Efficient classical simulation of quantum computations beyond Wigner positivity

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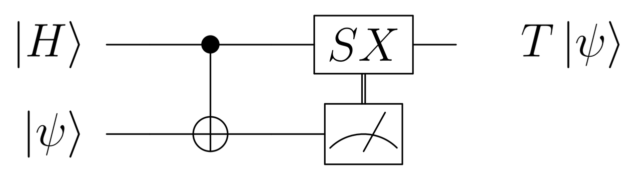
Abstract: We present the generalization of the CNC formalism, based on closed and noncontextual sets of Pauli observables, to the setting of odd-prime-dimensional qudits. By introducing new CNC-type phase space point operators, we construct a quasiprobability representation for quantum computation which is covariant with respect to the Clifford group and positivity preserving under Pauli measurements, and whose nonnegative sector strictly contains the subtheory of quantum theory described by nonnegative Wigner functions. This allows for a broader class of magic state quantum circuits to be efficiently classically simulated than those covered by the stabilizer formalism and Wigner function methods. For more details, see the preprint Ref. [6].

Magic state quantum computation (QCM)

QCM [1] is a model of quantum computation in which:

- Allowed operations are restricted to stabilizer operations—Clifford gates and Pauli measurements.
- These alone are not universal for quantum computation, and they can be efficiently simulated classically.
- Universality is restored by additional nonstabilizer quantum states at the input of the circuit.

For example, a T gate is implemented by the following circuit



where $|H\rangle = (|0\rangle + e^{i\pi/4}|1\rangle)/\sqrt{2}$.

Definitions

- Let $\mathrm{Herm}(\mathbb{C}^{d^n})$ denote the Hermitian operators on $\mathbb{C}^{d^n}$, and $\mathcal{S}_n$ the set of n-qudit stabilizer states.
- Measurements are n-qudit Pauli observables which we label by $\mathbb{Z}_d^{2n}$. Eigenvalues of Paulis are ω^k, $k \in \mathbb{Z}_d$ where $\omega = \exp(2\pi i/d)$. For any Pauli T_a, $a \in \mathbb{Z}_d^{2n}$, the projector for measurement outcome $s \in \mathbb{Z}_d$ is denoted Π_a^s.
- The Clifford group $\mathcal{C}\ell$ is the group of unitary gates that map Pauli operators to Pauli operators under conjugation.

Quasiprobability simulation methods

Define a set $\{A_\alpha \mid \alpha \in \mathcal{V}\} \subset \mathrm{Herm}(\mathbb{C}^{d^n})$ with the properties:

1. $\mathrm{span}(\{A_\alpha \mid \alpha \in \mathcal{V}\}) = \mathrm{Herm}(\mathbb{C}^{d^n})$,
2. $\mathrm{Tr}(A_\alpha) = 1 \quad \forall \alpha \in \mathcal{V}$,
3. For any Clifford gate $g \in \mathcal{C}\ell$,
$$gA_\alpha g^\dagger = A_{g\cdot\alpha},$$
4. For any Pauli projector Π_a^s,
$$\Pi_a^s A_\alpha \Pi_a^s = \sum_\beta q_{\alpha,a}(\beta,s) A_\beta$$
with $q_{\alpha,a}(\beta,s) \geq 0$ and $\sum_{\beta,s} q_{\alpha,a}(\beta,s) = 1$.

Then in the quasiprobability representation:

- States are represented as
$$\rho = \sum_{\alpha\in\mathcal{V}} W_\rho(\alpha) A_\alpha$$
with $\sum_\alpha W_\rho(\alpha) = 1$ (follows from properties 1 & 2)
- Computational dynamics represented by properties 3 & 4

Classical simulation of QCM

When $W_\rho \geq 0$, it is a probability distribution over $\mathcal{V}$.

$\rho \Longrightarrow$ $\boxed{\text{Clifford gates+ Pauli Measurements}}$ $\Longrightarrow$ Measurement outcomes

To simulate a circuit with input state $W_\rho \geq 0$,

1: sample $\alpha \in \mathcal{V}$ from W_ρ, propagate α through circuit
2: **while** the end of the circuit has not been reached **do**
3: **if** a Clifford gate $g \in \mathcal{C}\ell$ is encountered **then**
4: update phase space point according to $\alpha \longleftarrow g\cdot\alpha$
5: **if** a Pauli measurement $a \in \mathbb{Z}_d^{2n}$ is encountered **then**
6: sample $(\beta, s) \in \mathcal{V}\times\mathbb{Z}_d$ according to $q_{\alpha,a}$
7: **return** s as the outcome of the measurement
8: update phase space point according to $\alpha \longleftarrow \beta$

This returns samples from the distribution of measurement outcomes for the circuit which agree with the predictions of quantum theory.

