Crank-Nicolson Method for 1D Diffusion-Advection Equation

Solving the Equation:

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = d\frac{\partial^2 C}{\partial x^2}$$

C = C(x,t), u = dL/dt, and d is a parameter.

Crank-Nicolson:

$$\frac{\partial C}{\partial t} \approx \frac{C_i^{n+1} - C_i^n}{\tau}$$

$$\frac{\partial^2 C}{\partial x^2} \approx \frac{1}{2h^2}\left((C_{i+1}^{n+1} - 2C_i^{n+1} + C_{i-1}^{n+1}) + (C_{i+1}^n - 2C_i^n + C_{i-1}^n)\right)$$

$$\frac{\partial C}{\partial x} \approx \frac{1}{2}\left(\left(\frac{C_{i+1}^{n+1} - C_{i-1}^{n+1}}{2h}\right) + \left(\frac{C_{i+1}^n - C_{i-1}^n}{2h}\right)\right)$$

Substitute these terms into the diffusion-advection equation:

$$\frac{C_i^{n+1} - C_i^n}{\tau} + \frac{u}{2}\left(\left(\frac{C_{i+1}^{n+1} - C_{i-1}^{n+1}}{h}\right) + \left(\frac{C_{i+1}^n - C_{i-1}^n}{h}\right)\right) = \frac{d}{2h^2}\left((C_{i+1}^{n+1} - 2C_i^{n+1} + C_{i-1}^{n+1}) + (C_{i+1}^n - 2C_i^n + C_{i-1}^n)\right)$$

Isolate the terms depending on the new time (n+1) on the right-hand-side:

$$\frac{C_i^{n+1}}{\tau} + \frac{u}{4h}(C_{i+1}^{n+1} - C_{i-1}^{n+1}) - \frac{d}{2h^2}(C_{i+1}^{n+1} - 2C_i^{n+1} + C_{i-1}^{n+1}) = \frac{C_i^n}{\tau} - \frac{u}{4h}(C_{i+1}^n - C_{i-1}^n) + \frac{d}{2h^2}(C_{i+1}^n - 2C_i^n + C_{i-1}^n)$$

Letting $a \equiv \frac{\tau u}{4h}$, $b \equiv \frac{\tau d}{2h^2}$, and **grouping** like terms:

$$(1+2b)C_i^{n+1} + (a-b)C_{i+1}^{n+1} - (a+b)C_{i-1}^{n+1} = (1-2b)C_i^n + (b-a)C_{i+1}^n + (a+b)C_{i-1}^n$$

Formation as Matrix Equation

$$\begin{bmatrix} 1+2b & a-b & & & & \\ -a-b & 1+2b & a-b & & & \\ & -a-b & 1+2b & a-b & & \\ & & . & . & . & \\ & & & . & . & . \\ & & & & -a-b & 1+2b \end{bmatrix} \begin{bmatrix} C_1^{n+1} \\ C_2^{n+1} \\ . \\ . \\ C_N^{n+1} \end{bmatrix} =$$

$$\begin{bmatrix} 1-2b & b-a & & & & \\ b+a & 1-2b & b-a & & & \\ & b+a & 1-2b & b-a & & \\ & & . & . & . & \\ & & & . & . & . \\ & & & & b+a & 1-2b \end{bmatrix} \begin{bmatrix} C_1^{n} \\ C_2^{n} \\ . \\ . \\ C_N^{n} \end{bmatrix},$$

where N is the number of grid points.

Solution

Call the left-hand matrix A and the right hand matrix B.

To find C at all grid points at a given time, we iterate:

$$\boxed{\vec{C}^{n+1} = A^{-1} B \vec{C}^{n}}$$