# MA 1521

calculus for computing

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# LIMITS

## CONTINUITY

CASE 1 : Interior point

(i) $\lim_{x \to c} f(x)$ exists (ii) $\lim_{x \to c} f(x) = f(c)$

CASE 2 : Left end-point

(i) $\lim_{x \to c^+} f(x)$ exists (ii) $\lim_{x \to c^+} f(x) = f(c)$

CASE 3 : Right end-point

(i) $\lim_{x \to c^-} f(x)$ exists (ii) $\lim_{x \to c^-} f(x) = f(c)$

## LIMITS AT INFINITY

Indeterminate forms

$\frac{0}{0}$ $\left( \lim_{x \to c} \frac{f(x)}{g(x)} \right.$, $f(x) \to 0$ & $g(x) \to 0$ as $x \to c$ )

$\frac{\infty}{\infty}$ $\left( \lim_{x \to c} \frac{f(x)}{g(x)} \right.$, $f(x) \to \infty$ & $g(x) \to \infty$ as $x \to c$ )

$$\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)} = \lim_{x \to \pm\infty} \frac{Ax^{\alpha} + \dots}{Bx^{\beta} + \dots} = \begin{cases} 0 & \text{if } \alpha < \beta \\ \frac{A}{B} & \text{if } \alpha = \beta \\ \underbrace{\infty \text{ or } -\infty}_{\text{sign of } A \text{ and } B} & \text{if } \alpha > \beta \end{cases}$$

## WAYS TO SOLVE LIMITS

**REPLACEMENT RULE** factorise to ensure denominator is non-zero.

e.g. $$\lim_{x \to 6} \frac{x^2-7x+6}{36-x^2} = \frac{(x-1)\cancel{(x-6)}}{-(6+x)\cancel{(x-6)}}$$

**sin & tan**

if $\lim_{x \to c} g(x) = 0$,

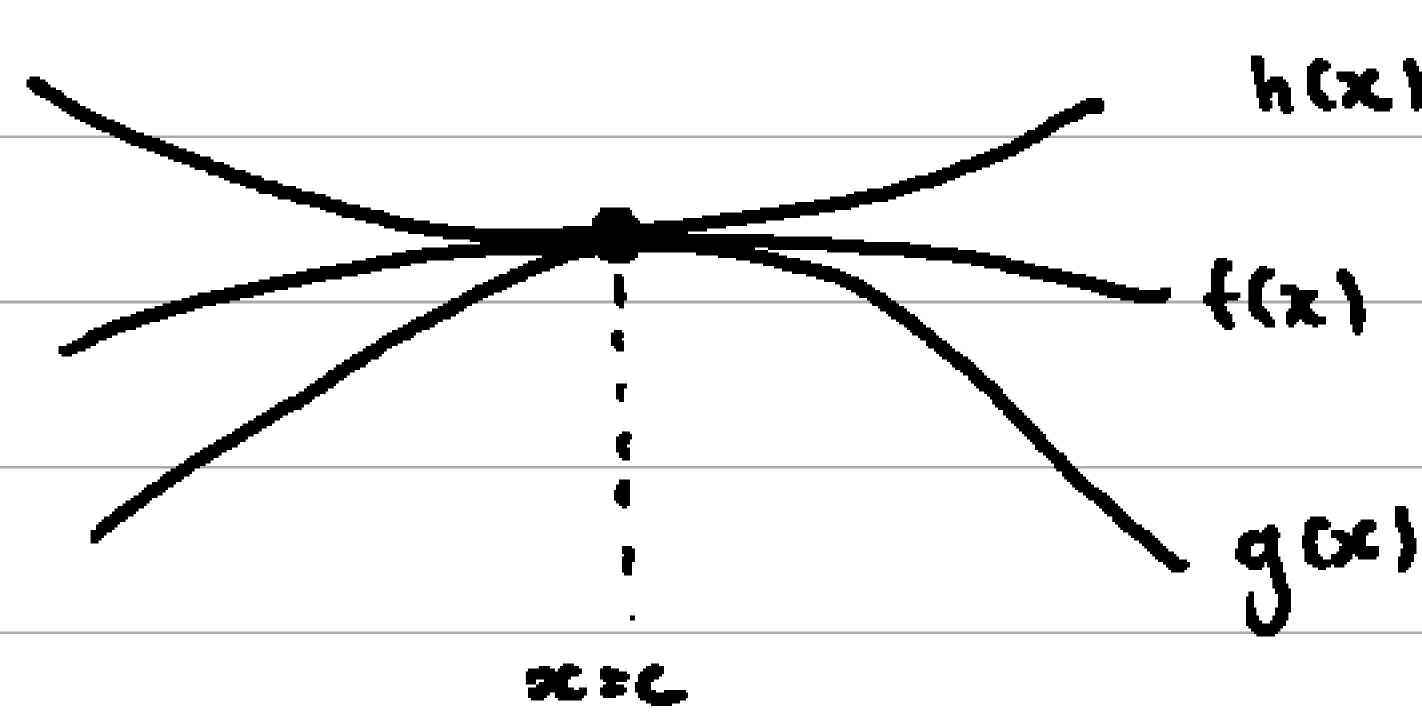
$$\lim_{x \to c} \frac{\sin(g(x))}{g(x)} = \lim_{x \to c} \frac{g(x)}{\sin(g(x))} = 1$$

