## Hidden Variable Model for Quantum Computation with Magic States on Any Number of Qudits of Any Dimension

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#### Abstract

It was recently shown that a hidden variable model can be constructed for universal quantum computation with magic states on qubits. Here we show that this result can be extended, and a hidden variable model can be defined for quantum computation with magic states on any number of qudits with any local Hilbert space dimension. This model leads to a classical simulation algorithm for universal quantum computation.

#### 1 Introduction

The field of quantum computation has seen an explosion of interest in recent years. It is widely believed that the era of quantum advantage is upon us and that we are entering the realm of so-called Noisy Intermediate-Scale Quantum (NISQ) computation. This view is evidenced by the impressive performance of quantum devices in recent hardware demonstrations [1-10].

But in spite of the age of the field and the recent surge in interest, a key question at the heart of quantum computation remains without an entirely satisfying answer: what is the essential quantum resource that provides the computational speedup for quantum computation over classical computation? This is clearly an important question as an answer to it could inform the development of quantum hardware and the design of quantum computer architectures. It is also a significant research program and has been an active area of research for a number of years (e.g. Refs. [11–28]).

One inroad to approaching this question comes from the model of quantum computation with magic states (QCM) [29, 30]. QCM, a universal model of quantum computation closely related to the circuit model, is one of the leading candidates for fault-tolerant quantum computation [31]. In QCM, the allowed operations are restricted to a subset of unitary gates forming the so-called Clifford group, as well as arbitrary Pauli measurements. These operations alone are not universal for quantum computation. In fact, any quantum circuit consisting of only these operations can be simulated efficiently on a classical computer [32, 33], and so with these operations by themselves no quantum computational speedup is possible. Universality of quantum computation is restored in QCM through the inclusion of additional nonstabilizer quantum states to the input of the circuit. Focusing on this model of quantum computation allows us to refine the question posed above. Instead of asking broadly which nonclassical resources are required for a computational speedup, we can focus on the quantum states and ask: which quantum states could provide a quantum computational speedup in the model of QCM?

A partial answer to this question is provided by the study of quasiprobability representations like the discrete Wigner function. The Wigner function has many of the same properties as a probability distribution, but unlike a regular probability distribution it can take negative values, making it a quasiprobability distribution. When adapted to finite-dimensional quantum mechanics, the setting relevant for quantum

computation, quantum states are represented by a discrete Wigner function [34, 11–15]—a quasiprobability distribution over a finite set (or generalized phase space).

Veitch et al. [16] showed that a necessary condition for a computational speedup in the model of QCM on odd-dimensional qudits (quantum systems with odd Hilbert space dimension) is that the discrete Wigner function of the input state of the quantum circuit must take negative values. In particular, the amount of negativity in the discrete Wigner function quantifies the cost of classical simulation of a quantum computation [35] with simulation being efficient if the Wigner function is nonnegative everywhere. Since nonnegativity of the discrete Wigner function also implies the existence of a classical (noncontextual) hidden variable model (HVM) describing the computation [36, 37], this provides a satisfying alignment of two notions of classicality for quantum systems, one computational and one foundational. Negativity (or equivalently, the failure of a classical hidden variable model) thus heralds a possible quantum advantage over classical computation.

Similar necessary conditions for a quantum speedup based on negativity in quasiprobability representations have been proven for systems of qubits (Hilbert space dimension two) [20–25]. In all cases, negativity is required in the representation of states or operations in order to describe universal quantum computation.

Recently a hidden variable model was defined which bucks this trend by representing all quantum states, operations, and measurements relevant for QCM on qubits using only classical (nonnegative) probabilities [38]. This representation is structurally similar to previously defined quasiprobability representations for quantum computation (absence of negativity notwithstanding) and leads to a classical simulation method for universal quantum computation based on sampling from the defining probability distributions. In this paper we show that this result can be significantly extended, and a nonnegative hidden variable model can be constructed for quantum computation with magic states on any number of qudits of any dimension. We also show that many of the properties of the qubit hidden variable model also apply in the qudit case, for example, this model leads to a classical simulation algorithm for universal quantum computation with magic states, and it encompasses previously defined quasiprobability representations of QCM.

The remainder of this paper is structured as follows. In Section 2 we review some background material on quantum computation with magic states and the stabilizer formalism for qudits. In Section 3 we define a hidden variable model that is capable of representing all of the components of quantum computation with magic states—Clifford gates, Pauli measurements, and the magic states themselves. In Section 4 we present a classical simulation algorithm for quantum computation with magic states based on the hidden variable model. In Section 5 we characterize a subset of the hidden variables that define the model. Finally, in Section 6 we show how hidden variables of the n-qudit model can be constructed from hidden variables of a m-qudit model with m < n. We conclude with a discussion in Section 7.

## 2 Background

We are interested in the model of quantum computation with magic states (QCM) [29, 30] on multiple qudits—d-level quantum systems. This is a universal model of quantum computation in which computation proceeds through the application a sequence of Clifford gates and Pauli measurements on an initially prepared "magic state". Formally, for a system of n d-dimensional qudits, the measurements are associated with elements of the generalized Pauli group,  $P = \langle \omega \mathbb{1}, X_k, Z_k \mid 1 \leq k \leq n \rangle$  where  $\omega = \exp(2\pi i/d)$  is a primitive  $d^{th}$  root of unity and the local Pauli operators are the d-dimensional generalization of the standard qubit Pauli operators [39], given by

$$X = \sum_{j=1}^{d} |j+1 \mod d\rangle \langle j|$$
 and  $Z = \sum_{j=1}^{d} \omega^{j} |j\rangle \langle j|$ .

With phases modded out we have  $\mathcal{P} := P/\langle \omega \mathbb{1} \rangle \cong \mathbb{Z}_d^{2n}$  and we can parametrize the Pauli operators as

$$T_a = \mu^{\phi(a)} Z(a_Z) X(a_X), \quad \forall a = (a_Z, a_X) \in \mathbb{Z}_d^n \times \mathbb{Z}_d^n =: E$$
 (1)

where  $Z(a_Z) = \bigotimes_{k=1}^n Z^{a_Z[k]}$  and  $X(a_X) = \bigotimes_{k=1}^n X^{a_X[k]}$ . Here  $\mu = \omega$  when d is odd, and  $\mu = \sqrt{\omega}$  when d is even. The phase function  $\phi: E \to \mathbb{Z}_{\mu}$  can be chosen freely subject to the constraint  $(T_a)^d = \mathbb{I}$  for all  $a \in E$ .

This constraint forces the eigenvalues of the operators to be in the set  $\{\omega^j: j\in\mathbb{Z}_d\}$ . For concreteness we can choose the phase function to be

$$\phi(a) = \begin{cases} -\langle a_Z | a_X \rangle \cdot 2^{-1} & \text{if } d \text{ is odd} \\ -\langle a_Z | a_X \rangle & \text{if } d \text{ is even} \end{cases}$$

where the inner product  $\langle a_Z | a_X \rangle := \sum_{k=1}^n a_Z[k] a_X[k]$  is computed mod d if d is odd, and mod 2d if d is even. The gates of the model are the Clifford gates, which are drawn from the normalizer of the Pauli group in the unitary group (up to overall phases):  $\mathcal{C}\ell = \mathcal{N}(\mathcal{P})/U(1)$ . The Clifford gates are not required for quantum computational universality of this model since they can always be propagated past the Pauli measurements, conjugating them into other Pauli measurements [20, 40]. After they are propagated past the final Pauli measurements they can be dropped since they no longer affect the statistics of the measurements. In the following we include the Clifford gates anyway for completeness.

The last primitive required for QCM is the preparation of so-called "magic" input states. It is these states which allow for the universality of QCM. If the set of input states in QCM were restricted to only include n-qudit stabilizer states then the model would not be universal, and in fact any circuit of this type could be efficiently simulated on a classical computer. This is the result of the Gottesman-Knill theorem [32, 39]. Magic states are any nonstabilizer states which allow for universality in the QCM model.

Before proceeding we need to introduce some additional notation. The symplectic inner product  $[\cdot,\cdot]$ :  $E \times E \longrightarrow \mathbb{Z}_d$  defined by

$$[a,b] := \langle a_Z | b_X \rangle - \langle a_X | b_Z \rangle \tag{2}$$

tracks the commutator of the generalized Pauli operators in the sense

$$[T_a, T_b] := T_a T_b T_a^{-1} T_b^{-1} = \omega^{[a,b]} \mathbb{1}.$$
 (3)

Because of this correspondence we will say that elements  $a, b \in E$  commute when [a, b] = 0. The Pauli group with phases modded out forms a normal subgroup of the Clifford group such that  $\mathcal{C}\ell/\mathcal{P} \cong \operatorname{Sp}(E)$  is the group of symplectic transformations on E. The Clifford group acts on the Pauli group by conjugation as

$$UT_aU^{\dagger} = \omega^{\tilde{\Phi}_U(a)}T_{S_U(a)} \quad \forall U \in \mathcal{C}\ell \ \forall a \in E$$
(4)

where  $S_U \in \operatorname{Sp}(E)$  and the function  $\tilde{\Phi}_U : E \to \mathbb{Z}_d$  tracks the extra phases that get picked up.

A function  $\beta: E \times E \to \mathbb{Z}_d$  tracks how Pauli operators compose through the relation

$$T_a T_b = \omega^{-\beta(a,b)} T_{a+b}. \tag{5}$$

An explicit form for  $\beta$  along with some of its properties are given in Appendix A. The functions  $\tilde{\Phi}$  and  $\beta$ have a cohomological interpretation elucidated in Ref. [41] (also see Ref. [42]).

Recall from Ref. [25] a few definitions.

**Definition 1** A set  $\Omega \subset E$  is closed under inference if for every pair of elements  $a, b \in \Omega$  satisfying [a, b] = 0, it holds that  $a+b\in\Omega$ . The closure under inference of a set  $\Omega\subset E$ , denoted  $\Omega$ , is the smallest subset of E which is closed under inference and contains  $\Omega$ .

**Definition 2** A set  $\Omega \subset E$  is called noncontextual if there exists a noncontextual value assignment for the closure of that set, i.e., there exists a function  $\gamma: \bar{\Omega} \to \mathbb{Z}_d$  that satisfies  $\omega^{-\gamma(\mathbf{0})}T_{\mathbf{0}} = \mathbb{1}$  and

$$\gamma(a) + \gamma(b) - \gamma(a+b) = -\beta(a,b) \quad \forall a,b \in \Omega \text{ such that } [a,b] = 0. \tag{6}$$

A set which is both closed under inference and noncontextual we call cnc for short.

For any isotropic subgroup  $I \subset E$ , i.e. any subgroup I of E on which the symplectic product vanishes, and any noncontextual value assignment  $r: I \to \mathbb{Z}_d$ , the operator

$$\Pi_I^r = \frac{1}{|I|} \sum_{b \in I} \omega^{-r(b)} T_b$$

is the projector onto the simultaneous +1-eigenspace of the operators  $\{\omega^{-r(b)}T_b \mid b \in I\}$ . This represents a measurement of the Pauli observables labeled by I yielding measurement outcomes  $\omega^{r(b)}$ . In particular, for a single Pauli measurement  $I = \langle a \rangle$  is generated by a single element  $a \in E$ , and when  $|I| = d^n$ ,  $\Pi_I^r$  is a projector onto a stabilizer state. See Appendix B for more background on the stabilizer formalism for qudits.

## 3 Hidden variable model for quantum computation with magic states

In this section we define a hidden variable model that represents all components of quantum computation with magic states by a family of probability distributions. This is in contrast to previous quasiprobability representations which required negativity in the representation of either the states or the operations of QCM in order to represent universal quantum computation. The main result of this section is Theorem 1.

