beta}}}.
\end{aligned}
\tag{32}
$$

The mean stress effect term in Chaboche model is $s_{-1}\left(1-3\dfrac{\sigma_H}{\sigma_u}\right)$, where the fatigue limit at zero mean stress s_{-1} is reduced in the presence of σ_H. In our model, the yield limit decreases with positive mean stress. So we adopt similar method and here $M(\sigma_H) = \left(1-3\dfrac{\sigma_H}{\sigma_u}\right)$.

We have the damage accumulation deduced in Eq.(21):

$$
g_{n+1} = g_n + n_0(1-\alpha)(\gamma+1)\frac{\dot{W}dt}{W_F} = g_n + n_0(1-\alpha)(\gamma+1)\frac{W}{W_F},
$$

with $g_n = \left[1-(1-D_n)^{\gamma+1}\right]^{1-\alpha}$, thus $D_n = \left[1-\left(1-g_n^{\frac{1}{1-\alpha}}\right)^{\frac{1}{\gamma+1}}\right]$.

Now we are able to put these formula into numerical tests.

5 Method verification

5.1. One dimensional application to simple cyclic data(from Cetim)

The test is first performed on a sinusoidal axial load $\Sigma = Csin(t)$ with parameters in Table.4, giving a deviatoric amplitude $S_{max} = dev\Sigma = \sqrt{\dfrac{2}{3}}Csin(t)$.

We use matlab to realize our analytical method. We plot $\left\| \underline{\underline{S}} - \underline{\underline{b}} \right\|_{trial}$ and $\left\| \underline{\underline{S}} - \underline{\underline{b}} \right\|$ at two different scales($s_{33} = 4.20$ and $s_{40} = 9.68$) in Figure 3.

The damage evolves like in Figure 4, where we compare the damage evolution as predicted by the cycle accumulation Eq.(7) and by the numerical strategy of section 4.

Now we compare the result to the one demonstrated in Figure 2. We can see from Figure 5 the difference between cyclic load calculation and numerical method as function of time steps n. From the relative difference figure we conclude that the two methods converge.

The cyclic load calculation is only valid for very simple such as proportional loading in fatigue, nevertheless it can still be used as a comparison group to verify the numerical results. The outcome seems satisfactory. Hence, to be more general for any loading history, we adopt the numerical method.

Parameters	**Value**
Load	$\Sigma = 5e8sin(t)$ Pa
Young's modulus	$E = 2e11$ Pa
Hardening parameter	$k = 6e8$ Pa
Weakening scales distribution exponent	$\beta = 3$
Hydrostatic pressure sensitivity	$\lambda = 0.5$
Macroscopic yield stress	$\sigma_y = 6.38e8$ Pa
Material parameter from Chaboche law(Wohler curve exponent)	$\gamma = 0.5$
Non-linearity of damage accumulation	$\alpha = 0.5$
Initial damage	$D = 0$
Initial time	$t = 0$ s
Dissipated energy to failure per unit volume	$W_F = 3e6$ J
Looping step	1e-4 s

Table 1: Material parameters in a simple cyclic load

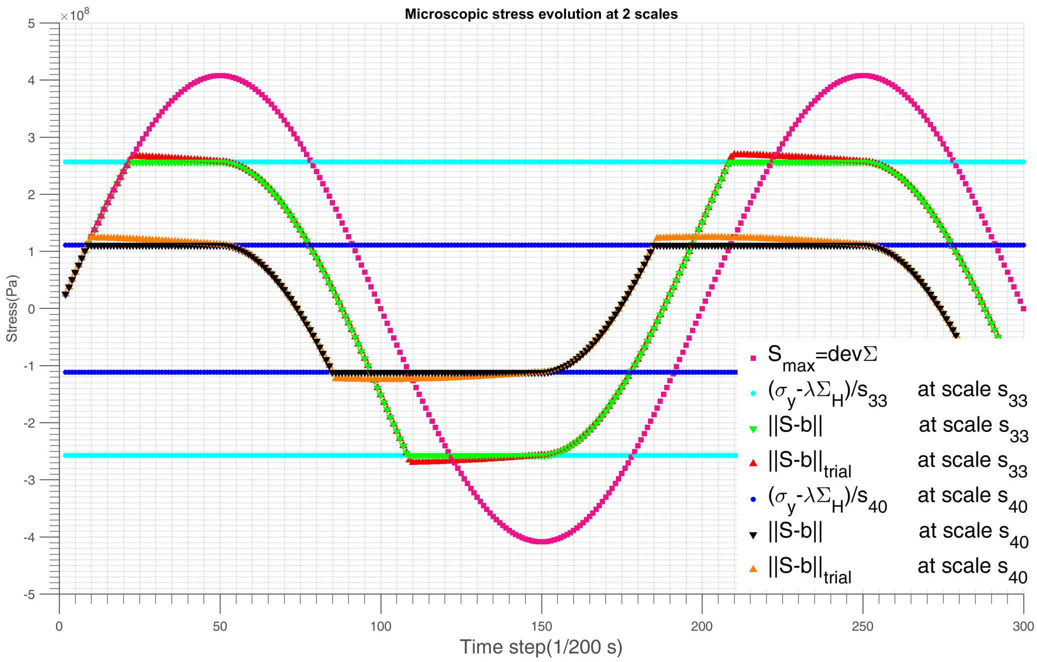


Figure 3: Microscopic $\left(\underline{\underline{S}} - \underline{\underline{b}}\right)_{trial}$ and $\left(\underline{\underline{S}} - \underline{\underline{b}}\right)$ evolution with time under different weakening scales in sinusoidal load

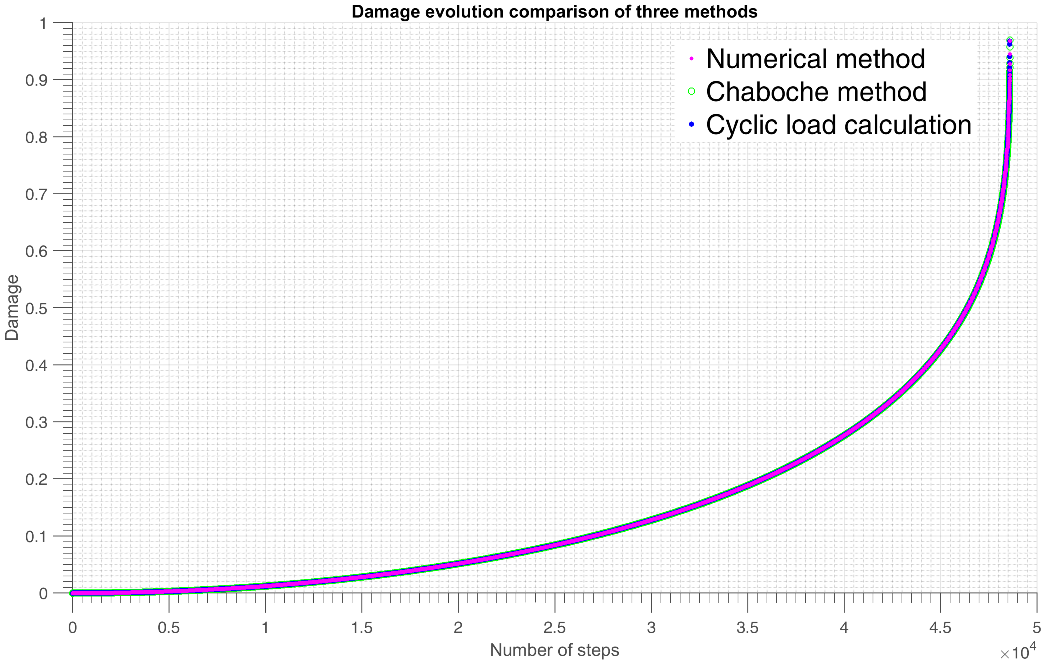


Figure 4: Damage evolution with time under sinusoidal load with two different methods