$$\lim_{x \to c} \frac{\tan(g(x))}{g(x)} = \lim_{x \to c} \frac{g(x)}{\tan(g(x))} = 1$$

**SQUEEZE THEOREM**

$$g(x) \le f(x) \le h(x)$$
$\forall$ x in some open interval except possibly at c.

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L \rightarrow \lim_{x \to c} f(x) = L$$



# DERIVATIVES

$$\frac{d}{dx} f(x) = f'(x) = \frac{dy}{dx} = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f$ is differentiable @ $x = x_0 \Rightarrow f$ is continuous @ $x = x_0$

## STANDARD DERIVATIVES

| Function | Derivative |
|---|---|
| $x^n$ | $nx^{n-1}$ |
| $\cos(x)$ | $-\sin(x)$ |
| $\sin(x)$ | $\cos(x)$ |
| $\tan(x)$ | $\sec^2(x)$ |
| $\sec(x)$ | $\sec(x)\tan(x)$ |
| $\csc(x)$ | $-\csc(x)\cot(x)$ |
| $\cot(x)$ | $-\csc^2(x)$ |
| $e^x$ | $e^x$ |
| $\ln(x)$ | $\frac{1}{x}$ |
| $\sin^{-1}(x)$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\cos^{-1}(x)$ | $\frac{-1}{\sqrt{1-x^2}}$ |
| $\tan^{-1}(x)$ | $\frac{1}{1+x^2}$ |
| $\cot^{-1}(x)$ | $-\frac{1}{1+x^2}$ |
| $\sec^{-1}(x)$ | $\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$, $\lvert x\rvert > 1$ |
| $\csc^{-1}(x)$ | $-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$, $\lvert x\rvert > 1$ |

QUOTIENT RULE: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$

## DERIVATIVES OF INVERSE FUNCTIONS

$b = f^{-1}(a) \qquad a = f(b)$

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))} = \frac{1}{f'(b)}$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

## PARAMETRIC EQUATIONS

$x = f(t)$ & $y = g(t)$

$$\frac{dy}{dt} = \frac{dy}{dx}\left(\frac{dx}{dt}\right) \qquad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \div \frac{dx}{dt}$$

**CIRCLES**

$$(x-x_0)^2 + (y-y_0)^2 = r^2$$

**ELLIPSES**

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

**HYPERBOLAS**

$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1$$

$$\frac{(y-y_0)^2}{a^2} - \frac{(x-x_0)^2}{b^2} = 1$$

## MISCELLANOUS EXAMPLES

$y = f(x)^{g(x)}$

Step 1: Find derivative of $\ln y$

2: Solve for $\frac{dy}{dx}$

$\ln y = g(x) \ln f(x)$

$\frac{1}{y}\frac{dy}{dx} = g'(x) \ln f(x) + \frac{f'(x)g(x)}{f(x)}$

....

