MA1521 Cheatsheet
for AY23-24, Sem 1

m. **zaidan**

$$\lim_{x\to\infty}\frac{Ax^a+...}{Bx^b+...}=\begin{cases} 0 & a<b \\ \frac{A}{B} & a=b \\ \infty/-\infty & a>b\end{cases}$$

$\lim_{x\to c} g(x)=0$,
$\lim_{x\to c}\frac{sin(g(x))}{g(x)}=1 \mid \lim_{x\to c}\frac{tan(g(x))}{g(x)}=1$

if $g(x)\le f(x)\le h(x)$ for all x for some (a,b),
$\lim_{x\to c} g(x)=\lim_{x\to c} h(x)=L$,
$\lim_{\mathbf{x}\to\mathbf{c}}\mathbf{f(x)}=\mathbf{L}$

Derivatives

$\frac{d}{dx}\sin(x)=\cos(x)$ $\quad \frac{d}{dx}\sin^{-1}(x)=\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\cos(x)=-\sin(x)$ $\quad \frac{d}{dx}\cos^{-1}(x)=\frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\tan(x)=\sec^2(x)$ $\quad \frac{d}{dx}\tan^{-1}(x)=\frac{1}{1+x^2}$

$\frac{d}{dx}\sec(x)=\sec(x)\tan(x)$

$\frac{d}{dx}\csc(x)=-\csc(x)\cot(x)$

$\frac{d}{dx}\cot(x)=-\csc^2(x)$

$\frac{d}{dx}\cot^{-1}(x)=\frac{-1}{1+x^2}$

$\frac{d}{dx}\sec^{-1}(x)=\frac{1}{|x|\sqrt{x^2-1}}$

$\frac{d}{dx}\csc^{-1}(x)=\frac{-1}{|x|\sqrt{x^2-1}}$

$\frac{d}{dx}g(y)=g'(y)\frac{dy}{dx}$

$(f^{-1})(f(x)))=\frac{1}{f'(x)}$

Quotient rule: $\frac{d}{dx}\frac{v}{u}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

Parametric Equations $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \mid \frac{d^2y}{dx^2}=\frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$

$y=f(x)^{g(x)}$: get $\frac{d}{dx}ln(y)$ and solve for dy/dx

if f is differentiable on (a,b) continuous on $[a,b]$
$f'(x)>0\to$ **increasing** | $f'(x)<0\to$ **decreasing**
$f''(c)>0\to$ **concave upward** on c
$f''(c)<0\to$ **concave downward** on c
$f''(c)=0\to$ **point of inflection** on $(c,f(c))$

Maximum & Minimum

$\forall x\in dom(f), f(x)\le f(c)$: **absolute maximum**
$\forall x\in dom(f), f(x)\ge f(c)$: **absolute minimum**
$\forall x\in(a,b), c\in(a,b), f(x)\le f(c)$: **local maximum**
$\forall x\in(a,b), c\in(a,b), f(x)\ge f(c)$: **local minimum**

Extreme Value Theorem: if f is continuous on $[a,b]$, f attains an absolute max and min somewhere.
Critical Point: not an end point & $f'(c)=0\vee f'(c)$ does not exist

1st Derivative for Absolute Extrema

$f'(x)>0\,\forall\, x<c \quad f'(x)<0\,\forall\, x>c$: **abs max**
$f'(x)<0\,\forall\, x<c \quad f'(x)>0\,\forall\, x>c$: **abs min**

1st Derivative for Local Extrema

f' changes from $+$ to $-$: **local max**
f' changes from $-$ to $+$: **local min**

2nd Derivative for Local Extrema

$f'(c)=0 \quad f''(c)<0$: **local max**
$f'(c)=0 \quad f''(c)>0$: **local min**

L'Hopital Rule if $\lim_{x\to c}\frac{f(x)}{g(x)}=0$ or ∞,
$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$