Let  $\operatorname{Herm}(d^n)$  be the space of  $d^n$ -dimensional Hermitian operators,  $\operatorname{Herm}_1(d^n)$  be the affine subspace of this space obtained by fixing the trace of the operators to be 1, and let  $\operatorname{Herm}_1^{\succeq 0}(d^n)$  be the subset of  $\operatorname{Herm}_1(d^n)$  consisting of positive semidefinite operators. Let  $\mathcal S$  denote the set of pure n-qudit stabilizer states. The state space of the hidden variable model is based on the set

$$\Lambda = \{ X \in \operatorname{Herm}_{1}(d^{n}) \mid \operatorname{Tr}(|\sigma\rangle \langle \sigma| X) \ge 0 \ \forall |\sigma\rangle \in \mathcal{S} \}. \tag{7}$$

The elements of  $\Lambda$  are much like density operators in that they are Hermitian operators with unit trace, but unlike density operators they are not necessarily positive semidefinite. In order to define the hidden variable model we first need to establish some basic properties of  $\Lambda$ .

**Lemma 1** For any number of qudits  $n \in \mathbb{N}$  of any dimension  $d \in \mathbb{N}$ , (i)  $\Lambda$  is convex, and (ii)  $\Lambda$  is compact.

To prove the lemma, we will use the concept of polar duality for objects in the affine space  $\operatorname{Herm}_1(d^n)$  (see Appendix C and Ref. [43] for a discussion). If  $P = \operatorname{conv}\{X_1, \dots, X_m\} \subset \operatorname{Herm}_1(d^n)$  is a polytope, then define its *polar dual* as

$$P^* = \{Y \in \text{Herm}_1(d^n) \mid \text{Tr}(X_i Y) \ge 0, i = 1, \dots, m\}.$$

Thus,  $\Lambda = SP^*$  for SP being the stabilizer polytope:  $SP := conv\{|\sigma\rangle \langle \sigma| \mid \sigma \in \mathcal{S}\}.$ 

To prove that the set  $\Lambda$  is bounded, it will suffice to show that SP contains a set M, whose dual  $M^*$  is bounded (see Eq. (30), Appendix C). Additionally, we will make us of the concept of *dilation* [44, Chap. 9]: for a set  $M \subset \operatorname{Herm}_1(d^n)$  define its dilation centered at the maximally mixed state via

$$c \cdot M := \left\{ \frac{1}{d^n} \mathbb{1} + c\pi(X) \mid X \in M \right\}$$
 (8)

where  $\pi: \operatorname{Herm}_1(d^n) \to \operatorname{Herm}_0(d^n)$  is the projection that maps  $X \in \operatorname{Herm}_1(d^n)$  to  $X - \frac{1}{d^n}\mathbb{1}$ . The dilation has the following property:

**Lemma 2** The dilation of a set  $M \subset Herm_1(d^n)$  satisfies

$$(c \cdot M)^* = \frac{1}{c} \cdot M^*.$$

A proof of Lemma 2 will be given in Appendix C. Having introduced all necessary concepts, we will proceed with the proof of Lemma 1.

Proof of Lemma 1— By definition,  $\Lambda$  is a polyhedron and therefore convex and closed. To prove that  $\Lambda$  is bounded, the previous discussion implies that it suffices to find a set  $M \subset SP$  such that  $M^*$  is bounded. The object we will choose here will be a dilation of the full-dimensional simplex

$$\Delta_{\mathrm{Herm}_1} := \mathrm{conv} \left\{ A_{\gamma} := \frac{1}{d^n} \sum_{u \in E} \omega^{\gamma(u)} T_u \mid \gamma : E \to \mathbb{Z}_d, \ \gamma(u+v) = \gamma(u) + \gamma(v) \right\}.$$

The simplex  $\Delta_{\mathrm{Herm_1}}$  is the Wigner simplex for d being an odd prime [16, 19]. For every d, it is a full-dimensional polytope as the convex-hull of  $d^{2n}$  affinely independent vertices  $A_{\gamma}$  in the  $(d^{2n}-1)$ -dimensional affine space  $\mathrm{Herm_1}(d^n)$ . Due to  $\mathrm{Tr}(A_{\gamma}A_{\gamma'})=\delta_{\gamma=\gamma'}$  for additive functions  $\gamma,\gamma':E\to\mathbb{Z}_d$ , one can easily verify that the simplex  $\Delta_{\mathrm{Herm_1}}$  has the following hyperplane description:

$$\Delta_{\mathrm{Herm}_1} = \{ X \in \mathrm{Herm}_1(d^n) \mid \mathrm{Tr}(A_{\gamma}X) \ge 0 \},$$

which makes it a self-dual simplex, i.e.  $\triangle_{\mathrm{Herm_1}} = \triangle_{\mathrm{Herm_1}}^*$ . Now, Lemma 2 implies that for c > 0 the simplex  $(c \cdot \triangle_{\mathrm{Herm_1}})^*$  is bounded, since

$$(c \cdot \Delta_{\mathrm{Herm}_1})^* = \frac{1}{c} \Delta_{\mathrm{Herm}_1}^* = \frac{1}{c} \Delta_{\mathrm{Herm}_1}.$$

Hence, it suffices to show that  $c \cdot \Delta_{\operatorname{Herm}_1} \subset \operatorname{SP}$  for some c > 0, because then

$$\Lambda = \mathrm{SP}^* \subset (c \cdot \Delta_{\mathrm{Herm}_1})^* = \frac{1}{c} \Delta_{\mathrm{Herm}_1}$$

implies that  $\Lambda$  is bounded. Therefore, we will show that dilations of the vertices of  $\triangle_{\text{Herm}_1}$  are contained in SP, i.e. there is c > 0 such that

$$\frac{1}{d^n}\mathbb{1} + c(A_\gamma - \frac{1}{d^n}\mathbb{1}) \in SP$$

for all additive maps  $\gamma$ .

To achieve this, we will write  $1/d^n\mathbb{1} + c(A_{\gamma} - 1/d^n\mathbb{1})$  as a convex combination of normalized projectors of the form  $\frac{|\langle a \rangle|}{d^n} \Pi^r_{\langle a \rangle} \in SP$  for  $a \in E$  with noncontextual value assignments  $r : \langle a \rangle \to \mathbb{Z}_d$ . Due to Corollary 2 in Appendix B, the elements  $\frac{|\langle a \rangle|}{d^n} \Pi^r_{\langle a \rangle}$  are indeed contained in SP. A noncontextual value assignment on a line  $\langle a \rangle$  is always additive, that is

$$r(ka) = kr(a) \quad \forall k \in \mathbb{Z}_d$$
 (9)

because of  $\beta(a,ka)=0$  for all  $a\in E$  (see Appendix A, Eq. (24)). Let  $C=\{a_1,\ldots,a_N\}\subset E$  be a set that "covers" E, that is

$$E = \bigcup_{a \in C} \langle a \rangle.$$

For each subset  $\mathcal{I} \subset [N] := \{1, \dots, N\}$ , there is an  $a_{\mathcal{I}} \in E$  such that

$$\langle a_{\mathcal{I}} \rangle := \bigcap_{k \in \mathcal{I}} \langle a_k \rangle.$$

First, we will write  $A_{\gamma}$  as a linear combination of stabilizer code projectors  $\Pi_{(a)}^r$ . We claim that

$$A_{\gamma} = \frac{1}{d^n} \sum_{\mathcal{I} \subset [N]} (-1)^{|\mathcal{I}|+1} \cdot |\langle a_{\mathcal{I}} \rangle| \cdot \Pi_{\langle a_{\mathcal{I}} \rangle}^{\gamma|\langle a_{\mathcal{I}} \rangle}. \tag{10}$$

Since  $\gamma$  is additive, the restriction  $\gamma|_{\langle a_{\mathcal{I}} \rangle} : \langle a_{\mathcal{I}} \rangle \to \mathbb{Z}_d$  satisfies (9) and defines a noncontextual value assignment on  $\langle a_{\mathcal{I}} \rangle$ . We rewrite the right hand side of (10) in the following way:

$$\frac{1}{d^n} \sum_{\mathcal{I} \subset [N]} (-1)^{|\mathcal{I}|+1} \cdot |\langle a_{\mathcal{I}} \rangle| \cdot \Pi_{\langle a_{\mathcal{I}} \rangle}^{\gamma|_{\langle a_{\mathcal{I}} \rangle}} = \frac{1}{d^n} \sum_{\mathcal{I} \subset [N]} (-1)^{|\mathcal{I}|+1} \sum_{b \in \langle a_{\mathcal{I}} \rangle} \omega^{\gamma(b)} T_b = \frac{1}{d^n} \sum_{b \in E} \Big( \sum_{\mathcal{I} \subset [N] : b \in \langle a_{\mathcal{I}} \rangle} (-1)^{|\mathcal{I}|+1} \Big) \omega^{\gamma(b)} T_b.$$

Thus, it suffices to show that

$$\sum_{\mathcal{I}\subset[N]:b\in\langle a_I\rangle}(-1)^{|\mathcal{I}|+1}=1\quad\forall b\in E.$$

However, this is a consequence of the inclusion-exclusion principle [45], that is

$$1 = \delta_{b \in E} = \delta_{b \in \cup_{a \in C} \langle a \rangle} = \sum_{\mathcal{I} \subset [N]} (-1)^{|\mathcal{I}|+1} \delta_{b \in \cap_{k \in I} \langle a_k \rangle} = \sum_{\mathcal{I} \subset [N]} (-1)^{|\mathcal{I}|+1} \delta_{b \in \langle a_{\mathcal{I}} \rangle} = \sum_{\mathcal{I} \subset [N] : b \in \langle a_{\mathcal{I}} \rangle} (-1)^{|\mathcal{I}|+1}.$$

Finally, we will show that there is c>0 such that we can write  $\frac{1}{d^n}\mathbb{1}+c(A_\gamma-\frac{1}{d^n}\mathbb{1})$  for every  $\gamma$  as a convex combination of the operators  $\frac{|\langle a_{\mathcal{I}} \rangle|}{d^n} \Pi_{\langle a_{\mathcal{I}} \rangle}^{\gamma_{|\langle a_{\mathcal{I}} \rangle}} \in SP$ .

Observe that the identity (10) is equivalent to

$$A_{\gamma} - \frac{1}{d^n} \mathbb{1} = \sum_{\mathcal{I} \subset [N]} (-1)^{|\mathcal{I}|+1} \cdot \frac{|\langle a_{\mathcal{I}} \rangle|}{d^n} \cdot \prod_{\langle a_{\mathcal{I}} \rangle^*}^{\gamma_{|\langle a_{\mathcal{I}} \rangle}}$$
(11)

with

$$\Pi^{r}_{\langle a \rangle^{*}} := \frac{1}{|\langle a \rangle|} \sum_{b \in \langle a \rangle \setminus \{0\}} \omega^{r(b)} T_{b} \in \operatorname{Herm}_{0}(d^{n}), \qquad r : \langle a \rangle \to \mathbb{Z}_{d}.$$

Moreover, due to Lemma 9, we have

$$\mathbb{1} = \sum_r \Pi^r_{\langle a \rangle} \quad \Longrightarrow \quad 0 = \sum_r \Pi^r_{\langle a \rangle^*}$$

for all  $a \in E$ ,  $\frac{|\langle a \rangle|}{d^n} \Pi^r_{\langle a \rangle} \in SP$ , where r ranges over all noncontextual value assignments  $r : \langle a \rangle \to \mathbb{Z}_d$ . This implies that

$$(-1) \cdot \Pi_{\langle a_{\mathcal{I}} \rangle^*}^{\gamma|_{\langle a_{\mathcal{I}} \rangle}} = \sum_{r \neq \gamma|_{\langle a \rangle}} \Pi_{\langle a_{\mathcal{I}} \rangle^*}^r.$$

As a consequence, every summand in the right hand side of Eq. (11) can be written as a conic combination of elements  $\Pi_{(a)^*}^r$ . By properly properly rescaling, we can find c > 0 such that

$$c(A_{\gamma} - \frac{1}{d^n}\mathbb{1}) = \sum_a \alpha_a \Pi^{r_a}_{\langle a \rangle^*} \quad \text{with} \quad \alpha_a \ge 0, \ \sum_a \alpha_a = 1,$$

which is equivalent to

$$\frac{1}{d^n}\mathbb{1} + c(A_{\gamma} - \frac{1}{d^n}\mathbb{1}) = \sum_a \alpha_a (\Pi_{\langle a \rangle^*}^{r_a} + \frac{1}{d^n}\mathbb{1}) = \sum_a \alpha_a \Pi_{\langle a \rangle}^{r_a} \in SP.$$

This proves that  $c \cdot \triangle_{\text{Herm}_1} \subset \text{SP}$  for some c > 0, which remained to be shown.  $\square$ 

 $\Lambda$  can be interpreted as a subset of a real affine space defined by the intersection of a finite number of linear inequalities, i.e., it is a polyhedral set. Since  $\Lambda$  is compact, it is a polytope, and so by the Minkowski-Weyl theorem [43] it can equivalently be described as the convex hull of finitely many vertices. Let  $\{A_{\alpha} \mid \alpha \in \mathcal{V}\}$  denote the (finite) set of vertices of  $\Lambda$ . Then we have the following result, which is a generalization of [38, Theorem 1] to qudits of arbitrary local Hilbert space dimension d.