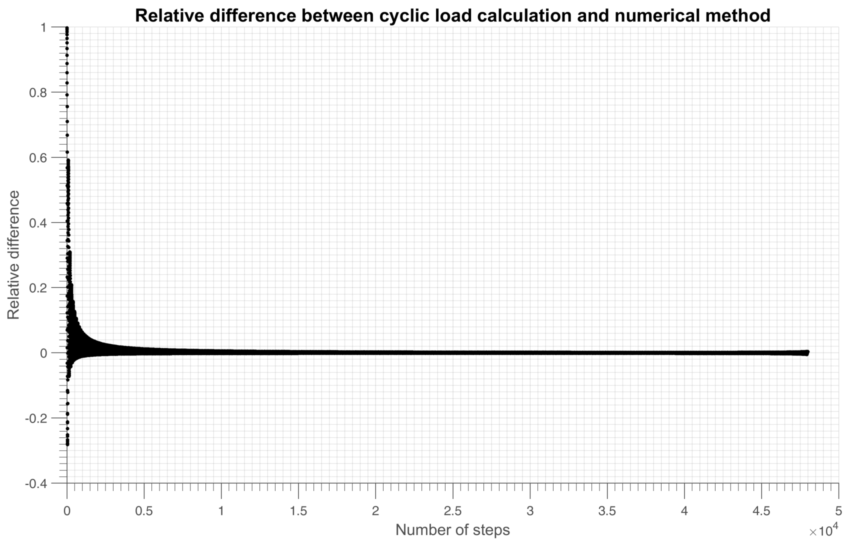
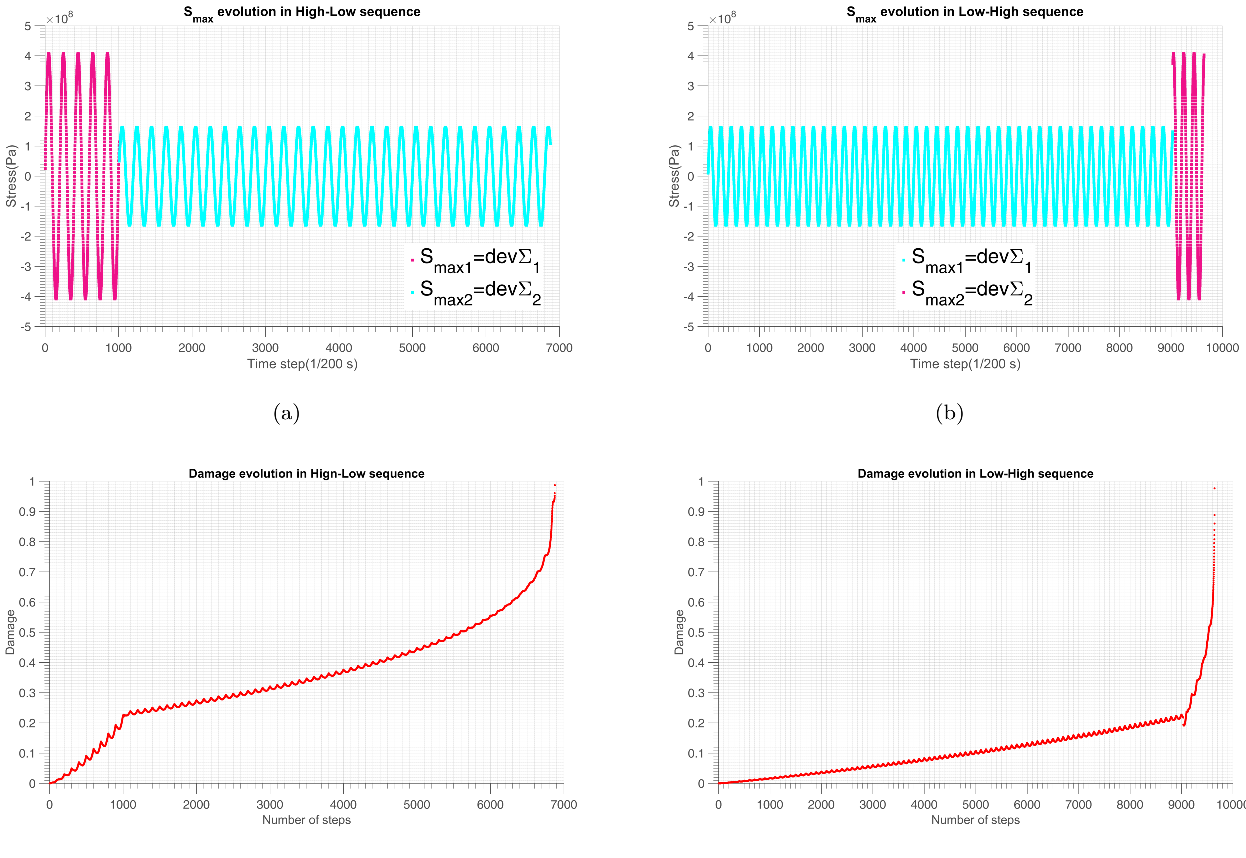


Figure 5: $\dfrac{D_{cyclic} - D_{numerical}}{D_{cyclic}}$ evolution with time

5.1.1. Sequence effect

We adopt the parameter α to take into account the sequence effect. The high-low loading sequence clearly reduces the fatigue life, as depicted in Figure 6. In order to cover this phenomenon, we let α change with time. Here a is the sequence effect sensitivity, we take $a = 0.5$.

$$\alpha = 1 - \left(1 - \frac{S_{max}}{\sigma_y\,(1 - \lambda \Sigma_H)}\right)^a .$$



(a) (b) (c) (d)

Figure 6: Two level sequence effect.

6 Experimental verification

6.1. Parameter identification on 30NCD16 steel

The 30NCD16 steel considered herein is tempered steel. [Dubar 1992] had realized fatigue tests on this material which is supplied by the EUROCOPTER Company in the form of several batches of substantially different mechanical and dynamic characteristics. Mechanical properties were determined by [Dubar 1992] by monotonic traction test per batch. The parameters are shown in Table.3.

E [GPa]	γ	k [GPa]	$\sigma_{y0.2\%}$ [MPa]	σ_u [MPa]	σ_{-1} [MPa]	τ_{-1} [MPa]
191	0.38	1	1080	1200	690	428

where

E	:	Young's modulus,
γ	:	Poisson's ratio
k	:	hardening parameter
$\sigma_{y0.2\%}$	:	elastic limit to 0.2% plastic deformation,
σ_u	:	maximum tensile strength,
σ_{-1}	:	fatigue limit in symmetrical alternating bending,
τ_{-1}	:	fatigue limit in symmetrical alternating torsion.

Applying $W_{cyc} = \dfrac{4(E-k)(1+\nu)\,(\beta-1)}{E(E+k\nu)\beta\,(\beta+1)} \dfrac{S_{max}^{\beta+1}}{(\sigma_y - \lambda\Sigma_H)^{\beta-1}}$ to pure bending and torsion test we can find the value of W_F and β as depicted in Figure 7.

Tests (R=-1)	**N [Cycles]**	**Pure bending [MPa]**	**Pure torsion [MPa]**
1	51000	820	527
2	80000	795	505
3	90000	790	500
4	95000	785	497
5	100000	780	495
6	120000	765	482
7	140000	752	470
8	200000	725	450
9	210000	720	446
10	230000	715	445
11	250000	708	440

Table 2: 30NCD16 steel fully reversed bending and torsion tests

Parameters	**Bending**	**Torsion**
Weakening scales distribution exponent	$\beta = 8.716$	$\beta = 7.043$
Dissipated energy to failure per unit volume	$W_F = 1.657e8$ J	$W_F = 1.243e9$ J