Handling negative quasiprobabilities

- When $W_\rho \ngeq 0$, we simulate using the procedure from Ref. [2] by sampling from $P(\alpha) := W_\rho(\alpha)/||W_\rho||_1$.
- The cost of classical simulation (number of samples to achieve a given probability of error) scales with $||W_\rho||_1^2$.

The Wigner function

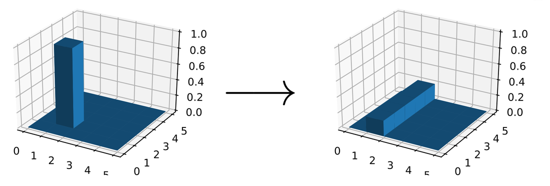
The Wigner function is a quasiprobability representation for quantum computation on odd-dimensional qudits.

- Points in phase space are identified with noncontextual assignments $\gamma : \mathbb{Z}_d^{2n} \to \mathbb{Z}_d$ — functions satisfying
$$T_aT_b = T_bT_a \implies \omega^{-\gamma(a)-\gamma(b)}T_aT_b = \omega^{-\gamma(a+b)}T_{a+b} \quad \forall a,b$$
- In odd dimensions, this is equivalent to $\gamma \in (\mathbb{Z}_d^{2n})^*$
- Phase space points come with phase space point operators
$$A^\gamma = \frac{1}{d^n}\sum_{b\in\mathbb{Z}_d^{2n}} \omega^{-\gamma(b)}T_b.$$
- The Wigner function [3] of state ρ is defined by the expansion coefficients in
$$\rho = \sum_\gamma W_\rho(\gamma)A^\gamma$$
- By orthogonality of A^γ operators, $W_\rho(\gamma) = \frac{1}{d^n}\mathrm{Tr}(\rho A^\gamma)$.

Phase space simulation method

When $W_\rho \geq 0$, it is a probability distribution over $\mathbb{Z}_d^{2n}$.

1. Sample γ according to W_ρ
2. Propagate A^γ through the circuit. Computational dynamics map points to distributions over phase space:



3. For Pauli measurements a, return $\gamma(a)$ as the outcome.

No-go results for multiqubit Wigner functions

No Wigner function on qubits can be defined which has the properties required to simulate quantum computations:

1. No Wigner function on even-dimensional qudits is Clifford covariant [4; Theorem 5]
2. No Wigner function on even-dimensional qudits preserves nonnegativity under Pauli measurements [4; Theorem 8]

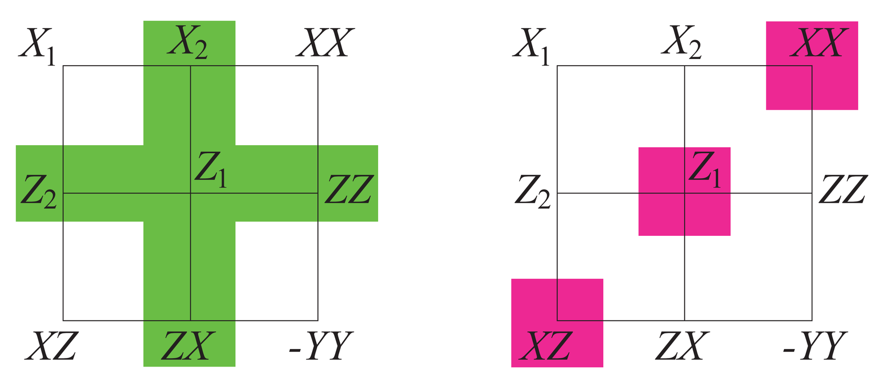
The CNC model circumvents these no-go results by relaxing the assumptions that define a Wigner function to get a new quasiprobability representation.