**Theorem 1** For any number  $n \in \mathbb{N}$  of qudits with any local Hilbert space dimension  $d \in \mathbb{N}$ ,

1. For any quantum state  $\rho \in Herm_{\mathbb{T}}^{\succeq 0}(d^n)$ , there is a probability function  $p_{\rho}: \mathcal{V} \to \mathbb{R}_{>0}$  such that

$$\rho = \sum_{\alpha \in \mathcal{V}} p_{\rho}(\alpha) A_{\alpha}. \tag{12}$$

- 2. For any vertex  $A_{\alpha}$  of  $\Lambda$  and any Clifford unitary  $U \in \mathcal{C}\ell$ ,  $UA_{\alpha}U^{\dagger} =: A_{U \cdot \alpha}$  is a vertex of  $\Lambda$ .
- 3. For update under Pauli measurements it holds that for any isotropic subgroup  $I \subset E$ , any noncontextual value assignment  $r: I \to \mathbb{Z}_d$ , and any vertex  $A_{\alpha}$ ,

$$\Pi_I^r A_\alpha \Pi_I^r = \sum_{\beta \in \mathcal{V}} q_{\alpha,I}(\beta, r) A_\beta, \tag{13}$$

where  $q_{\alpha,I}(\beta,r) \geq 0$  for all  $\beta \in \mathcal{V}$  and  $\sum_{\beta,r} q_{\alpha,I}(\beta,r) = 1$ .

4. The Born rule takes the form

$$Tr(\Pi_I^r \rho) = \sum_{\alpha \in \mathcal{V}} p_\rho(\alpha) Q_I(r \mid \alpha)$$
(14)

where Q is given by

$$Q_I(r \mid \alpha) = \sum_{\beta \in \mathcal{V}} q_{\alpha, I}(\beta, r). \tag{15}$$

This theorem defines a hidden variable model which represents all of the primitives of quantum computation with magic states using only probability distributions for states and measurements, and probabilistic update rules for dynamics. It has a similar structure to previous hidden variable models based on quasiprobability representations [13–15, 20, 35, 37, 23, 46, 25, 28, 38], with a key difference being that in this model every state can be represented by a probability distribution, no negativity is required. Unlike the model of Beltrametti and Bugajski [47] which requires a hidden variable for each pure quantum state, this model has a finite number of hidden variables for any number of qudits.

The proof of Theorem 1 requires the following lemma.

#### **Lemma 3** The polytope $\Lambda$ has the following properties:

- 1. For any density operator  $\rho \in Herm_1^{\succeq 0}(d^n)$  representing an n-qudit quantum state,  $\rho \in \Lambda$ .
- 2. For any  $X \in \Lambda$  and any Clifford unitary  $U \in \mathcal{C}\ell$ ,  $UXU^{\dagger} \in \Lambda$ ,
- 3. For any  $X \in \Lambda$ , any isotropic subgroup  $I \subset E$ , and any noncontextual value assignment  $r: I \to \mathbb{Z}_d$ , if  $\operatorname{Tr}(\Pi_I^r X) > 0$  then

$$\frac{\Pi_I^r X \Pi_I^r}{\operatorname{Tr}(\Pi_I^r X)} \in \Lambda.$$

Proof of Lemma 3—We will prove the three properties in order. For the first property note that for any density operator  $\rho \in \operatorname{Herm}_{1}^{\succeq 0}(d^{n})$ ,  $\rho$  is positive semidefinite. Therefore, for any pure quantum state  $|\psi\rangle$ ,  $\operatorname{Tr}(|\psi\rangle\langle\psi|\rho) \geq 0$ . This holds in particular for any pure stabilizer state. Therefore,  $\rho$  satisfies all of the defining inequalities of the polytope in eq. (7) and so  $\rho \in \Lambda$ .

For the second property, let  $X \in \Lambda$  and  $U \in \mathcal{C}\ell$ . Then for any stabilizer state  $|\sigma\rangle \in S$ ,

$$\operatorname{Tr}(|\sigma\rangle\langle\sigma|(UXU^{\dagger})) = \operatorname{Tr}((U^{\dagger}|\sigma\rangle\langle\sigma|U)X) = \operatorname{Tr}(|\sigma'\rangle\langle\sigma'|X) \ge 0.$$

Here the first equality follows from the cyclic property of the trace, the second equality from the fact that Clifford unitaries map stabilizer states to stabilizer states (see Lemma 10 in Appendix B), and the last inequality from the assumption  $X \in \Lambda$ .

Now we can prove the third property of Lemma 3. Let  $I, J \subset E$  be isotropic subgroups with noncontextual value assignments  $r: I \to \mathbb{Z}_d$  and  $s: J \to \mathbb{Z}_d$ . By Lemma 11 in Appendix B,

$$\Pi_I^r \Pi_J^s \Pi_I^r = \delta_{r|_{I \cap J} = s|_{I \cap J}} \frac{|J \cap I^\perp|}{|J|} \Pi_{I+J \cap I^\perp}^{r \star s}$$

where  $r \star s$  is the unique noncontextual value assignment on the set  $I + J \cap I^{\perp}$  such that  $r \star s|_{I} = r$  and  $r \star s|_{J \cap I^{\perp}} = s|_{J \cap I^{\perp}}$ . For any  $X \in \Lambda$  and any Pauli projector  $\Pi_{I}^{r}$ , if  $\text{Tr}(\Pi_{I}^{r}X) > 0$ , then for any projector onto a stabilizer state  $\Pi_{I}^{s}$ ,

$$\operatorname{Tr}\left(\Pi_{J}^{s} \frac{\Pi_{I}^{r} X \Pi_{I}^{r}}{\operatorname{Tr}(\Pi_{I}^{r} X)}\right) = \frac{\operatorname{Tr}((\Pi_{I}^{r} \Pi_{J}^{s} \Pi_{I}^{r}) X)}{\operatorname{Tr}(\Pi_{I}^{r} X)}$$
$$= \delta_{r|_{I \cap J} = s|_{I \cap J}} \frac{|J \cap I^{\perp}|}{|J|} \frac{\operatorname{Tr}\left(\Pi_{I+J \cap I^{\perp}}^{r \star s} X\right)}{\operatorname{Tr}(\Pi_{I}^{r} X)} \geq 0.$$

Here the first line follows from linearity and the cyclic property of the trace. The last inequality follows from the fact that by Lemma 9 in Appendix B,  $\Pi_{I+J\cap I^{\perp}}^{r\star s}$  can be written as a conic combination of projectors onto stabilizer states, and from the assumption  $X \in \Lambda$ . Therefore, for any  $X \in \Lambda$ , any Pauli projector  $\Pi_I^r$ , and any stabilizer state  $|\sigma\rangle \in \mathcal{S}$ , if  $\text{Tr}(\Pi_I^r X) > 0$ , then

$$\operatorname{Tr}\left(\left|\sigma\right\rangle\left\langle\sigma\right|\frac{\Pi_{I}^{r}X\Pi_{I}^{r}}{\operatorname{Tr}\left(\Pi_{I}^{r}X\right)}\right)\geq0$$

and so  $\Pi_I^T X \Pi_I^T / \operatorname{Tr}(\Pi_I^T X) \in \Lambda$ . This proves the third statement of the lemma.  $\square$ 

We can now prove the main result of this section.

*Proof of Theorem 1*—We will prove the four statements of the theorem in order. First, as shown in Lemma 3,  $\Lambda$  contains all density matrices corresponding to physical n-qudit quantum states. Therefore, by the Krein-Milman theorem [43] any state can be written as a convex combination of the vertices of  $\Lambda$ . This proves the first statement of the theorem.

The second property from Lemma 3 shows that for any Clifford unitary  $U \in \mathcal{C}\ell$  and any vertex  $A_{\alpha}$  of  $\Lambda$ , we have  $A_{U\cdot\alpha} := UA_{\alpha}U^{\dagger} \in \Lambda$ . It remains to show that  $A_{U\cdot\alpha}$  is a vertex of  $\Lambda$ .

Let  $S_{\alpha} = \{ |\sigma\rangle \in \mathcal{S} \mid \text{Tr}(|\sigma\rangle \langle \sigma| A_{\alpha}) = 0 \}$  be the set of stabilizer states with projectors orthogonal to vertex  $A_{\alpha}$  with respect to the Hilbert-Schmidt inner product. By Theorem 18.1 of Ref. [48], since  $A_{\alpha}$  is a vertex of  $\Lambda$ ,  $A_{\alpha}$  is the unique solution in  $\text{Herm}_{1}(d^{n})$  of the system

$$\begin{cases} \operatorname{Tr}(|\sigma\rangle \langle \sigma| X) = 0 & \forall |\sigma\rangle \in \mathcal{S}_{\alpha} \\ \operatorname{Tr}(|\sigma\rangle \langle \sigma| X) \ge 0 & \forall |\sigma\rangle \in \mathcal{S} \setminus \mathcal{S}_{\alpha}. \end{cases}$$
(16)

For any stabilizer state  $|\sigma\rangle \in \mathcal{S}_{\alpha}$ ,

$$\operatorname{Tr}(|\sigma\rangle \langle \sigma| X) = \operatorname{Tr}(U |\sigma\rangle \langle \sigma| U^{\dagger} U X U^{\dagger})$$
$$= \operatorname{Tr}(|U \cdot \sigma\rangle \langle U \cdot \sigma| U X U^{\dagger}).$$

Therefore, under conjugation by  $U \in \mathcal{C}\ell$ , solutions to the system eq. (16) are mapped bijectively to solutions of the system

$$\begin{cases} \operatorname{Tr}(|\sigma\rangle \langle \sigma| X) = 0 & \forall |\sigma\rangle \in \mathcal{S}_{U \cdot \alpha} \\ \operatorname{Tr}(|\sigma\rangle \langle \sigma| X) \ge 0 & \forall |\sigma\rangle \in \mathcal{S} \setminus \mathcal{S}_{U \cdot \alpha} \end{cases}$$
(17)

where  $S_{U\cdot\alpha} := \{U \mid \sigma \rangle \mid |\sigma \rangle \in S_{\alpha}\}$ . In particular,  $A_{U\cdot\alpha} := UA_{\alpha}U^{\dagger}$  is the unique solution to this system, so by Theorem 18.1 of Ref. [48], it is a vertex of  $\Lambda$ .