Change base formula:

LOG LAW

↳ $\log_a x = \frac{\ln x}{\ln a}$

# APPLICATIONS OF DIFFERENTIATION

## TANGENTS & NORMAL

Tangent: $y - f(x_0) = m(x - x_0)$ $\quad (m = f'(x_0))$

Normal: $y - f(x_0) = -\frac{1}{m}(x - x_0)$

## INCREASING & DECREASING

$f$ is increasing if $\forall x_1, x_2 \in I \ (x_2 > x_1 \rightarrow f(x_2) > f(x_1))$

decreasing if $\forall x_1, x_2 \in I \ (x_2 > x_1 \rightarrow f(x_1) > f(x_2))$

INCREASING on $[a,b]$ if $f'(x) > 0 \ \forall x$ in $(a,b)$

DECREASING on $[a,b]$ if $f'(x) < 0 \ \forall x$ in $(a,b)$

Increasing / Decreasing $\longrightarrow$ Injective

## CONCAVE UPWARD & DOWNWARD

$f$ is CONCAVE UPWARD if $f''(c) > 0$

CONCAVE DOWNWARD if $f''(c) < 0$

**POINT OF INFLECTION:**

$(c, f(c))$ is a point of inflection & $f''(c)$ exists $\rightarrow f''(c) = 0$

## EXTREMA

**Absolute** maximum at $x=c$ if $f(x) \leq f(c)$ $\forall x \in \mathrm{dom}(f)$

minimum at $x=c$ if $f(x) \geq f(c)$ $\forall x \in \mathrm{dom}(f)$

**Local** maximum at $x=c$ if $f(x) \leq f(c)$ $\forall x \in (a,b)$, $c \in (a,b)$

minimum at $x=c$ if $f(x) \leq f(c)$ $\forall x \in (a,b)$, $c \in (a,b)$

### EXTREME VALUE THEOREM

$f$ is continuous on $[a,b]$ $\rightarrow$ $f$ has a absolute maximum / minimum at some point in $[a,b]$

$f$ is differentiable on $(a,b)$
& $f$ has a local maximum | minimum) $\rightarrow$ $f'(c)=0$

### CRITICAL POINT if

(i) not end-point

(ii) $f'(c)=0$ or $f'(c)$ does not exist

Local maximum / minimum $\longrightarrow$ critical point

# DERIVATIVE TESTS FOR EXTREMA

## FIRST DERIVATIVE TESTS FOR ABSOLUTE EXTREMA

Absolute **MAXIMUM** $f'(x) > 0 \ \forall x < c$ & $f'(x) < 0 \ \forall x > c$

**MINIMUM** $f'(x) < 0 \ \forall x < c$ & $f'(x) > 0 \ \forall x > c$

## FIRST DERIVATIVE TESTS FOR LOCAL EXTREMA

Local **MAXIMUM** $f'$ changes from +ve to -ve

**MINIMUM** $f'$ changes from -ve to +ve

**NO LOCAL EXTREMA** $f'$ does not change sign

## SECOND DERIVATIVE TESTS FOR LOCAL EXTREMA

Local **MAXIMUM** $f'(c) = 0$ & $f''(c) < 0$

**MINIMUM** $f'(c) = 0$ & $f''(c) > 0$

**NO CONCLUSION** $f''(c) = 0$

## L'HOPITAL RULE

if $\lim_{x \to c} f(x) = 0 = \lim_{x \to c} g(x)$ or

$\lim_{x \to c} f(x) = \infty = \lim_{x \to c} g(x)$,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