Table 3: Parameters identification on 30NCD16 steel in bending

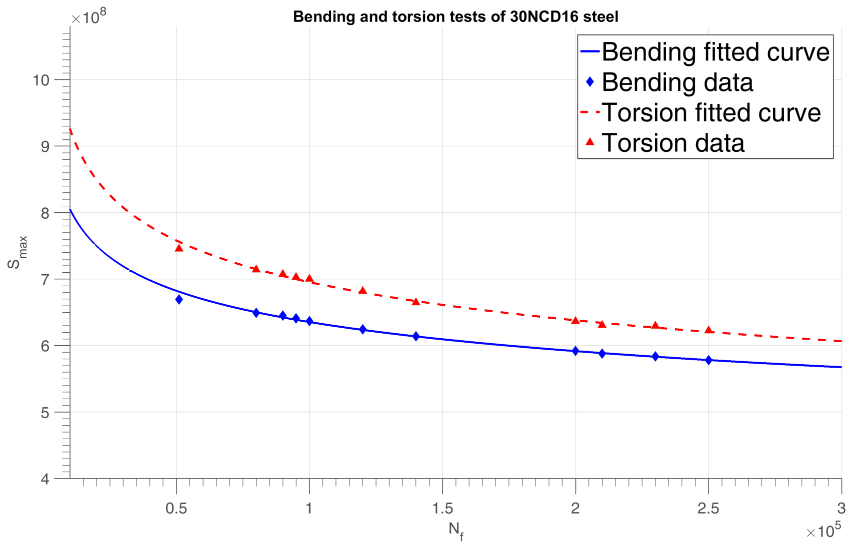


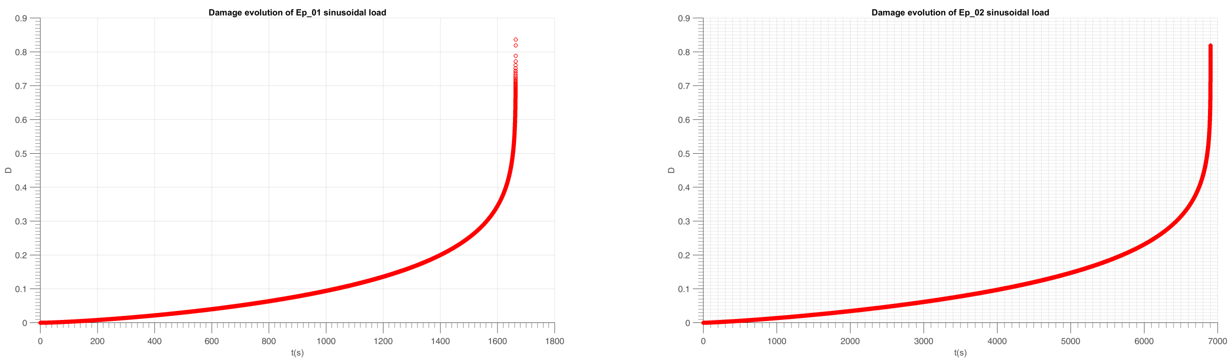
Figure 7: Calibrated S-N curve of 30NCD16 steel under fully reversed bending and torsion tests

6.2. Sinusoidal 1D data from Cetim

The stress of Ep_{01} and Ep_{02} are respectively 131.9MPa and 97.0MPa. We found the same parameters feasible to both loading cases.

Parameters	**Value**
Load	$\Sigma = 131.9sin(t)\&97sin(t)$ MPa
Young's modulus	$E = 72$ GPa
Hardening parameter	$k = 6$ MPa
Hydrostatic pressure sensitivity	$\lambda = 0$
Macroscopic yield stress	$\sigma_y = 230$ MPa
Material parameter from Chaboche law(Wohler curve exponent)	$\gamma = 6$
Non-linearity of damage accumulation	$\alpha = 1 - \left(1 - \dfrac{S_{max}}{\sigma_y\,(1 - \lambda\Sigma_H)}\right)^2$
Weakening scales distribution exponent	$\beta = 3.928$
Dissipated energy to failure per unit volume	$W_F = 3.219$ GJ

Table 4: Material parameters in a simple cyclic load



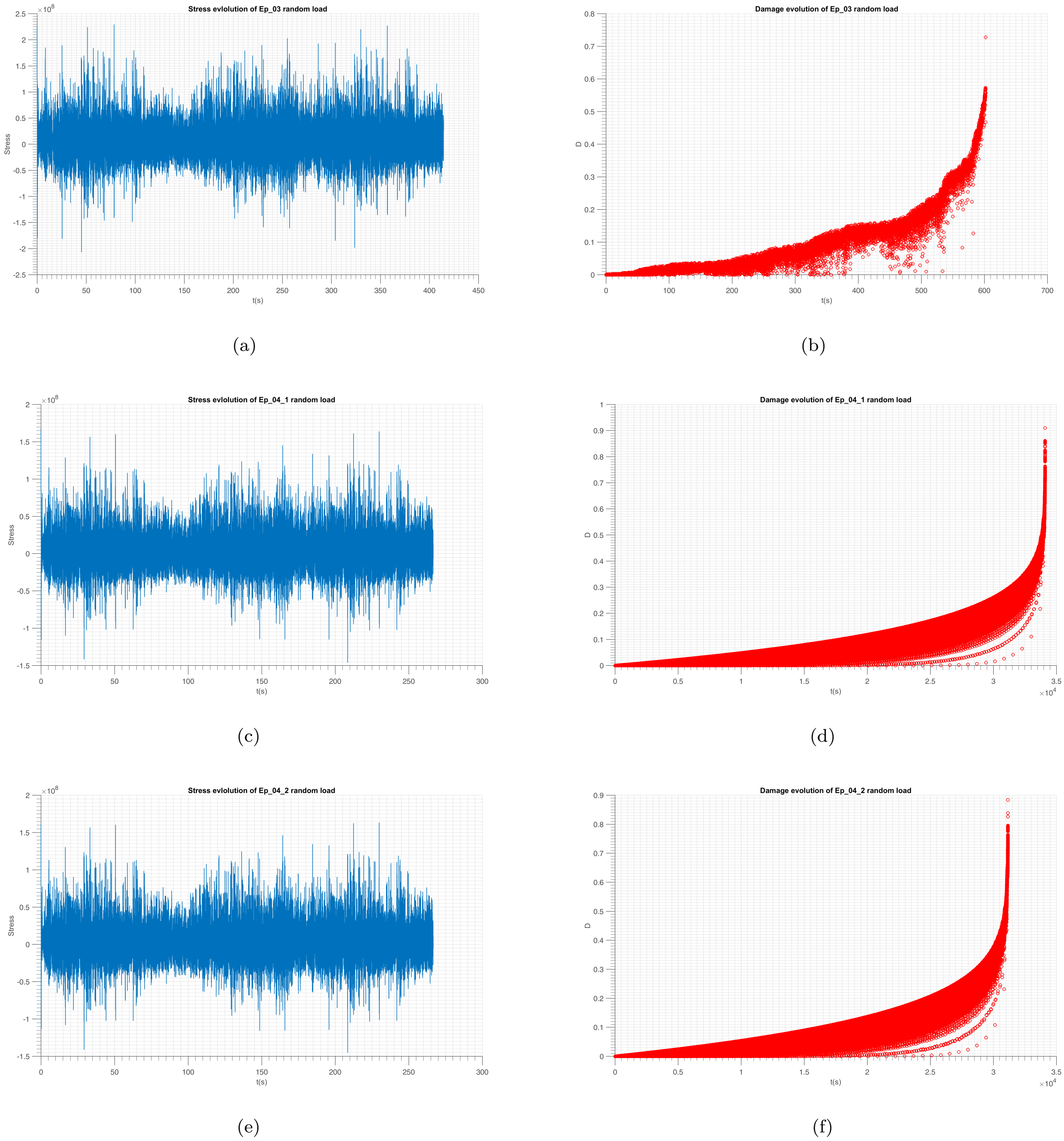
(a)

(b)

Figure 8: Damage evolution of aluminum alloy(sample 1 and 2) at constant loading with $R = -1$.

6.3. Random 1D data from Cetim

	Ep_03	Ep_04_1	Ep_04_2
λ	0.3	0.3	0.3
n_0	254	12	13



(a) (b)

(c) (d)

(e) (f)

Figure 9: Damage evolution of aluminum alloy(sample 3 and 4) at random loading.

6.4. 1D data from PSA

In this test, we reconstruct a unidimensional macroscopic stress history from recorded force data proposed by PSA group.