By the third statement of Lemma 3, for any vertex  $A_{\alpha}$  of  $\Lambda$  and any Pauli projector  $\Pi_{I}^{r}$  such that  $\text{Tr}(\Pi_{I}^{r}A_{\alpha}) > 0$ , we have  $\Pi_{I}^{r}A_{\alpha}\Pi_{I}^{r}/\text{Tr}(\Pi_{I}^{r}A_{\alpha}) \in \Lambda$ , and so there exists a decomposition of  $\Pi_{I}^{r}A_{\alpha}\Pi_{I}^{r}/\text{Tr}(\Pi_{I}^{r}A_{\alpha})$  as a convex combination of the vertices of  $\Lambda$ . Therefore, there exist nonnegative coefficients  $q_{\alpha,I}(\beta,r)$  such that

$$\Pi_I^r A_\alpha \Pi_I^r = \sum_{\beta \in \mathcal{V}} q_{\alpha, I}(\beta, r) A_\beta.$$

Taking a trace of this equation and adding the corresponding equations for all noncontextual value assignments r of I we have on the left hand side

$$\sum_r \operatorname{Tr}(\Pi_I^r A_\alpha \Pi_I^r) = \operatorname{Tr}\left[\left(\sum_r \Pi_I^r\right) A_\alpha\right] = \operatorname{Tr}(A_\alpha) = 1$$

and on the right hand side

$$\sum_{\beta,r} q_{\alpha,I}(\beta,r) \operatorname{Tr}(A_{\beta}) = \sum_{\beta,r} q_{\alpha,I}(\beta,r).$$

Therefore,  $\sum_{\beta,r} q_{\alpha,I}(\beta,r) = 1$ . This proves the third statement of the Theorem. Finally, we calculate

$$\begin{aligned} \operatorname{Tr}(\Pi_{I}^{r}\rho) &= \sum_{\alpha \in \mathcal{V}} p_{\rho}(\alpha) \operatorname{Tr}(\Pi_{I}^{r}A_{\alpha}) \\ &= \sum_{\alpha \in \mathcal{V}} p_{\rho}(\alpha) \sum_{\beta \in \mathcal{V}} q_{\alpha,I}(\beta,r) \\ &= \sum_{\alpha \in \mathcal{V}} p_{\rho}(\alpha) Q_{I}(r \mid \alpha) \end{aligned}$$

and we obtain the fourth statement of the theorem.  $\square$ 

# 4 Classical simulation algorithm for quantum computation with magic states

Theorem 1 shows that all of the components of quantum computation with magic states can be described by a hidden variable model which represents all relevant states and dynamical operations by probabilities. This leads to a classical simulation algorithm for quantum computation with magic states, Algorithm 1, based on sampling from these probability distributions.

In short, a vertex  $A_{\alpha}$  of  $\Lambda$  is sampled according to the probability distribution of eq. (12) representing the input state of the quantum circuit. This vertex is then propagated through the circuit. When a Clifford gate  $U \in \mathcal{C}\ell$  is encountered, we have a deterministic update rule: the vertex is updated as  $A_{\alpha} \to U A_{\alpha} U^{\dagger}$  according to the second statement of Theorem 1. When a Pauli measurement  $a \in E$  is encountered, the third and fourth statements of Theorem 1 give a way of determining probabilities for measurement outcomes, as well as a probabilistic update rule for the vertex  $A_{\alpha}$ . That is, we sample a pair  $(\beta, r)$  according to the probability distribution  $q_{\alpha,\langle a\rangle}$  where  $\beta \in \mathcal{V}$  and  $r:\langle a\rangle \to \mathbb{Z}_d$  is a noncontextual value assignment. Then r(a) is returned as the measurement outcome and the vertex is updated as  $\alpha \to \beta$ . This process continues until the end of the circuit is reached.

A proof of the correctness of this simulation algorithm is given below.

```
Input: p_{\rho_{in}}
 1: sample a point \alpha \in \mathcal{V} according to the probability distribution p_{\rho_{in}}
 2: while end of circuit has not been reached do
       if a Clifford unitary U \in \mathcal{C}\ell is encountered then
          update \alpha \leftarrow U \cdot \alpha
 4:
       end if
       if a Pauli measurement T_a, a \in E is encountered then
          sample (\beta, r) according to the probability distribution q_{\alpha, \langle a \rangle}
 7:
          Output: r(a) as the outcome of the measurement
          update \alpha \leftarrow \beta
 9:
10:
       end if
11: end while
```

Algorithm 1: One run of the classical simulation algorithm for quantum computation with magic states based on the hidden variable model of Theorem 1. The algorithm provides samples from the joint probability distribution of the Pauli measurements in a quantum circuit consisting of Clifford unitaries and Pauli measurements applied to an input state  $\rho_{in}$ .

**Theorem 2** The classical simulation algorithm, Algorithm 1, correctly reproduces the predictions of quantum theory.

Proof of Theorem 2—Without loss of generality, a QCM circuit can be represented as a sequence  $U_1, I_1, U_2, I_2, \ldots$  with  $U_1, U_2, \cdots \in \mathcal{C}\ell$  specifying the Clifford unitaries to be applied, and  $I_1, I_2, \cdots \subset E$  specifying the Pauli measurements to be performed. First, consider a single layer of this circuit comprising a Clifford unitary  $U \in \mathcal{C}\ell$  followed by Pauli measurements corresponding to a isotropic subgroup  $I \subset E$ .

Using the classical simulation algorithm, Algorithm 1, the conditional probability of obtaining measurement outcomes specified by the noncontextual value assignment  $r: I \to \mathbb{Z}_d$  for the measurements given the state  $\alpha \in \mathcal{V}$  is  $Q_I(r \mid U \cdot \alpha)$ . Therefore, the probability of obtaining outcomes r given the unitary U is applied to the state  $\rho$  followed by the measurements of the Pauli observables in I is given by

$$P_{\rho,U,I}^{(Sim)}(r) = \sum_{\alpha \in \mathcal{V}} p_{\rho}(\alpha) Q_I(r \mid U \cdot \alpha).$$

The corresponding outcome probability predicted by the Born rule,  $P_{\rho,U,I}^{(QM)}(r)$ , is

$$\operatorname{Tr}(\Pi_I^r U \rho U^{\dagger}) = \sum_{\alpha \in \mathcal{V}} p_{\rho}(\alpha) \operatorname{Tr}(\Pi_I^r U A_{\alpha} U^{\dagger})$$
$$= \sum_{\alpha \in \mathcal{V}} p_{\rho}(\alpha) \operatorname{Tr}(\Pi_I^r A_{U \cdot \alpha})$$
$$= \sum_{\alpha \in \mathcal{V}} p_{\rho}(\alpha) Q_I(r \mid U \cdot \alpha).$$

Here in the first line we use the expansion of  $\rho$  in the vertices of  $\Lambda$ , eq. (12), in the second line we use the second statement of Theorem 1, and in the last line we use the fourth statement of Theorem 1. This agrees with the outcome probability predicted by the classical simulation algorithm.

Now consider the postmeasurement state  $\rho'$ . According to quantum mechanics, the postmeasurement state is

$$\rho' = \frac{\Pi_I^r U \rho U^{\dagger} \Pi_I^r}{\text{Tr}(\Pi_I^r U \rho U^{\dagger})}.$$

Here the numerator is

$$\begin{split} \Pi_I^r U \rho U^\dagger \Pi_I^r &= \sum_{\alpha \in \mathcal{V}} p_\rho(\alpha) \Pi_I^r U A_\alpha U^\dagger \Pi_I^r \\ &= \sum_{\alpha \in \mathcal{V}} p_\rho(\alpha) \Pi_I^r A_{U \cdot \alpha} \Pi_I^r \\ &= \sum_{\alpha \in \mathcal{V}} p_\rho(\alpha) \sum_{\beta \in \mathcal{V}} q_{U \cdot \alpha, I}(\beta, r) A_\beta \end{split}$$

and the denominator is

$$\operatorname{Tr}\left(\Pi_{I}^{r}U\rho U^{\dagger}\Pi_{I}^{r}\right) = \sum_{\alpha \in \mathcal{V}} p_{\rho}(\alpha) \sum_{\beta \in \mathcal{V}} q_{U \cdot \alpha, I}(\beta, r)$$
$$= \sum_{\alpha \in \mathcal{V}} p_{\rho}(\alpha) Q_{I}(r \mid U \cdot \alpha),$$

so the postmeasurement state predicted by quantum theory is

$$\rho'^{(QM)} = \frac{\sum_{\alpha} p_{\rho}(\alpha) \sum_{\beta} q_{U \cdot \alpha, I}(\beta, r) A_{\beta}}{\sum_{\alpha} p_{\rho}(\alpha) Q_{I}(r \mid U \cdot \alpha)}.$$

Using the classical simulation algorithm, the probability of obtaining outcomes r and state  $A_{\beta}$  given a Clifford unitary U followed by measurements of the Pauli observables I on state  $\rho$  is  $P_{\rho,U,I}(\beta,r) = P_{\rho,U,I}(\beta|r)P_{\rho,U,I}(r)$ . But  $P_{\rho,U,I}(\beta,r) = \sum_{\alpha} p_{\rho}(\alpha)P_{U,I}(\beta,r|\alpha) = \sum_{\alpha} p_{\rho}(\alpha)q_{U\cdot\alpha,I}(\beta,r)$ , and  $P_{\rho,U,I}(\beta|r) = p_{\rho'}(\beta)$ . Therefore, the postmeasurement state predicted by the classical simulation algorithm is

$$\rho'^{(Sim)} = \sum_{\beta \in \mathcal{V}} p_{\rho'}(\beta) A_{\beta} = \sum_{\beta \in \mathcal{V}} \frac{P_{\rho,U,I}(\beta,r)}{P_{\rho,U,I}(r)} A_{\beta}$$
$$= \sum_{\beta \in \mathcal{V}} \frac{\sum_{\alpha} p_{\rho}(\alpha) q_{U \cdot \alpha,I}(\beta,r)}{\sum_{\alpha} p_{\rho}(\alpha) Q_{I}(r \mid U \cdot \alpha)} A_{\beta}.$$

This agrees with the postmeasurement state predicted by quantum mechanics. Therefore, the classical simulation algorithm correctly reproduces the outcome probabilities and the postmeasurement state predicted by quantum mechanics for a single layer of a QCM circuit.

Now let  $\rho(t)$  denote the state after t-1 layers of the circuit. Then the above shows that the classical simulation algorithm correctly reproduces the Born rule probabilities  $P_{\rho,U_t,I_t}(r_t \mid r_1, r_2, \dots, r_{t-1})$  as well as the postmeasurement state  $\rho(t+1)$ . Therefore, by induction, the simulation algorithm correctly reproduces the outcome probabilities predicted by the Born rule for any QCM circuit.  $\square$ 

#### 5 Partial characterization of vertices of $\Lambda$

In Ref. [28] a classical simulation algorithm for quantum computation with magic states is introduced based on sampling from a quasiprobability function. Points in the generalized phase space over which the quasiprobability function is defined are associated with pairs  $(\Omega, \gamma)$  where  $\Omega \subset E$  is a cnc set and  $\gamma : \Omega \to \mathbb{Z}_d$  is a noncontextual value assignment for  $\Omega$ . For each point in phase space there is a corresponding phase point operator defined as

$$A_{\Omega}^{\gamma} = \frac{1}{d^n} \sum_{b \in \Omega} \omega^{-\gamma(b)} T_b. \tag{18}$$

For qubits, if  $\Omega$  is a maximal cnc set then phase point operators  $A_{\Omega}^{\gamma}$  of the form eq. (18) are vertices of  $\Lambda$  [46, 25]. Vertices of the type eq. (18) we call enc-type vertices.

A similar result holds for odd-prime-dimensional qudits. Namely, phase point operators of the form  $A_E^{\gamma}$  define facets of the stabilizer polytope [12, 16], or equivalently, by polar duality,  $A_E^{\gamma}$  are vertices of  $\Lambda$ . Here we show that this also holds for any odd-dimensional qudits. This is the result of the following theorem:

**Theorem 3** For any number n of qudits of any odd local Hilbert space dimension d, the phase space point operators of the form

$$A_E^{\gamma} = \frac{1}{d^n} \sum_{b \in E} \omega^{-\gamma(b)} T_b \tag{19}$$

are vertices of  $\Lambda$ .

Note that the phase point operators of the form eq. (19) only exist when the local Hilbert space dimension d is odd since noncontextual value assignments on E exist only when d is odd [41].

The proof of this statement requires the following lemma.

**Lemma 4** For any vertex  $A_{\alpha}$  of  $\Lambda$  and any Pauli operator  $T_a$ ,  $|\operatorname{Tr}(T_aA_{\alpha})| \leq 1$ .