### INDETERMINATE FORMS

$$0^0, \ \infty^0, \ 1^\infty$$

## ROLLE'S THEOREM

$f$ continuous on $[a,b]$ & differentiable on $[a,b]$

$$f(a) = f(b), \ \therefore \ \exists \ c \in (a,b) \ (f'(c) = 0)$$

### MEAN VALUE THEOREM

$$\exists \, c \in (a,b) \ \left( f'(c) = \frac{f(b) - f(a)}{b-a} \right)$$

# INTEGRATION

## STANDARD INTEGRALS

1. $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C \quad (n \neq -1)$
2. $\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln |ax+b| + C$
3. $\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$
4. $\int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + C$
5. $\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + C$
6. $\int \tan(ax+b) \, dx = \frac{1}{a} \ln |\sec(ax+b)| + C$
7. $\int \sec(ax+b) \, dx = \frac{1}{a} \ln |\sec(ax+b) + \tan(ax+b)| + C$
8. $\int \csc(ax+b) \, dx = -\frac{1}{a} \ln |\csc(ax+b) + \cot(ax+b)| + C$
9. $\int \cot(ax+b) \, dx = -\frac{1}{a} \ln |\csc(ax+b)| + C$
10. $\int \sec^2(ax+b) \, dx = \frac{1}{a} \ln |\tan(ax+b)| + C$
11. $\int \csc^2(ax+b) \, dx = -\frac{1}{a} \cot(ax+b) + C$
12. $\int \sec(ax+b) \tan(ax+b) \, dx = \frac{1}{a} \sec(ax+b) + C$
13. $\int \csc(ax+b) \cot(ax+b) \, dx = -\frac{1}{a} \csc(ax+b) + C$
14. $\int \frac{1}{a^2+(x+b)^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x+b}{a}\right) + C$
15. $\int \frac{1}{\sqrt{a^2-(x+b)^2}} \, dx = \sin^{-1}\left(\frac{x+b}{a}\right) + C$
16. $\int \frac{-1}{\sqrt{a^2-(x+b)^2}} \, dx = \cos^{-1}\left(\frac{x+b}{a}\right) + C$
17. $\int \frac{1}{a^2-(x+b)^2} \, dx = \frac{1}{2a} \ln \left| \frac{x+b+a}{x+b-a} \right| + C$
18. $\int \frac{1}{(x+b)^2-a^2} \, dx = \frac{1}{2a} \ln \left| \frac{x+b-a}{x+b+a} \right| + C$
19. $\int \frac{1}{\sqrt{(x+b)^2+a^2}} \, dx = \ln \left| (x+b) + \sqrt{(x+b)^2+a^2} \right| + C$

20. $\int \frac{1}{\sqrt{(x+b)^2-a^2}}\,dx = \ln\left|(x+b) + \sqrt{(x+b)-a^2}\right| + c$

21. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

22. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

## IMPORTANT TRIGONOMETRIC IDENTITIES

$\sec^2 x - 1 = \tan^2 x$ $\qquad$ $\csc^2 x - 1 = \cot^2 x$

$\sin A\cos A = \frac{1}{2}\sin 2A$ $\qquad$ $\cos^2 A = \frac{1}{2}(1+\cos 2A)$ $\qquad$ $\sin^2 A = \frac{1}{2}(1-\cos 2A)$

$\sin A\cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$

$\cos A\sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$

$\cos A\cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$

$\sin A\sin B = -\frac{1}{2}(\cos(A+B) - \cos(A-B))$

## PARTIAL FRACTIONS

When $\int \frac{P(x)}{Q(x)}\,dx$, where degree of $P(x)$ > degree of $Q(x)$

| Factors of $Q(x)$ | Partial fractions |
|---|---|
| $ax+b$ | $\frac{A}{ax+b}$ |
| $(ax+b)^2$ | $\frac{A}{ax+b} \quad \frac{B}{(ax+b)^2}$ |
| $ax^2+bx+c$, $b^2-4ac<0$ | $\frac{Ax+B}{ax^2+bx+c}$ |

