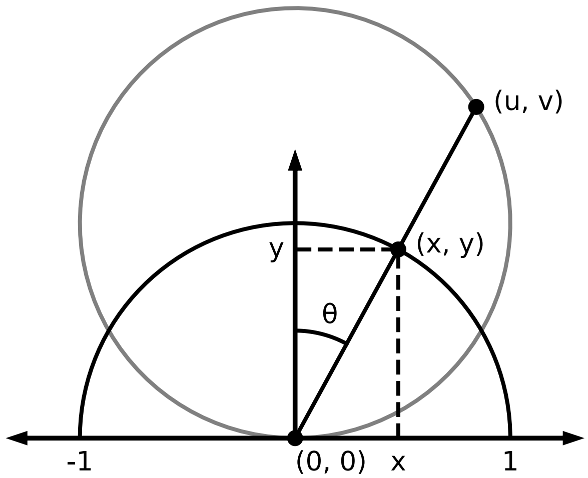
Sampling a disc in 2D and projecting vertically to a hemisphere is equivalent to sampling uniformly in a sphere and normalizing down to the hemisphere. We will sketch the proof in 2D (because we only care about latitude θ and not longitude).

OOPS: Turns out this wasn't quite right, see https://twitter.com/self_shadow/status/980521956066775040 and reply...



Sampling a 2D disc gives a cosine weighted distribution

So disregarding longitude, sampling a disc in 2D and projecting to a hemisphere is analagous to sampling x uniformly in $[-1, 1]$ and projecting to a semicircle.

The uniform PDF for x is simply $p(x) = c$ on the domain $[-1, 1]$ for some constant c (which we know to be $\frac{1}{2}$ but who cares).

Then projection onto the circle says $y = \sqrt{1 - x^2}$. Solving for latitude we find that $\theta = \arcsin(x)$ so $x = \sin(\theta)$ and also $y = \cos(\theta)$.

The PDF for θ is obtained via change of variables:

$$p(\theta) = p(x)\frac{dx}{d\theta} = c\,cos(\theta)$$

So $p(\theta) \propto cos(\theta)$.

Sampling a sphere tangent to the plane gives a cosine weighted distribution

Again, disregarding longitude, uniformly sampling a sphere tangent to the plane and normalizing onto the unit hemisphere is analagous to sampling a circle tangent to the x-axis at 0 and projecting onto the upper unit semicircle.

For any given θ, the original sample on the sphere could have come from any point in some line connecting $(0,0)$ to a point (u,v) on the boundary of the circle, where

$$u^2 + (v-1)^2 = 1$$

Also since (x,y) is the result of normalizing (u,v) onto the upper unit semicircle we know that

$$u = kx = k\sin(\theta), \quad \text{and} \quad v = ky = k\cos(\theta)$$

for some normalizing constant k. If we put these equations together we can solve for $k = 2\cos(\theta)$. Finally, since we are sampling the circle uniformly, we know the probability of any particular angle θ must be proportional to the length of the line connecting $(0,0)$ and (u,v), which tells us

$$\begin{aligned}
p(\theta) &\propto \sqrt{u^2+v^2} \\
&= \sqrt{4\cos^2(\theta)\sin^2(\theta) + 4\cos^4(\theta)} \\
&= \sqrt{4\cos^2(\theta)\left(\sin^2(\theta)+\cos^2(\theta)\right)} \\
&= \sqrt{4\cos^2(\theta)} \\
&= 2\cos(\theta)
\end{aligned}$$

So once again $p(\theta) \propto \cos(\theta)$.