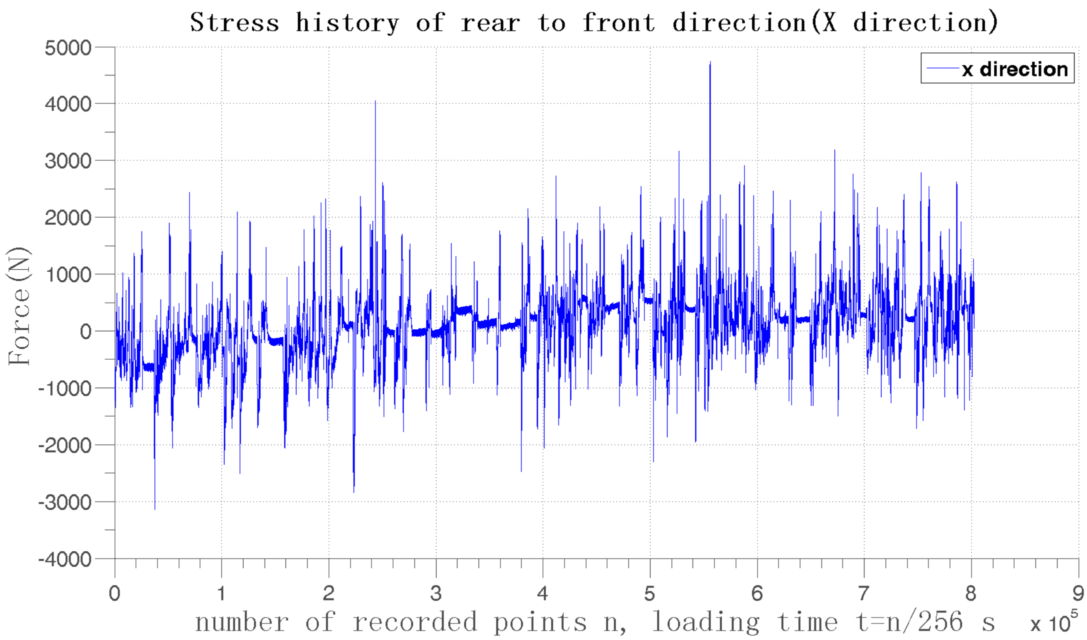


Figure 10: Loading history of X direction, force vs the record index n, with 256 sample recorded per second

The sample recording rate is 256 per second. In order to accumulate damage using small steps, we have created 1 additional points between every 2 recorded points by linear interpolation. So the sample rate is $256 * 2$ per second.

The force on wheel is firstly considered as under uniaxial loading F_x. Here we temporally set $\Sigma_x = F_x/A$ where $A = \dfrac{1}{1e6}m^2$ is the area of force, and $W_F = 3e6J$. The other data are as Table.4. The plot of $\left\|\underline{\underline{S}} - \underline{\underline{b}}\right\|_{trial}$ and $\left\|\underline{\underline{S}} - \underline{\underline{b}}\right\|$ at 2 different scales are shown in Figure 11. The damage evolves like Figure 13.

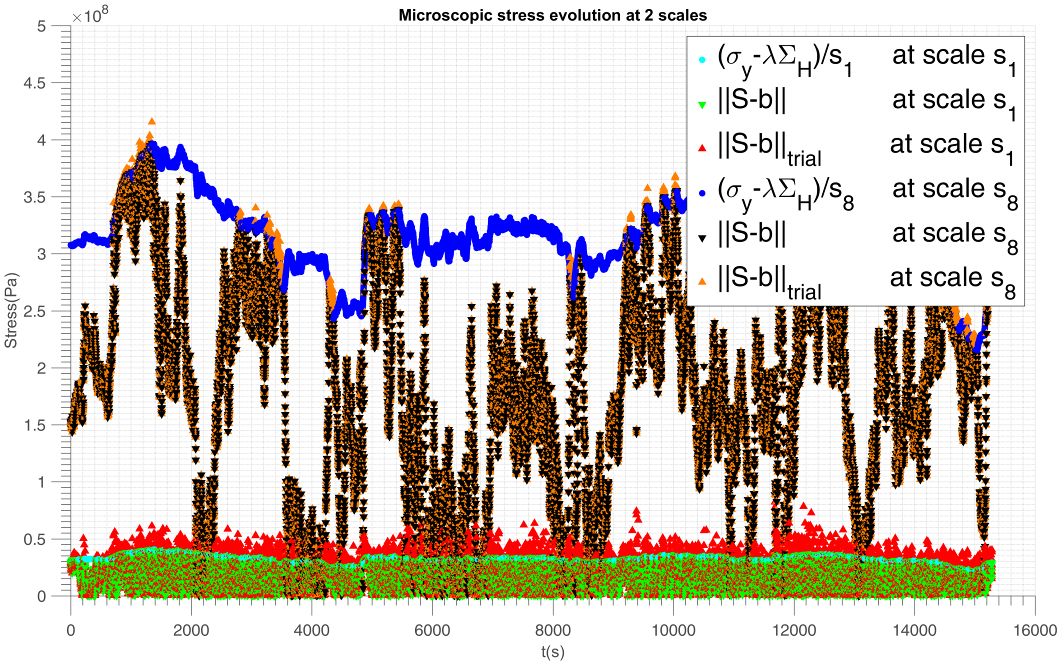


Figure 11: $\left\|\underline{\underline{S}}-\underline{\underline{b}}\right\|_{trial}$ and $\left\|\underline{\underline{S}}-\underline{\underline{b}}\right\|$ evolution with time under different weakening scales in PSA load history

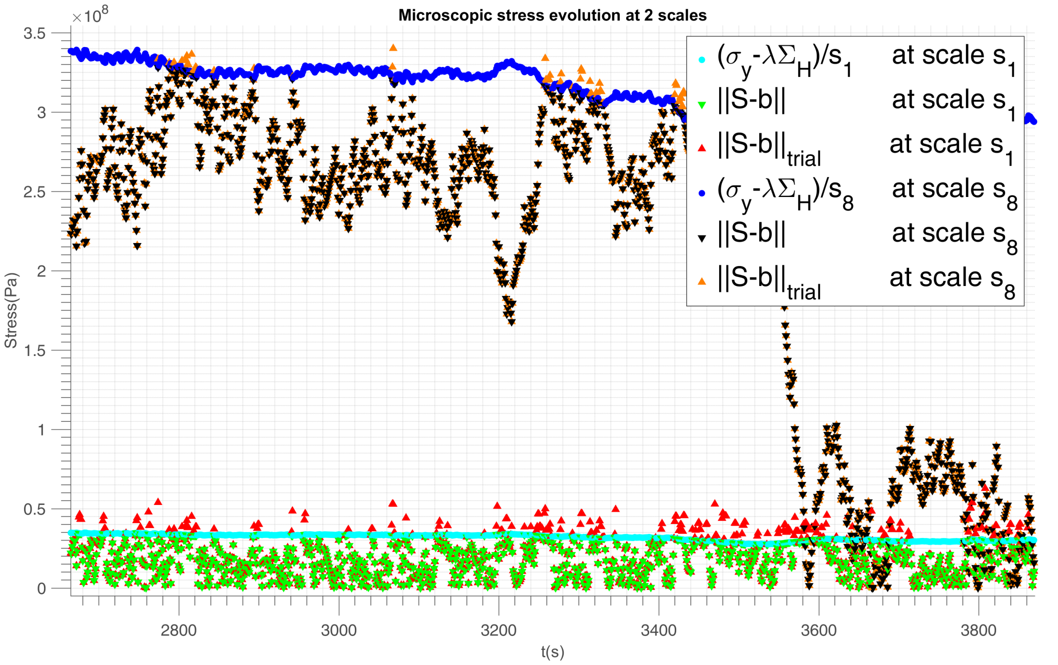


Figure 12: Circled area magnification in Figure 11 where there is more $\left\|\underline{\underline{S}}-\underline{\underline{b}}\right\|_{trial} > \left(\sigma_y-\lambda\Sigma_H\right)$(plasticity) at s_1 than at s_8