Proof of Lemma 4—For any  $\alpha \in \mathcal{V}$  and any  $a \in E$ ,

$$\operatorname{Tr}(T_a A_\alpha) = \sum_{s \in \Gamma[\langle a \rangle]} \operatorname{Tr}\left(\Pi^s_{\langle a \rangle} A_\alpha\right) \omega^{s(a)}$$

where  $\Gamma[\langle a \rangle]$  is the set of noncontextual value assignments on  $\langle a \rangle$  and

$$\Pi_{\langle a \rangle}^s = \frac{1}{d^n} \sum_{b \in \langle a \rangle} \omega^{-s(b)} T_b$$

is the projector onto the eigenspace of the Pauli observable  $T_a$  with eigenvalue  $\omega^{s(a)}$ . The set  $\{\Pi^s_{\langle a \rangle} \mid s \in \Gamma[\langle a \rangle]\}$  is a projection-valued measure, i.e.  $\sum_{s \in \Gamma[\langle a \rangle]} \Pi^s_{\langle a \rangle} = 1$ . Therefore,

$$\sum_{s \in \Gamma[\langle a \rangle]} \operatorname{Tr} \left( \Pi_{\langle a \rangle}^s A_{\alpha} \right) = \operatorname{Tr} \left[ \left( \sum_{s \in \Gamma[\langle a \rangle]} \Pi_{\langle a \rangle}^s \right) A_{\alpha} \right] = \operatorname{Tr} (A_{\alpha}) = 1$$

and so

$$|\operatorname{Tr}(T_aA_\alpha)| \leq \sum_{s \in \Gamma[\langle a \rangle]} \left| \operatorname{Tr} \left( \Pi^s_{\langle a \rangle} A_\alpha \right) \omega^{s(a)} \right| \leq \sum_{s \in \Gamma[\langle a \rangle]} \left| \operatorname{Tr} \left( \Pi^s_{\langle a \rangle} A_\alpha \right) \right| = 1.$$

We can now prove Theorem 3.

Proof of Theorem 3—First we need to show that the phase point operators of eq. (19) are in  $\Lambda$ . This requires checking that the Hilbert-Schmidt inner product of  $A_E^{\gamma}$  with the projector onto any stabilizer state

is nonnegative. For any maximal isotropic subgroup I with any noncontextual value assignment  $r: I \to \mathbb{Z}_d$ ,

$$\operatorname{Tr}(\Pi_I^r A_E^{\gamma}) = \frac{1}{d^{2n}} \sum_{a \in I} \sum_{b \in E} \omega^{-r(a) - \gamma(b)} \operatorname{Tr}(T_a T_b)$$
$$= \frac{1}{d^n} \sum_{a \in I} \omega^{-r(a) - \gamma(-a)}$$

Here we have two cases.

Case 1:  $r|_{I} = \gamma|_{I}$ . In this case we have

$$\operatorname{Tr}(\Pi_I^r A_{\Omega}^{\gamma}) = \frac{1}{d^n} \sum_{a \in I} \omega^{-\gamma(a) - \gamma(-a)} = \frac{|I|}{d^n} = 1.$$

Here we use property 2 of Corollary 1 in Appendix A so that  $\omega^{-\gamma(a)-\gamma(-a)}=1$  for all  $a\in I$ .

Case 2:  $r|_I \neq \gamma|_I$ . In this case, there exists a  $c \in I$  such that  $r(c) \neq \gamma(c)$ . Since I is an isotropic subgroup, it can be decomposed into a disjoint union of cosets as

$$I = J \sqcup c + J \sqcup 2c + J \sqcup \cdots \sqcup (d-1)c + J$$

where J is an isotropic subgroup with  $c \notin J$ . Then

$$\operatorname{Tr}(\Pi_I^r A_E^{\gamma}) = \frac{1}{d^n} \sum_{a \in I} \omega^{-r(a)-\gamma(-a)}$$

$$= \frac{1}{d^n} \sum_{k \in \mathbb{Z}_d} \sum_{b \in J} \omega^{-r(kc+b)+\gamma(kc+b)}$$

$$= \frac{1}{d^n} \sum_{k \in \mathbb{Z}_d} \sum_{b \in J} \omega^{-r(kc)-r(b)-\beta(kc,b)+\gamma(kc)+\gamma(b)+\beta(kc,b)}$$

$$= \frac{1}{d^n} \sum_{b \in J} \omega^{-r(b)+\gamma(b)} \sum_{k \in \mathbb{Z}_d} \omega^{-r(kc)+\gamma(kc)}$$

$$= \frac{1}{d^n} \sum_{b \in J} \omega^{-r(b)+\gamma(b)} \sum_{k \in \mathbb{Z}_d} \left[ \omega^{-r(c)+\gamma(c)} \right]^k = 0.$$

Here the last equality follows from the fact that the inner sum is a uniformly weighted sum over roots of unity and so it is zero.

Therefore, in both cases  $\text{Tr}(\Pi_I^r A_E^{\gamma}) \geq 0$ . This proves that phase point operators of eq. (19) are in  $\Lambda$ . Since  $A_E^{\gamma} \in \Lambda$ , there exists an expansion of  $A_E^{\gamma}$  as a convex combination of the vertices of  $\Lambda$ :

$$A_E^{\gamma} = \sum_{\alpha \in \mathcal{V}} p(\alpha) A_{\alpha}. \tag{20}$$

Consider a Pauli operator  $T_a$ ,  $a \in E$ . We have  $\text{Tr}(T_a A_E^{\gamma}) = \omega^{\gamma(a)}$  so multiplying eq. (20) by  $T_a$  and taking a trace we get

$$\omega^{\gamma(a)} = \sum_{\alpha \in \mathcal{V}} p(\alpha) \operatorname{Tr}(T_a A_\alpha)$$

Taking the absolute value of this equation we get

$$\left|\omega^{\gamma(a)}\right| = 1 = \left|\sum_{\alpha \in \mathcal{V}} p(\alpha) \operatorname{Tr}(T_a A_\alpha)\right| \le \sum_{\alpha \in \mathcal{V}} p(\alpha) |\operatorname{Tr}(T_a A_\alpha)| \le \sum_{\alpha \in \mathcal{V}} p(\alpha) = 1.$$

Here the first inequality is the triangle inequality and the second inequality follows from Lemma 4. The second inequality is strict if  $|\operatorname{Tr}(T_aA_\alpha)| < 1$  for any  $A_\alpha$  with  $p(\alpha) > 0$ . If this were the case then this would lead to a contradiction: 1 < 1. Therefore,  $|\operatorname{Tr}(T_aA_\alpha)| = 1$  for all  $A_\alpha$  with  $p(\alpha) > 0$ .

Now consider the real part of the equation above:

$$\operatorname{Re}\left[\omega^{\gamma(a)}\right] = \sum_{\alpha \in \mathcal{V}} p(\alpha) \operatorname{Re}\left[\operatorname{Tr}(T_a A_\alpha)\right].$$

Since the coefficients are nonnegative and sum to one, this implies that  $\operatorname{Re}\left[\operatorname{Tr}(T_aA_\alpha)\right]=\operatorname{Re}\left[\omega^{\gamma(a)}\right]$  for every  $\alpha$  with  $p(\alpha)>0$ . Thus,  $\operatorname{Tr}(T_aA_\alpha)=\omega^{\gamma(a)}$  for all  $\alpha$  with  $p(\alpha)>0$ . I.e. each  $A_\alpha$  that appears with nonzero weight in the expansion of  $A_E^\gamma$  in eq. (20) must agree with  $A_E^\gamma$  on the expectation of  $T_a$  for all  $a\in E$ . There is exactly one such operator in  $\operatorname{Herm}_1(d^n)$ , namely  $A_E^\gamma$ . Therefore,  $A_E^\gamma$  is a vertex of  $\Lambda$ .  $\square$ 

## 6 Mapping vertices of $\Lambda_m$ to $\Lambda_n$

In this section we introduce a version of the  $\Phi$ -map [49], that embeds the m-qudit polytope  $\Lambda_m$  as a subpolytope of the n-qudit polytope  $\Lambda_n$  where  $n \geq m$ , that works over qudits for an arbitrary  $d \geq 2$  (in this section we use subscripts to specify the number of qudits to avoid ambiguity).

Before stating the definition of the map let us point out some differences between the qubit and the qudit cases. For arbitrary d, maximal isotropic subgroups in E are not necessarily isomorphic as abelian groups. For example, for d=4 and n=1 the subgroups  $\langle x_1 \rangle$  and  $\langle 2x_1, 2z_1 \rangle$  are maximal isotropic subgroups in  $E_1$  but the first one is isomorphic to  $\mathbb{Z}_4$  whereas the second one is isomorphic to  $(\mathbb{Z}_2)^2$ . Although these groups have the same order (size) they are not isomorphic. One immediate consequence of this observation is that Witt's lemma, which says that the group of symplectic transformations act transitively on the set of maximal isotropic subspaces in the case of prime local dimensions (i.e. when d is a prime), fails. It is known that for arbitrary d every maximal isotropic subgroup of  $E_n$  has order  $d^n$  (see Appendix B). Another point to be cautious about is that the Pauli operators are not Hermitian for arbitrary d. This introduces some signs which requires more care.

The proof of our generalization follows closely the qubit analog described in Ref. [49] but suitably modified by taking account these differences mentioned above.

**Theorem 4** Let  $\Pi_J^r$  denote an (n-m)-qudit stabilizer projector and U denote an n-qudit Clifford unitary. The linear map

$$\Phi^r_{U,I}: Herm_1(d^m) \to Herm_1(d^n), \quad X \mapsto U(X \otimes \Pi^r_I)U^{\dagger}$$

satisfies the following properties:

- 1. The image of  $\Phi^r_{U,J}$  is given by  $\{(\Pi^{r'}_{J'}Y\Pi^{r'}_{J'})/Tr(Y\Pi^{r'}_{J'})|\ Y\in\Lambda_n\ and\ Tr\Big(Y\Pi^{r'}_{J'}\Big)\neq 0\}$  where  $\Pi^{r'}_{J'}=U(\mathbb{1}\otimes\Pi^r_J)U^{\dagger}$ .
- 2.  $\Phi_{U,I}^r$  is injective and maps a vertex  $X \in \Lambda_m$  to a vertex of  $\Lambda_n$ .
- 3. A vertex X is of the form given in Eq. (18) if and only if  $\Phi_{U,I}^r(X)$  is of the form given in Eq. (18).

We will regard  $E_m$  as a subgroup of  $E_n$  by identifying it with  $\langle x_1, \cdots, x_m, z_1, \cdots, z_m \rangle$ . We will write  $E_{n-m}^{(n)}$  for the subgroup  $\langle x_{m+1}, \cdots, x_n, z_{m+1}, \cdots, z_n \rangle$ . These two subgroups intersect at the zero element and they generate the whole group. We regard J as a subgroup of  $E_{n-m}^{(n)}$ . To be able to distinguish Pauli operators we will write  $T_a^{(n)}$  to indicate an n-qudit Pauli operator. With the choice of phase function  $\phi$  in eq. (1) we have (1)  $T_0 = \mathbbm{1}$ , (2)  $\beta(a, ka) = 0$  for all  $a \in E_n$  and  $k \in \mathbb{Z}_d$  and (3)  $\beta(a, b) = 0$  for all  $a \in E_m$  and  $b \in E_{n-m}^{(n)}$  (see Appendix A for details).

First we set  $\hat{U}$  to be the identity unitary operator  $\mathbb{1}$  and prove some results about  $\Phi_J^r = \Phi_{1,J}^r$ .

**Lemma 5** Let I be a maximal isotropic subgroup of  $E_n$ . Then

$$\mathrm{Tr}(\Pi_I^s\Phi_J^r(X))=\delta_{r|_{K\cap J},s|_{K\cap J}}\frac{|K|}{2^n}\,\mathrm{Tr}\big(\Pi_{K\cap E_m,s|_{K\cap E_m}}X\big)$$

where  $K = (E_m + J) \cap I$ . In particular, if  $X \in \Lambda_m$  then  $\Phi_J^r(X) \in \Lambda_n$ .