## INTEGRATION BY SUBSTITUTION

Let $u = g(x)$, $\therefore du = g'(x)\,dx$

$\therefore \int f(g(x))\,g'(x)\,dx = \int f(u)\,du$

## INTEGRATION BY PARTS

$\int f(x)$ , let $f(x) = g(x)\,h'(x)$

$$\int f(x)dx = \int g(x)h'(x)dx$$
$$= g(x)h(x) - \int g'(x)h(x)dx$$

## GENERAL RULES ON FUNCTION TO INTEGRATE / DIFFERENTIATE

| | |
|---|---|
| Logarithmic | differentiate |
| Inverse trigonometric | differentiate |
| Algebraic ( polynomials ) | differentiate |
| Trigonometric | differentiate |
| Combination of trigonometric | integrate |
| Exponential | integrate |

## RIEMANN SUM

for sufficiently large $n$,

$$\int_a^b f(x)dx \approx \sum_{k=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + k\left(\frac{b-a}{n}\right)\right)$$

## FUNDAMENTAL THEOREM OF CALCULUS

**FTC 1**

If $f$ is continuous on $[a,b]$, & $F$ is anti-derivative of $f$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\Longleftrightarrow \int_a^b F'(x)\,dx = F(b) - F(a)$$

**FTC 2**

If $f$ is continuous on $[a,b]$,

$$g(x) = \int_a^x f(t)\,dt \quad , \quad a \le x \le b$$

↑ continuous & differentiable on $(a,b)$

$$g'(x) = f(x)$$

$$\therefore \quad \frac{d}{dx} \int_a^x f(t)\,dt = f(x)$$

## IMPROPER INTEGRALS

**TYPE 1**: Integrals with infinite limits of integration

$f(x)$ continuous on $[a, \infty)$

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

$f(x)$ continuous on $(-\infty, b]$

$$\int_{-\infty}^b f(x)\,dx = \lim_{a \to \infty} \int_a^b f(x)\,dx$$

$f(x)$ continuous on $(-\infty, \infty)$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

TYPE II: Integrals of functions that become infinite at a point within the interval of integration

If $f(x)$ continuous on $(a,b]$, discontinuous at $a$,

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx$$

$f(x)$ continuous on $[a,b)$, discontinuous at $b$,

$$\int_a^b f(x)dx = \lim_{c \to b^+} \int_a^c f(x)dx$$

$f(x)$ discontinuous at $c$ with $a < c < b$,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

# APPLICATIONS OF INTEGRATION

## AREA BETWEEN CURVES

$f(x) \geq g(x) \quad \forall x \text{ in } [a,b]$

$$A = \int_a^b f(x) - g(x) \, dx$$

else :

$$A = \int_a^b |f(x) - g(x)| \, dx$$

y-axis

$f(y) \geq g(y) \quad \forall y \text{ in } [a,b]$

$$A = \int_a^b f(y) - g(y) \, dy$$

else :

$$A = \int_a^b |f(y) - g(y)| \, dy$$

x-axis

## VOLUME OF SOLID OF REVOLUTION

### DISK METHOD

$f(x) > g(x) \quad \forall x \text{ in } [a,b]$

$$V = \pi \left( \int_a^b f(x)^2 - g(x)^2 \, dx \right)$$

### CYLINDRICAL METHOD

$$V = 2\pi \int_a^b x \, |f(x) - g(x)| \, dx$$

## ARC LENGTH OF A CURVE

$$\int_a^b \sqrt{1 + f'(x)^2} \, dx$$

# SEQUENCES AND SERIES

## LIMITS OF SEQUENCES

if $\{a_n\}$ such that $f(n) = a_n$,

$$\lim_{x\to\infty} f(x) = L \quad \rightarrow \quad \lim_{n\to\infty} a_n = L$$

## SQUEEZE THEOREM FOR SEQUENCE

$$a_n \le b_n \le c_n \quad \wedge \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

$$\rightarrow \quad \lim_{n\to\infty} b_n = L.$$

## SERIES LIMIT

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$$

if $\lim_{n\to\infty} S_n \in \mathbb{R}$ (converges)