Standard integration

$\int\sin(x)dx=-\cos(x)+C$
$\int\cos(x)dx=\sin(x)+C$
$\int\tan(x)dx=\ln|\sec(x)|+C$
$\int\sec(x)dx=\ln|\sec(x)+\tan(x)|+C$
$\int\csc(x)dx=-\ln|\csc(x)+\cot(x)|+C$
$\int\cot(x)dx=-\ln|\csc(x)|+C$
$\int\sec^2(x)dx=\ln|\tan(x)|+C$
$\int\csc^2(x)dx=-\cot(x)+C$
$\int\sec(x)\tan(x)dx=\ln|\sec(x)|+C$
$\int\csc(x)\cot(x)dx=\ln|\csc(x)|+C$
$\int\frac{1}{a^2+(x+b)^2}dx=\frac{1}{a}\tan^{-1}(\frac{x+b}{a})+C$
$\int\frac{1}{\sqrt{a^2-(x+b)^2}}dx=\sin^{-1}(\frac{x+b}{a})+C$
$\int-\frac{1}{\sqrt{a^2-(x+b)^2}}dx=\cos^{-1}(\frac{x+b}{a})+C$
$\int\frac{1}{a^2-(x+b)^2}dx=\frac{1}{2a}\ln|\frac{x+b+a}{x+b-a}|+C$
$\int\frac{1}{(x+b)^2-a^2}dx=\frac{1}{2a}\ln|\frac{x+b-a}{x+b+a}|+C$
$\int\frac{1}{\sqrt{(x+b)^2+a^2}}dx=\ln|(x+b)+\sqrt{(x+b)^2+a^2}|+C$
$\int\frac{1}{\sqrt{(x+b)^2-a^2}}dx=\ln|(x+b)+\sqrt{(x+b)^2-a^2}|+C$
$\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$
$\int\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}|+C$

Trigonometric Identities

$\sec^2x-1=\tan^2x$ $\quad \sin(\frac{\pi}{2}-A)=\cos A$
$\csc^2x-1=\cot^2x$ $\quad \cos(\frac{\pi}{2}-A)=\sin A$
$\sin A\cos A=\frac{1}{2}\sin 2A$ $\quad \tan(\frac{\pi}{2}-A)=\cot A$
$\cos^2A=\frac{1}{2}(1+\cos 2A)$
$\sin^2A=\frac{1}{2}(1-\cos 2A)$
$\sin A\cos B=\frac{1}{2}(\sin(A+B)+\sin(A-B))$
$\cos A\sin B=\frac{1}{2}(\sin(A+B)-\sin(A-B))$
$\cos A\cos B=\frac{1}{2}(\cos(A+B)+\cos(A-B))$
$\sin A\sin B=\frac{1}{2}(\cos(A-B)-\cos(A+B))$
$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$
$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$
$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

Substitution: Let $u=g(x),\therefore du=g'(x)dx$,
$\int f(g(x))g'x(dx)=\int f(u)du$
Integration: Let $f(x)=g(x)h'(x)$,
$\int f(x)dx=g(x)h(x)-\int g'(x)h(x)dx$

Partial fractions:

$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^2$	$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$
$ax^2+bx+c, b^2-4ac<0$	$\frac{Ax+B}{ax^2+bx+c}$

Rules for integration:

Logarithmic	Differentiate
Inverse trigonometric	Differentiate
Algebraic (Polynomials)	Differentiate
Trigonometric	Differentiate
Combination of Trigonometric	Integrate
Exponential	Integrate

Riemann Sum: for sufficiently large n,
$\int_a^b f(x)dx\approx\sum_{k+1}^{n}(\frac{b-a}{n})+f(a+k(\frac{b-a}{n}))$

FTC1: If f is continuous on $[a,b]$ & F is anti-derivative of f, $\int_a^b F'(x)dx=F(b)-F(a)$
FTC2: If f is continuous on $[a,b]$ &
$g(x)=\int_a^x f(t)dt, a\le x\le b, f(x)=g'(x)$
$\frac{d}{dx}\int_a^x f(t)dt=f(x)$