Proof of Lemma 5— For  $X \in \text{Herm}(d^m)$  we can write

$$X = \frac{1}{d^m} \sum_{a \in E_m} \alpha_a T_a^{(m)}$$

which gives

$$\Phi_J^r(X) = X \otimes \Pi_J^r = \frac{1}{d^n} \sum_{a+b \in E_m + J} \alpha_a \omega^{-r(b)} T_{a+b}^{(n)}.$$
 (21)

We see that  $\operatorname{Tr}(\Phi_J^r(X)) = \alpha_0 = 1$  since  $\operatorname{Tr}(X) = 1$ . Next we compute

$$\begin{split} \operatorname{Tr}(\Pi_I^s \Phi_J^r(X)) &= \frac{1}{d^{2n}} \sum_{c \in I} \sum_{a+b \in E_m + J} \alpha_a \omega^{-r(b) + s(c)} \underbrace{\operatorname{Tr}(T_c^\dagger T_{a+b})}_{d^n \delta_{c,a+b}} \\ &= \frac{1}{d^n} \sum_{a+b \in (E_m + J) \cap I} \alpha_a \omega^{-r(b) + s(a+b)} \\ &= \frac{|K \cap J|}{d^n} \sum_{a \in K \cap E_m} \alpha_a \omega^{s(a)} \left( \frac{1}{|K \cap J|} \sum_{b \in K \cap J} \omega^{-r(b)} \omega^{s(b)} \right) \\ &= \delta_{r|_{K \cap J}, s|_{K \cap J}} \frac{|K \cap J|}{d^n} \sum_{a \in K \cap E_m} \alpha_a \omega^{s(a)} \\ &= \delta_{r|_{K \cap J}, s|_{K \cap J}} \frac{|K|}{d^n} \operatorname{Tr}(\Pi_{K \cap E_m, s|_{K \cap E_m}} X) \geq 0. \end{split}$$

This shows that  $\Phi_J^r(X) \in \Lambda_n$ .  $\square$ 

**Lemma 6** For a Hermitian operator  $Y \in Herm(d^n)$  of trace zero, expressed as  $Y = \left(\sum_{0 \neq a \in E_n} z_a T_a^{(n)}\right)/d^n$  in the Pauli basis, define an operator in  $Herm(d^m)$  of trace zero by setting  $\tilde{Y} = \left(\sum_{0 \neq a \in E_m} \tilde{z}_a T_a^{(m)}\right)/d^m$  where  $\tilde{z}_a = \left(\sum_{b \in J} z_{a+b} \omega^{r(b)}\right)/|J|$ . Let  $I' \subset E_m$  be a maximal isotropic subgroup and  $s' : I' \to \mathbb{Z}_d$  be a value assignment. Then

$$\operatorname{Tr}(\Pi_{I'+J,s'*r}Y) = \operatorname{Tr}\left(\Pi_{I',s'}\tilde{Y}\right) \tag{22}$$

where  $s' * r : (I' + J) \to \mathbb{Z}_d$  is the value assignment defined by s' \* r(a + b) = s'(a) + r(b).

Proof of Lemma 6— We calculate

$$\begin{aligned} \operatorname{Tr} \Big( \Pi_{I'+J}^{s'*r} Y \Big) &= \frac{1}{|I'||J|} \sum_{a \in I'} \sum_{b \in J} z_{a+b} \omega^{s'(a)+r(b)} \\ &= \frac{1}{|I'|} \sum_{a \in I'} \omega^{s'(a)} \frac{1}{|J|} \sum_{b \in J} z_{a+b} \omega^{r(b)} \\ &= \frac{1}{|I'|} \sum_{a \in I'} \omega^{s'(a)} \tilde{z}_a \\ &= \operatorname{Tr} \Big( \Pi_{I',s'} \tilde{Y} \Big). \end{aligned}$$

Proof of Theorem 4— We start with the case  $U=\mathbb{1}$ . Part (1) follows immediately from comparing  $\Pi_J^r Y \Pi_J^r$  after normalization with Eq. (21). For part (2) note that Lemma 5 says that  $\Phi_J^r(\Lambda_m) \subset \Lambda_n$ . Let X be a vertex of  $\Lambda_m$ . As in the Theorem 1 in [49] Lemma 5 and Lemma 6 can be used to show that  $\{\Phi_J^r(X) \pm Y\} \not\subset \Lambda_n$  for any  $Y \in \operatorname{Herm}(d^n)$  of trace zero. Therefore  $\Phi_J^r(X) \in \Lambda_n$  is a vertex whenever  $X \in \Lambda_m$  is a vertex. This proves part (2). For the general case, that is when U is an arbitrary element in  $\mathcal{C}\ell_n$ , part (1) follows from the observation that  $\Phi_{U,J}^r(X) = U\Phi_{1,J}^r(X)U^{\dagger}$  and that we can run though  $UYU^{\dagger}$  where  $Y \in \Lambda_n$ . Part (2) follows from the fact that  $\mathcal{C}\ell_n$  maps vertices of  $\Lambda_n$  to vertices, this is statement 2 of Theorem 1. For part (3)

observe that if  $X = A_{\Omega}^{\gamma}$  then the image  $\Phi_{J}^{r}(A_{\Omega}^{\gamma})$  has the form  $A_{\Omega+J}^{\gamma*r}$ . Conversely, if  $\Phi_{J}^{r}(X)$  has the form  $A_{\tilde{\Omega}}^{\tilde{\gamma}}$  then  $X = A_{\Omega}^{\gamma}$  where  $\Omega = E_{m} \cap \tilde{\Omega}$  and  $\gamma = \tilde{\gamma}|_{\Omega}$ . This extends to  $\Phi_{U,J}^{r}$  since conjugating by a Clifford unitary preserves the form given in Eq. (18).  $\square$ 

#### 7 Discussion

In this paper, we have presented a hidden variable model (Theorem 1 in Section 3) which represents all aspects of quantum computation with magic states using only classical (nonnegative) probabilities. In this model, magic states are represented by a probability distribution over a finite set according to eq. (12). Clifford gates are represented by a deterministic update rule—a map from the set of hidden variables to itself. Pauli measurements are represented by a probabilistic update rule—a map from hidden variables to probability distributions over the set of hidden variables according to eq. (13). This model is similar in form to many previously defined quasiprobability representations of quantum computation with magic states [13, 16, 20, 23, 25], but with the distinguishing feature that in our model all states can be represented by a probability distribution. No "negative probabilities" are required. This is the generalization of the hidden variable model of Ref. [28] to qudits of arbitrary local Hilbert space dimension.

Since everything in this model is represented probabilistically, the model leads to a classical simulation algorithm for quantum computation with magic states based on sampling from the defining probability distributions. This is Algorithm 1 presented in Section 4. This algorithm is similar in structure to simulation algorithms based on sampling from quasiprobability distributions like the discrete Wigner function [16, 23, 25, 38], except that those algorithms are limited in their scope. They can only simulate quantum circuits for which the input state of the circuit is represented by a probability distribution. Since in our model any state can be represented by a probability distribution as in eq. (12), we have no such limitation. Note, however, that although Algorithm 1 can simulate any quantum computation, we make no claims that the simulation is efficient in general. In fact, if a quantum computational speedup over classical computation is possible at all, as many people believe, then the algorithm must be inefficient in general.

There are, however, some important cases where the simulation is efficient. For example, in the qubit case, it is known that the (efficient) classical simulation algorithm of Ref. [25] is a special case of this more general model/simulation algorithm. This is a result of the fact that the phase point operators of the form eq. (18) are vertices of the polytope  $\Lambda$  [46, 28]. These are the cnc-type vertices. In Section 5 we show that for higher-dimensional qudits as well there are vertices of  $\Lambda$  of cnc-type. We conjecture that the update of these vertices under Clifford gates and Pauli measurements will be efficiently computable classically. This would result in the simulation algorithm, Algorithm 1, being efficient whenever the support of the probability distribution of eq. (12) representing the input state of the circuit is restricted to vertices of this type.

Of course not all vertices of  $\Lambda$  are cnc-type. For example, Ref. [50] characterizes all vertices of the two-qubit  $\Lambda$  polytope. Under the action of the Clifford group there are eight orbits of vertices, only two of which are cnc-type vertices. Characterizing the remaining non-cnc-type vertices of  $\Lambda$  for arbitrary n and d is an open problem which could expand the scope of efficiency of Algorithm 1. This has already been partially achieved in the qubits case: Ref. [49] provides an efficient description, along with update rules under Pauli measurements, for one of the non-cnc-type orbits of the two qubit  $\Lambda$  polytope. As a result of the  $\Phi$ -map, which maps vertices of the m-qubit polytope to vertices of the n-qubit polytope with m < n, this also provides a description of a non-cnc-type orbit of vertices for the  $\Lambda$  polytope on any number of qubits. Ref. [49] also provides a reduction of the simulation algorithm that shows that the  $\Phi$ -map does not significantly increase the cost of classical simulation. Therefore, the scope of efficiency of Algorithm 1 in the qubit case goes beyond states supported on cnc-type vertices. In section 6 we show that a version of the  $\Phi$ -map also holds for qudits of arbitrary dimension. We conjecture that a version of the corresponding reduction of the simulation algorithm with the resource theory perspective in which preparation of stabilizer states is considered a "free" operation, along with Clifford gates and Pauli measurements [19].

We have seen evidence that the  $\Lambda$  polytopes could prove to be of independent interest. A subset of vertices, namely the cnc-type vertices, have already found a handful of applications [46, 51, 25]. The remaining vertices are less well studied but have proven useful in certain areas as well [50, 49]. Therefore, we conclude by proposing the family of  $\Lambda$  polytopes for arbitrary n and d as a subject of further study.

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## A Properties of $\beta$

**Lemma 7** Viewed as a function from  $E \times E \to \mathbb{Q}$ , for any number of qudits, the function  $\beta$  is given by

$$\beta(a,b) = \begin{cases} (\langle a_Z | b_X \rangle - \langle a_X | b_Z \rangle) \cdot 2^{-1} & \text{if } d \text{ is odd} \\ -\frac{\langle a_Z | b_X \rangle - \langle a_X | b_Z \rangle}{2} & \text{if } d \text{ is even.} \end{cases}$$
 (23)

Here, the inner products  $\langle a_Z|b_X\rangle$ ,  $\langle a_X|b_Z\rangle$  are not computed modulo d. In the odd case,  $2^{-1} \in \mathbb{Z}$  is the integer representation of the inverse of 2 modulo d. In the case of even d, the division is done by the usual 2.

If d is even and the symplectic inner product of a, b is  $[a, b] = 0 \pmod{2}$ , then  $\beta(a, b) \in \mathbb{Z}$  and by slight abuse of notation, we consider  $\beta(a, b)$  as a value in  $\mathbb{Z}_d$  for the noncontextual value assignment condition (6).

Proof of Lemma 7—For  $a=(a_Z,a_X), b=(b_Z,b_X)\in\mathbb{Z}_d^{2n}$  we have the identities

$$X(a_X)X(b_X) = X(a_X + b_X),$$
  

$$Z(a_Z)Z(b_Z) = Z(a_Z + b_Z),$$
  

$$X(a_X)Z(b_Z) = \omega^{-\langle a_X|b_Z\rangle}Z(b_Z)X(a_X).$$

Since  $\omega = \mu^2$  for even d it follows for  $T_a = \mu^{\phi(a)} Z(a_Z) X(a_X)$ ,  $T_b = \mu^{\phi(b)} Z(b_Z) X(b_X)$  that

$$T_a T_b = \omega^{-\beta(a,b)} T_{a+b} = \begin{cases} \mu^{\phi(a)+\phi(b)-\langle a_X|b_Z\rangle - \phi(a+b)} T_{a+b}, & \text{if } d \text{ is odd} \\ \mu^{\phi(a)+\phi(b)-2\langle a_X|b_Z\rangle - \phi(a+b)} T_{a+b}, & \text{if } d \text{ is even} \end{cases}$$

and using that

$$\phi(a+b) = \begin{cases} \phi(a) + \phi(b) - (\langle b_Z | a_X \rangle - \langle a_Z | b_X \rangle) \cdot 2^{-1}, & \text{if } d \text{ is odd} \\ \phi(a) + \phi(b) - \langle b_Z | a_X \rangle - \langle a_Z | b_X \rangle, & \text{if } d \text{ is even} \end{cases}$$

we obtain

$$\beta(a,b) = \begin{cases} (\langle a_Z | b_X \rangle - \langle a_X | b_Z \rangle) \cdot 2^{-1} & \text{if } d \text{ is odd} \\ -\frac{\langle a_Z | b_X \rangle - \langle a_X | b_Z \rangle}{2} & \text{if } d \text{ is even,} \end{cases}$$

where the inner products  $\langle a_Z|b_X\rangle$ ,  $\langle a_X|b_Z\rangle$  are not computed modulo d.  $\square$ 

**Corollary 1** *The function*  $\beta$  *has the following properties:* 

1. For any  $a \in \mathbb{Z}_d^{2n}$  and any  $k \in \mathbb{Z}_d$ ,  $\beta(a, ka) = 0,$ and in particular,  $\beta(a, -a) = 0.$  (24)

- 2. For any noncontextual value assignment  $\gamma$ ,  $\gamma(-a) = -\gamma(a)$ .
- 3. For two noncontextual value assignments  $\gamma$  and  $\nu$ , and any  $k \in \mathbb{Z}_d$ ,  $\gamma(ka) \nu(ka) = k\gamma(a) k\nu(a)$ .