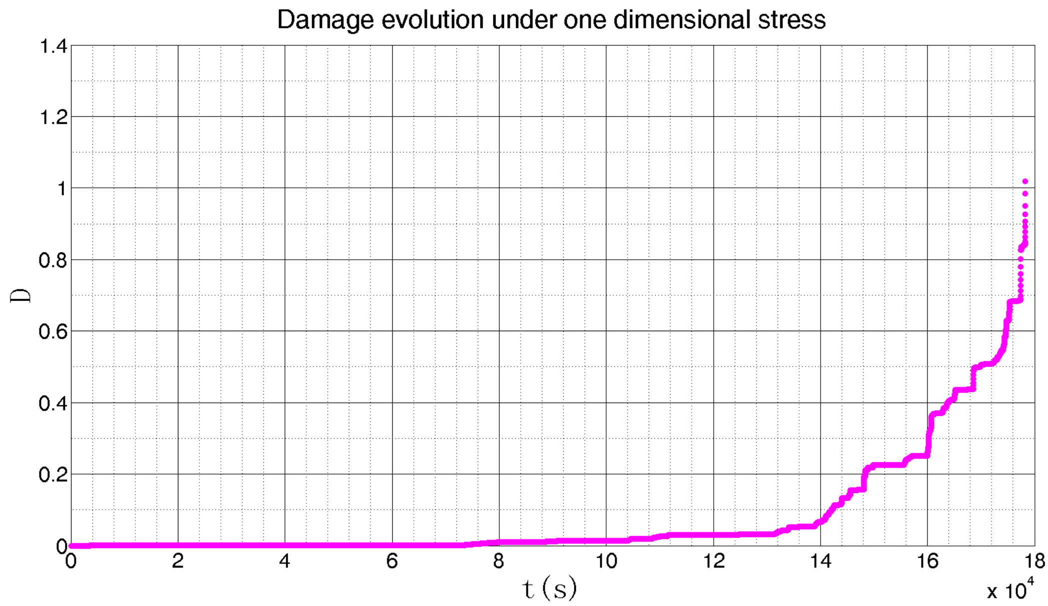


Figure 13: Damage evolution with time at one dimension PSA load history

6.5. Multi-dimensional application to PSA data

We now consider a situation where we have force recorded measured in 3 different directions as shown in Figure 14. In real case, the vertical force F_z is much larger than the axial and horizontal forces F_x and F_y,

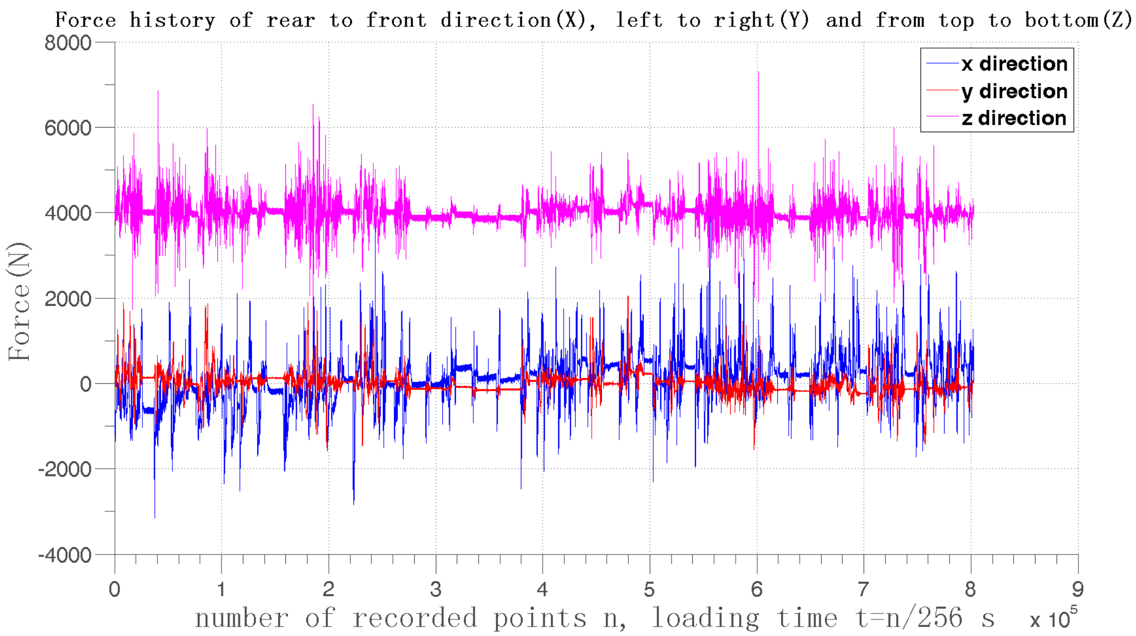


Figure 14: Loading history of 3 different directions

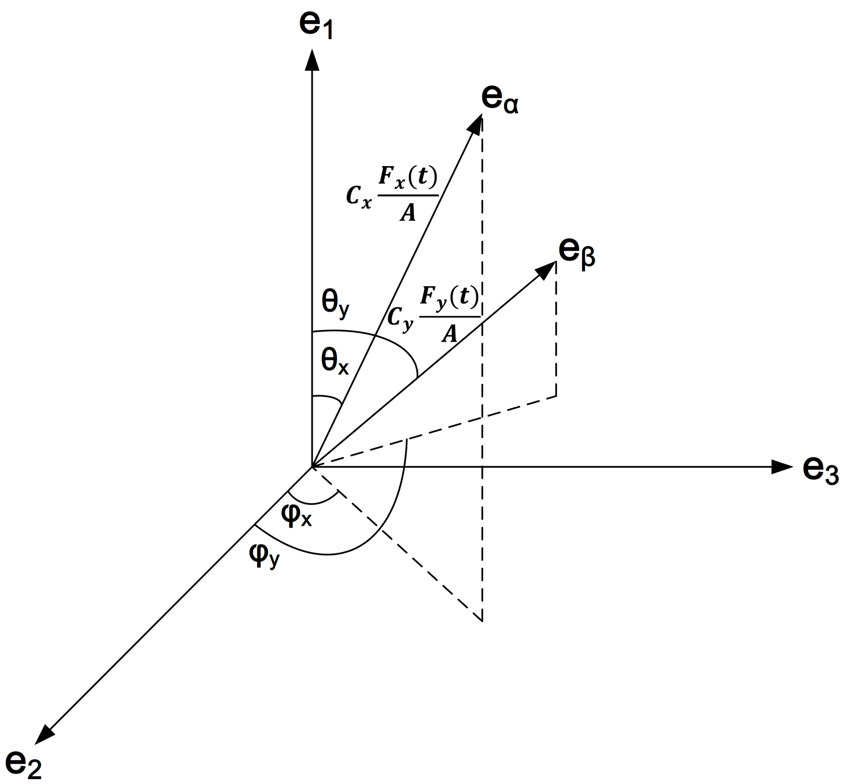


Figure 15: Loading in 3 different directions

as shown in Figure 14. However, in order to investigate large domains of interest, we first scale the axial and horizontal forces to reach comparable impact and transform them in principal stresses $c_x \frac{F_x}{A}$ applied along the stress principle vector $\underline{e}_\alpha$(respectively $\underline{e}_\beta$) that we choose randomly(Figure 15). We therefore consider the following macroscopic stress tensor:

$$\underline{\underline{\Sigma}} = \frac{F_z(t)}{A}\underline{e}_1 \otimes \underline{e}_1 + c_x \frac{F_x(t)}{A}\underline{e}_\alpha \otimes \underline{e}_\alpha + c_y \frac{F_y(t)}{A}\underline{e}_\beta \otimes \underline{e}_\beta \tag{33}$$

where $\underline{e}_\alpha$ and $\underline{e}_\beta$ are principal vectors whose spherical coordinate are θ_x, φ_x, θ_y and φ_y respectively:

$$\underline{e}_\alpha = cos\theta_x \underline{e}_1 + sin\theta_x cos\varphi_x \underline{e}_2 + sin\theta_x sin\varphi_x \underline{e}_3,$$

$$\underline{e}_\beta = cos\theta_y \underline{e}_1 + sin\theta_y cos\varphi_y \underline{e}_2 + sin\theta_y sin\varphi_y \underline{e}_3.$$

Here $F_x(t)$, $F_y(t)$, $F_z(t)$ are from test data, and θ_x, φ_x, θ_y, φ_y are structural parameters to be chosen randomly.The physical data are the same with parameters in Table.4. The structural data we choose is shown in Table.5.