Type I: Infinite limits of integration

$f(x)$ continuous on $[a,\infty)$:
$\int_a^\infty f(x)dx=lim_{b\to\infty}\int_a^b f(x)dx$
$f(x)$ continuous on $(-\infty,b]$:
$\int_{-\infty}^b f(x)dx=lim_{a\to-\infty}\int_a^b f(x)dx$
$f(x)$ continuous on $(-\infty,\infty)$:
$\int_{-\infty}^\infty f(x)dx=\int_{-\infty}^c f(x)dx+\int_c^\infty f(x)dx$

Type II: Integrals that become ∞ somewhere

$f(x)$ continuous on $(a,b]$, discontinuous at a:

$\int_a^b f(x)dx = lim_{c\to a^+}\int_c^b f(x)dx$

$f(x)$ continuous on $[a,b)$, discontinuous at b:

$\int_a^b f(x)dx = lim_{c\to b^+}\int_a^c f(x)dx$

$f(x)$ discontinuous at $c, a<c<b$:

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Area between curves

$A = \int_a^b |f(x)-g(x)|dx$

$A = \int_a^b |f(y)-g(y)|dy$

Volume of Solid of Revolution

Disk $f(x) > g(x) \forall x \in [a,b]$,

$V = \pi(\int_a^b f(x)^2 - g(x)^2)dx$

Cylindrical $V = 2\pi(\int_a^b x|f(x)-g(x)|)dx$

Arc Length of a Curve $\int_a^b \sqrt{1+f'(x)^2}dx$

Squeeze Theorem for Sequence

$a_n \le b_n \le c_n \wedge \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

$\to lim_{n\to\infty} b_n = L$

Series Limit $\sum_{n=1}^\infty a_n = \lim_{n\to\infty} S_n$

$\in \mathbb{R}$ (converges), $\notin \mathbb{R}$ (diverges)

Test for Convergence & Divergence $\sum_{n=1}^\infty a_n$

n^{th} term	$lim_{n\to\infty} a_n \ne 0$ div, else inconclusive
integral $f(n) = a_n \ge 0$	$lim_{b\to\infty} \int_1^b f(x)$ check for convergence ($\ne \infty$)
comparison $0 \le a_n \le b_n$	$\sum_{n+1}^\infty b_n$ conv $\Rightarrow \sum_{n+1}^\infty a_n$ conv $\sum_{n+1}^\infty a_n$ div $\Rightarrow \sum_{n+1}^\infty b_n$ div
geometric series *p-series*	$\sum_{n=1}^\infty ar^{n-1}$ conv $\iff |r|<1$ $\sum_{n=1}^\infty \frac{1}{n^p}$ conv $\iff p<1$
ratio/root $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ $L = \lim_{n\to\infty} \sqrt[n]{|a_n}$	$L<1 \iff \sum_{n=1}^\infty a_n$ conv $L>1 \iff \sum_{n=1}^\infty a_n$ div inconclusive otherwise.
alternating series $b_n \ge 0, b_n \ge b_{n+1}$	$\lim_{n\to\infty} b_n = 0 \Rightarrow$ $\sum_{n=1}^\infty (-1)^{n-1} b_n$ conv

Power Series $\sum_{n=0}^\infty C_n(x-a)^n$

Radius of convergence $R: \lim_{n\to\infty} \frac{C_{n+1}}{C_n} = \frac{1}{R}$

$|x-a| > R \to$ conv, $|x-a| < R \to$ div

Taylor series $f(x) = \sum_{n=0}^\infty \frac{f^n(a)}{n!}(x-a)^n$

$\frac{1}{1-x} = \sum_{n=0}^\infty x^n, |x|<1$

$\frac{1}{1+x} = \sum_{n=0}^\infty (-1)^n x^n, |x|<1$

$e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$

$\sin(x) = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$\cos(x) = \sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{(2n)!}$