*Proof of Corollary 1*—The first property follows immediately from the explicit form for  $\beta$  found in Lemma 7. With eq. (6) we have

$$\gamma(a) + \gamma(-a) - \gamma(0) = -\beta(a, -a)$$

With the phase convention chosen,  $T_0 = 1$  so  $\gamma(0) = 0$  and with property 1 above  $\beta(a, -a) = 0$ . Therefore,  $\gamma(-a) = -\gamma(a)$ . This proves the second property.

By iteratively applying the formula  $\gamma(a) + \gamma(b) - \gamma(a+b) = -\beta(a,b)$  which holds for any noncontextual value assignment  $\gamma$ , it can be shown that

$$\gamma(ka) = k\gamma(a) + \beta((k-1)a, a) + \beta((k-2)a, a) + \dots + \beta(a, a).$$

Therefore, for two noncontextual value assignments  $\gamma$  and  $\nu$ , we have

$$\gamma(ka) - \nu(ka) = [k\gamma(a) + \beta((k-1)a, a) + \beta((k-2)a, a) + \dots + \beta(a, a)] - [k\nu(a) + \beta((k-1)a, a) + \beta((k-2)a, a) + \dots + \beta(a, a)] = k\gamma(a) - k\nu(a).$$

This proves the third property.  $\Box$ 

## B The qudit stabilizer formalism

The stabilizer formalism [52] describes a large family of quantum error correcting codes, as well as a broader framework for describing quantum error correction and fault-tolerant quantum computation. In this appendix we review some features of the stabilizer formalism for systems of qudits [39] which are useful for some of the proofs in the main text.

A stabilizer code is specified by a pair (I, r) where I is an isotropic subgroup of E, and  $r: I \to \mathbb{Z}_d$  is a noncontextual value assignment for I. The codespace of the code is the simultaneous +1-eigenspace of the Pauli operators  $S_I^r = \{\omega^{-r}T_a \mid a \in I\}$ .  $S_I^r$  is called the stabilizer group of the code. The projector onto this eigenspace, or equivalently, the projector onto the eigenspace of the Pauli observables labeled by I corresponding to eigenvalues  $\{\omega^{r(a)} \mid a \in I\}$  is

$$\Pi_I^r = \frac{1}{|I|} \sum_{b \in I} \omega^{-r(b)} T_b.$$

The dimension of this eigenspace is  $d^n/|I|$  [53]. In particular, we have the following lemma.

**Lemma 8** The order of a maximal isotropic subgroup of E is  $d^n$ .

For the proof of Lemma 8 see Theorem 1 in Ref. [53]. If I is a maximal isotropic subgroup of E,  $|I| = d^n$ , and there is a unique quantum state fixed by a stabilizer group  $S_I^r$ . Such states are called stabilizer states.

Stabilizer code projectors can be constructed from products of stabilizer code projectors of lower rank. If  $\{I_k \mid I_k \subset I\}$  are such that  $I = \bigcup_k I_k$ , and the value assignments  $r_k : I_k \to \mathbb{Z}_d$  satisfy  $r_k = r|_{I_k}$ , then

$$\Pi_I^r = \prod_k \Pi_{I_k}^{r_k}.$$

Stabilizer measurements can also be coarse-grained to give stabilizer projectors of lower rank. This is the result of the following lemma.

**Lemma 9** If I is a non-maximal isotropic subgroup and  $\Pi_I^r = \frac{1}{|I|} \sum_{b \in I} \omega^{-r(b)} T_b$  a stabilizer code projector for some noncontextual value assignment  $r: I \to \mathbb{Z}_d$ , then for any isotropic subgroup I' containing I

$$\sum_{r'\in\Gamma}\Pi_{I'}^{r'}=\Pi_{I}^{r},\tag{25}$$

where  $\Gamma$  is the set of all noncontextual value assignments on I' such that  $r'|_{I} = r$ .

Proof of Lemma 9— The proof is obtained by adapting the proof of Ref. [25, Lemma 1]. Let I, I' be isotropic subgroups of E such that  $I \subseteq I'$ . Let  $r: I \to \mathbb{Z}_d$  be a noncontextual value assignment for I and  $\Gamma$  be the set of all noncontextual value assignments on I' satisfying

$$r'|_{I} = r \quad \forall r' \in \Gamma.$$

Then

$$\sum_{r' \in \Gamma} \Pi_{I'}^{r'} = \sum_{r' \in \Gamma} \frac{1}{|I'|} \sum_{a \in I'} \omega^{-r'(a)} T_a = \frac{1}{|I'|} \sum_{a \in I'} \left[ \sum_{r' \in \Gamma} \omega^{-r'(a)} \right] T_a.$$

We have two cases for the inner sum in the last expression. If  $a \in I$  then r'(a) = r(a) for all  $r' \in \Gamma$ . Therefore,

$$\sum_{r' \in \Gamma} \omega^{-r'(a)} = |\Gamma| \omega^{-r(a)}.$$

In the second case,  $a \notin I$ .  $\Gamma$  is the coset of a vector space, the proof of this is analogous to the proof of Ref. [25, Lemma 2] which applies only to qubits. Therefore, by character orthogonality,

$$\sum_{r' \in \Gamma} \omega^{-r'(a)} = 0.$$

Thus,

$$\sum_{r' \in \Gamma} \Pi_{I'}^{r'} = \frac{1}{|I'|} \sum_{a \in I'} |\Gamma| \delta_{a \in I} \omega^{-r(a)} T_a = \frac{|\Gamma|}{|I'|} \sum_{a \in I} \omega^{-r(a)} T_a.$$

We have  $|\Gamma| = |I'|/|I|$ . Therefore,

$$\sum_{r' \in \Gamma} \Pi_{I'}^{r'} = \frac{1}{|I|} \sum_{a \in I} \omega^{-r(a)} T_a = \Pi_I^r.$$

Corollary 2 The stabilizer polytope is

$$SP = conv \left\{ \frac{|I|}{d^n} \Pi_I^r \mid \Pi_I^r \text{ stabilizer code projector} \right\}.$$
 (26)

The above lemmas describe structural properties of stabilizer code projectors. It will also be useful for us to describe how stabilizer projectors behave under the dynamical operations of quantum computation with magic states—Clifford gates and Pauli measurements. First, we have the following result regarding Clifford gates.

**Lemma 10** For any Clifford group element  $U \in \mathcal{C}\ell$ , and any stabilizer state  $|\sigma\rangle \in \mathcal{S}$ ,

$$U |\sigma\rangle \langle \sigma| U^{\dagger} = |\sigma'\rangle \langle \sigma'|$$

where  $|\sigma'\rangle \in \mathcal{S}$  is a stabilizer state. I.e. Clifford group elements map stabilizer states to stabilizer states.

Proof of Lemma 10—The action of the Clifford group on the Pauli operators is defined in eq. (4). With this equations, for a projector onto a stabilizer state  $|\sigma\rangle$  corresponding to maximal isotropic subgroup  $I \subset E$  and noncontextual value assignment  $r: I \to \mathbb{Z}_d$  we have

$$U\left|\sigma\right\rangle\left\langle\sigma\right|U^{\dagger} = U\Pi_{I}^{r}U^{\dagger} = \frac{1}{\left|I\right|}\sum_{a\in I}\omega^{-r(a)}UT_{a}U^{\dagger} = \frac{1}{\left|I\right|}\sum_{a\in I}\omega^{-r(a)+\Phi_{U}(a)}T_{S_{U}(a)} = \frac{1}{\left|I\right|}\sum_{a\in U\cdot I}\omega^{-U\cdot r(a)}T_{a} = \Pi_{U\cdot I}^{U\cdot r(a)}$$

where  $U \cdot I = \{S_U(a) \mid a \in I\}$  and  $U \cdot r$  is defined by the relation

$$U \cdot r(S_U(a)) = r(a) - \Phi_U(a), \quad \forall a \in I.$$

In order to show that  $U | \sigma \rangle \langle \sigma | U^{\dagger}$  is a projector onto a stabilizer state is suffices to show that  $U \cdot I$  is a maximal isotropic subgroup of E and  $U \cdot r : U \cdot I \to \mathbb{Z}_d$  is a noncontextual value assignment for  $U \cdot I$ .

 $U \cdot I$  is isotropic since I is isotropic and  $S_U$  is a symplectic operation. Also,  $|U \cdot I| = |I| = d^n$  so by Lemma 8,  $U \cdot I$  is a maximal isotropic subgroup.

Since  $T_0 \propto 1$ 

$$UT_0U^{\dagger} = T_0$$

so  $\Phi_U(0) = 0$  for any  $U \in \mathcal{C}\ell$ . Therefore,

$$\omega^{-U \cdot r(0)} T_0 = \omega^{-r(0) + \Phi_U(0)} T_0 = \omega^{-r(0)} T_0 = 1.$$

For any  $a, b \in U \cdot I$ , there exist  $c, d \in I$  such that  $S_U(c) = a$  and  $S_U(d) = b$ . Computing the product  $UT_cT_dU^{\dagger}$  in two different ways we have

$$UT_{c}T_{d}U^{\dagger} = UT_{c}U^{\dagger}UT_{d}U^{\dagger} = \omega^{\Phi_{U}(c) + \Phi_{U}(d)}T_{S_{U}(c)}T_{S_{U}(d)} = \omega^{\Phi_{U}(c) + \Phi_{U}(d) - \beta(a,b)}T_{a+b}$$

and

$$UT_cT_dU^{\dagger} = \omega^{-\beta(c,d)}UT_{c+d}U^{\dagger} = \omega^{-\beta(c,d)+\Phi_U(c+d)}T_{a+b}.$$

Therefore,  $-\Phi_U(c) - \Phi_U(d) + \Phi_U(c+d) = \beta(c,d) - \beta(a,b)$  and so

$$U \cdot r(a) + U \cdot r(b) - U \cdot r(a+b) = r(c) + r(d) - r(c+d) - \Phi_U(c) - \Phi_U(d) + \Phi_U(c+d) = -\beta(a,b).$$

Thus,  $U \cdot r$  satisfies eq. (6) and  $\Pi_{U,I}^{U \cdot r}$  is a projector onto a stabilizer state.  $\square$ 

The update of stabilizer states under Pauli measurements is probabilistic in general. It is shown in the following lemma.

**Lemma 11** For any isotropic subgroups  $I, J \subset E$  and any noncontextual value assignments  $r: I \to \mathbb{Z}_d$  and  $s: J \to \mathbb{Z}_d$ ,

1. if  $r|_{I \cap I} = s|_{I \cap I}$  then

$$\mathrm{Tr}(\Pi_J^s\Pi_I^r) = \frac{|I\cap J|}{|I||J|}d^n > 0 \quad and \quad \frac{\Pi_I^r\Pi_J^s\Pi_I^r}{\mathrm{Tr}(\Pi_I^s\Pi_I^r)} = \frac{|J\cap I^\perp|}{|J|}\Pi_{I+J\cap I^\perp}^{r\star s}.$$

where  $r \star s$  is the unique noncontextual value assignment on the set  $I + J \cap I^{\perp}$  such that  $r \star s|_{I} = r$  and  $r \star s|_{J \cap I^{\perp}} = s|_{J \cap I^{\perp}}$ .