The underlying assumption is that a unit load on wheel in direction $\underline{e}_x$ creates a stress tensor at point

Parameter	A(m^2)	c_x	c_y	θ_x	φ_x	θ_y	φ_y
Value	1/6e4	10	60	0.5	0.3	0.6	0.4

Table 5: The structural data in 3D analysis

$\boldsymbol{M}$ given by:

$$c_x \frac{F_x(t)}{A} \underline{e}_\alpha \otimes \underline{e}_\alpha,$$

where $\underline{e}_\alpha \otimes \underline{e}_\alpha$ defines the local structural response of the vehicle.

Replacing $\underline{e}_\alpha$ and $\underline{e}_\beta$ in Eq.(33) we get the stress tensor in Eq.(A.9) in the annex.

The plot of $\left\|\underline{\underline{S}} - \underline{\underline{b}}\right\|_{trial}$ and $\left\|\underline{\underline{S}} - \underline{\underline{b}}\right\|$ under 2 different scales are shown in Figure 16.

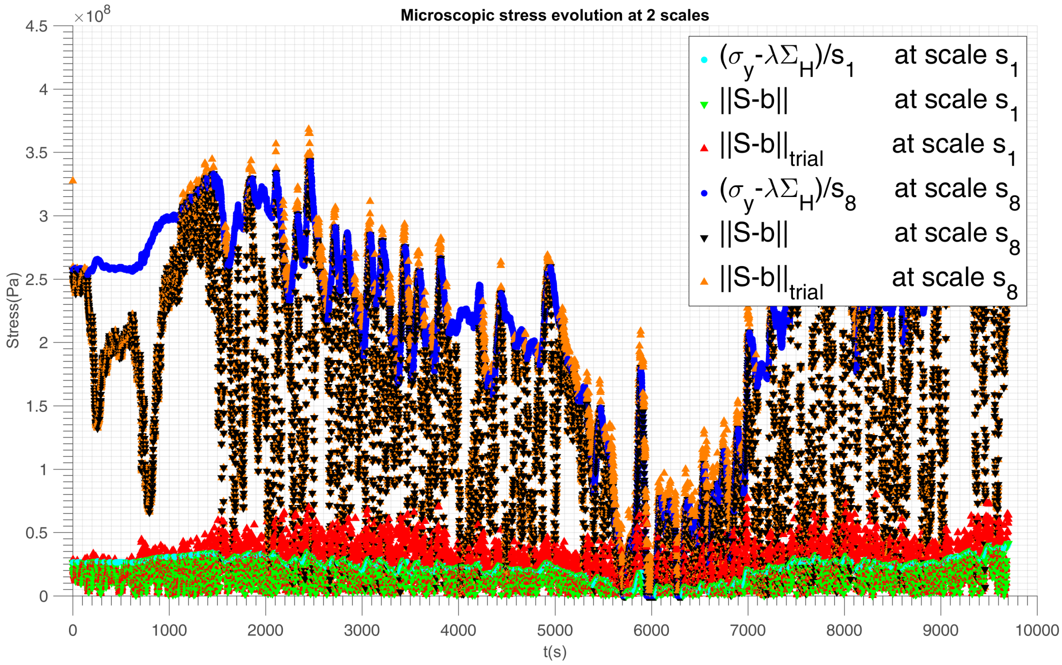


Figure 16: $\left\|\underline{\underline{S}} - \underline{\underline{b}}\right\|_{trial}$ and $\left\|\underline{\underline{S}} - \underline{\underline{b}}\right\|$ evolution with time under different weakening scales in PSA load history

In the load history, when $\left\|\underline{\underline{S}} - \underline{\underline{b}}\right\|_{trial} > (\sigma_y - \lambda\Sigma_H)$, the damage accumulates. However, at scale s_8, there are much less damage accumulation than at scale s_1. In this way we do not neglect the small influences in load history and the big fluctuation in stress is magnified which reflects the real situation.

The damage evolves like in Figure 17.

We can improve the result by inserting more arithmetic sequence points between every 2 recorded points. As is shown in Table.6 :

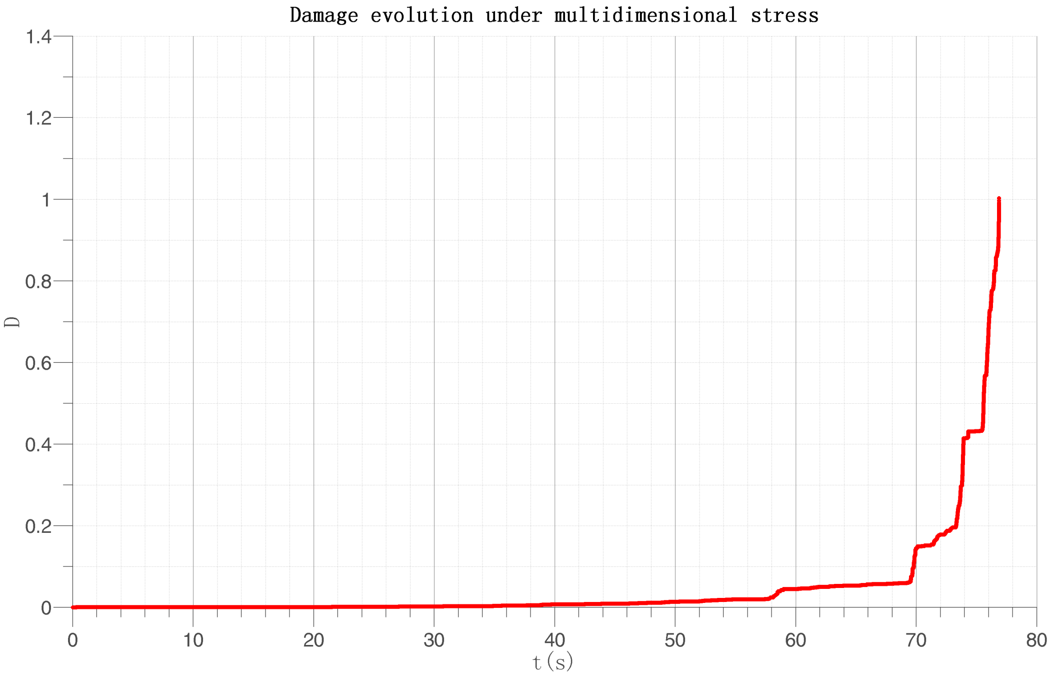


Figure 17: Damage evolution under multidimensional stress

Table 6: Arithmetic sequence points density effect

Arithmetic sequence points between every two points	**Total time to failure(s)**
10	78.63711
20	72.24630
30	70.25793
50	68.69148
100	67.49223

7 Discussion

We work on the stress tensor directly in 3D analysis in stead of using the multidimensional equivalent stress. The strategy can be made more complex by introducing a local space averaging process in the calculation of the local damage, and by taking more general plastic flows. The energy based fatigue approach takes into account impurities and hardness in the material and is applicable to any type of micro plasticity law and multiaxial load geometry. The time implicit strategy gets rid of cycle counting which is hardly applicable to complex loading, big fluctuation is magnified which reflects the real situation.

Further research of energy based failure criteria should be focused on the following aspects:

1. The accommodation law might be more elaborate than kinematic hardening.
2. The differentiation of shear stress and normal stress effect on fatigue life should be clarified.
3. The non-linearity parameter α contains the stress σ, so it can evolve with time. But for complex loading history, should it change at every time step?

Acknowledgments

We are grateful for the financial and technical support of Chaire PSA.

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Appendices

Appendix A DETAILED EXPLOITATION

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A DETAILED DESCRIPTION OF ANALYTICAL EXPLOITATION ON UNIAXIAL CYCLE

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Phase 1: The deviatoric stress amplitude increases from σ_y/s to S_{max}.

The material is in local plastic regime, then $\dot{\varepsilon}^p > 0$ and $\dot{\sigma} - \dot{b} = 0 \Rightarrow \dot{\Sigma} - \frac{E}{1+\nu}\dot{\varepsilon}^p = \frac{kE}{E-k}\dot{\varepsilon}^p \Rightarrow$

$$\dot{\varepsilon}^p = \frac{(E-k)(1+\nu)}{E(E+k\nu)}\dot{\Sigma}.$$

$\Rightarrow \dot{\varepsilon}^p$ varies from 0 to $\frac{(E-k)(1+\nu)(S_{max}-\sigma_y/s)}{E(E+k\nu)}$.