$\tan^{-1}(x) = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{2n+1}$

$\ln(1-x) = \sum_{n=0}^\infty \frac{x^{n+1}}{n+1}, |x|<1$

$\ln(1+x) = \sum_{n=0}^\infty (-1)^n \frac{x^{n+1}}{n+1}, |x|<1$

Vector $< x_2-x_1, y_2-y_1, z_2-z_1 >$

$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

Length $||\vec{a}||$

Standard Vectors

$\mathbf{i} =< 1,0,0 >, \mathbf{j} =< 0,1,0 >, \mathbf{k} =< 0,0,1 >$

Dot Product $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$

$a \cdot b = ||a|| \cdot ||b|| \cdot \cos\theta$

$\theta = 0$ (same), $\theta = \pi$ (opp), $\theta = \frac{\pi}{2}$ $(\perp)$

$a \cdot b = 0 \iff a \perp b$

Eqn. of Plane & Line

Line: $r(t) =< x_0, y_0, z_0 > + t(a,b,c)$

Plane: $ax + by + cz = d, (r - r_0) \cdot < a,b,c >= 0$

Distance: $\frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$

Cross Product $a \times b =$

$(a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$

$a \times b \perp a \wedge a \times b \perp b$

$||a|| \cdot ||b|| \cdot \sin\theta = ||a \times b||$

Vector Function $r'(t) =< f'(t), g'(t), h'(t) >$

Arc Length $= s = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}$

$f(x,y)$: $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$

Clairaut's Theorem: $f_{xy} = f_{yx}$

$\nabla \mathbf{f} =< f_x, f_y >$

$D_\mathbf{u} f =< f_x, f_y > \cdot \mathbf{u}$

Tangent Plane:

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$f(x,y,z)$: $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$, , $\frac{\partial f}{\partial z} = f_z$

$\nabla \mathbf{f} =< f_x, f_y, f_z >$

$D_\mathbf{u} f =< f_x, f_y, f_z > \cdot \mathbf{u}$

Tangent Plane:

$z = \nabla f(a,b,c) \cdot < x-a, y-b, z-c >= 0$

Chain Rule: $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

$x = g(s,t), y = h(s,t)$

$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ & $\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$

$\frac{\partial z}{\partial x} = \frac{-F_x(x,y,z)}{F_z(x,y,z)}$ & $\frac{\partial z}{\partial y} = \frac{-F_y(x,y,z)}{F_z(x,y,z)}$

$\triangle z = f(x + \triangle x, y + \triangle y) - f(x,y)$

$dz = f_x(x,y)dx + f_y(x,y)dy$

2^{nd} derivative test: $f_x(a,b) = 0 = f_y(a,b)$

$D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$

$D > 0 \wedge f_{xx}(a,b) > 0$ **(Local min)**

$D > 0 \wedge f_{xx}(a,b) < 0$ **(Local max)**

$D < 0$ **(saddle point)**, $D = 0$ **(no conclusion)**

$\int\int_R f(x)g(y)dA = \int_a^b f(x)dx \int_c^d g(y)dy$

$R: \{a < x < b, c < x < d\}$

Type I: $D: \{(x,y): a \le x \le b, g_1(x) \le y \le g_2(x)\}$

$\int\int_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$

Type II: $D: \{(x,y): g_1(y)a \le x \le g_2(y), a \le y \le b\}$

$\int\int_D f(x,y)dA = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)dxdy$

Polar Coordinates $x^2 + y^2 = r^2$

$x = r\cos\theta, y = r\sin\theta$

$R = \{(r,\theta): 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$

$\int\int_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r dr d\theta$

Surface Area $\int\int_D dS = \int\int_D \sqrt{f_x^2 + f_y^2 + 1} dA$

Reduction to Separable Form $y' = g(\frac{y}{x})$, let $v = \frac{y}{x}$

$y' = v + xv'$

First Order ODE $\frac{dy}{dx} + P(x)y = Q(x)$

$I(X) = e^{\int P(x)dx}$

$y = \frac{\int I(x) \cdot Q(x)dx}{I(x)}$

Bernoulli Equation $\frac{dy}{dx} + P(x)y = Q(x)y^n, u = y^{1-n}$

$u' + (1-n)(p(x)u = (1-n)q(x)$