2. If  $r|_{I \cap J} \neq s|_{I \cap J}$  then

$$\mathrm{Tr}(\Pi_J^s\Pi_I^r)=0 \quad and \quad \Pi_I^r\Pi_J^s\Pi_I^r=0.$$

Proof of Lemma 11—Let  $I, J \subset E$  be isotropic subgroups with noncontextual value assignments  $r: I \to \mathbb{Z}_d$  and  $s: J \to \mathbb{Z}_d$ .

Case 1:  $r|_{I\cap J} = s|_{I\cap J}$ . Let  $r\star s$  denote the unique noncontextual value assignment on the set  $I+J\cap I^{\perp}$  such that  $r\star s|_{I} = r$  and  $r\star s|_{J\cap I^{\perp}} = s|_{J\cap I^{\perp}}$ . We calculate

$$\Pi_{I}^{r}\Pi_{J}^{s}\Pi_{I}^{r} = \frac{1}{|I|^{2}|J|} \sum_{a,c\in I} \sum_{b\in J} \omega^{-r(a)-r(c)-s(b)} T_{a} T_{b} T_{c} 
= \frac{1}{|I|^{2}|J|} \sum_{a,c\in I} \sum_{b\in J} \omega^{-r(a)-r(c)-s(b)-\beta(a,c)+[b,c]} T_{a+c} T_{b} 
= \frac{1}{|I|^{2}|J|} \sum_{a,c\in I} \sum_{b\in J} \omega^{-r(a+c)-s(b)+[b,c]} T_{a+c} T_{b} 
= \frac{1}{|I||J|} \Pi_{I}^{r} \sum_{b\in J} \omega^{-s(b)} \left[ \sum_{c\in I} \omega^{[b,c]} \right] T_{b} 
= \frac{1}{|J|} \prod_{b\in J\cap I^{\perp}} \omega^{-s(b)} T_{b} 
= \frac{1}{|I||J|} \sum_{a\in I} \sum_{b\in J\cap I^{\perp}} \omega^{-r(a)-s(b)-\beta(a,b)} T_{a+b} 
= \frac{1}{|I||J|} \sum_{a\in I} \sum_{b\in J\cap I^{\perp}} \omega^{-r\star s(a+b)} T_{a+b}.$$
(27)

Each Pauli operator in the above sum appears with the same multiplicty. For an element  $a \in I + J \cap I^{\perp}$ , let  $\mu(a)$  denote the number of distinct pairs  $(b,c) \in I \times J \cap I^{\perp}$  such that b+c=a. We have  $\mu(a)=\mu(0)$  for any  $a \in I + J \cap I^{\perp}$ . To see this, suppose  $a \in I + J \cap I^{\perp}$  and let  $(c_1,d_1),(c_2,d_2),\ldots,(c_{\mu(a)},d_{\mu(a)})$  be the distinct pairs in  $I \times J \cap I^{\perp}$  such that  $c_j+d_j=a$ . Then the pairs  $(c_j-c_1,d_j-d_1) \in I \times J \cap I^{\perp}$  for  $j=2,\ldots,\mu(a)$  together with the pair  $(0,0) \in I \times J \cap I^{\perp}$  show that  $\mu(0) \geq \mu(a)$ . Therefore,  $\mu(0) \geq \mu(a)$  for any  $a \in I + J \cap I^{\perp}$ .

Now let  $(c_1, d_1), (c_2, d_2), \ldots, (c_{\mu(0)}, d_{\mu(0)})$  denote the  $\mu(0)$  distinct pairs in  $I \times J \cap I^{\perp}$  such that  $c_j + d_j = 0$ , and  $(c, d) \in I \times J \cap I^{\perp}$  be such that c + d = a. Then the pairs  $(c_j + c, d_j + d), j = 1, 2, \ldots, \mu(0)$  show that  $\mu(a) \geq \mu(0)$ . Therefore,  $\mu(a) = \mu(0)$  for any  $a \in I + J \cap I^{\perp}$ . Here we can see  $\mu(0) = |I \cap J|$ 

Then we have

$$\Pi_{I}^{r}\Pi_{J}^{s}\Pi_{I}^{r} = \frac{1}{|I||J|} \sum_{a \in I} \sum_{b \in J \cap I^{\perp}} \omega^{-r \star s(a+b)} T_{a+b}$$

$$= \frac{\mu(0)}{|I||J|} \sum_{a \in I+J \cap I^{\perp}} \omega^{-r \star s(a)} T_{a} = \frac{|I \cap J||I+J \cap I^{\perp}|}{|I||J|} \Pi_{I+J \cap I^{\perp}}^{r \star s}.$$

Since  $|I + J \cap I^{\perp}| = |I| \cdot |J \cap I^{\perp}|/|I \cap J|$ ,

$$\Pi_I^r \Pi_J^s \Pi_I^r = \frac{|J \cap I^{\perp}|}{|J|} \Pi_{I+J \cap I^{\perp}}^{r \star s}.$$

Taking a trace of this equation we get

$$\operatorname{Tr}(\Pi_I^r\Pi_J^s) = \operatorname{Tr}(\Pi_I^r\Pi_J^s\Pi_I^r) = \frac{|J \cap I^{\perp}|}{|J|} \operatorname{Tr}\left(\Pi_{I+J \cap I^{\perp}}^{r \star s}\right) = \frac{|J \cap I^{\perp}|}{|J|} \cdot \frac{d^n}{|I+J \cap I^{\perp}|} = \frac{|I \cap J|}{|I||J|} d^n > 0.$$

Case 2:  $r|_{I\cap J} \neq s|_{I\cap J}$ . Denote by  $S_J^s$  the stabilizer group  $\{\omega^{-s(b)}T_b \mid b \in J\}$  and denote by  $V_J^s$  the subspace of the Hilbert space  $\mathbb{C}^{d^n}$  stabilized by  $S_J^s$ . By assumption, there exists a  $c \in I \cap J$  such that

 $r(c) \neq s(c)$ . Therefore, for any vector  $|\psi\rangle \in V_J^s$ ,

$$\begin{split} \Pi_I^r \left| \psi \right\rangle = & \Pi_I^r \omega^{-s(c)} T_c \left| \psi \right\rangle \\ = & \frac{1}{d^n} \sum_{a \in I} \omega^{-r(a) - s(c)} T_a T_c \left| \psi \right\rangle \\ = & \frac{1}{d^n} \sum_{a \in I} \omega^{-r(a) - s(c) - \beta(a,c)} T_{a+c} \left| \psi \right\rangle \\ = & \frac{1}{d^n} \sum_{a \in I} \omega^{-r(a+c) - r(c) - s(c)} T_{a+c} \left| \psi \right\rangle \\ = & \frac{1}{d^n} \sum_{a \in I} \omega^{-r(a) - r(c) - s(c)} T_a \left| \psi \right\rangle \\ = & \omega^{-r(c) - s(c)} \Pi_I^s \left| \psi \right\rangle. \end{split}$$

That is,  $\Pi_I^s |\psi\rangle = \omega^{-r(c)-s(c)} \Pi_I^s |\psi\rangle$ . But since  $r(c) \neq s(c)$ ,  $\omega^{-r(c)-s(c)} \neq 1$  so this relation can be true only if  $\Pi_I^r |\psi\rangle = 0$ .

We can write

$$\Pi_{J}^{s} = \sum_{i=1}^{\dim(V_{J}^{s})} \left| \psi_{i} \right\rangle \left\langle \psi_{i} \right|$$

where  $\{|\psi_i\rangle \mid 1 \leq i \leq \dim(V_J^s)\}$  is a basis for  $V_J^s$ . Therefore,

$$\Pi_{I}^{r}\Pi_{J}^{s}\Pi_{I}^{r} = \sum_{i=1}^{\dim(V_{J}^{s})} \Pi_{I}^{r} \left| \psi_{i} \right\rangle \left\langle \psi_{i} \right| \Pi_{I}^{r} = 0.$$

Taking a trace of this equation we get

$$\operatorname{Tr}(\Pi_I^r\Pi_I^s\Pi_I^r) = \operatorname{Tr}(\Pi_I^r\Pi_I^s) = 0.$$

## C Polar duality in the space of Hermitian matrices of trace one

For a Hilbert space  $\mathcal{H}$  with inner product  $\langle \cdot, \cdot \rangle$  define the polar dual [43] of a set  $P \subset \mathcal{H}$  as

$$P^* = \left\{ x \in \mathbb{R}^N \mid \langle x, y \rangle \ge -\frac{1}{d^n} \text{ for all } y \in P \right\}.$$
 (28)

If  $P = \operatorname{conv}\{v_1, \dots, v_k\}$  is a polytope, then this simplifies to

$$P^* = \left\{ x \in \mathbb{R}^N \mid \langle x, v_i \rangle \ge -\frac{1}{d^n}, \ i, \dots, k \right\}.$$
 (29)

If  $P \subset Q \subset \mathcal{H}$ , then obviously

$$Q^* \subset P^*. \tag{30}$$

In our setting, we are interested in objects living in the affine space of matrices of trace one  $\operatorname{Herm}_1(d^n)$ . We can project  $\operatorname{Herm}_1$  into the linear subspace of  $\operatorname{Hermitian}$  matrices of trace zero  $\operatorname{Herm}_0(d^n)$  via the transformation

$$\pi: X \in \operatorname{Herm}_1(d^n) \to \operatorname{Herm}_0(d^n), \qquad X \mapsto X - \frac{1}{d^n} \mathbb{1}.$$
 (31)

<sup>&</sup>lt;sup>1</sup>The usual definition differs slightly by an irrelevant scaling factor [43].

Now, observe that for  $X, Y \in \text{Herm}_1(d^n)$  we have the relation

$$\operatorname{Tr}(XY) \ge 0 \iff \operatorname{Tr}(\pi(X)\pi(Y)) = \operatorname{Tr}(XY) - \operatorname{Tr}\left(X\frac{1}{d^n}\mathbb{1}\right) \ge -\frac{1}{d^n}.$$
 (32)

Hence, by associating  $\operatorname{Herm}_0(d^n)$  and  $\operatorname{Herm}_1(d^n)$ , we define for a set  $M \subset \operatorname{Herm}_1(d^n)$ 

$$M^* := \{ X \in \operatorname{Herm}_1(d^n) \mid \operatorname{Tr}(XY) \ge 0 \text{ for all } Y \in M \}.$$
 (33)

Next, we will explain the relation of dilation (see Eq. (8)) and duality in  $\operatorname{Herm}_1(d^n)$ .

**Lemma 12** The dilation of a set  $M \subset Herm_1(d^n)$  satisfies

$$(c \cdot M)^* = \frac{1}{c} \cdot M^*.$$

Proof of Lemma 2— Observe that every  $A, Y \in \text{Herm}_1(d^n)$  can be written as

$$Y = \frac{1}{d^n} \mathbb{1} + c\pi(X), \quad A = \frac{1}{d^n} \mathbb{1} + \frac{1}{c}\pi(B)$$

for suitable  $B, X \in \text{Herm}_1(d^n)$ . Then

$$\operatorname{Tr}(AY) = \frac{1}{d^{2n}}\operatorname{Tr}(\mathbb{1}) + \operatorname{Tr}(\pi(B)\pi(X)) = \frac{1}{d^n} + \operatorname{Tr}\left(B(X - \frac{1}{d^n}\mathbb{1})\right) + \operatorname{Tr}\left(-\frac{1}{d^n}\mathbb{1}(X - \frac{1}{d^n}\mathbb{1})\right)$$
(34)

$$=\frac{1}{d^n} - \frac{1}{d^n} + \text{Tr}(BX) \tag{35}$$

$$= Tr(BX). (36)$$

Hence, if  $Y \in c \cdot M$  with  $X \in M$ , then  $A \in (c \cdot M)^*$  if and only if

$$0 \le \operatorname{Tr}(AY) = \operatorname{Tr}(BX) \quad \Longleftrightarrow \quad B \in M^*, \tag{37}$$

which is equivalent to  $A \in \frac{1}{c} \cdot M^*$