From Taylor-Lin scale transition model:

$$\dot{\sigma} = \dot{\Sigma} - \frac{E}{1+\nu}\dot{\varepsilon}_p = \dot{\Sigma} - \frac{E-k}{E-\nu k}\dot{\Sigma} = \frac{k(1-\nu)}{E-k\nu}\dot{\Sigma}.$$

$\Rightarrow \sigma$ varies from σ_y/s to $\sigma_y/s + \frac{k(1-\nu)(S_{max}-\sigma_y/s)}{E-k\nu}$.

$$\dot{b} = \dot{\Sigma} - \frac{E}{1+\nu}\dot{\varepsilon}_p = \dot{\Sigma} - \frac{E-k}{E-\nu k}\dot{\Sigma} = \frac{k(1-\nu)}{E-k\nu}\dot{\Sigma}.$$

$\Rightarrow b$ varies from 0 to $\frac{k(1-\nu)(S_{max}-\sigma_y/s)}{E-k\nu}$.

So the energy dissipation rate is:

$$(\sigma - b)\dot{\varepsilon}^p = \frac{\sigma_y}{s}\dot{\varepsilon}^p = \frac{\sigma_y}{s}\frac{(E-k)(1+\nu)}{E(E+k\nu)}\dot{\Sigma}.$$

The energy dissipation is:

$$(\sigma - b)\Delta\varepsilon^p = \frac{\sigma_y}{s}\frac{(E-k)(1+\nu)(S_{max}-\sigma_y/s)}{E(E+k\nu)}.$$

Phase 2: The deviatoric stress amplitude decreases from S_{max} to $S_{max} - 2\sigma_y/s$.

The material is in local elastic regime, then $\dot{\varepsilon}^p = 0$ and $\dot{\sigma} - \dot{b} = 0 \Rightarrow$

$\dot{b} = 0$, $\dot{\sigma} = \dot{\Sigma} - \frac{E}{1+\nu}\dot{\varepsilon}_p = \dot{\Sigma}$.

σ varies from $\sigma_y/s + \frac{k(1-\nu)(S_{max}-\sigma_y/s)}{E-k\nu}$ to $-\sigma_y/s + \frac{k(1-\nu)(S_{max}-\sigma_y/s)}{E-k\nu}$.

$\sigma - b$ varies from σ_y/s to $-\sigma_y/s$.

The energy dissipation rate is:

$$(\sigma - b)\dot{\varepsilon}^p = 0.$$

Phase 3: The deviatoric stress amplitude decreases from $S_{max} - 2\sigma_y/s$ to $-S_{max}$.

The material is in local plastic regime, then $\dot{\varepsilon}^p > 0$ and $\dot{\sigma} - \dot{b} = 0 \Rightarrow$

$$\dot{\varepsilon}^p = \frac{(E-k)(1+\nu)}{E(E+k\nu)}\dot{\Sigma}$$

as opposite to phase 1 for $\dot{\Sigma} < 0$.

$\Rightarrow \varepsilon^p$ varies from $\dfrac{(E-k)(1+\nu)(S_{max}-\sigma_y/s)}{E(E+k\nu)}$ to
$\dfrac{(E-k)(1+\nu)(S_{max}-\sigma_y/s - S_{max} - (S_{max}-2\sigma_y/s))}{E(E+k\nu)} = -\dfrac{(E-k)(1+\nu)(S_{max}-\sigma_y/s)}{E(E+k\nu)}$.

From Taylor-Lin scale transition model:

$$\dot{\sigma} = \dot{\Sigma} - \frac{E}{1+\nu}\dot{\varepsilon}_p = \dot{\Sigma} - \frac{E-k}{E-\nu k}\dot{\Sigma} = \frac{k(1-\nu)}{E-k\nu}\dot{\Sigma}.$$

$\Rightarrow \sigma$ varies from $-\sigma_y/s + \dfrac{k(1-\nu)(S_{max}-\sigma_y/s)}{E-k\nu}$ to $-\sigma_y/s - \dfrac{k(1-\nu)(S_{max}-\sigma_y/s)}{E-k\nu}$.

$$\dot{b} = \dot{\Sigma} - \frac{E}{1+\nu}\dot{\varepsilon}_p = \dot{\Sigma} - \frac{E-k}{E-\nu k}\dot{\Sigma} = \frac{k(1-\nu)}{E-k\nu}\dot{\Sigma}.$$

$\Rightarrow b$ varies from $\dfrac{k(1-\nu)(S_{max}-\sigma_y/s)}{E-k\nu}$ to $-\dfrac{k(1-\nu)(S_{max}-\sigma_y/s)}{E-k\nu}$.

So the energy dissipation rate is:

$$(\sigma - b)\dot{\varepsilon}^p = -\frac{\sigma_y}{s}\dot{\varepsilon}^p = -\frac{\sigma_y}{s}\frac{(E-k)(1+\nu)}{E(E+k\nu)}\dot{\Sigma}.$$

The energy dissipation is:

$$(\sigma - b)\Delta\varepsilon^p = -\frac{\sigma_y}{s}\frac{(E-k)(1+\nu)(-2S_{max}+2\sigma_y/s)}{E(E+k\nu)} = \frac{2\sigma_y}{s}\frac{(E-k)(1+\nu)(S_{max}-\sigma_y/s)}{E(E+k\nu)}.$$

Phase 4: The deviatoric stress amplitude increases from $-S_{max}$ to $-S_{max} + 2\sigma_y/s$.

The material is in local elastic regime, then $\dot{\varepsilon}^p = 0$ and $\dot{\sigma} - \dot{b} = 0 \Rightarrow$

$\dot{b} = 0$, $\dot{\sigma} = \dot{\Sigma} - \dfrac{E}{1+\nu}\dot{\varepsilon}_p = \dot{\Sigma}$.

σ varies from $-\sigma_y/s - \dfrac{k(1-\nu)(S_{max}-\sigma_y/s)}{E-k\nu}$ to $\sigma_y/s - \dfrac{k(1-\nu)(S_{max}-\sigma_y/s)}{E-k\nu}$.

$\sigma - b$ varies from $-\sigma_y/s$ to σ_y/s.

So the energy dissipation rate is:

$$(\sigma - b)\dot{\varepsilon}^p = 0.$$

Phase 5: The deviatoric stress amplitude increases from $-S_{max} + 2\sigma_y/s$ to σ_y/s.

The material is in local plastic regime, then $\dot{\varepsilon}^p > 0$ and $\dot{\sigma} - \dot{b} = 0 \Rightarrow$

$$\dot{\varepsilon}^p = \frac{(E-k)(1+\nu)}{E(E+k\nu)}\dot{\Sigma}$$

as in phase 1.

$\Rightarrow \dot{\varepsilon}^p$ varies from $-\dfrac{(E-k)(1+\nu)(S_{max}-\sigma_y/s)}{E(E+k\nu)}$ to 0.

$$\dot{\sigma} = \dot{\Sigma} - \frac{E}{1+\nu}\dot{\varepsilon}_p = \dot{\Sigma} - \frac{E-k}{E-\nu k}\dot{\Sigma} = \frac{k(1-\nu)}{E-k\nu}\dot{\Sigma}.$$

$\Rightarrow \sigma$ varies from $\sigma_y/s - \dfrac{k(1-\nu)(S_{max}-\sigma_y/s)}{E-k\nu}$ to σ_y/s.

$$\dot{b} = \dot{\Sigma} - \frac{E}{1+\nu}\dot{\varepsilon}_p = \dot{\Sigma} - \frac{E-k}{E-\nu k}\dot{\Sigma} = \frac{k(1-\nu)}{E-k\nu}\dot{\Sigma}.$$

$\Rightarrow b$ varies from $-\dfrac{k(1-\nu)(S_{max}-\sigma_y/s)}{E-k\nu}$ to 0.