$\notin \mathbb{R}$ (diverges

# TEST FOR CONVERGENCE & DIVERGENCE

$$\sum_{n=1}^{\infty} a^n$$

**$n^{th}$ Term Test**

$\lim_{n\to\infty} a_n \neq 0 \Rightarrow$ diverges

$\lim_{n\to\infty} a_n = 0 \Rightarrow$ inconclusive

**Integral Test**

$f(n) = a_n \geq 0$

$\lim_{b\to\infty} \int_1^b f(x)dx$ converges $\Rightarrow$ converges

$\lim_{b\to\infty} \int_1^b f(x)dx$ diverges $\Rightarrow$ diverges

**Comparison Test**

$0 \leq a_n \leq b_n$

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

$\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} b_n$ diverges

**GEOMETRIC SERIES** $\sum_{n=1}^{\infty} ar^{n-1}$ converges $\Longleftrightarrow |r| < 1$

**p-SERIES** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\Longleftrightarrow p > 1$

**Ratio Test**

$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

or

**Root Test**

$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

$L < 1 \Rightarrow \sum_{n=1}^{\infty} |a_n|$ converges

$L > 1 \Rightarrow \sum_{n=1}^{\infty} |a_n|$ diverges

$L = 1$ : inconclusive

**Alternating Series Test**

$b_n \geq 0$ , $b_n \geq b_{n+1}$ ,

$\lim_{n\to\infty} b_n = 0 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges

# POWER SERIES

$$\sum_{n=0}^{\infty} C_n (x-a)^n$$

Radius of convergence $R$ $\qquad \lim_{n\to\infty} \left| \frac{C_{n+1}}{C_n} \right| = \frac{1}{R}$

$|x-a| < R$ converges

$|x-a| > R$ diverges

Taylor series $\qquad f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$

## TAYLOR SERIES

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \quad |x|<1$$

# VECTORS

**DISTANCE**

$P_1(x_1, y_1, z_1)$ & $P_2(x_2, y_2, z_2)$

$$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)}$$

**SPHERE**

center $(x_1, y_1, z_1)$, Radius $r$

$$(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 = r^2$$

**LENGTH**

$\|\vec{a}\|$

**STANDARD VECTORS**

$i = \langle 1,0,0\rangle$ $j = \langle 0,1,0\rangle$, $k = \langle 0,0,1\rangle$

**VECTOR FROM POINT-TO-POINT**

Given points $(x_1, y_1, z_1)$ & $(x_2, y_2, z_2)$ $a = \langle x_2-x_1, y_2-y_1, z_2-z_1\rangle$

**DOT PRODUCT**

$$a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$$

$\langle a_1, a_2, a_3\rangle$
$\langle b_1, b_2, b_3\rangle$

$$a \cdot b = \|a\| \|b\| \cos\theta$$

$\theta = 0 \Longleftrightarrow$ same direction
$\theta = \pi \Longleftrightarrow$ opposite direction
$\theta = \frac{\pi}{2} \Longleftrightarrow$ perpendicular

$$a \cdot b = 0 \Longleftrightarrow a \perp b$$

## EQUATION OF PLANE & LINE

LINE: $x = x_0 + t$, $y = y_0 + t$, $z = z_0 + t$

$$r(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$$

PLANE: $ax + by + cz = d$

$$(r - r_0) \cdot \langle a, b, c \rangle = 0$$

DISTANCE OF LINE FROM PLANE

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

## CROSS PRODUCT

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2 b_3 - a_3 b_2) i - (a_1 b_3 - a_3 b_1) j + (a_1 b_2 - a_2 b_1) k$$