The CNC model

Consider a set of Pauli observables $\Omega \subset \mathbb{Z}_2^{2n}$ satisfying

1. Closure under inference: $\forall a, b \in \Omega$,
$$T_aT_b = T_bT_a \implies a + b \in \Omega$$
2. Noncontextuality: $\exists\gamma : \Omega \to \mathbb{Z}_d$ such that $\forall a, b \in \Omega$,
$$T_aT_b = T_bT_a \implies \omega^{-\gamma(a)-\gamma(b)}T_aT_b = \omega^{-\gamma(a+b)}T_{a+b}$$

The set of operators
$$\left\{ A_\Omega^\gamma := \frac{1}{2^n}\sum_{b\in\Omega}\omega^{-\gamma(b)}T_b \;\middle|\; \forall\Omega,\gamma \right\}$$
define a quasiprobability representation [5].



- For qubits, CNC sets cannot include state-independent proofs of contextuality like the Mermin square.
- In odd dimensions, there are no state-independent proofs of contextuality on Pauli observables [4]

   $\Rightarrow$ CNC phase space is characterized by sets that are closed under inference (see Theorem 2).

Multiqubit CNC model

The multiqubit CNC operators can be characterized:

Theorem 1 ([5]) *A set $\Omega \subset \mathbb{Z}_2^{2n}$ is CNC iff*
$$\Omega = \bigcup_{k=1}^{\xi}\langle a_k, I\rangle$$
where T_a pair-wise commute $\forall a \in I$, all T_{a_k} commute with all $T_a, a \in I$, and a_k satisfy $T_{a_i}T_{a_j} = (-1)^{1-\delta_{i,j}}T_{a_j}T_{a_i}$.

Equivalently, for any CNC operator A_Ω^γ,
$$A_\Omega^\gamma = g(A_{\tilde{\Omega}}^{\tilde{\gamma}} \otimes |\sigma\rangle\langle\sigma|)g^\dagger$$
where $g \in \mathcal{C}\ell$, $|\sigma\rangle$ is a stabilizer state, and
$$A_{\tilde{\Omega}}^{\tilde{\gamma}} = \frac{1}{2^n}\sum_{b\in\tilde{\Omega}}(-1)^{\tilde{\gamma}(b)}T_b, \text{ with } \{T_a, T_b\} = 2\delta_{a,b}\ \forall a,b \in \tilde{\Omega}.$$

Multiqudit CNC model

The multiqudit CNC phase space points can be classified:

Theorem 2 *For any number of qudits n of any odd-prime dimension d, a set $\Omega \subset \mathbb{Z}_d^{2n}$ is closed under inference iff*

(i) Ω is a subspace of $\mathbb{Z}_d^{2n}$, or

(ii) Ω has the form
$$\Omega = \bigcup_{k=1}^{\xi}\langle a_k, I\rangle$$
where T_a pair-wise commute $\forall a \in I$, all T_{a_k} commute with all $T_a, a \in I$, and the a_k satisfy $[T_{a_i}, T_{a_j}] \neq 0$.

- For the case $\Omega = \mathbb{Z}_d^{2n}$, the CNC operators A_Ω^γ are the Wigner function phase space point operators.

   $\Rightarrow$ CNC phase space contains the usual phase space
- When $\Omega = \bigcup_k\langle a_k, I\rangle$, there are nonlinear noncontextual assignments $\gamma : \Omega \to \mathbb{Z}_d$.
   - These look like the multiqubit CNC points.
   - In this case the operators A_Ω^γ are not Wigner function phase space points or convex mixtures of them.

Conclusion

In this work, we present the generalization of the CNC construction [5] to the setting of odd-prime-dimensional qudits. We provide a characterization of the CNC phase space in this setting, and we describe its relation to other models like the Wigner function [3] and the Λ polytope models. The phase space of this model contains the phase space of the Wigner function, but it also includes new phase space points which cannot be described as convex mixtures of Wigner function phase space points. We also show that all vertices of the Λ polytopes with coefficients of absolute value equal to one when expanded in the Pauli basis are CNC-type phase space point operators.

We introduce a classical simulation algorithm for quantum computation with magic states based on sampling from probability distributions over the CNC phase space. Since the CNC construction outperforms the Wigner function and stabilizer methods in terms of the volume of states that can be positively represented, this new method allows a broader class of magic state quantum circuits to be efficiently classically simulated.

For more details, see the preprint Ref. [6].

References

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