So the energy dissipation rate is:

$$(\sigma-b)\dot{\varepsilon}^p = \frac{\sigma_y}{s}\dot{\varepsilon}^p = \frac{\sigma_y}{s}\frac{(E-k)(1+\nu)}{E(E+k\nu)}\dot{\Sigma}.$$

The energy dissipation is:

$$(\sigma-b)\Delta\varepsilon^p = \frac{\sigma_y}{s}\frac{(E-k)(1+\nu)(S_{max}-\sigma_y/s)}{E(E+k\nu)}.$$

From the three phase analysis in local plastic regime, the dissipated energy is like $dW(phase1) = \frac{1}{2}dW(phase3) = dW(phase5)$ and the dissipation rate is like $d\dot{W}(phase1) = d\dot{W}(phase3) = d\dot{W}(phase5)$.

$$d\dot{W} = \frac{(E-k)(1+\nu)}{E(E-k\nu)}\left(\frac{\sigma_y}{s}\right)|\dot{\Sigma}| \tag{A.1}$$

****

MULTI-DIMENSIONAL PLASTIC AND ELASTIC REGIME ANALYSIS

****

At a certain scale s_i, after elimination of $\dot{\underline{\underline{\varepsilon}}}^p$, there are

$$\dot{\underline{\underline{S}}} - \dot{\underline{\underline{b}}} = dev\dot{\underline{\underline{\Sigma}}} - E\gamma\left(\frac{1}{1+\nu} + \frac{k}{E-k}\right)\frac{\underline{\underline{S}} - \underline{\underline{b}}}{\left\|\underline{\underline{S}} - \underline{\underline{b}}\right\|}.$$

If we are at yield limit at (t+dt), we get on the other hand:

$$\left(\underline{\underline{S}} - \underline{\underline{b}}\right)(t+dt) = \left(\underline{\underline{S}} - \underline{\underline{b}}\right)(t) + \left(\dot{\underline{\underline{S}}} - \dot{\underline{\underline{b}}}\right)dt,$$
$$\left\|\left(\underline{\underline{S}} - \underline{\underline{b}}\right)(t+dt)\right\| = \left(\sigma_y - \lambda\sigma_m\right)/s_i. \tag{A.2}$$

Replacing $\left(\dot{\underline{\underline{S}}} - \dot{\underline{\underline{b}}}\right)$ in the integration by its expression we get:

$$\left(\underline{\underline{S}} - \underline{\underline{b}}\right)(t+dt) = \left(\underline{\underline{S}} - \underline{\underline{b}}\right)(t) + dev\dot{\underline{\underline{\Sigma}}}dt - E\gamma dt\left(\frac{1}{1+\nu} + \frac{k}{E-k}\right)\frac{\left(\underline{\underline{S}} - \underline{\underline{b}}\right)(t+dt)}{\left\|\underline{\underline{S}} - \underline{\underline{b}}\right\|(t+dt)} \tag{A.3}$$

Putting all terms with $\left(\underline{\underline{S}} - \underline{\underline{b}}\right)(t+dt)$ on the left hand side, we get:

$$\left(\underline{\underline{S}} - \underline{\underline{b}}\right)(t+dt)(1+\eta) = \left(\underline{\underline{S}} - \underline{\underline{b}}\right)(t) + dev\dot{\underline{\underline{\Sigma}}}dt = \left(\underline{\underline{S}} - \underline{\underline{b}}\right)_{trial}(t+dt) \tag{A.4}$$

with

$$\eta = \frac{E\gamma dt}{\left\|\underline{\underline{S}} - \underline{\underline{b}}\right\|(t+dt)}\left(\frac{1}{1+\nu} + \frac{k}{E-k}\right). \tag{A.5}$$

To see whether the structure is in elastic or plastic regime at each time step, we use $\left(\underline{\underline{S}} - \underline{\underline{b}}\right)_{trial}(t+dt)$ to compare with the yield stress at the same scale s_i, thus to give a value to $\left(\underline{\underline{S}} - \underline{\underline{b}}\right)(t+dt)$.

Since $\left(\underline{\underline{S}} - \underline{\underline{b}}\right)(t+dt)$ is in the same direction as $\left(\underline{\underline{S}} - \underline{\underline{b}}\right)_{trial}(t+dt)$, we have

$$\left(\underline{\underline{S}} - \underline{\underline{b}}\right)(t+dt) = \left(\sigma_y - \lambda\sigma_m\right)/s\frac{\left(\underline{\underline{S}} - \underline{\underline{b}}\right)_{trial}(t+dt)}{\left\|\underline{\underline{S}} - \underline{\underline{b}}\right\|_{trial}(t+dt)} \tag{A.6}$$

We now compare Eq.(A.4) and Eq.(A.6), the only solution is to have:

$$1+\eta = \frac{\left\|\underline{\underline{S}} - \underline{\underline{b}}\right\|_{trial}}{\left(\sigma_y - \lambda\sigma_m\right)/s} \tag{A.7}$$

that is:

$$\eta = \frac{\left\|\underline{\underline{S}} - \underline{\underline{b}}\right\|_{trial}}{\left(\sigma_y - \lambda\sigma_m\right)/s} - 1 \tag{A.8}$$

which is positive in plastic regime.

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3D STRESS TENSOR

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\begin{aligned}
\underline{\underline{\Sigma}} =& \frac{F_z}{A}\underline{e}_1 \otimes \underline{e}_1 + c_x\frac{F_x}{A}\underline{e}_\alpha \otimes \underline{e}_\alpha + c_y\frac{F_y}{A}\underline{e}_\beta \otimes \underline{e}_\beta \\
=& \frac{F_z}{A}\underline{e}_1 \otimes \underline{e}_1 + c_x\frac{F_x}{A}\left(cos\theta_x\underline{e}_1 + sin\theta_x cos\varphi_x\underline{e}_2 + sin\theta_x sin\varphi_x\underline{e}_3\right) \otimes \left(cos\theta_x\underline{e}_1 + sin\theta_x cos\varphi_x\underline{e}_2 + sin\theta_x sin\varphi_x\underline{e}_3\right) \\
&+ c_y\frac{F_y}{A}\left(cos\theta_y\underline{e}_1 + sin\theta_y cos\varphi_y\underline{e}_2 + sin\theta_y sin\varphi_y\underline{e}_3\right) \otimes \left(cos\theta_y\underline{e}_1 + sin\theta_y cos\varphi_y\underline{e}_2 + sin\theta_y sin\varphi_y\underline{e}_3\right) \\
=& \left(\frac{F_z}{A} + c_x\frac{F_x}{A}cos^2\theta_x + c_y\frac{F_zy}{A}cos^2\theta_y\right)\underline{e}_1 \otimes \underline{e}_1 \\
&+ \left(c_x\frac{F_x}{A}cos\theta_x sin\theta_x cos\varphi_x + c_y\frac{F_y}{A}cos\theta_y sin\theta_y cos\varphi_y\right)\left(\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1\right) \\
&+ \left(c_x\frac{F_x}{A}cos\theta_x sin\theta_x sin\varphi_x + c_y\frac{F_y}{A}cos\theta_y sin\theta_y sin\varphi_y\right)\left(\underline{e}_1 \otimes \underline{e}_3 + \underline{e}_3 \otimes \underline{e}_1\right) \\
&+ \left(c_x\frac{F_x}{A}sin^2\theta_x cos^2\varphi_x + c_y\frac{F_y}{A}sin^2\theta_y cos^2\varphi_y\right)\underline{e}_2 \otimes \underline{e}_2 \\
&+ \left(c_x\frac{F_x}{A}sin^2\theta_x cos\varphi_x sin\varphi_x + c_y\frac{F_y}{A}sin^2\theta_y cos\varphi_y sin\varphi_y\right)\left(\underline{e}_2 \otimes \underline{e}_3 + \underline{e}_3 \otimes \underline{e}_2\right) \\
&+ \left(c_x\frac{F_x}{A}sin^2\theta_x sin^2\varphi_x + c_y\frac{F_y}{A}sin^2\theta_y sin^2\varphi_y\right)\underline{e}_3 \otimes \underline{e}_3
\end{aligned}
\tag{A.9}
$$