$a \times b \perp a \wedge a \times b \perp b$

$$\|a\| \, \|b\| \sin \theta = \|a \times b\|$$

# FUNCTIONS
## OF SEVERAL VARIABLES

### VECTOR FUNCTION

$r(t) = \langle f(t), g(t), h(t) \rangle$

$r'(t) : \langle f'(t), g'(t), h'(t) \rangle$

### ARC LENGTH

$r(t) = \langle f(t), g(t), h(t) \rangle \quad a \leq t \leq b$

$$s = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt$$

$$= \int_a^b \| r'(t) \| \, dt$$

### FUNCTION OF 2 VARIABLES

$f(x,y)$ $\quad \frac{\partial f}{\partial x} = f_x \quad \frac{\partial f}{\partial y} = f_y$

Clairaut's Theorem $\quad f_{xy} = f_{yx}$

$\nabla f \quad \langle f_x, f_y \rangle$

$D_{\hat{u}} f \quad \langle f_x, f_y \rangle \cdot \hat{u}$

Tangent Plane $\quad z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

## FUNCTIONS OF 3 VARIABLES

$f(x,y,z)$ $\quad \frac{\partial f}{\partial x} = f_x \quad \frac{\partial f}{\partial y} = f_y \quad \frac{\partial f}{\partial z} = f_z$

$\nabla f$ $\quad \langle f_x, f_y, f_z \rangle$

$D_{\hat{u}} f$ $\quad \langle f_x, f_y, f_z \rangle \cdot \hat{u}$

Tangent Plane $\quad \nabla f(a,b,c) \cdot \langle x-a, y-b, z-c \rangle = 0$

## CHAIN RULE

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

If $x = g(s,t)$, $y = h(s,t)$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \qquad \frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

IMPLICIT DIFFERENTIATION

$$\frac{\partial z}{\partial x} = \frac{-F_x(x,y,z)}{F_z(x,y,z)} \qquad \frac{\partial z}{\partial y} = \frac{-F_y(x,y,z)}{F_z(x,y,z)}$$

## INCREMENTS & DIFFERENTIALS

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

## SECOND DERIVATIVE TEST

$f_x(a,b) = f_y(a,b) = 0 \qquad D = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$

$D > 0$ & $f_{xx}(a,b) > 0$ LOCAL MIN

$< 0$ MAX

$D < 0$ saddle point

$D = 0$ no conclusion

# DOUBLE INTEGRALS

## INTEGRAL AS A FUNCTION OF X · FUNCTION OF Y

$$\iint_R f(x)g(y)\, dA = \int_a^b f(x)dx \int_c^d g(y)dy$$

R: $a < x < b$, $c < x < d$

## GENERAL REGION

TYPE 1: $D = \{(x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$

$$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dydx$$

TYPE 2: $D = \{(x,y) : g_1(y) \le x \le g_2(y), a \le \cancel{x}y \le b\}$

$$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(\ )}^{g_2(\ )} f(x,y)\, dxdy$$

## POLAR COORDINATES

$x^2 + y^2 = r^2$

$x = r\cos\theta$, $y = r\sin\theta$

$R : \{(r,\theta) : 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, drd\theta$$

## SURFACE AREA

$$\iint_D dS = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

# ORDINARY DIFFERENTIAL EQUATION

## REDUCTION TO SEPARABLE FORM

$y' = g\left(\frac{y}{x}\right)$, let $v = \frac{y}{x}$

$y' = v + xv'$

## FIRST ORDER ODE

$\frac{dy}{dx} + P(x)y = Q(x)$

$I(x) = e^{\int P(x)dx}$

$y = \frac{\int I(x) \cdot Q(x)\,dx}{I(x)}$

## BERNOULLI EQUATION

$\frac{dy}{dx} + P(x)y = Q(x)y^n$

$u = y^{1-n}$

$u' + (1-n)\,p(x)u = (1-n)\,q(x)$